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Cones and matrix invariance: a short survey

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Abstract. *In this survey we collect and revisit some notions and results regarding the theory of cones and matrices admitting an invariant cone. The aim is to provide a self-contained treatment to form a convenient background to further researches. In doing this, we introduce some new intermediate concepts and propose several new proofs.*

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1. Introduction

In the framework of Linear Algebra, the description of the eigenvalues of an endomorphism of a vector space is one of the most classical problems.

A sufficient condition for the existence of a *leading* eigenvalue equal to the spectral radius was determined in 1907, in the real and finite dimensional case, by the mile-stone Theorem of Perron [11, 12], giving an affirmative answer as far as a positive matrix (associated to the endomorphism) is concerned.

In 1912, Frobenius [5] extended this result to irreducible nonnegative matrices. From then, the so called *Perron–Frobenius Theory* played a very important role within matrix theory, leading to several applications in Probability, Dynamical Systems, Economics, etc.

In the subsequent decades, this theory admitted a wide development, together with several generalizations which, in turn, have been applied to other branches of Mathematics and to applied sciences such as Physics, Social Sciences, Biology, etc.

The observation that a real positive $d \times d$ matrix corresponds to an endomorphism of $\mathbb{R}^d$ mapping the positive orthant into itself has naturally led to investigate endomorphisms admitting an *invariant cone* (the natural generalization of the orthant). In this context we mention, in particular, the generalization of the Perron–Frobenius Theorem due to Birkhoff [1] and the work by Vandergraft [17], where necessary and sufficient conditions on a matrix to have an invariant cone are given.

In this survey we collect some known notions and revisit several results regarding the theory of cones and matrices admitting an invariant cone.

The aim is to provide a convenient background to our papers [3, 2].

In doing this, on the one hand we introduce some new intermediate concepts. On the other hand, in order to provide a self-contained treatment, we fill in some gaps and, hence, we propose several new proofs.

2. Notation

We refer to $\mathbb{R}^d$ as a real vector space endowed with the Euclidean scalar product, denoted by $x^T y$ for any $x, y \in \mathbb{R}^d$. The metric and topological structures of this Euclidean space are induced by this pairing.

In this framework, if U is a nonempty subset of $\mathbb{R}^d$, we denote by $\mathrm{cl}(U)$ its *closure*, by $\mathrm{conv}(U)$ its *convex hull*, by $\mathrm{int}(U)$ its *interior* and by ∂U its *boundary* as a subset of $\mathbb{R}^d$. We also denote by $\mathrm{span}(U)$ the smallest vector subspace containing U. Finally, we set

$$\mathbb{R}_+ U := \left\{ \alpha x \;\middle|\; \alpha \geq 0 \text{ and } x \in U \right\}$$

and

$$U^\perp := \left\{ h \in \mathbb{R}^d \;\middle|\; h^T x = 0 \text{ for all } x \in U \right\}$$

denotes the *orthogonal set* of U.

In particular, if H is a (vector) hyperplane of $\mathbb{R}^d$ (i.e., a linear subspace of $\mathbb{R}^d$ of dimension $d-1$), then $H = \{h\}^\perp$ for a suitable vector $h \in \mathbb{R}^d \setminus \{0\}$, unique up to a scalar.

The hyperplane H splits $\mathbb{R}^d$ into two parts, say the *positive* and the *negative* semi-space

$$S_+^h := \{x \in \mathbb{R}^d \mid h^T x \geq 0\} \qquad \text{and} \qquad S_-^h := \{x \in \mathbb{R}^d \mid h^T x \leq 0\},$$

respectively. Clearly,

$$\mathrm{int}\left(S_+^h\right) = \{x \in \mathbb{R}^d \mid h^T x > 0\} \qquad \text{and} \qquad \mathrm{int}\left(S_-^h\right) = \{x \in \mathbb{R}^d \mid h^T x < 0\},$$

$$\mathbb{R}^d = \mathrm{int}\left(S_+^h\right) \cup H \cup \mathrm{int}\left(S_-^h\right) \qquad \text{and} \qquad \partial S_+^h = \partial S_-^h = H.$$

3. Cones and duality

The notion of *proper cone* is standard enough in the literature (see, e.g., Tam [16], Schneider and Tam [14] and Rodman, Seyalioglu and Spitkovsky [13]). The more general notion of *cone* is, instead, not universally shared: accordingly to the various authors, it involves a variable subset (or even all, see Schneider and Vidyasagar [15]) of the requirements for proper cones.

In this survey we shall deal with proper cones, as defined in the standard way, and with cones that verify a particular subset of the possible properties. We shall also find it useful to consider a weaker instance of our definition of cone, that we refer to as *quasi-cone*.

DEFINITION 3.1. *Let K be a nonempty closed and convex set of* $\mathbb{R}^d$ *and consider the following conditions:*

c1) $\mathbb{R}_+ K \subseteq K$ *(i.e., K is* positively homogeneous*);*

c2) $K \cap -K = \{0\}$ *(i.e., K is* pointed *or* salient*);*

c3) $\operatorname{span}(K) = \mathbb{R}^d$ *(i.e., K is* full *or* solid*).*

We say that K is a quasi-cone *if it satisfies (c1). If, in addition, it satisfies (c2), we say that K is a* cone. *Finally, if it satisfies all the above properties, we say that K is a* proper cone.

If a quasi-cone K is not solid, we also say that it is a degenerate quasi-cone.

The most known example of proper cone is the *positive orthant*

$$\mathbb{R}^d_+ = \{x \in \mathbb{R}^d \mid x_i \geq 0,\ i = 1, \ldots, d\}.$$

In this section we recall some of the basic properties of quasi-cones. Most is well known and we refer the reader, e.g., to Fenchel [4], Schneider and Vidyasagar [15] and Tam [16].

The following invariants of a quasi-cone measure, in some sense, how far it is from being either pointed or full, respectively.

