On small energy stabilization in the NLS with a trapping potential

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Abstract

We describe the asymptotic behavior of small energy solutions of an NLS with a trapping potential generalizing work of Soffer and Weinstein, and of Tsai and Yau. The novelty is that we allow generic spectra associated to the potential. This is yet a new application of the idea of interpreting the *nonlinear Fermi Golden Rule* as a consequence of the Hamiltonian structure.

1 Introduction

We consider the initial value problem

$$\mathrm{i}u_t = Hu + |u|^2 u,\ (t,x) \in \mathbb{R}^{1+3}, \quad u(0) = u_0 \tag{1.1}$$

where $H = -\Delta + V$. For $f, g : \mathbb{R}^3 \to \mathbb{C}$ we introduce the bilinear form

$$\langle f, g\rangle = \int_{\mathbb{R}^3} f(x)g(x)dx. \tag{1.2}$$

We assume the following.

(H1) $V \in \mathcal{S}(\mathbb{R}^3)$, where $\mathcal{S}(\mathbb{R}^3)$ is the space of Schwartz functions.

(H2) $\sigma_p(H) = \{e_1 < e_2 < e_3 \cdots < e_n < 0\}$. Here we assume that all the eigenvalues have multiplicity 1. 0 is neither an eigenvalue nor a resonance (that is, if $(-\Delta + V)u = 0$ with $u \in C^\infty$ and $|u(x)| \le C|x|^{-1}$ for a fixed C, then $u = 0$).

(H3) There is an $N \in \mathbb{N}$ with $N > |e_1|(\min\{e_i - e_j : i > j\})^{-1}$ s.t. if $\mu \in \mathbb{Z}^n$ satisfies $|\mu| \le 4N + 8$ and $\mathbf{e} := (e_1, \dots, e_n)$, then we have

$$\mu \cdot \mathbf{e} := \mu_1 e_1 + \cdots + \mu_n e_n = 0 \iff \mu = 0\ .$$

(H4) The following Fermi Golden Rule (FGR) holds: the expression

$$\sum_{L\in\Lambda} \langle \delta(H - L)\overline{G}_L(\zeta), G_L(\zeta)\rangle,$$

which is defined in the course of the paper (for $\Lambda \subset \mathbb{R}_+$ see (6.25) and for G_L see (6.44)) and which is always nonnegative, satisfies formula (6.47).

To each e_j we associate an eigenfunction ϕ_j. We choose them s.t. $\langle \phi_j, \overline{\phi}_k \rangle = \delta_{jk}$. Since we can, we also choose the ϕ_j to be all real valued. To each ϕ_j we associate nonlinear bound states.

Proposition 1.1 (Bound states)**.** *Fix $j \in \{1, \cdots, n\}$. Then $\exists a_0 > 0$ s.t. $\forall z \in B_{\mathbb{C}}(0, a_0)$, there is a unique $Q_{jz} \in \mathcal{S}(\mathbb{R}^3, \mathbb{C}) := \cap_{t \geq 0} \Sigma_t(\mathbb{R}^3, \mathbb{C})$ (where for the spaces Σ_t see Sect. 2.1), s.t.*

$$HQ_{jz} + |Q_{jz}|^2 Q_{jz} = E_{jz} Q_{jz} \quad , \quad Q_{jz} = z\phi_j + q_{jz}, \ \langle q_{jz}, \overline{\phi}_j \rangle = 0, \tag{1.3}$$

and s.t. we have for any $r \in \mathbb{N}$:

(1) $(q_{jz}, E_{jz}) \in C^\infty(B_{\mathbb{C}}(0, a_0), \Sigma_r \times \mathbb{R})$; we have $q_{jz} = z\widehat{q}_j(|z|^2)$, with $\widehat{q}_j(t^2) = t^2 \widetilde{q}_j(t^2)$, $\widetilde{q}_j(t) \in C^\infty((-a_0{}^2, a_0{}^2), \Sigma_r(\mathbb{R}^3, \mathbb{R}))$ and $E_{jz} = E_j(|z|^2)$ with $E_j(t) \in C^\infty((-a_0{}^2, a_0{}^2), \mathbb{R})$;

(2) $\exists \ C > 0$ s.t. $\|q_{jz}\|_{\Sigma_r} \leq C|z|^3$, $|E_{jz} - e_j| < C|z|^2$.

For the proof of Proposition 1.1 see Appendix A.

Definition 1.2. Let $b_0 > 0$ be sufficiently small so that for $z \in B_{\mathbb{C}^n}(0, b_0)$ and $z = (z_1 \cdots, z_n)$, Q_{jz_j} exists for all $j \in \{1, \cdots, n\}$. For such z and for D_{jI} and D_{jR} defined in Sect. 2.1, we set

$$\mathcal{H}_c[z] := \left\{ \eta \in L^2 : \ \mathrm{Re} \left\langle \mathrm{i}\, \overline{\eta}, D_{jR} Q_{jz_j} \right\rangle = \mathrm{Re} \left\langle \mathrm{i}\, \overline{\eta}, D_{jI} Q_{jz_j} \right\rangle = 0 \ \forall \ j \right\}. \tag{1.4}$$

In particular as an elementary consequence of (1.4) and Proposition 1.1 we have

$$\mathcal{H}_c[0] = \left\{ \eta \in L^2; \ \langle \overline{\eta}, \phi_j \rangle = 0 \text{ for all } j \right\}. \tag{1.5}$$

We denote by P_c the orthogonal projection of L^2 onto $\mathcal{H}_c[0]$.

A pair (p, q) is *admissible* when

$$2/p + 3/q = 3/2 \,, \quad 6 \geq q \geq 2 \,, \quad p \geq 2. \tag{1.6}$$

The following theorem is our main result.

Theorem 1.3. *Assume* (H1)–(H4)*. Then there exist $\epsilon_0 > 0$ and $C > 0$ such that for $\epsilon = \|u(0)\|_{H^1} < \epsilon_0$ the solution $u(t)$ of* (1.1) *can be written uniquely for all times as*

$$u(t) = \sum_{j=1}^n Q_{jz_j(t)} + \eta(t) \ \textit{with} \ \eta(t) \in \mathcal{H}_c[z(t)], \tag{1.7}$$

s.t. there exist a unique j_0, a $\rho_+ \in [0, \infty)^n$ with $\rho_{+j} = 0$ for $j \neq j_0$, s.t. $|\rho_+| \leq C\|u(0)\|_{H^1}$ and an $\eta_+ \in H^1$ with $\|\eta_+\|_{H^1} \leq C\|u(0)\|_{H^1}$, s.t.

$$\lim_{t \to +\infty} \|\eta(t, x) - e^{\mathrm{i}t\Delta} \eta_+(x)\|_{H^1_x} = 0 \quad , \quad \lim_{t \to +\infty} |z_j(t)| = \rho_{+j}. \tag{1.8}$$

Furthermore we have $\eta = \widetilde{\eta} + A(t, x)$ s.t. for all admissible pairs (p, q)

$$\begin{aligned} &\|z\|_{L^\infty_t(\mathbb{R}_+)} + \|\widetilde{\eta}\|_{L^p_t(\mathbb{R}_+, W^{1,q}_x)} \leq C\|u(0)\|_{H^1} \ , \\ &\|\dot{z}_j + \mathrm{i}e_j z_j\|_{L^\infty_t(\mathbb{R}_+)} \leq C\|u(0)\|^2_{H^1} \end{aligned} \tag{1.9}$$

and s.t. $A(t, \cdot) \in \Sigma_2$ for all $t \geq 0$ and

$$\lim_{t \to +\infty} \|A(t, \cdot)\|_{\Sigma_2} = 0. \tag{1.10}$$

As an interesting corollary to Theorem 1.3 we show rather simply that the excited states are *orbitally unstable.* We recall that $e^{-\mathrm{i}tE_{jz}}Q_{jz}$ is called *orbitally stable* in $H^1(\mathbb{R}^3)$ for (1.1) if

$$\forall\, \varepsilon > 0 \ \exists\, \delta > 0 \text{ s.t. } \|u_0 - Q_{jz}\|_{H^1(\mathbb{R}^3)} < \delta \Rightarrow \sup_{t\in\mathbb{R}} \inf_{\vartheta\in\mathbb{R}} \|u(t) - e^{\mathrm{i}\vartheta} e^{-\mathrm{i}tE_{jz}} Q_{jz}\|_{H^1(\mathbb{R}^3)} < \varepsilon \tag{1.11}$$

and is orbitally unstable if (1.11) does not hold. We prove what follows.

Theorem 1.4. *Assume* (H1)–(H4). *Then there exists $\epsilon_0 > 0$ such that if $j \geq 2$ and for $|z| < \epsilon_0$ the standing wave $e^{-\mathrm{i}tE_{jz}}Q_{jz}$ is orbitally unstable. Furthermore $e^{-\mathrm{i}tE_{1z}}Q_{1z}$ is orbitally stable.*

Notice that [29, 30, 31, 25, 12, 13, 18, 23] contain only very partial proofs of the instability of the 2nd excited state. Theorem 1.4 will be proved in Sect.7 and until then, and in particular in the sequel of this introduction, we will focus only on Theorem 1.3.

We recall that [17] proved Theorem 1.3, for $|u|^2u$ replaced by more general functions, in the case when H has one eigenvalue (for the NLS with an electromagnetic potential we refer to [21]). The case of two eigenvalues is discussed in the series [28, 29, 30] and in [25], under more stringent conditions on the initial data, which are such that $\|u_0\|_{H^{k,s}}$ is small for $k > 2$ and some s large enough in [25] and $\|u_0\|_{H^1\cap L^{2,s}}$ small for $s > 3$ in [28, 29, 30] . A crucial restriction in these papers is that $2e_2 > e_1$. They then prove versions of Theorem 1.3 involving also rates of decay of $|z(t)|$, of $\|\eta(t)\|_{L^\infty(\mathbb{R}^3)}$ and of $\|\eta(t)\|_{L^{2,s}(\mathbb{R}^3)}$ for appropriate $s > 0$.

The ideas used in proofs such as in [28, 29, 30, 25] appear very difficult to extend to operators with more than 2 eigenvalues, where only partial results like in [23] are known, and for initial data small only in H^1. On one hand, the Poincaré Dulac normal form argument in these papers seems not suited to discuss the higher order FGR needed when $2e_2 < e_1$. Furthermore, in these papers there is a subdivision of the evolution in distinct phases, which the solution enters in a somewhat irreversible fashion and which are considered one by one. This division in distinct phases might become unclear in cases when $u(t)$ oscillates from one phase to the other, as it is not unlikely to happen in the H^1 case, or when the passage from one phase to the other is very slow, as is certainly true in the H^1 case. Moreover, an increase in the number of eigenvalues of H increases also the number of distinct phases that need to be accounted for and the complexity of the argument. So, any hope of proving Theorem 1.3 should rely on an argument which yields the asymptotics in a single stroke and which does not distinguish distinct cases. This is what we do, see for example in the second part of Sect. 6. We did not check if our method yields the decay estimates of [28, 29, 30, 25] under more stringent conditions on u_0.

In the present paper we give a yet new application of the interpretation of the FGR in terms of the Hamiltonian structure of the equation. This interpretation was first introduced in [9] and was then applied in [1] to generalize the result of [26]. It was later applied to the problem of asymptotic stability of ground states of the NLS, first not allowing translation symmetries in [5], and then with translation in [6], see also [4].

The link between FGR and Hamiltonian structure rests in the fact that the latter yields algebraic identities between coefficients of different coordinates in the system (compare the r.h.s. in (6.13) with the second line in (6.27)). These allow to show that some other coefficients in the equations of the z_j's have a square power structure and have a fixed sign (in the case of the NLS), see Lemma 6.8. This then yields decay of the z_j's, except at most for one of the j's here. We refer to pp. 287–288 in [5] for the original intuition behind this approach to the FGR, which views the FGR as a simple consequence of Schwartz's Lemma on mixed derivatives, and which has made possible papers such as [9, 1, 5, 6, 4], as well as others. For other applications of this theory we refer to the references in [4], [10]. We refer also to [8], whose treatment of the FGR is similar to the one in this paper. Earlier treatments of FGR, are in [28, 29, 30, 25] and, still earlier, in [3, 26], but they seem to work only in relatively simple cases, because they run into trouble if the normal form argument requires more than very few steps. For more references and comments see [5].

As we will see below, the FGR can be seen relatively easily after one finds an appropriate effective Hamiltonian in the right system of coordinates. This coordinate system is obtained by a normal form argument. Right from the beginning though, it is crucial to choose the right ansatz and system of coordinates. For example, since H has eigenvalues, it would seem natural to split the NLS (1.1) into a system using the coordinates of the spectral decomposition of H, see (4.2). However this would not be a good choice for our nonlinear system. Following [17], it is better to pick as coordinates the z_j's of Prop.1.1, complementing them with an appropriate continuous coordinate. There is the natural ansatz (2.1) (the same used in [25]) which, following [17], can be used to obtain the continuous coordinate, here denoted η and introduced in Lemma 2.4.

Once we have coordinates (z, η) with $z = (z_1, ..., z_n)$, where z_1 is the ground state coordinate, z_j for $j > 1$ the excited states coordinates and η the radiation coordinate, Theor. 1.3 can be loosely paraphrased as follows:

$$\eta(t) \to 0 \text{ in } H^1_{loc} \text{ and } z_j(t) \to 0 \text{ except at most for one } j. \tag{1.12}$$

In particular, if $z(t) \to 0$ the solution $u(t)$ of (1.1) scatters like a solution of $\mathrm{i}\dot{u} = -\Delta u$ in H^1. Otherwise there is one j such that $u(t)$ scatters to a $e^{\mathrm{i}\vartheta(t)} Q_{z_{+j}}$, with $\vartheta(t)$ a phase term which we do not control here. We have convergence by scattering to a ground state if $j = 1$, and to an excited state if $j > 1$. The latter presumably occurs for the $u(t)$ whose trajectory is contained in an appropriate manifold, see [31, 2, 18].

It is not easy to see (1.12) in the initial coordinate system. So we need a Birkhoff normal form argument to identify an effective Hamiltonian, like in [1]. Unlike [1] and like in [5], the initial coordinates, while quite natural from the point of view of the NLS (1.1), are not Darboux coordinates for the natural symplectic form Ω in the problem, see (4.1). Hence before doing normal forms, we have first to implement the Darboux theorem to diagonalize the problem (of course the coordinates arising from the spectral decomposition of H, see (4.2), are Darboux coordinates, but as we wrote they are not suited for our nonlinear asymptotic analysis). So in this paper we need to perform a number of coordinate changes: first a Darboux Theorem and then normal form analysis. At the end of the process we get new coordinates $(z_1, ..., z_n, \eta)$ where the Hamiltonian is sufficiently simple that we can prove (1.12) relatively easily using the FGR (which tells us that all z_j's, except at most one, are damped) and a semilinear NLS for η which shows scattering of η because of linear dispersion. In the context of the theory developed in [1, 5] and other literature, the work in the last system of coordinates, that is all the material in Sect.6, is rather routine.

Having proved (1.12) for the last system of coordinates (z, η), the obvious question is why (1.12) should hold, as Theorem 1.3 is saying, also for the initial coordinates, which we now denote by (z', η'), to distinguish them from the final coordinates (z, η). Keeping in mind that all coordinate changes are small nonlinear perturbations of the identity, the only simple reason why this might happen is that different coordinates must be related in the form

$$\begin{aligned} &z'_1 = z_1 + O(z\eta) + O(\eta^2) + \sum_{i\neq j} O(z_i z_j), ..., z'_n = z_n + O(z\eta) + O(\eta^2) + \sum_{i\neq j} O(z_i z_j), \\ &\eta' = \eta + O(z\eta) + O(\eta^2) + \sum_{i\neq j} O(z_i z_j). \end{aligned} \tag{1.13}$$

This relation between any two systems of coordinates forbids relations like $z'_1 = z_1 + z_2^2$ etc. Indeed, with the latter relations it would not be true (except for the case $z(t) \to 0$) that (1.12) for (z, η) implies (1.12) for (z', η'). So our main strategy is to prove (1.12) for the final (z, η) with some relatively standard method using FGR and linear dispersion, and to be careful to implement only coordinate changes like in (1.13). This latter point is the novel problem we need to face in this paper. It is not obvious from the outset that (1.13) should hold.

As we wrote above, [17] suggests a very natural choice of functions z_j, based on Proposition 1.1 which can be completed in a system of independent coordinates. Loosely speaking, the z_j's have the problem that they are defined somewhat independently to each other. This shows up in the expansion of the Hamiltonian in Lemma 3.1, with a certain lack of decoupling inside the energy between distinct z_j's, see (3.9) and Remark 3.2. This leads in (3.3) (see the 2nd line) to terms whose elimination in a normal form argument would seem incompatible with coordinate changes satisfying (1.13). These bad terms of the Energy can be better seen in (4.45): they are the $l = 0$ terms in the 2nd line. Other additional bad terms arise in the course of the Darboux theorem transformation. Bad terms in the differential form Γ in (4.17) (used in the classical formula (4.40)) are those in I_1 in (4.22). Specifically they are the first term in the r.h.s. of (4.22). The r.h.s. of (4.28) is also filled with bad terms in the sense that they yield a coordinate change $\mathfrak{F}$ in Lemma 4.8 leading to more $l = 0$ terms in the 2nd line in (4.45). Specifically, they originate from the pullback $\mathfrak{F}^* \sum_{j=1}^n E(Q_{jz_j})$ of the 1st term in the r.h.s. of (3.3) (more bad terms seem to arise if we use Ω_0', see (4.8) rather than the slightly more complicated Ω_0, see (4.13), as local model of Ω). In a somewhat empirical fashion, for which we don't have a simple conceptual reason, a plain and simple computation shows that all the bad terms cancel out and that there are no $l = 0$ terms in (4.45). This is proved in the Cancelation Lemma 4.11, which is the main new ingredient in the paper. This lemma proves that the change of coordinates designed to diagonalize Ω, is also decoupling the discrete coordinates inside the Hamiltonian. From that point on, the structure (1.13) for the coordinate changes is automatic and the various steps of the proof of Theorem 1.3 are similar to arguments such as [4, 8] which have been repeated in a number of papers. So they are fairly standard, even though we are able to discuss them only in a rather technical way. We have to go into the details of the proof, rather than refer to the references, because of some technical novelties required by the fact that in general $z \not\to 0$, and what converges to 0 is instead the vector $\mathbf{Z}$ introduced in Def.2.2, whose components are products of distinct components of z.

In the second part of Sect. 6 the FGR and the asymptotics of the z_j's in the final coordinate system are rather simple to see in a single stroke. Furthermore, Theorem 6.1 is more or less the same of [5, 8].

One limitation in our present paper is that we do not generate examples of equations which satisfy Hypothesis (H4). Notice though that our result, for solutions only in H^1, is new even in the 2 eigenvalues case of [28, 29, 30, 25] where our FGR is the same. Still, we believe that (H4) holds for generic V. And even if it fails at one stage, this is not necessarily a problem: the strict positive sign in the FGR is only an obstruction at performing further the normal form argument, so if there is a 0, in principle it is enough to proceed with some further coordinate change until, after a finite number of steps, there will finally be a positive sign in the FGR, and so the stabilization will occur, just at a slower rate. And if the FGR is always 0, then maybe this is because the NLS has a special structure, see p.69 [26] for some thoughts.

Prop. 2.2 [1] proves validity in general of the FGR. Transposing here that proof would require replacing the cubic nonlinearity with a more general nonlinearity $\beta(|u|^2)u$. This seems rather simple to do because the cubic power is only used to simplify the discussion in Lemma 3.1. But it is not so clear how to offset here the absence of a meaningful mass term m^2u, which in [1] pp. 1444–1445, by choosing m generic, is used to move some appropriate spheres in phase space. Adding to the NLS a term m^2u would not change the spheres here.

We reiterate that Proposition 1.1 is valid for small $z_j \in \mathbb{C}$. As z_j increases there are interesting symmetry breaking bifurcation phenomena, see [20, 19] and therein and see also [11, 15, 24] and therein for the semiclassical NLS. Notice that Theorem 1.3 should allow to prove asymptotic breakdown of the beating motion in the case $\mu_\infty = 0$ in [15]. [14, 22] consider finite dimensional approximations of the solutions at energies close to the symmetry breaking point of [20] and prove the long time existence of interesting patterns for the full NLS. Unfortunately, it is beyond the scope of our analysis, and it remains an interesting open problem, to understand the eventual asymptotic behavior of the solutions in [14, 22].

2 Notation, coordinates and resonant sets

2.1 Notation

- We denote by $\mathbb{N} = \{1, 2, ...\}$ the set of natural numbers and set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.
- We denote $z = (z_1, \dots, z_n)$, $|z| := \sqrt{\sum_{j=1}^n |z_j|^2}$.
- Given a Banach space X, $v \in X$ and $\delta > 0$ we set $B_X(v, \delta) := \{x \in X \mid \|v - x\|_X < \delta\}$.
- Let A be an operator on $L^2(\mathbb{R}^3)$. Then $\sigma_p(A) \subset \mathbb{C}$ is the set of eignvalues of A and $\sigma_e(A) \subset \mathbb{C}$ is the essential spectrum of A.
- For $\mathbb{K} = \mathbb{R}, \mathbb{C}$ we denote by $\Sigma_r = \Sigma_r(\mathbb{R}^3, \mathbb{K})$ for $r \in \mathbb{N} \cup \{0\}$ the Banach spaces defined by the completion of $C_c(\mathbb{R}^3, \mathbb{K})$ by the norms
$$\|u\|_{\Sigma_r}^2 := \sum_{|\alpha| \le r} (\|x^\alpha u\|_{L^2(\mathbb{R}^3)}^2 + \|\partial_x^\alpha u\|_{L^2(\mathbb{R}^3, \mathbb{K})}^2).$$
For $m < 0$ we consider the topological dual $\Sigma_m = (\Sigma_{-m})'$. Notice, see [6], that the spaces Σ_r can be equivalently defined using for $r \in \mathbb{R}$ the norm $\|u\|_{\Sigma_r} := \|(1 - \Delta + |x|^2)^{\frac{r}{2}} u\|_{L^2}$.
- $\mathcal{S}(\mathbb{R}^3) = \cap_{m \ge 0} \Sigma_m$ is the space of Schwartz functions; $\mathcal{S}'(\mathbb{R}^3) = \cup_{m \le 0} \Sigma_m$ is the space of tempered distributions.
- We set $z_j = z_{jR} + \mathrm{i} z_{jI}$ for $z_{jR}, z_{jI} \in \mathbb{R}$.
- For $f : \mathbb{C}^n \to \mathbb{C}$ set $D_{jR} f(z) := \frac{\partial}{\partial z_{jR}} f(z)$, $D_{jI} f(z) := \frac{\partial}{\partial z_{jI}} f(z)$.
- We set $\partial_l := \partial_{z_l}$ and $\partial_{\bar{l}} := \partial_{\bar{z}_l}$. Here as customary $\partial_{z_l} = \frac{1}{2}(D_{lR} - \mathrm{i} D_{lI})$ and $\partial_{\bar{z}_l} = \frac{1}{2}(D_{lR} + \mathrm{i} D_{lI})$.
- Occasionally we use a single index $\ell = j, \bar{j}$. To define $\bar{\ell}$ we use the convention $\bar{\bar{j}} = j$. We will also write $z_{\bar{j}} = \bar{z}_j$.
- We will consider vectors $z = (z_1, ..., z_n) \in \mathbb{C}^n$ and for vectors $\mu, \nu \in (\mathbb{N} \cup \{0\})^n$ we set $z^\mu \bar{z}^\nu := z_1^{\mu_1} ... z_n^{\mu_n} \bar{z}_1^{\nu_1} ... \bar{z}_n^{\nu_n}$. We will set $|\mu| = \sum_j \mu_j$.
- We have $dz_j = dz_{jR} + \mathrm{i} dz_{jI}$, $d\bar{z}_j = dz_{jR} - \mathrm{i} dz_{jI}$.
- We consider the vector $\mathbf{e} = (e_1, ..., e_n)$ whose entries are the eigenvalues of H.
- P_c is the orthogonal projection of L^2 onto $\mathcal{H}_c[0]$.
- Given two Banach spaces X and Y we denote by $B(X, Y)$ the space of bounded linear operators $X \to Y$ with the norm of the uniform operator topology.

2.2 Coordinates

The first thing we need is an ansatz. This is provided by the following lemma.

Lemma 2.1. *There exists $c_0 > 0$ s.t. there exists a $C > 0$ s.t. for all $u \in H^1$ with $\|u\|_{H^1} < c_0$, there exists a unique pair $(z, \Theta) \in \mathbb{C}^n \times (H^1 \cap \mathcal{H}_c[z])$ s.t.*

$$u = \sum_{j=1}^{n} Q_{jz_j} + \Theta \text{ with } |z| + \|\Theta\|_{H^1} \le C\|u\|_{H^1}. \tag{2.1}$$

Finally, the map $u \to (z, \Theta)$ is $C^\infty(B_{H^1}(0, c_0), \mathbb{C}^n \times H^1)$ and satisfies the gauge property

$$z(e^{i\vartheta}u) = e^{i\vartheta}z(u) \text{ and } \Theta(e^{i\vartheta}u) = e^{i\vartheta}\Theta(u) \ . \tag{2.2}$$

Proof. We consider the functions

$$F_{jA}(u, z) := \mathrm{Re}\langle u - \sum_{l=1}^{n} Q_{lz_l}, \mathrm{i}\, \overline{D_{jA}Q_{jz_j}}\rangle \text{ for } A = R, I.$$

We have $F_{jR}(0, 0) = F_{jI}(0, 0) = 0$. These functions are smooth in $L^2 \times B_{\mathbb{C}^n}(0, b_0)$ for the b_0 in Def. 1.2. We have $F_{jR}(0, z) = \mathrm{Im}\, z_j + O(z^3)$ and $F_{jI}(0, z) = \mathrm{Re}\, z_j + O(z^3)$ by Proposition 1.1. By the implicit function theorem there is a map $u \to z$ which is $C^\infty(B_{L^2}(0, c_0), \mathbb{C}^n)$ for a $c_0 > 0$ sufficiently small. Set $\Theta := u - \sum_{j=1}^{n} Q_{jz_j}$. Then $\Theta \in C^\infty(B_{H^1}(0, c_0), H^1)$. The inequalities follow from $|z(u)| \le C\|u\|_{H^1}$ which follows from $z \in C^1$ and $z(0) = 0$. Formula (2.2) follows from

$$e^{i\vartheta}u = \sum_{j=1}^{n} e^{i\vartheta}Q_{jz_j} + e^{i\vartheta}\Theta = \sum_{j=1}^{n} Q_{je^{i\vartheta}z_j} + e^{i\vartheta}\Theta$$

and from the fact that $\Theta \in \mathcal{H}_c[z]$ implies $e^{i\vartheta}\Theta \in \mathcal{H}_c[z']$ where $z' = e^{i\vartheta}z$. This last fact is elementary. Indeed, setting only for this proof $z_j = x_j + \mathrm{i}y_j$ and $z'_j = x'_j + \mathrm{i}y'_j$, we have

$$\mathrm{Re}\left\langle \mathrm{i}\overline{e^{i\vartheta}\Theta}, \partial_{x'_j}Q_{jz'_j}\right\rangle = \partial_{x'_j}x_j\, \mathrm{Re}\left\langle \mathrm{i}\overline{e^{i\vartheta}\Theta}, e^{i\vartheta}\partial_{x_j}Q_{jz_j}\right\rangle + \partial_{x'_j}y_j\, \mathrm{Re}\left\langle \mathrm{i}\overline{e^{i\vartheta}\Theta}, e^{i\vartheta}\partial_{y_j}Q_{jz_j}\right\rangle = 0$$

if $\Theta \in \mathcal{H}_c[z]$. Similarly, $\mathrm{Re}\left\langle \mathrm{i}\overline{e^{i\vartheta}\Theta}, \partial_{y'_j}Q_{jz'_j}\right\rangle = 0$. Hence $\Theta \in \mathcal{H}_c[z]$ implies $e^{i\vartheta}\Theta \in \mathcal{H}_c[e^{i\vartheta}z]$.

□

Definition 2.2. Given $z \in \mathbb{C}^n$, we denote by $\widehat{Z}$ the vector with entries $(z_i\overline{z}_j)$ with $i, j \in [1, n]$ ordered in lexicographic order. We denote by $\mathbf{Z}$ the vector with entries $(z_i\overline{z}_j)$ with $i, j \in [1, n]$ ordered in lexicographic order but only with pairs of indexes with $i \neq j$. Here $\mathbf{Z} \in L$ with L the subspace of $\mathbb{C}^{n_0} = \{(a_{i,j})_{i,j=1,\dots,n} : i \neq j\}$ where $n_0 = n(n-1)$, with $(a_{i,j}) \in L$ iff $a_{i,j} = \overline{a}_{j,i}$ for all i, j. For a multi index $\mathbf{m} = \{m_{ij} \in \mathbb{N}_0 : i \neq j\}$ we set $\mathbf{Z}^{\mathbf{m}} = \prod(z_i\overline{z}_j)^{m_{ij}}$ and $|\mathbf{m}| := \sum_{i,j} m_{ij}$.

We need a system of independent coordinates, which the (z, Θ) in (2.1) are not. The following lemma is used to complete the z with a continuous coordinate.

Lemma 2.3. *There exists $d_0 > 0$ such that for any $z \in \mathbb{C}$ with $|z| < d_0$ there exists a $\mathbb{R}$–linear operator $R[z] : \mathcal{H}[0] \to \mathcal{H}_c[z]$ such that $P_c|_{\mathcal{H}_c[z]} = R[z]^{-1}$, with P_c the orthogonal projection of L^2 onto $\mathcal{H}_c[0]$, see Def. 1.2. Furthermore, for $|z| < d_0$ and $\eta \in \mathcal{H}_c[0]$, we have the following properties.*

(1) $R[z] \in C^\infty(B_{\mathbb{C}^n}(0, d_0), B(H^1, H^1))$, with $B(H^1, H^1)$ the Banach space of $\mathbb{R}$–linear bounded operators from H^1 into itself.

(2) For any $r>0$, we have $\|(R[z]-1)\eta\|_{\Sigma_r} \le c_r|z|^2\|\eta\|_{\Sigma_{-r}}$ for a fixed c_r.

(3) We have the covariance property $R[e^{i\vartheta}z] = e^{i\vartheta}R[z]e^{-i\vartheta}$.

(4) We have, summing on repeated indexes,

$$R[z]\eta = \eta + (\alpha_j[z]\eta)\phi_j \text{ with } \alpha_j[z]\eta = \langle B_j(z),\eta\rangle + \langle C_j(z),\overline{\eta}\rangle \tag{2.3}$$

where $B_j(z) = \widehat{B}_j(\widehat{Z})$ and $C_j(z) = z_i z_\ell \widehat{C}_{i\ell j}(\widehat{Z})$, for $\widehat{B}$ and $\widehat{C}_{i\ell j}$ smooth and the $\widehat{Z}$ of Def. 2.2.

(5) We have for $r\in\mathbb{R}$ with $\boldsymbol{Z}$ as in Def. 2.2

$$\|B_j(z) + \partial_{\overline{z}_j}\overline{q}_{jz_j}\|_{\Sigma_r} + \|C_j(z) - \partial_{\overline{z}_j}q_{jz_j}\|_{\Sigma_r} \le c_r|\boldsymbol{Z}|^2. \tag{2.4}$$

Proof. Summing over repeated indexes, we search for a map $R[z]: L^2 \to \mathcal{H}_c[z]$ of the form

$$R[z]f = f + (\alpha_j[z]f)\phi_j \text{ with } \alpha_j[z]f = \langle B'_j(z), f\rangle + \langle C_j(z),\overline{f}\rangle$$

such that $R[z]f \in \mathcal{H}_c[z]\ \forall f\in L^2$. The latter condition can be expressed as

$$\mathrm{Re}\left\langle \overline{f}, \mathrm{i}D_{lA}Q_{lz_l} + \langle\phi_j, \mathrm{i}D_{lA}Q_{lz_l}\rangle\overline{B}'_j - \langle\phi_j,\mathrm{i}D_{lA}\overline{Q}_{lz_l}\rangle C_j\right\rangle = 0 \text{ for all } f\in L^2.$$

This and the following equalities

$$\begin{aligned}
&\langle\phi_j, \mathrm{i}D_{lR}Q_{lz_l}\rangle = \mathrm{i}\delta_{jl} + \langle\phi_j,\mathrm{i}D_{lR}q_{lz_l}\rangle\ , \quad \langle\phi_j,\mathrm{i}D_{lI}Q_{lz_l}\rangle = -\delta_{jl} + \langle\phi_j,\mathrm{i}D_{lI}q_{lz_l}\rangle,\\
&\langle\phi_j, \mathrm{i}D_{lR}\overline{Q}_{lz_l}\rangle = \mathrm{i}\delta_{jl} + \langle\phi_j,\mathrm{i}D_{lR}\overline{q}_{lz_l}\rangle\ , \quad \langle\phi_j,\mathrm{i}D_{lI}\overline{Q}_{lz_l}\rangle = \delta_{jl} + \langle\phi_j,\mathrm{i}D_{lI}\overline{q}_{lz_l}\rangle,
\end{aligned}$$

yield the equalities

$$\begin{aligned}
&D_{lR}Q_{lz_l} + (\delta_{jl} + \langle\phi_j, D_{lR}q_{lz_l}\rangle)\overline{B}'_j - (\delta_{jl} + \langle\phi_j, D_{lR}\overline{q}_{lz_l}\rangle)C_j = 0,\\
&\mathrm{i}D_{lI}Q_{lz_l} + (-\delta_{jl} + \mathrm{i}\langle\phi_j, D_{lI}q_{lz_l}\rangle)\overline{B}'_j - (\delta_{jl} + \mathrm{i}\langle\phi_j, D_{lI}\overline{q}_{lz_l}\rangle)C_j = 0.
\end{aligned}$$

They can be rewritten as

$$\begin{aligned}
&\phi_l + \partial_l q_{lz_l} + (\delta_{jl} + \mathrm{i}\langle\phi_j,\partial_l q_{lz_l}\rangle)\overline{B}'_j - \langle\phi_j,\partial_l\overline{q}_{lz_l}\rangle C_j = 0,\\
&\partial_{\overline{l}} q_{lz_l} + \langle\phi_j,\partial_{\overline{l}} q_{lz_l}\rangle\overline{B}'_j - (\delta_{jl} + \langle\phi_j,\partial_{\overline{l}}\overline{q}_{lz_l}\rangle)C_j = 0.
\end{aligned} \tag{2.5}$$

For $z^2 = \{z_j^2\delta_{ij}\}$ and $\overline{z}^2 = \{\overline{z}_j^2\delta_{ij}\}$ two $n\times n$ matrices, the solution of this system is of the form

$$\begin{pmatrix}\overline{B}'\\ C\end{pmatrix} = \sum_{m=0}^{\infty}(-1)^m\begin{pmatrix}\mathbf{A}_1 & \overline{z}^2\mathbf{A}_2\\ z^2\mathbf{A}_3 & \mathbf{A}_4\end{pmatrix}^m\begin{pmatrix}u_1\\ z^2u_2\end{pmatrix} \tag{2.6}$$

where $\mathbf{A}_l = \mathbf{A}_l(|z_1|^2,...,|z_n|^2)$ are $n\times n$ matrices and $u_l = u_l(|z_1|^2,...,|z_n|^2)$ are $n\times 1$ matrices for $l=1$ (resp. $l=2$) with entries $\phi_j + \partial_j q_{jz_j}$ (resp. $\partial_{\overline{j}}q_{jz_j}$) as $j=1,...,n$. This yields the structure $\overline{B}'(z) = \widehat{B'}(\widehat{Z})$ and $C_j(z) = z_i z_\ell\widehat{C}_{i\ell j}(\widehat{Z})$.
Using $\langle\phi_j, q_{jz_j}\rangle = 0$, we can rewrite (2.5) in the form

$$\begin{aligned}
&\overline{B}'_l = -\phi_l - \partial_l q_{lz_l} - \sum_{j\neq l}(\mathrm{i}\langle\phi_j,\partial_l q_{lz_l}\rangle\overline{B}'_j - \langle\phi_j,\partial_l\overline{q}_{lz_l}\rangle C_j),\\
&C_l = \partial_{\overline{l}}q_{lz_l} + \sum_{j\neq l}(\langle\phi_j,\partial_{\overline{l}}q_{lz_l}\rangle\overline{B}'_j - \langle\phi_j,\partial_{\overline{l}}\overline{q}_{lz_l}\rangle)C_j.
\end{aligned} \tag{2.7}$$

By Proposition 1.1 this implies

$$\|\overline{B}_l' + \phi_l\|_{\Sigma_r} + \|C_l\|_{\Sigma_r} \le C|z_l|^2. \tag{2.8}$$

Reiterating this estimate, from (2.7) and for B_l defined by the following formula, we get

$$\begin{aligned}&\|\overbrace{\overline{B}_l' + \phi_l - \sum_{j\neq l} \mathrm{i}\langle \phi_j, \partial_l q_{l z_l}\rangle \phi_j}^{\overline{B}_l} + \partial_l q_{l z_l}\|_{\Sigma_r} \le C|\mathbf{Z}|^2 \\ &\|C_l - \partial_{\overline{l}} q_{l z_l}\|_{\Sigma_r} \le C|\mathbf{Z}|^2.\end{aligned}$$

This yields (2.4). Claim (3) follows by

$$\alpha_j[e^{\mathrm{i}\vartheta} z]\eta = e^{\mathrm{i}\vartheta}\alpha_j[z]e^{-\mathrm{i}\vartheta}\eta, \tag{2.9}$$

which in turn follows by claim (4). Indeed

$$\begin{aligned}\alpha_j[e^{\mathrm{i}\vartheta} z]\eta &= \langle \widehat{B}_j(\widehat{Z}), \eta\rangle + \langle e^{2\mathrm{i}\vartheta} z_i z_\ell \widehat{C}_{i\ell j}(\widehat{Z}), \overline{\eta}\rangle \\ =&e^{\mathrm{i}\vartheta}\langle \widehat{B}_j(\widehat{Z}), e^{-\mathrm{i}\vartheta}\eta\rangle + e^{\mathrm{i}\vartheta}\langle z_i z_\ell \widehat{C}_{i\ell j}(\widehat{Z}), \overline{e^{-\mathrm{i}\vartheta}\eta}\rangle = e^{\mathrm{i}\vartheta}\alpha_j[z]e^{-\mathrm{i}\vartheta}\eta.\end{aligned}$$

□

We are now able to define a system of coordinates near the origin in L^2.

