Finite group actions on 3-manifold and cyclic branched covers of knots

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Abstract

As a consequence of a general result about finite group actions on 3-manifolds, we show that a hyperbolic 3-manifold can be the cyclic branched cover of at most fifteen inequivalent knots in $\mathbf{S}^3$ (in fact, a main motivation of the present paper is to establish the existence of such a universal bound). A similar, though weaker, result holds for arbitrary irreducible 3-manifolds: an irreducible 3-manifold can be a cyclic branched cover of odd prime order of at most six knots in $\mathbf{S}^3$. We note that in most other cases such a universal bound does not exist.

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1 Introduction

A classical, much considered class of closed orientable 3-manifolds is constituted by the cyclic branched covers of knots in the 3-sphere. In the present paper we try to understand in how many different ways a given 3-manifold can occur as a cyclic branched cover of a knot. For a fixed branching order, the problem is well understood ([V], [MonW], [Z2], [Re] and [Gr]); in particular, a hyperbolic 3-manifold is a 2-fold branched cover of at most nine knots in S^3 and, for $n > 2$, an n-fold cyclic branched cover of at most two knots. For different branching orders, the situation turned out to be much more difficult, and even in the hyperbolic case the existence of a universal bound on the number of different branching orders was not known; the existence of such a universal bound for hyperbolic 3-manifolds is the first main result of the present paper.

Theorem 1. *A closed hyperbolic 3-manifold is a cyclic branched cover of at most fifteen inequivalent knots in S^3.*

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We call two knots equivalent if one is mapped to the other by an orientation-preserving diffeomorphism of S^3. In the present paper, all manifolds are closed, connected and orientable, and all maps are smooth and orientation-preserving.

The orientation-preserving isometry group of a closed hyperbolic 3-manifold M is finite, and every finite group occurs for some hyperbolic M. Suppose that M is a cyclic branched cover of a knot in S^3; then the group of covering transformations acting on M is generated by a hyperelliptic rotation: We call a periodic diffeomorphism of a closed 3-manifold a *hyperelliptic rotation* if all of its powers have connected, non-empty fixed point set (a simple closed curve), and its quotient (orbit) space is S^3. By the geometrization of 3-orbifolds, or of finite group actions on 3-manifolds ([BLP], [BoP], [DL]), the group of covering transformations is conjugate to a subgroup of the isometry group of M. Hence establishing a universal upper bound for hyperbolic 3-manifolds as in Theorem 1 is equivalent to bounding the number of conjugacy classes of cyclic subgroups generated by hyperelliptic rotations of the isometry group of a hyperbolic 3-manifold. Now Theorem 1 is a consequence of the following more general result on finite group actions on closed 3-manifolds.

Theorem 2. *Let M be a closed 3-manifold not homeomorphic to $\mathbf{S}^3$. Let G be a finite group of orientation-preserving diffeomorphisms of M. Then G contains at most fifteen conjugacy classes of cyclic subgroups generated by a hyperelliptic rotation (at most six for cyclic subgroups whose order is not a power of two).*

Note that the 3-sphere is the n-fold cyclic branched cover of the trivial knot for any integer $n \geq 2$ (and, by the solution to the Smith conjecture, only of the trivial knot). It is well-known that, for any branching order n, a 3-manifold can be the n-fold cyclic branched cover of an arbitrary number of non-prime knots (such a manifold is not irreducible), and that an irreducible 3-manifold can be the 2-fold branched cover of arbitrarily many prime knots (see [V]).

For irreducible 3-manifolds, the following holds:

Theorem 3. *Let M be a closed, irreducible 3-manifold. Then there are at most six inequivalent knots in $\mathbf{S}^3$ having M as a cyclic branched cover of odd prime order.*

The proof of Theorem 3 uses Theorem 2 in connection with the equivariant torus-decomposition of irreducible 3-manifolds into geometric pieces, see [BoP], [BLP], [CHK] and [KL]. For arbitrary 3-manifolds, as a direct consequence of Theorem 3 as well as the equivariant prime decomposition for 3-manifolds [MSY], the following remains true.

Corollary 1. *Let M be a closed 3-manifold not homeomorphic to $\mathbf{S}^3$. Then M is a p-fold cyclic branched cover of a knot in $\mathbf{S}^3$ for at most six distinct odd prime numbers p.*

We note that, in both Theorem 3 and Corollary 1, there is no universal bound for nonprime orders (see Proposition 5 in Section 6.1).

Another consequence of Theorem 2 is the following characterization of the 3-sphere which generalizes the main result of [BPZ] from the case of homology 3-spheres to arbitrary closed 3-manifolds.

Corollary 2. *A closed* 3*-manifold* M *is homeomorphic to* $\mathbf{S}^3$ *if and only if there is a finite group* G *of orientation-preserving diffeomorphisms of* M *such that* G *contains sixteen conjugacy classes of subgroups generated by a hyperelliptic rotation.*

An interesting aspect of the proof of Theorem 2 is the substantial use of finite group theory, in particular of the classification of the finite simple groups. Since, as already observed, every finite group acts on some closed 3-manifold (see also Section 7), the classification seems to be intrinsic to the proofs of our results. We have tried to separate the algebraic, purely group theoretical parts of the proof (Section 4) from the topological parts (Sections 3 and 5), so they can be read independently.

We note that, although the upper bounds in our results are quite small, we do not know if they are really optimal; in fact, a main point of our paper was to establish the existence of such universal upper bounds.

The paper is organized as follows. In Section 3 we consider hyperelliptic rotations and their properties. Section 4 contains the main group-theoretical part of the paper, Section 5 the proof of Theorem 2 and Section 6 the proof of Theorem 3 for the irreducible case. Finally, in an Appendix we prove that every finite group acts non-freely on some rational homology sphere (proved in [CL] for free actions).

2 Sketch of the proof of Theorem 2

The proof of Theorem 2 is based on a series of preliminary results, some of them rather technical, which are presented in Sections 3 and 4. Our choice to present the group-theoretical part of the proof in a dedicated section (Section 4) allows a topologist to avoid the reading of the most specialized details of the algebraic part, and implies that the group-theoretical results in Section 4 can be read independently of the other parts of the paper. On the other hand this choice probably makes less clear the general strategy of the proof, so we decide to provide the reader with a road map.

We begin with a more detailed definition of hyperelliptic rotation. Note that all through the paper, unless otherwise stated, 3*-manifold* will mean orientable, connected, closed 3-manifold. Also, all finite group actions by diffeomorphisms will be faithful and orientation preserving.

Definition 1. Let $\psi : M \longrightarrow M$ be a finite order diffeomorphism of a 3-manifold M. We shall say that ψ is a *rotation* if it preserves the orientation of M, $Fix(\psi)$ is non-empty and connected, and $Fix(\psi) = Fix(\psi^k)$ for all non-trivial powers ψ^k of ψ. $Fix(\psi)$ will be referred to as the *axis of the rotation.* Note that if ψ is a periodic diffeomorphism of prime order, then ψ is a rotation if and only if $Fix(\psi) = \mathbf{S}^1$. We shall say that a rotation ψ is *hyperelliptic* if the space of orbits M/ψ of its action is $\mathbf{S}^3$.

Let M be a closed 3-manifold and G be a finite group of diffeormorphism of M. The main steps to obtain the proof of Theorem 2 are the following:

1. The special case of hyperellitpic rotations of order a power of two is well understood. To bound the number of conjugacy classes of cyclic groups

generated by a hyperelliptic rotation of order a power of two, we can suppose that G is a 2-group. If G contains such hyperelliptic rotation, then M is a $\mathbb{Z}_2$-homology sphere and G is isomorphic to a subgroup of $SO(4)$ (see []). By using these facts is not too difficult to prove that there are at most 9 conjugacy classes of cyclic groups generated by a hyperelliptic rotations of order a power of two (see [Re] and [Mec1]).

We exclude this case and we will consider cyclic groups generated by a hyperelliptic rotation of order not a power of two; for the sake of brevity, we call these groups *general hyperelliptic.*

2. Section 3 presents two ingredients of the proof. First of all the presence of a hyperelliptic rotation in a finite group G acting on a 3-manifold provides some "local" constraints on the structure of the group. In particular the structure of the normalizer of a general hyperelliptic group is known (it is the subgroup of a generalized dihedral subgroup of rank 2). Moreover, if p is an odd prime dividing the order of a hyperellitpic rotation, then a Sylow p-subgroup of G is abelian of rank at most two and contains at most two non-trivial cyclic subgroups acting non-freely. This is enough to prove that there are at most two conjugacy classes of general hyperelliptic groups of the same order (see [Z2]). As in the case of rotations of order a power of two, if we fix the branching order we can work with a single Sylow p-subgroup. The situation of our general case is much more difficult because to prove the existence of the upper bound we have to reconstruct the whole structure of the group.

   The second ingredient is the upper bound (three) on the number of general hyperelliptic groups commuting pairwise. This result implies the existence of the universal bound in the solvable case. In fact if the group G is solvable, the presence of Hall subgroups in G assures us that all general hyperelliptic groups commute up to conjugacy, implying that there at most three conjugacy classes of such groups.

3. The most intricate case concerns non-solvable groups where local approaches fail. We need a global description of the groups, which is provided in Section 4. We choice in this case a purely algebraic approach: we characterize the general hyperelliptic groups in terms of their algebraic properties and a collection of cyclic groups satisfying these properties is called algebraic hyperelliptic. We prove that a non-solvable finite group generated by an algebraic hyperelliptic collection is of very special type, in particular it has a factor by a normal solvable subgroup isomorphic to the direct product of two simple groups.

   The successive step is to "cover" these groups by a bounded number of solvable subgroups (i.e. to find a collection of solvable subgroups such that each element of odd prime order that is contained in a general hyperelliptic group can be conjugate in one of these solvable subgroups). The last part of Section 4 is devoted to the algebraic results supporting this strategy.

4. In Section 5 we conclude the proof of Theorem 2. First of all we replace G with a more suitable group. Roughly speaking, we can suppose that G is generated by an algebraic hyperelliptic collection and that the maximal normal solvable subgroup of G, which has order coprime with any odd

prime dividing the order of a hyperellitpic rotation, is trivial. Successively the proof for the solvable case is briefly completed and we divide the argument for the non-solvable groups in two cases.

In the first one we suppose that no rotation of order two is contained in G. We obtain that, up to a factor by a solvable subgroup, the group G is a simple group. Moreover, in the commutator of every general hyperelliptic group there exists another general hyperelliptic group of the same order. These two facts, together with the results about the solvable covers presented in the last part of Section 4, give the upper bound. The presence of two commuting general hyperellitpic groups of the same order has a nice geometric interpretation. The general hyperelliptic subgroups of G are the covering groups of the cyclic branched covers of knots yielding M as total space. If G does not contain any rotation of order two, we have that all these knots have a periodic symmetry of order equal to the branching order. Moreover if we quotient S^3 by this symmetry, the projection of the knot and of the axis of the symmetry give a link with two components, and this link has a symmetry inverting the two components. For a more detailed description of this type of knots and its role in the determination of the finite groups acting on 3-manifolds see [Mec2].