DEFINITION 3.2. *For any quasi-cone K, we denote by $L(K)$ the largest vector subspace included in K, called the* lineality space *of K, and by $l(K)$ the dimension of $L(K)$.*

Moreover, we denote by $d(K)$ the dimension of $\operatorname{span}(K)$, *called the* (linear) dimension *of K.*

REMARK 3.3: If K is a quasi-cone, it is clear that:

(i) $L(K) = K \cap -K$;

(ii) K is pointed if and only if $l(K) = 0$;

(iii) K is solid if and only if $d(K) = d$ or, equivalently, if and only if $\operatorname{int}(K) \neq \emptyset$.

If K is degenerate, then it is solid in the linear space $\mathrm{span}(K) \cong \mathbb{R}^{d(K)}$. So we can give the following definition.

Definition 3.4. *If K is a quasi-cone, its interior as a subset of* $\mathrm{span}(K)$ *is called the* relative interior *of K and is denoted by* $\mathrm{int}_{rel}(K)$.

Note that, if K is a quasi-cone, then $l(K) \leq d(K)$ and the equality holds if and only if one of the following equivalent conditions is satisfied:

(i) $L(K) = K$;

(ii) $K = \mathrm{span}(K)$;

(iii) K is a linear subspace;

(iv) $\mathrm{int}_{rel}(K) = K$.

The next notion is well known.

Definition 3.5. *Given a hyperplane H, we say that a nonempty positively homogeneous set $U \subset \mathbb{R}^d$ is* supported by H *(or, briefly, H-*supported *) if*

$$U \subseteq S_+^h \quad \text{or} \quad U \subseteq S_-^h.$$

Moreover, we say that U is strictly supported by H *(or, briefly,* strictly H-supported *) if*

$$U \setminus \{0\} \subseteq \mathrm{int}\left(S_+^h\right) \quad \text{or} \quad U \setminus \{0\} \subseteq \mathrm{int}\left(S_-^h\right).$$

Remark 3.6: Let K be a cone and H be a hyperplane. Then K is strictly H-supported if and only if $K \cap H = \{0\}$.

Proposition 3.7. *If $K \neq \mathrm{span}(K)$ is a quasi-cone of $\mathbb{R}^d$, then there exists a hyperplane H which supports K and*

$$H \cap \mathrm{int}_{rel}(K) = \emptyset.$$

Proof. First assume that K is solid. In this case, there exists a hyperplane H which supports K. (see Fenchel [4] (Corollary 1)).

If there exists $v \in H \cap \mathrm{int}(K)$, then we can consider a d-dimensional ball U_v, centered in v and contained in $\mathrm{int}(K)$. Clearly, U_v meets both $\mathrm{int}(S_+^h)$ and $\mathrm{int}(S_-^h)$, against the fact that K is H-supported .

Otherwise, if K is degenerate, let $S := \mathrm{span}(K)$, $s := d(K)$ its dimension and let T be a $(d-s)$-dimensional subspace such that $S \oplus T = \mathbb{R}^d$. Clearly,

K is solid in S and, so, from the previous case, we obtain the existence of a hyperplane V of S which supports K and $V \cap \mathrm{int}_{rel}(K) = \emptyset$. Now set $H := V \oplus T$, so that K is clearly H-supported and

$$H \cap \mathrm{int}_{rel}(K) = H \cap S \cap \mathrm{int}_{rel}(K) = V \cap \mathrm{int}_{rel}(K) = \emptyset,$$

as required. □

DEFINITION 3.8. *Given a nonempty set $U \subset \mathbb{R}^d$, the intersection of all the quasi-cones containing U (i.e., the smallest quasi-cone containing U) is called the* quasi-cone generated by U *and we denote it by* $\mathrm{qcone}(U)$.

Note that, while $\mathrm{qcone}(U)$ is defined for any set U, the smallest cone containing U may well not exist. Anyway, if it does exist, then it coincides with $\mathrm{qcone}(U)$.

DEFINITION 3.9. *Consider a nonempty set $U \subset \mathbb{R}^d$ and assume that* $\mathrm{qcone}(U)$ *is a cone. Then we denote it by* $\mathrm{cone}(U)$ *and call it the* cone generated by U.

The quasi-cone generated by U can be represented explicitly in formula by the aid of the following properties, whose proofs are straightforward.

PROPOSITION 3.10. *Let $U \subset \mathbb{R}^d$ be a nonempty set. Then*

(i) $\mathrm{conv}\,(\mathbb{R}_+ U) = \mathbb{R}_+ \mathrm{conv}\,(U)$;

(ii) $\mathrm{cl}\,(\mathbb{R}_+ U) \supseteq \mathbb{R}_+ \mathrm{cl}\,(U)$ *and, consequently,* $\mathrm{cl}\,(\mathbb{R}_+ U) = \mathbb{R}_+ \mathrm{cl}\,(\mathbb{R}_+ U)$;

(iii) $\mathrm{cl}\,(\mathrm{conv}\,(U)) \supseteq \mathrm{conv}\,(\mathrm{cl}\,(U))$ *and, consequently,* $\mathrm{cl}\,(\mathrm{conv}\,(U)) = \mathrm{conv}\,(\mathrm{cl}\,(\mathrm{conv}(U)))$.

COROLLARY 3.11. *For any nonempty set $U \subset \mathbb{R}^d$, we have*

$$\mathrm{qcone}(U) = \mathrm{cl}\,(\mathrm{conv}\,(\mathbb{R}_+ U)) = \mathrm{cl}\,(\mathbb{R}_+ \mathrm{conv}\,(U))\,. \tag{1}$$

Proof. The second equality in (1) is obtained just by taking the closure of both sides of (i) in Proposition 3.10.

Concerning the first equality, note that $\mathrm{cl}\,(\mathrm{conv}\,(\mathbb{R}_+ U))$ contains U, is convex (by (iii) in Proposition 3.10) and positively homogeneous (by (i) and (ii) in Proposition 3.10). Thus, by Definitions 3.1 and 3.8, we obtain $\mathrm{qcone}(U) \subseteq \mathrm{cl}\,(\mathrm{conv}\,(\mathbb{R}_+ U))$.

