Accepted Manuscript

International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems

Article Title:	Partial representations of orderings
Author(s):	Asier Estevan, Gianni Bosi, Magali Zuanon
DOI:	10.1142/S0218488518500228
Received:	23 November 2016
Accepted:	08 August 2017
To be cited as:	Asier Estevan, Gianni Bosi, Magali Zuanon, Partial representations of orderings, *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, doi: 10.1142/S0218488518500228
Link to final version:	https://doi.org/10.1142/S0218488518500228

International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems
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PARTIAL REPRESENTATIONS OF ORDERINGS

G. BOSI

Dipartimento de Scienze Economiche, Aziendali, Mathematiche e Statistiche
Università degli Studi di Trieste
Piazzale Europa 1, I-34127
Trieste, Italia.

A. ESTEVAN*

Departamento de Matemáticas, Universidad Pública de Navarra, Campus Arrosadía
Pamplona, 31006, Spain
asier.mugertza@unavarra.es

M. ZUANON

Dipartimento di Economia e Management, Università degli Studi di Brescia, Contrada Santa Chiara 50, 25122 Brescia, Italy.

Received (received date)
Revised (revised date)

In the present paper a new concept of representability is introduced, which can be applied to not total and also to intransitive relations (semiorders in particular). This idea tries to represent the orderings in the simplest manner, avoiding any unnecessary information. For this purpose, the new concept of representability is developed by means of partial functions, so that other common definitions of representability (i.e. (Richter-Peleg) multi-utility, Scott-Suppes representability,...) are now particular cases in which the partial functions are actually functions. The paper also presents a collection of examples and propositions showing the advantages of this kind of representations, particularly in the case of partial orders and semiorders, as well as some results showing the connections between distinct kinds of representations.

Keywords: partial representability; multi-utility; preorders; semiorders; intransitivity.

1. Introduction and motivation

Different kinds of representations of preferences have been recently proposed in the literature in order to consider general situations when completeness is not required. It is well known that in this case more than one function must be used. In some sense, the best way of representing transitive preferences which are not necessarily total is to invoke the multi-utility approach, since it provides a characterization of them.

*Corresponding author.

In this paper we generalize this classical approach in order to allow nontransitivity of the preferences. This is done by simply allowing each function to be defined not on the whole space of the alternatives but on some subset of it.

Given a preorder $\precsim$ on X, a real function $u\colon X \to \mathbb{R}$ is said to be *isotonic* or *increasing* if for every $x, y \in X$ the implication $x \precsim y \Rightarrow u(x) \leq u(y)$ holds true. In addition, if it also holds true that $x \prec y$ implies $u(x) < u(y)$, then u is said to be a *Richter-Peler utility representation*.

In the case of a total preorder $\precsim$ on X, it is said to be *representable* if there is a real-valued function $u\colon X \to \mathbb{R}$ that is *strictly isotonic* or *strictly increasing* (also known as *order-preserving*), so that, for every $x, y \in X$, it holds that $x \precsim y \iff u(x) \leq u(y)$. The map u is said to be an *order-monomorphism* (also known as a *utility function* for $\precsim$).

A (not necessarily total) preorder $\precsim$ on a set X is said to have a *multi-utility representation* if there exists a family $\mathcal{U}$ of isotonic real functions such that for all points $x, y \in X$ the equivalence

$$x \precsim y \Leftrightarrow \forall u \in \mathcal{U}\ (u(x) \leq u(y)) \tag{1}$$

holds. This kind of representation, whose main feature is to fully characterize the preorder, was first introduced by Vladimir L. Levin [22] (see also [23]), who called *functionally closed* a preorder admitting a multi-utility representation. However, the first systematic study of multi-utility representations is due to Ozgur Evren and Efe A. Ok [16], who presented different conditions for the existence of continuous multi-utility representations.

While a multi-utility representation exists for every not necessarily total preorder $\precsim$ on X (see Evren and Ok [16]), its application is in some sense limited, since if we start from a binary relation $\precsim$ on set X and it admits the representation above, then it must be necessarily a preorder (i.e., a reflexive and transitive binary relation). Nevertheless, it is interesting to search for a *continuous multi-utility representation* of a preorder $\precsim$ when the set X is endowed with a topology τ (cf., for instance, Evren and Ok [16], Bosi and Herden [5] and Alcantud et al. [1]). The existence of a finite multi-utility representation was studied by Ok [29] and Kaminski [20], who refers to *representation by means of multi-objective functions*.

We recall that a particular case of the previous representation is the so called *Richter-Peleg multi-utility representation* (see Minguzzi [26]), which holds when all the functions of the family $\mathcal{U}$ in representation (1) are *order-preserving* for the preorder $\precsim$ (i.e., for all $u \in \mathcal{U}$, and $x, y \in X$, $x \prec y$ implies that $u(x) < u(y)$). It is well known that in this case the family $\mathcal{U}$ also represents the *strict part* $\prec$ of $\precsim$ (see Alcantud et al. [1]), in the sense that, for all $x, y \in X$, $x \prec y$ if and only if $u(x) < u(y)$ for all $u \in \mathcal{U}$.

Therefore if we want to represent a binary relation $\precsim$ which is reflexive and not necessarily transitive (like *interval orders* or *semiorders*, for example), we cannot use the multi-utility approach. In order to remove this restriction, Nishimura and Ok [28] introduced very recently the following representation of a necessarily reflexive binary relation $\precsim$, which allows intransitivity: for a set $\mathbb{U}$ of sets $\mathcal{U}$ of real-valued functions u on X, and all points $x, y \in X$,

$$x \precsim y \Leftrightarrow \sup_{\mathcal{U} \in \mathbb{U}} \inf_{u \in \mathcal{U}} (u(y) - u(x)) \geq 0. \tag{2}$$

While this *maxmin multi-utility representation* fully characterizes the binary relation $\precsim$ in the general case when it is neither total nor transitive, we can consider that this latter representation is much demanding and difficult to perform, since it requires, in some sense, a lot of *information*, represented by a set of sets of real-valued functions.

In this paper, we introduce the concept of *partial multi-utility representation* of a preorder as a coherent and practical representation that characterizes the order structure. This means that we refer to the multi-utility approach (1) in the much more general case when the functions u are not required to be defined on the whole set X, but they are only *partial* (in the sense that, generally speaking, they are defined on subsets of X).

Needless to say, this generalization leads to a new representation which is compatible both with incompleteness and intransitivity. Indeed, the characteristic feature of the present work is to allow intransitivity in a multi-utility fashion.

Referring to the multi-utility representation (1), usually it is not easy or even possible (when continuity or at least upper semicontinuity of the functions is required) to construct a representation of a binary relation through functions that assign a value for each element of the set. Actually, since these relations fail to be total in general, it has not too much sense to impose a value to each element by each function. So, it seems consistent to provide the representation a degree of uncertainty or 'undefinition': if we cannot compare a pair (x, y), maybe we can avoid mapping x and y with each function of the representation (i.e., with each function $u \in \mathcal{U}$).

This uncertainty or 'undefinition' allows us to construct representations more easily (even when the order structure is not representable in the usual manner), and on the other hand it facilitates the continuity of the representation.