Lemma 2.4. *For the $d_0 > 0$ of Lemma 2.3 the map $(z, \eta) \to u$ defined, by*

$$u = \sum_{j=1}^{n} Q_{j z_j} + R[z]\eta \text{ for } (z, \eta) \in B_{\mathbb{C}^n}(0, d_0) \times (H^1 \cap \mathcal{H}_c[0]) \tag{2.10}$$

is with values in H^1 and is C^∞. Furthermore, there is a $d_1 > 0$ such that for $(z, \eta) \in B_{\mathbb{C}^n}(0, d_1) \times (B_{H^1}(0, d_1) \cap \mathcal{H}_c[0])$ the above map is a diffeomorphism and

$$|z| + \|\eta\|_{H^1} \sim \|u\|_{H^1}. \tag{2.11}$$

Finally, we have the gauge properties $u(e^{\mathrm{i}\vartheta} z, e^{\mathrm{i}\vartheta}\eta) = e^{\mathrm{i}\vartheta} u(z, \eta)$ and

$$z(e^{\mathrm{i}\vartheta} u) = e^{\mathrm{i}\vartheta} z(u) \text{ and } \eta(e^{\mathrm{i}\vartheta} u) = e^{\mathrm{i}\vartheta}\eta(u)\ . \tag{2.12}$$

Proof. The smoothness follows from the smoothness in z in Proposition 1.1 and Lemma 2.3. Property $u(e^{\mathrm{i}\vartheta} z, e^{\mathrm{i}\vartheta}\eta) = e^{\mathrm{i}\vartheta} u(z, \eta)$ and its equivalent formula (2.12) follow from (2.2) and claim (3) in Lemma 2.3. Notice that $u = u(z, \eta)$ is the inverse of the smooth map $u \to (z, \Theta) \to (z, P_c\Theta)$. Formula (2.11) follows by the estimates in Prop. 1.1 and by claim (2) in Lemma 2.3.

□

2.3 Resonant sets

Definition 2.5. Consider the set of multiindexes $\mathbf{m}$ as in Def. 2.2 and for any $k \in \{1, ..., n\}$ the set

$$\begin{aligned}&\mathcal{M}_k(r) = \{\mathbf{m} : \sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}(e_i - e_j) - e_k < 0 \text{ and } |\mathbf{m}| \le r\} \\ &\mathcal{M}_0(r) = \{\mathbf{m} : \sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}(e_i - e_j) = 0 \text{ and } |\mathbf{m}| \le r\}.\end{aligned} \tag{2.13}$$

Set now

$$\begin{aligned}&M_k(r) = \{(\mu,\nu) \in \mathbb{N}_0^n \times \mathbb{N}_0^n : \exists \mathbf{m} \in \mathcal{M}_k(r) \text{ s.t. } z^\mu \overline{z}^\nu = \overline{z}_k \mathbf{Z}^\mathbf{m}\}, \\ &M(r) = \cup_{k=1}^n M_k(r) \quad \text{ and } M = M(2N+4)\end{aligned} \tag{2.14}$$

Lemma 2.6. *Assuming (H3) we have the following facts.*

(1) For $\mathbf{Z}^\mathbf{m} = z^\mu \overline{z}^\nu$, then $\mathbf{m} \in \mathcal{M}_0(2N+4)$ implies $\mu = \nu$. In particular $\mathbf{m} \in \mathcal{M}_0(2N+4)$ implies $\mathbf{Z}^\mathbf{m} = |z_1|^{2l_1}...|z_n|^{2l_n}$ for some $(l_1, ..., l_n) \in \mathbb{N}_0^n$.

(2) For $|\boldsymbol{m}| \le 2N+3$ and any j we have $\sum_{a,b}(e_a - e_b)m_{ab} - e_j \neq 0$.

Proof. First of all, if $\mu = \nu$ then $z^\mu \overline{z}^\nu = |z_1|^{2\mu_1}...|z_n|^{2\mu_n}$. So the first sentence in claim (1) implies the second sentence in claim (1). We have

$$\mathbf{Z}^\mathbf{m} = \prod_{i,l=1}^n (z_i \overline{z}_l)^{m_{il}} = \prod_{i=1}^n z_i^{\sum_{l=1}^n m_{il}} \overline{z}_i^{\sum_{l=1}^n m_{li}} = z^\mu \overline{z}^\nu.$$

The pair (μ, ν) satisfies $|\mu| = |\nu| \le 2N+4$ by

$$|\mu| = \sum_l \mu_l = \sum_{i,l} m_{il} = |\nu|.$$

We have $(\mu - \nu) \cdot \mathbf{e} = 0$ by $\mathbf{m} \in \mathcal{M}_0(2N+4)$ and

$$\sum_i \mu_i e_i - \sum_l \nu_l e_l = \sum_{i,l} m_{il}(e_i - e_l) = 0.$$

We conclude by (H3) that $\mu - \nu = 0$. This proves the 1st sentence of claim (1).
The proof of claim (2) is similar. Set

$$\mathbf{Z}^\mathbf{m} \overline{z}_j = \prod_{i,l=1}^n (z_i \overline{z}_l)^{m_{il}} \overline{z}_j = \prod_{i=1}^n z_i^{\sum_{l=1}^n m_{il}} \overline{z}_i^{\sum_{l=1}^n m_{li}} \overline{z}_j = z^\mu \overline{z}^\nu$$

We have

$$(\mu - \nu) \cdot \mathbf{e} = \sum_i \mu_i e_i - \sum_l \nu_l e_l = \sum_{i,l} m_{il}(e_i - e_l) - e_j.$$

We have

$$|\mu| = \sum_l \mu_l = \sum_{i,l} m_{il} = |\nu| - 1. \tag{2.15}$$

If $(\mu - \nu) \cdot \mathbf{e} = 0$ then by $|\mu - \nu| \le 4N+5$ and by (H3) we would have $\mu = \nu$, impossible by (2.15). □

Lemma 2.7. *We have the following facts.*

(1) Consider $\mathbf{m} = (m_{ij}) \in \mathbb{N}_0^{n_0}$ s.t. $\sum_{i<j} m_{ij} > N$ for $N > |e_1|(\min\{e_j - e_i : j > i\})^{-1}$, see (H3). Then for any eigenvalue e_k we have

$$\sum_{i<j} m_{ij}(e_i - e_j) - e_k < 0. \tag{2.16}$$

(2) Consider $\mathbf{m} \in \mathbb{N}_0^{n_0}$ *with* $|\mathbf{m}| \geq 2N+3$ *and the monomial* $z_j \mathbf{Z}^{\mathbf{m}}$*. Then* $\exists\, \mathbf{a}, \mathbf{b} \in \mathbb{N}_0^{n_0}$ *s.t.*

$$\begin{aligned} &\sum_{i<j} a_{ij} = N+1 = \sum_{i<j} b_{ij}, \\ &a_{ij} = b_{ij} = 0 \text{ for all } i > j \text{ and } a_{ij} + b_{ij} \leq m_{ij} + m_{ji} \text{ for all } (i,j) \end{aligned} \tag{2.17}$$

and moreover there are two indexes (k,l) *s.t.*

$$\sum_{i<j} a_{ij}(e_i - e_j) - e_k < 0 \text{ and } \sum_{i<j} b_{ij}(e_i - e_j) - e_l < 0 \tag{2.18}$$

and such that for $|z| \leq 1$

$$|z_j \mathbf{Z}^{\mathbf{m}}| \leq |z_j|\ |z_k \mathbf{Z}^{\mathbf{a}}|\ |z_l \mathbf{Z}^{\mathbf{b}}|. \tag{2.19}$$

(3) For $\mathbf{m}$ *with* $|\mathbf{m}| \geq 2N+3$ *there exist* (k,l)*,* $\mathbf{a} \in \mathcal{M}_k$ *and* $\mathbf{b} \in \mathcal{M}_l$ *s.t.* (2.19) *holds.*

Proof. (2.16) follows immediately from

$$\sum_{i<j} m_{ij}(e_i - e_j) - e_k \leq -\min\{e_j - e_i : j > i\} N - e_1 < 0,$$

where the latter inequality follows by the definition of N.

Given $\mathbf{a}, \mathbf{b} \in \mathbb{N}_0^{n_0}$ satisfying (2.17), by claim (1) they satisfy (2.18) for any pair of indexes (k,l). Consider now the monomial $z_j \mathbf{Z}^{\mathbf{m}}$. Since $|\mathbf{m}| \geq 2N+3$, there are vectors $\mathbf{c}, \mathbf{d} \in \mathbb{N}_0^{n_0}$ s.t. $|\mathbf{c}| = |\mathbf{d}| = N+1$ with $c_{ij} + d_{ij} \leq m_{ij}$ for all (i,j). Furthermore we have

$$z_j \mathbf{Z}^{\mathbf{m}} = z_j z^{\mu} \overline{z}^{\nu} \mathbf{Z}^{\mathbf{c}} \mathbf{Z}^{\mathbf{d}} \text{ with } |\mu| > 0 \text{ and } |\nu| > 0. \tag{2.20}$$

So, for z_k a factor of z^{μ} and $\overline{z}_l$ a factor of $\overline{z}^{\nu}$, and for

$$a_{ij} = \begin{cases} c_{ij} + c_{ji} \text{ for } i < j \\ \quad 0 \text{ for } i > j \end{cases}, \qquad b_{ij} = \begin{cases} d_{ij} + d_{ji} \text{ for } i < j \\ \quad 0 \text{ for } i > j \end{cases} \tag{2.21}$$

for $|z| \leq 1$ we have from (2.20)

$$|z_j \mathbf{Z}^{\mathbf{m}}| \leq |z_j|\ |z_k \mathbf{Z}^{\mathbf{c}}|\ |\ z_l \mathbf{Z}^{\mathbf{d}}| = |z_j|\ |z_k \mathbf{Z}^{\mathbf{a}}|\ |\ z_l \mathbf{Z}^{\mathbf{b}}|.$$

Furthermore, (2.17) is satisfied.
Since our $(\mathbf{a}, \mathbf{b})$ satisfy $\mathbf{a} \in \mathcal{M}_k$ and $\mathbf{b} \in \mathcal{M}_l$, claim (3) is a consequence of claim (2).

□

We end this section exploiting the notation introduced in claim (5) of Lemma 2.3 to introduce two classes of functions. First of all notice that the linear maps $\eta \to \langle \eta, \phi_j \rangle$ extend into bounded linear maps $\Sigma_r \to \mathbb{R}$ for any $r \in \mathbb{R}$. We set

$$\Sigma_r^c := \{\eta \in \Sigma_r :\ \langle \eta, \phi_j \rangle = 0,\ j = 1, \cdots, n\}. \tag{2.22}$$

The following two classes of functions will be used in the rest of the paper. Recall that in Def. 2.2 we introduced the space L with $\dim L = n(n-1)$. In Definitions 2.8–2.9 by $\mathbf{Z}$ we denote an auxiliary variable independent of z which takes values in L

Definition 2.8. Let $\mathfrak{B}$ be an open subset of a Banach space. We will say that $F(t, \mathfrak{b}, z, \mathbf{Z}, \eta) \in C^M(I \times \mathfrak{B} \times \mathcal{A}, \mathbb{R})$, with I a neighborhood of 0 in $\mathbb{R}$ and $\mathcal{A}$ a neighborhood of 0 in $\mathbb{C}^n \times L \times \Sigma^c_{-K}$ is $F = \mathcal{R}^{i,j}_{K,M}(t, \mathfrak{b}, z, \mathbf{Z}, \eta)$, if there exists a $C > 0$ and a smaller neighborhood $\mathcal{A}'$ of 0 s.t.

$$|F(t, \mathfrak{b}, z, \mathbf{Z}, \eta)| \le C(\|\eta\|_{\Sigma_{-K}} + |\mathbf{Z}|)^j(\|\eta\|_{\Sigma_{-K}} + |\mathbf{Z}| + |z|)^i \text{ in } I \times \mathfrak{B} \times \mathcal{A}'. \tag{2.23}$$

We will specify $F = \mathcal{R}^{i,j}_{K,M}(t, \mathfrak{b}, z, \mathbf{Z})$ if

$$|F(t, \mathfrak{b}, z, \mathbf{Z}, \eta)| \le C|\mathbf{Z}|^j|z|^i \tag{2.24}$$

and $F = \mathcal{R}^{i,j}_{K,M}(t, \mathfrak{b}, z, \eta)$ if

$$|F(t, \mathfrak{b}, z, \mathbf{Z}, \eta)| \le C\|\eta\|^j_{\Sigma_{-K}}(\|\eta\|_{\Sigma_{-K}} + |z|)^i. \tag{2.25}$$

We will omit t or $\mathfrak{b}$ if there is no dependence on such variables.

We write $F = \mathcal{R}^{i,j}_{K,\infty}$ if $F = \mathcal{R}^{i,j}_{K,m}$ for all $m \ge M$. We write $F = \mathcal{R}^{i,j}_{\infty,M}$ if for all $k \ge K$ the above F is the restriction of an $F(t, \mathfrak{b}, z, \eta) \in C^M(I \times \mathfrak{B} \times \mathcal{A}_k, \mathbb{R})$ with $\mathcal{A}_k$ a neighborhood of 0 in $\mathbb{C}^n \times L \times \Sigma^c_{-k}$ and which is $F = \mathcal{R}^{i,j}_{k,M}$. Finally we write $F = \mathcal{R}^{i,j}_{\infty,\infty}$ if $F = \mathcal{R}^{i,j}_{k,\infty}$ for all k.

Definition 2.9. We will say that an $T(t, \mathfrak{b}, z, \eta) \in C^M(I \times \mathfrak{B} \times \mathcal{A}, \Sigma_K(\mathbb{R}^3, \mathbb{C}))$, with the above notation, is $T = \mathbf{S}^{i,j}_{K,M}(t, \mathfrak{b}, z, \mathbf{Z}, \eta)$, if there exists a $C > 0$ and a smaller neighborhood $\mathcal{A}'$ of 0 s.t.

$$\|T(t, \mathfrak{b}, z, \mathbf{Z}, \eta)\|_{\Sigma_K} \le C(\|\eta\|_{\Sigma_{-K}} + |\mathbf{Z}|)^j(\|\eta\|_{\Sigma_{-K}} + |\mathbf{Z}| + |z|)^i \text{ in } I \times \mathfrak{B} \times \mathcal{A}'. \tag{2.26}$$

We use notations $\mathbf{S}^{i,j}_{K,M}(t, \mathfrak{b}, z, \mathbf{Z})$, $\mathbf{S}^{i,j}_{K,M}(t, \mathfrak{b}, z, \eta)$ etc. as above.

Notice that we have the elementary formulas

$$\mathcal{R}^{a,b}_{K,M}\mathbf{S}^{i,j}_{K,M} = \mathbf{S}^{i+a,j+b}_{K,M} \text{ and } \mathcal{R}^{a,b}_{K,M}\mathcal{R}^{i,j}_{K,M} = \mathcal{R}^{i+a,j+b}_{K,M}. \tag{2.27}$$

Remark 2.10. For functions $F(t, \mathfrak{b}, z, \eta)$ and $T(t, \mathfrak{b}, z, \eta)$ we write $F(t, \mathfrak{b}, z, \eta) = \mathcal{R}^{i,j}_{K,M}(t, \mathfrak{b}, z, \mathbf{Z}, \eta)$ and $T(t, \mathfrak{b}, z, \eta) = \mathbf{S}^{i,j}_{K,M}(t, \mathfrak{b}, z, \mathbf{Z}, \eta)$ when the equality holds restricting the variable $\mathbf{Z}$ to $\mathbf{Z} = (z_i\overline{z}_j)_{i,j=1,\dots,n}$ where $i \neq j$, for symbols satisfying Definitions 2.8–2.9.
Furthermore, later, when we write $\mathcal{R}^{i,j}_{K,M}$ and $\mathbf{S}^{i,j}_{K,M}$, we mean $\mathcal{R}^{i,j}_{K,M}(z, \mathbf{Z}, \eta)$ and $\mathbf{S}^{i,j}_{K,M}(z, \mathbf{Z}, \eta)$.
Notice that $F = \mathcal{R}^{i,j}_{K,M}(z, \mathbf{Z})$ or $T = \mathbf{S}^{i,j}_{K,M}(z, \mathbf{Z})$ do not mean independence of the variable η.

3 Invariants

Equation (1.1) admits the energy and mass invariants, defined as follows:

$$\begin{aligned} &E(u) := E_K(u) + E_P(u), \text{ where } E_K(u) := \langle Hu, \overline{u}\rangle \text{ and} \\ &E_P(u) = \frac{1}{2}\int_{\mathbb{R}^3} |u(x)|^4\, dx \ ; \quad Q(u) := \langle u, \overline{u}\rangle. \end{aligned} \tag{3.1}$$

We have $E \in C^\infty(H^1(\mathbb{R}^3, \mathbb{C}), \mathbb{R})$ and $Q \in C^\infty(L^2(\mathbb{R}^3, \mathbb{C}), \mathbb{R})$. We denote by dE the Frechét derivative of E. We define $\nabla E \in C^\infty(H^1(\mathbb{R}^3, \mathbb{C}), H^{-1}(\mathbb{R}^3, \mathbb{C}))$ by $dEX = \mathrm{Re}\langle \nabla E, \overline{X}\rangle$ for any $X \in H^1$. We define also $\nabla_u E$ and $\nabla_{\overline{u}} E$ by

$$dEX = \langle \nabla_u E, X\rangle + \langle \nabla_{\overline{u}} E, \overline{X}\rangle \text{ that is } \nabla_u E = 2^{-1}\overline{\nabla E} \text{ and } \nabla_{\overline{u}} E = 2^{-1}\nabla E.$$

Notice that $\nabla E = 2Hu + 2|u|^2 u$. Then equation (1.1) can be interpreted as

$$\mathrm{i}\dot{u} = \nabla_{\overline{u}} E(u). \tag{3.2}$$

Lemma 3.1. *Consider the coordinates $(z, \eta) \to u$ in Lemma 2.4. Then there exists some functions as in Definitions 2.8 and 2.9 s.t. for $(z, \eta) \in B_{\mathbb{C}^n}(0, d_0) \times (B_{H^1}(0, d_0) \cap \mathcal{H}_c[0])$ we have for any preassigned $r_0 \in \mathbb{N}$ the expansion (where c.c. means complex conjugate)*

$$\begin{aligned} E(u) &= \sum_{j=1}^{n} E(Q_{jz_j}) + \langle H\eta, \overline{\eta}\rangle + \mathcal{R}^{1,2}_{r_0,\infty}(z,\eta) \\ &+ \sum_{j\neq k} [E_{jz_j}(\mathrm{Re}\langle q_{jz_j}, \overline{z}_k \phi_k\rangle + \mathrm{Re}\langle q_{kz_k}, \overline{z}_j \phi_j\rangle) + \mathrm{Re}\langle |Q_{kz_k}|^2 Q_{kz_k}, \overline{z}_j \phi_j\rangle] \\ &+ \mathcal{R}^{0,2N+5}_{r_0,\infty}(z, \boldsymbol{Z}) + \sum_{j=1}^{n} \sum_{l=0}^{2N+3} \sum_{|\boldsymbol{m}|=l+1} \boldsymbol{Z}^{\boldsymbol{m}} a_{j\boldsymbol{m}}(|z_j|^2) \\ &+ \mathrm{Re}\langle \boldsymbol{S}^{0,2N+4}_{r_0,\infty}(z,\boldsymbol{Z}), \overline{\eta}\rangle + \sum_{j,k=1}^{n} \sum_{l=0}^{2N+3} \sum_{|\boldsymbol{m}|=l} (\overline{z}_j \boldsymbol{Z}^{\boldsymbol{m}} \langle G_{jk\boldsymbol{m}}(|z_k|^2), \eta\rangle + c.c.) + \\ &\sum_{i+j=2} \sum_{|\boldsymbol{m}|\le 1} \boldsymbol{Z}^{\boldsymbol{m}} \langle G_{2\boldsymbol{m}ij}(z), \eta^i \overline{\eta}^j\rangle + \sum_{d+c=3} \sum_{i+j=d} \langle G_{dij}(z), \eta^i \overline{\eta}^j\rangle \mathcal{R}^{0,c}_{r_0,\infty}(z,\eta) + E_P(\eta) \text{ where:} \end{aligned} \tag{3.3}$$

- $(a_{j\boldsymbol{m}}, G_{jk\boldsymbol{m}}) \in C^\infty(B_{\mathbb{R}}(0, d_0), \mathbb{C} \times \Sigma_{r_0}(\mathbb{R}^3, \mathbb{C}))$;
- $(G_{2\boldsymbol{m}ij}, G_{dij}) \in C^\infty(B_{\mathbb{C}^n}(0, d_0), \Sigma_{r_0}(\mathbb{R}^3, \mathbb{C}) \times \Sigma_{r_0}(\mathbb{R}^3, \mathbb{C}))$;
- *For $|\boldsymbol{m}| = 0$, where in particular we have $G_{2\boldsymbol{0}ij}(0) = 0$, we have*

$$\sum_{i+j=2} \langle G_{2\boldsymbol{0}ij}(z), \eta^i \overline{\eta}^j\rangle = \sum_{j=1}^{n} \langle |Q_{jz_j}|^2 \eta, \overline{\eta}\rangle + 2 \sum_{j=1}^{n} \mathrm{Re}\langle Q_{jz_j} \mathrm{Re}(Q_{jz_j} \overline{\eta}), \overline{\eta}\rangle; \tag{3.4}$$

- $\mathcal{R}^{1,2}_{r_0,\infty}(e^{\mathrm{i}\vartheta} z, e^{\mathrm{i}\vartheta} \eta) = \mathcal{R}^{1,2}_{r_0,\infty}(z,\eta)$ *for all $\vartheta \in \mathbb{R}$ for the 3rd term in the r.h.s. of* (3.3).

Remark 3.2. In formula (3.3) the terms of the second line could potentially derail our proof. They appear in (3.7)–(3.9). Similarly problematic is the first term in the r.h.s. in (4.18) later. All these terms are tied up. Indeed, in Lemma 4.11 we will show that in a system of coordinates better suited to search an effective Hamiltonian the problematic terms in the expansion of E cancel out.

In the proof of Lemma 3.1 we use the following lemma.

Lemma 3.3. *For we have for $j \neq k$ and $\delta E_{jz_j} := E_{jz_j} - e_j$*

$$E_{jz_j} \langle q_{kz_k}, \phi_j\rangle + \langle |Q_{kz_k}|^2 Q_{kz_k}, \phi_j\rangle = E_{kz_k} \langle q_{kz_k}, \phi_j\rangle + \delta E_{jz_j} \langle q_{kz_k}, \phi_j\rangle. \tag{3.5}$$

Proof. We apply $\langle \ , \phi_j\rangle$ to

$$H q_{kz_k} + |Q_{kz_k}|^2 Q_{kz_k} = z_k \delta E_{kz_k} \phi_k + E_{kz_k} q_{kz_k}$$

to get the following equality which from $e_j = E_{jz_j} - \delta E_{jz_j}$ yields (3.5):

$$e_j \langle q_{kz_k}, \phi_j\rangle + \langle |Q_{kz_k}|^2 Q_{kz_k}, \phi_j\rangle = E_{kz_k} \langle q_{kz_k}, \phi_j\rangle \ .$$

$\square$

Proof of Lemma 3.1. First of all, we have the Taylor expansion

$$E(u) = E(\sum_{j=1}^{n} Q_{jz_j}) + \mathrm{Re}\langle \nabla E(\sum_{j=1}^{n} Q_{jz_j}), \overline{R[z]\eta}\rangle \tag{3.6}$$
$$+ 2^{-1}\,\mathrm{Re}\langle \nabla^2 E(\sum_{j=1}^{n} Q_{jz_j}) R[z]\eta, \overline{R[z]\eta}\rangle + E_3(\eta) \text{ with } E_3(\eta) :=$$
$$\int_0^1 (1-t)\,\mathrm{Re}\langle \big[\nabla^2 E_P(\sum_{j=1}^{n} Q_{jz_j} + tR[z]\eta) - \nabla^2 E_P(\sum_{j=1}^{n} Q_{jz_j})\big] R[z]\eta, \overline{R[z]\eta}\rangle dt$$

Step 1. We consider the expansion of the 1st term in the r.h.s of (3.6). We have

$$|\sum Q_{jz_j}|^4 = \sum |Q_{jz_j}|^4 + 4\sum_{j\neq k} |Q_{jz_j}|^2\,\mathrm{Re}(Q_{jz_j}\overline{Q}_{kz_k})$$
$$+ 2\sum_{j<k} |Q_{jz_j}|^2 |Q_{kz_k}|^2 + \sum_{j\neq k,\ j'\neq k'} \mathrm{Re}(Q_{jz_j}\overline{Q}_{kz_k})\,\mathrm{Re}(Q_{j'z_{j'}}\overline{Q}_{k'z_{k'}}) + 4\sum_{k<l,\ j\neq k,l} |Q_{jz_j}|^2 \mathrm{Re}(Q_{kz_k}\overline{Q}_{lz_l}).$$

All terms are invariant by change of variable $z \rightsquigarrow e^{\mathrm{i}\vartheta} z$. The 2nd line is $O(|\mathbf{Z}|^2)$. We conclude that

$$E(\sum_{j=1,\dots,n} Q_{jz_j}) = \sum_{j,k} \langle HQ_{jz_j}, \overline{Q_{kz_k}}\rangle + \frac{1}{2}\int |\sum_{j=1,\dots,n} Q_{jz_j}|^4 = \sum_{j=1,\dots,n} E(Q_{jz_j}) + R_1$$
$$+ \sum_{j\neq k} [\mathrm{Re}\langle HQ_{jz_j}, \overline{Q_{kz_k}}\rangle + 2\,\mathrm{Re}\langle |Q_{jz_j}|^2 Q_{jz_j}, \overline{Q}_{kz_k}\rangle], \tag{3.7}$$

where

$$R_1 := \sum_{j<k} \int |Q_{jz_j}|^2 |Q_{kz_k}|^2 + \frac{1}{2}\sum_{j\neq k,\ j'\neq k'} \int \mathrm{Re}(Q_{jz_j}\overline{Q}_{kz_k})\,\mathrm{Re}(Q_{j'z_{j'}}\overline{Q}_{k'z_{k'}})$$
$$+ 2\sum_{k<l,\ j\neq k,l} \int |Q_{jz_j}|^2 \mathrm{Re}(Q_{kz_k}\overline{Q}_{lz_l}) = O(|\mathbf{Z}|^2).$$

By Prop. 1.1 and by (3.5) the summation in the last line of (3.7) equals

$$\sum_{j\neq k} [E_{jz_j}\,\mathrm{Re}\langle Q_{jz_j}, \overline{Q_{kz_k}}\rangle + \mathrm{Re}\langle |Q_{jz_j}|^2 Q_{jz_j}, \overline{Q}_{kz_k}\rangle]$$
$$= \sum_{j\neq k} [E_{jz_j}(\mathrm{Re}\langle q_{jz_j}, \overline{z}_k\phi_k\rangle + \mathrm{Re}\langle q_{kz_k}, \overline{z}_j\phi_j\rangle) + \mathrm{Re}\langle |Q_{kz_k}|^2 Q_{kz_k}, \overline{z}_j\phi_j\rangle] + R_2, \tag{3.8}$$

where

$$R_2 := \sum_{j\neq k} E_{jz_j}\,\mathrm{Re}\left\langle q_{jz_j}, \overline{q_{kz_k}}\right\rangle + \mathrm{Re}\left\langle |Q_{kz_k}|^2 Q_{kz_k}, \overline{q_{jz_j}}\right\rangle = O(|\mathbf{Z}|^2).$$

The summation in (3.8) is $O(|z|^2\ |\mathbf{Z}|)$ and not of the form $O(|\mathbf{Z}|^2)$. Indeed, in the particular case when $z_k = \rho_k$ and $z_j = \rho_j$ are real numbers, we have what follows, which is not $O(\rho_k^2\rho_j^2)$,

$$E_{jz_j}\,\mathrm{Re}\langle q_{jz_j}, \overline{z}_k\phi_k\rangle + E_{kz_k}\,\mathrm{Re}\langle q_{kz_k}, \overline{z}_j\phi_j\rangle + \mathrm{Re}\left\langle |Q_{kz_k}|^2 Q_{kz_k}, \overline{z}_j\phi_j\right\rangle$$
$$= \rho_k\rho_j [E_{j\rho_j}\rho_j^2\langle \widetilde{q}_j(\rho_j^2), \phi_k\rangle + E_{k\rho_k}\rho_k^2\langle \widetilde{q}_k(\rho_k), \phi_j\rangle + \rho_k^2\left\langle (\phi_k + \hat{q}_k(\rho_k^2))^3, \phi_j\right\rangle] . \tag{3.9}$$

Finally, we observe that the $R_1 + R_2 = O(|\mathbf{Z}|^2)$ summed up together yield the 3rd line of (3.3). Indeed, since $R_1 + R_2$ is gauge invariant, by Lemma B.3 in Appendix B we have

$$R_1 + R_2 = \sum_{j=1}^{n} \sum_{l=1}^{2N+3} \sum_{|\mathbf{m}|=l+1} \mathbf{Z}^{\mathbf{m}} b_{j\mathbf{m}}(|z_j|^2) + O(|\mathbf{Z}|^{2N+5}). \tag{3.10}$$

with $O(|\mathbf{Z}|^{2N+5})$ smooth in z, independent of η and gauge invariant.

We have discussed the contribution to (3.3) of the 1st term in the expansion (3.6). Now we consider the other terms in (3.6).

Step 2. We consider the expansion of the 2nd term in the r.h.s of (3.6).

By $\mathrm{Re}\langle \nabla E(Q_{jz_j}), \overline{R[z]\eta}\rangle = 2\,\mathrm{Re}\, E_{jz_j}\langle Q_{jz_j}, \overline{R[z]\eta}\rangle = 0$, which follows by $R[z]\eta \in \mathcal{H}_c[z]$ and by $\mathrm{i}Q_{jz_j} = -z_{jI} D_{jR} Q_{jz_j} + z_{jR} D_{jI} Q_{jz_j}$, see (11) in [17] (and which is an immediate consequence of $Q_{jz_j} = e^{\mathrm{i}\theta} Q_{j|z_j|}$ for $z_j = e^{\mathrm{i}\theta}|z_j|$), we have

$$\begin{aligned}
&\mathrm{Re}\langle \nabla E(\sum_{j=1}^{n} Q_{jz_j}), \overline{R[z]\eta}\rangle = \overbrace{\mathrm{Re}\langle \nabla E(Q_{1z_1}), \overline{R[z]\eta}\rangle}^{0} \\
&+ \int_0^1 \partial_t \,\mathrm{Re}\langle \nabla E(Q_{1z_1} + t\sum_{j>1} Q_{jz_j}), \overline{R[z]\eta}\rangle dt = \mathrm{Re}\langle \nabla E(\sum_{j>1} Q_{jz_j}), \overline{R[z]\eta}\rangle \\
&+ \int_{[0,1]^2} \partial_s\partial_t \,\mathrm{Re}\langle \nabla E_P(sQ_{1z_1} + t\sum_{l>1} Q_{lz_l}), \overline{R[z]\eta}\rangle dt ds \\
&= \sum_{j=1}^{n-1} \int_{[0,1]^2} \partial_s\partial_t \,\mathrm{Re}\langle \nabla E_P(sQ_{jz_j} + t\sum_{l>j} Q_{lz_l}), \overline{R[z]\eta}\rangle dt ds,
\end{aligned} \tag{3.11}$$

where the last line is obtained repeating the argument in the first three lines. For $\widehat{Q}_j = \sum_{l>j} Q_{lz_l}$ and by $\nabla E_P(u) = 2|u|^2 u$, the last line of (3.11) is, in the notation of Lemma 2.3,

$$2\sum_{j=1}^{n-1} \mathrm{Re}\left\langle 2Q_{jz_j}|\widehat{Q}_j|^2 + 2|Q_{jz_j}|^2\widehat{Q}_j + Q_{jz_j}^2\overline{\widehat{Q}_j} + \overline{Q}_{jz_j}\widehat{Q}_j^2,\ \overline{\eta} + \phi_j(\langle \overline{\widehat{B}}_j(\widehat{Z}), \overline{\eta}\rangle + \langle \overline{z}_i\overline{z}_\ell \overline{\widehat{C}}_{i\ell j}(\widehat{Z}), \eta\rangle)\right\rangle.$$

Further expanding $\widehat{Q}_j = \sum_{l>j} Q_{lz_l}$ and using $Q_{lz_l} = z_l(\phi_l + \widehat{q}_l(|z_l|^2)$, the above term is of the form

$$\sum_{j=1}^{n} \sum_{|\mathbf{m}|=1} (\overline{z}_j \mathbf{Z}^{\mathbf{m}} \langle G_{j\mathbf{m}}(\widehat{Z}), \eta\rangle + \text{c.c.}).$$

As in Step 1, by Lemma B.4, this can be expanded into

$$\sum_{j=1}^{n} \sum_{1\le|\mathbf{m}|\le 2N+3} \left(\overline{z}_j \mathbf{Z}^{\mathbf{m}} \langle G_{jk\mathbf{m}}(|z_k|^2), \eta\rangle + \text{c.c.}\right) + \sum_{|\mathbf{m}|=2N+4} (\mathbf{Z}^{\mathbf{m}} \langle G_{\mathbf{m}}(z), \eta\rangle + \text{c.c.})\,. \tag{3.12}$$

Thus the last line in (3.11) can be absorbed in the 4th line of (3.3).

Step 3. We consider the expansion of the 3rd term in the r.h.s of (3.6). Using $\nabla^2 E_K(u) = 2H$

and proceeding as for (3.6), we obtain

$$
\begin{aligned}
&2^{-1}\operatorname{Re}\langle \nabla^2 E(\sum_{j=1}^n Q_{jz_j})R[z]\eta, \overline{R[z]\eta}\rangle \\
&= 2^{-1}\operatorname{Re}\langle \nabla^2 E_K(\sum_{j=1}^n Q_{jz_j})R[z]\eta, \overline{R[z]\eta}\rangle + 2^{-1}\sum_{j=1}^n \operatorname{Re}\langle \nabla^2 E_P(Q_{jz_j})R[z]\eta, \overline{R[z]\eta}\rangle \\
&+ 2^{-1}\sum_{j=1}^{n-1}\int_{[0,1]^2}\partial_s\partial_t \operatorname{Re}\langle \nabla^2 E_P(sQ_{jz_j} + t\sum_{l=j+1}^n Q_{lz_l})R[z]\eta, \overline{R[z]\eta}\rangle dtds.
\end{aligned}
$$

The 3rd line is absorbed in the $\mathbf{Z^m}\langle G_{2\mathbf{m}ij}(z), \eta^i\overline{\eta}^j\rangle + \mathcal{R}^{1,2}_{r_0,\infty}(z,\eta)$ with $|\mathbf{m}| = 1$ terms in (3.3). From the 2nd line, using (2.3)–(2.4) and in particular $\alpha_j[z]\eta = \mathcal{R}^{1,1}_{r_0,\infty}(z,\eta)$ for the last equality, we have

$$
\begin{aligned}
2^{-1}\operatorname{Re}\langle \nabla^2 E_K(\sum_{j=1}^n Q_{jz_j})R[z]\eta, \overline{R[z]\eta}\rangle = \langle HR[z]\eta, \overline{R[z]\eta}\rangle = \langle H\eta, \overline{\eta}\rangle + 2\sum_{j=1}^n \operatorname{Re}\left[(\alpha_j[z]\eta)\langle H\phi_j, \overline{\eta}\rangle\right] \\
+ \sum_{j,k=1}^n e_j|\alpha_j[z]\eta|^2 = \langle H\eta, \overline{\eta}\rangle + \mathcal{R}^{1,2}_{r_0,\infty}(z,\eta),
\end{aligned}
$$

which yield the 2nd and 3rd terms in the r.h.s. of (3.3). For

$$
2^{-1}\sum_{j=1}^n \nabla^2 E_P(Q_{jz_j})\eta = \sum_{j=1}^n |Q_{jz_j}|^2\eta + 2\sum_{j=1}^n Q_{jz_j}\operatorname{Re}(Q_{jz_j}\overline{\eta})
$$

we have for $G_{2\mathbf{0}ij}(z)$ as in (3.4)

$$
2^{-1}\sum_{j=1}^n \operatorname{Re}\langle \nabla^2 E_P(Q_{jz_j})R[z]\eta, \overline{R[z]\eta}\rangle = \mathcal{R}^{1,2}_{r_0,\infty}(z,\eta) + \sum_{i+j=2}\langle G_{2\mathbf{0}ij}(z), \eta^i\overline{\eta}^j\rangle. \tag{3.13}
$$

This $\mathcal{R}^{1,2}_{r_0,\infty}(z,\eta)$ defines the 3rd term in the r.h.s. of (3.3). Notice that $\mathcal{R}^{1,2}_{r_0,\infty}(e^{\mathrm{i}\vartheta}z, e^{\mathrm{i}\vartheta}\eta) = \mathcal{R}^{1,2}_{r_0,\infty}(z,\eta)$ because this invariance is satisfied both by the l.h.s. of (3.13) (by the invariance of E, (2.2) and by Lemma 2.3) and by the last summation in the r.h.s. of (3.13), by formula (3.4).