In the second case we suppose that G contains a rotation of order two. The group acting on 3-manifold containing such an involution are studied in [Mec2]. We combine the characterization of non-solvable groups generated by a hyperelliptic algebraic collection and the characterization obtained in [Mec2]. We obtain that G, up to a factor by a solvable subgroup, is isomorphic to $\mathbb{A}_8$, or to $PSL_2(q)$, or to $PSL_2(q) \times PSL_2(q')$. For these groups we explicitly find a solvable cover with a bounded number of elements and we obtain the universal bound.

3 Rotations and their properties

In this section we shall establish some properties of the hyperelliptic rotation.

Remark 1. Assume that ψ is a hyperelliptic rotation acting on a 3-manifold M then:

1. The natural projection from M to the space of orbits M/ψ of ψ is a cyclic cover of $\mathbf{S}^3$ branched along a knot $K = Fix(\psi)/\psi$. The converse is also true, that is any deck transformation generating the automorphism group of a cyclic covering of $\mathbf{S}^3$ branched along a knot is a hyperelliptic rotation.

2. If the order of ψ is a prime p, then M is a $\mathbb{Z}/p$-homology sphere [Go].

Remark 2. By Smith theory, if f is a periodic diffeomorphism of order p acting on a $\mathbb{Z}/p$-homology sphere and p is a prime number, then f either acts freely or is a rotation.

We start with a somehow elementary remark which is however central to determine constraints on finite groups acting on 3-manifolds.

Remark 3. Let $G \subset Diff^+(M)$ be a finite group of diffeomorphisms acting on a 3-manifold M. One can choose a Riemannian metric on M which is invariant by G and with respect to which G acts by isometries. Let now $\psi \in G$ be a rotation. Since the normaliser $\mathcal{N}_G(\langle\psi\rangle)$ of ψ in G consists precisely of those diffeomorphisms that leave the circle $Fix(\psi)$ invariant, we deduce that $\mathcal{N}_G(\langle\psi\rangle)$ is a finite subgroup of $\mathbb{Z}/2 \ltimes (\mathbb{Q}/\mathbb{Z} \oplus \mathbb{Q}/\mathbb{Z})$, where the nontrivial element in $\mathbb{Z}/2$ acts by conjugation sending each element of $\mathbb{Q}/\mathbb{Z} \oplus \mathbb{Q}/\mathbb{Z}$ to its inverse. Note that the elements of $\mathcal{N}_G(\langle\psi\rangle)$ are precisely those that rotate about $Fix(\psi)$, translate along $Fix(\psi)$, or invert the orientation of $Fix(\psi)$; in the last case the elements have order 2 and non-empty fixed-point set meeting $Fix(\psi)$ in two points.

Note that if $M \neq \mathbf{S}^3$ and ψ is a hyperelliptic rotation of order $n > 2$, then its centraliser $\mathcal{C}_G(\langle\psi\rangle)$ in G satisfies $1 \longrightarrow \langle\psi\rangle \longrightarrow \mathcal{C}_G(\langle\psi\rangle) \longrightarrow H \longrightarrow 1$, where H is cyclic, possibly trivial. This follows easily from the positive solution to the Smith conjecture which implies that any group of symmetries of a non-trivial knot K (that is, any finite group of orientation-preserving diffeomorphisms of $\mathbf{S}^3$ acting on the pair $(\mathbf{S}^3, K)$) is either cyclic or dihedral. Moreover, since the symmetries of a knot not acting freely have connected fixed-point set, the possible elements of $\mathcal{N}_G(\langle\psi\rangle) \setminus \mathcal{C}_G(\langle\psi\rangle)$ are rotations of order two.

Definition 2. With the notation of the above remark, we shall call $Fix(\psi)$-*rotations* the elements of $\mathcal{N}_G(\langle\psi\rangle)$ that preserve the orientation of $Fix(\psi)$ and $Fix(\psi)$-*inversions* those that reverse it.

Lemma 1. *Let φ and ψ be two rotations contained in a finite group of orientation preserving diffeomorphisms of a 3-manifold M.*

1. *A non-trivial power of φ commutes with a non-trivial power of ψ, both of orders different from 2, if and only if φ and ψ commute.*

2. *Assume $M \neq \mathbf{S}^3$. If φ and ψ are hyperelliptic and $Fix(\varphi) = Fix(\psi)$, then $\langle\varphi\rangle = \langle\psi\rangle$ (in particular they have the same order).*

3. *Assume $M \neq \mathbf{S}^3$. If φ and ψ are hyperelliptic, then $\langle\varphi\rangle$ and $\langle\psi\rangle$ are conjugate if and only if some non-trivial power of φ is conjugate to some non-trivial power of ψ.*

Proof.
Part 1 The sufficiency of the condition being obvious, we only need to prove the necessity. Remark that we can assume that both rotations act as isometries for some fixed Riemannian metric on the manifold. Denote by g and f the non trivial powers of φ and ψ, respectively. Note that, by definition, $Fix(\psi) = Fix(g)$ and $Fix(\varphi) = Fix(f)$. Since g and f commute, g leaves invariant $Fix(\varphi) = Fix(f)$ and thus normalises every rotation about $Fix(\varphi)$. Moreover g and φ commute, for the order of g is not 2 (see Remark 3). In particular, φ leaves $Fix(\psi) = Fix(g)$ invariant and normalises every rotation about $Fix(\psi)$. The conclusion follows.
Part 2 Reasoning as in Part 1, one sees that the two rotations commute. Assume, by contradiction, that the subgroups they generate are different. Under this assumption, at least one of the two subgroups is not contained in the other. Without loss of generality we can assume that $\langle\varphi\rangle \not\subset \langle\psi\rangle$. Take the quotient of M by the action of ψ. The second rotation φ induces a non-trivial rotation of $\mathbf{S}^3$ which leaves the quotient knot $K = Fix(\psi)/\psi \subset \mathbf{S}^3$ invariant. Moreover, this induced rotation fixes pointwise the knot K. The positive solution to the Smith

conjecture implies now that K is the trivial knot and thus $M = \mathbf{S}^3$, against the hypothesis.
Part 3 follows from 2. $\square$

We notice also that a conjugate of a hyperelliptic rotation is a hyperelliptic rotation.

There is a natural bound on the number of cyclic subgroups generated by hyperelliptic rotations of order not a power of two which commute pairwise; we begin analysing the situation of the symmetry group of a knot.

Definition 3. A *rotation of a knot* K in $\mathbf{S}^3$ is a rotation ψ of $\mathbf{S}^3$ such that $\psi(K) = K$ and $K \cap Fix(\psi) = \emptyset$. We shall say that ψ is a *full rotation* if K/ψ in $\mathbf{S}^3 = \mathbf{S}^3/\psi$ is the trivial knot.

Remark 4. Let ψ and φ be two commuting rotations acting on some manifold M whose orders are different from 2 and whose axes are distinct. Assume that ψ is hyperelliptic. In this case φ induces a rotation ϕ of $K = Fix(\psi)/\psi$, for $Fix(\psi) \cap Fix(\varphi) = \emptyset$ (see Remark 3). We have that φ is hyperelliptic if and only if ϕ is a full rotation. This can be seen by considering the quotient of M by the action of the group generated by ψ and φ. This quotient is $\mathbf{S}^3$ and the projection onto it factors through M/φ, which can be seen as a cyclic cover of $\mathbf{S}^3$ branched along K/ϕ. By the positive solution to the Smith conjecture, M/φ is $\mathbf{S}^3$ if and only if K/ϕ is the trivial knot.

The following finiteness result about commuting rotations of a non-trivial knot in $\mathbf{S}^3$ is one of the main ingredients in the proof of Theorem 2 (see [BoPa, Proposition 2], and [BoPa, Theorem 2] for a stronger result where commutativity is not required).

Proposition 1. *Let K be a non-trivial knot in $\mathbf{S}^3$. Let us consider a set of pairwise commuting full rotations in $Diff^+(S^3, K)$, they generate at most two pairwise distinct cyclic groups.*

Proof.
Assume by contradiction that there are three pairwise distinct cyclic groups generated by commuting full rotations of K, φ, ψ and ρ respectively. If two of them -say φ, ψ- have the same axis, then by hypothesis they cannot have the same order. Fix the one with smaller order -say ψ-: the quotient K/ψ is the trivial knot, and φ induces a rotation of K/ψ which is non-trivial since φ commutes with ψ and its order is distinct from that of ψ. The axis A of this induced symmetry is the image of $Fix(\psi)$ in the quotient $\mathbf{S}^3/\psi$ by the action of ψ. In particular K/ψ and A form a Hopf link and K is the trivial knot: this follows from the equivariant Dehn lemma, see [Hil].

We can thus assume that the axes are pairwise disjoint. Since the rotations commute, even if one of them has order 2, it cannot act as a strong inversion on the axes of the other rotations. Therefore we would have that the axis of ρ, which is a trivial knot, admits two commuting rotations, φ and ψ, with distinct axes, which is impossible: this follows, for instance, from the fact (see [EL, Thm 5.2]) that one can find a fibration of the complement of the trivial knot which is equivariant with respect to the two symmetries. $\square$

Observe that the proof of the proposition shows that two commuting full rotations of a non-trivial knot either generate the same cyclic group or have disjoint axes.

Remark 5. If a knot $K \subset \mathbf{S}^3$ admits a full rotation, then it is a prime knot, see [BoPa, Lemma 2].

Lemma 2. *Let $\{H_1, \dots, H_m\}$ be a set of cyclic subgroups of G generated by hyperelliptic rotations of order not a power of two. Suppose that there exists an abelian subgroup of G, containing at least an element of odd prime order of each H_i, then $m \leq 3$. Moreover either the orders of the H_i are pairwise distinct or $m \leq 2$.*

Proof.

By Lemma 1 we obtain that the groups H_i commute. Consider the cyclic branched covering $M \to M/H_1 \cong S^3$ over the knot $Fix(H_1)/H_1$. By projecting H_i with $i \geq 2$ to M/H_1 we obtain $n-1$ full rotations of $Fix(H_1)/H_1$. By Proposition 1 we obtain $n-1 \leq 2$. Note that, by the positive solution to the Smith conjecture, a non-trivial knot cannot admit two distinct and commuting cyclic groups of symmetries of the same order. This proves the latter part of the lemma.

□

This lemma implies directly the following corollary.

Corollary 3. *Let p be an odd prime and assume that $H \cong \mathbb{Z}/p \oplus \mathbb{Z}/p$ acts faithfully by orientation preserving diffeomorphisms on a 3-manifold M, the group H contains at most two distinct cyclic subgroups generated by nontrivial powers of hyperelliptic rotations.*

4 Group theoretical results

In this section we provide some algebraic Lemmas which we need in the proof of Theorem 2. In this section G is a finite group.

Definition 4. A collection C_i of subgroups of G of odd prime order p_i is said to be *algebraically hyperelliptic* if the following conditions are satisfied:

1. the centralizer of C_i is abelian of rank at most two and has index at most two in the normalizer of C_i;
2. each element that is in the normalizer of C_i but not in the centralizer inverts by conjugation each element in the centralizer;
3. if C_i is contained in a Sylow p_i-subgroup S_i of G, then S_i contains at most two distinct conjugates of C_i.

We remark that in this definition the primes p_i are not necessary pairwise distinct.