Conversely, since $\mathrm{qcone}(U)$ is positively homogeneous, $\mathrm{qcone}(U) \supseteq \mathbb{R}_+ U$. Moreover, it is convex and, hence, $\mathrm{qcone}(U) \supseteq \mathrm{conv}\,(\mathbb{R}_+ U)$. The fact that $\mathrm{qcone}(U)$ is also closed completes the proof. □

PROPOSITION 3.12. *A nonempty set* $U \subset \mathbb{R}^d$ *is contained in a closed semispace* S^h_+ *if and only if* $\mathrm{qcone}(U) \neq \mathbb{R}^d$.

Proof. It is clear that $U \subseteq S^h_+$ if and only if $\mathrm{qcone}(U) \subseteq S^h_+$. On the other hand, by Proposition 3.7, this condition is equivalent to $\mathrm{qcone}(U) \neq \mathbb{R}^d$. □

The notion of *duality* is essential in the study of cones. Now we summarize a few basic definitions and properties.

DEFINITION 3.13. *Let* U *be a nonempty set of* $\mathbb{R}^d$. *Then*

$$U^* := \{h \in \mathbb{R}^d \mid h^T x \geq 0 \quad \forall x \in U\}$$

is called the dual set *of* U. *By convention, we also define* $\emptyset^* := \mathbb{R}^d$.

REMARK 3.14: If U is a subset of $\mathbb{R}^d$, then it is clear that $U \subseteq S^h_+$ if and only if $h \in U^* \setminus \{0\}$.

The proofs of the following relationships are straightforward.

PROPOSITION 3.15. *Let* U *and* V *be nonempty sets of* $\mathbb{R}^d$. *Then we have:*

(i) $U \subseteq U^{**}$;

(ii) $U \subseteq V$ *implies* $U^* \supseteq V^*$;

(iii) $(U \cup V)^* = U^* \cap V^*$;

(iv) $(U \cap V)^* \supseteq U^* \cup V^*$.

REMARK 3.16: Note that $\{0\}^* = \mathbb{R}^d$, $(\mathbb{R}^d)^* = \{0\}$ and, if $x \in \mathbb{R}^d \setminus \{0\}$, then

$$\{x\}^* = \{h \in \mathbb{R}^d \mid h^T x \geq 0\} = S^x_+$$

is the positive semi-space determined by x. Consequently, if U is a nonempty subset of $\mathbb{R}^d$, then

$$U^* = \bigcap_{x \in U} S^x_+ \, .$$

Hence, U^* is closed, convex and positively homogeneous, i.e., U^* is a quasi-cone.

The above observation shows that the notion of dual of a set is deeply related to that of quasi-cone, as is evident also from the following fact.

PROPOSITION 3.17. *Let* U *be a subset of* $\mathbb{R}^d$ *and* U^* *be its dual set. Then*

$$U^* = (\mathrm{qcone}(U))^*.$$

Proof. Since for any $V \subseteq \mathbb{R}^d$ we easily have $V^* = (\mathrm{cl}(V))^*$, $V^* = (\mathrm{conv}(V))^*$ and $V^* = (\mathbb{R}_+ V)^*$, the claim follows immediately from (1). □

DEFINITION 3.18. *If K is a quasi-cone of $\mathbb{R}^d$, the set*

$$K^* = \{h \in \mathbb{R}^d \mid h^T x \geq 0 \quad \forall x \in K\}$$

is called the dual quasi-cone *of K.*

As we saw in Proposition 3.15, $*$ is not completely a "geometric duality" on the subsets of $\mathbb{R}^d$. Namely, even if it is compatible with the union and contravariant with respect to the inclusion, a generic subset is not reflexive. Besides the category of vector subspaces of $\mathbb{R}^d$, that of quasi-cones fulfils the reflexivity, too. To this purpose, we recall that, for any quasi-cone K, we have

$$K^{**} = K \tag{2}$$

(see [4], Corollary to Theorem 3). Consequently, using the general implication in Proposition 3.15-(ii), we obtain

$$K^{(1)} \subseteq K^{(2)} \iff (K^{(1)})^* \supseteq (K^{(2)})^* \tag{3}$$

for any pair $K^{(1)}$ and $K^{(2)}$ of quasi-cones.

REMARK 3.19: Let $K \neq \mathbb{R}^d$ be a quasi-cone. Then, thanks to Proposition 3.7, it is supported by some hyperplane H. As observed in Remark 3.14, this fact is equivalent to $K^* \neq \{0\}$.

The following key-fact can be found in Fenchel [4] (Theorem 5 and its Corollary).

PROPOSITION 3.20. *Let K be a quasi-cone of $\mathbb{R}^d$. Then*

$$d(K) + l(K^*) = d \qquad \text{and} \qquad d(K^*) + l(K) = d. \tag{4}$$

Remark 3.3 and Proposition 3.20 immediate yield the next consequence.

COROLLARY 3.21. *Let K be a quasi-cone. Then K is pointed if and only if K^* is solid and, dually, K^* is pointed if and only if K is solid. In particular, K is a proper cone if and only if K^* is a proper cone.*

Moreover, $K = \mathrm{span}(K)$ if and only if $K^* = \mathrm{span}(K^*)$.

This observation allows us to describe the lineality space of a quasi-cone in terms of its dual quasi-cone.

Lemma 3.22. *Let K be a quasi-cone. Then*

$$L(K) = (K^*)^\perp. \tag{5}$$

Proof. Let us first show that $L(K) \subseteq (K^*)^\perp$. To this purpose, let $h \in K^*$. Since $L(K) \subseteq K$, for each $z \in L(K)$ we have $h^T z \geq 0$. Since $L(K)$ is a vector space, it also contains $-z$ and, hence, $h^T(-z) \geq 0$. Therefore, $h^T z = 0$ for each $z \in L(K)$ and, so, $L(K) \subseteq \{h\}^\perp$.