Step 4. We now turn to the $E_3(\eta)$ term in (3.6). By elementary computations

$$
\begin{aligned}
E_3(\eta) &= \int_{[0,1]^2} t(1-t)d^3E_P(\sum_{j\geq 1} Q_{jz_j} + stR[z]\eta)\cdot(R[z]\eta)^3 dtds = E_P(R[z]\eta) \\
&+ \int_{[0,1]^3} t(1-t)d^4E_P(\tau\sum_{j\geq 1} Q_{jz_j} + stR[z]\eta)\cdot(R[z]\eta)^3 \sum_{j\geq 1} Q_{jz_j} dtdsd\tau,
\end{aligned} \tag{3.14}
$$

with $d^3E_P(u)\cdot v^3$ the trilinear differential form applied to (v,v,v) and $d^4E_P(u)\cdot v^3w$ the 4–linear differential form applied to (v,v,v,w).
In particular we have used the fact that since $d^jE_P(0) = 0$ for $0 \leq j \leq 2$ we have

$$
E_P(R[z]\eta) = \int_{[0,1]^2} t\ (1-t)d^3E_P(stR[z]\eta)\cdot(R[z]\eta)^3 dtds. \tag{3.15}
$$

For $\beta(u) = |u|^4$ and using the fact that $d^4\beta(u) \in B^4(\mathbb{C},\ R)$ is constant in u, the 2nd line of (3.14) is

$$\frac{1}{12}\int_{\mathbb{R}^3} d^4\beta \cdot ((R[z]\eta)(x))^3 \sum_{j\geq 1} Q_{jz_j}(x)dx,$$

and can be absorbed in the $\langle G_{dij}(z), \eta^i\overline{\eta}^j\rangle \mathcal{R}^{0,c}_{r_0,\infty}(z,\eta)$ terms in (3.3). We expand $E_P(R[z]\eta)$ as a sum of similar terms and of $E_P(\eta)$. □

In order to extract from the functional in (3.3) an effective Hamiltonian well suited for the FGR and dispersive estimates, we need to implement a Birkhoff normal form argument, see Sect.5. This requires an intermediate change of coordinates, which will partially normalize the symplectic form Ω defined in (4.1) below, and diagonalize the homological equations. Notice that, as a bonus, this change of coordinates erases the bad terms in the expansion of E in (3.3) discussed in Remark 3.2.

4 Darboux Theorem

System (3.2) is Hamiltonian with respect to the symplectic form in $H^1(\mathbb{R}^3,\mathbb{C})$

$$\Omega(X,Y) := \mathrm{i}\langle X,\overline{Y}\rangle - \mathrm{i}\langle \overline{X},Y\rangle = 2\,\mathrm{Im}\langle \overline{X},Y\rangle. \tag{4.1}$$

In terms of the spectral decomposition of H (recall $\overline{\phi}_j = \phi_j$)

$$X = \sum_{j=1}^{n}\langle X,\phi_j\rangle\phi_j + P_cX \tag{4.2}$$

$$\Omega(X,Y) = \mathrm{i}\sum_{j=1}^{n}\left(\langle X,\phi_j\rangle\langle \overline{Y},\phi_j\rangle - \langle \overline{X},\phi_j\rangle\langle Y,\phi_j\rangle\right) + \mathrm{i}\langle P_cX, P_c\overline{Y}\rangle - \mathrm{i}\langle P_c\overline{X}, P_cY\rangle. \tag{4.3}$$

However, in terms of the coordinates in Lemma 2.4, Ω admits a quite more complicated representation, as we shall see. This will require us to adjust these coordinates.

Our first observation is that for the coordinates in Lemma 2.4 we have the following facts.

Lemma 4.1. *The Frechét derivative of $\eta(u)$ and dz_j is given by the following formula:*

$$d\eta(u) = -\sum_{j=1,\dots,n}\ \sum_{A=I,R} P_cD_{jA}q_{jz_j}dz_{jA} + P_c, \tag{4.4}$$

$$dz_j = \langle\quad,\phi_j\rangle - \sum_{k:k\neq j}\ \sum_{A=I,R}\langle D_{kA}q_{kz_k},\phi_j\rangle dz_{kA} - \sum_{k=1}^{n}\sum_{A=I,R} D_{kA}\alpha_j[z]\eta dz_{kA} - \alpha_j[z]\circ d\eta. \tag{4.5}$$

Analogous formulas for dz_{jR} and dz_{jI} are obtained applying Re *and* Im *to* (4.5).

Proof. We start with (4.4). By the independence of z and η, we have

$$d\eta\frac{\partial}{\partial z_{jR}} = d\eta\frac{\partial}{\partial z_{jI}} = 0, \tag{4.6}$$

where

$$\frac{\partial}{\partial z_{jA}} = D_{jA}Q_{jz_j} + \sum_{k=1}^{n} D_{jA}\left(\alpha_k[z]\eta\right)\phi_k. \tag{4.7}$$

Next, for $\xi \in \mathcal{H}_c[0]$ we have what follows, which implies $d\eta R[z]P_c = 1|_{\mathcal{H}_c[0]}$:

$$d\eta R[z]P_c\xi = \frac{d}{dt}\Big|_{t=0} \eta(Q_{jz_j} + R[z](\eta + t\xi)) = \xi.$$

So $d\eta = \sum(a_j dz_{jR} + b_j dz_{jI}) + P_c$, where we used $P_c R[z] = 1$. a_j and b_j can be computed applying $\sum(a_j dz_{jR} + b_j dz_{jI}) + P_c$ to the vectors (4.7) and using (4.6). Finally (4.5) follows by

$$z_j(u) = \langle u - \sum_{k=1}^{n} q_{kz_k} - R[z]\eta, \phi_j\rangle = \langle u - \sum_{k:k\neq j} q_{kz_k}, \phi_j\rangle - \alpha_j[z]\eta.$$

□

We consider the function $\overline{\eta}(u)$. Notice that $d\overline{\eta}(u)X = \frac{d}{dt}\overline{\eta}(u+tX)_{|t=0} = \overline{d\eta(u)X}$. Now we introduce a new symplectic form. Notice that our final choice of symplectic form is not the Ω_0' defined right here in (4.8), but rather the Ω_0 defined in (4.13) further down.

Lemma 4.2. *Set*

$$\begin{aligned} \Omega_0' &:= 2\sum_{j=1}^{n} dz_{jR}\wedge dz_{jI} + \mathrm{i}\,\langle d\eta, d\overline{\eta}\rangle - \mathrm{i}\,\langle d\overline{\eta}, d\eta\rangle \quad \textit{and} \\ B_0' &:= \sum_{j=1}^{n}(z_{jR}dz_{jI} - z_{jI}dz_{jR}) - \frac{\mathrm{i}}{2}(\langle \overline{\eta}, d\eta\rangle - \langle \eta, d\overline{\eta}\rangle). \end{aligned} \tag{4.8}$$

Then $dB_0' = \Omega_0'$ and $\Omega = \Omega_0'$ at $u = 0$ for the Ω of (4.1). Furthermore

$$\Phi^* B_0' = B_0' \textit{ for } \Phi(u) = e^{\mathrm{i}\vartheta}u \textit{ for any fixed } \vartheta\in\mathbb{R}. \tag{4.9}$$

Proof. The equality $dB_0' = \Omega_0'$ is elementary. Indeed $d(z_{jR}dz_{jI} - z_{jI}dz_{jR}) = 2dz_{jR}\wedge dz_{jI}$ and for a pair of constant vectorfields X and Y, by $d^2\eta(X,Y) = d^2\eta(Y,X)$, we have

$$d\,\langle\overline{\eta}, d\eta\rangle\,(X,Y) = X\,\langle\overline{\eta}, d\eta Y\rangle - Y\,\langle\overline{\eta}, d\eta X\rangle = \langle d\overline{\eta}X, d\eta Y\rangle - \langle d\overline{\eta}Y, d\eta X\rangle\,.$$

This yields $d\,\langle\overline{\eta}, d\eta\rangle = \langle d\overline{\eta}, d\eta\rangle - \langle d\eta, d\overline{\eta}\rangle$ and also $d\,\langle\eta, d\overline{\eta}\rangle = -d\,\langle\overline{\eta}, d\eta\rangle = \langle d\eta, d\overline{\eta}\rangle - \langle d\overline{\eta}, d\eta\rangle$

To compute Ω_0' at $u=0$ we observe that by Lemma 4.1 we have $d\eta = P_c$ at $u=0$, so that

$$\mathrm{i}\,\langle d\eta X, d\overline{\eta}Y\rangle - \mathrm{i}\,\langle d\overline{\eta}X, d\eta Y\rangle = \mathrm{i}\langle P_cX, P_c\overline{Y}\rangle - \mathrm{i}\langle P_c\overline{X}, P_cY\rangle \text{ at } u=0. \tag{4.10}$$

By Lemma 4.1 and Proposition 1.1, at $u=0$ we have $dz_{jR} = \mathrm{Re}\langle\ \ ,\phi_j\rangle$ and $dz_{jI} = \mathrm{Im}\langle\ \ ,\phi_j\rangle$. Summing on repeated indexes, we have

$$\begin{aligned} &\mathrm{i}\left(\langle X,\phi_j\rangle\langle\overline{Y},\phi_j\rangle - \langle\overline{X},\phi_j\rangle\langle Y,\phi_j\rangle\right) = -2\,\mathrm{Im}\left(\langle X,\phi_j\rangle\langle\overline{Y},\phi_j\rangle\right) = \\ &2(\mathrm{Re}\langle X,\phi_j\rangle\,\mathrm{Im}\langle Y,\phi_j\rangle - \mathrm{Re}\langle Y,\phi_j\rangle\,\mathrm{Im}\langle X,\phi_j\rangle) = \\ &2\,\mathrm{Re}\langle\ ,\phi_j\rangle\wedge\mathrm{Im}\langle\ ,\phi_j\rangle(X,Y) = 2dz_{jR}\wedge dz_{jI}|_{u=0}(X,Y). \end{aligned} \tag{4.11}$$

By (4.10)–(4.11) we get $\Omega = \Omega_0'$ at $u=0$. Finally, (4.9) follows immediately by

$$B_0' := \sum_{j=1}^{n}\mathrm{Im}(\overline{z}_j dz_j) + \mathrm{Im}\,\langle\overline{\eta}, d\eta\rangle\,. \tag{4.12}$$

$\square$

Summing on repeated indexes and using the notation in Prop.1.1, we introduce the differential forms:

$$\begin{aligned}&\Omega_0 := \Omega_0' + \mathrm{i}\gamma_j(|z_j|^2) dz_j \wedge d\overline{z}_j \text{ where}\\ &\gamma_j(|z_j|^2) := \left\langle \widehat{q}_j(|z_j|^2), \widehat{q}_j(|z_j|^2)\right\rangle + 2|z_j|^2 \left\langle \widehat{q}_j(|z_j|^2), \widehat{q}_j'(|z_j|^2)\right\rangle ,\\ &B_0 := B_0' - \mathrm{Im}\left\langle D_{jA}\overline{q}_{jz_j}, q_{jz_j}\right\rangle dz_{jA}.\end{aligned} \tag{4.13}$$

with $\widehat{q}_j'(t) = \frac{d}{dt}\widehat{q}_j$. We have the following lemma.

Lemma 4.3. *We have $\gamma_j(|z_j|^2) = \mathcal{R}^{2,0}_{\infty,\infty}(|z_j|^2)$. We have $dB_0 = \Omega_0$ and*

$$\Phi^* B_0 = B_0 \text{ for } \Phi(u) = e^{\mathrm{i}\vartheta} u \text{ for any fixed } \vartheta \in \mathbb{R} \tag{4.14}$$

Proof. $\gamma_j(|z_j|^2) = \mathcal{R}^{2,0}_{\infty,\infty}(|z_j|^2)$ is elementary from Prop. 1.1 and Def. 2.8. $dB_0 = \Omega_0$ follows by $dB_0' = \Omega_0'$ and by

$$\begin{aligned}&- d\,\mathrm{Im}\left\langle D_{jA}\overline{q}_{jz_j}, q_{jz_j}\right\rangle dz_{jA} = \mathrm{Im}\left\langle D_{jA}\overline{q}_{jz_j}, D_{jB} q_{jz_j}\right\rangle dz_{jA}\wedge dz_{jB} =\\ &2\,\mathrm{Im}\left\langle D_{jR}\overline{q}_{jz_j}, D_{jI} q_{jz_j}\right\rangle dz_{jR}\wedge dz_{jI} = 2\gamma(|z_j|^2) dz_{jR}\wedge dz_{jI}\\ &= \mathrm{i}\gamma_j(|z_j|^2) dz_j \wedge d\overline{z}_j\end{aligned}$$

where $q_{jz_j} = z_j \widehat{q}_j(|z_j|^2)$.

Turning to the proof of (4.14), we have

$$\Phi^*\left(\mathrm{i}\gamma_j(|z_j|^2) dz_j\wedge d\overline{z}_j\right) = \mathrm{i}\gamma_j(|z_j|^2) d\left(\Phi^* z_j\right)\wedge d\left(\Phi^*\overline{z}_j\right) = \mathrm{i}\gamma_j(|z_j|^2) dz_j \wedge d\overline{z}_j.$$

$\square$

Lemma 4.4. *We have $dB = \Omega$ with B the differential form in the manifold H^1 defined by*

$$B(u)X := \mathrm{Im}\langle \overline{u}, X\rangle \tag{4.15}$$

Consider for $u \in B_{H^1}(0, d_0)$ for the $d_0 > 0$ of Lemma 2.3 the function $\psi \in C^\infty(B_{H^1}(0, d_0), \mathbb{R})$ and the differential form $\Gamma(u)$ defined as follows:

$$\psi(u) := \sum_{j=1}^n \mathrm{Im}\langle \overline{q}_{jz_j}, u\rangle + \sum_{j=1}^n \mathrm{Im}\left(\alpha_j[z]\eta\ \overline{z}_j\right) \tag{4.16}$$

$$\Gamma(u) := B(u) - B_0(u) + d\psi(u). \tag{4.17}$$

Then the map $(z, \eta) \to \Gamma(u(z, \eta))$, for $u(z, \eta)$ the r.h.s. of (2.10), which is initially defined in $B_{\mathbb{C}^n}(0, d_0) \times (H^1 \cap \mathcal{H}_c[0])$, extends to $B_{\mathbb{C}^n}(0, d_0) \times \Sigma^c_{-r}$ for any $r \in \mathbb{N}$. In particular, we have $\Gamma = \Gamma_{jA} dz_{jA} + \langle \Gamma_\eta, d\eta\rangle + \langle \Gamma_{\overline{\eta}}, d\overline{\eta}\rangle$ with, in the sense of Remark 2.10,

$$\Gamma_{jA} = \mathcal{R}^{1,1}_{\infty,\infty}(z, \boldsymbol{Z}, \eta) \text{ and } \Gamma_\xi = \mathbf{S}^{1,1}_{\infty,\infty}(z, \boldsymbol{Z}, \eta) \text{ for } \xi = \eta, \overline{\eta}. \tag{4.18}$$

Furthermore, Γ satisfies the invariance property in $B_{H^1}(0, d_0)$:

$$\Phi^*\Gamma = \Gamma \text{ for } \Phi(u) = e^{\mathrm{i}\vartheta} u \text{ for any fixed } \vartheta \in \mathbb{R}. \tag{4.19}$$

Proof. By the definition of the exterior differential, and focusing on constant vectorfields X and Y,

$$dB(X,Y) = XB(u)Y - YB(u)X = \mathrm{Im}\langle \overline{X}, Y\rangle - \mathrm{Im}\langle \overline{Y}, X\rangle = \Omega(X,Y).$$

This is enough to prove $dB = \Omega$. Next, using $R[z]\eta = \eta + \sum_j \alpha_j[z]\eta\,\phi_j$, we expand

$$\begin{aligned} B(u) &= \sum_j \mathrm{Im}\langle \overline{Q}_{jz_j},\ \rangle + \mathrm{Im}\langle \overline{R[z]\eta},\ \rangle = \sum_j \mathrm{Im}\langle \overline{z}_j\phi_j,\ \rangle + \mathrm{Im}\langle \overline{\eta},\ \rangle \\ &+ \sum_j \mathrm{Im}\langle \overline{q}_{jz_j},\ \rangle + \sum_j \mathrm{Im}(\overline{\alpha_j[z]\eta}\ \ \langle \phi_j,\ \rangle)\ . \end{aligned} \tag{4.20}$$

By the definition of B_0 in (4.13) we have

$$B - B_0 = I_1 + I_2 + I_3 + \sum_{j,A} \mathrm{Im}\langle D_{jA}\overline{q}_{jz_j}, q_{jz_j}\rangle dz_{jA} + \sum_j \mathrm{Im}\langle \overline{q}_{jz_j},\ \rangle\ , \tag{4.21}$$

$$I_1 := \sum_j \mathrm{Im}\left[\overline{z}_j\left(\langle \phi_j,\ \rangle - dz_j\right)\right]\ ,\quad I_2 := -\,\mathrm{Im}\left\langle \overline{\eta}, d\eta - P_c\quad\right\rangle\ ,\quad I_3 := \sum_j \mathrm{Im}\left[\overline{\alpha_j[z]\eta}\ \langle \phi_j,\ \rangle\right]\ .$$

We substitute $d\eta$ with (4.4) and $\langle \phi_j,\ \rangle$ with (4.5). For $\alpha_j[z]\circ d\eta$ the linear operator defined by $\alpha_j[z]\circ d\eta(X) := \alpha_j[z]d\eta(X)$ we then get

$$\begin{aligned} I_1 &= \mathrm{Im}\langle D_{jA}q_{jz_j}, \overline{z}_k\phi_k\rangle dz_{jA} + \mathrm{Im}(\overline{z}_j D_{kA}\alpha_j[z]\eta)dz_{kA} + \mathrm{Im}\left(\overline{z}_j\alpha_j[z]\circ d\eta\right) \\ &= \sum_{jA} \mathcal{R}^{1,1}_{\infty,\infty} dz_{jA} + \mathrm{Im}\left(\overline{z}_j\alpha_j[z]\circ d\eta\right), \end{aligned} \tag{4.22}$$

where, as anticipated in Remark 2.10, here we set $\mathcal{R}^{i,j}_{K,M} = \mathcal{R}^{i,j}_{K,M}(z,\mathbf{Z},\eta)$ and $\mathbf{S}^{i,j}_{K,M} = \mathbf{S}^{i,j}_{K,M}(z,\mathbf{Z},\eta)$, where here $\mathbf{Z} = (z_i\overline{z}_j)_{i,j=1,\dots,n}$ with $i \neq j$.

The second term in the last line of the last formula is incorporated in (4.23). We have

$$I_2 = \mathrm{Im}\langle \overline{\eta}, D_{jA}q_{jz_j}\rangle dz_{jA} = \sum_{jA} \mathcal{R}^{2,1}_{\infty,\infty} dz_{jA}.$$

Substituting with (4.5) we have

$$I_3 = \sum_{jA} \mathcal{R}^{2,1}_{\infty,\infty} dz_{jA} + \langle \mathbf{S}^{1,1}_{\infty,\infty}, d\eta\rangle + \langle \mathbf{S}^{1,1}_{\infty,\infty}, d\overline{\eta}\rangle.$$

Hence we get

$$B - B_0 = \sum_j \mathrm{Im}\left(\overline{z}_j\alpha_j[z]\circ d\eta\right) \tag{4.23}$$

$$+ \sum_{jA} \mathcal{R}^{1,1}_{\infty,\infty} dz_{jA} + \langle \mathbf{S}^{1,1}_{\infty,\infty}, d\eta\rangle + \langle \mathbf{S}^{1,1}_{\infty,\infty}, d\overline{\eta}\rangle \tag{4.24}$$

$$+ \sum_{jA} \mathrm{Im}\langle D_{jA}\overline{q}_{jz_j}, q_{jz_j}\rangle dz_{jA} + \sum_j \mathrm{Im}\langle \overline{q}_{jz_j},\ \rangle\ . \tag{4.25}$$

Set now $\widetilde{\psi}(u) := -\sum_{j=1}^n \mathrm{Im}\langle \overline{q}_{jz_j}, u\rangle$. Then it is elementary that we have

$$d\widetilde{\psi} = -\sum_{j=1}^n \mathrm{Im}\langle \overline{q}_{jz_j}, \rangle - \sum_{j,A} \mathrm{Im}\langle D_{jA}\overline{q}_{jz_j}, q_{jz_j}\rangle dz_{jA} + \sum_{j,A} \mathcal{R}^{1,1}_{\infty,\infty} dz_{jA}. \tag{4.26}$$

By the Leibnitz rule we have

$$\operatorname{Im}\left(\overline{z}_j\alpha_j[z]\circ d\eta\right)=d\operatorname{Im}\left(\overline{z}_j\alpha_j[z]\ \eta\right)-\operatorname{Im}\left(d(\overline{z}_j\alpha_j[z])\ \eta\right). \tag{4.27}$$

The contribution to (4.23) of the last term in the r.h.s. of (4.27) can be absorbed in (4.24). Then

$$B-B_0+d\psi=\sum_{jA}\mathcal{R}^{1,1}_{\infty,\infty}dz_{jA}+\langle\mathbf{S}^{1,1}_{\infty,\infty},d\eta\rangle+\langle\mathbf{S}^{1,1}_{\infty,\infty},d\overline{\eta}\rangle.$$

Here we have used: the first two terms in the r.h.s. of (4.26) cancel with (4.25); there is a cancelation between the contribution to the r.h.s. of (4.23) of the first term on the r.h.s. of (4.27) and the differential of the last term in (4.16). This yields (4.18).

Finally we consider (4.19). We have $\Phi^*B_0=B_0$ by (4.14), while $\Phi^*B=B$ follows immediately from the definition of B in (4.15). Finally $\Phi^*\psi=\psi$ follows immediately from $\Phi^*\langle\overline{q}_{jz_j},u\rangle=\langle\overline{q}_{jz_j},u\rangle$, which follows from $q_{jz_j}(e^{\mathrm{i}\vartheta}z)=e^{\mathrm{i}\vartheta}q_{jz_j}(z)$, and from (2.9) and (2.12) which impy

$$\Phi^*\left(\overline{z}_j\alpha_j[z]\eta\right)=e^{-\mathrm{i}\vartheta}\overline{z}_j\alpha_j[e^{\mathrm{i}\vartheta}z]e^{\mathrm{i}\vartheta}\eta=\overline{z}_j\alpha_j[z]\eta.$$

□

Lemma 4.5. *Consider the differential form $\Omega-\Omega_0$, which is defined in $B_{H^1}(0,d_0)$ for the $d_0>0$ of Lemma 2.3. Then, summing on repeated indexes, we have*

$$\Omega-\Omega_0=\widetilde{\Omega}_{ijAB}dz_{iA}\wedge dz_{jB}+\sum_{\xi=\eta,\overline{\eta}}dz_{iA}\wedge\langle\widetilde{\Omega}_{iA\xi},d\xi\rangle+\sum_{\xi,\xi'=\eta,\overline{\eta}}\langle\widetilde{\Omega}_{\xi'\xi}d\xi,d\xi'\rangle \tag{4.28}$$

where, expressed as functions of (z,η), the coefficients extend into functions defined $B_{\mathbb{C}^n}(0,d_0)\times\Sigma^c_{-r}$ for any $r\in\mathbb{N}$ and in particular we have $\widehat{\Omega}_{iA\xi}=\boldsymbol{S}^{1,0}_{\infty,\infty}(z,\boldsymbol{Z},\eta)$, $\widehat{\Omega}_{ijAB}=\mathcal{R}^{1,0}_{\infty,\infty}(z,\boldsymbol{Z},\eta)$ in the sense of Remark 2.10 and $\widetilde{\Omega}_{\xi'\xi}=\partial_\xi\boldsymbol{S}^{1,1}_{\infty,\infty}(z,\boldsymbol{Z},\eta)-(\partial_{\xi'}\boldsymbol{S}^{1,1}_{\infty,\infty}(z,\boldsymbol{Z},\eta))^*$ (with two distinct $\mathbf{S}$'s). We furthermore have

$$\Phi^*(\Omega-\Omega_0)=\Omega-\Omega_0 \text{ for } \Phi(z,\eta)=(e^{\mathrm{i}\vartheta}z,e^{\mathrm{i}\vartheta}\eta) \text{ for any fixed } \vartheta\in\mathbb{R}. \tag{4.29}$$

Proof. We have

$$\Omega-\Omega_0=d\Gamma=d\sum_{j,A}\mathcal{R}^{1,1}_{\infty,\infty}dz_{jA}+d\sum_{\xi}\langle\mathbf{S}^{1,1}_{\infty,\infty},d\xi\rangle.$$

Summing over k,B,ξ we have

$$d(\mathcal{R}^{1,1}_{\infty,\infty}dz_{jA})=\partial_{z_{kB}}\mathcal{R}^{1,1}_{\infty,\infty}dz_{kB}\wedge dz_{jA}+\langle\partial_\xi\mathcal{R}^{1,1}_{\infty,\infty},d\xi\rangle\wedge dz_{jA}$$

with the $\partial_\xi\mathcal{R}^{1,1}_{\infty,\infty}\in\mathcal{H}_c[0]$ defined, summing on repeated indexes and for F with values in $\mathbb{R}$, by

$$dFX=\partial_{z_{kB}}F\ dz_{kB}X+\langle\partial_\xi F,d\xi X\rangle \text{ for any } X\in L^2(\mathbb{R}^3,\mathbb{C}).$$

It is easy to see that $\partial_\xi\mathcal{R}^{1,1}_{\infty,\infty}=\mathbf{S}^{1,0}_{\infty,\infty}$ and $\partial_{z_{kB}}\mathcal{R}^{1,1}_{\infty,\infty}=\mathcal{R}^{1,0}_{\infty,\infty}$.
Furthermore, summing on repeated indexes we have

$$\begin{aligned}d\langle\mathbf{S}^{1,1}_{\infty,\infty},d\xi\rangle&=dz_{kB}\wedge\langle\partial_{z_{kB}}\mathbf{S}^{1,1}_{\infty,\infty},d\xi\rangle+\langle\partial_{\xi'}\mathbf{S}^{1,1}_{\infty,\infty}d\xi',d\xi\rangle-\langle d\xi,\partial_{\xi'}\mathbf{S}^{1,1}_{\infty,\infty}d\xi'\rangle\\&=dz_{kB}\wedge\langle\partial_{z_{kB}}\mathbf{S}^{1,1}_{\infty,\infty},d\xi\rangle+\langle\partial_{\xi'}\mathbf{S}^{1,1}_{\infty,\infty}d\xi',d\xi\rangle-\langle(\partial_{\xi'}\mathbf{S}^{1,1}_{\infty,\infty})^*d\xi,d\xi'\rangle,\end{aligned} \tag{4.30}$$

where, for $T\in C^1(U_{L^2},L^2)$ for U_{L^2} open subset in L^2, $\partial_\xi T\in B(\mathcal{H}_c[0],L^2)$ is defined by

$$dTX=\partial_{z_{kB}}T\ dz_{kB}X+\partial_\xi Td\xi X \text{ for any } X\in L^2(\mathbb{R}^3,\mathbb{C}).$$

Summing on ξ in (4.30) we get terms which are absorbed in the last two terms of (4.28).
Formula (4.29) follows from (4.19), $\Omega_0=dB_0$ and $\Omega=dB$. □

Lemma 4.6. *Consider the form $\Omega_t := \Omega_0 + t(\Omega - \Omega_0)$ and set $i_X\Omega_t(Y) := \Omega_t(X,Y)$. For any preassigned $r \in \mathbb{N}$ recall by (4.8), (4.13) and Lemmas 4.4 and 4.5 that $(\Omega - \Omega_0)$ and Γ extend to forms defined in $B_{\mathbb{C}^n}(0,d_0) \times \Sigma^c_{-r}$. Then there is $\delta_0 \in (0,d_0)$ s.t. for any $(t,z,\eta) \in (-4,4) \times B_{\mathbb{C}^n}(0,\delta_0) \times B_{\Sigma^c_{-r}}(0,\delta_0)$ there exists exactly one solution $\mathcal{X}^t(z,\eta) \in L^2$ of the equation $i_{\mathcal{X}^t}\Omega_t = -\Gamma$. Furthermore, we have the following facts.*

(1) $\mathcal{X}^t(z,\eta) \in \Sigma_r$ and if we set $\mathcal{X}^t_{jA}(z,\eta) = dz_{jA}\mathcal{X}^t(z,\eta)$ and $\mathcal{X}^t_\eta(z,\eta) = d\eta\mathcal{X}^t(z,\eta)$, we have $\mathcal{X}^t_{jA}(z,\eta) = \mathcal{R}^{1,1}_{r,\infty}(t,z,\boldsymbol{Z},\eta)$ and $\mathcal{X}^t_\eta(z,\eta) = \boldsymbol{S}^{1,1}_{r,\infty}(t,z,\boldsymbol{Z},\eta)$ in the sense of Remark 2.10.

(2) For $\mathcal{X}^t_j := dz_j\mathcal{X}^t$ and $\mathcal{X}^t_\eta := d\eta\mathcal{X}^t$, we have $\mathcal{X}^t_j(e^{i\vartheta}z, e^{i\vartheta}\eta) = e^{i\vartheta}\mathcal{X}^t_j(z,\eta)$ and $\mathcal{X}^t_\eta(e^{i\vartheta}z, e^{i\vartheta}\eta) = e^{i\vartheta}\mathcal{X}^t_\eta(z,\eta)$.

Proof. We define Y such that $i_Y\Omega'_0 = -\Gamma$, which yields $Y_{jR} = -\frac{1}{2}\Gamma_{jI}$, $Y_{jI} = \frac{1}{2}\Gamma_{jR}$ (both $\mathcal{R}^{1,1}_{\infty,\infty}$), $Y_\eta = -i\Gamma_{\overline{\eta}}$ and $Y_{\overline{\eta}} = i\Gamma_\eta$ (both $\mathbf{S}^{1,1}_{\infty,\infty}$) . We use $i_{K_tX}\Omega'_0 = i_X(\Omega_0 - \Omega'_0 + t\widehat{\Omega})$, where $\widehat{\Omega} := \Omega - \Omega_0$, to define in L^2 the operator K_t . We claim the following lemma.

Lemma 4.7. *For appropriate symbols $\mathcal{R}^{1,0}_{\infty,\infty}(t,z,\boldsymbol{Z},\eta)$ and $\mathbf{S}^{1,0}_{\infty,\infty}(t,z,\boldsymbol{Z},\eta)$ which differ from one term to the other and for $\mathbf{Z} = (z_i\overline{z}_j)_{i,j=1,\dots,n}$ with $i \neq j$, we have*

$$\begin{aligned}(K_tX)_{jA} &= \sum_{lB}\mathcal{R}^{1,0}_{\infty,\infty}X_{lB} + \sum_{\xi=\eta,\overline{\eta}}\langle \mathbf{S}^{1,0}_{\infty,\infty}, X_\xi\rangle,\\ (K_tX)_\xi &= \sum_{lB}\mathbf{S}^{1,0}_{\infty,\infty}X_{lB} + \sum_{\xi'=\eta,\overline{\eta}}(\partial_{\xi'}\boldsymbol{S}^{1,1}_{\infty,\infty}(t,z,\boldsymbol{Z},\eta) - (\partial_\xi\boldsymbol{S}^{1,1}_{\infty,\infty}(t,z,\boldsymbol{Z},\eta))^*X_{\xi'}.\end{aligned} \tag{4.31}$$

We assume for a moment Lemma 4.7 and complete the proof of Lemma 4.6. $i_{\mathcal{X}^t}\Omega_t = -\Gamma$ becomes $\mathcal{X}^t + K_t\mathcal{X}^t = Y$. Indeed, suppose $\mathcal{X}^t + K_t\mathcal{X}^t = Y$ holds. Then, by definition of K_t, we have

$$i_{\mathcal{X}_t}(\Omega_t - \Omega') = i_{K_t\mathcal{X}_t}\Omega'_0 \text{ and so } i_{\mathcal{X}_t}\Omega_t = i_{\mathcal{X}_t}\Omega'_0 + i_{K_t\mathcal{X}_t}\Omega'_0 = -\Gamma.$$

By Lemma 4.7, in coordinates and for $\xi = \eta, \overline{\eta}$ the last equation is schematically of the form

$$\begin{aligned}&\mathcal{X}^t_{jA} + \sum_{\ell,B}\mathcal{R}^{1,0}_{r,\infty}\mathcal{X}^t_{\ell B} + \sum_{\xi=\eta,\overline{\eta}}\langle \mathbf{S}^{1,1}_{r,\infty}, \mathcal{X}^t_\xi\rangle = \mathcal{R}^{1,1}_{r,\infty}\\ &\mathcal{X}^t_\xi + \sum_{\ell B}\mathbf{S}^{1,0}_{r,\infty}\mathcal{X}^t_{\ell B} + \sum_{\xi'=\eta,\overline{\eta}}(\partial_{\xi'}\mathbf{S}^{1,1}_{\infty,\infty}(t,z,\mathbf{Z},\eta) - (\partial_\xi\mathbf{S}^{1,1}_{\infty,\infty}(t,z,\mathbf{Z},\eta))^*\mathcal{X}^t_{\xi'} = \mathbf{S}^{1,1}_{r,\infty}.\end{aligned} \tag{4.32}$$

Notice that $(\partial_\xi\mathbf{S}^{1,1}_{\infty,\infty})\mathbf{S}^{1,1}_{r,\infty}$ is C^∞ in $(t,z,\mathbf{Z},\eta)$ with values in Σ_r. We have

$$\|(\partial_\xi\mathbf{S}^{1,1}_{\infty,\infty})\mathbf{S}^{1,1}_{r,\infty}\|_{\Sigma_r} \le \|\partial_\xi\mathbf{S}^{1,1}_{\infty,\infty}\|_{B(\Sigma_{-r},\Sigma_r)}\|\mathbf{S}^{1,1}_{r,\infty}\|_{\Sigma_r}.$$

By (2.26) we have $\partial_\xi\mathbf{S}^{1,1}_{\infty,\infty}(t,0,0,0)$. This implies

$$\|\partial_\xi\mathbf{S}^{1,1}_{\infty,\infty}\|_{B(\Sigma_{-r},\Sigma_r)} \le C\|\eta\|_{\Sigma_{-K}} + |\mathbf{Z}| + |z| \tag{4.33}$$

and so

$$\|(\partial_\xi\mathbf{S}^{1,1}_{\infty,\infty})\mathbf{S}^{1,1}_{r,\infty}\|_{\Sigma_r} \le C(\|\eta\|_{\Sigma_{-K}} + |\mathbf{Z}|)(\|\eta\|_{\Sigma_{-K}} + |\mathbf{Z}| + |z|)^{i2}.$$

So $(\partial_\xi\mathbf{S}^{1,1}_{\infty,\infty})\mathbf{S}^{1,1}_{r,\infty} = \mathbf{S}^{2,1}_{r,\infty}$.
Inequality (4.33), a Neumann expansion and formulas (2.27) yield claim (1) in Lemma 4.6.