Although multi-utility representations deserve their interest in economics for the aforementioned reasons, they also appear in computer science, especially in *distributed systems* (see e.g Lamport [21], Estevan [11], Fidge [17], Raynal and Singhal [32] and Mattern [25]) and even in physics (see e.g. Panangaden [30]). In computer science the terminology (*labelings, random structures, clocks, ...*) is quite different with respect to the field of economics, but the ideas are essentially the same. Moreover, the idea of partial functions is not strange at all in computation so, this new theory may be quite useful for this field.

The structure of the paper goes as follows: After the introduction and the motivation, a section of notation and preliminaries is included. In Section 3 the new concept of *partial representability* is introduced, as well as some examples and propositions showing its advantages for the case of preorders. In this section, we also show some connections between distinct kinds of representations. Finally, in Section 4 we focus our attention on the usefulness of the partial representability for intransitive relations, and in particular, we deepen the study of semiorders presenting a partial version of the Scott-Suppes representation. A Section 5 of further comments closes the paper. There, incidentally, it is shown that a Theorem of Evren and Ok [16] is incorrect.

2. Notation and preliminaries

From now on X will denote a nonempty set.

Definition 1. A binary relation $\mathcal{R}$ on X is a subset of the Cartesian product $X \times X$. Given two elements $x, y \in X$, we will use the standard notation $x\mathcal{R}y$ to express that the pair (x, y) belongs to $\mathcal{R}$.

Associated to a binary relation $\mathcal{R}$ on a set X, we consider its *negation* (respectively, its *dual*) as the binary relation $\mathcal{R}^c$ (respectively, $\mathcal{R}^t$) on X given by $(x, y) \in \mathcal{R}^c \iff (x, y) \notin \mathcal{R}$ for every $x, y \in X$ (respectively, given by $(x, y) \in \mathcal{R}^t \iff (y, x) \in \mathcal{R}$, for every $x, y \in X$). We also define the *codual* $\mathcal{R}^a$ of the given relation $\mathcal{R}$, as $\mathcal{R}^a = (\mathcal{R}^t)^c$.

A binary relation $\mathcal{R}$ defined on a set X is said to be:

(i) *reflexive* if $x\mathcal{R}x$ holds for every $x \in X$,
(ii) *irreflexive* if $x\mathcal{R}^c x$ holds for every $x \in X$,
(iii) *symmetric* if $\mathcal{R}$ and $\mathcal{R}^t$ coincide,
(iv) *antisymmetric* if $\mathcal{R} \cap \mathcal{R}^t \subseteq \Delta = \{(x, x) : x \in X\}$,
(v) *asymmetric* if $\mathcal{R} \cap \mathcal{R}^t = \varnothing$,
(vi) *total* if $\mathcal{R} \cup \mathcal{R}^t = X \times X$,
(vii) *transitive* if $x\mathcal{R}y$ and $y\mathcal{R}z \Rightarrow x\mathcal{R}z$ for every $x, y, z \in X$.

In the particular case of a nonempty set where some kind of *ordering* (e.g., preorder, interval order, biorder, etc.) has been defined, the standard notation is different. We include it here for sake of completeness, and we will use it throughout the present manuscript.

In what follows $\precsim$ denotes a reflexive binary relation on X. Given a reflexive binary relation $\precsim$, then as usual we denote the associated *asymmetric* relation by $\prec$ and the associated *indifference* relation by $\sim$ and these are defined, respectively, by $[x \prec y \iff (x \precsim y)$ and $(y \precsim^c x)]$ and $[x \sim y \iff (x \precsim y)$ and $(y \precsim x)]$. If two elements are not comparable, that is, if it holds true that $x \precsim^c y$ as well as $y \precsim^c x$ for some $x, y \in X$, then we shall denote that by $x \bowtie y$.

Definition 2. A *preorder* $\precsim$ on X is a binary relation on X which is reflexive and transitive. An antisymmetric preorder is said to be an *order*. A *total preorder* $\precsim$ on a set X is a preorder such that if $x, y \in X$ then $(x \precsim y)$ or $(y \precsim x)$. A total order is also called a *linear order*, and a totally ordered set $(X, \precsim)$ is also said to be a *chain*. Usually, an order that fails to be total is also said to be a *partial order*. If $\precsim$ is a preorder on X, then the associated indifference relation $\sim$ is actually an equivalence relation. The asymmetric part of a linear order (respectively, of a total preorder) is said to be a *strict linear order* (respectively, a strict total preorder). Usually, in the case of partial orders (in particular dealing on finite sets), the relation $\precsim$ is also denoted by $\sqsubseteq$, and the corresponding strict part by $\sqsubset$.

In case of not total relations defined on a set X, if one element is not related or comparable to any other of the set, this element is said to be an *isolated point* [6]. Given a preorder $\precsim$ on X, a set $Y \subseteq X$ is called an *antichain* if $\prec_{|Y} = \emptyset$. The *width* (denoted by $w(X, \precsim)$) of a preordered set is the cardinality of the largest antichain Y contained in X. A partial order $\precsim$ on X is *near-complete* if and only if $w(X, \precsim) < \infty$.

For every $x \in X$ we define the following subsets of X:

$$l(x) = \{y \in X \mid y \prec x\}, \;\; r(x) = \{z \in X \mid x \prec z\},$$

$$d(x) = \{y \in X \mid y \precsim x\}, \;\; i(x) = \{z \in X \mid x \precsim z\}.$$

Definition 3. A preorder $\precsim$ on a topological space (X, τ) is *regular* if and only if for each $x \in X$ sets $i(x)$ and $d(x)$ are closed.

Next Definition 4 introduces the notion of representability[1]. The idea behind representability corresponds to the possibility of converting qualitative scales (preferences) into quantitative ones, comparing real numbers instead of, just, elements of a nonempty set.

Definition 4. A total preorder $\precsim$ on X is called *representable* if there is a real-valued function $u \colon X \to \mathbb{R}$ that is order-preserving, so that, for every $x, y \in X$, it holds that $[x \precsim y \iff u(x) \leq u(y)]$. The map u is said to be an *order-monomorphism* (also known as a *utility function* for $\precsim$).

Now we introduce the definitions of some intransitive relations, namely interval orders and semiorders [2,6,18,24,35].

Definition 5. An *interval order* $\precsim$ on a set X is a reflexive binary relation on X which in addition satisfies the following condition for all $x, y, z, w \in X$:

$$(x \precsim z) \textit{ and } (y \precsim w) \Rightarrow (x \precsim w) \textit{ or } (y \precsim z).$$

A *semiorder* $\precsim$ on a set X is a binary relation on X which is an interval order and in addition verifies the following condition for all $x, y, z, w \in X$:

$$(x \precsim y) \textit{ and } (y \precsim z) \Rightarrow (x \precsim w) \textit{ or } (w \precsim z).$$

Definition 6. An asymmetric relation $\prec$ defined on a set X is called *regular with respect to sequences* if there are no $x, y \in X$, and sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \subseteq X$,

[1]The notion of representability of some orderings is not unique (see [2,27,13]).

such that $x \prec \ldots \prec x_{n+1} \prec x_n \prec \ldots \prec x_1$ happens, or, dually, $y_1 \prec \ldots \prec y_n \prec y_{n+1} \prec \ldots \prec y$ holds.