To prove the equality, it is enough to observe that $(K^*)^\perp = (\mathrm{span}(K^*))^\perp$. Hence, $\dim((K^*)^\perp) = d - d(K^*) = l(K)$, where the second equality follows from (4). □

Proposition 3.23. *If $K \neq \mathrm{span}(K)$ is a quasi-cone, then*

$$L(K) \cap \mathrm{int}_{rel}(K) = \emptyset.$$

Proof. On one hand, by Proposition 3.7, there exists a hyperplane H supporting K such that $H \cap \mathrm{int}_{rel}(K) = \emptyset$. On the other hand, by Lemma 3.22 and Remark 3.19, we have that $L(K) \subseteq H$. □

Lemma 3.24 ([4], Theorem 12). *If K is a quasi-cone and $h \in K^* \setminus \{0\}$, then*

$$h \in \mathrm{int}_{rel}(K^*) \iff K \cap \{h\}^\perp = L(K). \tag{6}$$

Note that, if $K = \mathrm{span}(K)$, then it is clear that $K^* = K^\perp$ and Lemma 3.24 just says that $K \cap \{h\}^\perp = K$ for each $h \in K^* \setminus \{0\}$.

Now we are in a position to prove a stronger version of Proposition 3.7.

Proposition 3.25. *Let K be a quasi-cone. Then it is a cone if and only if it is strictly supported by some hyperplane H.*

Proof. Assume that K is a cone. So, by Corollary 3.21, its dual K^* is solid. Then just take $h \in \mathrm{int}(K^*)$ and set $H = \{h\}^\perp$. By Lemma 3.24, we have $K \cap H = \{0\}$ and, hence, by Remark 3.6, K is strictly H-supported .

Conversely, if K is strictly H-supported for some H, then $K \cap H = \{0\}$. Thus, $K \cap -K = \{0\}$ and, by Remark 3.3, K is pointed. □

The above discussion allows us to show the inclusions opposite to (ii) and (iii) of Proposition 3.10 hold in some particular cases.

LEMMA 3.26. *Let X be a bounded subset of $\mathbb{R}^d$. Then* $\mathrm{conv}\,(\mathrm{cl}\,(X))$ *is closed and, hence,*

$$\mathrm{cl}(\mathrm{conv}(X)) = \mathrm{conv}(\mathrm{cl}(X)). \tag{7}$$

In addition, if $0 \notin \mathrm{cl}(X)$*, then also* $\mathbb{R}_+\mathrm{cl}\,(X)$ *is closed and, hence,*

$$\mathrm{cl}\,(\mathbb{R}_+X) = \mathbb{R}_+\mathrm{cl}\,(X)\,. \tag{8}$$

Proof. The first claim is well known. Hence, since $\mathrm{conv}(X) \subseteq \mathrm{conv}(\mathrm{cl}(X))$, we have that $\mathrm{cl}(\mathrm{conv}(X)) \subseteq \mathrm{conv}(\mathrm{cl}(X))$. Therefore, equality (7) follows from Proposition 3.10-(iii).

Now let $Y := \mathrm{cl}(X)$ and let $x \in \partial(\mathbb{R}_+Y)\setminus\{0\}$. Then there exists a sequence $(x_n)_n \subset \mathbb{R}_+Y$ converging to x and, so, there exists $M > 0$ such that definitively

$$\| x_n \| \leq M.$$

On the other hand, we can write

$$x_n = \lambda_n a_n$$

where $\lambda_n \in \mathbb{R}_+$ and $a_n \in Y$ for all n.

Since Y is compact, the sequence $(a_n)_n$ (or a suitable subsequence) converges to a point, say a, of Y. Necessarily, $a \neq 0$ because $0 \notin Y$. Thus, there exists $\mu > 0$ such that definitively

$$\| a_n \| \geq \mu > 0.$$

Since $\| x_n \| = |\lambda_n| \; \| a_n \|$, we then obtain definitively

$$\lambda_n \leq M/\mu.$$

Therefore, the sequence $(\lambda_n)_n$ (or a suitable subsequence) converges to a certain $\lambda \in \mathbb{R}_+$.

Finally, we obtain that (a suitable subsequence of) $(x_n)_n$ converges to λa. This implies that $x = \lambda a \in \mathbb{R}_+Y$. So $\mathbb{R}_+Y$ is closed. By using Proposition 3.10-(ii), similarly as before (8) follows. $\square$

PROPOSITION 3.27. *Let $X \subset \mathbb{R}^d$ be positively homogeneous and such that* $\mathrm{cl}(X)$ *is strictly supported by some hyperplane H. Then*

$$\mathrm{cl}(\mathrm{conv}(X)) = \mathrm{conv}(\mathrm{cl}(X)).$$

Proof. Denote by S the unit d-sphere of $\mathbb{R}^d$ and consider the compact set $\mathrm{cl}(X) \cap S$. Therefore, by Lemma 3.26, $\mathrm{conv}(\mathrm{cl}(X) \cap S)$ is closed.

Moreover, observe that $0 \notin \mathrm{conv}(\mathrm{cl}(X) \cap S)$ since $\mathrm{cl}(X)$ is strictly H-supported by assumption. Thus, by the second part of Lemma 3.26, we obtain that $\mathbb{R}_+\mathrm{conv}(\mathrm{cl}(X) \cap S)$ is closed.

On the other hand, $\mathrm{cl}(X)$ is positively homogeneous. Therefore, as is easy to see, $\mathbb{R}_+(\mathrm{cl}(X) \cap S) = \mathrm{cl}(X)$. Hence,

$$\mathrm{conv}(\mathbb{R}_+(\mathrm{cl}(X) \cap S)) = \mathrm{conv}(\mathrm{cl}(X))$$

and, so, Proposition 3.10-(i) yields

$$\mathbb{R}_+\mathrm{conv}(\mathrm{cl}(X) \cap S) = \mathrm{conv}(\mathrm{cl}(X)).$$

Therefore, $\mathrm{conv}(\mathrm{cl}(X))$ is closed and, using Proposition 3.10-(iii), like in the first part of Lemma 3.26 we get the thesis. □

COROLLARY 3.28. *Consider a nonempty set $U \subset \mathbb{R}^d$ and assume that* $\mathrm{qcone}(U)$ *is a cone. Then*

$$\mathrm{cone}(U) = \mathrm{conv}\left(\mathrm{cl}\left(\mathbb{R}_+U\right)\right) = \mathrm{cl}\left(\mathrm{conv}\left(\mathbb{R}_+U\right)\right) = \mathrm{cl}\left(\mathbb{R}_+\mathrm{conv}\left(U\right)\right). \tag{9}$$

Proof. Note first that

$$\mathrm{cl}\left(\mathbb{R}_+U\right) \subseteq \mathrm{cl}\left(\mathrm{conv}\left(\mathbb{R}_+U\right)\right) = \mathrm{cone}(U),$$

where the equality follows from Corollary 3.11. Therefore, $\mathrm{cl}\left(\mathbb{R}_+U\right)$ is strictly supported by some hyperplane H by Proposition 3.25.