Claim (2) in Lemma 4.6 follows from

$$i_{\Phi_*^{-1}\mathcal{X}^t}\Phi^*\Omega_t = -\Phi^*\Gamma = -\Gamma = i_{\mathcal{X}^t}\Omega_t = i_{\Phi_*^{-1}\mathcal{X}^t}\Omega_t,$$

where $\Phi^*\Gamma = \Gamma$ is (4.19) and we use (4.14) and (4.29) to conclude $\Phi^*\Omega_t = \Omega_t$. Then $\Phi_*^{-1}\mathcal{X}^t = \mathcal{X}^t$, which is equivalent to $\Phi_*\mathcal{X}^t = \mathcal{X}^t$. For the other formulas in claim (2) we have for instance

$$\begin{aligned}&\mathcal{X}_j^t(e^{\mathrm{i}\vartheta}z, e^{\mathrm{i}\vartheta}\eta) = \mathcal{X}_j^t(\Phi(u)) = dz_j(\mathcal{X}^t(\Phi(u))) = dz_j(\Phi_*\mathcal{X}^t(u))\\ &= d(z_j\circ\Phi)(\mathcal{X}^t(u)) = e^{\mathrm{i}\vartheta}\mathcal{X}_j^t(u).\end{aligned}$$

This ends the proof of Lemma 4.6, assuming Lemma 4.7.

□

Proof of Lemma 4.7. By (4.13) and summing over the indexes (j, A, B) we can write

$$\Omega_0 - \Omega_0' = \mathcal{R}^{4,0}_{\infty,\infty} dz_{jA}\wedge dz_{jB} \Rightarrow i_X(\Omega_0 - \Omega_0') = \mathcal{R}^{4,0}_{\infty,\infty}X_{jR}dz_{jI} + \mathcal{R}^{4,0}_{\infty,\infty}X_{jI}dz_{jR}. \tag{4.34}$$

So if we define $K'X$ setting $i_{K'X}\Omega_0' = i_X(\Omega_0 - \Omega_0')$, by comparing (4.34) with

$$i_{K'X}\Omega_0' = 2(K'X)_{jR}dz_{jI} - 2(K'X)_{jI}dz_{jR} + \mathrm{i}\langle (K'X)_\eta, X_{\overline{\eta}}\rangle - \mathrm{i}\langle (K'X)_{\overline{\eta}}, X_\eta\rangle,$$

we obtain

$$(K'X)_{jA} = \mathcal{R}^{4,0}_{\infty,\infty}X_{jA} \text{ and } (K'X)_\xi = 0 \text{ for } \xi = \eta, \overline{\eta}. \tag{4.35}$$

Summing on (j, l, A, B, ξ, ξ') we have

$$t\widehat{\Omega} = \mathcal{R}^{1,0}_{\infty,\infty}dz_{jA}\wedge dz_{lB} + dz_{jA}\wedge\langle \mathbf{S}^{1,0}_{\infty,\infty}, d\xi\rangle + t\langle[\partial_\xi \mathbf{S}^{1,1}_{\infty,\infty}(z,\mathbf{Z},\eta) - (\partial_{\xi'}\mathbf{S}^{1,1}_{\infty,\infty}(z,\mathbf{Z},\eta))^*]d\xi, d\xi'\rangle.$$

Hence

$$t\ i_X\widehat{\Omega} = \mathcal{R}^{1,0}_{\infty,\infty}X_{jA}dz_{lB} + \langle \mathbf{S}^{1,0}_{\infty,\infty}, X_\xi\rangle dz_{jA} + X_{jA}\langle \mathbf{S}^{1,0}_{\infty,\infty}, d\xi\rangle + \langle[\partial_\xi \mathbf{S}^{1,1}_{\infty,\infty} - (\partial_{\xi'}\mathbf{S}^{1,1}_{\infty,\infty})^*]X_\xi, d\xi'\rangle.$$

So, if we define $K''X$ setting $i_{K''X}\Omega_0' = t\ i_X\widehat{\Omega}$, we obtain

$$\begin{aligned}(K''X)_{jA} &= \sum_{\ell B}\mathcal{R}^{1,0}_{\infty,\infty}X_{\ell B} + \sum_{\xi=\eta,\overline{\eta}}\langle \mathbf{S}^{1,0}_{\infty,\infty}, X_\xi\rangle,\\ (K''X)_\xi &= \sum_{lB}\mathbf{S}^{1,0}_{\infty,\infty}X_{lB} + \sum_{\xi=\eta,\overline{\eta}}[\partial_{\xi'}\mathbf{S}^{1,1}_{\infty,\infty} - (\partial_\xi \mathbf{S}^{1,1}_{\infty,\infty})^*]X_{\xi'}.\end{aligned} \tag{4.36}$$

Since $K_t = K' + K''$, summing up (4.35) and (4.36) we get (4.31) and so Lemma 4.7.

□

Having established that $\mathcal{X}^t(z,\eta)$ has components which are restrictions of symbols as in Definitions 2.8 and 2.9 we have the following result.

Lemma 4.8. *Fix an $r\in\mathbb{N}$ and for the $\delta_0 > 0$ and the $\mathcal{X}^t(z,\eta)$ of Lemma 4.6, consider the following system, which is well defined in $(t, z, \eta)\in(-4,4)\times B_{\mathbb{C}^n}(0,\delta_0)\times B_{\Sigma_k^c}(0,\delta_0)$ for all $k\in\mathbb{Z}\cap[-r,r]$:*

$$\dot{z}_j = \mathcal{X}_j^t(z,\eta) \text{ and } \dot{\eta} = \mathcal{X}_\eta^t(z,\eta). \tag{4.37}$$

Then the following facts hold.

(1) For $\delta_1 \in (0,\delta_0)$ sufficiently small system (4.37) generates flows

$$\mathfrak{F}^t \in C^\infty((-2,2)\times B_{\mathbb{C}^n}(0,\delta_1)\times B_{\Sigma_k^c}(0,\delta_1), B_{\mathbb{C}^n}(0,\delta_0)\times B_{\Sigma_k^c}(0,\delta_0)) \text{ for all } k\in\mathbb{Z}\cap[-r,r]$$
$$\mathfrak{F}^t \in C^\infty((-2,2)\times B_{\mathbb{C}^n}(0,\delta_1)\times B_{H^1\cap\mathcal{H}_c[0]}(0,\delta_1), B_{\mathbb{C}^n}(0,\delta_0)\times B_{H^1\cap\mathcal{H}_c[0]}(0,\delta_0)). \quad (4.38)$$

In particular for $z_j^t := z_j\circ\mathfrak{F}^t(z,\eta)$ and $\eta^t := \eta\circ\mathfrak{F}^t(z,\eta)$ we have

$$z_j^t = z_j + S_j(t,z,\eta) \text{ and } \eta^t = \eta + S_\eta(t,z,\eta) \quad (4.39)$$

with $S_j(t,z,\eta) = \mathcal{R}^{1,1}_{r,\infty}(t,z,\boldsymbol{Z},\eta)$ and $S_\eta(t,z,\eta) = \boldsymbol{S}^{1,1}_{r,\infty}(t,z,\boldsymbol{Z},\eta)$ in the sense of Remark 2.10.

(2) $\mathfrak{F}=\mathfrak{F}^1$ is a local diffeomorphism of H^1 into itself near the origin s.t. $\mathfrak{F}^*\Omega=\Omega_0$.

(3) We have $S_j(t,e^{\mathrm{i}\vartheta}z,e^{\mathrm{i}\vartheta}\eta)=e^{\mathrm{i}\vartheta}S_j(t,z,\eta)$, $S_\eta(t,e^{\mathrm{i}\vartheta}z,e^{\mathrm{i}\vartheta}\eta)=e^{\mathrm{i}\vartheta}S_\eta(t,z,\eta)$.

Proof. The first sentence has been established in Lemma 4.6. Elementary theory of ODE's yields (4.38). The rest of claim (1) is a special case of a more general result, see Lemma 4.9 below. We get claim (2) by the classical formula, for L_X the Lie derivative,

$$\partial_t(\mathfrak{F}^{t*}\Omega_t) = \mathfrak{F}^{t*}(L_{\mathcal{X}^t}\Omega_t + \partial_t\Omega_t) = \mathfrak{F}^{t*}(di_{\mathcal{X}^t}\Omega_t + d\Gamma) = 0. \quad (4.40)$$

Notice that (4.40) is well defined here, while it has no clear meaning for the NLS with translation treated in [4, 6], where the flows $\mathfrak{F}^t$ are not differentiable (see [4] for a rigorous argument on how to offset this problem). The symmetry in claim (3) is elementary and we skip it.

□

Lemma 4.9. *Consider a system*

$$\dot z_j = X_j(t,z,\eta) \text{ and } \dot\eta = X_\eta(t,z,\eta), \quad (4.41)$$

where $X_j=\mathcal{R}^{a,b}_{r,m}(t,z,\mathbf{Z},\eta)\ \forall\ j$ and $X_\eta=\mathbf{S}^{c,d}_{r,m}(t,z,\mathbf{Z},\eta)$, for fixed pairs (r,m), (a,b) and (c,d). Assume $m,b,d\ge1$, with possibly $m=\infty$, and $r\ge0$. Then for the flow $(z^t,\eta^t)=\mathfrak{F}^t(z,\eta)$ we have

$$z_j^t = z_j + S_j(t,z,\eta) \text{ and } \eta^t = \eta + S_\eta(t,z,\eta) \quad (4.42)$$

for appropriate functions $S_j=\mathcal{R}^{a,b}_{r,m}(t,z,\mathbf{Z},\eta)$ and $S_\eta=\mathbf{S}^{c,d}_{r,m}(t,z,\mathbf{Z},\eta)$ in the sense of Remark 2.10.

Proof. Consider the vectors $\mathbf{Z}=(z_i\overline{z}_j)_{i,j=1,\dots,n}$ with $i\neq j$. Notice that $\dot{\mathbf{Z}}=\mathcal{R}^{a+1,b}_{r,m}(t,z,\mathbf{Z},\eta)$, and this equation can be extended to a whole neighborhood of 0 in the space L. Pairing the latter equation with equations (4.42), a system remains defined which has a flow $\mathfrak{F}^t(z,\mathbf{Z},\eta)$ which is C^m in $(t,z,\mathbf{Z},\eta)$ and which reduces to the flow in (4.41) when we restrict to vectors $\mathbf{Z}\in\{(z_i\overline{z}_j)_{i,j=1,\dots,n} : i\neq j\}$, by construction. The inequalities (2.23) and (2.26), required to prove $S_j=\mathcal{R}^{a,b}_{r,m}$ and $S_\eta=\mathbf{S}^{c,d}_{r,m}$, can be obtained as follows. We have for all $|k|\le r$

$$|z^t-z| \le \int_0^t |\mathcal{R}^{a,b}_{r,m}(s,z^s,\mathbf{Z}^s,\eta^s)|ds \le C\int_0^t (\|\eta^s\|_{\Sigma_{-r}}+|\mathbf{Z}^s|)^b(\|\eta^s\|_{\Sigma_{-r}}+|\mathbf{Z}^s|+|z^s|)^a ds,$$
$$\|\eta^t-\eta\|_{\Sigma_k} \le \int_0^t \|\mathbf{S}^{c,d}_{r,m}(s,z^s,\mathbf{Z}^s,\eta^s)\|_{\Sigma_k} ds \le C\int_0^t (\|\eta^s\|_{\Sigma_{-r}}+|\mathbf{Z}^s|)^d(\|\eta^s\|_{\Sigma_{-r}}+|\mathbf{Z}^s|+|z^s|)^c ds$$
$$|\mathbf{Z}^t-\mathbf{Z}| \le \int_0^t |\mathcal{R}^{a,b}_{r,m}(s,z^s,\mathbf{Z}^s,\eta^s)|ds \le C\int_0^t (\|\eta^s\|_{\Sigma_{-r}}+|\mathbf{Z}^s|)^b(\|\eta^s\|_{\Sigma_{-r}}+|\mathbf{Z}^s|+|z^s|)^{a+1} ds. \quad (4.43)$$

By Gronwall inequality we get that $|\mathbf{Z}^t|$ and $\|\eta^t\|_{\Sigma_{-r}}$ are bounded by $C(|\mathbf{Z}| + \|\eta\|_{\Sigma_{-r}})$. Plugging this in the r.h.s. of (4.43), we obtain the last part of the statement.

□

We discuss the pullback of the energy E by the map $\mathfrak{F} := \mathfrak{F}^1$ in claim (2) of Lemma 4.8. We set $H_2(z,\eta) = \sum_{j=1}^n e_j |z_j|^2 + \langle H\eta, \overline{\eta}\rangle$. Our first preliminary result is the following one.

Lemma 4.10. *Consider the $\delta_1 > 0$ of Lemma 4.8, the $\delta_0 > 0$ of Lemma 4.6 and set $r = r_0$ with r_0 the index in Lemma 3.1. Then for the map $\mathfrak{F}$ in claim (2) of Lemma 4.8 we have*

$$\mathfrak{F}(B_{\mathbb{C}^n}(0,\delta_1) \times (B_{H^1}(0,\delta_1) \cap \mathcal{H}_c[0])) \subset B_{\mathbb{C}^n}(0,\delta_0) \times (B_{H^1}(0,\delta_0) \cap \mathcal{H}_c[0]) \tag{4.44}$$

and $\mathfrak{F}|_{B_{\mathbb{C}^n}(0,\delta_1)\times(B_{H^1}(0,\delta_1)\cap\mathcal{H}_c[0])}$ is a diffeomorphism between domain and an open neighborhood of the origin in $\mathbb{C}^n \times (H^1 \cap \mathcal{H}_c[0])$. Furthermore, the functional $K := E \circ \mathfrak{F}$ admits an expansion

$$\begin{aligned} K(z,\eta) &= H_2(z,\eta) + \sum_{j=1,\dots,n} \lambda_j(|z_j|^2) \\ &+ \sum_{l=0}^{2N+3} \sum_{|\boldsymbol{m}|=l+1} \boldsymbol{Z}^{\boldsymbol{m}} a_{\boldsymbol{m}}^{(1)}(|z_1|^2,\dots,|z_n|^2) + \sum_{j=1}^{n} \sum_{l=0}^{2N+3} \sum_{|\boldsymbol{m}|=l} (\overline{z}_j \boldsymbol{Z}^{\boldsymbol{m}} \langle G_{j\boldsymbol{m}}^{(1)}(|z_j|^2), \eta\rangle + c.c.) \\ &+ \mathcal{R}^{1,2}_{r_1,\infty}(z,\eta) + \mathcal{R}^{0,2N+5}_{r_1,\infty}(z,\boldsymbol{Z},\eta) + \mathrm{Re}\langle \boldsymbol{S}^{0,2N+4}_{r_1,\infty}(z,\boldsymbol{Z},\eta), \overline{\eta}\rangle \\ &+ \sum_{i+j=2} \sum_{|\boldsymbol{m}|\le 1} \boldsymbol{Z}^{\boldsymbol{m}} \langle G^{(1)}_{2\boldsymbol{m}ij}(z,\eta), \eta^i \overline{\eta}^j\rangle + \sum_{d+c=3} \sum_{i+j=d} \langle G^{(1)}_{dij}(z), \eta^i\overline{\eta}^j\rangle \mathcal{R}^{0,c}_{r,\infty}(z,\eta) + E_P(\eta) , \end{aligned} \tag{4.45}$$

where: $r_1 = r_0 - 2$; $G^{(1)}_{j\boldsymbol{m}}$, $G^{(1)}_{2\boldsymbol{m}ij}$ and $G^{(1)}_{dij}$ are $\boldsymbol{S}^{0,0}_{r_1,\infty}$; $a^{(1)}_{\boldsymbol{m}}(|z_1|^2,\dots,|z_n|^2) = \mathcal{R}^{0,0}_{\infty,\infty}(z)$; c.c. means complex conjugate; $\lambda_j(|z_j|^2) = \mathcal{R}^{2,0}_{\infty,\infty}(|z_j|^2)$. For $|\boldsymbol{m}| = 0$, $G^{(1)}_{2\boldsymbol{m}ij}(z,\eta) = G_{2\boldsymbol{m}ij}(z)$ is the same of (3.4). Finally, we have the invariance $\mathcal{R}^{1,2}_{r_1,\infty}(e^{i\vartheta}z, e^{i\vartheta}\eta) \equiv \mathcal{R}^{1,2}_{r_1,\infty}(z,\eta)$.

Proof. Consider the expansion (3.3) for $E(u(z',\eta'))$, and substitute the formulas $z'_j = z_j + S_j(z,\eta)$ and $\eta' = \eta + S_\eta(z,\eta)$, with $S_\ell(z,\eta) = S_\ell(1,z,\eta)$ for $\ell = j,\overline{j},\eta,\overline{\eta}$, with $S_{\overline{\ell}} = \overline{S}_\ell$. By $S_j(z,\eta) = \mathcal{R}^{1,1}_{r_0,\infty}(z,\mathbf{Z},\eta)$ and $S_\eta(z,\eta) = \mathbf{S}^{1,1}_{r_0,\infty}(z,\mathbf{Z},\eta)$ it is elementary to see that the last three lines of (3.3) yield terms that can be absorbed in last three lines of (4.45) (with $l \ge 1$ in the 2nd line). Notice that the z dependence of the $a^{(1)}_{\mathbf{m}}$ in terms of $(|z_1|^2,\dots,|z_n|^2)$ follows by Lemmas 4.8 and B.3. The z dependence of the $G^{(1)}_{j\mathbf{m}}$ is obtained by Lemma B.4. Notice also that if an $\mathcal{R}^{i,0}_{r,\infty}(z)$ depends only on z, then it is an $\mathcal{R}^{i,0}_{\infty,\infty}(z)$.

We have $\mathcal{R}^{1,2}_{r_0,\infty}(z',\eta') = \mathcal{R}^{1,2}_{r_0,\infty}(z,\mathbf{Z},\eta)$. Notice that by the invariance of $\mathcal{R}^{1,2}_{r_0,\infty}(z,\eta)$ and by claim (3) in Lemma 4.8 we have $\mathcal{R}^{1,2}_{r_0,\infty}(e^{i\vartheta}z,\mathbf{Z},e^{i\vartheta}\eta) \equiv \mathcal{R}^{1,2}_{r_0,\infty}(z,\mathbf{Z},\eta)$. By Taylor expansion (using the conventions under (3.14))

$$\mathcal{R}^{1,2}_{r_0,\infty}(z,\mathbf{Z},\eta) = \mathcal{R}^{1,2}_{r_0,\infty}(z,\mathbf{Z},0) + d_\eta \mathcal{R}^{1,2}_{r_0,\infty}(z,\mathbf{Z},0)\eta + \int_0^1 (1-t)\partial_\eta^2 \mathcal{R}^{1,2}_{r_0,\infty}(z,\mathbf{Z},t\eta)dt \cdot \eta^2. \tag{4.46}$$

Each of the terms in the r.h.s. is invariant by change of variables $(z,\eta) \rightsquigarrow (e^{i\vartheta}z, e^{i\vartheta}\eta)$. We have

$$\mathcal{R}^{1,2}_{r_0,\infty}(z,\mathbf{Z},\eta)|_{\eta=0} = \mathcal{R}^{1,2}_{\infty,\infty}(z,\mathbf{Z}) = \sum_{k\le 2N+4} \frac{1}{k!} d^k_{\mathbf{Z}} \mathcal{R}^{1,2}_{\infty,\infty}(z,0)\mathbf{Z}^k + \mathcal{R}^{1,2N+5}_{\infty,\infty}(z,\mathbf{Z}) =$$

$$\mathcal{R}^{1,2N+5}_{\infty,\infty}(z,\mathbf{Z}) + \sum_{l=2}^{2N+4} \sum_{|\mathbf{m}|=l+1} \mathbf{Z}^{\mathbf{m}} c_{\mathbf{m}}(z) = \mathcal{R}^{1,2N+5}_{\infty,\infty}(z,\mathbf{Z}) + \sum_{l=2}^{2N+4} \sum_{|\mathbf{m}|=l+1} \mathbf{Z}^{\mathbf{m}} \sum_{j=1}^{n} c_{j\mathbf{m}}(|z_j|^2),$$

where, as in step 1 in Lemma 3.1, the last equality is obtained by the invariance w.r.t $(z,\eta) \rightsquigarrow (e^{\mathrm{i}\vartheta}z, e^{\mathrm{i}\vartheta}\eta)$ and by smoothness. We have proceeding like above

$$d_\eta \mathcal{R}^{1,2}_{r_0,\infty}(z,\mathbf{Z},0)\eta = \mathrm{Re}\langle \mathbf{S}^{1,1}_{r_0,\infty}(z,\mathbf{Z}),\overline{\eta}\rangle = \sum_{k\le 2N+3}\frac{1}{k!}\mathrm{Re}\langle d^k_{\mathbf{Z}}\mathbf{S}^{1,1}_{r_0,\infty}(z,0),\overline{\eta}\rangle \mathbf{Z}^k$$

$$+\mathrm{Re}\langle \mathbf{S}^{1,2N+4}_{r_0,\infty}(z,\mathbf{Z},\eta),\overline{\eta}\rangle = \mathrm{Re}\langle \mathbf{S}^{1,2N+4}_{r_0,\infty}(z,\mathbf{Z},\eta),\overline{\eta}\rangle + \sum_{j=1}^n\sum_{l=1}^{2N+3}\sum_{|\mathbf{m}|=l}(\overline{z}_j\mathbf{Z}^{\mathbf{m}}\langle A_{j\mathbf{m}}(|z_j|^2),\eta\rangle + c.c.),$$

Finally, for a $\mathcal{R}^{1,2}_{r_0,\infty}(e^{\mathrm{i}\vartheta}z, e^{\mathrm{i}\vartheta}\eta) \equiv \mathcal{R}^{1,2}_{r_0,\infty}(z,\eta)$ we have, see Definition 2.8,

$$\int_0^1 (1-t)\partial^2_\eta \mathcal{R}^{1,2}_{r_0,\infty}(z,\mathbf{Z},t\eta)dt\eta^2 = \mathcal{R}^{1,2}_{r_0,\infty}(z,\eta).$$

By (4.46) and the subsequent formulas we see that $\mathcal{R}^{1,2}_{r_0,\infty}(z',\eta')$ is absorbed in last three lines of (4.45) (with $l \ge 1$ in the 2nd line). The term $\langle H\eta',\overline{\eta}'\rangle = \langle H\eta,\overline{\eta}\rangle + \mathcal{R}^{1,2}_{r_0-2,\infty}(z,\mathbf{Z},\eta)$ behaves similarly, recalling that $r_1 = r_0 - 2$. Here too we have $\mathcal{R}^{1,2}_{r_0-2,\infty}(e^{\mathrm{i}\vartheta}z,\mathbf{Z},e^{\mathrm{i}\vartheta}\eta) \equiv \mathcal{R}^{1,2}_{r_0-2,\infty}(z,\mathbf{Z},\eta)$. This function can be treated like the $\mathcal{R}^{1,2}_{r_0,\infty}(z,\mathbf{Z},\eta)$ discussed earlier.

The terms $E(Q_{jz_j})$ and, for $j \ne k$, $\mathrm{Re}\langle q_{jz_j},\overline{z}_k\phi_k\rangle = \mathcal{R}^{1,1}_{\infty,\infty}(z,\mathbf{Z})$ can be expanded similarly. But this time we need $l = 0$ in the 2nd line.

□

The expansion in Lemma 4.10 is too crude. We have the following additional and crucial fact.

Lemma 4.11 (Cancellation Lemma)**.** *In the 2nd line of* (4.45) *all the terms with* $l = 0$ *are zeros.*

Proof. We first observe that the terms in the 2nd line of (4.45) with $l = 0$ can be written as

$$\sum_{k=1}^n\sum_{j\ne k}\sum_{A=R,I} z_{jA}b_{kjA}(z_k) + \sum_{k=1}^n \mathrm{Re}\langle \mathbf{A}_k(z_k),\overline{\eta}\rangle. \tag{4.47}$$

Indeed they are

$$\sum_{|\mathbf{m}|=1}\mathbf{Z}^{\mathbf{m}}a^{(1)}_{\mathbf{m}}(|z_1|^2,...,|z_n|^2) + \sum_{j=1}^n(\overline{z}_j\langle G^{(1)}_{j\mathbf{0}}(|z_j|^2),\eta\rangle + c.c.), \tag{4.48}$$

and it is obvious that the 2nd term of (4.48) is the second term of (4.47). Arguing as in Lemma 3.1, the first term of (4.48) can be written as

$$\sum_{k=1}^n\sum_{|\mathbf{m}|=1}\mathbf{Z}^{\mathbf{m}}a^{(1)}_{k\mathbf{m}}(|z_k|^2)$$

Further, for $\mathbf{Z}^{\mathbf{m}} = z_i\overline{z}_j$, we can assume that i or j must equal to k, because if not, it can be absorbed in the terms with $l \ge 1$. Set $\mathcal{N}_k := \{\mathbf{m} \mid |\mathbf{m}| = 1,\ m_{i,j} = 0 \text{ if } i \ne k \text{ and } j \ne k\}$. We have

$$\sum_{k=1}^n\sum_{|\mathbf{m}|=1}\mathbf{Z}^{\mathbf{m}}a^{(1)}_{k\mathbf{m}}(|z_k|^2) = \sum_{k=1}^n\sum_{\mathbf{m}\in\mathcal{N}_k}\mathbf{Z}^{\mathbf{m}}a^{(1)}_{k\mathbf{m}}(|z_k|^2) = \sum_{k=1}^n\sum_{j\ne k}(z_j\overline{z}_k a^{(1)}_{km_{jk}}(|z_k|^2) + z_k\overline{z}_j a^{(1)}_{km_{jk}}(|z_k|^2)).$$

So, we can write the term in the form of the first term of (4.47).

Next, notice that for $p_k = (0, \cdots, 0, z_k, \cdots, 0; 0)$,

$$b_{kjA}(z_k) = \partial_{z_{jA}} K(z,\eta)\big|_{p_k} \text{ and } \mathbf{A}_k(z_k) = \nabla_\eta K(p_k). \tag{4.49}$$

Therefore, it suffices to show the r.h.sides in (4.49) are both zero. Recall $u(z,\eta) = \sum_{j=1}^n Q_{jz_j} + R[z]\eta$. We have

$$\begin{aligned}\partial_{z_{jA}} K(z,\eta)\big|_{p_k} &= \partial_{z_{jA}} E(u(z'(z,\eta),\eta'(z,\eta)))\big|_{p_k}\\ &= \mathrm{Re}\langle \nabla E(u(z'(p_k),\eta'(p_k))), \overline{\partial_{z_{jA}} u(z'(z,\eta),\eta'(z,\eta))\big|_{p_k}}\rangle.\end{aligned}$$

By Lemma 4.8 we have

$$(z'(p_k), \eta'(p_k)) = p_k. \tag{4.50}$$

So

$$\nabla E(u(z'(p_k),\eta'(p_k))) = \nabla E(Q_{kz_k}) = 2E_{kz_k} Q_{kz_k}.$$

By Prop. 1.1 and by (4.50), for $z_k = e^{\mathrm{i}\vartheta_k}\rho_k$ we have

$$-\mathrm{i}\mathfrak{F}_* \frac{\partial}{\partial\vartheta_k}|_{p_k} = -\mathrm{i}\frac{\partial}{\partial\vartheta_k}(\sum_{j=1}^n Q_{jz'_j} + R[z']\eta')|_{p_k} = -\mathrm{i}\frac{\partial}{\partial\vartheta_k} Q_{kz_k} = -\mathrm{i}\frac{\partial}{\partial\vartheta_k} e^{\mathrm{i}\vartheta_k} Q_{k\rho_k} = Q_{kz_k},$$

where the 1st equality follows by definition of push forward, the 2nd by (4.50) and the 3rd by Prop.1.1. Similarly, by the definition of push forward, we have

$$\partial_{z_{jA}} u(z'(z,\eta),\eta'(z,\eta))\big|_{p_k} = \mathfrak{F}_* \,\partial_{z_{jA}}\big|_{p_k}.$$

Therefore $b_{kjA}(z_k) = 0$ follows by

$$\partial_{z_{jA}} K(z,\eta)\big|_{p_k} = 2E_{kz_k} \,\mathrm{Im}\langle \mathfrak{F}_*\partial_{\vartheta_k}|_{p_k}, \overline{\mathfrak{F}_*\partial_{z_{jA}}}|_{p_k}\rangle = -E_{kz_k}\,\Omega_0(\partial_{\vartheta_k}, \partial_{z_{j,A}})\big|_{p_k} = 0.$$

To get $\mathbf{A}_k(z_k) = 0$, fix $\Xi \in \mathcal{H}_c[0]$ and set $p_{k,\Xi}(t) := (0, \cdots, 0, z_k, 0, \cdots, 0; t\Xi)$. Then $\forall\,\Xi$

$$\begin{aligned}\mathrm{Re}\langle \nabla K(p_k), \Xi\rangle &= \frac{d}{dt}\,K(p_{k,\Xi}(t))|_{t=0} = \frac{d}{dt}\,E(u(z'(p_{k,\Xi}(t)),\eta'(p_{k,\Xi}(t))))|_{t=0}\\ &= \mathrm{Re}\langle \nabla E(Q_{kz_k}), \overline{\frac{d}{dt}\,u(z'(p_{k,\Xi}(t)),\eta'(p_{k,\Xi}(t)))|_{t=0}}\rangle\\ &= 2E_{kz_k}\,\mathrm{Im}\langle \mathfrak{F}_*\frac{\partial}{\partial\vartheta_k}|_{p_k}, \overline{\mathfrak{F}_*\Xi}\rangle = -E_{kz_k}\,\Omega_0(\frac{\partial}{\partial\vartheta_k},\Xi)\bigg|_{p_k} = 0 \Rightarrow \mathbf{A}_k(z_k) = 0.\end{aligned}$$

□

5 Birkhoff normal form

In this section, where we search the effective Hamiltonian, the main result is Theorem 5.9. We consider the symplectic form Ω_0 introduced in (4.13). We introduce an index $\ell = j, \overline{j}$, for $\overline{\overline{j}} = j$ with $j = 1, ..., n$. We write $\partial_j = \partial_{z_j}$ and $\partial_{\overline{j}} = \partial_{\overline{z}_j}$, $z_{\overline{j}} = \overline{z}_j$. With this notation, summing on j, by (4.8) and (4.34) for $\gamma_j(|z_j|^2) = \mathcal{R}^{2,0}_{\infty,\infty}(|z_j|^2)$ we have

$$\Omega_0 = \mathrm{i}(1+\gamma_j(|z_j|^2))dz_j \wedge d\overline{z}_j + \mathrm{i}\,\langle d\eta, d\overline{\eta}\rangle - \mathrm{i}\,\langle d\overline{\eta}, d\eta\rangle\,. \tag{5.1}$$

Given $F \in C^1(U, \mathbb{R})$ with U an open subset of $\mathbb{C}^n \times \Sigma_r^c$, its Hamiltonian vector field X_F is defined by $i_{X_F}\Omega_0 = dF$. We have summing on j

$$\begin{aligned} i_{X_F}\Omega_0 &= \mathrm{i}(1+\gamma_j(|z_j|^2))((X_F)_j d\overline{z}_j - (X_F)_{\overline{j}} dz_j) + \mathrm{i}\,\langle (X_F)_\eta, d\overline{\eta}\rangle - \mathrm{i}\,\langle (X_F)_{\overline{\eta}}, d\eta\rangle \\ &= \partial_j F dz_j + \partial_{\overline{j}} F d\overline{z}_j + \langle \nabla_\eta F, d\eta\rangle + \langle \nabla_{\overline{\eta}} F, d\overline{\eta}\rangle \,. \end{aligned}$$

So comparing the components of the two sides we get for $1 + \varpi_j(|z_j|^2) = (1+\gamma_j(|z_j|^2))^{-1}$ where $\varpi_j(|z_j|^2) = \mathcal{R}^{2,0}_{\infty,\infty}(|z_j|^2)$:

$$\begin{aligned} &(X_F)_j = -\mathrm{i}(1+\varpi_j(|z_j|^2))\partial_{\overline{j}} F \ , \quad (X_F)_{\overline{j}} = \mathrm{i}(1+\varpi_j(|z_j|^2))\partial_j F \\ &(X_F)_\eta = -\mathrm{i}\nabla_{\overline{\eta}} F \ , \quad (X_F)_{\overline{\eta}} = \mathrm{i}\nabla_\eta F. \end{aligned} \tag{5.2}$$

Given $G \in C^1(U, \mathbb{R})$ and $F \in C^1(U, \mathbf{E})$ with $\mathbf{E}$ a Banach space, we set $\{F, G\} := dF X_G$.

Definition 5.1 (Normal Form)**.** Recall Def. 2.5 and in particular (2.13). Fix $r \in \mathbb{N}_0$. A real valued function $Z(z, \eta)$ is in normal form if $Z = Z_0 + Z_1$ where Z_0 and Z_1 are finite sums of the following type for $\mathbf{l} \geq 1$ and for $\mathbf{Z} = (z_i \overline{z}_j)_{i,j=1,\ldots,n}$ where $i \neq j$:

$$Z_1(z, \mathbf{Z}, \eta) = \sum_{j=1}^{n} \sum_{\substack{|\mathbf{m}|=\mathbf{l} \\ \mathbf{m}\in\mathcal{M}_j(\mathbf{l})}} \left(\overline{z}_j \mathbf{Z}^{\mathbf{m}} \langle G_{j\mathbf{m}}(|z_j|^2), \eta\rangle + \text{c.c.}\right) , \text{ where } G_{j\mathbf{m}}(|z_j|^2) = \mathbf{S}^{0,0}_{r,\infty}(|z_j|^2) \tag{5.3}$$

and where c.c. means complex conjugate; for $a_{\mathbf{m}}(|z_1|^2, \ldots, |z_n|^2) = \mathcal{R}^{0,0}_{r,\infty}(|z_1|^2, \ldots, |z_n|^2)$

$$Z_0(z, \mathbf{Z}) = \sum_{\substack{|\mathbf{m}|=\mathbf{l}+1 \\ \mathbf{m}\in\mathcal{M}_0(\mathbf{l}+1)}} \mathbf{Z}^{\mathbf{m}} a_{\mathbf{m}}(|z_1|^2, \ldots, |z_n|^2) \ . \tag{5.4}$$

Remark 5.2. By Lemma 2.6, $\mathbf{Z}^{\mathbf{m}} = |z_1|^{2m_1} \ldots |z_n|^{2m_n}$ $\forall$ $\mathbf{m} \in \mathcal{M}_0(2N+4)$ for an $m \in \mathbb{N}_0^n$ with $2|m| = |\mathbf{m}|$. By Lemma 2.6 for $|\mathbf{m}| \leq 2N+3$ either $\sum_{a,b}(e_a - e_b)m_{ab} - e_j > 0$ or $\sum_{a,b}(e_a - e_b)m_{ab} - e_j < 0$.

For $\mathbf{l} \leq 2N+4$ we will consider flows associated to Hamiltonian vector fields X_χ with real valued functions χ of the following form, with $b_{\mathbf{m}} = \mathcal{R}^{0,0}_{\mathbf{r},\infty}(|z_1|^2, \ldots, |z_n|^2)$ and $B_{j\mathbf{m}} = \mathbf{S}^{0,0}_{\mathbf{r},\infty}(|z_j|^2)$ for some $\mathbf{r} \in \mathbb{N}$ defined in $B_{\mathbb{C}^n}(0, \mathbf{d})$ for some $\mathbf{d} > 0$:

$$\chi = \sum_{\substack{|\mathbf{m}|=\mathbf{l}+1 \\ \mathbf{m}\notin\mathcal{M}_0(\mathbf{l}+1)}} \mathbf{Z}^{\mathbf{m}} b_{\mathbf{m}}(|z_1|^2, \ldots, |z_n|^2) + \sum_{j=1}^{n} \sum_{\substack{|\mathbf{m}|=\mathbf{l} \\ \mathbf{m}\notin\mathcal{M}_j(\mathbf{l})}} (\overline{z}_j \mathbf{Z}^{\mathbf{m}} \langle B_{j\mathbf{m}}(|z_j|^2), \eta\rangle + \text{c.c.}) \ . \tag{5.5}$$

The Hamiltonian vector field X_χ can be explicitly computed using (5.2). We have

$$(X_\chi)_j = (Y_\chi)_j + (\tilde{Y}_\chi)_j \ , \quad (X_\chi)_\eta = -\mathrm{i} \sum_{j=1}^{n} \sum_{\substack{|\mathbf{m}|=\mathbf{l} \\ \mathbf{m}\notin\mathcal{M}_j(\mathbf{l})}} z_j \overline{\mathbf{Z}}^{\mathbf{m}} \overline{B}_{j\mathbf{m}}(|z_j|^2), \tag{5.6}$$

where

$$\begin{aligned}
(Y_\chi)_j(z,\eta) :=& - \mathrm{i}(1+\varpi_j(|z_j|^2))\big[\sum_{|\mathbf{m}|=\mathbf{l}+1} b_{\mathbf{m}}(|z_1|^2,\cdots,|z_n|^2)\partial_{\overline{j}}\mathbf{Z}^{\mathbf{m}} \\
&+ \sum_{k=1}^{n}\sum_{|\mathbf{m}|=\mathbf{l}} (\langle B_{k\mathbf{m}}(|z_k|^2),\eta\rangle \partial_{\overline{j}}(\overline{z}_k\mathbf{Z}^{\mathbf{m}}) + \langle \overline{B}_{k\mathbf{m}}(|z_k|^2),\overline{\eta}\rangle \partial_{\overline{j}}(z_k\overline{\mathbf{Z}}^{\mathbf{m}})\big], \\
(\tilde{Y}_\chi)_j(z,\eta) :=& - \mathrm{i}(1+\varpi_j(|z_j|^2))\big[\sum_{|\mathbf{m}|=\mathbf{l}+1} \partial_{|z_j|^2} b_{\mathbf{m}}(|z_1|^2,\cdots,|z_n|^2) z_j\mathbf{Z}^{\mathbf{m}} \\
&+ \sum_{|\mathbf{m}|=\mathbf{l}} (\langle B'_{j\mathbf{m}}(|z_j|^2),\eta\rangle |z_j|^2\mathbf{Z}^{\mathbf{m}} + \langle \overline{B}'_{j\mathbf{m}}(|z_j|^2),\overline{\eta}\rangle z_j^2\overline{\mathbf{Z}}^{\mathbf{m}}\big].
\end{aligned} \tag{5.7}$$

Notice that $(Y_\chi)_j = \mathcal{R}^{1,\mathbf{l}}_{\mathbf{r},\infty}$, $(\tilde{Y}_\chi)_j = \mathcal{R}^{1,\mathbf{l}+1}_{\mathbf{r},\infty}$ and $(X_\chi)_\eta = \mathbf{S}^{1,\mathbf{l}}_{\mathbf{r},\infty}$. We introduce now a new space.