Proposition 2. *Let S be a Sylow p_i-subgroup where p_i is the order of a group C_i belonging to an algebraically hyperelliptic collection.*

1. S is either cyclic or the product of two cyclic groups, and

2. *$\mathcal{N}_G(S)$ contains with index at most 2 the normalizer of a conjugate of C_i, and contains an abelian subgroup of rank at most 2 with index a divisor of 4. In particular $\mathcal{N}_G(S)$ is solvable.*

Proof.
Up to conjugation we can suppose that S contains C_i. By property 1 and 2 in Definition 4, the normalizer $\mathcal{N}_S(C_i)$ is abelian of rank at most two. Property 3 in Definition 4 implies that $\mathcal{N}_S(\mathcal{N}_S(C_i)) = \mathcal{N}_S(C_i)$. Since S is a p_i-subgroup we obtain that $S = \mathcal{N}_G(C_i)$ and we get the first part of the thesis.

Since the subgroup $\mathcal{N}_G(S)$ normalizes the maximal elementary abelian subgroup of S, we obtain also the second part of the thesis.

□

The following general observation will be useful in the sequel.

Remark 6. We work under the hypotheses of Proposition 2 and we suppose that C_i is contained in the Sylow p_i-subgroup S. The normaliser of S contains the normaliser of C_i with index 2 if and only there exist an element $g \in N_G(S)$ such that $C_i^g \neq C_i$. This case happens if and only if $\mathcal{N}_G(S)$ contains elements of order a power of 2 which do not act in the same way on all elements of order p_i in S. Indeed, all elements in $\mathcal{N}_G(C_i)$ either commute with all elements of order p_i or act dihedrally. On the other hand, any element h in $\mathcal{N}_G(S) \setminus \mathcal{N}_G(C_i)$ conjugates C_i to C_i^g. If f is a genarator of C_i and $f' = f^g$ (this element is a generator of C_i^g, we obtain that $hf'fh^{-1} = f'f$ and $hf'f^{-1}h^{-1} = (f'f^{-1})^{-1}$, i.e. h acts dihedrally on some elements of order p_i while it commutes with others.

Lemma 3. *Let G be a solvable group containing an algebraically hyperelliptic collection $\{C_1, \cdots, C_m\}$ of subgroups of odd prime order. Then there exists an abelian subgroup of G containing a conjugate of C_i, for each $i = 1, \ldots, m$. In particular the subgroups C_i commute pairwise up to conjugacy.*

Proof. Let π be the set of the orders of the C_i and let B be a Hall π-subgroup of G. Each C_i is conjugate to a subgroup of B. Since π contains only odd primes, Definition 4 yields that centraliser and normaliser of every Sylow p-subgroup of B coincide. By Burnside's p-complement theorem (see [Su, Theorem 2.10 page 144]), every Sylow p-subgroup of B has a normal complement, and hence B is abelian. □

Remark 7. Suppose that N is a normal subgroup of G and H is a p-subgroup of G. If the order of N is coprime with p, then the normalizer of the projection of H to G/N is the projection of the normalizer of H in G, that is

$$\mathcal{N}_{G/N}(HN/N) = \mathcal{N}_G(H)N/N.$$

The inclusion $\supseteq$ holds trivially. We prove briefly the other inclusion. Let fN be an element of $\mathcal{N}_{G/N}(HN/N)$, then $H^f \subseteq HN$. Both H^f and H are Sylow p-subgroups of HN and by the second Sylow Thorem they are conjugate by an element $hn \in HN$. We obtain that $H^{fhn} = H$, and hence $f \in \mathcal{N}_G(H)N$.

Analogously if f is an element of prime order coprime with the order of N, then $\mathcal{C}_{G/N}(fN) = \mathcal{C}_G(f)N/N$.

Recall that a finite group Q is *quasisimple* if it is perfect (the abelianised group is trivial) and the factor group $Q/\mathcal{Z}$ of Q by its centre $\mathcal{Z}$ is a nonabelian simple group (see [Su, chapter 6.6]). A group E is *semisimple* if it is perfect and the factor group $E/\mathcal{Z}(E)$ is a direct product of nonabelian simple groups. A semisimple group E is a central product of quasisimple groups which are uniquely determined. Any finite group G has a unique maximal semisimple normal subgroup $E(G)$ (maybe trivial), which is characteristic in G. The subgroup $E(G)$ is called the *layer* of G and the quasisimple factors of $E(G)$ are called the *components* of G.

The maximal normal nilpotent subgroup of a finite group G is called the *Fitting subgroup* and is usually denoted by $F(G)$. The Fitting subgroup commutes elementwise with the layer of G. The normal subgroup generated by $E(G)$ and by $F(G)$ is called the *generalised Fitting subgroup* and is usually denoted by $F^*(G)$. The generalised Fitting subgroup has the important property to contain its centraliser in G, which thus coincides with the centre of $F^*(G)$. For further properties of the generalised Fitting subgroup see [Su, Section 6.6.].

Lemma 4. *Suppose that G is generated by the algebraically hyperelliptic collection $\mathcal{H} := \{C_1, \cdots, C_m\}$. Denote by p_i the order of C_i and by A the maximal normal solvable subgroup of order coprime with every p_i.*

If G is non-solvable, then the following properties hold:

1. *every p_i divides the order of any component of G/A;*
2. *either G/A is the direct product of a cyclic group of odd order and a simple group or it is the direct product of two simple groups;*
3. *if in addition G does not contain any involution acting dihedrally on some C_i, then $E(G/A)$ is simple, every p_i divides the order of $F(G/A)$ and a Sylow p_i-subgroup of G contains exactly two conjugates of C_i.*

Proof. Let π be the set of the primes p_i.

By Remark 7 we can suppose that A is trivial and $F(G)$ is a π-group.

Claim 1. *$F(G)$ is cyclic and $E(G)$ is not trivial*

Suppose first that $F(G)$ contains an abelian p_i-subgroup S of rank two. Then S, being the maximal elementary abelian p_i-subgroup contained in $F(G)$, is normal in G and contains C_i. This implies that G is solvable and we get a contradiction. Hence $F(G)$ is cyclic.

If $E(G)$ is trivial, then $F(G) = F^*(G)$ and $G/F(G)$ is isomorphic to a subgroup of $\mathrm{Aut}F(G)$. Since $F(G)$ is cyclic $\mathrm{Aut}F(G)$ is solvable, and we get again a contradiction.

Claim 2 *Each p_i divides the order of any component of G. Moreover the components of G are simple groups and are at most two.*

Since the Sylow p_i-subgroups are abelian and A is trivial, by [Su, Exercise 1, page 161] the components of G have trivial centre.

Now we prove that each component of G is normalised by any C_i. Let f_i be a generator of C_i and Q a component of G. Suppose by contradiction that Q is not normalised by f_i. We define the following subgroup:

$$Q_c = \{x f_i x f_i^{-1} \dots f_i^{p_i-1} x f_i^{p_i-1} \mid x \in Q\}.$$

Since the components of G commute elementwise, Q_c is a subgroup of G isomorphic to Q. Moreover, each element of Q_c commutes with f_i and this gives a contradiction.

We have that C_i normalizes Q but cannot centralize it, so the action by conjugation of f_i on Q is not trivial.

Assume that Q is either sporadic or alternating. Since the order of the outer automorphism group of any such simple group is a (possibly trivial) power of 2 (see [GLS, Section 5.2 and 5.3]), we conclude that f must induce an inner automorphism of Q. In particular p_i divides the order of Q.

We can thus assume that Q is a simple group of Lie type.

Recall that, by [GLS, Theorem 2.5.12], $Aut(Q)$ is the semidirect product of a normal subgroup $Inndiag(Q)$, containing the subgroup $Inn(Q)$ of inner automorphisms, and a group $\Phi\Gamma$, where, roughly speaking, Φ is the group of automorphisms of Q induced by the automorphisms of the defining field and Γ is the group of automorphisms of Q induced by the symmetries of the Dynkin diagram associated to Q (see [GLS] for the exact definition). By [GLS, Theorem 2.5.12.(c)], every prime divisor of $|Inndiag(Q)|$ divides $|Q|$. Thus we can assume that the automorphism induced by f_i on Q is not contained in $Inndiag(Q)$ and its projection θ on $Aut(Q)/Inndiag(Q) \cong \Phi\Gamma$ has order p_i. We will find a contradiction showing that in this case the centraliser of f_i in Q is not abelian.

Write $\theta = \phi\gamma$, with $\phi \in \Phi$ and $\gamma \in \Gamma$. If $\phi = 1$, then γ is nontrivial and f_i induces a graph automorphism according to [GLS, Definition 2.5.13]. Since p_i is odd, the only possibility is that Q is $D_4(q)$ and $p_i = 3$ (see [GLS, Theorem 2.5.12 (e)]). The centraliser of f_i in Q is nonabelian by [GLS, Table 4.7.3 and Proposition 4.9.2.]. If $\phi \neq 1$ and Q is not isomorphic to the group ${}^3D_4(q)$, then the structure of the centraliser of f_i in Q is described by [GLS, Theorem 4.9.1], and it is nonabelian. Finally, if $\phi \neq 1$ and $Q \cong {}^3D_4(q)$, the structure of the non abelian centraliser of f_i in Q follows from [GLS, Proposition 4.2.4]. We proved that the automorphism induced by f_i is contained in $Inndiag(Q)$ and p_i divides $|Q|$.

Since p_i divides the order of any component, G has at most two components.

Claim 3 $G = E(G)F(G)$.

We prove first that $G = E(G)\mathcal{C}_G(E(G))$. Let us assume by contradiction that there exists C_i with trivial intersection with $E(G)\mathcal{C}_G(E(G))$ and denote by f a generator of C_i. Since p_i divides the order of every component of G and the Sylow p_i-subgroup has rank at most 2, we get that $E(G)$ has only one component which we denote by Q. The Sylow p_i-subgroups of Q are cyclic. Moreover, by the first part of the proof, the automorphism induced by f on Q is inner-diagonal. If it is inner, we obtain f as a product of an element that centralises Q and an element in Q, a contradiction to our assumption; otherwise, we get again contradiction, since, by [GLS, Theorem 2.5.12] and [A, (33.14)], a group of Lie type with cyclic Sylow p_i-subgroup cannot have a diagonal automorphism of order p_i.

Hence, all the subgroups C_i are contained in $E(G)\mathcal{C}_G(E(G))$ and, since they generate G, we obtain that $G = E(G)\mathcal{C}_G(E(G))$.

Now if $F(G) = 1$, then $F^*(G) = E(G)$ and hence $\mathcal{C}_G(E(G)) = \mathcal{C}_G(F^*(G)) = \mathcal{Z}(E(G)) = 1$ and the claim is proved. So suppose that $F(G) \neq 1$. Then, since $F(G)$ is a π-group, there is at least one subgroup C_i that is contained in

$E(G)F(G)$. Hence there is a subgroup T_1 of $E(G)$ with order p_i and a subgroup T_2 of $F(G)$ with order p_i such that $C_i \leq T_1T_2$. Since $F(G)$ is cyclic, T_2 is normal in G and so $\mathcal{C}_G(E(G)) \leq \mathcal{N}_G(T_1T_2)$. But $\mathcal{N}_G(T_1T_2)$ has an abelian normal subgroup of index a divisor of 4. This shows that every Sylow p-subgroup of $\mathcal{C}_G(E(G))$ for p odd is contained in $F(G)$, whence it follows that $G = E(G)F(G)$.