Definition 7. A semiorder $\precsim$ defined on a nonempty set X, is said to be *regular* (with respect to sequences) if its corresponding strict preference $\prec$ is regular with respect to sequences in the sense of Definition 6.

Definition 8. A semiorder $\precsim$ defined on X is said to be *SS-representable* (that is *representable in the sense of Scott and Suppes*) if there exists a real-valued map $u\colon X \to \mathbb{R}$ (now again called a *utility function*) such that $x \precsim y \iff u(x) \leq u(y) + 1$ $(x, y \in X)$ (see Scott and Suppes [35]).

(In this case, the pair $(u, 1)$ is said to be a *Scott-Suppes representation* of the semiorder $\prec$).

Remark 1. It is well known that regularity is a necessary condition for the existence of a Scott-Suppes representation (see e.g. [7]).

3. Partial representability of preorders

In the present section we introduce the main definitions and some propositions illustrating the advantages of dealing with partial representability instead of the usual representability. Now we focus our attention on preorders, leaving a further study concerning intransitive relations to the next section. We also recover some results (Proposition 1 and Proposition 3) introduced in [1], but now adapting and proving them for this new theory of partial representability.

The knowledge on preorders, in particular in finite partially ordered sets, is a strong tool in computer sciences. That is the reason why we include some notions (e.g. labelings, random structure, ... [34]) related to this field. Furthermore, these concepts are strongly related to similar ideas on economics (e.g. utilities, Richter-Peleg multi-utility representations, ... [1,5,16,31,33]). Besides, partial functions are quite common in computing so, dealing with partial functions in order to represent orderings could be a good technique.

In the following lines we introduce some basic definitions.

Definition 9. A *partial function* from X to Y (written as $f\colon X \nrightarrow Y$) is a function $f\colon X' \to Y$, for some subset X' of X. It generalizes the concept of a function $f\colon X \to Y$ by not forcing f to map every element of X to an element of Y (only some subset X' of X). If $X' = X$, then f is called a total function and is equivalent to a function.

Definition 10. A partial function $f\colon (X, \tau_X) \nrightarrow (Y, \tau_Y)$ is *continuous* if the corresponding total function $f\colon (X', \tau_{X'}) \to (Y, \tau_Y)$ is continuous (here X' is the biggest subset of X where the partial function f is defined and $\tau_{X'}$ is the topology on X' inherited from (X, τ_X)).

Definition 11. Let $\precsim$ be a preorder on a nonempty set X. We will say that the preorder is *partial multi-utility representable* (or *multi-utility representable through partial functions*) if there exists a family of real partial functions $\mathcal{U}$ on X such that for any pair $x \precsim y$ there exists $u \in \mathcal{U}$ such that $u(x) \leq u(y)$, as well as $v(x) \leq v(y)$ for any $v \in \mathcal{U}$ which is defined on both x and y.

Remark 2. With definition above, notice that:
(*i*) $x \prec y$ if and only if there exists $u \in \mathcal{U}$ such that $u(x) < u(y)$, as well as $v(x) \leq v(y)$ for any $v \in \mathcal{U}$ which is defined on both x and y.

(*ii*) $x \sim y$ if and only if there exists $u \in \mathcal{U}$ such that $u(x) = u(y)$, as well as $v(x) = v(y)$ for any $v \in \mathcal{U}$ which is defined on both x and y.

(*iii*) $x \bowtie y$ if and only if there exists $u, v \in \mathcal{U}$ such that $u(x) < u(y)$ and $v(x) > v(y)$, or there is no $v \in \mathcal{U}$ defined on both x and y.

Definition 12. Let $\precsim$ be a preorder on a nonempty set X. We will say that the preorder is *partial Richter-Peleg multi-utility representable* (or *Ritcher-Peleg multi-utility representable through partial functions*) if there exists an *isotonic* partial multi-utility representation $\mathcal{U}$ on X. This means that there exists a partial multi-utility representation $\mathcal{U}$ on X such that for any pair $x \prec y$ there exists $u \in \mathcal{U}$ such that $u(x) < u(y)$, as well as $v(x) < v(y)$ for any $v \in \mathcal{U}$ which is defined on both x and y.

Remark 3. With definition above, notice that the corresponding indifference and incomparability are characterized like in the previous case of partial multi-utility.

Remark 4.
(1) In the definitions above we say 'partial' because the representation is made by means of partial functions, but this kind of representations totally characterizes the order structure.
(2) If the partial functions of the definition above are functions (that is, all of them are defined on all the set X) then we recover the definition of (Richter-Peleg) multi-utility representation. That is, the partial representation generalizes the concept of representability.
(3) Notice that, given a partial (Richter-Peleg) multi-utility representation and an element x of the set, if there is no function defined on this point x, then this element is an isolated point.

Proposition 1. *Let $\precsim$ be a preorder on X and A the set of all isolated points. Assume that there exists a (continuous) partial multi-utility representation $\{v_i\}_{i\in I}$. If there exists a (continuous) Richter-Peleg utility of the preorder on a subset Y such that $X \setminus A \subseteq Y$, then there exists a (continuous) partial Richter-Peleg multi-utility representation.*

Proof. Let $\mathcal{V} = \{v_i\}_{i\in I}$ be a (continuous) partial multi-utility representation of $\precsim$, and let f be a (continuous) Richter-Peleg representation of $\precsim_{|Y}$. Then it is easily checked that $\mathcal{U} = \{v + \alpha f : v \in \mathcal{V}, \alpha \in \mathbb{Q}, \alpha > 0\}$[2] is a (continuous) partial Richter-Peleg multi-utility representation of $\precsim$. The continuity of the functions of $\mathcal{U}$ arises from the continuity of the functions of $\mathcal{V}$ and from the continuity of f (see [36]).

This argument serves for the corresponding equivalence under upper/lower semicontinuity too. □

These new partial utilities migth be useful for dealing with random structures (see Schellekens [34]). These structures are studied in computation, with several questions unsolved yet. In the following lines we include some basic definitions related to this topic.

Definition 13. Let $(X, \sqsubseteq)$ be a finite partially ordered set with $|X| = n$. We define a *labeling* of the partial order as a function $u\colon (X, \sqsubseteq) \to \{1, ..., n\}$ such that for any $x \sqsubset y$ it holds that $u(x) < u(y)$, $x, y \in X$.

Definition 14. Let $(X, \sqsubseteq)$ be a finite partially ordered set. The collection $\mathcal{U} = \{u_i\}_{i \in I}$ of all possible labelings is called the *random structure* of the partial order, and it is also denoted by $\mathcal{R}_{\mathcal{L}}(X, \sqsubseteq)$[3].

Remark 5. (1) Notice that a labeling is simply a linear order extending the strict preference and the random structure is simply the set of all such extensions.
(2) There is a unique correspondence between partial orders and random structures: each of ones defines the other (see [34]).
(3) Notice that the concept of random structure implies a Richter-Peleg multi-utility representation.

Example 1. Let $(X, \sqsubseteq)$ be the partially ordered set defined by $\{x_1 \sqsubset x_2 \sqsubset x_4, x_3 \sqsubset x_4\}$. The corresponding random structure is shown in Figure 1, whereas a partial multi-utility Richter-Peleg representation is shown in Figure 2.