Definition 5.3. We denote by $X_{\mathbf{r}}(\mathbf{l})$ the space formed by

$$\begin{aligned}
&\{(b,B) := (\{b_{\mathbf{m}}\}_{\mathbf{m}\in\mathcal{A}(\mathbf{l})}, \{B_{j\mathbf{n}}\}_{j\in 1,\cdots,n,\mathbf{n}\in\mathcal{B}_j(\mathbf{l})}) :\ b_{\mathbf{m}}\in\mathbb{C},\ B_{j\mathbf{n}}\in\Sigma^c_{\mathbf{r}} \\
&\quad \text{and } \chi(b,B) \text{ is real valued for all } z\in B_{\mathbb{C}^n(0,\mathbf{d})}\},\ \text{where} \\
&\mathcal{A}(\mathbf{l}) := \{\mathbf{m} :\ |\mathbf{m}| = \mathbf{l}+1, \quad \mathbf{m}\notin\mathcal{M}_0(\mathbf{l}+1)\}, \\
&\mathcal{B}_j(\mathbf{l}) := \{\mathbf{n} :\ |\mathbf{n}| = \mathbf{l}, \quad \mathbf{n}\notin\mathcal{M}_j(\mathbf{l}+1)\},
\end{aligned}$$

where we have assigned some order in the coordinates and where

$$\chi(b,B) = \sum_{\mathbf{m}\in\mathcal{A}(\mathbf{l})} \mathbf{Z}^{\mathbf{m}} b_{\mathbf{m}} + \sum_{j=1}^{n}\sum_{\mathbf{m}\in\mathcal{B}_j(\mathbf{l})} (\overline{z}_j\mathbf{Z}^{\mathbf{m}}\langle B_{j\mathbf{m}},\eta\rangle + \text{c.c.}) .$$

We provide $X_{\mathbf{r}}(\mathbf{l})$ with the norm

$$\|(b,B)\|_{X_{\mathbf{r}}(\mathbf{l})} = \sum_{\mathbf{m}\in\mathcal{A}(\mathbf{l})} |b_{\mathbf{m}}| + \sum_{j=1}^{n}\sum_{\mathbf{m}\in\mathcal{B}_j(\mathbf{l})} \|B_{j\mathbf{m}}\|_{\Sigma_{\mathbf{r}}}.$$

Set $\varrho(z) = (\varrho_1(z),....,\varrho_n(z))$ with $\varrho_j(z) = |z_j|^2$.

Lemma 5.4. *Consider the* χ *in* (5.5) *for fixed* $\mathbf{r}>0$ *and* $\mathbf{l}\geq 1$, *with coefficients* $(b(\varrho(z)),B(\varrho(z)))\in C^2(B_{\mathbb{C}^n}(0,\mathbf{d}),X_{\mathbf{r}}(\mathbf{l}))$ *and with* $B_{j\mathbf{m}}(\varrho(z)) = B_{j\mathbf{m}}(\varrho_j(z))$. *Consider the system*

$$\dot{z}_j = (X_\chi)_j(z,\eta) \text{ and } \dot{\eta} = (X_\chi)_\eta(z,\eta),$$

which is defined in $(t,z)\in\mathbb{R}\times B_{\mathbb{C}^n}(0,\mathbf{d})$ *and* $\eta\in\Sigma^c_k$ *for all* $k\in\mathbb{Z}\cap[-\mathbf{r},\mathbf{r}]$ *(or* $\eta\in H^1\cap\mathcal{H}_c[0]$*)* . *Let* $\delta\in(0,\min(\mathbf{d},\delta_1))$ *with* δ_1 *the constant of Lemma 4.8. Then the following properties hold.*

(1) If the following inequality holds,

$$4(\mathbf{l}+1)\delta\|(b(\varrho(z)),B(\varrho(z)))\|_{W^{1,\infty}(B_{\mathbb{C}^n}(0,\mathbf{d}),X_{\mathbf{r}}(\mathbf{l}))} < 1, \tag{5.8}$$

then for all $k\in\mathbb{Z}\cap[-\mathbf{r},\mathbf{r}]$ *for the flow* $\phi^t(z,\eta)$ *we have*

$$\begin{aligned}
&\phi^t\in C^\infty((-2,2)\times B_{\mathbb{C}^n}(0,\delta/2)\times B_{\Sigma^c_k}(0,\delta/2), B_{\mathbb{C}^n}(0,\delta)\times B_{\Sigma^c_k}(0,\delta)) \text{ and} \\
&\phi^t\in C^\infty((-2,2)\times B_{\mathbb{C}^n}(0,\delta/2)\times B_{H^1\cap\mathcal{H}_c[0]}(0,\delta/2), B_{\mathbb{C}^n}(0,\delta)\times B_{H^1\cap\mathcal{H}_c[0]}(0,\delta)).
\end{aligned} \tag{5.9}$$

In particular for $z_j^t := z_j \circ \phi^t(z,\eta)$ *and* $\eta^t := \eta \circ \phi^t(z,\eta)$ *and in the sense of Remark 2.10*

$$\begin{aligned} &z_j^t = z_j + S_j(t,z,\eta) \text{ and } \eta^t = \eta + S_\eta(t,z,\eta) \\ &\text{with } S_j(t,z,\eta) = \mathcal{R}^{1,\mathbf{l}}_{\mathbf{r},\infty}(t,z,\mathbf{Z},\eta) \text{ and } S_\eta(t,z,\eta) = \boldsymbol{S}^{1,\mathbf{l}}_{\mathbf{r},\infty}(t,z,\mathbf{Z},\eta). \end{aligned} \tag{5.10}$$

(2) *We have* $S_j(t, e^{\mathrm{i}\vartheta}z, e^{\mathrm{i}\vartheta}\eta) = e^{\mathrm{i}\vartheta}S_j(t,z,\eta)$, $S_\eta(t, e^{\mathrm{i}\vartheta}z, e^{\mathrm{i}\vartheta}\eta) = e^{\mathrm{i}\vartheta}S_\eta(t,z,\eta)$.

(3) *The flow* ϕ^t *is canonical, that is* $\phi^{t*}\Omega_0 = \Omega_0$ *in* $B_{\mathbb{C}^n}(0,\delta/2) \times B_{H^1\cap\mathcal{H}_c[0]}(0,\delta/2)$.

Proof. Claim (2) is elementary. The same is true for (3) given that ϕ^t is a standard sufficiently regular flow. In claim (1), (5.10) and the following sentence are a consequence of Lemma 4.9. The first part of claim (1) follows from elementary estimates such as

$$\begin{aligned} &|(X_\chi)_j(z,\eta)| = |(1+\varpi_j(|z_j|^2))\partial_{\bar{j}}\chi(z,\eta)| \\ &\le (1+\|\varpi_j\|_{L^\infty(B_{\mathbb{C}}(0,\delta_0))})(\mathbf{l}+1)\|(b,B)\|_{W^{1,\infty}(B_{\mathbb{C}^n}(0,\delta_0),X_{\mathbf{r}}(\mathbf{1}))}\delta_0^{\mathbf{l}+1} \end{aligned}$$

for $(z,\eta) \in B_{\mathbb{C}^n}(0,\delta) \times B_{\Sigma^c_{-r}}(0,\delta)$. Notice that taking δ_0 sufficiently small in Lemma 4.6, we can arrange $\|\varpi_j\|_{L^\infty(B_{\mathbb{C}}(0,\delta_0))} < 1$. We also have

$$\|(X_\chi)_\eta(z,\eta)\|_{\Sigma_{\mathbf{r}}} \le \|(0,B)\|_{L^\infty(B_{\mathbb{C}^n}(0,\delta_0),X_{\mathbf{r}}(\mathbf{1}))}\delta_0^{\mathbf{l}+1}.$$

Then if (5.8) holds we obtain (5.9). ☐

The main part of ϕ^t will be given by the following lemma.

Lemma 5.5. *Consider a function* χ *as in* (5.5). *For a parameter* $\varrho \in [0,\infty)^n$ *consider the field* W_χ *defined as follows (notice that* $W_\chi(z,\eta,\varrho(z)) = Y_\chi(z,\eta)$*):*

$$\begin{aligned} &(W_\chi)_j(z,\eta,\varrho) := -\mathrm{i}(1+\varpi_j(\varrho_j))\Big[\sum_{|\boldsymbol{m}|=\mathbf{l}+1} b_{\boldsymbol{m}}(\varrho)\partial_{\bar{j}}\boldsymbol{Z}^{\boldsymbol{m}} + \\ &\sum_{k=1}^{n}\sum_{|\boldsymbol{m}|=\mathbf{l}}(\langle B_{k\boldsymbol{m}}(\varrho_k),\eta\rangle\partial_{\bar{j}}(\bar{z}_k\boldsymbol{Z}^{\boldsymbol{m}}) + \langle \overline{B}_{k\boldsymbol{m}}(\varrho_k),\overline{\eta}\rangle z_k\partial_{\bar{j}}\overline{\boldsymbol{Z}}^{\boldsymbol{m}}\Big], \\ &(W_\chi)_\eta(z,\eta,\varrho) := -\mathrm{i}\sum_{k=1}^{n}\sum_{|\boldsymbol{m}|=\mathbf{l}} z_k\overline{\boldsymbol{Z}}^{\boldsymbol{m}}\overline{B}_{k\boldsymbol{m}}(\varrho_k). \end{aligned} \tag{5.11}$$

Denote by $(w^t,\sigma^t) = \phi_0^t(z,\eta)$ *the flow associated to the system*

$$\begin{aligned} &\dot{w}_j = (W_\chi)_j(w,\sigma,\varrho(z)), \quad w_j(0) = z_j, \\ &\dot{\sigma} = (W_\chi)_\sigma(w,\sigma,\varrho(z)), \quad \sigma(0) = \eta. \end{aligned} \tag{5.12}$$

Let $\delta \in (0,\min(\mathbf{d},\delta_1))$ *like in Lemma 5.4. Then the following facts hold.*

(1) *If* (5.8) *holds, then, for* $B(\varrho(z)) = (B_{j\mathbf{m}}(\varrho_j(z))_{j\mathbf{m}}$,

$$w_j^t = z_j + T_j(t,b(\varrho(z)),B(\varrho(z)),z,\eta) \text{ and } \sigma^t = \eta + T_\eta(t,b(\varrho(z)),B(\varrho(z)),z,\eta) \tag{5.13}$$

$$T_j \text{ (resp. } T_\eta) \ C^\infty \text{ for } (t,b,B,z,\eta) \in (-2,2)\times B_{X_{\mathbf{r}}}(0,c)\times B_{\mathbb{C}^n}(0,\delta)\times B_{\Sigma_{-\mathbf{r}}}(0,\delta) \text{ with values in } \mathbb{C} \text{ (resp. } \Sigma_{\mathbf{r}}). \tag{5.14}$$

Furthermore, we have

$$\begin{aligned} &T_j(t,b,B,z,\eta) = \mathcal{R}^{1,\mathbf{l}}_{\mathbf{r},\infty}(t,b,B,z,\mathbf{Z},\eta) \\ &T_\eta(t,b,B,z,\eta) = \mathbf{S}^{1,\mathbf{l}}_{\mathbf{r},\infty}(t,b,B,z,\mathbf{Z},\eta). \end{aligned} \tag{5.15}$$

(2) We have the gauge covariance for any fixed $\vartheta \in \mathbb{R}$

$$\begin{aligned} T_j(t, b, B, e^{i\vartheta} z, e^{i\vartheta}\eta) &= e^{i\vartheta} T_j(t, b, B, z, \eta) \\ T_\eta(t, b, B, e^{i\vartheta} z, e^{i\vartheta}\eta) &= e^{i\vartheta} T_\eta(t, b, B, z, \eta). \end{aligned} \tag{5.16}$$

(3) Consider the Hamiltonian flow $(z^t, \eta^t) = \phi^t(z, \eta)$ associated to χ, see Lemma 5.4. Then

$$z^t - w^t = \mathcal{R}^{1,\mathbf{l}+1}_{\mathbf{r},\infty}(t, z, \mathbf{Z}, \eta) \quad , \quad \eta^t - \sigma^t = \mathbf{S}^{1,\mathbf{l}+1}_{\mathbf{r},\infty}(t, z, \mathbf{Z}, \eta) \tag{5.17}$$

Proof. We have (5.13)–(5.14) by standard ODE theory. For $\mathbf{W} = (w_i \overline{w}_j)_{i \neq j}$ like the $\mathbf{Z}$ in (2.2)

$$\begin{aligned} w_j^t = z_j - i(1 + \varpi_j(\varrho_j(z)))\big[\sum_{|\mathbf{m}|=\mathbf{l}+1} b_{\mathbf{m}}(\varrho(z)) \int_0^t (\partial_{\overline{j}} \mathbf{W}^{\mathbf{m}})^s ds + \\ \sum_{k=1}^{n} \sum_{|\mathbf{m}|=\mathbf{l}} (\langle B_{k\mathbf{m}}(\varrho_k(z)), \int_0^t \sigma^s (\partial_{\overline{j}}(\overline{w}_k \mathbf{W}^{\mathbf{m}}))^s ds\rangle + \langle \overline{B}_{k\mathbf{m}}(\varrho_k(z)), \int_0^t \overline{\sigma}^s w_k^s (\partial_{\overline{j}} \overline{\mathbf{W}^{\mathbf{m}}})^s ds\rangle \big]. \end{aligned} \tag{5.18}$$

where $(\partial_{\overline{j}} \overline{\mathbf{W}^{\mathbf{m}}})^s = \partial_{\overline{j}} \overline{\mathbf{W}^{\mathbf{m}}}|_{w=w^s}$. Similarly we have

$$\sigma^t = \eta - i \sum_{k=1}^{n} \sum_{|\mathbf{m}|=\mathbf{l}} \overline{B}_{k\mathbf{m}}(\varrho_k(z)) \int_0^t w_k^s (\overline{\mathbf{W}^{\mathbf{m}}})^s ds. \tag{5.19}$$

Like in Lemma 4.9, we have also $\mathbf{W}^t = \mathbf{Z} + \int_0^t \mathcal{R}^{1,\mathbf{l}}_{\mathbf{r},\infty}(s, b(\varrho(z)), B(\varrho(z)), z, \mathbf{Z}, \eta) ds$. We can apply Gronwall inequality like in Lemma 4.9 on these formulas to obtain (5.15). This yields claim (1).
$(W_\chi)_j(e^{i\vartheta} w, e^{i\vartheta}\sigma, \varrho(z)) = e^{i\vartheta}(W_\chi)_j(w, \sigma, \varrho(z))$ and $(W_\chi)_\eta(e^{i\vartheta} w, e^{i\vartheta}\sigma, \varrho(z)) = e^{i\vartheta}(W_\chi)_\eta(w, \sigma, \varrho(z))$ yield claim (2).
Consider claim (3). Observe that (5.17) holds replacing $\mathbf{l}+1$ by $\mathbf{l}$. By (5.6), we have for a fixed C

$$\begin{aligned} |\dot{z} - \dot{w}| &\le |(W_\chi)_j(z, \eta) - (W_\chi)_j(w, \sigma)| + |\mathcal{R}^{1,\mathbf{l}+1}_{\mathbf{r},\infty}(t, z, \mathbf{Z}, \eta)| \\ &\le C|z - w| + C\|\eta - \sigma\|_{\Sigma_{-r}} + |\mathcal{R}^{1,\mathbf{l}+1}_{\mathbf{r},\infty}(t, z, \mathbf{Z}, \eta)|. \end{aligned}$$

Similarly we have

$$\|\dot{\eta} - \dot{\sigma}\|_{\Sigma_{\mathbf{r}}} \le \|(W_\chi)_\eta(z, \eta, \varrho(z)) - (W_\chi)_\eta(w, \sigma, \varrho(z))\|_{\Sigma_r} \le C|z - w| + C\|\eta - \sigma\|_{\Sigma_{-\mathbf{r}}}.$$

We then conclude by Gronwall's inequality

$$|z^t - w^t| + \|\eta^t - \sigma^t\|_{\Sigma_{\mathbf{r}}} \le |\mathcal{R}^{1,\mathbf{l}+1}_{\mathbf{r},\infty}(t, z, \mathbf{Z}, \eta)|$$

which, along with (5.17) with $\mathbf{l}+1$ replaced by $\mathbf{l}$, yields (5.17) ending Lemma 5.5.

□

Using Lemma 5.5, we expand ϕ^1 given in Lemma 5.4.

Lemma 5.6. *Let $(z', \eta') = \phi^1(z, \eta)$, where ϕ^t is the canonical flow given in Lemma 5.4. We have:*

(1) for $\mathcal{T}_j(b, B, z, \eta) = \mathcal{R}^{3,2\mathbf{l}-1}_{\mathbf{r},\infty}$, $\mathcal{T}_\eta(b, B, z, \eta) = \mathbf{S}^{3,2\mathbf{l}-1}_{\mathbf{r},\infty}$ and $\mathcal{T}_j$, $\mathcal{T}_\eta$ smooth in (b, B, z, η),

$$\begin{aligned} z_j' &= z_j + (Y_\chi)_j(z, \eta) + \mathcal{T}_j(b(\varrho(z)), B(\varrho(z)), z, \eta) + \mathcal{R}^{1,\mathbf{l}+1}_{\mathbf{r},\infty}, \\ \eta' &= \eta + (X_\chi)_\eta(z, \eta) + \mathcal{T}_\eta(b(\varrho(z)), B(\varrho(z)), z, \eta) + \mathcal{S}^{1,\mathbf{l}+1}_{\mathbf{r},\infty}; \end{aligned} \tag{5.20}$$

(2) for $\tilde{\mathcal{T}}_j(b,B,z,\eta)=\mathcal{R}^{1,2\mathbf{l}}_{\mathbf{r},\infty}$ *smooth in* (b,B,z,η),

$$|z_j'|^2=|z_j|^2+\overline{z}_j(Y_\chi)_j(z,\eta)+z_j\overline{(Y_\chi)_j(z,\eta)}+\tilde{\mathcal{T}}_j(b(\varrho(z)),B(\varrho(z)),z,\eta)+\mathcal{R}^{1,2\mathbf{l}+1}_{\mathbf{r},\infty}. \tag{5.21}$$

Remark 5.7. For $\mathbf{l}\geq 2$, $\mathcal{T}_j$ and $\mathcal{T}_\eta$ are absorbed in $\mathcal{R}^{1,\mathbf{l}+1}_{\mathbf{r},\infty}$ and $\mathcal{S}^{1,\mathbf{l}+1}_{\mathbf{r},\infty}$ and do not appear in the *homological equations* in Theorem 5.9. But if $\mathbf{l}=1$ they do, although as small perturbations.

Proof. First of all by (5.7) and by Definition 5.3 we have $\overline{z}_j(\tilde{Y}_\chi)_j+z_j\overline{(\tilde{Y}_\chi)_j}=2\,\mathrm{Re}\left(\overline{z}_j(\tilde{Y}_\chi)_j\right)=0$. So, using the following formula to define $\mathcal{Y}_j$, we have

$$\frac{d}{dt}|z_j|^2=\overline{z}_j(X_\chi)_j+z_j\overline{(X_\chi)_j}=\overline{z}_j(Y_\chi)_j+z_j\overline{(Y_\chi)_j}=:\mathcal{Y}_j(z,\eta). \tag{5.22}$$

Notice that $\mathcal{Y}_j$ is $\mathcal{R}^{0,\mathbf{l}+1}_{\mathbf{r},\infty}$. Therefore, we have

$$|z_j^s|^2-|z_j|^2=\mathcal{R}^{0,\mathbf{l}+1}_{\mathbf{r},\infty}. \tag{5.23}$$

This implies

$$b(\varrho(z^s))-b(\varrho(z))=\mathcal{R}^{0,\mathbf{l}+1}_{\mathbf{r},\infty}\text{ and }B(\varrho(z^s))-B(\varrho(z))=\mathcal{S}^{0,\mathbf{l}+1}_{\mathbf{r},\infty}. \tag{5.24}$$

Similarly, see right before (5.2), we have

$$\varpi_j(|z_j^s|^2)-\varpi_j(|z_j|^2)=\mathcal{R}^{2,\mathbf{l}+1}_{\mathbf{r},\infty} \tag{5.25}$$

Now we show (1). By (5.6) and (5.11), using (5.24) and (5.25), we have

$$(Y_\chi)_j(z^s,\eta^s)-(W_\chi)_j(z^s,\eta^s,\varrho(z))=\mathcal{R}^{1,2\mathbf{l}+1}_{\mathbf{r},\infty} \tag{5.26}$$

By (5.6), (5.10), (5.17) and (5.26), we have

$$\begin{aligned}
z_j'=&z_j+\int_0^1(W_\chi)_j(z^s,\eta^s,\varrho(z))ds+\int_0^1((Y_\chi)_j(z^s,\eta^s)-(W_\chi)_j(z^s,\eta^s,\varrho(z)))\,ds+\int_0^1(\tilde{Y}_\chi)_j(z^s,\eta^s)ds\\
=&z_j+\int_0^1(W_\chi)_j(w^s+\mathcal{R}^{1,\mathbf{l}+1}_{\mathbf{r},\infty},\sigma^s+S^{1,\mathbf{l}+1}_{\mathbf{r},\infty},\varrho(z))\,ds+\mathcal{R}^{1,\mathbf{l}+1}_{\mathbf{r},\infty}\\
=&z_j+\int_0^1(W_\chi)_j(w^s,\sigma^s,\varrho(z))\,ds+\mathcal{R}^{1,\mathbf{l}+1}_{\mathbf{r},\infty}=z_j+(W_\chi)_j(z,\eta,\varrho(z))+\mathcal{T}_j+\mathcal{R}^{1,\mathbf{l}+1}_{\mathbf{r},\infty},
\end{aligned}$$

where $\mathcal{T}_j=\int_0^1(W_\chi)_j(w^s,\sigma^s,\varrho(z))\,ds-(W_\chi)_j(z,\eta,\varrho(z))$ and the last $\mathcal{R}^{1,\mathbf{l}+1}_{\mathbf{r},\infty}$ in the 2nd line is different from the $\mathcal{R}^{1,\mathbf{l}+1}_{\mathbf{r},\infty}$ in the 3rd line. Finally, by (1) of Lemma 5.5 and the fact $(W_\chi)_j=\mathcal{R}^{1,\mathbf{l}}_{\mathbf{r},\infty}$, we have $\mathcal{T}_j=\mathcal{R}^{1,2\mathbf{l}-1}_{\mathbf{r},\infty}$ with $\mathcal{T}_j$ smooth in (t,b,B,z,η). The argument for η' is similar.
We next show (2). Set $\tilde{\mathcal{Y}}_j(z,\eta,\varrho):=\overline{z}_j(W_\chi)_j(z,\eta,\varrho)+z_j\overline{(W_\chi)_j(z,\eta,\varrho)}$. As in (5.23)–(5.24) we have

$$\tilde{\mathcal{Y}}_j(z^s,\eta^s,\varrho(z))-\mathcal{Y}_j(z^s,\eta^s)=\mathcal{R}^{0,2\mathbf{l}+2}_{\mathbf{r},\infty}$$

where $\mathcal{Y}_j$ is defined in (5.22). So we have

$$\begin{aligned}
|z_j'|^2&=|z_j|^2+\int_0^1\mathcal{Y}_j(z^s,\eta^s)\,ds=|z_j|^2+\int_0^1\tilde{\mathcal{Y}}_j(z^s,\eta^s,\varrho(z))\,ds+\mathcal{R}^{0,2\mathbf{l}+2}_{\mathbf{r},\infty}\\
&=|z_j|^2+\int_0^1\tilde{\mathcal{Y}}_j(w^s,\sigma^s,\varrho(z))\,ds+\mathcal{R}^{1,2\mathbf{l}+1}_{\mathbf{r},\infty}=|z_j|^2+\tilde{\mathcal{Y}}_j(z,\eta)+\tilde{\mathcal{T}}_j+\mathcal{R}^{1,2\mathbf{l}+1}_{\mathbf{r},\infty},
\end{aligned}$$

where $\tilde{\mathcal{T}}_j=\int_0^1\tilde{\mathcal{Y}}_j(w^s,\sigma^s,\varrho(z))\,ds-\tilde{\mathcal{Y}}_j(z,\eta)$. As in (1), we see $\tilde{\mathcal{T}}_j=\mathcal{R}^{1,2\mathbf{l}}_{\mathbf{r},\infty}$ and $\tilde{T}$ is C^∞ for (b,B,z,η). □

After a coordinate change $\phi=\phi^1$ as in Lemma 5.4 the Hamiltonian expands like in (4.45).

Lemma 5.8 (Structure Lemma). *Consider a function K which admits an expansion as in* (4.45) *defined for $(z,\eta)\in B_{\mathbb{C}^n}(0,\delta)\times(B_{H^1}(0,\delta)\cap\mathcal{H}_c[0])$ for some small $\delta>0$ and with r_1 is replaced by a r'. Suppose also that the $l=0$ terms in the first two lines are zero. Consider a function χ such as in* (5.5) *with $1\le \mathbf{l}\le 2N+4$ with $\|(b,B)\|_{W^{1,\infty}(B_{\mathbb{C}^n}(0,\delta),X_r(\mathbf{l}))}\le \underline{C}$ and with $\underline{C}$ a preassigned number. Suppose also that $2c_2(2N+4)\delta\underline{C}<1$ with c_2 the constant of Lemma 5.4. Denote by $\phi=\phi^1$ the corresponding flow Then claims (1)–(5) of Lemma 5.4 hold and for $(z,\eta)\in B_{\mathbb{C}^n}(0,\delta/2)\times(B_{H^1}(0,\delta/2)\cap\mathcal{H}_c[0])$ and for $r=r'-2$ for $\mathbf{Z}=(z_i\overline{z}_j)_{i,j=1,\dots,n}$ where $i\neq j$ we have an expansion*

$$\begin{aligned}
&K\circ\phi(z,\eta)=H_2(z,\eta)+\sum_{j=1}^{n}\lambda_j(|z_j|^2)\\
&+\sum_{l=1}^{2N+3}\sum_{|\boldsymbol{m}|=l+1}\boldsymbol{Z}^{\boldsymbol{m}}a_{\boldsymbol{m}}(|z_1|^2,\dots,|z_n|^2)+\sum_{j=1}^{n}\sum_{l=1}^{2N+3}\sum_{|\boldsymbol{m}|=l}(\overline{z}_j\boldsymbol{Z}^{\boldsymbol{m}}\langle G_{j\boldsymbol{m}}(|z_j|^2),\eta\rangle+c.c.)\\
&+\mathcal{R}^{1,2}_{r,\infty}(z,\eta)+\mathcal{R}^{0,2N+5}_{r,\infty}(z,\boldsymbol{Z},\eta)+\mathrm{Re}\langle\boldsymbol{S}^{0,2N+4}_{r,\infty}(z,\boldsymbol{Z},\eta),\overline{\eta}\rangle\\
&+\sum_{i+j=2}\sum_{|\boldsymbol{m}|\le 1}\boldsymbol{Z}^{\boldsymbol{m}}\langle G_{2\boldsymbol{m}ij}(z,\eta),\eta^i\overline{\eta}^j\rangle+\sum_{d+c=3}\sum_{i+j=d}\langle G_{dij}(z,\eta),\eta^i\overline{\eta}^j\rangle\mathcal{R}^{0,c}_{r,\infty}(z,\eta)+E_P(\eta)\ ,
\end{aligned}\tag{5.27}$$

where $G_{j\boldsymbol{m}}$, $G_{2\boldsymbol{m}ij}$ and G_{dij} are $\boldsymbol{S}^{0,0}_{r,\infty}$ and the $a_{\boldsymbol{m}}$ are $\mathcal{R}^{0,0}_{\infty,\infty}$. Furthermore, for $|\boldsymbol{m}|=0$ we have $G_{2\boldsymbol{m}ij}(z,\eta)=G_{2\boldsymbol{m}ij}(z)$ are the functions in (3.4) *and the $\lambda_j(|z_j|^2)$ are the same of* (4.45). *Furthermore the 1st function in the 3rd line of* (5.27) *satisfies $\mathcal{R}^{1,2}_{r,\infty}(e^{i\vartheta}z,e^{i\vartheta}\eta)\equiv\mathcal{R}^{1,2}_{r,\infty}(z,\eta)$.*

Proof. Like in Lemma 4.10 we consider the expansion (4.45) for $K(z',\eta')$, and substitute the formulas $z_j'=z_j+S_j(z,\eta)$ and $\eta'=\eta+S_\eta(z,\eta)$. Proceeding like in Lemma 4.10 we have

$$\begin{aligned}
&\mathcal{R}^{1,2}_{r',\infty}(z',\eta')=\mathcal{R}^{1,2}_{r',\infty}(z,\eta)+\mathcal{R}^{1,2N+5}_{r',\infty}(z,\mathbf{Z},\eta)+\mathrm{Re}\langle\mathbf{S}^{1,2N+4}_{r',\infty}(z,\mathbf{Z},\eta),\overline{\eta}\rangle\\
&+\ \text{terms like in the 2nd line of (5.27)},
\end{aligned}\tag{5.28}$$

Similarly we have

$$\begin{aligned}
&\langle H\eta',\overline{\eta}'\rangle=\langle H\eta,\overline{\eta}\rangle+\mathcal{R}^{1,\mathbf{l}+1}_{r'-2,\infty}(z,\mathbf{Z},\eta)=\langle H\eta,\overline{\eta}\rangle+\mathcal{R}^{1,\mathbf{l}+1}_{r'-2,\infty}(z,\eta)+\mathcal{R}^{1,\mathbf{l}+1}_{r'-2,\infty}(z,\mathbf{Z})\\
&+\mathrm{Re}\langle\mathbf{S}^{1,\mathbf{l}}_{r'-2,\infty}(z,\mathbf{Z},\eta),\overline{\eta}\rangle=\langle H\eta,\overline{\eta}\rangle+\mathcal{R}^{1,\mathbf{l}+1}_{r'-2,\infty}(z,\eta)+\mathcal{R}^{1,2N+5}_{r'-2,\infty}(z,\mathbf{Z},\eta)+\mathrm{Re}\langle\mathbf{S}^{1,2N+4}_{r'-2,\infty}(z,\mathbf{Z},\eta),\overline{\eta}\rangle\\
&+\ \text{terms like in the 2nd line of (5.27)}
\end{aligned}\tag{5.29}$$

Consider an $\lambda_j(|z_j|^2)$ in (4.45). Then by (5.21) we have

$$\lambda(|z_j'|^2)=\lambda\left(|z_j|^2+\mathcal{R}^{0,\mathbf{l}+1}_{r,\infty}(z,\mathbf{Z},\eta)\right)=\mu(|z_j|^2)+\mathcal{R}^{1,\mathbf{l}+1}_{r,\infty}(z,\mathbf{Z},\eta).\tag{5.30}$$

The latter admits an expansion like in and below formula (4.46).

The term $\mathcal{R}^{1,2}_{r,\infty}(z,\eta)$ in the 3rd line of (5.27) is either the first in the r.h.s in (5.28) for $l>1$ in Lemma 4.8, or the sum of the latter with the $\mathcal{R}^{1,l+1}_{r'-2,\infty}(z,\eta)$ originating from (5.29)–(5.30) for $l=1$ in Lemma 4.8. In either case it satisfies $\mathcal{R}^{1,2}_{r,\infty}(e^{i\vartheta}z,e^{i\vartheta}\eta)\equiv\mathcal{R}^{1,2}_{r,\infty}(z,\eta)$. Other terms in (4.45) computed at (z',η') and by similar elementary expansions are similarly absorbed in (5.27). ☐

All of the above lemmas are preparatory for the following result, which will give us an effective Hamiltonian by picking $\iota=2N+4$.

Theorem 5.9 (Birkhoff normal form). *For any $\iota \in \mathbb{N} \cap [2, 2N+4]$ there are a $\delta_\iota > 0$, a polynomial χ_ι as in (5.5) with $\mathbf{l} = \iota$, $\mathbf{d} = \delta_\iota$ and $\mathbf{r} = r_\iota = r_0 - 2(\iota+1)$ s.t. for all $k \in \mathbb{Z} \cap [-r(\iota), r(\iota)]$ we have for each χ_ι a flow (for $\delta_1 > 0$ the constant in Lemma 4.10)*

$$\begin{aligned} &\phi^t_\iota \in C^\infty((-2,2) \times B_{\mathbb{C}^n}(0,\delta_\iota) \times B_{\Sigma^c_k}(0,\delta_\iota), B_{\mathbb{C}^n}(0,\delta_{\iota-1}) \times B_{\Sigma^c_k}(0,\delta_{\iota-1})) \text{ and} \\ &\phi^t_\iota \in C^\infty((-2,2) \times B_{\mathbb{C}^n}(0,\delta_\iota) \times B_{H^1 \cap \mathcal{H}_c[0]}(0,\delta_\iota), B_{\mathbb{C}^n}(0,\delta_{\iota-1}) \times B_{H^1 \cap \mathcal{H}_c[0]}(0,\delta_{\iota-1})) \end{aligned} \tag{5.31}$$

and s.t.,for $\mathfrak{F}^{(\iota)} := \mathfrak{F} \circ \phi_2 \circ \ldots \circ \phi_\iota$, $\mathfrak{F}$ the transformation in Lemma 4.8 and $\phi_j = \phi^1_\iota$, then for $(z,\eta) \in B_{\mathbb{C}^n}(0,\delta_\iota) \times (B_{H^1}(0,\delta_\iota) \cap \mathcal{H}_c[0])$ and for $\mathbf{Z} = (z_i \overline{z}_j)_{i,j=1,\ldots,n}$, where $i \neq j$, we have

$$\begin{aligned} &H^{(\iota)}(z,\eta) := E \circ \mathfrak{F}^{(\iota)}(z,\eta) = H_2(z,\eta) + \sum_{j=1}^{n} \lambda_j(|z_j|^2) + Z^{(\iota)}(z,\mathbf{Z},\eta) \\ &+ \sum_{l=\iota}^{2N+3} \sum_{|\boldsymbol{m}|=l+1} \boldsymbol{Z}^{\boldsymbol{m}} a^{(\iota)}_{\boldsymbol{m}}(|z_1|^2,\ldots,|z_n|^2) + \sum_{j=1}^{n} \sum_{l=\iota}^{2N+3} \sum_{|\boldsymbol{m}|=l} (\overline{z}_j \boldsymbol{Z}^{\boldsymbol{m}} \langle G^{(\iota)}_{j\boldsymbol{m}}(|z_j|^2),\eta\rangle + c.c.) \\ &+ \mathcal{R}^{1,2}_{r_\iota,\infty}(z,\eta) + \mathcal{R}^{0,2N+5}_{r_\iota,\infty}(z,\boldsymbol{Z},\eta) + \mathrm{Re}\langle \boldsymbol{S}^{0,2N+4}_{r_\iota,\infty}(z,\boldsymbol{Z},\eta),\overline{\eta}\rangle + \\ &\sum_{i+j=2} \sum_{|\boldsymbol{m}|\leq 1} \boldsymbol{Z}^{\boldsymbol{m}} \langle G^{(\iota)}_{2\boldsymbol{m}ij}(z,\eta),\eta^i\overline{\eta}^j\rangle + \sum_{d+c=3} \sum_{i+j=d} \langle G^{(\iota)}_{dij}(z,\eta),\eta^i\overline{\eta}^j\rangle \mathcal{R}^{0,c}_{r_\iota,\infty}(z,\eta) + E_P(\eta) \end{aligned} \tag{5.32}$$

where, for coefficients like in Def. 5.1 for $(r,m) = (r_\iota,\infty)$,

$$Z^{(\iota)} = \sum_{\mathbf{m} \in \mathcal{M}_0(\iota)} \boldsymbol{Z}^{\boldsymbol{m}} a_{\boldsymbol{m}}(|z_1|^2,\ldots,|z_n|^2) + \sum_{j=1}^{n} \Big(\sum_{\mathbf{m} \in \mathcal{M}_j(\iota-1)} \overline{z}_j \boldsymbol{Z}^{\boldsymbol{m}} \langle G_{j\boldsymbol{m}}(|z_j|^2),\eta\rangle + c.c.\Big). \tag{5.33}$$

We have $\mathcal{R}^{1,2}_{r_\iota,\infty} = \mathcal{R}^{1,2}_{r_2,\infty}$ and $\mathcal{R}^{1,2}_{r_2,\infty}(e^{i\vartheta}z, e^{i\vartheta}\eta) \equiv \mathcal{R}^{1,2}_{r_2,\infty}(z,\eta)$.
In particular we have for $\delta_f := \delta_{2N+4}$ and for the δ_0 in Lemma 4.6,

$$\mathfrak{F}^{(2N+4)}(B_{\mathbb{C}^n}(0,\delta_f) \times (B_{H^1}(0,\delta_f) \cap \mathcal{H}_c[0])) \subset B_{\mathbb{C}^n}(0,\delta_0) \times (B_{H^1}(0,\delta_0) \cap \mathcal{H}_c[0]) \tag{5.34}$$

with $\mathfrak{F}|_{B_{\mathbb{C}^n}(0,\delta_f) \times (B_{H^1}(0,\delta_f) \cap \mathcal{H}_c[0])}$ a diffeomorphism between its domain and an open neighborhood of the origin in $\mathbb{C}^n \times (H^1 \cap \mathcal{H}_c[0])$.
Furthermore, for $r = r_0 - 4N - 10$ there is a pair $\mathcal{R}^{1,1}_{r,\infty}$ and $\mathbf{S}^{1,1}_{r,\infty}$ s.t. for $(z',\eta') = \mathfrak{F}^{(2N+4)}(z,\eta)$

$$z' = z + \mathcal{R}^{1,1}_{r,\infty}(z,\mathbf{Z},\eta) \quad \eta' = \eta + \mathbf{S}^{1,1}_{r,\infty}(z,\mathbf{Z},\eta). \tag{5.35}$$

Furthermore, by taking all the $\delta_\iota > 0$ sufficiently small, we can assume that all the symbols in the proof, i.e. the symbols in (5.35) and the symbols in the expansions (5.32), satisfy the estimates of Definitions 2.8 and 2.9 for $|z| < \delta_\iota$ and $\|\eta\|_{\Sigma_{-r(\iota)}} < \delta_\iota$ for their respective ι's.