Claim 4. *If no involution of G acts dihedrally on some C_i, then $E(G)$ is simple, and, for every i, p_i divides the order of $F(G)$ and a Sylow p_i-subgroup of G contains exactly two conjugates of C_i.*

Suppose no involution of G acts dihedrally on some C_i. By Definition 4, it follows that $\mathcal{N}_G(C_i) = \mathcal{C}_G(C_i)$ for every $i = 1, \ldots, m$. Let Q be a component of $E(G)$ and suppose by contradiction that Q contains a Sylow p_i-subgroup S of rank two, for some i. Up to conjugation we can suppose that S contains C_i. By Definition 4, $\mathcal{N}_Q(S)$ contains with index at most two the abelian group $\mathcal{C}_Q(C_i)$. Since Q is simple, Burnside's p-complement theorem (see [Su, Theorem 2.10 page 144]) yields that $\mathcal{N}_Q(S)$ is not abelian. Therefore, $\mathcal{N}_Q(S)$ contains with index two $\mathcal{N}_Q(C_i)$ and the elements of $\mathcal{N}_Q(S) \setminus \mathcal{N}_Q(C_i)$ conjugate C_i to a cyclic subgroup distinct by C_i. By using [Su, Exercise 1, page 161] and the fact that Q is perfect we get again a contradiction. Hence, for every $i \in \{1, \ldots, m\}$, the Sylow p_i-subgroups of Q are cyclic. Now, as above foe every Sylow subgroup S of Q we have $\mathcal{N}_Q(S) \neq \mathcal{C}_Q(S)$. Since $\mathcal{N}_G(C_i) = \mathcal{C}_G(C_i)$ for every $i = 1, \ldots, m$, that implies that the Sylow p_i-subgroups of G are not cyclic and hence p_i divides the order of $F(G)$ for every i. □

To bound by above the number of conjugacy classes of hyperelliptic rotations, our strategy will consist of conjugating hyperelliptic rotations in solvable subgroups of G, where they are forced to commute, hence we introduce the notion of solvable normal π-cover and we prove the following Lemma.

Definition 5. Let G be a finite group. Let π be a set of primes dividing $|G|$. We will call a collection $\mathcal{C}$ of subgroups of G a *solvable normal π-cover* of G if every element of G of prime order p belonging to π is contained in an element of $\mathcal{C}$ and for every $g \in G$, $H \in \mathcal{C}$ we have that $H^g \in \mathcal{C}$. We denote by $\gamma^s_\pi(G)$ the smallest number of conjugacy classes of subgroups in a solvable normal π-cover of G. Note that, since Sylow subgroups are clearly solvable, $\gamma^s_\pi(G) \leq |\pi|$.

Lemma 5. *Let G be a finite nonabelian simple group. If π is the set of odd primes p such that G has cyclic Sylow p-subgroup, the centralizer of $C_G(g)$ is abelian for every element $g \in G$ of order p and the normaliser of any subgroup of order p contains with index at most two its centralizer, then $\gamma^s_\pi(G) \leq 4$.*

Proof.

If G is a sporadic group, the primes dividing the order of the group do not satisfy the condition on the normaliser.

If G is isomorphic to the alternating group $\mathbb{A}_n$ and $p \in \pi$, then the condition on the centralizer of the elements of order p implies that that $p > n - 4$ and $\gamma^s_\pi(G) \leq |\pi| \leq 2$.

The only remaining case is that of groups of Lie type.

Let $G \cong \Sigma_n(q)$ or $G \cong {}^d\Sigma_n(q)$, where q is a power of a prime s. Here we use the same notation as in [GLS]: the symbol $\Sigma(q)$ (resp. ${}^d\Sigma(q)$) may refer to finite groups in different isomorphism classes, each of them is an untwisted (resp. twisted) finite group of Lie type with root system Σ (see [GLS, Remark

2.2.5]). Any finite group of Lie type is quasisimple with the exception of the following groups: $A_1(2)$, $A_1(3)$, ${}^2A_2(2)$, ${}^2B_2(2)$, $B_2(2)$, $G_2(2)$, ${}^2F_4(2)$ and ${}^2G_2(3)$ (see [GLS, Theorem 2.2.7]).

If $s \in \pi$, then by [GLS, Theorem 3.3.3], either $s = 3$ and $G \cong ({}^2G_2(3))'$ or $G \cong A_1(s)$. In the former case the order of ${}^2G_2(3)'$ is divided only by two odd primes, thus $\gamma^s_\pi(G) \leq 2$; in the latter case we have $\gamma^s_\pi(G) \leq 2$ (see for example [H]).

Assume now that $s \notin \pi$. By [GLS, Paragraph 4.10], since the Sylow subgroups are cyclic, every element of order $p \in \pi$ is contained in a maximal torus of G, and clearly a maximal torus is abelian.

Therefore, we need only to bound the number of conjugacy classes of cyclic maximal tori in G with abelian centraliser. Note that the number of conjugacy classes of maximal tori in G is bounded by the number of different cyclotomic polynomials evaluated in q appearing as factors of $|G|$. Moreover the power of a cyclotomic polynomial in the order of G gives the rank of the corresponding maximal torus (except possibly when the prime divides the order of the centre but in this case the Sylow subgroup is not cyclic, see [A, (33.14)])

Recall Σ is the root system associated to G as in [GLS, 2.3.1]; let $\Pi = \{\alpha_1, \dots, \alpha_n\}$ be a fundamental system for Σ as in Table 1.8 in [GLS], α_* be the lowest root relative to Π as defined in [GLS, Paragraph 1.8] and set $\Pi^* = \Pi \cup \{\alpha_*\}$. We recall that $|G|$ can be deduced from [GLS, Table 2.2] and the Dynkin diagrams can be found in [GLS, Table 1.8]. Observe that, by [GLS, Proposition 2.6.2], if Σ_0 is a root subsystem of Σ, then G contains a subsystem subgroup H, which is a central product of groups of Lie type corresponding to the irreducible constituents of Σ_0. In order to prove the lemma, we shall show that for every group G and for every element g of order a prime r lying in a maximal torus belonging to any but four conjugacy classes of maximal tori, either the Sylow r-subgroup is not cyclic or we find a subsystem subgroup H that is a central product of two groups H_1 and H_2 such that H_1 contains g and H_2 is not abelian. Note that for every prime power q, $A_1(q)$ is a non-abelian group (see [GLS, Theorem 2.2.7]).

We treat the case $G \cong A_n(q)$ in details as an example. All other cases can be dealt similarly. Assume $G \cong A_n(q)$. Let m be the minimum index i such that r divides $q^{i+1}-1$ and let Σ_0 be generated by $\Pi^* \setminus \{\alpha_1, \alpha_n\}$. Then the corresponding subsystem subgroup is $H = H_1 \cdot H_2$, where $H_1 \cong A_{n-2}(q)$ and $H_2 \cong A_1(q)$. Thus if $m \leq n-1$, then H_1 contains an element g of order r and $C_G(g)$ contains H_2 which is not abelian. Therefore, since g has an abelian centraliser, r may divide only $(q^n-1)(q^{n+1}-1)$, that is r divides $\Phi_n(q)\Phi_{n+1}(q)$. Hence we have at most two conjugacy classes of maximal tori with abelian centraliser. □

5 Proof of Theorem 2

Let G be a finite group of orientation preserving diffeomorphisms of M, a closed orientable connected 3-manifold which is not homeomorphic to $\mathbf{S}^3$.

By [Re] and [Mec1] there at most nine conjugacy classes of cyclic groups generated by a hyperelliptic rotation of order 2^n, so we concentrate on hyperelliptic rotations of order not a power of two and we will prove that there are at most six conjugacy classes of cyclic groups generated by such rotations.

Let $\mathcal{S} = \{C_1, \ldots, C_m\}$ be the set of cyclic subgroups of odd prime orders that are generated by powers of the hyperelliptic rotations of G. We recall that the conjugate of a power of a hyperelliptic rotation is the power of a hyperelliptic rotation too. By Remark 3 and Corollary 3, $\mathcal{S}$ is an algebraically hyperelliptic collection. For $i \in \{1, \ldots, m\}$, let p_i be the prime order of C_i. We denote by G_0 the subgroup generated by the subgroups C_i. Let A be the maximal normal solvable subgroup of G_0 of order coprime with all p_i. We denote by $\overline{G_0}$ the quotient group G_0/A and by $\overline{C}_i$ the projection of C_i to $\overline{G_0}$.

Case 1. If G_0 is solvable, Lemma 2 and Lemma 3 imply that there are at nost three conjugacy classes and that two hyperelliptic rotations of order not a power of two commute up to conjugacy. The solvable case is interesting in its own right, in particular it plays an important role in the proof of Theorem3, and then we summarize the situation in the following proposition.

Proposition 3. *Let M be a closed 3-manifold not homeomorphic to $\mathbf{S}^3$. Let G be a finite group of orientation preserving diffeomorphisms of M. If G is solvable, it contains at most three conjugacy classes of cyclic groups generated by a hyperelliptic rotation of order not a power, and two of such subgroups commute up to conjugacy.*

Case 2. Suppose G_0 is not solvable and it has no rotation of order 2 outside A. Then, by Remark 3, G_0 has no involution acting dihedrally on some C_i and by Lemma 4, $E(\overline{G_0})$ is simple and for each p_i any Sylow p_i-subgroup contains exactly two distinct conjugates of $\bar{C}_i$. By Remark 3 any hyperelliptic rotation commuting with one of these two groups of order p_i commute also with the other one. From this fact, Lemma 3 and Lemma 2, it follows that $\gamma^s_\pi(G_0)$ bounds from above the number of conjugacy classes of groups generated by hyperelliptic rotations of order not a power of two. It is easy to see that $\gamma^s_\pi(G_0) \leq \gamma^s_\pi(E(\overline{G_0}))$. By Lemma 5 we get the thesis in this case.

Case 3. Suppose G_0 is not solvable and it has a rotation of order 2 not contained in A. The groups containing a rotation of order two are studied in [Mec2] where the following result was proved.

Theorem 4. *[Mec2] Let D be a finite group of orientation-preserving diffeomorphisms of a closed orientable 3-manifold. Let O be the maximal normal subgroup of odd order and $E(\tilde{D})$ be the layer of $\tilde{D} = D/O$. Suppose that D contains an involution which is a rotation.*

1. *If $E(\tilde{D})$ is trivial, there exists a normal subgroup H of D such that H is solvable and D/H is isomorphic to a subgroup of $\mathbb{A}_8$, the alternating group on 8 letters.*

2. *If the semisimple group $E(\tilde{D})$ is not trivial, it has at most two components and the factor group of $\tilde{D}/E(\tilde{D})$ is solvable.*

Moreover if D contains a rotation of order 2 such that its projection is contained in $E(\tilde{D})$, then $E(\tilde{D})$ is isomorphic to one of the following groups:

$$PSL_2(q), \;\; SL_2(q) \times_{\mathbb{Z}/2} SL_2(q')$$

where q and q' are odd prime powers greater than 4.

Applying Theorem 4 and Lemma 4 to G_0, we get that $E(\overline{G_0})$ is isomorphic either to a subgroup of $\mathbb{A}_8$, or to $PSL_2(q)$, or to $PSL_2(q) \times PSL_2(q')$. In the first case, there are at most three odd primes dividing the order of $E(\overline{G_0})$ and the thesis follows from Lemma 2 and Lemma 3.