Consequently, $\mathbb{R}_+U$ satisfies the assumptions on the set X of Proposition 3.27 which, in turn, gives the second equality in (9).

Finally, (1) gives the third equality. □

A more detailed study of the notion of dual of a quasi-cone leads us to the forthcoming Proposition 3.30.

LEMMA 3.29. *If $K \neq \mathrm{span}(K)$ is a quasi-cone and $h \in \mathbb{R}^d \setminus \{0\}$, then the following conditions are equivalent:*

(i) $h \in K^*$ *and* $K \cap \{h\}^\perp = L(K)$*;*

(ii) $h^T x > 0$ *for all* $x \in K \setminus L(K)$*.*

Proof. $(i) \Rightarrow (ii)$ Since $h \in K^*$, then $h^T x \geq 0$ for all $x \in K$. Now, if $x \in K \setminus L(K)$, then (i) implies that $x \notin \{h\}^\perp$, i.e., $h^T x \neq 0$.

$(ii) \Rightarrow (i)$ By Proposition 3.23 we have that $K \setminus L(K) \supseteq \mathrm{int}_{rel}(K)$ and, hence, the assumption implies that $h^T x > 0$ for all $x \in \mathrm{int}_{rel}(K)$. Therefore, the continuity of the scalar product proves that $h^T x \geq 0$ for all $x \in K$, i.e., $h \in K^*$. In turn, this fact implies that $K \cap \{h\}^\perp \supseteq L(K)$ holds (see (5)). So we are left to show that $K \cap \{h\}^\perp \subseteq L(K)$. If $x \in K$ and $h^T x = 0$, then necessarily $x \notin K \setminus L(K)$ by assumption, and this proves the requested inclusion. □

PROPOSITION 3.30. *Let K be a quasi-cone of $\mathbb{R}^d$. Then we have:*

(i)
$$\mathrm{int}_{rel}(K^*) = \{h \in \mathbb{R}^d \mid h^T x > 0 \quad \forall x \in K \setminus L(K)\}$$
and, if K is a cone, then
$$\mathrm{int}(K^*) = \{h \in \mathbb{R}^d \mid h^T x > 0 \quad \forall x \in K \setminus \{0\}\}.$$

(ii)
$$K^* \setminus L(K^*) = \{h \in \mathbb{R}^d \mid h^T x > 0 \quad \forall x \in \mathrm{int}_{rel}(K)\}$$
and, if K is solid, then
$$K^* \setminus \{0\} = \{h \in \mathbb{R}^d \mid h^T x > 0 \quad \forall x \in \mathrm{int}(K)\}.$$

Proof. (i) The first equality follows immediately from Lemmas 3.24 and 3.29. In particular, if K is a cone, then $L(K) = 0$ and the second equality is also proved.
(ii) It is clear that (i) implies
$$K \setminus L(K) \subseteq \{x \in \mathbb{R}^d \mid h^T x > 0 \quad \forall h \in \mathrm{int}_{rel}(K^*)\}.$$
Conversely, let $x \in \mathbb{R}^d$ be such that $h^T x > 0$ for all $h \in \mathrm{int}_{rel}(K^*)$. Then $x \notin \{h\}^\perp$ and, hence, $x \notin L(K)$ by (5). Moreover, still by the continuity of the scalar product, we also get $h^T x \geq 0$ for all $h \in K^*$. This means that $x \in K^{**} = K$. In this way we have shown that
$$K \setminus L(K) = \{x \in \mathbb{R}^d \mid h^T x > 0 \quad \forall h \in \mathrm{int}_{rel}(K^*)\}.$$
Exchanging the role of K and K^* and applying the reflexivity of the quasi-cones (see (2)), we obtain the requested equality.

Finally, if K is solid, then $L(K^*) = \{0\}$. □

A straightforward consequence of the above proposition follows.

COROLLARY 3.31. *If $K^{(1)}$ and $K^{(2)}$ are quasi-cones, then*
$$K^{(1)} \setminus L(K^{(1)}) \subseteq \mathrm{int}_{rel}(K^{(2)}) \quad \Longrightarrow \quad \mathrm{int}_{rel}((K^{(1)})^*) \supseteq (K^{(2)})^* \setminus L((K^{(2)})^*).$$

The last part of this section is devoted to some properties concerning the quasi-cone generated by a finite union of quasi-cones.

LEMMA 3.32. *Let $K^{(1)}, \dots, K^{(r)}$ be quasi-cones of $\mathbb{R}^d$ and $U := \bigcup_{i=1}^{r} K^{(i)}$. Then*

$$(\mathrm{qcone}\,(U))^* = U^* = \bigcap_{i=1}^{r} \left(K^{(i)}\right)^* .$$

Moreover, the above set, which is a quasi-cone, is $\neq \{0\}$ if and only if U is supported by some hyperplane H.

Proof. The first equality follows from Proposition 3.17 and the second from Proposition 3.15-(iii). Moreover,

$$(\mathrm{qcone}\,(U))^* \neq \{0\} \quad \Longleftrightarrow \quad \mathrm{qcone}\,(U) \neq \mathbb{R}^d,$$

and this is equivalent to U being H-supported (see Proposition 3.12). □

DEFINITION 3.33. *Let $K^{(1)}, \dots, K^{(r)}$ be quasi-cones. Their* sum *is defined as*

$$K^{(1)} + \cdots + K^{(r)} := \{x_1 + \cdots + x_r \mid x_i \in K^{(i)},\ i = 1, \dots, r\}.$$

LEMMA 3.34. *Let $K^{(1)}, \dots, K^{(r)}$ be quasi-cones. Then*

$$K^{(1)} + \cdots + K^{(r)} = \mathrm{conv}(K^{(1)} \cup \cdots \cup K^{(r)}) \tag{10}$$

and

$$\mathrm{cl}(K^{(1)} + \cdots + K^{(r)}) = \mathrm{qcone}(K^{(1)} \cup \ldots \cup K^{(r)}). \tag{11}$$

Proof. Equality (10) proved in Kusraev and Kutateladze [9], 1.1.8.