Proof. Notice that the functional K in Lemma 4.10 satisfies case $\iota = 1$. The proof will be by induction on ι. We assume that $H^{(\iota)}$ satisfies the statement for $\iota \geq 1$ and prove that there is a $\phi_{\iota+1}$ such that $H^{(\iota+1)} := H^{(\iota)} \circ \phi_{\iota+1}$ satisfies the statement for $\iota+1$. We consider the representation (5.27) for $H^{(\iota)}$, which is guaranteed by the Structure Lemma 5.8. Using (5.27) we set $\mathbf{h} = H^{(\iota)}(z,\mathbf{Z},\eta)$ interpreting $(z,\mathbf{Z},\eta)$ as independent variables. Then we have for $\mathbf{l} = \iota$

$$a^{(\mathbf{l})}_{\mathbf{m}}(|z_1|^2,\ldots,|z_n|^2) = \frac{1}{\mathbf{m}!} \partial^{\mathbf{m}}_{\mathbf{Z}} \mathbf{h}|_{(z,\eta,\mathbf{Z})=(z;0,0)}, \quad |\mathbf{m}| \leq 2N+4, \tag{5.36}$$

$$\overline{z}_j G^{(\mathbf{l})}_{j\mathbf{m}}(|z_j|^2) = \frac{1}{\mathbf{m}!} \partial^{\mathbf{m}}_{\mathbf{Z}} \nabla_\eta \mathbf{h}|_{(z,\eta,\mathbf{Z})=(0,\ldots,z_j,0,\ldots 0;0,0)}, \quad |\mathbf{m}| \leq 2N+3. \tag{5.37}$$

The inductive hypothesis on $H^{(\iota)}$ is a statement on the Taylor coefficients in (5.36)–(5.37), that is that, for $\mathbf{l}=\iota$ (see Def. 2.5 and Remark 5.2)

$$\partial_{\mathbf{Z}}^{\mathbf{m}}\mathbf{h}|_{(z,\eta,\mathbf{Z})=(z;0,0)}=0 \text{ for all } \mathbf{m}\notin\mathcal{M}_0(\mathbf{l}), \tag{5.38}$$
$$\partial_{\mathbf{Z}}^{\mathbf{m}}\nabla_\eta\mathbf{h}|_{(z,\eta,\mathbf{Z})=(0,\dots,z_j,0,\dots 0;0,0)}=0 \text{ for all } (j,\mathbf{m}) \text{ with } \mathbf{m}\notin\mathcal{M}_j(\mathbf{l}-1). \tag{5.39}$$

We consider now a yet unknown χ as in (5.5) with $\mathbf{l}=\iota$, $\mathbf{r}=r_\iota$ and a yet to be determined $\mathbf{d}=\delta>0$. Set $\phi:=\phi^1$, where ϕ^t is the flow of Lemma 5.4. We are seeking χ such that $H^{(\iota)}\circ\phi$ satisfies the conclusions of Theorem 5.9 for $\iota+1$, i.e. that using again Lemma 5.8 and setting this time $\mathbf{h}=(H^{(\iota)}\circ\phi)(z,\eta,\mathbf{Z})$, we will have (5.38)–(5.39) for $\mathbf{l}=\iota+1$. Notice that for any χ, (5.38)–(5.39) are automatically true for $\mathbf{l}=\iota$. This because $H^{(\iota)}(z,\eta,\mathbf{Z})$ and $(H^{(\iota)}\circ\phi)(z,\eta,\mathbf{Z})$ have same derivatives in (5.36) for $|\mathbf{m}|\le\iota$ and in (5.37) for $|\mathbf{m}|\le\iota-1$. So it is enough to consider (5.38) for $|\mathbf{m}|=\iota+1$ and (5.39) for $|\mathbf{m}|=\iota$. This will be true for a specific choice of χ whose coefficients solve the *Homological Equations*, which we set up in the sequel.

By (5.20) and by $G^{(\iota)}_{\mathbf{20}ij}(z,\eta)=G_{\mathbf{20}ij}(z)$ we have

$$\begin{aligned}H^{(\iota)}(z',\eta')&=H_2(z',\eta')+\sum_{j=1}^n\lambda_j(|z_j'|^2)+Z^{(\iota)}(z',\mathbf{Z}',\eta')+\mathcal{R}^{1,2}_{r,\infty}(z',\eta')+\sum_{i+j=2}\langle G_{\mathbf{20}ij}(z'),\eta'^i\overline{\eta}'^j\rangle\\&+(*)+\sum_{|\mathbf{m}|=\iota+1}\mathbf{Z}^{\mathbf{m}}a^{(\iota)}_{\mathbf{m}}(|z|^2)+\sum_{j=1}^n\sum_{|\mathbf{m}|=\iota}(\overline{z}_j\mathbf{Z}^{\mathbf{m}}\langle G^{(\iota)}_{j\mathbf{m}}(|z_j|^2),\eta\rangle+\text{ c.c.}),\end{aligned} \tag{5.40}$$

where $\mathbf{h}:=(*)(z,\eta,\mathbf{Z})$ satisfies (5.38)–(5.39) for $\mathbf{l}=\iota+1$. In the sequel we will use $(*)$ with this meaning. Let $(z',\eta')=\phi(z,\eta)$. We have

$$\begin{aligned}&\sum_{j=1}^n e_j\left(\overline{z}_j(Y_\chi)_j(z,\eta)+z_j(Y_\chi)_{\overline{j}}(z,\eta)\right)=\sum_{|\mathbf{m}|=\iota+1}\mathrm{i}\widetilde{\mathbf{e}}\cdot(\mu(\mathbf{m})-\nu(\mathbf{m}))b_{\mathbf{m}}(|z_1|^2,\dots,|z_n|^2)\mathbf{Z}^{\mathbf{m}}\\&+\sum_j\sum_{|\mathbf{m}|=\iota}\left(\mathrm{i}\widetilde{\mathbf{e}}\cdot(\tilde{\mu}_j(\mathbf{m})-\tilde{\nu}_j(\mathbf{m}))\left\langle B_{j\mathbf{m}}(|z_j|^2),\eta\right\rangle\overline{z}_j\mathbf{Z}^{\mathbf{m}}+\text{c.c.}\right)\text{ for}\end{aligned} \tag{5.41}$$

$$\begin{aligned}&\mathbf{Z}^{\mathbf{m}}=z^{\mu(\mathbf{m})}\overline{z}^{\nu(\mathbf{m})},\ \overline{z}_j\mathbf{Z}^{\mathbf{m}}=z^{\tilde{\mu}_j(\mathbf{m})}\overline{z}^{\tilde{\nu}_j(\mathbf{m})},\\&\widetilde{\mathbf{e}}(z):=(e_1(1+\varpi_1(|z_1|^2)),\dots,e_n(1+\varpi_n(|z_n|^2))),\end{aligned} \tag{5.42}$$

and, summing on repeated indexes,

$$\langle H\eta,(X_\chi)_{\overline{\eta}}(z,\eta)\rangle+\langle H(X_\chi)_\eta(z,\eta),\overline{\eta}\rangle=\mathrm{i}\overline{z}_j\mathbf{Z}^{\mathbf{m}}\left\langle HB_{j,\mathbf{m}}(|z_j|^2),\eta\right\rangle+\text{c.c.}\quad. \tag{5.43}$$

So, by Lemma 5.6, (5.41)–(5.43) and using the notation in(5.42), we have

$$\begin{aligned}H_2(z',\eta')=&\sum_{j=1}^n e_j|z_j'|^2+\langle H\eta',\overline{\eta}'\rangle=H_2(z,\eta)+\sum_{\substack{|\mathbf{m}|=\mathbf{l}+1\\ \mathbf{m}\notin\mathcal{M}_0(\mathbf{l}+1)}}\mathrm{i}\widetilde{\mathbf{e}}\cdot(\mu(\mathbf{m})-\nu(\mathbf{m}))b_{\mathbf{m}}(|z_1|^2,\dots,|z_n|^2)\mathbf{Z}^{\mathbf{m}}\\&+\sum_j\sum_{\substack{|\mathbf{m}|=\mathbf{l}\\ \mathbf{m}\notin\mathcal{M}_j(\mathbf{l})}}\left(\mathrm{i}\left\langle(\widetilde{\mathbf{e}}\cdot(\tilde{\mu}_j(\mathbf{m})-\tilde{\nu}_j(\mathbf{m}))+H)\,B_{j\mathbf{m}}(|z_j|^2),\eta\right\rangle\overline{z}_j\mathbf{Z}^{\mathbf{m}}+\text{c.c.}\right)\\&+\mathcal{R}^{2,2\iota}_{\mathbf{r},\infty}(b,B,z,\mathbf{Z},\eta)+(*),\end{aligned} \tag{5.44}$$

where c.c. refers only to the second line and in the last line

$$\mathcal{R}^{2,2\iota}_{\mathbf{r},\infty}(b,B,z,\mathbf{Z},\eta)=\sum_{j=1}^{n}e_j\tilde{\mathcal{T}}_j+\left\langle H\eta,\overline{\mathcal{T}}_\eta\right\rangle+\langle H\mathcal{T}_\eta,\overline{\eta}\rangle+\left\langle H\mathcal{T}_\eta,\overline{\mathcal{T}}_\eta\right\rangle,$$

where here and in the sequel of this proof we abuse notation denoting by (b,B) the element in $X_r(\iota)$, see Def. 5.3, with entries $b_{\mathbf{m}}(|z_1|^2,...,|z_n|^2)$ and $B_{j\mathbf{m}}(|z_j|^2)$. $\mathcal{R}^{2,2\iota}_{\mathbf{r},\infty}(b,B,z,\mathbf{Z},\eta)$ can be absorbed in $(*)$ if $\iota\ge 2$ but if $\iota=1$ needs to be considered explicitly. By $\lambda_j(|z_j|^2)=\mathcal{R}^{2,0}_{\infty,\infty}$ and (5.21) we have

$$\lambda_j(|z_j'|^2)=\lambda_j(|z_j|^2)+\mathcal{R}^{2,\iota+1}_{\mathbf{r},\infty}(b,B,z,\mathbf{Z},\eta)+(*). \tag{5.45}$$

Next, we claim

$$Z^{(\iota)}(z',\mathbf{Z}',\eta')=Z^{(\iota)}(z,\mathbf{Z},\eta)+\mathcal{R}^{2,\iota+1}_{\mathbf{r},\infty}(b,B,z,\mathbf{Z},\eta)+(*). \tag{5.46}$$

Let us take a term $\mathbf{Z}^{\mathbf{m}}a_{\mathbf{m}}(\varrho(z))$ in the sum (5.33). Notice that by Lemma 2.6 we have necessarily $|\mathbf{m}|\ge 2$. Furthermore, by (5.21) it is easy to see that we can omit the factor $a_{\mathbf{m}}(\varrho(z))$. For definiteness let $\mathbf{Z}^{\mathbf{m}}=|z_1|^2|z_2|^2$ (so $|\mathbf{m}|=2$; the case $|\mathbf{m}|>2$ is simpler). By (5.21) we have

$$|z_1'|^2|z_2'|^2=(|z_1|^2+\mathcal{R}^{0,\iota+1}_{\mathbf{r},\infty})(|z_2|^2+\mathcal{R}^{0,\iota+1}_{\mathbf{r},\infty})=|z_1|^2|z_2|^2+R^{2,\iota+1}_{\mathbf{r},\infty}(b,B,z,\mathbf{Z},\eta),$$

where we used information such as $\tilde{\mathcal{T}}_j=\mathcal{R}^{1,2\iota}_{\mathbf{r},\infty}$ contained in Lemma 5.6 and the fact, easy to check, that $\overline{z}_j(Y_\chi)_j(z,\eta)+z_j(Y_\chi)_{\overline{j}}(z,\eta)=R^{0,\iota+1}_{\mathbf{r},\infty}(b,B,z,\mathbf{Z},\eta)$.
To complete the proof of (5.46) let us take now a term of the form $\overline{z}_2\mathbf{Z}^{\mathbf{m}}\langle G(|z_2|^2),\eta\rangle$. Here we can write $G=G(|z_2|^2)$ ignoring the dependence on $|z_2|^2$ and we can focus on $|\mathbf{m}|=1$. For definiteness let $\mathbf{Z}^{\mathbf{m}}=z_1\overline{z}_2$. By Lemma 5.6

$$z_1'(\overline{z}_2')^2\langle G,\eta'\rangle=(z_1+\mathcal{R}^{1,\iota}_{\mathbf{r},\infty})(\overline{z}_2+\mathcal{R}^{1,\iota}_{\mathbf{r},\infty})^2\langle G,\eta+S^{1,\iota}_{\mathbf{r},\infty}\rangle.$$

which for $\iota>1$ is of the form $z_1\overline{z}_2^2\langle G,\eta\rangle+(*)$ and for $\iota=1$ using formula (5.20) yields (5.46).

By claim (1) in Lemma 5.4 and $d_\eta\mathcal{R}^{1,2}_{r,\infty}(z,\eta)\cdot\mathbf{S}^{1,\iota}_{\mathbf{r},\infty}(b,B,z,\eta)=\mathcal{R}^{2,\iota+1}_{\mathbf{r},\infty}(b,B,z,\mathbf{Z},\eta)$ we get

$$\begin{aligned}\mathcal{R}^{1,2}_{r,\infty}(z',\eta')&=\mathcal{R}^{1,2}_{r,\infty}(z,\eta')+(*)=\mathcal{R}^{1,2}_{r,\infty}(z,\eta)+(*)\\&+\int_0^1 d_\eta\mathcal{R}^{1,2}_{r,\infty}(z,\eta+\tau\mathbf{S}^{1,\iota}_{\mathbf{r},\infty}(b,B,z,\eta))\cdot\mathbf{S}^{1,\iota}_{\mathbf{r},\infty}(b,B,z,\eta)d\tau\\&=\mathcal{R}^{1,2}_{r,\infty}(z,\eta)+d_\eta\mathcal{R}^{1,2}_{r,\infty}(z,\eta)\cdot\mathbf{S}^{1,\iota}_{\mathbf{r},\infty}(b,B,z,\eta)+(*).\end{aligned}\tag{5.47}$$

Like in (5.47) and using (5.20) and $G_{\mathbf{20}ij}(z)=\mathcal{R}^{2,0}_{\infty,\infty}(z)$, see (3.4), we have

$$\begin{aligned}\sum_{i+j=2}\langle G_{\mathbf{20}ij}(z'),\eta'^i\overline{\eta}'^j\rangle&=\sum_{i+j=2}\langle G_{\mathbf{20}ij}(z),\eta'^i\overline{\eta}'^j\rangle+(*)\\&=\sum_{i+j=2}\langle G_{\mathbf{20}ij}(z),\eta^i\overline{\eta}^j\rangle+\mathcal{R}^{3,\iota+1}_{\mathbf{r},\infty}(b,B,z,\mathbf{Z},\eta)+(*).\end{aligned}\tag{5.48}$$

Therefore, we seek χ_ι s.t. the following holds, with $\varrho(z)=(|z_1|^2,...,|z_n|^2)$ and the notation in (5.42):

$$\begin{aligned}(*)=&\sum_{\substack{|\mathbf{m}|=\iota+1\\ \mathbf{m}\notin\mathcal{M}_0(\iota+1)}}\mathrm{i}\widetilde{\mathbf{e}}\cdot(\mu(\mathbf{m})-\nu(\mathbf{m}))b_{\mathbf{m}}(\varrho(z))\mathbf{Z}^{\mathbf{m}}\\ &+\sum_j\sum_{\substack{|\mathbf{m}|=\iota\\ \mathbf{m}\notin\mathcal{M}_j(\iota)}}\big(\mathrm{i}\left\langle(\widetilde{\mathbf{e}}\cdot(\mu_j(\mathbf{m})-\nu_j(\mathbf{m}))+H)\,B_{j\mathbf{m}}(|z_j|^2),\eta\right\rangle\overline{z}_j\mathbf{Z}^{\mathbf{m}}+\text{c.c.}\big)\\ &+\mathcal{R}^{2,\iota+1}_{\mathbf{r},\infty}(b,B,z,\mathbf{Z},\eta)+\sum_{\substack{|\mathbf{m}|=\iota+1\\ \mathbf{m}\notin\mathcal{M}_0(\iota+1)}}\mathbf{Z}^{\mathbf{m}}a^{(\iota)}_{\mathbf{m}}(\varrho(z))+\sum_{j=1}^n\sum_{\substack{|\mathbf{m}|=\iota\\ \mathbf{m}\notin\mathcal{M}_j(\iota)}}(\overline{z}_j\mathbf{Z}^{\mathbf{m}}\langle G^{(\iota)}_{j\mathbf{m}}(|z_j|^2),\eta\rangle+\text{ c.c.}).\end{aligned}\tag{5.49}$$

By a Taylor expansion we can write

$$\begin{aligned}\mathcal{R}^{2,\iota+1}_{\mathbf{r},\infty}(b,B,z,\mathbf{Z},\eta)=&(*)+\sum_{\substack{|\mathbf{m}|=\iota+1\\ \mathbf{m}\notin\mathcal{M}_0(\iota+1)}}\mathbf{Z}^{\mathbf{m}}\alpha_{\mathbf{m}}(b,B,\varrho(z))\\ &+\sum_{j=1}^n\sum_{\substack{|\mathbf{m}|=\iota\\ \mathbf{m}\notin\mathcal{M}_j(\iota)}}(\overline{z}_j\mathbf{Z}^{\mathbf{m}}\langle\Gamma_{j\mathbf{m}}(b(0,...,|z_j|^2,0,...,0),B(0,...,|z_j|^2,0,...,0),|z_j|^2),\eta\rangle+\text{ c.c.})\end{aligned}$$

where $\alpha_{\mathbf{m}}(b,B,\varrho(z))=\mathcal{R}^{1,0}_{\mathbf{r},\infty}(b,B,\varrho(z))$ and

$$\begin{aligned}\text{where }&\Gamma_{j\mathbf{m}}(b(0,...,|z_j|^2,0,...,0),B(0,...,|z_j|^2,0,...,0),|z_j|^2)\\ &=S^{1,0}_{\mathbf{r},\infty}(b(0,...,|z_j|^2,0,...,0),B(0,...,|z_j|^2,0,...,0),|z_j|^2).\end{aligned}$$

Furthermore, by (5.42) and $\varpi_j(|z_j|^2)=\mathcal{R}^{2,0}_{r_0,\infty}(|z_j|^2)$ the 2nd line of (5.49) has an expansion

$$\sum_j\sum_{\substack{|\mathbf{m}|=\iota\\ \mathbf{m}\notin\mathcal{M}_j(\iota)}}\big(\mathrm{i}\left\langle\big(\mathbf{e}\cdot(\mu_j(\mathbf{m})-\nu_j(\mathbf{m}))+\mathcal{R}^{1,0}_{r_0,\infty}(|z_j|^2)+H\big)\,B_{j\mathbf{m}}(|z_j|^2),\eta\right\rangle\overline{z}_j\mathbf{Z}^{\mathbf{m}}+\text{c.c.}\big)+(*).$$

Then we reduce to the following system:

$$\begin{aligned}b_{\mathbf{m}}(\varrho(z))&=\frac{\mathrm{i}}{\widetilde{\mathbf{e}}(z)\cdot(\mu(\mathbf{m})-\nu(\mathbf{m}))}[a^{(\iota)}_{\mathbf{m}}(\varrho(z))+\alpha_{\mathbf{m}}((b_{\mathbf{n}}(\varrho(z)))_{\mathbf{n}},(B_{j\mathbf{n}}(\varrho_j(z)))_{j\mathbf{n}},\varrho(z))],\\ B_{j\mathbf{m}}(|z_j|^2)&=\mathrm{i}R_H(\mathbf{e}\cdot(\mu_j(\mathbf{m})-\nu_j(\mathbf{m}))+\mathcal{R}^{1,0}_{r_0,\infty}(|z_j|^2))[G^{(\iota)}_{j\mathbf{m}}(|z_j|^2)\\ &\quad+\Gamma_{j\mathbf{m}}(b(0,...,|z_j|^2,0,...,0),B(0,...,|z_j|^2,0,...,0),|z_j|^2)\end{aligned}\tag{5.50}$$

The $b_{\mathbf{m}}(\varrho(z))$, $B_{j\mathbf{m}}(|z_j|^2)$ can be found by implicit function theorem for $|z|<\delta'_\iota$ for δ'_ι sufficiently small. This gives us the desired polynomial χ yielding $H^{(\iota+1)}$. Formulas (5.31) for the flow ϕ^t of χ are obtained choosing $\delta_\iota>0$ sufficiently small by claim (1) in Lemma 5.4. For the composition $\mathcal{F}^{(2N+4)}$ we obtain (5.34) as a consequence of (5.31) and of (4.44).

□

6 Dispersion

We apply Theorem 5.9, set $\mathcal{H}=H^{(2N+4)}$ so that

$$\mathcal{H}(z,\eta)=H_2(z,\eta)+\sum_{j=1}^n\lambda_j(|z_j|^2)+Z^{(2N+4)}(z,\mathbf{Z},\eta)+\mathcal{R}\tag{6.1}$$

$$\mathcal{R} := \mathcal{R}^{1,2}_{r,\infty}(z,\eta) + \mathcal{R}^{0,2N+5}_{r,\infty}(z,\mathbf{Z},\eta) + \mathrm{Re}\langle \mathbf{S}^{0,2N+4}_{r,\infty}(z,\mathbf{Z},\eta), \overline{\eta}\rangle$$
$$+ \sum_{i+j=2}\sum_{|\mathbf{m}|\le 1} \mathbf{Z}^{\mathbf{m}} \langle G_{2\mathbf{m}ij}(z,\eta), \eta^i\overline{\eta}^j\rangle + \sum_{d+c=3}\sum_{i+j=d} \langle G_{dij}(z,\eta), \eta^i\overline{\eta}^j\rangle \mathcal{R}^{0,c}_{r,\infty}(z,\eta) + E_P(\eta). \tag{6.2}$$

Using formula (5.33) for $\iota = 2N+4$ we have

$$\begin{aligned} &\sum_{j=1}^n \lambda_j(|z_j|^2) + Z^{(2N+4)}(z,\mathbf{Z},\eta) = Z_0(z) + \sum_{j=1}^n \Big(\sum_{\mathbf{m}\in\mathcal{M}_j(2N+3)} \overline{z}_j \mathbf{Z}^{\mathbf{m}} \langle G_{j\mathbf{m}}(|z_j|^2), \eta\rangle + \text{c.c.}\Big), \\ &Z_0(z) := \sum_{j=1}^n \lambda_j(|z_j|^2) + \sum_{\mathbf{m}\in\mathcal{M}_0(2N+4)} \mathbf{Z}^{\mathbf{m}} a_{\mathbf{m}}(|z_1|^2,...,|z_n|^2) = \mathcal{Z}_0(|z_1|^2,...,|z_n|^2), \end{aligned} \tag{6.3}$$

where the last equality holds for some $\mathcal{Z}_0(|z_1|^2,...,|z_n|^2)$ by Lemma 2.6.

Theorem 6.1 (Main Estimates)**.** *There exist $\epsilon_0 > 0$ and $C_0 > 0$ s.t. if the constant $0 < \epsilon$ of Theorem 1.3 satisfies $\epsilon < \epsilon_0$, for $I = [0,\infty)$ and $C = C_0$ we have:*

$$\|\eta\|_{L^p_t(I,W^{1,q}_x)} \le C\epsilon \text{ for all admissible pairs } (p,q), \tag{6.4}$$
$$\|z_j \mathbf{Z}^{\mathbf{m}}\|_{L^2_t(I)} \le C\epsilon \text{ for all } (j,\mathbf{m}) \text{ with } \mathbf{m}\in\mathcal{M}_j(2N+4), \tag{6.5}$$
$$\|z_j\|_{W^{1,\infty}_t(I)} \le C\epsilon \text{ for all } j\in\{1,\dots,n\}\ . \tag{6.6}$$

Furthermore, there exists $\rho_+ \in [0,\infty)^n$ s.t. there exist a j_0 with $\rho_{+j} = 0$ for $j \ne j_0$, and there exists $\eta_+ \in H^1$ s.t. $|\rho_+ - |z(0)|| \le C\epsilon$ and $\eta_+ \in H^1$ with $\|\eta_+\|_{H^1} \le C\epsilon$, such that

$$\lim_{t\to+\infty} \|\eta(t,x) - e^{\mathrm{i}t\Delta}\eta_+(x)\|_{H^1_x} = 0 \quad , \quad \lim_{t\to+\infty} |z_j(t)| = \rho_{+j}. \tag{6.7}$$

Proof that Theor.6.1 implies Theor.1.3. Denote by (z',η') the initial coordinate system. By (5.35)

$$z' = z + \mathcal{R}^{1,1}_{r,\infty}(z,\mathbf{Z},\eta)\,, \quad \eta' = \eta + \mathbf{S}^{1,1}_{r,\infty}(z,\mathbf{Z},\eta).$$

Notice that (6.7) and $\lim_{t\to+\infty}\mathbf{Z}(t) = 0$ and that by standard arguments for $s > 3/2$ we have

$$\lim_{t\to+\infty} \|e^{t\Delta}\eta_+\|_{L^{2,-s}(\mathbb{R}^3)} = 0 \text{ for any } \eta_+ \in L^2. \tag{6.8}$$

These two limits, Definitions 2.8–2.9 and (6.7) imply

$$\lim_{t\to+\infty} \mathcal{R}^{1,1}_{r,\infty}(z,\mathbf{Z},\eta) = 0 \text{ in } \mathbb{C}^n \text{ and } \quad \lim_{t\to+\infty} \mathbf{S}^{1,1}_{r,\infty}(z,\mathbf{Z},\eta) = 0 \text{ in } \Sigma_r.$$

This means that

$$\lim_{t\to+\infty} \|\eta'(t,x) - e^{\mathrm{i}t\Delta}\eta_+(x)\|_{H^1_x} = 0 \quad , \quad \lim_{t\to+\infty} |z'_j(t)| = \rho_{+j}. \tag{6.9}$$

so that (1.8) is true. Notice also that if we set $\widetilde{\eta} = \eta$ and $A(t,x) = \mathbf{S}^{1,1}_{r,\infty}(z,\mathbf{Z},\eta)$ we obtain the desired decomposition of η' satisfying (1.9) and (1.10). Finally we have

$$\dot{z}'_j + \mathrm{i}e_j z'_j = \dot{z}_j + \mathrm{i}e_j z_j + \frac{d}{dt}\mathcal{R}^{1,1}_{r,\infty}(z,\mathbf{Z},\eta) + \mathcal{R}^{1,1}_{r,\infty}(z,\mathbf{Z},\eta) = O(\epsilon^2),$$

where $\dot{z}_j + \mathrm{i}e_j z_j = O(\epsilon^2)$ by (6.27) below, $\mathcal{R}^{1,1}_{r,\infty}(z,\mathbf{Z},\eta) = O(\epsilon^2)$ by (2.23) and $\frac{d}{dt}\mathcal{R}^{1,1}_{r,\infty}(z,\mathbf{Z},\eta) = O(\epsilon^2)$. To check the latter, we write (it is easy that $d_w\mathcal{R}^{1,1}_{r,\infty}(z,\mathbf{Z},\eta) = \mathcal{R}^{1,0}_{r,\infty}(z,\mathbf{Z},\eta)$ for $w = z, \mathbf{Z}$)

$$\frac{d}{dt}\mathcal{R}^{1,1}_{r,\infty}(z,\mathbf{Z},\eta) = \mathcal{R}^{1,0}_{r,\infty}(z,\mathbf{Z},\eta)\dot{z} + \mathcal{R}^{1,0}_{r,\infty}(z,\mathbf{Z},\eta)\dot{\mathbf{Z}} + d_\eta\mathcal{R}^{1,1}_{r,\infty}(z,\mathbf{Z},\eta)\cdot\dot{\eta},$$

with $d_\eta \mathcal{R}^{1,1}_{r,\infty}$ the partial derivative in η. By a simple use of Taylor expansions and Def. 2.8

$$\|d_\eta \mathcal{R}^{1,1}_{r,\infty}(z,\mathbf{Z},\eta)\|_{\Sigma^c_{-r}\to\Sigma^c_r} \le C(|z| + \|\eta\|_{\Sigma_{-r}}).$$

Then by equations (6.12) and (6.27) below, we have $\frac{d}{dt}\mathcal{R}^{1,1}_{r,\infty}(z,\mathbf{Z},\eta) = O(\epsilon^2)$. This yields the inequality claimed in the second line in (1.9).

□

By a standard argument (6.4)–(6.6) for $I = [0,\infty)$ are a consequence of the following Proposition.

Proposition 6.2. *There exists a constant $c_0 > 0$ such that for any $C_0 > c_0$ there is a value $\epsilon_0 = \epsilon_0(C_0)$ such that if the inequalities (6.4)–(6.6) hold for $I = [0,T]$ for some $T > 0$, for $C = C_0$ and for $0 < \epsilon < \epsilon_0$, then in fact for $I = [0,T]$ the inequalities (6.4)–(6.6) hold for $C = C_0/2$.*

6.1 Proof of Proposition 6.2

Lemma 6.3. *Assume the hypotheses of Prop. 6.2 and take the M of Def. 2.5. Then ∃ a fixed c s.t.*

$$\|\eta\|_{L^p_t([0,T],W^{1,q})} \le c\epsilon + c \sum_{(\mu,\nu)\in M} |z^\mu \overline{z}^\nu|_{L^2_t(0,T)} \text{ for all admissible pairs } (p,q). \tag{6.10}$$

Proof. First of all, for $|z| < \delta_f$ and $\|\eta\|_{H^1\cap\mathcal{H}_c[0]} < \delta_f$ defining the domain of the Hamiltonian $\mathcal{H}(z,\eta)$ in (6.1), we will pick $\epsilon_0 \in (0,\delta_f)$ sufficiently small. Let $\epsilon \in (0,\epsilon_0)$, where $\epsilon = \|u(0)\|_{H^1}$. By (2.11) we have $|z'(0)| + \|\eta'(0)\|_X \le c_1\epsilon$, where $(z'(0),\eta'(0))$ are the coordinates in the initial system of coordinates introduced in Lemma 2.4. Let $(z(0),\eta(0))$ be the corresponding coordinates in the final system of coordinates. Then by the relation (5.35), if ϵ_0 is sufficiently small we conclude that

$$|z(0)| + \|\eta(0)\|_{H^1} \le c_1'\epsilon \tag{6.11}$$

for some other fixed constant c_1'. We now turn to the equation of η. We have for $\overline{G}_{j\mathbf{m}} = \overline{G}_{j\mathbf{m}}(0)$

$$\begin{aligned} &\mathrm{i}\dot\eta = \mathrm{i}\{\eta,\mathcal{H}\} = H\eta + \sum_{j=1}^{n}\sum_{l=1}^{2N+3}\sum_{|\mathbf{m}|=l} z_j \overline{\mathbf{Z}}^{\mathbf{m}}\overline{G}_{j\mathbf{m}} + \mathbb{A} \text{ where} \\ &\mathbb{A} := \sum_{j=1}^{n}\sum_{l=1}^{2N+3}\sum_{|\mathbf{m}|=l} z_j \overline{\mathbf{Z}}^{\mathbf{m}}[\overline{G}_{j\mathbf{m}}(|z_j|^2) - \overline{G}_{j\mathbf{m}}] + \nabla_{\overline{\eta}}\mathcal{R}. \end{aligned} \tag{6.12}$$

We rewrite

$$\sum_{j=1}^{n}\sum_{l=1}^{2N+3}\sum_{|\mathbf{m}|=l} z_j \overline{\mathbf{Z}}^{\mathbf{m}}\overline{G}_{j\mathbf{m}} = \sum_{(\mu,\nu)\in M} \overline{z}^\mu z^\nu \overline{G}_{\mu\nu}. \tag{6.13}$$

Notice that (6.5) is the same as

$$\|z^\mu \overline{z}^\nu\|_{L^2_t(I)} \le C\epsilon \text{ for all } (\mu,\nu)\in M\ . \tag{6.14}$$

Suppose we can show that for $I_T := [0,T]$

$$\|\mathbb{A}\|_{L^2(I_T,H^{1,S})+L^1(I_T,H^1)} \le C(S,C_0)\epsilon^2. \tag{6.15}$$

Then, if ϵ_0 is small enough and $\epsilon \in (0,\epsilon_0)$, we obtain (6.10) by $H^{1,S}(\mathbb{R}^3) \hookrightarrow W^{1,\frac{6}{5}}(\mathbb{R}^3)$, by (6.11), (6.14) and (6.15) and by the Strichartz estimates, which, for P_c the orthogonal projection of L^2 onto $\mathcal{H}[0]$, are valid for P_cH by [33] (here notice that all the terms in (6.12) belong to $\mathcal{H}[0]$).