In the remaining cases, we will use a solvable normal π-cover to bound the number of conjugacy classes. We have that $\gamma^s_\pi(PSL_2(q)) \leq 2$. In fact the upper triangular matrices form a solvable subgroup of $SL_2(q)$ of order $(q-1)q$, moreover $SL_2(q)$ contains a cyclic subgroup of order $q+1$ (see [H]). The conjugates of the projections of these two subgroups to $PSL_2(q)$ give a solvable normal π-cover of $PSL_2(q)$. It is easy to see that if $E(\overline{G_0})$ is isomorphic to $PSL_2(q)$, then $\gamma^s_\pi(G) \leq 2$. As above, by Lemma 3 and Lemma 2, we get the thesis. Finally, if $E(\overline{G})$ is isomorphic to $PSL_2(q) \times PSL_2(q')$, then $\gamma^s_\pi(E(\overline{G_0})) \leq 4$, and hence $\gamma^s_\pi(G_0) \leq 4$. As in Case 2, using Remark 6, one can see that any Sylow p_i-subgroup contains exactly two distinct and conjugate cyclic subgroups which are generated by the power of a hyperelliptic rotation and thus $\gamma^s_\pi(G_0)$ bounds from above the number of conjugacy classes of groups generated by hyperelliptic rotations of order not a power of two. This concludes the proof. □

6 Proof of Theorem 3

The statement of Theorem 3 is equivalent to the following:

Theorem 5. *Let M be a closed, orientable, connected, irreducible 3-manifold which is not homeomorphic to $\mathbf{S}^3$, then the group $Diff^+(M)$ of orientation preserving diffeomorphisms of M contains at most six conjugacy classes of cyclic subgroups generated by a hyperelliptic rotation of odd prime order.*

Notice that generically one expects that two hyperelliptic rotations in the group $Diff^+(M)$ generate an infinite subgroup.

6.1 Proof of Theorem 5 for Seifert manifolds

In this section we prove Proposition 4 which implies Theorem 5 for closed Seifert fibred 3-manifolds. We also show that the assumption that the hyperelliptic rotations have odd prime orders cannot be avoided in general by exhibiting examples of closed Seifert fibred 3-manifolds M such that $Diff^+(M)$ contains an arbitrarily large number of conjugacy classes of cyclic subgroups generated by hyperelliptic rotations of odd, but not prime, orders.

Proposition 4. *Let M be a closed Seifert fibred 3-manifold which is not homeomorphic to $\mathbf{S}^3$. Then the group $Diff^+(M)$ of orientation preserving diffeomorphisms of M contains at most one conjugacy class of cyclic subgroups generated by a hyperelliptic rotation of odd prime order except if M is a Brieskorn integral homology sphere with 3 exceptional fibres. In this latter case $Diff^+(M)$ contains at most three non conjugate cyclic subgroups generated by hyperelliptic rotations of odd prime orders.*

Proof.

By hypothesis M is a cyclic cover of S^3 branched over a knot, so it is orientable and a rational homology sphere by Remark 1. Notably, M cannot be

$\mathbf{S}^1 \times \mathbf{S}^2$ nor a Euclidean manifold, except for the Hantzsche-Wendt manifold, see [Or, Chap. 8.2]. In particular, since M is prime it is also irreducible.

Consider a hyperelliptic rotation ψ on M of odd prime order p and let K be the image of $Fix(\psi)$ in the quotient $\mathbf{S}^3 = M/\psi$ by the action of ψ. The knot K must be hyperbolic or a torus knot, otherwise its exterior would be toroidal and have a non-trivial JSJ-collection of essential tori which would lift to a non-trivial JSJ-collection of tori for M, since the order of ψ is $p > 2$ (see [JS, J] and [BS]). By the orbifold theorem (see [BoP], [CHK]), the cyclic branched cover with order $p \geq 3$ of a hyperbolic knot is hyperbolic, with a single exception for $p = 3$ when K is the the figure-eight knot and M is the Hantzsche-Wendt Euclidean manifold. But then, by the orbifold theorem and the classification of 3-dimensional christallographic groups, ψ is the unique, up to conjugacy, Euclidean hyperelliptic rotation on M, see for example [Dun], [Z1].

So we can assume that M is the p-fold cyclic cover of $\mathbf{S}^3$ branched along a non-trivial torus knot K of type (a, b), where $a > 1$ and $b > 1$ are coprime integers. Then M is a Brieskorn-Pham manifold $M = V(p, a, b) = \{z^p + x^a + y^b = 0 \text{ with } (z, x, y) \in \mathbb{C}^3 \text{ and } |z|^2 + |x|^2 + |y|^2 = 1\}$. A simple computation shows that M admits a Seifert fibration with 3, p or $p+1$ exceptional fibres and base space $\mathbf{S}^2$, see [Ko, Lem. 2], or [BoPa, Lemma 6 and proof of Lemma 7]. In particular M has a unique Seifert fibration, up to isotopy: by [Wa], [Sco] and [BOt] the only possible exception with base $\mathbf{S}^2$ and at least 3 exceptional fibres is the double of a twisted I-bundle, which is not a rational homology sphere, since it fibers over the circle. We distinguish now two cases:

Case 1: *The integers a and b are coprime with p, and there are three singular fibres of pairwise relatively prime orders a, b and p.* By the orbifold theorem any hyperelliptic rotation of M of order > 2 is conjugate into the circle action $S^1 \subset Diff^+(M)$ inducing the Seifert fibration, hence the uniqueness of the Seifert fibration, up to isotopy, implies that M admits at most 3 non conjugate cyclic groups generated by hyperelliptic rotations with odd prime orders belonging to $\{a, b, p\}$. Indeed M is a Brieskorn integral homology sphere, see [BPZ].

Case 2: *Either $a = p$ and M has p singular fibres of order b, or $a = a'p$ with $a' > 1$, and M has p singular fibres of order b and one extra singular fibre of order a'.* In both situations, there are $p \geq 3$ exceptional fibres of order b which are cyclically permuted by the hyperelliptic rotation ψ. As before, M has a unique Seifert fibration, up to isotopy. Therefore, up to conjugacy, ψ is the only hyperelliptic rotation of order p on M, and by the discussion above M cannot admit a hyperelliptic rotation of odd prime order $q \neq p$. □

Remark 8. The requirement that the rotations are hyperelliptic is essential in the proof of Proposition 4. The Brieskorn homology sphere $\Sigma(p_1, \ldots, p_n)$, $n \geq 4$, with $n \geq 4$ exceptional fibres admits n rotations of pairwise distinct prime orders but which are not hyperelliptic.

The hypothesis that the orders of the hyperelliptic rotations are $\neq 2$ cannot be avoided either.

Indeed, Montesinos' construction of fibre preserving hyperelliptic involutions on Seifert fibered rational homology spheres [Mon1], [Mon2], (see also [BS, Appendix A], [BZH, Chapter 12]), shows that for any given integer n there are infinitely many closed orientable Seifert fibred 3-manifolds with at least n conjugacy classes of hyperelliptic rotations of order 2.

On the other hand, the hypothesis that the orders are odd primes is sufficient but not necessary: A careful analysis of the Seifert invariants shows that if $M \neq \mathbf{S}^3$ is a Seifert rational homology sphere, then M can be the cyclic branched cover of a knot in $\mathbf{S}^3$ of order > 2 in at most three ways.

The hypotheses of Proposition 4 cannot be relaxed further, though: Proposition 5 below shows that there exist closed 3-dimensional circle bundles with arbitrarily many conjugacy classes of hyperelliptic rotations of odd, but not prime, orders.

Proposition 5. *Let N be an odd prime integer. For any integer $1 \leq q < \frac{N}{2}$ the Brieskorn-Pham manifold $M = V((2^q+1)(2^{(N-q)}+1), 2^q+1, 2^{(N-q)}+1)$ is a circle bundle over a closed surface of genus $g = 2^{N-1}$ with Euler class ± 1. Hence, up to homeomorphism (possibly reversing the orientation), M depends only on the integer N and admits at least $\frac{N-1}{2}$ conjugacy classes of cyclic subgroups generated by hyperelliptic rotations of odd orders.*

Proof. We remark first that the integers q and $N-q$ are relatively prime, because N is prime. If k is a common prime divisor of 2^q+1 and $2^{(N-q)}+1$, by Bezout identity we have $2^1 = 2^{aq+b(N-q)} \equiv (-1)^{a+b} \mod k$ which implies that $k=3$. But then $(-1)^q \equiv (-1)^{(N-q)} \equiv -1 \mod 3$ and thus $(-1)^N \equiv 1 \mod 3$ which is impossible since N is odd. Hence 2^q+1 and $2^{(N-q)}+1$ are relatively prime.

So the Brieskorn-Pham manifold M is the $(2^q+1)(2^{(N-q)}+1)$-fold cyclic cover of S^3 branched over the torus knot $K_q = T(2^q+1, 2^{(N-q)}+1)$. It is obtained by Dehn filling the $(2^q+1)(2^{(N-q)}+1)$-fold cyclic cover of the exterior of the torus knot K_q along the lift of its meridian. The $(2^q+1)(2^{(N-q)}+1)$-fold cyclic cover of the exterior of K_q is a trivial circle bundle over a once punctured surface of genus $g = 2^N - 1$. On the boundary of the torus-knot exterior the algebraic intersection between a meridian and a fibre of the Seifert fibration of the exterior is ± 1 (the sign depends on a choice of orientation, see for example [BZH, Chapter 3]). So on the torus boundary of the $(2^q+1)(2^{(N-q)}+1)$-fold cyclic cover the algebraic intersection between the lift of a meridian of the torus knot and a $\mathbf{S}^1$-fiber is again ± 1. Hence the circle bundle structure of the $(2^q+1)(2^{(N-q)}+1)$-fold cyclic cover of the exterior of the torus knot K_q can be extended with Euler class ± 1 to the Dehn filling along the lift of the meridian. So M is a circle bundle over a closed surface of genus $g = 2^{N-1}$ with Euler class ± 1.

Since the torus knots $K_q = T(2^q+1, 2^{(N-q)}+1)$ are pairwise inequivalent for $1 \leq q \leq \frac{N-1}{2}$, the cyclic subgroups generated by the hyperelliptic rotations corresponding to the $(2^q+1)(2^{(N-q)}+1)$-fold cyclic branched covers of the knots K_q are pairwise not conjugate in $Diff^+(M)$. □

Remark 9. Note that the Seifert manifolds M and their hyperelliptic rotations constructed in Proposition 5 enjoy the following properties: If $N > 8$, then no hyperelliptic rotation can commute up to conjugacy with all the remaining ones (see Proposition 3 and [BoPa, Theorem 2]). If $N > 14$ no finite subgroup of $Diff^+(M)$ can contain up to conjugacy all hyperelliptic rotations of M, according to Theorem 2.

6.2 Reduction to the finite group action case

The fact that Theorem 5 implies Corollary 1 follows from the existence of a decomposition of a closed, orientable 3-manifold M as a connected sum of prime manifolds and the observation that a hyperelliptic rotation on M induces a hyperelliptic rotation on each of its prime summands. A 3-manifold admitting a hypereliptic rotation must be a rational homology sphere, and so M cannot have $\mathbf{S}^2 \times \mathbf{S}^1$ summands. Hence all prime summands are irreducible and at least one is not homeomorphic to $\mathbf{S}^3$, since M itself is not homeomorphic to $\mathbf{S}^3$. This is enough to conclude.