Equality (11) immediately follows from (10). In fact, since the quasi-cones $K^{(i)}$ are positively homogeneous, equality (1) implies that $\mathrm{qcone}(K^{(1)} \cup \cdots \cup K^{(r)}) = \mathrm{cl}(\mathrm{conv}(K^{(1)} \cup \cdots \cup K^{(r)}))$. □

We recall that the notion of separatedness of two closed convex subsets of $\mathbb{R}^d$ has to be slightly modified (e.g., following Klee [7]) to adapt it to the case of cones.

DEFINITION 3.35. *Two cones $K^{(1)}$ and $K^{(2)}$ of $\mathbb{R}^d$ are said to be* separated *if there exists a hyperplane $H = \{h\}^\perp$ such that*

$$K^{(1)} \setminus \{0\} \subseteq \mathrm{int}\left(S^h_+\right) \quad \text{and} \quad K^{(2)} \setminus \{0\} \subseteq \mathrm{int}\left(S^h_-\right).$$

Moreover, we say that such an H is a separating hyperplane *for $K^{(1)}$ and $K^{(2)}$.*

Let us mention two well know results, the first of which is the "cone version" of a "separation-type" theorem, obtained directly from Klee [7], Theorem 2.7 (see also Holmes [6]).

THEOREM 3.36. *Two cones $K^{(1)}$ and $K^{(2)}$ of $\mathbb{R}^d$ are separated if and only if $K^{(1)} \cap K^{(2)} = \{0\}$.*

In other words, $K^{(1)} \cap -K^{(2)} = \{0\}$ if and only if $K^{(1)} \cup K^{(2)}$ is strictly supported by some hyperplane H. So the next statement immediately comes from Klee [7], Proposition 2.1.

PROPOSITION 3.37. *Let $K^{(1)}$ and $K^{(2)}$ be two cones of $\mathbb{R}^d$. If $K^{(1)} \cup K^{(2)}$ is strictly supported by some hyperplane H, then $K^{(1)} + K^{(2)}$ is closed.*

Let $U \subset \mathbb{R}^d$. Clearly, if $K = \mathrm{qcone}(U)$ is strictly H-supported , then $U \setminus \{0\} \subseteq \mathrm{int}(S^h_+)$. The converse is false as long as U is a generic set. For instance, let $U \subset \mathbb{R}^2$ be the unit open ball centered in the point $(0, 1)$. Clearly, $U = U \setminus \{0\}$ is contained in $\mathrm{int}(S^h_+)$, where $h = (0, 1)$, but, at the same time, $\mathrm{qcone}(U) = S^h_+$.

Nevertheless, the converse is true whenever U is a finite union of cones.

PROPOSITION 3.38. *Let $K^{(1)}, \ldots, K^{(r)}$ be cones of $\mathbb{R}^d$, H a hyperplane and*

$$K := \mathrm{qcone}(K^{(1)} \cup \ldots \cup K^{(r)}).$$

Then the following statements are equivalent:

(i) K is strictly H-supported ;

(ii) $K^{(1)} + \cdots + K^{(r)}$ is strictly H-supported and, hence, closed;

(iii) $K^{(1)} \cup \ldots \cup K^{(r)}$ is strictly H-supported .

In this case, $K = K^{(1)} + \cdots + K^{(r)}$ is a cone, too.

Proof. With reference to (ii), we begin by observing that, if $K^{(1)} + \cdots + K^{(r)}$ is strictly H-supported , then it is closed. In fact, this can be easily proved by induction on r using Proposition 3.37.
$(i) \Rightarrow (ii)$ By (11).
$(ii) \Rightarrow (iii)$ By (10).
$(iii) \Rightarrow (ii)$ From the assumption, there exists h such that $\{h\}^\perp = H$ and $h^T z > 0$ for all $z \in K^{(1)} \cup \ldots \cup K^{(r)}$, $z \neq 0$. Hence, $h^T(z_1 + \cdots + z_r) > 0$ for all $z_i \in K^{(i)}$, $i = 1, \ldots, r$, such that $z_1 + \cdots + z_r \neq 0$.
$(ii) \Rightarrow (i)$ Since $K^{(1)} + \cdots + K^{(r)}$ is closed, then it coincides with K by (11) and, hence, K is strictly H-supported as well. □

4. Matrices with invariant cones

Let $\mathbb{F}$ denote either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. Throughout this paper we denote by $\mathbb{F}^{d\times d}$ the space of the $d \times d$ matrices on $\mathbb{F}$.

If $A \in \mathbb{F}^{d\times d}$, we identify it with the corresponding endomorphism

$$f_A : \mathbb{F}^d \to \mathbb{F}^d$$

defined by $f_A(x) = Ax$. Hence, the kernel and the image of f_A will be simply denoted by $\ker(A)$ and $\mathrm{range}(A)$, respectively, and, if U is a subset of $\mathbb{F}^d$, its image will be denoted by $A(U)$.

Nevertheless, the preimage of a subset V of $\mathbb{F}^d$ will be explicitly denoted by $f_A^{-1}(V)$.

Definition 4.1. *A subset U of $\mathbb{R}^d$ is said to be* invariant under the action of the matrix A *on $\mathbb{R}^d$ (in short,* invariant for A*) if $A(U) \subseteq U$.*

Assumption 4.1. *In order to avoid trivial cases, from now on we assume that A is a nonzero matrix.*

If $\lambda \in \mathbb{F}$ and a nonzero vector $v \in \mathbb{F}^d$ are such that $Av = \lambda v$, then they are called *eigenvalue* and *eigenvector* of A, respectively.

The set $V_\lambda(A)$, or simply V_λ, consisting of such eigenvectors and of the zero vector, is a linear subspace called the *eigenspace* corresponding to λ. Obviously, V_λ is invariant under the action of A.