So now we prove (6.15). We have for $r-1 \geq S > 9/2$

$$\begin{aligned} &\|z_j \overline{\mathbf{Z}}^{\mathbf{m}} [\overline{G}_{j\mathbf{m}}(|z_j|^2) - \overline{G}_{j\mathbf{m}}]\|_{L^2(I_T, H^{1,S})} \leq \|z_j \overline{\mathbf{Z}}^{\mathbf{m}}\|_{L^2(I_T,\mathbb{C})} \|\overline{G}_{j\mathbf{m}}(|z_j|^2) - \overline{G}_{j\mathbf{m}}\|_{L^\infty(I_T,H^{1,S})} \\ &\leq C_0 \epsilon \sup\{\|G'_{j\mathbf{m}}(|z_j|^2)\|_{\Sigma_r} : |z_j| \leq \delta_0\} \|z_j^2\|_{L^\infty(I_T,\mathbb{C})} \leq C C_0^3 \epsilon^3 < c\epsilon. \end{aligned} \tag{6.16}$$

We have for a fixed $c_1 > 0$

$$\|\nabla_\eta E_P(\eta)\|_{L^1(I_T,H^1)} = 2\||\eta|^2\eta\|_{L^1(I_T,H^1)} \leq c_1 \|\eta\|_{L^\infty(I_T,H^1)} \|\eta\|^2_{L^2(I_T,L^6)} \leq c_1 C_0^3 \epsilon^3. \tag{6.17}$$

We finally show that for an arbitrarily preassigned $S > 2$

$$\|R_1\|_{L^2(I_T,H^{1,S})} \leq C(S, C_0)\epsilon^2 \text{ for } R_1 = \nabla_\eta(\mathcal{R} - E_P(\eta)). \tag{6.18}$$

R_1 is a sum of various term obtained from the expansion (6.2). Let us start by showing

$$\|\nabla_{\overline{\eta}} \mathcal{R}^{1,2}_{r,\infty}(z,\eta)\|_{L^2(I_T,H^{1,S})} \leq C(S, C_0)\epsilon^2. \tag{6.19}$$

Recalling (2.25), it is elementary to show that $\nabla_{\overline{\eta}} \mathcal{R}^{1,2}_{r,\infty}(z,\eta) = \mathbf{S}^{1,1}_{r,\infty}(z,\eta)$ and

$$\begin{aligned} &\|\mathbf{S}^{1,1}_{r,\infty}(z,\eta)\|_{L^2(I_T,H^{1,S})} \leq C_1 \|(\|\eta\|_{\Sigma_{-r}} + |z|)\|_{L^\infty(I_T)} \|\eta\|_{L^2(I_T,\Sigma_{-r})} \\ &\leq C_2 \|(\|\eta\|_{H^1} + |z|)\|_{L^\infty(I_T)} \|\eta\|_{L^2(I_T,L^6)} \leq C(S, C_0)\epsilon^2. \end{aligned}$$

We next show

$$\|\nabla_{\overline{\eta}} \mathcal{R}^{0,2N+5}_{r,\infty}(z,\mathbf{Z},\eta)\|_{L^2(I_T,H^{1,S})} \leq C(S, C_0)\epsilon^2. \tag{6.20}$$

We have, for a reminder $\|O(\|\eta\|^2_{\Sigma_{-r}})\|_{\Sigma_r} \leq C\|\eta\|^2_{\Sigma_{-r}}$ easily shown to satisfy an inequality like (6.20),

$$\begin{aligned} &\nabla_{\overline{\eta}} \mathcal{R}^{0,2N+5}_{r,\infty}(z,\mathbf{Z},\eta) = \mathbf{S}^{0,2N+4}_{r,\infty}(z,\mathbf{Z},\eta) \\ &= \mathbf{S}^{0,2N+4}_{r,\infty}(z,\mathbf{Z}) + d_\eta \mathbf{S}^{0,2N+4}_{r,\infty}(z,\mathbf{Z},0) \cdot \eta + O(\|\eta\|^2_{\Sigma_{-r}}). \end{aligned}$$

We have by Lemma 2.7

$$\begin{aligned} &\|\mathbf{S}^{0,2N+4}_{r,\infty}(z,\mathbf{Z})\|_{L^2(I_T,H^{1,S})} \leq C_1 \sup_{|z|\leq C_0\epsilon} \|\mathbf{S}^{0,0}_{r,\infty}(z,\mathbf{Z})\|_{\Sigma_{M'}} \||\mathbf{Z}|^{2N+4}\|_{L^2(I_T)} \\ &\leq C_2 \|z\|_{L^\infty(I)} \sum_j \sum_{(\mu,\nu)\in M_j(N+1)} \|z^\mu \overline{z}^\nu\|_{L^\infty(I_T)} \|z^\mu \overline{z}^\nu\|_{L^2(I_T)} \leq C(S, C_0)\epsilon^3. \end{aligned}$$

We have

$$\begin{aligned} &\|d_\eta \mathbf{S}^{0,2N+4}_{r,\infty}(z,\mathbf{Z},0)\cdot\eta\|_{L^2(I_T,H^{1,S})} \leq C_1(S)\|\eta\|_{L^2(I_T,\Sigma_{-r})} \sup_{|z|\leq C_0\epsilon} \|d_\eta \mathbf{S}^{0,2N+4}_{r,\infty}(z,\mathbf{Z},0)\|_{\Sigma_{-r}\to\Sigma_r} \\ &\leq C_2(S)\|\eta\|_{L^2(I_T,L^6)} \sup_{|z|\leq C_0\epsilon} |\mathbf{Z}|^{2N+3} \leq C(S, C_0)\epsilon^2 \end{aligned}$$

Hence (6.20) is proved. Other terms in R_1 can be bounded with similarly elementary arguments, yielding (6.18). Then (6.16), (6.17) and (6.18) imply (6.15).

□

Setting $M = M(2N+4)$, see Def. 2.5, we now introduce a new variable g setting

$$g = \eta + Y \text{ with } Y := \sum_{(\alpha,\beta)\in M} \overline{z}^\alpha z^\beta R^+_H(\mathbf{e}\cdot(\beta-\alpha))\overline{G}_{\alpha\beta}. \tag{6.21}$$

Lemma 6.4. *Assume the hypotheses of Prop.* (6.2) *and fix* $S > 9/2$. *Then there is a* $c_1(S) > 0$ *s.t. for any* C_0 *there is a* $\epsilon_0 = \epsilon_0(C_0, S) > 0$ *such that for* $\epsilon \in (0, \epsilon_0)$ *in Theor.1.3 we have*

$$\|g\|_{L^2([0,T],L^{2,-S})} \le c_1(S)\epsilon. \tag{6.22}$$

Proof. We have

$$\mathrm{i}\dot g = Hg + \mathbb{A} + \mathbf{T} \text{ where } \mathbf{T} := \sum_j \left[\partial_{z_j} Y(\mathrm{i}\dot z_j - e_j z_j) + \partial_{\overline{z}_j} Y(\mathrm{i}\dot{\overline{z}}_j + e_j \overline{z}_j)\right]. \tag{6.23}$$

We then have

$$g(t) = e^{-\mathrm{i}Ht}\eta(0) + e^{-\mathrm{i}Ht}Y(0) - \mathrm{i}\int_0^t e^{-\mathrm{i}H(t-s)}(\mathbb{A}(s) + \mathbf{T}(s))ds. \tag{6.24}$$

We have for fixed constants by (6.11) and (6.15) the following inequalities:

$$\|e^{-\mathrm{i}Ht}\eta(0)\|_{L^2([0,T],L^{2,-S})} \le c_2\|e^{-\mathrm{i}Ht}\eta(0)\|_{L^2([0,T],L^6)} \le c_2'\|\eta(0)\|_{L^2} \le c_3\epsilon;$$
$$\|\int_0^t e^{-\mathrm{i}H(t-s)}\mathbb{A}(s)ds\|_{L^2([0,T],L^{2,-S})} \le c_2\|\mathbb{A}\|_{L^2([0,T],H^{1,S})+L^1([0,T],H^1)} \le C(C_0,S)\epsilon^2.$$

For a proof of the following standard lemma see for instance to the proof of Lemma 5.4 [7].

Lemma 6.5. *Let* Λ *be a compact subset of* $(0,\infty)$ *and let* $S > 9/2$. *Then there exists a fixed* $c(S,\Lambda)$ *s.t. for every* $t \ge 0$ *and* $\lambda \in \Lambda$

$$\|e^{-\mathrm{i}Ht}R_H^+(\lambda)P_c v_0\|_{L^{2,-S}(\mathbb{R}^3)} \le c(S,\Lambda)\langle t\rangle^{-\frac{3}{2}}\|P_c v_0\|_{L^{2,S}(\mathbb{R}^3)} \textit{ for all } v_0 \in L^{2,S}(\mathbb{R}^3).$$

□

By Lemma 6.5, by (6.11) and by $G_{\alpha\beta} = P_c G_{\alpha\beta}$ we have

$$\|e^{-\mathrm{i}Ht}Y(0)\|_{L^2([0,T],L^{2,-S})} \le \sum_{(\alpha,\beta)\in M} |z^\alpha(0)z^\beta(0)|\ \|e^{-\mathrm{i}Ht}R_H^+(\mathbf{e}\cdot(\beta-\alpha))\overline{G}_{\alpha\beta}\|_{L^2([0,T],L^{2,-S})}$$
$$\le (\sharp M)c_2\epsilon^2\|\langle t\rangle^{-\frac{3}{2}}\|_{L^2(0,T)}c(S,\Lambda)\|\overline{G}_{\alpha\beta}\|_{L^{2,S}} \le C(N,C_0,S)\epsilon^2$$

with $\sharp M$ the cardinality of M and a fixed c_2 and where the following set Λ is as in Lemma 6.5,

$$\Lambda := \{(\nu-\mu)\cdot\mathbf{e} : (\mu,\nu)\in M\}. \tag{6.25}$$

We finally consider, for definiteness (the term $\partial_{\overline{z}_j} Y(\mathrm{i}\dot{\overline{z}}_j + e_j\overline{z}_j)$ can be treated similarly)

$$\begin{aligned}
&\|\int_0^t e^{-\mathrm{i}H(t-s)}R_H^+(\mathbf{e}\cdot(\beta-\alpha))\overline{G}_{\alpha\beta}\partial_{z_j}(\overline{z}^\alpha z^\beta)(s)(\mathrm{i}\dot z_j - e_j z_j)(s)ds\|_{L^2([0,T],L^{2,-S})}\\
&\le c(S,\Lambda)\sum_{(\alpha,\beta)\in M}\|G_{\alpha\beta}\|_{L^{2,S}}\beta_j\|\int_0^t\langle t-s\rangle^{-\frac{3}{2}}|\frac{\overline{z}^\alpha(s)z^\beta(s)}{z_j(s)}(\mathrm{i}\dot z_j - e_j z_j)(s)|ds\|_{L^2(0,T)}\\
&\le c(S,\Lambda)c_2\sum_{(\alpha,\beta)\in M}\beta_j\|\frac{\overline{z}^\alpha(s)z^\beta}{z_j}(\mathrm{i}\dot z_j - e_j z_j)\|_{L^2(0,T)},
\end{aligned} \tag{6.26}$$

for fixed c_2. We have

$$\begin{aligned}
\mathrm{i}\dot z_j &= (1+\varpi_j(|z_j|^2))(e_j z_j + \partial_{\overline{z}_j}\mathcal{Z}_0(|z_1|^2,...,|z_n|^2) + \partial_{\overline{z}_j}\mathcal{R})\\
&+ (1+\varpi_j(|z_j|^2))[\sum_{(\mu,\nu)\in M} \nu_j \frac{z^\mu \overline{z}^\nu}{\overline{z}_j}\langle \eta, G_{\mu\nu}\rangle + \sum_{(\mu',\nu')\in M} \mu'_j \frac{z^{\nu'}\overline{z}^{\mu'}}{\overline{z}_j}\langle \overline{\eta}, \overline{G}_{\mu'\nu'}\rangle] \\
&+ (1+\varpi_j(|z_j|^2))[\sum_{\mathbf{m}\in\mathcal{M}_j(2N+3)} |z_j|^2 \mathbf{Z}^{\mathbf{m}}\left\langle G'_{j\mathbf{m}}, \eta\right\rangle + z_j^2 \overline{\mathbf{Z}}^{\mathbf{m}}\left\langle \overline{G}'_{j\mathbf{m}}, \overline{\eta}\right\rangle].
\end{aligned} \tag{6.27}$$

To bound (6.26) we substitute $(\mathrm{i}\dot z_j - e_j z_j)$ by the other terms in (6.27) in the last line of (6.26) . So for example we have $\partial_{\overline{z}_j}\mathcal{Z}_0(|z_1|^2,...,|z_n|^2)\sim z_j O(\epsilon)$ which by (6.14) yields

$$\beta_j\|\frac{\overline{z}^\alpha z^\beta}{z_j}\partial_{\overline{z}_j}\mathcal{Z}_0(|z_1|^2,...,|z_n|^2)\|_{L^2(0,T)} \le C(C_0)\epsilon\|\overline{z}^\alpha z^\beta\|_{L^2(0,T)} \le C(C_0)C_0\epsilon^2.$$

For $(\mu,\nu)\in M$ we have in $(0,T)$

$$\beta_j\nu_j\|\frac{\overline{z}^\alpha z^\beta}{z_j}\frac{z^\mu\overline{z}^\nu}{\overline{z}_j}\langle\eta, G_{\mu\nu}\rangle\|_{L^2_t} \le \beta_j\nu_j\|\frac{\overline{z}^\alpha z^\beta}{z_j}\frac{z^\mu\overline{z}^\nu}{\overline{z}_j}\|_{L^\infty_t}\|G_{\mu\nu}\|_{L^{\frac{6}{5}}}\|\eta\|_{L^\infty_t L^6} \le C(C_0)\epsilon^2.$$

A similar argument works for the terms in the 2nd summation in the 2nd line of (6.27). Finally

$$\beta_j\|\frac{\overline{z}^\alpha z^\beta}{z_j}\partial_{\overline{z}_j}\mathcal{R}\|_{L^2(0,T)} \le \beta_j\|\frac{\overline{z}^\alpha z^\beta}{z_j}\|_{L^\infty(0,T)}\|\partial_{\overline{z}_j}\mathcal{R}\|_{L^2(0,T)} \le C(C_0)\epsilon^3$$

is a consequence of the bound

$$\|\partial_{\overline{z}_j}\mathcal{R}\|_{L^p(0,T)} \le C(C_0)\epsilon^2 \text{ for any } p\in[1,\infty]. \tag{6.28}$$

Here we need to check (6.28) term by term for the sum in the r.h.s. of (6.2). This is straightforward using (2.23), (2.25) and (2.26) and the fact, stated in Lemma 5.8, that $G_{2\mathbf{m}ij}$ and G_{dij} are $\mathbf{S}^{0,0}_{r,\infty}$. □

We turn now to the Fermi Golden Rule (FGR). We substitute (6.21) in (6.27) getting

$$\begin{aligned}
\mathrm{i}\dot z_j &= (1+\varpi_j(|z_j|^2))(e_j z_j + \partial_{\overline{z}_j}\mathcal{Z}_0(|z_1|^2,...,|z_n|^2))\\
&- \sum_{\substack{(\mu,\nu)\in M\\(\alpha,\beta)\in M}} \nu_j \frac{z^{\mu+\beta}\overline{z}^{\nu+\alpha}}{\overline{z}_j}\langle R^+_H(\mathbf{e}\cdot(\beta-\alpha))\overline{G}_{\alpha\beta}, G_{\mu\nu}\rangle\\
&- \sum_{\substack{(\mu',\nu')\in M\\(\alpha',\beta')\in M}} \mu'_j \frac{z^{\nu'+\alpha'}\overline{z}^{\mu'+\beta'}}{\overline{z}_j}\langle R^-_H(\mathbf{e}\cdot(\beta'-\alpha'))G_{\alpha'\beta'}, \overline{G}_{\mu'\nu'}\rangle + \mathcal{F}_j, \text{ where}
\end{aligned} \tag{6.29}$$

$$\begin{aligned}
\mathcal{F}_j &:= (1+\varpi_j(|z_j|^2))\partial_{\overline{z}_j}\mathcal{R} + \varpi_j(|z_j|^2)[\sum_{(\mu,\nu)\in M}\nu_j\frac{z^\mu\overline{z}^\nu}{\overline{z}_j}\langle\eta, G_{\mu\nu}\rangle + \sum_{(\mu',\nu')\in M}\mu'_j\frac{z^{\nu'}\overline{z}^{\mu'}}{\overline{z}_j}\langle\overline{\eta},\overline{G}_{\mu'\nu'}\rangle]\\
&+ \sum_{(\mu,\nu)\in M}\nu_j\frac{z^\mu\overline{z}^\nu}{\overline{z}_j}\langle g, G_{\mu\nu}\rangle + \sum_{(\mu',\nu')\in M}\mu'_j\frac{z^{\nu'}\overline{z}^{\mu'}}{\overline{z}_j}\langle\overline{g},\overline{G}_{\mu'\nu'}\rangle\\
&+ (1+\varpi_j(|z_j|^2))[\sum_{\mathbf{m}\in\mathcal{M}_j(2N+3)} |z_j|^2\mathbf{Z}^{\mathbf{m}}\left\langle G'_{j\mathbf{m}},\eta\right\rangle + z_j^2\overline{\mathbf{Z}}^{\mathbf{m}}\left\langle\overline{G}'_{j\mathbf{m}},\overline{\eta}\right\rangle].
\end{aligned} \tag{6.30}$$

We now introduce the new variable ζ defined by

$$
\begin{aligned}
z_j - \zeta_j = &- \sum_{\substack{(\mu,\nu)\in M\\(\alpha,\beta)\in M}} \frac{\nu_j z^{\mu+\beta}\overline{z}^{\nu+\alpha}}{((\mu-\nu)\cdot\mathbf{e}-(\alpha-\beta)\cdot\mathbf{e})\overline{z}_j}\langle R_H^+(\mathbf{e}\cdot(\beta-\alpha))\overline{G}_{\alpha\beta}, G_{\mu\nu}\rangle \\
&- \sum_{\substack{(\mu',\nu')\in M\\(\alpha',\beta')\in M}} \frac{\mu'_j z^{\nu'+\alpha'}\overline{z}^{\mu'+\beta'}}{((\alpha'-\beta')\cdot\mathbf{e}-(\mu'-\nu')\cdot\mathbf{e})\overline{z}_j}\langle R_H^-(\mathbf{e}\cdot(\beta'-\alpha'))G_{\alpha'\beta'}, \overline{G}_{\mu'\nu'}\rangle
\end{aligned}
\tag{6.31}
$$

where we are summing only on pairs where the formula makes sense (i.e. only on pairs not in a same set M_L for an $L\in\Lambda$, see (6.33) below). It is easy to see that

$$
\|\zeta - z\|_{L^2(0,T)} \le c(N,C_0)\epsilon^2 \text{ and } \|\zeta - z\|_{L^\infty(0,T)} \le c(N,C_0)\epsilon^2. \tag{6.32}
$$

Recall now the set $\Lambda = \{(\nu-\mu)\cdot\mathbf{e} : (\mu,\nu)\in M\}$ defined in (6.25). For any $L\in\Lambda$ set

$$
M_L := \{(\mu,\nu)\in M : (\nu-\mu)\cdot\mathbf{e} = L\}. \tag{6.33}
$$

We then get

$$
\begin{aligned}
\mathrm{i}\dot{\zeta}_j = &(1+\varpi(|z_j|^2))(e_j\zeta_j + \partial_{\overline{j}}\mathcal{Z}_0(|\zeta_1|^2,...,|\zeta_n|^2)) \\
&- \sum_{L\in\Lambda}\sum_{\substack{(\mu,\nu)\in M_L\\(\alpha,\beta)\in M_L}} \nu_j \frac{\zeta^{\mu+\beta}\overline{\zeta}^{\nu+\alpha}}{\overline{\zeta}_j}\langle R_H^+(\mathbf{e}\cdot(\beta-\alpha))\overline{G}_{\alpha\beta}, G_{\mu\nu}\rangle \\
&- \sum_{L\in\Lambda}\sum_{\substack{(\mu',\nu')\in M_L\\(\alpha',\beta')\in M_L}} \mu'_j \frac{\zeta^{\nu'+\alpha'}\overline{\zeta}^{\mu'+\beta'}}{\overline{\zeta}_j}\langle R_H^-(\mathbf{e}\cdot(\beta'-\alpha'))G_{\alpha'\beta'}, \overline{G}_{\mu'\nu'}\rangle + \mathcal{G}_j,
\end{aligned}
\tag{6.34}
$$

where for some $A_{k\alpha\beta\mu\nu}, B_{k\alpha\beta\mu\nu}$ we have

$$
\begin{aligned}
\mathcal{G}_j = &\mathcal{F}_j + (1+\varpi(|z_j|^2))[\partial_{\overline{j}}\mathcal{Z}_0(|z_1|^2,...,|z_n|^2) - \partial_{\overline{j}}\mathcal{Z}_0(|\zeta_1|^2,...,|\zeta_n|^2)] \\
&- e_j\varpi(|z_j|^2)[\sum_{\substack{(\mu,\nu)\in M\\(\alpha,\beta)\in M}} \frac{\nu_j z^{\mu+\beta}\overline{z}^{\nu+\alpha}}{((\mu-\nu)\cdot\mathbf{e}-(\alpha-\beta)\cdot\mathbf{e})\overline{z}_j}\langle R_H^+(\mathbf{e}\cdot(\beta-\alpha))\overline{G}_{\alpha\beta}, G_{\mu\nu}\rangle \\
&+ \sum_{\substack{(\mu',\nu')\in M\\(\alpha',\beta')\in M}} \frac{\mu'_j z^{\nu'+\alpha'}\overline{z}^{\mu'+\beta'}}{((\alpha'-\beta')\cdot\mathbf{e}-(\mu'-\nu')\cdot\mathbf{e})\overline{z}_j}\langle R_H^-(\mathbf{e}\cdot(\beta'-\alpha'))G_{\alpha'\beta'}, \overline{G}_{\mu'\nu'}\rangle] \\
&+ \sum_k \sum_{\substack{(\mu,\nu)\in M\\(\alpha,\beta)\in M}} (\mathrm{i}\dot{z}_k - e_k z_k)\frac{z^{\mu+\beta}\overline{z}^{\nu+\alpha}}{\overline{z}_j}A_{k\alpha\beta\mu\nu} + \overline{(\mathrm{i}\dot{z}_k - e_k z_k)}\frac{z^{\mu+\beta}\overline{z}^{\nu+\alpha}}{\overline{z}_j}B_{k\alpha\beta\mu\nu}.
\end{aligned}
\tag{6.35}
$$

Lemma 6.6. *There are fixed c_4 and $\epsilon_0 > 0$ such that for $\epsilon\in(0,\epsilon_0)$ we have*

$$
\|\mathcal{G}_j\overline{\zeta}_j\|_{L^1[0,T]} \le (1+C_0)c_4\epsilon^2. \tag{6.36}
$$

Proof. We consider separately the terms in the r.h.s. of (6.35) and (6.30). By (6.6), (6.28) (6.32)

$$\|\partial_{\overline{z}_j}\mathcal{R}\overline{\zeta}_j\|_{L^1_t[0,T]}\le C(C_0)\epsilon^3.$$

For fixed constants c_2 and c_3, by (6.4) and (6.22), we have

$$\|\frac{z^\mu\overline{z}^\nu\overline{\zeta}_j}{\overline{z}_j}\langle g,G_{\mu\nu}\rangle\|_{L^1[0,T]}\le c_2\|\frac{z^\mu\overline{z}^\nu\overline{\zeta}_j}{\overline{z}_j}\|_{L^2[0,T]}\|g\|_{L^2([0,T],L^{2,-S})}\le c_3C_0\epsilon^2. \tag{6.37}$$

To get (6.37) we exploit Lemma 6.4 and the following bound:

$$\begin{aligned}&\nu_j\|\frac{z^\mu\overline{z}^\nu\overline{\zeta}_j}{\overline{z}_j}\|_{L^2[0,T]}\le\nu_j\|z^\mu\overline{z}^\nu\|_{L^2[0,T]}+\nu_j\|\frac{z^\mu\overline{z}^\nu}{\overline{z}_j}\|_{L^\infty[0,T]}\|\zeta_j-\overline{z}_j\|_{L^2[0,T]}\\&\le c_2C_0\epsilon+C(C_0)\epsilon^3\end{aligned} \tag{6.38}$$

for fixed c_2, where we used (6.14) and (6.32). Terms such as (6.37), that is the terms from the 2nd term in the r.h.s. of (6.30), are the ones responsible for the $C_0c_4\epsilon^2$ in (6.36), where C_0 could be large. The other terms are $O(\epsilon^2)$ with fixed constants, if ϵ_0 is small enough.
By (6.4) and (6.5), for $\mathbf{m}\in\mathcal{M}_j(2N+4)$ we have

$$\||z_j|^2\mathbf{Z}^{\mathbf{m}}\left\langle G'_{j\mathbf{m}},\eta\right\rangle\overline{\zeta}_j\|_{L^1[0,T]}\le c_4\|z_j\zeta_j\|_{L^\infty}\|z_j\mathbf{Z}^{\mathbf{m}}\|_{L^2[0,T]}\|\eta\|_{L^2([0,T],L^{2,-S})}\le C(C_0)\epsilon^4. \tag{6.39}$$

It is easy to see by (6.32) that

$$\|\overline{\zeta}_j(\text{2nd–6th line of r.h.s.(6.35)})\|_{L^2[0,T]}\le C(C_0)\epsilon^3, \tag{6.40}$$

see Lemma 4.11 [8],

$$\|[\partial_{\overline{j}}\mathcal{Z}_0(|z_1|^2,...,|z_n|^2)-\partial_{\overline{j}}\mathcal{Z}_0(|\zeta_1|^2,...,|\zeta_n|^2)]\overline{\zeta}_j\|_{L^2[0,T]}\le C(C_0)\epsilon^3, \tag{6.41}$$

see Lemma 4.10 [8]. Finally we have for $(\mu,\nu)\in M$

$$\begin{aligned}&\|\varpi_j(|z_j|^2)\nu_j\frac{z^\mu\overline{z}^\nu}{\overline{z}_j}\langle\eta,G_{\mu\nu}\rangle\zeta_j\|_{L^1_t}\le\|\varpi_j(|z_j|^2)\nu_jz^\mu\overline{z}^\nu\langle\eta,G_{\mu\nu}\rangle\|_{L^1_t}\\&+\|\varpi_j(|z_j|^2)\nu_j\frac{z^\mu\overline{z}^\nu}{\overline{z}_j}\langle\eta,G_{\mu\nu}\rangle(\zeta_j-z_j)\|_{L^1_t}\le C(C_0)\epsilon^3\end{aligned}$$

by $\varpi_j(|z_j|^2)=O(|z_j|^2)$, (6.4)–(6.6) and (6.32). This completes the proof of Lemma 6.6. ☐

We now consider

$$\begin{aligned}&2^{-1}\frac{d}{dt}\sum_j|e_j|\;|\zeta_j|^2=-\sum_je_j\overbrace{\mathrm{Im}[(1+\varpi(|z_j|^2))e_j|\zeta_j|^2+\partial_{\overline{\zeta}_j}\mathcal{Z}_0(|\zeta_1|^2,...,|\zeta_n|^2)\overline{\zeta}_j]}^{0}-\sum_je_j\,\mathrm{Im}[\mathcal{G}_j\overline{\zeta}_j]\\&+\sum_{L\in\Lambda}\mathrm{Im}[\sum_{\substack{(\mu,\nu)\in M_L\\(\alpha,\beta)\in M_L}}\nu\cdot\mathbf{e}\;\zeta^{\mu+\beta}\overline{\zeta}^{\nu+\alpha}\langle R^+_H(L)\overline{G}_{\alpha\beta},G_{\mu\nu}\rangle\\&+\sum_{\substack{(\mu',\nu')\in M_L\\(\alpha',\beta')\in M_L}}\mu'\cdot\mathbf{e}\;\zeta^{\nu'+\alpha'}\overline{\zeta}^{\mu'+\beta'}\langle R^-_H(L)G_{\alpha'\beta'},\overline{G}_{\mu'\nu'}\rangle].\end{aligned} \tag{6.42}$$

We can now substitute $R^\pm_H(L)=P.V.\frac{1}{H-L}\pm\mathrm{i}\pi\delta(H-L)$.

Lemma 6.7. *The contributions to* (6.42) *from the* $P.V.\frac{1}{H-L}$ *cancel out:*

$$\begin{aligned}&\mathrm{Im}[\sum_{\substack{(\mu,\nu)\in M_L\\(\alpha,\beta)\in M_L}}\nu\cdot\mathbf{e}\;\zeta^{\mu+\beta}\overline{\zeta}^{\nu+\alpha}\langle P.V.\frac{1}{H-L}\overline{G}_{\alpha\beta},G_{\mu\nu}\rangle\\&+\sum_{\substack{(\mu',\nu')\in M_L\\(\alpha',\beta')\in M_L}}\mu'\cdot\mathbf{e}\;\zeta^{\nu'+\alpha'}\overline{\zeta}^{\mu'+\beta'}\langle P.V.\frac{1}{H-L}G_{\alpha'\beta'},\overline{G}_{\mu'\nu'}\rangle]=0.\end{aligned}\tag{6.43}$$

Proof. We set $(\alpha',\beta')=(\mu,\nu)$ and $(\mu',\nu')=(\alpha,\beta)$ in the 2nd line of (6.43). With these choices

$$\mu'\cdot\mathbf{e}\;\zeta^{\nu'+\alpha'}\overline{\zeta}^{\mu'+\beta'}\langle P.V.\frac{1}{H-L}G_{\alpha'\beta'},\overline{G}_{\mu'\nu'}\rangle=\alpha\cdot\mathbf{e}\;\zeta^{\mu+\beta}\overline{\zeta}^{\nu+\alpha}\langle P.V.\frac{1}{H-L}\overline{G}_{\alpha\beta},G_{\mu\nu}\rangle$$

Then 2 times the l.h.s. of (6.43) becomes

$$\begin{aligned}&2\,\mathrm{Im}[\sum_{\substack{(\mu,\nu)\in M_L\\(\alpha,\beta)\in M_L}}(\alpha+\nu)\cdot\mathbf{e}\;\zeta^{\mu+\beta}\overline{\zeta}^{\nu+\alpha}\langle P.V.\frac{1}{H-L}\overline{G}_{\alpha\beta},G_{\mu\nu}\rangle]=\sum_{\substack{(\mu,\nu)\in M_L\\(\alpha,\beta)\in M_L}}\mathrm{Im}[(\alpha+\nu)\cdot\mathbf{e}\;\zeta^{\mu+\beta}\overline{\zeta}^{\nu+\alpha}\times\\&\times\langle P.V.\frac{1}{H-L}\overline{G}_{\alpha\beta},G_{\mu\nu}\rangle+(\mu+\beta)\cdot\mathbf{e}\;\overline{\zeta}^{\mu+\beta}\zeta^{\nu+\alpha}\langle P.V.\frac{1}{H-L}\overline{G}_{\mu\nu},G_{\alpha\beta}\rangle]\\&=\mathrm{Im}[\sum_{\substack{(\mu,\nu)\in M_L\\(\alpha,\beta)\in M_L}}(\alpha+\nu)\cdot\mathbf{e}\left(\zeta^{\mu+\beta}\overline{\zeta}^{\nu+\alpha}\langle P.V.\frac{1}{H-L}\overline{G}_{\alpha\beta},G_{\mu\nu}\rangle+\text{c.c.}\right)]=0\end{aligned}$$

where we exploited the fact that if (μ,ν) and (α,β) both belong to M_L then $(\alpha+\nu)\cdot\mathbf{e}=(\mu+\beta)\cdot\mathbf{e}$. ☐

Lemma 6.8. *Set for any* $L\in\Lambda$

$$G_L(\zeta):=\sqrt{\pi}\sum_{(\mu,\nu)\in M_L}\zeta^{\mu}\overline{\zeta}^{\nu}G_{\mu\nu}.\tag{6.44}$$

Then we have

$$\begin{aligned}&\mathrm{Im}[\mathrm{i}\pi\sum_{\substack{(\mu,\nu)\in M_L\\(\alpha,\beta)\in M_L}}\nu\cdot\mathbf{e}\;\zeta^{\mu+\beta}\overline{\zeta}^{\nu+\alpha}\langle\delta(H-L)\overline{G}_{\alpha\beta},G_{\mu\nu}\rangle\\&-\mathrm{i}\pi\sum_{\substack{(\mu',\nu')\in M_L\\(\alpha',\beta')\in M_L}}\mu'\cdot\mathbf{e}\;\zeta^{\nu'+\alpha'}\overline{\zeta}^{\mu'+\beta'}\langle\delta(H-L)G_{\alpha'\beta'},\overline{G}_{\mu'\nu'}\rangle]=L\langle\delta(H-L)\overline{G}_L(\zeta),G_L(\zeta)\rangle\geq0.\end{aligned}\tag{6.45}$$

Proof. First of all the last inequality is a consequence of the formula

$$\langle F,\delta(H-L)\overline{G}\rangle=\frac{1}{2\sqrt{L}}\int_{|\xi|=\sqrt{L}}\widehat{F}(\xi)\overline{\widehat{G}}(\xi)d\sigma(\xi)$$

with $\widehat{F}$ and $\widehat{G}$ the Fourier transforms of F and G associated to H, see Prop. 2.2 Ch. 9 [27].

To prove the first equality in (6.45) set $(\alpha',\beta')=(\alpha,\beta)$ and $(\mu',\nu')=(\mu,\nu)$ in the 2nd line of (6.45). Then the l.h.s. of (6.45) equals

$$\pi\,\mathrm{Re}[\sum_{\substack{(\mu,\nu)\in M_L\\(\alpha,\beta)\in M_L}}\overbrace{(\nu-\mu)\cdot\mathbf{e}}^{L}\;\zeta^{\mu+\beta}\overline{\zeta}^{\nu+\alpha}\langle\delta(H-L)\overline{G}_{\alpha\beta},G_{\mu\nu}\rangle]=L\langle\delta(H-L)\overline{G}_L(\zeta),G_L(\zeta)\rangle.$$

□

From (6.42) and Lemmas 6.7–6.8 we obtain

$$2\sum_{L\in\Lambda} L\langle\delta(H-L)\overline{G}_L(\zeta), G_L(\zeta)\rangle = \frac{d}{dt}\sum_j |e_j|\ |\zeta_j|^2 + 2\sum_j e_j \operatorname{Im}[\mathcal{G}_j\bar{\zeta}_j]. \tag{6.46}$$

We are able to restate, precisely this time, hypothesis (H4).

(H4) We assume that for some fixed constants we have:

$$\sum_{L\in\Lambda}\langle\delta(H-L)\overline{G}_L(\zeta), G_L(\zeta)\rangle \sim \sum_{(\mu,\nu)\in M} |\zeta^{\mu+\nu}|^2 \quad \text{for all } \zeta\in\mathbb{C}^n \text{ with } |\zeta|\le 1. \tag{6.47}$$

We now complete the proof of Prop. 6.2. We *claim* we have for a fixed c

$$|\sum_j |e_j|(|\zeta_j(t)|^2 - |\zeta_j(0)|^2)| \le c\epsilon^2. \tag{6.48}$$

Indeed, first of all we have $|\zeta_j(0)| \le c'\epsilon$ by $\epsilon := \|u_0\|_{H^1}$. Observe that for (z',η') the initial coordinates in Lemma 2.4, by Proposition 1.1 and Lemma 2.3 it is easy to see that we have

$$\begin{aligned}\epsilon^2 > \|u_0\|^2_{L^2} = \|u(t)\|^2_{L^2} = \|(\sum_{j=1}^n z_j'(t)\phi_j + \eta'(t)) + (\sum_{j=1}^n q_{jz_j'(t)} + (R[z'(t)]-1)\eta'(t))\|^2_{L^2}\\ = \sum_{j=1}^n |z_j'(t)|^2 + \|\eta'(t)\|^2_{L^2} + O(|z'(t)|^6 + |z'(t)|^4\|\eta'(t)\|^2_{L^2}).\end{aligned}$$

This gives the following version of (2.11):

$$\sum_{j=1}^n |z_j'(t)|^2 + \|\eta'(t)\|^2_{L^2} \le 2\epsilon^2. \tag{6.49}$$

This yields an analogous formula for the last system of coordinates (z,η) in (5.35). Finally, this yields the following inequality for the variables ζ introduced in (6.31):

$$\sum_{j=1}^n |\zeta_j(t)|^2 \le 3\epsilon^2. \tag{6.50}$$

Hence the *claim* (6.48) is proved. By Lemma 6.6, by the hypothesis (6.47), by (6.32) and by (6.48), for $\epsilon\in(0,\epsilon_0)$ with ϵ_0 small enough we obtain for a fixed c

$$\sum_{(\mu,\nu)\in M} \|z^{\mu+\nu}\|^2_{L^2(0,t)} \le c\epsilon^2 + cC_0\epsilon^2. \tag{6.51}$$

(6.51) tells us that $\|z^{\mu+\nu}\|^2_{L^2(0,t)} \lesssim C_0^2\epsilon^2$ implies $\|z^{\mu+\nu}\|^2_{L^2(0,t)} \lesssim \epsilon^2 + C_0\epsilon^2$ for all $(\mu,\nu)\in M$. This means that we can take $C_0\sim 1$. This completes the proof of Prop. 6.2.