The remaining of this section is devoted to the proof that Theorem 2 implies Theorem 5.

We prove the following proposition:

Proposition 6. *If M is a closed, orientable, irreducible 3-manifold such that there are $k \geq 7$ conjugacy classes of cyclic subgroups of $Diff^+(M)$ generated by hyperelliptic rotations of odd prime order, then M is homeomorphic to $\mathbf{S}^3$.*

Proof.

Let M be a closed, orientable, irreducible 3-manifold such that $Diff^+(M)$ contains $k \geq 7$ conjugacy classes of cyclic subgroups generated by hyperelliptic rotations of odd prime orders.

According to the orbifold theorem (see [BoP], [BMP], [CHK]), a closed orientable irreducible manifold M admitting a rotation has geometric decomposition. This means that M can be split along a (possibly empty) finite collection of π_1-injective embedded tori into submanifolds carrying either a hyperbolic or a Seifert fibered structure. This splitting along tori is unique up to isotopy and is called the JSJ-decomposition of M, see for example [NS], [BMP, chapter 3]. In particular, if its JSJ-decomposition is trivial, M admits either a hyperbolic or a Seifert fibred structure.

First we see that M cannot be hyperbolic. Indeed, if the manifold M is hyperbolic then, by the orbifold theorem, any hyperelliptic rotation is conjugate into the finite group $Isom^+(M)$ of orientation preserving isometries of M. Hence, applying Theorem 2 to $G = Isom^+(M)$, we see that $k \leq 6$ against the hypothesis.

If the manifold M is Seifert fibred, it follows readily from Proposition 4 of the previous section that $M = \mathbf{S}^3$. So we are left to exclude the case where the JSJ-decomposition of M is not empty.

Consider the JSJ-decomposition of M: each geometric piece admits either a complete hyperbolic structure with finite volume or a Seifert fibred product structure with orientable base. Moreover, the geometry of each piece is unique, up to isotopy.

Let $\Psi = \{\psi_1, \ldots, \psi_k,\ k \geq 7\}$ be the set of hyperelliptic rotations which generate non conjugate cyclic subgroups in $Diff^+(M)$. By the orbifold theorem [BoP], [BMP], [CHK], after conjugacy, one can assume that each hyperelliptic rotation preserves the JSJ-decomposition, acts isometrically on the hyperbolic pieces, and respects the product structure on the Seifert pieces. We say that they are geometric.

Let Γ be the dual graph of the JSJ-decomposition: it is a tree, for M is a rational homology sphere (in fact, the dual graph of a manifold which is the cyclic branched cover of a knot is always a tree, regardless of the order of the covering). Let $H \subset Diff^+(M)$ be the group of diffeomorphisms of M generated by the set Ψ of geometric hyperelliptic rotations. By [BoPa, Thm 1], there is a subset $\Psi_0 \subset \Psi$ of $k_0 \geq 4$ hyperelliptic rotations with pairwise distinct odd prime order, say $\Psi_0 = \{\psi_i,\ i = 1, \dots, k_0\}$.

Let H_Γ denote the image of the induced representation of H in $Aut(\Gamma)$. Since rotations of finite odd order cannot induce an inversion on any edge of Γ, the finite group H_Γ must fix point-wise a non-empty subtree Γ_f of Γ.

The idea of the proof is now analogous to the ones in [BoPa] and [BPZ]: we start by showing that, up to conjugacy, the $k_0 \geq 4$ hyperelliptic rotations with pairwise distinct odd prime orders can be chosen to commute on the submanifold M_f of M corresponding to the subtree Γ_f. We consider then the maximal subtree corresponding to a submanifold of M on which these hyperelliptic rotations commute up to conjugacy and prove that such subtree is in fact Γ. Then the conclusion follows as in the proof of Theorem 2, by applying Proposition 1.

The first step of the proof is achieved by the following proposition:

Proposition 7. *The hyperelliptic rotations in Ψ_0 commute, up to conjugacy in $Diff^+(M)$, on the submanifold M_f of M corresponding to the subtree Γ_f.*

Proof.

Since the hyperelliptic rotations in Φ have odd orders, either Γ_f contains an edge, or it consists of a single vertex. We shall analyse these two cases.

Case 1: M_f contains an edge.

Claim 1. *Assume that Γ_f contains an edge and let T denote the corresponding torus. The hyperelliptic rotations in Ψ_0 commute, up to conjugacy in $Diff^+(M)$, on the geometric pieces of M adjacent to T.*

Proof.

The geometric pieces adjacent to T are left invariant by the hyperelliptic rotations in Ψ_0, since their orders are odd. Let V denote one of the two adjacent geometric pieces: each hyperelliptic rotation acts non-trivially on V with odd prime order. We distinguish two cases according to the geometry of V.

V **is hyperbolic.** In this case all rotations act as isometries and leave a cusp invariant. Since their order is odd, the rotations must act as translations along horospheres, and thus commute.

Note that, even in the case where a rotation has order 3, its axis cannot meet a torus of the JSJ-decomposition of M for each such torus is separating and cannot meet the axis in an odd number of points.

V **is Seifert fibred.** In this case we can assume that the hyperelliptic rotations in Ψ preserve the Seifert fibration with orientable base. Since their orders are odd and prime, each one preserves the orientation of the fibres and of the base. The conjugacy class of a fiber-preserving rotation of V with odd prime order depends only on its combinatorial behaviour, i.e. its translation action along the fibre and the induced permutation on cone points and boundary components of the base. In particular, two geometric rotations with odd prime order having the same combinatorial data are conjugate via a diffeomorphism isotopic to the identity.

Since the hyperelliptic rotations in Ψ_0 have pairwise distinct odd prime orders, an analysis of the different cases described in Lemma 6 below shows that at most one among these hyperelliptic rotations can induce a non-trivial action on the base of the fibration, and thus the remaining ones act by translation along the fibres and induce the idendity on the base. Since the translation along the fibres commutes with every fiber-preserving diffeomorphism of V, the hyperelliptic rotations in Ψ_0 commute on V.

Lemma 6 describes the Seifert fibred pieces of a manifold admitting a hyperelliptic rotation of odd prime order, as well as the action of the rotation on the pieces. Its proof can be found in [BoPa, Lemma 6 and proof of Lemma 7], see also [Ko, Lem. 2].

Lemma 6. *Let M be an irreducible 3-manifold admitting a non-trivial JSJ-decomposition. Assume that M admits a hyperelliptic rotation of prime odd order p. Let V be a Seifert piece of the JSJ-decomposition for M. Then the base B of V can be:*

1. *A disc with 2 cone points. In this case either the rotation freely permutes p copies of V or leaves V invariant and acts by translating along the fibres.*
2. *A disc with p cone points. In this case the rotation leaves V invariant and cyclically permutes the singular fibres, while leaving a regular one invariant.*
3. *A disc with $p+1$ cone points. In this case the rotation leaves V and a singular fibre invariant, while cyclically permuting the remaining p singular fibres.*
4. *An annulus with 1 cone point. In this case either the rotation freely permutes p copies of V or leaves V invariant and acts by translating along the fibres.*
5. *An annulus with p cone points. In this case the rotation leaves V invariant and cyclically permutes the p singular fibres.*
6. *A disc with $p-1$ holes and 1 cone point. In this case the rotation leaves V invariant and cyclically permutes all p boundary components, while leaving invariant the only singular fibre and a regular one.*
7. *A disc with k holes, $k \geq 2$. In this case either the rotation freely permutes p copies of V or leaves V invariant. In this latter case either the rotation acts by translating along the fibres, or $k = p-1$ and the rotation permutes all the boundary components (while leaving invariant two fibres), or $k = p$ and the rotation permutes p boundary components, while leaving invariant the remaining one and a regular fibre.*

□

We conclude that the rotations in Ψ_0 can be chosen to commute on the submanifold M_f of M corresponding to Γ_f by using inductively at each edge of Γ_f the gluing lemma below (see [Lemma 6][BPZ]). We give the proof for the sake of completeness.

Lemma 7. *If the rotations preserve a JSJ-torus T then they commute on the union of the two geometric pieces adjacent to T.*

Proof.

Let V and W be the two geometric pieces adjacent to T. By Claim 1, after conjugacy in $Diff^+(M)$, the rotations in Ψ_0 commute on V and W. Since they have pairwise distinct odd prime orders, their restrictions on V and W generate two cyclic groups of the same finite odd order. Let g_V and g_W be generators of these two cyclic groups. Since they have odd order, they both act by translation on T. We need the following result about the slope of translation for such periodic transformation of the torus:

Claim 2. *Let ψ be a periodic diffeomorphism of the product $T^2 \times [0,1]$ which is isotopic to the identity and whose restriction to each boundary torus $T \times \{i\}$, $i = 0, 1$, is a translation with rational slopes α_0 and α_1 in $H_1(T^2;\mathbb{Z})$. Then $\alpha_0 = \alpha_1$.*

Proof.

By Meeks and Scott [MS, Thm 8.1], see also [BS, Prop. 12], there is a Euclidean product structure on $T^2 \times [0,1]$ preserved by ψ such that ψ acts by translation on each fiber $T \times \{t\}$ with rational slope α_t. By continuity, the rational slopes α_t are constant. □

Now the the following claim shows that the actions of g_W and g_V can be glued on T.

Claim 3. *The translations $g_V|_T$ and $g_W|_T$ have the same slope in $H_1(T^2;\mathbb{Z})$.*

Proof.

Let $\Psi_0 = \{\psi_i,\, i = 1, \ldots, k_0\}$. Let p_i the order of ψ_i and $q_i = \Pi_{j\neq i} p_j$. Then the slopes α_V and α_W of $g_V|_T$ and $g_W|_T$ verify: $q_i\alpha_V = q_i\alpha_W$ for $i = 1, ..., k_0$, by applying Claim 2 to each ψ_i. Since the GCD of the q_i is 1, it follows that $\alpha_V = \alpha_W$. □

This finishes the proof of Lemma 7 and of Proposition 7 when M_f contains an edge. □

To complete the proof of Proposition 7 it remains to consider the case where Γ_f is a single vertex.

Case 2: M_f is a vertex.

Claim 4. *Assume that Γ_f consists of a single vertex and let V denote the corresponding geometric piece. Then the hyperelliptic rotations in Ψ_0 commute on V, up to conjugacy in $Diff^+(M)$.*

Proof.

We consider again two cases according to the geometry of V.

The case where V is Seifert fibred follows once more from Lemma 6.

We consider now the case where V is hyperbolic.

In this case, the hyperelliptic rotations in Ψ act non-trivially on V by isometries of odd prime orders. The restriction $H_{|V} \subset Isom^+(V)$ of the action of the subgroup H that they generate in $Diff^+(M)$ is finite.

If the action on V of the cyclic subgroups generated by two of the hyperelliptic rotations in Ψ are conjugate in $H_{|V}$, one can conjugate the actions in $Diff^+(M)$ to coincide on V, since any diffeomorphism in $H_{|V}$ extends to M.