Denoting by $\mu_a(\lambda)$ the *algebraic multiplicity* of λ (as root of the characteristic polynomial $\det(A-\lambda I)$) and by $\mu_g(\lambda)$ the *geometric multiplicity* of λ (i.e., $\dim_{\mathbb{F}}(V_\lambda)$), it is also well known that $\mu_g(\lambda) \leq \mu_a(\lambda)$. If the equality holds, then λ is called *nondefective*. Otherwise, it is called *defective*.

Definition 4.2. *Let λ be an eigenvalue of A and $k = \mu_a(\lambda)$. Then the linear space*

$$W_\lambda(A) := \ker((A-\lambda I)^k) \subseteq \mathbb{F}^d$$

is called generalized eigenspace *corresponding to λ and each of its nonzero elements which does not belong to V_λ is called* generalized eigenvector.

If no misunderstanding occurs, we shall simply write W_λ instead of $W_\lambda(A)$.

It is clear that W_λ is a linear subspace invariant for A and it is well known that $\dim_{\mathbb{F}}(W_\lambda) = \mu_a(\lambda)$ (see, e.g., Lax [10], Theorem 11). Therefore, $V_\lambda = W_\lambda$ if and only if λ is nondefective.

In this paper we shall deal with real matrices only. Clearly, if A is a real matrix, we can take $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$.

If $\lambda \in \mathbb{R}$, then W_λ is a linear subspace of $\mathbb{R}^d$ and $\dim_{\mathbb{R}}(W_\lambda) = \mu_a(\lambda)$.

Otherwise, if $\lambda \in \mathbb{C} \setminus \mathbb{R}$, take $\mathbb{F} = \mathbb{C}$ and consider $W_\lambda \subseteq \mathbb{C}^d$. Since the conjugate of λ is an eigenvalue as well, set $U_{\mathbb{C}}(\lambda, \bar{\lambda}) := W_\lambda \oplus W_{\bar{\lambda}} \subseteq \mathbb{C}^d$. With $k := \mu_a(\lambda) = \dim_{\mathbb{C}}(W_\lambda)$, it is clear that $\dim_{\mathbb{C}}(U_{\mathbb{C}}(\lambda, \bar{\lambda})) = 2k$. Setting also $U_{\mathbb{R}}(\lambda, \bar{\lambda}) := U_{\mathbb{C}}(\lambda, \bar{\lambda}) \cap \mathbb{R}^d$, it turns out that $\dim_{\mathbb{R}}(U_{\mathbb{R}}(\lambda, \bar{\lambda})) = 2k$ and that this linear space is spanned by the real and the imaginary parts of the vectors of W_λ. Clearly, $U_{\mathbb{R}}(\lambda, \bar{\lambda})$ is invariant for A.

Therefore, if $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$ and $\mu_1, \bar{\mu}_1, \ldots, \mu_s, \bar{\mu}_s \in \mathbb{C} \setminus \mathbb{R}$ are the distinct roots of the characteristic polynomial, then

$$\mathbb{R}^d = \bigoplus_{i=1}^{r} W_{\lambda_i} \oplus \bigoplus_{i=1}^{s} U_{\mathbb{R}}(\mu_i, \bar{\mu_i}). \tag{12}$$

Finally, recall that the set $\sigma(A)$ of the (real or complex) eigenvalues is called the *spectrum* of A and the nonnegative real number

$$\rho(A) := \max_{\lambda \in \sigma(A)} |\lambda|$$

is called the *spectral radius* of A.

It is well known that either $\rho(A) > 0$ or $A^d = 0$.

The eigenvalues whose modulus is $\rho(A)$ are called *leading eigenvalues* and the corresponding eigenvectors are called *leading eigenvectors*. (For the convenience of the reader, we recall that, in the literature, these objects are also known as *principal eigenvalues* and *principal eigenvectors*).

The remaining eigenvalues and eigenvectors are called *secondary eigenvalues* and *secondary eigenvectors*, respectively.

Remark 4.3: If the matrix A admits a real leading eigenvalue λ_1, we can write

$$\mathbb{R}^d = W_A \oplus H_A,$$

where

$$W_A := W_{\lambda_1} \qquad \text{and} \qquad H_A := \bigoplus_{i=2}^{r} W_{\lambda_i} \oplus \bigoplus_{i=1}^{s} U_{\mathbb{R}}(\mu_i, \bar{\mu_i}). \tag{13}$$

Observe that both W_A and H_A are linear subspaces invariant for A.

Proposition 4.4. *Let A be a matrix admitting a real leading eigenvalue $\lambda_1 > 0$ and let $x \in \mathbb{R}^d$. Then*

$$Ax \in H_A \implies x \in H_A.$$

Proof. Using (13), we can write $x = v + u$ for suitable $v \in W_A$ and $u \in H_A$ and thus $Ax = Av + Au$. Clearly, $Ax \in H_A$ by assumption and $Au \in H_A$ since H_A is invariant for A. Therefore, $Av \in W_A \cap H_A = \{0\}$ and, hence, $v \in \ker(A) = W_0$. But $W_A \cap W_0 = \{0\}$ since $\lambda_1 > 0$. Therefore, $v = 0$ and the proof is complete. $\square$

It is clear that, if $\lambda > 0$ is a real eigenvalue and $\dim(V_\lambda) = 1$, both the two half-lines which constitute V_λ are invariant for A. Therefore, it makes sense to extend the search of invariant sets from linear subspaces to cones.

In the case of cones the notion of invariance is the general one (see Definition 4.1), but it is useful to recall the following refinement.

Definition 4.5. *We say that a quasi-cone K is* strictly invariant under the action of the matrix *A on $\mathbb{R}^d$ (in short,* strictly invariant for *A) if*

$$A(K \setminus L(K)) \subseteq \mathrm{int}_{rel}(K).$$

In particular, if K is a cone, the above inclusion reads $A(K \setminus \{0\}) \subseteq \mathrm{int}_{rel}(K)$.