□

6.2 Proof of the asymptotics (6.9)

We write (6.12) in the form $\mathrm{i}\dot\eta = -\Delta\eta + V\eta + \mathbb{B}$. Then $\partial_t(e^{-\mathrm{i}\Delta t}\eta) = -\mathrm{i}e^{-\mathrm{i}\Delta t}(\eta + \mathbb{B})$ and so

$$e^{-\mathrm{i}\Delta t_2}\eta(t_2) - e^{-\mathrm{i}\Delta t_1}\eta(t_1) = -\mathrm{i}\int_{t_1}^{t_2} e^{-\mathrm{i}\Delta t}(V\eta(t) + \mathbb{B}(t))dt \text{ for } t_1 < t_2.$$

Then for a fixed c_2 by the Strichartz estimates

$$\|e^{-\mathrm{i}\Delta t_2}\eta(t_2) - e^{-\mathrm{i}\Delta t_1}\eta(t_1)\|_{H^1} \le c_2(\|\eta\|_{L^2(\mathbb{R}_+,W^{1,6})} + \|\mathbb{B}(t)\|_{L^1([t_1,t_2],H^1)+L^2([t_1,t_2],W^{\frac{6}{5}}))}).$$

Since we have

$$\mathbb{B} = \sum_{(\mu,\nu)\in M} \overline{z}^\mu z^\nu \overline{G}_{\mu\nu} + \mathbb{A},$$

and by (6.14) and (6.15), valid now in $[0,\infty)$, for a fixed C we have

$$\|\sum_{(\mu,\nu)\in M} \overline{z}^\mu z^\nu \overline{G}_{\mu\nu}\|_{L^2(\mathbb{R}_+,W^{1,\frac{6}{5}})} \le C\epsilon\,,\ \|\mathbb{A}\|_{L^2(\mathbb{R}_+,W^{1,\frac{6}{5}})+L^1(\mathbb{R}_+,H^1)} \le C\epsilon^2,$$

we conclude that there exists an $\eta_+ \in H^1$ with

$$\lim_{t\to+\infty} e^{-\mathrm{i}\Delta t}\eta(t) = \eta_+ \text{ in } H^1 \text{ with } \|\eta(t) - e^{\mathrm{i}\Delta t}\eta_+\|_{H^1} \le C\epsilon \text{ for all } t \ge 0\ .$$

So we have the first limit in (6.7) and the inequality $\|\eta_+\|_{H^1} \le C\|u(0)\|_{H^1}$ in Theorem 6.1.
We prove now the existence of z_+ and the facts about it in Theorem 6.1. First of all, from (6.27)

$$\frac{1}{2}\sum_j \frac{d}{dt}|z_j|^2 = \sum_j \mathrm{Im}\left[\partial_{\overline{j}}\mathcal{R}\overline{z}_j + \sum_{(\mu,\nu)\in M} \nu_j z^\mu \overline{z}^\nu \langle \eta, G_{\mu\nu}\rangle + \sum_{(\mu',\nu')\in M} \mu'_j z^{\nu'}\overline{z}^{\mu'}\langle \overline{\eta}, \overline{G}_{\mu'\nu'}\rangle\right].$$

Since the r.h.s. has $L^1(0,\infty)$ norm bounded by $C\epsilon^2$ for a fixed C, we conclude that the limit

$$\lim_{t\to+\infty}(|z_1(t)|, ..., |z_n(t)|) = (\rho_{+1}, ..., \rho_{+n}) \text{ exists with } |\rho_+| \le C\|u(0)\|_{H^1}.$$

By $\lim_{t\to+\infty}\mathbf{Z}(t) = 0$ we conclude that all but at most one of the ρ_{+j} are equal to 0.

7 Proof of Theorem 1.4

The stability of $e^{-\mathrm{i}tE_{1z}}Q_{1z}$ is known. By Theorem 1 [16] the stability of $e^{-\mathrm{i}tE_{1z}}Q_{1z}$, or equivalently of $e^{-\mathrm{i}tE_{1\rho_1}}Q_{1\rho}$ for $\rho > 0$, is a consequence of the following two points.

(1) The self–adjoint operator $L_{-\rho} := H - E_{1\rho} + |Q_{1\rho}|^2$ has kernel $\ker L_{-\rho} = \{Q_{1\rho}\}$ and $L_{-\rho} > 0$ in $\{Q_{1\rho}\}^\perp$.

(2) The self–adjoint operator $L_{+\rho} = H - E_{1\rho} + 3|Q_{1\rho}|^2$ is strictly positive: $L_{+\rho} > 0$.

If $|Q_{1\rho}(x)| > 0\ \forall\ x$, then (2) is an immediate consequence of (1). The fact that $\ker L_{-\rho} = \{Q_{1\rho}\}$ follows by the fact that $Q_{1\rho} \in \ker L_{-\rho}$ and by the fact that for $|\rho| < \epsilon_0$ with $\epsilon_0 > 0$ small, the number $E_{1\rho} \sim e_1$ is the smallest eigenvalue of $H + |Q_{1\rho}|^2$ since e_1 is the smallest eigenvalue of H.

We recall that [29, 30, 31, 25, 12, 13, 18, 23] give partial proofs of the instability of the 2nd excited state, and only for $2e_2 > e_1$. We now prove the instability of the excited states.
Fix a $j > 1$ and assume that Q_{jr} is orbitally stable. Then Q_{jr} is asymptotically stable by Theorem 1.3. So if $\|u(0) - Q_{jr}\|_{H^1} \ll 1$ then $\|u(t) - Q_{jz_j} - e^{\mathrm{i}\Delta t}\eta_+\|_{H^1} \to 0$ for $t \to \infty$ and $|z_j(t)| \to \rho$ with $\rho \neq 0$ and close to r. In this case we have

$$E(u(0)) = \lim_{t\to\infty} E(u(t)) = \lim_{t\to\infty} E(Q_{jz_j(t)} + e^{\mathrm{i}\Delta t}\eta_+),$$
$$\|u(0)\|_{L^2}^2 = \lim_{t\to\infty} \|Q_{jz_j(t)} + e^{\mathrm{i}\Delta t}\eta_+\|_{L^2}^2.$$

Since $\|e^{\mathrm{i}\Delta t}\eta_+\|_{L^2_t L^6_x} \lesssim \|\eta_+\|_{L^2}$ there exists $t_n \to \infty$ s.t. $\|e^{\mathrm{i}\Delta t_n}\eta\|_{L^6_x} \to 0$. So, since $\|e^{\mathrm{i}t_n\Delta}\eta_+\|_{L^4} \to 0$, $\int V|e^{\mathrm{i}t_n\Delta}\eta_+|^2\,dx \to 0$ and the cross terms in (3.3) disappear, we have

$$E(u(0)) = \lim_{n\to\infty} E(Q_{jz_j(t_n)} + e^{\mathrm{i}\Delta t_n}\eta_+) = E(Q_{j\rho}) + \|\nabla\eta_+\|_{L^2}^2,$$
$$\|u(0)\|_{L^2}^2 = \lim_{n\to\infty} \|Q_{jz_j(t_n)} + e^{\mathrm{i}\Delta t_n}\eta_+\|_{L^2}^2 = \|Q_{j\rho}\|_{L^2}^2 + \|\eta_+\|_{L^2}^2.$$

We claim that for $j \geq 2$ we can construct a curve on H^1 with the following property.

Lemma 7.1. *For sufficiently small δ, there exists a map $[0,\delta) \ni \varepsilon \mapsto \Psi(\varepsilon) \in H^1$ s.t.*

- $\Psi(0) = Q_{jr}$,
- $\|\Psi(\varepsilon)\|_{L^2}^2 = \|Q_{jr}\|_{L^2}^2$,
- $E(\Psi(\varepsilon)) < E(Q_{jr})$ *if* $\varepsilon > 0$.

Before proving the lemma we show that the assumption that Q_{jr} is asymptotically stable and the existence of Ψ lead to a contradiction.

Proof of instability. Since $\|Q_{jr}\|_{L^2}^2 = r^2 + O(r^6)$ by Proposition 1.1, $\|Q_{jr}\|_{L^2}^2$ is strictly increasing in r for r small. By Proposition 1.1 we have $E'(Q_{jr}) = (e_j + O(r^2))Q'(Q_{jr})$. This implies that $E(Q_{jr})$ is a strictly decreasing function of r. Setting $u(0) = \Psi(\varepsilon)$, we have

$$\|Q_{jr}\|_{L^2}^2 = \|\Psi(\varepsilon)\|_{L^2}^2 = \|Q_{j\rho}\|_{L^2}^2 + \|\eta_+\|_{L^2}^2.$$

Therefore we have $\|Q_{jr}\|_{L^2}^2 \geq \|Q_{j\rho}\|_{L^2}^2$. This implies $r \geq \rho$ and so $E(Q_{j\rho}) \geq E(Q_{jr})$. But looking at the energy we get the following contradiction which ends the proof of Theorem 1.4:

$$E(Q_{jr}) > E(\Psi(\varepsilon)) = E(Q_{j\rho}) + \|\nabla\eta_+\|_{L^2}^2 \geq E(Q_{j\rho}) \geq E(Q_{jr}).$$

□

We now construct the curve Ψ.

Proof of Lemma 7.1. We set $\Psi(\varepsilon) = \beta(\varepsilon)Q_{j,r} + \varepsilon\phi_1$ and choose $\beta(\varepsilon)$ to make $\|\Psi(\varepsilon)\|_{L^2}^2 = \|Q_{jr}\|_{L^2}^2$:

$$\|Q_{jr}\|_{L^2}^2\beta^2 + 2\varepsilon\,\langle Q_{jr}, \phi_1\rangle\,\beta + \varepsilon^2 - \|Q_{jr}\|_{L^2}^2 = 0.$$

So, we have

$$\begin{aligned}
\beta(\varepsilon) &= \frac{-\langle Q_{jr},\phi_1\rangle\varepsilon + \sqrt{\langle Q_{jr},\phi_1\rangle^2\varepsilon^2 - \|Q_{jr}\|_{L^2}^2(\varepsilon^2 - \|Q_{jr}\|_{L^2}^2)}}{\|Q_{jr}\|_{L^2}^2} = \sqrt{1-g_1(r)\varepsilon^2} + g_2(r)\varepsilon,\\
g_1(r) &:= \frac{1}{\|Q_{jr}\|_{L^2}^4}\left(\|Q_{jr}\|_{L^2}^2 - \langle Q_{jr},\phi_1\rangle^2\right) = \frac{1}{\|Q_{jr}\|_{L^2}^4}\left(\|Q_{jr}\|_{L^2}^2 - \langle q_{jr},\phi_1\rangle^2\right),\\
g_2(r) &:= -\frac{\langle Q_{jr},\phi_1\rangle}{\|Q_{jr}\|_{L^2}^2} = -\frac{\langle q_{jr},\phi_1\rangle}{\|Q_{jr}\|_{L^2}^2},
\end{aligned}$$

We now show $E(\Psi(\varepsilon)) < E(Q_{j,r})$ for $\varepsilon > 0$. It suffices to show $S_{E_{jr}}(\Psi(\varepsilon)) < S_{E_{jr}}(Q_{jr})$, where

$$S_{E_{jr}}(u) = E(u) - E_{jr}\|u\|_{L^2}^2.$$

Notice that we have $S'_{E_{jr}}(Q_{jr}) = 0$. Therefore, setting $\gamma(\varepsilon) = \beta(\varepsilon) - 1$, we have

$$\begin{aligned}
&S_{E_{jr}}(\Psi(\varepsilon)) = S_{E_{jr}}(Q_{jr} + \gamma(\varepsilon)Q_{jr} + \varepsilon\phi_1)\\
&= S_{E_{jr}}(Q_{jr}) + \frac{1}{2}\left\langle S''_{E_{jr}}(Q_{jr})\left(\gamma(\varepsilon)Q_{jr} + \varepsilon\phi_1\right), \gamma(\varepsilon)Q_{jr} + \varepsilon\phi_1\right\rangle + o\left(\|\gamma(\varepsilon)Q_{jr} + \varepsilon\phi_1\|_{H^1}^2\right)
\end{aligned}$$

If $g_2(r) = 0$ we have $\gamma(\varepsilon) = O(\varepsilon^2 r^{-2})$ and we conclude

$$\begin{aligned}
S_{E_{jr}}(\Psi(\varepsilon)) &= S_{E_{jr}}(Q_{jr}) + \varepsilon^2\left\langle S_{E_{jr}}(Q_{jr})\phi_1, \phi_1\right\rangle + o(\varepsilon^2)\\
&= S_{E_{jr}}(Q_{jr}) + \varepsilon^2(e_1 - e_j) + O(\varepsilon^2 r) + o(\varepsilon^2) < S_{E_{jr}}(Q_{jr}).
\end{aligned}$$

If $g_2(r) \neq 0$ we have $\gamma(\varepsilon) = O(r\varepsilon)$ and

$$S_{E_{jr}}(\Psi(\varepsilon)) = S_{E_{jr}}(Q_{jr}) + \varepsilon^2(e_1 - e_j) + O(r\varepsilon^2) < S_{E_{jr}}(Q_{jr}).$$

Therefore Lemma 7.1 is proved. This also completes the proof of Theorem 1.4.

□

A Appendix: a generalization of Proposition 1.1

For the reference purposes we generalize (1.1) as

$$\mathrm{i}u_t = -\Delta u + V(x)u + \beta(|u|^2)u\ , \quad (t,x) \in \mathbb{R}\times\mathbb{R}^3. \tag{A.1}$$

and assume that $\beta(0) = 0$, $\beta \in C^\infty(\mathbb{R},\mathbb{R})$ and further, there exists a $p \in (1,5)$ such that for every $k \geq 0$ there is a fixed C_k with

$$\left|\frac{d^k}{dv^k}\beta(v^2)\right| \leq C_k|v|^{p-k-1} \quad \text{if } |v| \geq 1.$$

Proposition A.1. *Fix $j \in \{1,\cdots,n\}$. Then $\exists a_0 > 0$ s.t. $\forall z_j \in B_{\mathbb{C}}(0,a_0)$ there is a unique $Q_{jz_j} \in \mathcal{S}(\mathbb{R}^3,\mathbb{C}) := \cap_{t\geq 0}\Sigma_t(\mathbb{R}^3,\mathbb{C})$ s.t.*

$$\begin{aligned}
&(-\Delta + V)Q_{jz_j} + \beta(|Q_{jz_j}|^2)Q_{jz_j} = E_{jz_j}Q_{jz_j},\\
&Q_{jz_j} = z_j\phi_j + q_{jz_j},\ \langle q_{jz_j}, \overline{\phi}_j\rangle = 0,
\end{aligned} \tag{A.2}$$

and s.t. we have for any $r \in \mathbb{N}$:

(1) $(q_{jz_j}, E_{jz_j}) \in C^\infty(B_{\mathbb{C}}(a_0), \Sigma_r \times \mathbb{R})$*; we have* $q_{jz_j} = z_j \widehat{q}_j(|z_j|^2)$ *, with* $\widehat{q}_j(t^2) = t^2 \widetilde{q}_j(t^2)$*,* $\widetilde{q}_j(t) \in C^\infty((-{a_0}^2, {a_0}^2), \Sigma_r(\mathbb{R}^3, \mathbb{R}))$ *and* $E_{jz_j} = E_j(|z_j|^2)$ *with* $E_j(t) \in C^\infty((-{a_0}^2, {a_0}^2), \mathbb{R})$*;*

(2) $\exists\, C > 0$ *s.t.* $\|q_{jz_j}\|_{\Sigma_r} \le C|z_j|^3$*,* $|E_{jz_j} - e_j| < C|z_j|^2$*.*

The rest of this section is devoted to the proof of Prop. A.1.
The first step is the following lemma, which follows by a direct computation.

Lemma A.2. *Let* $m \in \mathbb{N}_0$ *and* $k \in \{1, 2, 3\}$*. Then, we have*

$$\begin{aligned} &[-\Delta, |x|^{2m}] = -2m(2m+1)|x|^{2m-2} - 4m|x|^{2m-2} x \cdot \nabla \\ &[-\Delta, |x|^{2m} x_k] = -2m(2m+3)|x|^{2m-2} x_k - 4m x_k |x|^{2m-2} x \cdot \nabla - 2|x|^{2m} \partial_{x_k} \end{aligned} \tag{A.3}$$

□

Our second step is the following lemma.

Lemma A.3. *The eigenfunctions* ϕ_j *of* $-\Delta + V$ *satisfy* $\phi_j \in \mathcal{S}(\mathbb{R}^3)$*.*

Proof. First, $\phi_j \in L^2(\mathbb{R}^3)$, so we have $\phi_j \in H^2(\mathbb{R}^3)$ by

$$(-\Delta - e_j)\phi_j = -V\phi_j.$$

Furthermore, if we have $\phi_j \in H^{2m}(\mathbb{R}^3)$, then we have $\phi_j \in H^{2m+2}(\mathbb{R}^3)$. This implies $\phi_j \in \cap_{m=1}^\infty H^m$.
Next, by Lemma A.2, we have

$$(-\Delta - e_j) x_k \phi_j = -2\partial_{x_k}\phi_j - V x_k \phi_j,$$

for $k = 1, 2, 3$. Therefore, we have $x_k \phi_j \in H^2(\mathbb{R}^3)$. Again, by Lemma A.2, we have

$$(-\Delta - e_j)|x|^2 \phi_j = -6\phi_j - 4x \cdot \nabla \phi_j - V x_k \phi_j.$$

So, by $x \cdot \nabla \phi_j = \nabla(x\phi_j) - 3\phi_j \in L^2(\mathbb{R}^3)$, we have $|x|^2\phi_j \in H^2$.
Now, suppose $|x|^{2m}\phi_j \in H^2(\mathbb{R}^3)$. By Lemma A.2, we have

$$\begin{aligned} (-\Delta - e_j)|x|^{2m} x_k \phi_j = &- 2m(2m+3)|x|^{2m-2} x_k \phi_j - 4m x_k |x|^{2m-2} x \cdot \nabla \phi_j \\ &- 2|x|^{2m} \partial_{x_k} \phi_j - V|x|^{2m} x_k \phi_j. \end{aligned}$$

Since

$$|x|^{2m} \partial_{x_k} \phi_j = \partial_{x_k}\left(|x|^{2m}\phi_j\right) - 4m|x|^{2m-2} x_k \phi_j \in L^2(\mathbb{R}^3),$$

we have $|x|^{2m} x_k \phi_j \in H^2(\mathbb{R}^3)$. Finally, by

$$(-\Delta - e_j)|x|^{2m+2}\phi_j = -2(m+1)(2m+3)|x|^{2m}\phi_j - 4(m+1)|x|^{2m} x \cdot \nabla\phi_j - V|x|^{2m+2}\phi_j,$$

and $|x|^{2m} x \cdot \nabla \phi_j = \nabla \cdot (|x|^{2m} x \phi_j) - (4m+3)|x|^{2m}\phi_j \in L^2(\mathbb{R}^3)$, we have $|x|^{2m+2}\phi_j \in H^2(\mathbb{R}^2)$. By induction we have $\phi_j \in \Sigma_{2m}$ for any $m \ge 1$. □

The next step is the following lemma.

Lemma A.4. *Fix* $j \in \{1, \cdots, n\}$ *and* $r \in \mathbb{N}$ *with* $r \ge 2$*. Then* $\exists \delta_r > 0$ *s.t.* $\forall z_j \in B_{\mathbb{C}}(0, \delta_r)$ *there is a unique* $Q_{jz_j} \in \Sigma_r(\mathbb{R}^3, \mathbb{C})$ *satisfying* (1.3) *and claims (1) and (2) in Prop. 1.1.*

Proof. In this proof we write $g(u) := \beta(|u|^2)u$. Notice that it suffices to show the claim of Lemma A.4 for $z_j \in \mathbb{R}$ with $\mathbb{R}$ valued Q_{j,z_j}. Indeed, if we define

$$Q_{jz_j} = e^{\mathrm{i}\theta} Q_{j\rho}, \quad E_{jz_j} = E_{j\rho} \tag{A.4}$$

for $z = e^{\mathrm{i}\theta}\rho$, Q_{jz} and E_{jz} satisfies (1.3) if $Q_{j\rho}$ and $E_{j\rho}$ satisfy (1.3). Further, if $B_{\mathbb{R}}(0,\delta) \ni z \mapsto (Q_{jz}, E_{jz}) \in \Sigma_r \times \mathbb{R}$ is C^∞, then by (A.4), we have $B_{\mathbb{C}}(0,\delta) \ni z \mapsto (Q_{j,z}, E_{j,z}) \in \Sigma_r \times \mathbb{R}$ is C^∞.
Fix $j \in \{0, 1, \cdots, n\}$. For simplicity we set $z_j = z$, $e_j = e$ and $\phi_j = \phi$. Set

$$Q_{j,z} = z\left(\phi + |z|^2\psi(z)\right), \quad E_{j,z} = e + |z|^2 f(z).$$

We solve (1.3) under the above ansatz. Substituting the ansatz into (1.3), we have

$$H\psi + z^{-3} g(z(\phi + z^2\psi)) = e\psi + f\phi + z^2 f\psi. \tag{A.5}$$

Set $Pu = u - \langle u, \phi\rangle\phi$. Then, we have

$$H\psi + z^{-3} Pg(z(\phi + z^2\psi)) = e\psi + z^2 f\psi\,, \quad \langle z^{-3} g(z(\phi + z^2\psi)), \phi\rangle = f.$$

Therefore, it suffices to solve

$$(H-e)\psi = -z^{-3} Pg(z(\phi + z^2\psi)) + z^{-1}\langle g(z(\phi + z^2\psi)), \phi\rangle\psi. \tag{A.6}$$

Now, set $\tilde{\phi}(z) := \phi + z^2\psi(z)$. Then,

$$g(z\tilde{\phi}) = \beta(z^2\tilde{\phi})z\tilde{\phi} = z^3 \int_0^1 \beta'(sz^2\tilde{\phi}^2)\, ds\tilde{\phi}^3.$$

So, (A.6) can be rewritten as

$$(H-e)\psi = -P\left(\int_0^1 \beta'(sz^2\tilde{\phi}^2)\, ds\tilde{\phi}^3\right) + \langle \beta(z^2\tilde{\phi}^2)\tilde{\phi}, \phi\rangle\psi. \tag{A.7}$$

To show that $z \mapsto \psi(z) \in \Sigma_r$ exists and is C^∞, we use the inverse function theorem. Set

$$\Phi(z,\psi) := -(H-e)^{-1} P\left(\int_0^1 \beta'(sz^2\tilde{\phi}^2)\, ds\tilde{\phi}^3\right) + \langle \beta(z^2\tilde{\phi}^2)\tilde{\phi}, \phi\rangle (H-e)^{-1}\psi,$$

and

$$F(z,\psi) := \psi - \Phi(z,\psi).$$

Then, $F : \mathbb{R} \times P\Sigma_r \to P\Sigma_r$ is C^∞. Next, since

$$F(0,\psi) = \psi + \beta'(0)(H-e)^{-1} P\phi^3,$$

we have

$$F(0, -\beta'(0)(H-e)^{-1}P\phi^3) = 0.$$

We now compute $F_\psi(z,\psi)$.

$$\begin{aligned}\Phi_\psi(z,\psi)h =& -2z^4(H-e)^{-1}P\left(\int_0^1 \beta''(sz^2\tilde{\phi}^2)s\, ds\tilde{\phi}^4 h\right) - 3z^2(H-e)^{-1}P\left(\int_0^1 \beta'(sz^2\tilde{\phi}^2)\, ds\tilde{\phi}^2 h\right)\\ &+ 2z^4\langle \beta'(z^2\tilde{\phi}^2)\tilde{\phi}^2 h, \phi\rangle (H-e)\psi + z^2\langle \beta(z^2\tilde{\phi}^2)h, \phi\rangle (H-e)\psi + \langle \beta(z^2\tilde{\phi}^2)\tilde{\phi}, \phi\rangle (H-e)h.\end{aligned}$$

So, we have

$$F_\psi(0,\psi)h = h.$$

Therefore, by the inverse function theorem we have the conclusion of the Lemma. □

The final step is that the $\delta_r > 0$ can be chosen independent of r.

Lemma A.5. *Consider the Q_{jz_j} in Lemma A.4. Then $\exists$ a $\delta > 0$ s.t. $Q_{jz_j} \in \mathcal{S}(\mathbb{R}^3)$ for $|z_j| < \delta$.*

Proof. We use a bootstrap argument similar to the proof of Lemma A.3. We can consider the Q_{jz} given in Lemma A.4 with $r = 4$. It is enough to consider $z = \rho \in (0, \delta)$ with $\delta < \delta_4$. For $\delta > 0$ sufficiently small we also have $E_{j\rho} < \frac{1}{2} e_j < 0$. By (A.2) we have

$$(-\Delta - E_{j\rho}) Q_{j\rho} = -V Q_{j\rho} - \int_0^1 \beta'(s Q_{j\rho}^2)\, ds Q_{j\rho}^3. \tag{A.8}$$

We proceed as in Lemma A.3. Since the commutator term and $-VQ_{j\rho}$ are the same as in A.3,c we conclude that Lemma A.5 is a consequence of the following two simple facts for $m \geq 2$.

(i) If $Q_{j\rho} \in H^m$, then $\beta(Q_{j\rho}^2) Q_{j\rho} = \int_0^1 \beta'(s Q_{j\rho}^2)\, ds Q_{j\rho}^3 \in H^m$.

(ii) If $|x|^{2m} Q_{j\rho} \in L^2(\mathbb{R}^3)$, then $|x|^{2m+2} \int_0^1 \beta'(s Q_{j\rho}^2)\, ds Q_{j\rho}^3 \in L^2$.

(i) follows from the fact that $H^m(\mathbb{R}^3)$ is a ring for $m \geq 2$. We now look at (ii). Since $Q_{j\rho}$ is a continuous function with $Q_{j\rho}(x) \to 0$ as $|x| \to \infty$, the range of $Q_{j\rho}$ (i.e. $\{Q_{j\rho}(x) \in \mathbb{R} \mid x \in \mathbb{R}^3\}$) is relatively compact. So, since $t \to \int_0^1 \beta'(st^2)\, ds$ is a continuous function from $\mathbb{R} \to \mathbb{R}$, the range of $\int_0^1 \beta'(s Q_{j\rho}^2)\, ds$ is relatively compact. Therefore, we have $\int_0^1 \beta'(s Q_{j\rho}^2)\, ds \in L^\infty$. On the other hand, by $Q_{j\rho} \in \Sigma_4$ we have $|x| Q_{j\rho} \in \Sigma_3 \hookrightarrow L^\infty$. Therefore, we have

$$|x|^{2m+2} \int_0^1 \beta'(s Q_{j\rho}^2)\, ds Q_{j\rho}^3 = \int_0^1 \beta'(s Q_{j\rho}^2)\, ds \, (|x| Q_{j\rho})^2 \, |x|^{2m} Q_{j\rho} \in L^2(\mathbb{R}^3).$$

This proves (ii) and completes the proof of Lemma A.5. $\square$

Finally, Proposition A.1 is a consequence of Lemmas A.2–A.5.

B Appendix: expansions of gauge invariant functions

We prove here (3.10) and (3.12), which are direct consequences of Lemmas B.3 and B.4.

Lemma B.1. *Let $a(z) \in C^\infty(B_{\mathbb{C}}(0, \delta), \mathbb{R})$ and $a(e^{i\theta} z) = a(z)$ for any $\theta \in \mathbb{R}$. Then there exists $\alpha \in \mathbb{C}^\infty([0, \delta^2); \mathbb{R})$ s.t. $\alpha(|z|^2) = a(z)$.*

Proof. For $z = r e^{i\theta}$ we have $a(z) = a(r + i0)$. Since $x \to a(x + i0)$ is even and smooth, we have $a(x + i0) = \alpha(x^2)$ with $\alpha(x)$ smooth, see [32]. So $a(z) = \alpha(|z|^2)$. $\square$

Lemma B.2. *Let $\delta > 0$. Suppose $a \in C^\infty(B_{\mathbb{C}^n}(0, \delta); \mathbb{R})$ satisfies $a(e^{i\theta} z_1, \cdots, e^{i\theta} z_n) = a(z_1, \cdots, z_n)$ for all $\theta \in \mathbb{R}$ and $a(0, \cdots, 0) = 0$. Then, for any $M > 0$, there exists $b_{\mathbf{m}}$ s.t.*

$$a(z_1, \cdots, z_n) = \sum_{j=1}^n \alpha_j(|z_j|^2) + \sum_{|m|=1} \mathbf{Z}^{\mathbf{m}} b_{\mathbf{m}}(z_1, \cdots, z_n) + \mathcal{R}^{0,M}(z, \mathbf{Z}), \tag{B.1}$$

where $\alpha_j(|z_j|^2) = a(0, \cdots, 0, z_j, 0, \cdots, 0)$. Furthermore, $b_{\mathbf{m}} \in C^\infty(B_{\mathbb{C}^n}(0, \delta); \mathbb{R})$ and satisfies $b_{\mathbf{m}}(e^{i\theta} z_1, \cdots, e^{i\theta} z_n) = b_{\mathbf{m}}(z_1, \cdots, z_n)$ for all $\theta \in \mathbb{R}$.

Proof. First, we expand a as

$$a(z_1,\cdots,z_n)=a(z_1,0,\cdots,0)+\int_0^1\big(\sum_{j=2}^n\partial_j a(z_1,tz_2\cdots,tz_n)z_j+\partial_{\overline{j}}a(z_1,tz_2\cdots,tz_n)\overline{z}_j\big)\,dt.$$

Then, by

$$a(0,z_2,\cdots,z_n)=\int_0^1\big(\sum_{j=2}^n\partial_j a(0,tz_2\cdots,tz_n)z_j+\partial_{\overline{j}}a(0,tz_2\cdots,tz_n)\overline{z}_j\big)\,dt,$$

we have

$$\begin{aligned}&a(z_1,\cdots,z_n)=a(z_1,0,\cdots,0)+a(0,z_2,\cdots,z_n)\\&+\int_0^1\sum_{j=2}^n\Big[\left(\partial_j a(z_1,tz_2\cdots,tz_n)-\partial_j a(0,tz_2\cdots,tz_n)\right)z_j\\&\quad+\Big(\partial_{\overline{j}}a(z_1,tz_2\cdots,tz_n)-\partial_{\overline{j}}a(0,tz_2\cdots,tz_n)\Big)\,\overline{z}_j\Big]\,dt=a(z_1,0,\cdots,0)+a(0,z_2,\cdots,z_n)\\&+\sum_{j\geq 2}\int_0^1\int_0^1\Big[\left(\partial_1\partial_j a(sz_1,tz_2\cdots,tz_n)\right)z_1z_j+\left(\partial_{\overline{1}}\partial_j a(sz_1,tz_2\cdots,tz_n)\right)\overline{z}_1z_j\\&\qquad+\Big(\partial_1\partial_{\overline{j}}a(sz_1,tz_2\cdots,tz_n)\Big)\,\overline{z}_1z_j+\Big(\partial_{\overline{1}}\partial_{\overline{j}}a(sz_1,tz_2\cdots,tz_n)\Big)\,\overline{z}_1\overline{z}_j\Big]dsdt.\end{aligned}$$

Iterating this argument first for $a(0,z_2,\cdots,z_n)$ and then for $a(0,...,0,z_k,\cdots,z_n)$, we have

$$\begin{aligned}&a(z_1,\cdots,z_n)=a(z_1,0,\cdots,0)+a(0,z_2,0,\cdots,0)+\cdots+a(0,\cdots,0,z_n)\\&+\sum_{k=1}^{n-1}\sum_{j\geq k+1}\int_0^1\int_0^1[\left(\partial_k\partial_j a(0,\cdots,0,sz_k,tz_{k+1}\cdots,tz_n)\right)z_kz_j\\&+\left(\partial_{\overline{k}}\partial_j a(0,\cdots,0,sz_k,tz_{k+1}\cdots,tz_n)\right)\overline{z}_kz_j+\Big(\partial_k\partial_{\overline{j}}a(0,\cdots,0,sz_k,tz_{k+1}\cdots,tz_n)\Big)\,z_k\overline{z_j}\\&\quad+\Big(\partial_{\overline{k}}\partial_{\overline{j}}a(0,\cdots,0,sz_k,tz_{k+1}\cdots,tz_n)\Big)\,\overline{z}_k\overline{z}_j\Big]\,dsdt.\end{aligned}\tag{B.2}$$

By Lemma B.1, there exist smooth α_j s.t. $\alpha_j(|z_j|^2)=a(0,\cdots,0,z_j,0,\cdots,0)$. Furthermore, the 3rd line of (B.2) has the same form as the 2nd term in the r.h.s. of (B.1). So, it remains to handle the terms in the 2nd and 4th lines of (B.2). Since they can be treated similarly, we focus only the 2nd line of (B.2). Set

$$\beta_{jk}(z_k,\cdots,z_n)=\int_0^1\int_0^1\left(\partial_k\partial_j a(0,\cdots,0,sz_k,tz_{k+1}\cdots,tz_n)\right)\,dsdt,$$

with $j\geq k+1$. Notice that $\partial^\alpha\overline{\partial}^\beta a(0,...,0)\neq 0$ by the gauge invariance of a is easily shown to imply $|\alpha|=|\beta|$. This in particular implies $\beta_{jk}(0,\cdots,0)=0$. So as in (B.2) we have

$$\begin{aligned}&\beta_{jk}(z_k,\cdots,z_n)=\beta_{jk}(z_k,0,\cdots,0)+\beta_{jk}(0,z_{k+1},0,\cdots,0)+\cdots+\beta_{jk}(0,\cdots,0,z_n)\\&+\sum_{m=k}^{n-1}\sum_{l\geq m+1}\int_0^1\int_0^1[\left(\partial_m\partial_l\beta_{jk}(0,\cdots,0,sz_m,tz_{m+1}\cdots,tz_n)\right)z_mz_l\\&+\left(\partial_{\overline{m}}\partial_l\beta_{jk}(0,\cdots,0,sz_m,tz_{m+1}\cdots,tz_n)\right)\overline{z}_mz_l+\left(\partial_m\partial_{\overline{l}}\beta_{jk}(0,\cdots,0,sz_m,tz_{m+1}\cdots,tz_n)\right)z_m\overline{z}_l\\&+\left(\partial_{\overline{m}}\partial_{\overline{l}}\beta_{jk}(0,\cdots,0,sz_m,tz_{m+1}\cdots,tz_n)\right)\overline{z}_m\overline{z}_l\big]\,dsdt.\end{aligned}\tag{B.3}$$

Since $z_l^2\beta_{jk}(0,\cdots,0,z_l,0,\cdots,0)$ is gauge invariant by Lemma B.1 we have

$$z_l^2\beta_{jk}(0,\cdots,0,z_l,0,\cdots,0)=\tilde{\beta}_{jkl}(|z_l|^2)=\tilde{\beta}_{jkl}(0)+\tilde{\beta}'_{jkl}(0)|z_l|^2+\gamma_{jkl}(|z_l|^2)|z_l|^4,$$

for some smooth $\tilde{\beta}_{jkl}$ and γ_{jkl}. By the smoothness of β_{jk}, we have $\tilde{\beta}_{jkl}(0)=\tilde{\beta}'_{jkl}(0)=0$. Therefore,

$$\beta_{jk}(0,\cdots,0,z_l,0,\cdots,0)z_kz_j=\gamma_{jkl}(|z_l|^2)z_kz_j\overline{z}_l^2 \text{ with } k<\min\{j,l\}.$$

This can be absorbed in the 2nd term of the r.h.s. of (B.1). The same is true of the contribution of the last 2 lines of (B.3). The term

$$\int_0^1\int_0^1 \left(\partial_m\partial_l\beta_{jk}(0,\cdots,0,sz_m,tz_{m+1}\cdots,tz_n)\right)z_mz_lz_jz_k\,dsdt \tag{B.4}$$

does not have as factors components of $\mathbf{Z}=(z_i\overline{z}_j)_{i\neq j}$ but it is $O(|\mathbf{Z}|^2)$. Treating (B.4) the way we treated the 2nd line of (B.2). and repeating the procedure a sufficient number of times, we can express (B.4) as a sum of a summation like the 2nd in the r.h.s. of (B.1) and of a $O(|\mathbf{Z}|^M)$ for arbitrary M. Furthermore, notice that since we can think of the dependence on $\mathbf{Z}=(z_i\overline{z}_j)_{i\neq j}$ to be polynomial, and so the remainder term $R^{0,M}(z,\mathbf{Z})$ in (B.1) can be thought to depend polynomially on $\mathbf{Z}=(z_i\overline{z}_j)_{i\neq j}$, it can be thought as the restriction of a function in $\mathbf{Z}\in L$. □

Lemma B.3. *Take $a(z_1,\cdots,z_n)$ like in Lemma B.2. Then, for any $M>0$, there exist smooth a_j and $b_{j\mathbf{m}}$ s.t. for $\alpha_j(|z_j|^2)=a(0,\cdots,0,z_j,0,\cdots,0)$ we have*

$$a(z_1,\cdots,z_n)=\sum_{j=1}^n\alpha_j(|z_j|^2)+\sum_{1\leq|\mathbf{m}|\leq M-1}\mathbf{Z}^{\mathbf{m}}b_{j\mathbf{m}}(|z_j|^2)+\mathcal{R}^{0,M}(z,\mathbf{Z}). \tag{B.5}$$

Proof. To prove (B.5) one only has to repeatedly use Lemma B.2. □

Lemma B.4. *Suppose that $a:\mathbb{C}^n\to\mathcal{S}$ is smooth from $B_{\mathbb{R}^{2n}}(0,\delta_r)$ to Σ_r for arbitrary $r\in\mathbb{R}$ and satisfies $a(e^{\mathrm{i}\theta}z_1,\cdots,e^{\mathrm{i}\theta}z_n)=a(z_1,\cdots,z_n)$, $a(0,\cdots,0)=0$. Then, for any $M>0$, there exist smooth a_j and $b_{j\mathbf{m}}$ s.t. for $\alpha_j(|z_j|^2)=a(0,\cdots,0,z_j,0,\cdots,0)$ we have*

$$a(z_1,\cdots,z_n)=\sum_{j=1}^n\alpha_j(|z_j|^2)+\sum_{1\leq|\mathbf{m}|\leq M-1}\mathbf{Z}^{\mathbf{m}}G_{j\mathbf{m}}(|z_j|^2)+\mathcal{S}^{0,M}(z,\mathbf{Z}). \tag{B.6}$$

Proof. The proof is same as the proof of Lemmas B.1–B.3 □

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