Then by [BoPa, Lemma 10] these actions must coincide on M, contradicting the hypothesis that the conjugacy classes of cyclic subgroups generated by the hyperelliptic rotations in Ψ are pairwise distinct in $Diff^+(M)$. Hence, the cyclic subgroup generated by the $k \geq 7$ hyperelliptic rotations in Ψ are pairwise not conjugate in the finite group $H_{|V} \subset Isom^+(V)$.

Since the dual graph of the JSJ-decomposition of M is a tree, a boundary torus $T \subset \partial V$ is separating and bounds a component U_T of $M \setminus int(V)$. Since, by hypothesis, Γ_f consists of a single vertex, no boundary component T is setwise fixed by the finite group $H_{|V}$. This means that there is a hyperelliptic rotation $\psi_i \in \Psi$ of odd prime order p_i such that the orbit of U_T under ψ_i is the disjoint union of p_i copies of U_T. In particular U_T projects homeomorphically onto a knot exterior in the quotient $\mathbf{S}^3 = M/\psi_i$. Therefore on each boundary torus $T = \partial U_T \subset \partial V$, there is a simple closed curve λ_T, unique up to isotopy, that bounds a properly embedded incompressible and ∂-incompressible Seifert surface S_T in the knot exterior U_T.

By pinching the surface S_T onto a disc $\mathbf{D}^2$, in each component U_T of $M \setminus int(V)$, one defines a degree-one map $\pi : M \longrightarrow M'$, where M' is the rational homology sphere obtained by Dehn filling each boundary torus $T \subset V$ along the curve λ_T.

For each hyperelliptic rotation ψ_i in Ψ, of odd prime order p_i, the ψ_i-orbit of each component U_T of $M \setminus int(V)$ consists of either one or p_i elements. As a consequence, by [Sa], ψ_i acts equivariantly on the set of isotopy classes of curves $\lambda_T \subset \partial V$. Hence, each ψ_i extends to periodic diffeomorphism ψ_i' of order p_i on M'. Moreover, M' is a $\mathbb{Z}/p_i$-homology sphere, since so is M and $\pi : M \longrightarrow M'$ is a degree-one map. According to Smith theory, if $Fix(\psi')$ is non-empty on M', then ψ_i' is a rotation on M'. To see that $Fix(\psi') \neq \emptyset$ on M' it suffices to observe that either $Fix(\psi) \subset V$ or ψ_i is a rotation of some U_T; in this latter case, ψ_i' must have a fixed point on the disc $\mathbf{D}^2$ onto which the surface S_T is pinched. To show that ψ_i' is hyperelliptic it remains to show that the quotient M'/ψ_i' is homeomorphic to $\mathbf{S}^3$.

Since ψ_i acts equivariantly on the components U_T of $M \setminus int(V)$ and on the set of isotopy classes of curves $\lambda_T \subset \partial V$, the quotient $\mathbf{S}^3 = M/\psi_i$ is obtained from the compact 3-manifold V/ψ_i by gluing knot exteriors (maybe solid tori) to its boundary components, in such a way that the boundaries of the Seifert surfaces of the knot exteriors are glued to the curves $\lambda_T/\psi_i \subset \partial V/\psi_i$.

In the same way, the rotation ψ_i' acts equivariantly on the components $M' \setminus int(V)$ and on the set of isotopy classes of curves $\lambda_T \subset \partial V$. By construction, these components are solid tori, and either the axis of the rotation is contained in V or there exists a unique torus $T \in \partial V$ such that the solid torus glued to T to obtain M' contains the axis. In the latter case, by [EL, Cor. 2.2], the rotation ψ_i' preserves a meridian disc of this solid torus and its axis is a core of W_T. It follows that the images in the quotient M'/ψ_i' of the the solid tori glued to ∂V are again solid tori. Hence M'/ψ_i' is obtained from $\mathbf{S}^3$ by replacing each components of $\mathbf{S}^3 \setminus V/\psi_i$ by a solid torus, in such way that boundaries of meridian discs of the solid tori are glued to the curves $\lambda_T/\psi_i' \subset \partial V/\psi_i'$. It follows that M'/ψ_i' is again $\mathbf{S}^3$.

So far we have constructed a closed orientable 3-manifold M' with a finite subgroup of orientation preserving diffeomorphisms H_V that contains at least seven conjugacy classes of cyclic subgroups generated by hyperelliptic rotations of odd prime orders. Theorem 2 implies that M' must be $\mathbf{S}^3$, and thus by the orbifold theorem [BLP] H_V is conjugate to a finite subgroup of $SO(4)$. In particular the subgroup $H_0 \subset H_V$ generated by the subset Ψ_0 of at least 4 hyperelliptic rotations with pairwise distinct odd prime orders must be solvable. Therefore, by Proposition 3 the induced rotations commute on M' and, by restriction, the hyperelliptic rotations in Ψ_0 commute on V. □

In the final step of the proof we extend the commutativity on M_f to the whole manifold M. The proof of this step is analogous to the one given in [BPZ], since the proof there was not using the homology assumption. We give the argument for the sake of completeness.

Proposition 8. *The $k_0 \geq 4$ hyperelliptic rotations in Ψ_0 commute, up to conjugacy in $Diff^+(M)$, on M.*

Proof.
Let Γ_c be the largest subtree of Γ containing Γ_f, such that, up to conjugacy in $Diff^+(M)$, the rotations in Ψ_0 commute on the corresponding invariant submanifold M_c of M. We shall show that $\Gamma_c = \Gamma$. If this is not the case, we can choose an edge contained in Γ corresponding to a boundary torus T of M_c. Denote by U_T the submanifold of M adjacent to T but not containing M_c and by $V_T \subset U_T$ the geometric piece adjacent to T.

Let $H_0 \subset Diff^+(M)$ be the group of diffeomorphisms of M generated by the set of geometric hyperelliptic rotations $\Psi_0 = \{\psi_i, \, i = 1, \dots, k_0\}$. Since the rotations in Ψ_0 commute on M_c and have pairwise distinct odd prime orders, the restriction of H_0 on on M_c is a cyclic group with order the product of the orders of the rotations. Since $\Gamma_f \subset \Gamma_c$, the H_0-orbit of T cannot be reduced to only one element. Moreover each rotation $\psi \in \Psi_0$ either fixes T or acts freely on the orbit of T since its order is prime.

If no rotation in Ψ_0 leaves T invariant, the H_0-orbit of T contains as many elements as the product of the orders of the rotations, for they commute on M_c. In particular, only the identity (which extends to U) stabilises a torus in the H_0-orbit of T. Note that all components of ∂M_c are in the H_0-orbit of T and bound a manifold homeomorphic to U_T.

Since the rotation ψ_i acts freely on the H_0-orbit of U_T, U_T is a knot exterior in the quotient $M/\psi_i = \mathbf{S}^3$. Hence there is a well defined meridian-longitude system on $T = \partial U_T$ and also on each torus of the H_0-orbit of T. This set of meridian-longitude systems is cyclically permuted by each ψ_i and thus equivariant under the action of H_0.

Let M_c/H_0 be the quotient of M_c by the induced cyclic action of H_0 on M_c. Then there is a unique boundary component $T' \subset \partial(M_c/H_0)$ which is the image of the H_0-orbit of T. We can glue a copy of the knot exterior U_T to M_c/H_0 along T' by identifying the image of the meridian-longitude system on ∂U_T with the projection on T' of the equivariant meridian-longitude systems on the H_0-orbit of T. Denote by N the resulting manifold. For all $i = 1, \dots, k_0$, consider the cyclic (possibly branched) cover of N of order $q_i = \prod_{j \neq i} p_j$ which is induced by the cover $\pi_i : M_c/\psi_i \longrightarrow M_c/H_0$. This makes sense because $\pi_1(T') \subset \pi_{i*}(\pi_1(M_c/\psi_i))$. Call $\tilde{N}_i$ the total space of such covering. By construction it follows that $\tilde{N}_i$ is the quotient $(M_c \cup H_0 \cdot U_T)/\psi_i$. This implies that the ψ_i's commute on $M_c \cup H_0 \cdot U_T$ contradicting the maximality of Γ_c.

We can thus assume that some rotations fix T and some do not. Since all rotations commute on M_c and have pairwise distinct odd prime orders, we see that the orbit of T consists of as many elements as the product of the orders of the rotations which do not fix T and each element of the orbit is fixed by the rotations which leave T invariant. The rotations which fix T commute on the orbit of V_T according to Claim 1 and Lemma 7, and form a cyclic group generated by, say, γ. The argument for the previous case shows that the rotations acting freely on the orbit of T commute on the orbit of U_T and thus on the orbit of V_T, and form again a cyclic group generated by, say, η. To reach a contradiction to the maximality of M_c, we shall show that γ, after perhaps some conjugacy, commutes with η on the H_0-orbit of V_T, in other words that γ and $\eta\gamma\eta^{-1}$ coincide on $H_0 \cdot V_T$. Since η acts freely and transitively on the H_0-orbit of V_T there is a natural and well-defined way to identify each element of the orbit $H_0 \cdot V_T$ to V_T itself. Note that this is easily seen to be the case if V_T is hyperbolic: this follows from Claim 1 and Claim 2. We now consider the case when V_T is Seifert fibred.

Claim 5. *Assume that V_T is Seifert fibred and that the restriction of γ induces a non-trivial action on the base of V_T. Then γ induces a non-trivial action on the base of each component of the H_0-orbit of V_T. Moreover, up to conjugacy on $H_0 \cdot V_T \setminus V_T$ by diffeomorphisms which are the identity on $H_0 \cdot T$ and extend to M , we can assume that the restrictions of γ to these components induce the same permutation of their boundary components and the same action on their bases.*

Proof.

By hypothesis γ and $\eta\gamma\eta^{-1}$ coincide on ∂M_c. The action of γ on the base of V_T is non-trivial if and only if its restriction to the boundary circle of the base corresponding to the fibres of the torus T is non-trivial. Therefore the action of γ is non-trivial on the base of each component of $H_0 \cdot V_T$.

By Lemma 6 and taking into account the fact that V_T is a geometric piece in the JSJ-decomposition of the knot exterior U_T, the only situation in which the action of γ on the base of V_T is non-trivial is when the base of V_T consists of a disc with p holes, where p is the order of one of the rotations that generate γ, the only one whose action is non-trivial on the base of the fibration. Moreover, the restriction of γ to the elements of $H_0 \cdot V_T$ cyclically permutes their boundary components which are not adjacent to M_c. Up to performing Dehn twists, along vertical tori inside the components of $H_0 \cdot V_T \setminus V_T$, which permute the boundary components, we can assume that the restriction of γ induces the same cyclic permutations on the boundary components of each element of the orbit $H_0 \cdot V_T$. We only need to check that Dehn twists permuting two boundary components extend to the whole manifold M. This follows from the fact that the manifolds adjacent to these components are all homeomorphic and that Dehn twist act trivially on the homology of the boundary.