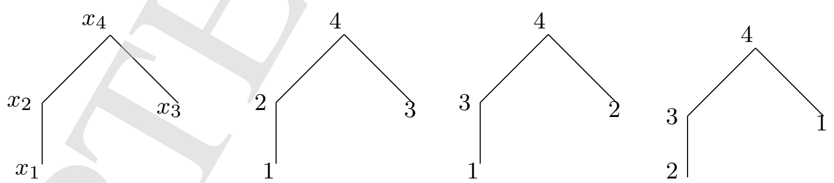

Fig. 1. Partially ordered set with its corresponding labelings.

[2]Here, if a function v or f is not defined on an element x, then we define the sum between v and f on x by $(v + f)(x) = \emptyset$, that is, as not defined. Otherwise, the sum is defined as usual: $(v + f)(x) = v(x) + f(x)$.
[3]Here, $\mathcal{L}$ denotes the set of labels where labeling functions take values. In this paper, $\mathcal{L}$ will be the set $\{1, ..., n\}$, where $n = |X|$. Therefore, we will omit this subscript.

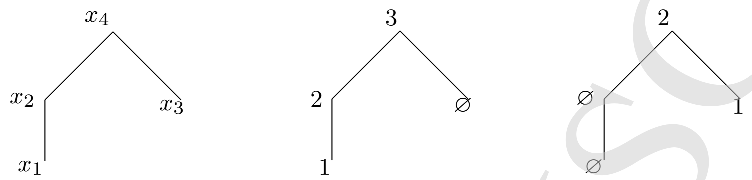

Fig. 2. Partially ordered set with a partial Richter-Peleg multi-utility.

Definition 15. Let $\sqsubseteq$ be a finite partial order on X. The *Scott topology* τ_S is defined by means of the basis $\{U_{\sqsubseteq}(x)\}_{x\in X} = \{\{y \in X \mid x \sqsubseteq y\}\}_{x\in X}$.

Remark 6. The Scott topology generated by a partial order is T_0 [8].

The following proposition is well known [8].

Proposition 2. *Let* $(X, \sqsubseteq)$ *and* $(Y, \sqsubseteq')$ *be two finite partially ordered sets endowed with the corresponding Scott topologies. Then, a function* $f : X \to Y$ *is continuous if and only if it is an order-preserving function, that is, if and only if* $x \sqsubseteq y$ *implies that* $f(x) \sqsubseteq' f(y)$*, for any* $x, y \in X$*.*

Corollary 1. *Let* $(X, \sqsubseteq)$ *be a finite partially ordered set* ($|X| = n$)*. Then, a function* $u: X \to \{1, ..., n\}$ *is a labeling if and only if it is continuous with respect to the corresponding Scott topologies.*

Remark 7. Let $\precsim$ be a preorder on X. Notice that if $\precsim$ is partially (Richter-Peleg) multi-utility representable by a family of partial functions $\mathcal{U}$, then the number of partial functions needed is less or equal than the number of functions needed for an hypothetical (Richter-Peleg) multi-utility representation $\mathcal{U}'$: $|\mathcal{U}| \leq |\mathcal{U}'|$.

Since a (Richter-Peleg) multi-utility representation is also a partial (Richter-Peleg) multi-utility representation, the inequality $|\mathcal{U}| \leq |\mathcal{U}'|$ is trivial. Moreover, Example 2 shows that in some cases the number of partial functions needed is strictly less than the number of functions needed for an hypothetical (Richter-Peleg) multi-utility representation, so $|\mathcal{U}| < |\mathcal{U}'|$.

Example 2.

Let $(X, \sqsubseteq)$ be the partially ordered set defined by $\{x_1 \sqsubset x_2\}$. The corresponding representations are shown in Figure 3.

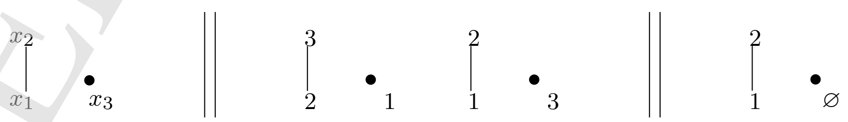

Fig. 3. The random structure and a partial Richter-Peleg multi-utility representation of a poset.

Remark 8.
(1) In the example and proposition before, it is shown that the partial multi-utility representations could be used in order to reduce the cardinal of the family of functions. This reduction may be interesting in other applied fields as computation, for example, dealing with distributed systems in order to reduce the number of clocks [25].

(2) The partial (Richter-Peleg) multi-utility representations can be studied too through permutations as it was done in [15]. This study searches properties of the order structure by means of the group properties of the corresponding set of permutations (which is directly defined by the corresponding partial (Richter-Peleg) multi-utility representation of the order structure).

(3) Moreover, partial multi-utility not only reduces the amount of functions needed, but also it allows us to represent some structures that cannot be represented by the usual multi-utility. That is, the set of orderings (and also for the particular case of partial orders) that can be represented through partial (Richter-Peleg) multi-utility is strictly bigger than this that can be represented through (Richter-Peleg) multi-utility. Since a (Richter-Peleg) multi-utility representation is also a partial (Richter-Peleg) multi-utility representation, the inequality is trivial. Moreover, Example 3 shows that there are partial Richter-Peleg multi-utility representable orderings that cannot be represented just through Richter-Peleg multi-utility.

Example 3. Let $\precsim$ be the following preorder defined on $X = \mathbb{Q} \times \{0, 1\}$:

$$(q,i) \prec (p,j) \iff \begin{cases} q < p & ; p, q \in X, \forall i, j. \\ q = p & ; i = 0, j = 1. \end{cases}$$

So, $(p,i) \sim (q,j)$ if and only if $p = q$ and $i = j$.

Then, since there is an infinite number of jumps, $((q,0),(q,1))$ for each $q \in \mathbb{Q}$, it is well known (see Bridges and Mehta [6]) that there does not exist a Richter-Peleg utility and, therefore, the preorder $\precsim$ (that actually is a total order) fails to be Richter-Peleg multi-utility representable.

However, we are able to construct a partial Richter-Peleg multi-utility representation by means of –at least– a countable number of partial functions:

Let ϕ be a bijection from $\mathbb{Q}$ to $\mathbb{N}$. Now, for each $n = \phi(q) \in \mathbb{N}$ we define the following two partial functions on X:

$$u_n((p,i)) = \begin{cases} p-1 & ; p \leq q,\ i = 0. \\ p & ; q < p,\ i = 0. \\ q & ; p = q,\ i = 1. \end{cases} \quad v_n((p,i)) = \begin{cases} p & ; p < q,\ i = 1. \\ p+1 & ; q \leq p,\ i = 1. \\ q & ; p = q,\ i = 0. \end{cases}$$

Moreover, it can be proved that, if X is endowed with the order topology $\tau_{\prec}$, then the partial representation is continuous.