Since the actions of the restrictions of γ on the bases of the elements of $H_0 \cdot V_T$ are combinatorially equivalent, after perhaps a further conjugacy by an isotopy, the different restrictions can be chosen to coincide on the bases. □

By Claim 1 and Claim 5 we can now deduce that the restrictions of γ and $\eta\gamma\eta^{-1}$ to the H_0-orbit of V_T commute, up to conjugacy of γ which is the identity on the H_0-orbit of T. Since γ and $\eta\gamma\eta^{-1}$ coincide on this H_0-orbit of T, we can conclude that they coincide on the H_0-orbit of V_T. This finishes the proof of Proposition 8. □

Since there are at least four hyperelliptic rotations with paiwise distinct odd prime orders in Ψ_0, Proposition 6 is consequence of Proposition 8 and Proposition 1, like in the solvable case. □

Remark 10. As we have seen, the strategy to prove that an irreducible manifold M with non-trivial JSJ-decomposition cannot admit more than six conjugacy classes of subgroups generated by hyperelliptic rotations of odd prime order inside $Diff^+(M)$ consists in modifying by conjugacy any given set of hyperelliptic rotations so that the new hyperelliptic rotations commute pairwise. Note that this strategy cannot be carried out in genral if the orders are not pairwise coprime (see, for instance, [BoPa, Section 4.1] where the case of two hyperelliptic rotations of the same odd prime order, generating non conjugate subgroups, is considered). Similarly, for hyperelliptic rotations of arbitrary orders > 2 lack of commutativity might arise on the Seifert fibred pieces of the decomposition, as it does for the Seifert fibred manifolds constructed in Proposition 5.

7 Appendix: non-free finite group actions on rational homolgy spheres

In this section we will show that every finite group G admits a faithful action by orientation preserving diffeomorphisms on some rational homology sphere so that some elements of G have non-empty fixed-point sets.

Cooper and Long's construction of G-actions on rational homolgy spheres in [CL] consists in starting with a G-action on some 3-manifold and then modifying the original manifold, notably by Dehn surgery, so that the new manifold inherits a G-action but has "smaller" rational homology. Since their construction does not require that the G-action is free, it can be used to prove the existence of non-free G-actions. We will thus start by exhibiting non-free G-actions on some 3-manifold before pointing out what need to be taken into account in this setting in order for Cooper and Long's construction to work.

Since every cyclic group acts as a group of rotations of $\mathbf{S}^3$, for simplicity we will assume that G is a finite non-cyclic group.

Claim 6. *Let G be a finite non-cyclic group. There is a closed, connected, orientable 3-manifold M on which G acts faitfully, by orientation preserving diffeomorphisms so that there are $g \in G \setminus \{1\}$ with the property that $Fix(g) \neq \emptyset$.*

Proof.

Let $k \geq 2$ and let $\{g_i\}_{1 \leq i \leq k+1}$ be a system of generators for G satisfying the following requirements:

- for all $1 \leq i \leq k+1$, the order of g_i is $n_i \geq 2$;
- $g_{k+1} = g_1 g_2 \cdots g_k$.

Since G is not cyclic these conditions are not difficult to fulfill, and actually they can be fulfilled even in the case when G is cyclic for an appropriate choice of the set $\{g_i\}_{1 \leq i \leq k+1}$.

Consider the free group of rank k that we wish to see as the fundamental group of a $(k+1)$-punctured 2-sphere: each generator x_i corresponds to a loop around a puncture of the sphere so that a loop around the $k+1$st puncture corresponds to the element $x_1 x_2 \dots x_k$. Build an orbifold $\mathcal{O}$ by compactifying the punctured-sphere with cone points so that the ith puncture becomes a cone point of order n_i. The resulting orbifold has (orbifold) fundamental group with the following presentation:

$$\langle x_1, x_2, \dots, x_k, x_{k+1} \mid \{x_i^{n_i}\}_{1 \le i \le k+1},\ x_1 \dots x_k x_{k+1}^{-1} \rangle.$$

Clearly this group surjects onto G. Such surjection gives rise to an orbifold covering $\Sigma \longrightarrow \mathcal{O}$, where Σ is an orientable surface on which G acts in such a way that each element g_i has non-empty fixed-point set. One can consider the 3-manifold $\Sigma \times \mathbf{S}^1$: the action of G on Σ extends to a product action of G on $\Sigma \times \mathbf{S}^1$ which is trivial on the $\mathbf{S}^1$ factor. □

To be able to repeat Cooper and Long's construction it is now easy to observe that it is always possible to choose G-equivariant families of simple closed curves in M so that they miss the fixed-point sets of elements of G and either their homology classes generate $H_1(M; \mathbb{Q})$ (so that the hypothesis of [CL, Lemma 2.3] are fulfilled when we choose X to be the exterior of such families) or the family is the G-orbit of a representative of some prescribed homology class (as in the proof of [CL, Proposition 2.5]).

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References

[A] M. Aschbacher, *Finite groups theory*, Cambridge studies in advanced mathematics **10**, Cambridge University Press, 1986.

[BMP] M. Boileau, S. Maillot, and J. Porti, *Three-dimensional orbifolds and their geometric structures*, Panoramas et Synthèses **15**, Société Mathématique de France, Paris, 2003.

[BLP] M. Boileau, B. Leeb and J. Porti, *Geometrization of* 3*-dimensional orbifolds*, Annals of Math. **162**, (2005), 195-290.

[BOt] M. Boileau and J.P. Otal. *Scindements de Heegaard et groupe des homéotopies des petites variétés de Seifert*, Invent. Math. **106**, (1991), 85-107.

[BoPa] M. Boileau and L. Paoluzzi, *On cyclic branched coverings of prime knots*, J. Topol. **1**, (2008), 557-583.

[BPZ] M. Boileau, L. Paoluzzi and B. Zimmermann, *A characterisation of* $\mathbf{S}^3$ *among homology spheres*, The Zieschang Gedenkschrift. Geom. Topol. Monogr. **14**, Geom. Topol. Publ., Coventry, (2008), 83-103.

[BoP] M. Boileau and J. Porti, *Geometrization of 3-orbifolds of cyclic type*, Astérisque Monograph, **272**, 2001.

[BS] F. Bonahon and L. Siebenmann, *The characteristic splitting of irreducible compact 3-orbifolds*, Math. Math. **278**, (1987), 441-479.

[BZH] G. Burde and H. Zieschang, M. Heusener, *Knots*, Third edition, De Gruyter Studies in Mathematics, 5. De Gruyter, Berlin, 2014.

[CL] D. Cooper and D. D. Long, *Free actions of finite groups on rational homology* 3*-spheres*, Topology Appl. **101**, (2000), 143148.

[CHK] D. Cooper, C. Hodgson, and S. Kerckhoff, *Three-dimensional orbifolds and cone-manifolds*, MSJ Memoirs **5**, 2000.

[DL] J. Dinkelbach and B. Leeb, *Equivariant Ricci flow with symmetry and applications to finite group actions on 3-manifolds*, Geom. Topol. **13**, (2009), 11291173.

[Dun] W. D. Dunbar, *Geometric Orbifolds*, Rev. Mat. Univ. Comp. Madrid, **1**, 1998, 67-99.

[EL] A. L. Edmonds and C. Livingston, *Group actions on fibered three-manifolds*, Comm. Math. Helv. **58**, (1983), 529-542.

[Gr] J. E. Greene, *Lattices, graphs, and Conway mutation.* Invent. Math. **192** (2013), no. 3, 717750.

[Go] C. McA. Gordon, *Some aspects of classical knot theory*, Knot Theory, Proceedings, Plans-sur-Bex, Switzerland (J.C. Hausmann ed.), Lect. Notes Math. **685**, (1977), Springer-Verlag, 1-60.

[GLS] D. Gorenstein, R. Lyons, and R. Solomon, *The classification of the finite simple groups, Number 3*, Math. Surveys Monogr. **40.3**, Amer. Math. Soc., Providence, RI, 1998.

[Hil] J. Hillman, *Links with infinitely many semifree periods are trivial*, Arch. Math. **42**, (1984), 568-572.

[H] B. Huppert, *Endlichen Gruppen I*, Springer-Verlag, New York, 1968.

[KL] B. Kleiner and J. Lott. *Local Collapsing, Orbifolds, and Geometrization*, Astérisque Monograph **365**, 2014.

[JS] W. H. Jaco and P. B. Shalen, *Seifert fibred spaces in 3-manifolds*, Mem. Amer. Math. Soc. **220**, 1979.

[J] K. Johannson, *Homotopy equivalence of 3-manifolds with boundary*, Lecture Notes in Math. **761**, Springer-Verlag, Berlin, 1979.

[Ko] S. Kojima, *Determining knots by branched covers*, in Low Dimensional Topology and Kleinian groups, London Math. Soc. Lecture Note Ser. **112**, Cambridge Univ. Press (1986), 193-207.

[Mec1] M. Mecchia, *How hyperbolic knots with homeomorphic cyclic branched coverings are related*, Topology Appl. **121**, (2002), 521-533.

[Mec2] M. Mecchia, *Finite groups acting on 3-manifolds and cyclic branched coverings of knots*, The Zieschang Gedenkschrift. Geom. Topol. Monogr. **14**, Geom. Topol. Publ., Coventry, (2008), 393-416.

[MSY] W. H. Meeks, L. Simon, and S.-T. Yau, *Embedded minimal surfaces, exotic spheres, and manifolds with positive Ricci curvature*, Ann. of Math. **116** (1983), 621659.

[MS] W. H. Meeks and P. Scott, *Finite group actions on 3-manifolds*, Invent. Math. **86**, (1986), 287-346.

[Mon1] J. M. Montesinos, *Variedades de Seifert que son recubridores ciclicos ramificados de dos hojas*, Bol. Soc. Mat. Mexicana **18**, (1973), 1-32.

[Mon2] J. M. Montesinos, *Revêtements ramifiés de noeuds, espaces fibrés de Seifert et scindements de Heegaard*, Publicaciones del Seminario Mathematico Garcia de Galdeano, Serie II, Seccion 3, (1984).

[MonW] J. M. Montesinos and W. Whitten *Constructions of two-fold branched covering spaces.* Pac. J. Math. **125**, 415446 (1986)

[NS] W. Neumann and G. Swarup, *Canonical decompositions of* 3*-manifolds*, Geom. Topol. **1**, (1998), 21-40.

[Or] P. Orlik, *Seifert manifolds*, Lecture Notes in Mathematics **291**, Springer, 1972.

[Re] M. Reni, *On π-hyperbolic knots with the same 2-fold branched coverings*, Math. Ann. **316**, (2000), no. 4, 681697

[Sa] M. Sakuma, *Uniqueness of symmetries of knots*, Math. Z. **192**, (1986), 225-242.

[Sco] P. Scott, *Homotopy implies isotopy for some Seifert fibre spaces*, Topology **24**, (1985), 341-351.

[Su] M. Suzuki, *Group theory II*, Grundlehren Math. Wiss. **248**, Springer-Verlag, New York, 1982.

[V] O. Ja. Viro *Nonprojecting isotopies and knots with homeomorphic coverings*, Zap. Nau. Semin. POMI **66**, 133147, 207208, Studies in topology, II (1976)

[Wa] F. Waldhausen, *Eine Klasse von 3-dimensionalen Mannigfaltigkeiten. I, II*, Invent. Math. **3**, (1967), 308333; ibid. **4** (1967), 87117.

[Z1] B. Zimmermann, *On the Hantzsche-Wendt Manifold*, Mh. Math. **110**, (1990) , 321-327.

[Z2] B. Zimmermann, *On hyperbolic knots with homeomorphic cyclic branched coverings*, Math. Ann. **311**, (1998), 665-673.

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