In the Example 4 it is shown that, even not continuous partial orders may be continuously represented through a finite partial Richter-Peleg multi-utility representation. This is impossible with multi-utility, as it was shown in Kaminski [20]. Furthermore, the partial order of Example 4 is connected, and since it is not total, it cannot be continuously multi-utility represented (see Proposition 5.2 of [1]). We summarize this idea in the following remark:

Remark 9. There are partial orders that fail to be continuously (Richter-Peleg) multi-utility representable, but that they can be continuously represented by means of a partial (Richter-Peleg) multi-utility representation. Since a (Richter-Peleg) multi-utility representation is also a partial (Richter-Peleg) multi-utility representation, then it is clear that any ordering that is continuous (Richter-Peleg) multi-utility representable it is also continuously representable by means of a partial (Richter-Peleg) multi-utility. The converse is not true, as it is shown in Example 4 (as well as in Example 5).

Example 4. Let $(X, \sqsubseteq)$ be the partially ordered set of Example 1 defined by $\{x_1 \sqsubset x_2 \sqsubset x_4,\, x_3 \sqsubset x_4\}$. The corresponding random structure is shown in Figure 2. Now we endow the codomain $\{1, 2, 3, 4\}$ with the Scott topology but, instead of endowing the set X with the corresponding Scott topology, assume that it is endowed with the topology $\tau_1 = \{\emptyset, \{x_4\}, \{x_2, x_3, x_4\}, \{x_1, x_2, x_4\}, \{x_2, x_4\}, X\}$ or with $\tau_2 = \{\emptyset, \{x_4\}, \{x_2, x_4\}, \{x_3, x_4\}, \{x_2, x_3, x_4\}, X\}$, which are coarser than the corresponding Scott topology. Notice that, as it is shown in Figure 5, τ_1 and τ_2 are related to two partial orders that refine the partial order $\sqsubseteq$ of the example.

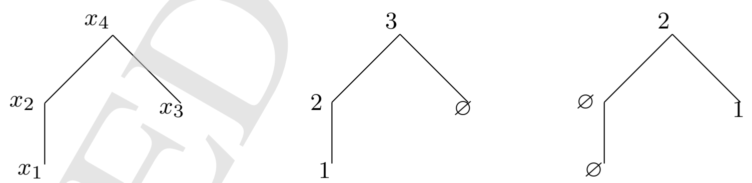

Fig. 4. A continuous partial Richter-Peleg multi-utility representation of a poset.

Since the topology on X is coarser than the corresponding Scott topology, by Corollary 1 the partially ordered set cannot be continuously Richter-Peleg multi-utility representable. However, it is continuous partial Richter-Peleg multi-utility representable (with respect to topology τ_1 and also with respect to topology τ_2) through the functions shown in Figure 4.

Remark 10. A study on the relations between partially ordered finite sets, T_0 finite topologies and permutations of the symmetric group is done in the paper [15] of Este-

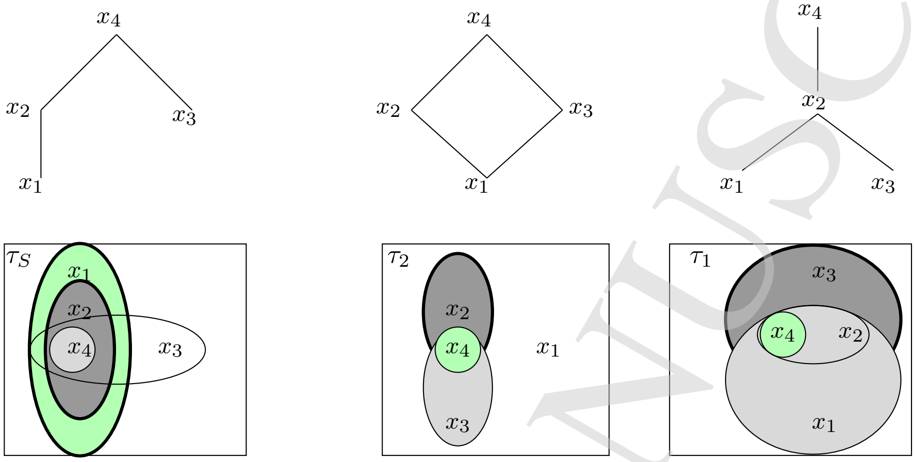

Fig. 5. Three partial orders and the corresponding Scott topologies.

van et al. entitled *Approximating SP-orders through total preorders: incomparability and transitivity through permutations.*

The following lemma of Schmeidler (1971) is well known in literature.

Lemma 1. *Let $\precsim$ be a nontrivial preorder on a connected topological space (X, τ). If for every $x \in X$ the sets $d(x)$ and $i(x)$ are closed and the sets $l(x)$ and $r(x)$ are open, then the preorder $\precsim$ is total.*

Proposition 3. *Let $\precsim$ be a preorder on a connected topological space (X, τ) without isolated points. If there exists a continuous partial Richter-Peleg multi-utility representation $\mathcal{U} = \{u_1, ..., u_n\}$, then $\precsim$ is total on X.*

Proof. It can be proved that if a preorder $\precsim$ on a topological space (X, τ) has a continuous partial multi-utility representation then both $d(x)$ and $i(x)$ are closed subsets of X for all $x \in X$. This proof is similar to the proof of Theorem 3.1 in Kaminski [20] (see also Proposition 5 in Bosi and Herden [5]) and it is included in the appendix with a lemma. Therefore, by using Lemma 1, it suffices to show that under our assumptions, both $l(x)$ and $r(x)$ are open subsets of X for all $x \in X$. To prove this fact we observe that, from Definition 12,
$l(x) = \{y \in X \mid y \prec x\} = \{y \in X \mid v_i(y) < v_i(x),$ for all $i \in \{1, ..., n\}$ s.t. v_i is defined on both$\} = \bigcap_{i=1}^n v_i^{-1}((-\infty, v_i(x))),$
$r(x) = \{y \in X \mid x \prec y\} = \{y \in X \mid v_i(x) < v_i(y),$ for all $i \in \{1, ..., n\}$ s.t. v_i is defined on both$\} = \bigcap_{i=1}^n v_i^{-1}((v_i(x), +\infty))$
for each $x \in X$. From these equalities and continuity of the functions v_i, the conclusion follows immediately. □

Remark 11. In the proposition above, if we allow the existence of isolated points, then the statement is false. We see that through the following example:

Let $\precsim$ be a preorder defined on $X = [0,1] \cup \{2\}$ by $x \precsim y$ if and only if $x \leq y$ for any $x, y \in [0,1]$, and $2 \bowtie x$ for any $x \in [0,1]$. Now, we endow the set with a topology τ such that it coincides with the usual topology on $[0,1]$ and such that the open neighbourhoods of 2 are the same of $0'5$ (that is, $\mathcal{O}_2 = \{(0'5-\epsilon, 0'5+\epsilon) \cup \{2\}\}_{\epsilon>0}$). Therefore, (X,τ) is a connected topological space. Under these assumptions, the function v defined by $v(x) = x$ (for any $x \in [0,1]$) and $v(2) = \emptyset$ is a continuous partial Richter-Peleg multi-utility representation. However, the preorder is not total.

By the way, notice that this preorder fails to be continuously multi-utility representable. To see that, observe that the constant sequence $\{0'5\}$ converges to 2 so, any function v of a continuous multi-utility must satisfy that $v(2) = v(0'5)$. Therefore, we arrive to the contradiction $0'5 \sim 2$. We can argue similarly for semicontinuity.

The following example shows another case in which it is not possible to achieve a continuous Richter-Peleg multi-utility, but which can be easily represented through a continuous partial Richter-Peleg multi-utility. In fact, it is proved that the preorder of this example does not admit a continuous multi-utility representation.

Example 5. Let X be the Cartesian product $\mathbb{R} \times \{0,1\}$ (or interpret that as the union of two real lines: $\mathbb{R}_0$ and $\mathbb{R}_1$) endowed with the following preorder:

$$(x,i) \prec (y,j) \iff \begin{cases} x < y, & \text{and } i = j; \\ \\ x \leq 0 \text{ and } 1 < y, & \text{with } i \neq j; \end{cases}$$

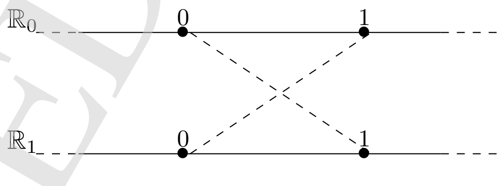

Fig. 6. Preorder defined on $\mathbb{R} \times \{0,1\}$.

It is trivial that the width of the preorder is two. We endow the set X with the order topology τ arised from the preorder $\precsim$, except for the points $(0,0)$ and $(0,1)$, whose open neighbourhood are as follows:

$$\mathcal{O}_{(0,0)} = \{(x,0); x \in (-\epsilon, +\epsilon)\} \cup \{(y,1); y \in (1, 1+\epsilon)\}$$
$$\mathcal{O}_{(0,1)} = \{(x,1); x \in (-\epsilon, +\epsilon)\} \cup \{(y,0); y \in (1, 1+\epsilon)\}$$

It is easy to check that this topology has a countable basis. Moreover, the contour set $i((0,0))$ is not closed (as well as $i((0,1))$), so the preorder is not regular and therefore, by Theorem 3.1. of [20], the preorder does not admit a continuous and finite multi-utility representation. Furthermore, $l((1'5,0))$ is not open either (as well as $l((1'5,1))$).

However, the following family of functions is a continuous partial Richter-Peleg multi-utility representation of the preorder:

$$u_1((x,i)) = \begin{cases} x \ ; i=0 \\ \varnothing \ ; \text{otherwise.} \end{cases} \quad u_2((x,i)) = \begin{cases} x \ ; i=1 \\ \varnothing \ ; \text{otherwise.} \end{cases}$$

$$u_3((x,i)) = \begin{cases} x \ ; x \leq 0 \text{ and } i=0 \\ x \ ; 1 < x \text{ and } i=1, \\ \varnothing \ ; \ \text{otherwise.} \end{cases} \quad u_4((x,i)) = \begin{cases} x \ ; x \leq 0 \text{ and } i=1, \\ x \ ; 1 < x \text{ and } i=0, \\ \varnothing \ ; \ \text{otherwise.} \end{cases}$$

Furthermore, it is possible to prove that there is no continuous Richter-Peleg multi-utility representation of this preorder, even through an infinite number of functions. To see that we can argue by contradiction.

If there was a continuous Richter-Peleg multi-utility representation $\{u_i\}_{i\in I}$, since $(0,0) \prec (1+\frac{1}{n},1)$ for any $n \in \mathbb{N}$, then each function would satisfy that $u_i((0,0)) < u_i((1+\frac{1}{n},1))$ for any $n \in \mathbb{N}$. Notice that the sequence $\{(1+\frac{1}{n},1)\}_{n\in\mathbb{N}}$ converges to $(0,0)$ (as well as to $(1,1)$) so, applying the limits we have that $\lim_{n\to+\infty}\{u_i((1+\frac{1}{n},1))\}_{n\in\mathbb{N}} = u_i(1,1) = u_i(0,0)$ for each function of the representation. Since $(0,1) \prec (0'5,1) \prec (1,1)$ and $\{u_i\}_{i\in I}$ is a Richter-Peleg multi-utility representation, we conclude that $u_i(0,1) < u_i(0,0)$ for any function of the representation, arriving to the desired contradiction, because $(0,0) \bowtie (0,1)$.

In fact, it is also impossible to obtain a continuous multi-utility, even through an infinite number of functions[4]. To see that we recover the sequence before and argue again by contradiction. If there was a continuous multi-utility representation $\{u_i\}_{i\in I}$, since $(0,0) \prec (1+\frac{1}{n},1)$ for any $n \in \mathbb{N}$, then each function would satisfy that $u_i((0,0)) \leq u_i((1+\frac{1}{n},1))$ for any $n \in \mathbb{N}$. Notice that the sequence $\{(1+\frac{1}{n},1)\}_{n\in\mathbb{N}}$ converges to $(0,0)$ (as well as to $(1,1)$) so, applying the limits we have that $\lim_{n\to+\infty}\{u_i((1+\frac{1}{n},1))\}_{n\in\mathbb{N}} = u_i(1,1) = u_i(0,0)$ for each function of the representation. Therefore, since $u_i(1,1) = u_i(0,0)$ for each function of the representation, we conclude that $(0,0) \precsim (0,1) \precsim (0,0)$, arriving to the desired contradiction, because $(0,0) \bowtie (0,1)$.

4. Partial representability of intransitive relations

In the present section we study the partial representability of intransitive relations. If a relation has a multi-utility representation, then it must be transitive (see the

[4]As we said, by Theorem 3.1. of [20], it is known that the preorder does not admit a continuous and finite multi-utility representation.

introduction). Therefore, multi-utility is not useful dealing with intransitive relations. But this is not the case of partial multi-utility, which allows us to characterize intransitive relations too, as it can be seen in Example 6 and Example 7.

In this section we also include a subsection related to the particular case of semiorders and Scott-Suppes representations. It is well known that semiorders fail to be transitive, and they are usually represented through a Scott-Suppes representation [2,6,24,35]. In this subsection we introduce the new concept of a partial Scott-Suppes representation, which generalizes the classical one. Before this new definition, in the following lines we show some examples of semiorders which are represented through a partial Richter-Peleg multi-utility representation.

Example 6. Let $\precsim$ be a semiorder on $\mathbb{Z}$ defined by $n \precsim m$ if and only if $n \leq m+1$. Then, the family of functions $\{u, v, w\}$ (see Figure 7) is a partial Richter-Peleg multi-utility representation:

$$u(m) = \begin{cases} n \ ; m = 3n \\ n \ ; m = 3n+1 \\ \varnothing \ ; \text{else} \end{cases} \quad v(m) = \begin{cases} n \ ; m = 3n+1 \\ n \ ; m = 3n+2 \\ \varnothing \ ; \text{else} \end{cases} \quad w(m) = \begin{cases} n \ , m = 3n+2 \\ n \ ; m = 3n+3 \\ \varnothing \ ; \text{else,} \end{cases}$$

where m and n are numbers from $\mathbb{Z}$.

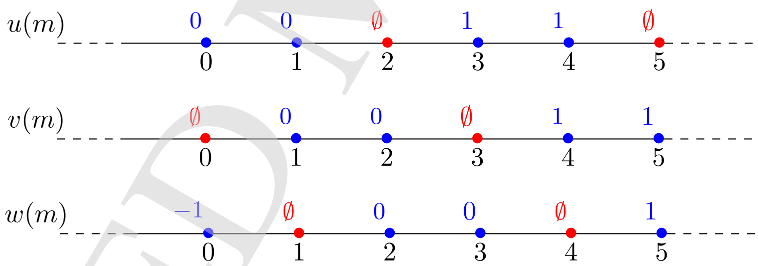

Fig. 7. A partial Richter-Peleg multi-utility representation of the semiorder.

Example 7. Let $\precsim$ be a semiorder defined on $\mathbb{R}$ as follows:

$$x \precsim y \iff x \leq y+1.$$

Then, the family of functions $\{u_r\}_{r\in[0,2)}$ is a partial Ritcher-Peleg multi-utility representation (see Figure 8):

$$u_r(x) = \begin{cases} 4n & ; x \in [4n+r, 4n+r+1) \\ \varnothing & ; x \in [4n+r+1, 4n+r+2) \\ 4n+2 & ; x \in [4n+r+2, 4n+r+3) \\ \varnothing & ; x \in [4n+r+3, 4n+r+4) \end{cases}$$

where $x \in \mathbb{R}$ and $n \in \mathbb{Z}$.

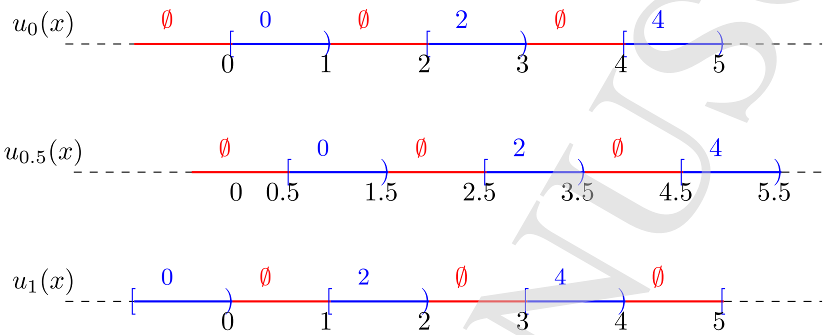

Fig. 8. Three partial functions of the partial Richter-Peleg multi-utility representation $\{u_r\}_{r\in[0,2)}$.

Remark 12. Notice that, since regularity of the semiorder is not necessary for this kind of partial multi utility representations, there are semiorders that cannot be represented by means of a Scott-Suppes representation but that they can be represented through a partial multi-utility Richter-Peleg representation.

4.1. *Partial Scott-Suppes representations*

As it has shown, partial (Richter-Peleg) multi-utility may be useful in order to represent intransitive relations and, in particular, semiorders. However, dealing with these last orderings, to keep the threshold could be a good technique if we want to simplify the representation (e.g. reduce the number of functions of the representation and facilitate their construction), but always trying to represent as much semiorders as possible. For this purpose (simplicity and usefulness), mixing partial representability and the threshold seems a plausible answer: partial SS-representability.

Since we keep the threshold, by means of this partial SS-representation we do not renounce at all to the usual Scott-Suppes representation, but now (as it is shown by means of examples) we can achieve some properties (mainly representability or (semi)continuity) in cases for which that was impossible in the usual manner.

In the following lines we introduce two new concepts of partial SS-representability: one in a 'multi-utility' manner (the weak one) and the other one in a 'Richter-Peleg' manner (the strict one).

Definition 16. Let $\precsim$ be a semiorder defined on a set X. Then, we say that the semiorder is *partially Scott-Suppes representable* (*partially SS-representable* for short) if there exists a family of partial functions $\mathcal{U}$ such that:

(*i*) $x \precsim y$ if and only if there exists $u \in \mathcal{U}$ such that $u(x) \leq u(y) + 1$ and, $v(x) \leq v(y) + 1$ for any $v \in \mathcal{U}$ defined in both x and y.

(*ii*) $x \prec y$ if and only if there exists $u \in \mathcal{U}$ such that $u(x) + 1 < u(y)$ and, $v(x) \leq v(y) + 1$ for any $v \in \mathcal{U}$ defined in both x and y.

Definition 17. Let $\precsim$ be a semiorder defined on a set X. Then, we say that the semiorder is *partially Richter-Peleg Scott-Suppes representable* (*partially RPSS-representable* for short) if there exists a family of partial functions $\mathcal{U}$ such that:

(*i*) $x \precsim y$ if and only if there exists $u \in \mathcal{U}$ such that $u(x) \leq u(y) + 1$ and, $v(x) \leq v(y) + 1$ for any $v \in \mathcal{U}$ defined in both x and y.

(*ii*) $x \prec y$ if and only if there exists $u \in \mathcal{U}$ such that $u(x) + 1 < u(y)$ and, $v(x) + 1 < v(y)$ for any $v \in \mathcal{U}$ defined in both x and y.

Since the functions of a partial RPSS-representation $\mathcal{U}$ are isotonic, that is, for any partial function $u \in \mathcal{U}$ defined in a pair $x, y \in X$ such that $x \prec y$ it must hold that $u(x) + 1 < u(y)$, it migth seem more successful to obtain a partial RPSS-representation instead of a partial SS-representation. However, notice that both kind of representations totally characterize the order structure and, since the concept of partial RPSS-representation is more restrictive, there are simple cases (as it is shown in Example 8) in which this representation is less 'comfortable' or useful than the partial SS-representation.

Example 8. Let $\precsim$ be a semiorder on $X = \{\frac{1}{n}\}_{n \in \mathbb{N}} \cup \{0\}$ defined as follows:

$$1 \prec \frac{1}{2} \prec \cdots \prec \frac{1}{n} \prec \frac{1}{n+1} \prec \cdots \prec 0, \quad n \in \mathbb{N}.$$

It is trivial that the semiorder is actually a total order. It is trivial too that this semiorder fails to be regular and so, it cannot be represented through a SS-representation.[5]

Nevertheless, it is easily partial SS-representable just by means of the following two partial functions:

$$u(x) = \begin{cases} 2n \ ; \ x = \frac{1}{n}, \ n \in \mathbb{N} \\ \varnothing \ \ ; \ x = 0 \end{cases} \qquad v(x) = \begin{cases} 0 \ ; \ x \in \{\frac{1}{n}\}_{n \in \mathbb{N}} \\ 2 \ ; \ x = 0 \end{cases}$$

The example before proves that, since regularity is not a necessary condition for the partial SS-representability, then there are partial SS-representable semiorders that fail to be SS-representable.

Proposition 4. *A SS-representable semiorder is also partial (Richter-Peleg) SS-representable. The converse is not true since there are partial (Richter-Peleg) SS-representable semiorders that fail to be SS-representable.*

[5]See [7,13,14] for more necessary conditions for representability and also for continuity.

However, as the following proposition shows, it is impossible to obtain a partial RPSS-representation through a finite number of partial functions of a non regular semiorder.

Proposition 5. *Let $\precsim$ be a non regular semiorder on X. If there exists a partial RPSS-representation $\mathcal{U}$ of the semiorder, then the cardinal of $\mathcal{U}$ is infinite.*

Proof. Since it is not regular, we may assume that there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ and an element $x \in X$ such that $x_1 \prec x_2 \prec \cdots \prec x_n \prec x_{n+1} \prec \cdots \prec x$, $n \in \mathbb{N}$.

By contradiction, we suppose that the cardinal of $\mathcal{U}$ is finite. Since $x_n \prec x$ for any $n \in \mathbb{N}$, for any $n \in \mathbb{N}$ there must exist a partial function $u_n \in \mathcal{U}$ such that $u_n(x_n)+1 < u_n(x)$. But, since the cardinal of $\mathcal{U}$ is finite, it implies that there must exists a partial function u defined on a infinite number of elements $\{x_{m_k}\}_{m_k\in M\subseteq\mathbb{N}}$ such that $x_{m_1} \prec x_{m_2} \prec \cdots \prec x_{m_s} \prec x_{m_{s+1}} \prec \cdots \prec x$, $s \in \mathbb{N}$, as well as on x. Therefore, we arrive to the desired contradiction, since the existence of this function implies that $u(x) > n$ for any $n \in \mathbb{N}$, what is not possible. □

Remark 13. Therefore, partial SS-representations allow us to represent not regular semiorder even through a finite number of functions, whereas they cannot be represented by means of the usual SS-representation. Not regular semiorders may be represented too by means of a partial RPSS-representation, but in this case an infinite number of partial functions is needed.

Moreover, the advantages of the partial (RP)SS-representations can be found dealing with continuity too:

Remark 14. Since any SS-representation of a semiorder is also a partial (Richter-Peleg) SS-representation, any continuous SS-representation of a semiorder is also continuous partial (Richter-Peleg) SS-representation. The converse is not true since there are continuous partial (Richter-Peleg) SS-representations of semiorders that fail to be continuously SS-representable (see Example 9).

Example 9. Let $\precsim$ be the usual semiorder on $X = \mathbb{R} \setminus (0, 0.5]$ defined as:

$$x \precsim y \iff x \leq y+1,\ x, y \in X.$$

If we endow the set X with the corresponding order topology $\tau_<$ of the euclidean order $\leq$ on X, it is well known [6] (see also [14] for more necessary conditions for continuity) that this semiorder is not continuously representable. However, we can construct a continuous partial RPSS-representation by means of the following three continuous partial functions:

$$u(x) = \begin{cases} x \ ; x \neq 0 \\ \varnothing \ ; x = 0 \end{cases} \quad v(x) = \begin{cases} x \ ; x \notin (0.5, 1] \\ \varnothing \ ; \ \text{else,} \end{cases} \quad w(x) = \begin{cases} 0.5 \ ; x = 0 \\ x \quad ; x \in (0.5, 1] \\ \varnothing \quad ; \ \text{else.} \end{cases}$$

5. Further comments

In this paper a new concept of representability has been introduced, always trying to match simplicity with usefulness as well as generalizing the classical ones. Through this partial representability the ordering is totally characterized and, besides, the amount of data needed for doing that is reduced. At the same time, we achieve to represent more orderings than by means of the usual concept of representability. In particular, this may be remarkable in the case of semiorders, where the usual Scott-Suppes representation seems to be too restrictive, since non regular simple semiorders cannot be represented.

Several recent papers have studied the problem of representability [1,2,13,7,16,20,26,29], as well as the corresponding continuity [4,5,14,12]. Although new results and some characterizations have been achieved, some questions are still open.

This new concept of partial representability increases the set of orderings that are now (continuously) representable (by means of partial functions) and so, it opens an analogue study in order to identify that set, as it was done in the papers cited before by using the classical approach.

Remark 15. We have shown in the example of Remark 11 that Theorem 3 in [16] is incorrect. It is easy to check that the preorder of Remark 11 satisfies the hypothesis of the theorem, however, it does not admit a semicontinuous multi-utility. The statement of this Theorem 3 is as follows:

Theorem 3 [16]: *Let X be a topological space with a countable basis. If $\precsim$ is a near-complete upper (lower) semicontinuous preorder on X; then it has an upper (lower) semicontinuous finite multi-utility representation.*

In particular, if we go into the details of the proof of Theorem 3 in [16] we notice that, after applying Dilworth's theorem in order to guarantee the existence of a partition $\{X_1, ..., X_n\}$ of X such that $\precsim \cap (X_i \times X_i)$ is a total preorder for each $i \in \{1, ..., n\}$, the authors define n total preorders $\precsim_i$ on X as follows: $x \precsim_i y$ if and only if $\{z \in X : z \precsim x\} \cap X_i \subseteq \{z \in X : z \precsim y\} \cap X_i$. If $\precsim$ is lower semicontinuous (i.e. $\{z \in X : z \precsim x\}$ is a closed subset of X for all $x \in X$), then $\precsim_i$ is a lower semicontinuous (not upper semicontinuous as the authors claim) total preorder on X_i, not on X in general, and therefore, when applying Rader's theorem to $\precsim_i$ on X_i, one can only guarantee the existence of a lower semicontinuous utility function u_i on X_i, so that $\{u_1, ..., u_n\}$ is no longer a lower semicontinuous representation of $\precsim$ on X, and the theorem cannot be proven.

Appendix

As we said, a similar proof –but for usual functions, not partial– of the following lemma was done in [20].

Lemma 2. *Let $\precsim$ be a preorder without isolated points defined on a topological space*

(X, τ). If there exists a continuous partial multi-utility representation $\{u_1, ..., u_n\}$, then both $d(x)$ and $i(x)$ are closed subsets of X for all $x \in X$.

Proof. First, notice that $d(x) = \{y \in X; y \precsim x\}$ coincides with the set $\{y \in X; u_k(y) \leq u_k(x), \forall u_k \text{ defined in both}\}$.

Consider an arbitrary net $(y_i)_{i \in I}$ convergent in X, such that $y_i \in d(x)$ for any $i \in I$. Let g be the limit of this net: $(y_i)_{i \in I} \longrightarrow g$. In order to show that $d(x)$ is closed it suffices to prove that $g \in d(x)$.

By definition there is no isolated points. Therefore, since the number of function is finite, there must be –at least– a function u_l defined in all the points of a subnet $(y_j)_{j \in J \subseteq I}$ (otherwise, there exists an index $i_0 \in I$ such that there is no function defined in y_i for any $i > i_0$, arriving to a contradiction). It is well known that the subnet converges to the same point of the net, that is, $(y_j)_{j \in J \subseteq I} \longrightarrow g$. Since functions u_k –in particular u_l– are continuous, $\{u_l(y_j)\}_{j \in J \subseteq I} \longrightarrow u_l(g)$. But $u_l(y_j) \leq u_l(x)$ for any $j \in J$ so, we deduce that $u_l(g) \leq u_l(x)$. Thus, $g \in d(x)$. Therefore, $d(x)$ is closed.

Analogous proof is done for $i(x)$. □

Acknowledgements

Asier Estevan acknowledges financial support from the Ministry of Economy and Competitiveness of Spain under grants MTM2012-37894-C02-02, MTM2015-63608-P and ECO2015-65031 as well as from the Basque Government under grant IT974-16. Gianni Bosi and Magalì E. Zuanon acknowledge financial support from the Istituto Nazionale di Alta Matematica "F. Severi" (Italy).

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