For example, the positive orthant $\mathbb{R}^d_+$ is invariant for a real matrix with nonnegative entries, whereas it is strictly invariant for a matrix with all strictly positive entries.

We recall that A and the *transpose matrix* A^T have the same eigenvalues with the same multiplicities. More precisely, for any eigenvalue $\lambda \in \mathbb{C}$ it holds that $\dim(V_\lambda(A)) = \dim(V_\lambda(A^T))$ and $\dim(W_\lambda(A)) = \dim(W_\lambda(A^T))$.

The following result is well known in the case of proper cones.

Proposition 4.6. *A quasi-cone K is invariant (respectively, strictly invariant) for a matrix A if and only if the dual quasi-cone K^* is invariant (respectively, strictly invariant) for the transpose matrix A^T.*

We recall the following well-known Perron-Frobenius theorems, which may be found, for instance, in Vandergraft [17].

Theorem 4.7. *Let a proper cone K be invariant for a nonzero matrix A. Then the following facts hold:*

(i) the spectral radius $\rho(A)$ is an eigenvalue of A;

(ii) the cone K contains an eigenvector v corresponding to $\rho(A)$.

Theorem 4.8. *Let a proper cone K be strictly invariant for a nonzero matrix A. Then the following facts hold:*

(i) the spectral radius $\rho(A)$ is a simple positive eigenvalue of A and $|\lambda| < \rho(A)$ for any other eigenvalue λ of A;

(ii) $\mathrm{int}(K)$ *contains the unique leading eigenvector v (corresponding to $\rho(A)$);*

(iii) the secondary eigenvectors and generalized eigenvectors of A do not belong to K.

Under the hypotheses of Theorem 4.7, in the next Theorem 4.10 we prove a stronger version of the analogous counterpart of Theorem 4.8-(iii). Moreover, following the same line, in Theorem 4.12 we then easily obtain a stronger version of Theorem 4.8-(iii) itself.

LEMMA 4.9. *Let A be a matrix having a real leading eigenvalue $\rho(A)$. Then $W_{A^T} = (H_A)^{\perp}$.*

Proof. Set $B := (A-\rho(A)I)^k$ and recall that $W_A = \ker(B)$ (see Definition 4.2). Moreover, H_A is invariant for B since it is invariant for A.

From Remark 4.3 we then obtain that $\mathrm{range}(B) = B(H_A) = H_A$, where the second equality holds since the matrix B is nonsingular on H_A.

Recalling that $\mathrm{range}(B) = (\ker(B^T))^{\perp}$, we get $H_A = (\ker(B^T))^{\perp}$ and, finally, the equality $W_{A^T} = \ker(B^T)$ concludes the proof. □

Note that, if A is a matrix having an invariant proper cone K, then $\lambda_1 = \rho(A)$ is a real leading eigenvalue by Theorem 4.7. So, keeping the notation of Remark 4.3, we have the following result.

THEOREM 4.10. *Let A be a matrix having an invariant proper cone K. Then*

$$\mathrm{int}(K) \cap H_A = \emptyset.$$

Proof. Let us consider $y \in \mathrm{int}(K) \cap H_A$. Then, by Proposition 3.30-(i) applied to K^*, we have that $y^T w > 0$ for all $w \in K^* \setminus \{0\}$.

Let us observe that $\rho(A) = \rho(A^T)$ and that K^* is a proper cone invariant for A^T (see Proposition 4.6). Therefore, by Theorem 4.7, there exists a leading eigenvector $\bar{w}$ of A^T which belongs to K^*, i.e., $\bar{w} \in W_{A^T} \cap K^*$.

Since, by Lemma 4.9, $W_{A^T} = (H_A)^{\perp}$, we have $y^T \bar{w} = 0$, which gives a contradiction. □

COROLLARY 4.11. *If $\rho(A) > 0$, in the assumptions of the previous theorem, we have*

$$\mathrm{int}(K) \cap \ker(A) = \emptyset.$$

Proof. If 0 is not an eigenvalue, the equality trivially holds. Otherwise, 0 is a secondary eigenvalue and, so, $W_0 \subseteq H_A$. On the other hand, $\ker(A) = V_0 \subseteq W_0$ and, thus, Theorem 4.10 concludes the proof. □

THEOREM 4.12. *If K is a strictly invariant proper cone for a matrix A, then*

$$K \cap H_A = \{0\}.$$

Proof. Let us consider $y \in K \cap H_A$. Then, by Proposition 3.30-(ii) applied to K^*, we have that $y^T w > 0$ for all $w \in \mathrm{int}(K^*)$.

The result is easily obtained by reasoning as in the proof of Theorem 4.10 by using Theorem 4.8 in place of Theorem 4.7. $\square$

The previous result may be also found, for example, in Krasnosel'skiĭ, Lifshits and Sobolev [8] with a different proof.

The analogue of Corollary 4.11 clearly holds.

COROLLARY 4.13. *In the assumptions of the previous theorem we also have*

$$K \cap \ker(A) = \{0\}.$$

We conclude this survey by considering a particular class of matrices, which turns out to be the only one we can meet in the strictly invariant case.

DEFINITION 4.14. *A matrix A is said to be* asymptotically rank-one *if the following conditions hold:*

(i) $\rho(A) > 0$*;*

(ii) exactly one between $\rho(A)$ and $-\rho(A)$ is an eigenvalue of A and, moreover, it is a simple eigenvalue;

(iii) $|\lambda| < \rho(A)$ *for any other eigenvalue λ of A.*

The unique leading eigenvalue of A will be denoted by λ_A.

REMARK 4.15: A matrix A is asymptotically rank-one if and only if A^T is so.

The term "asymptotically rank-one" is inspired by the following known fact.

PROPOSITION 4.16. *If A is an asymptotically rank-one matrix, then there exists*

$$\hat{A}^\infty := \lim_{k \to \infty} A^k / \lambda_A^k$$

and such a limit is the rank-one matrix

$$\hat{A}^\infty = (v_A^T h_A)^{-1} v_A h_A^T,$$

where v_A and h_A are the (unique) leading eigenvectors of A and A^T, respectively.

Proof. We need to observe that the Jordan canonical form $\hat{J}$ of the normalized matrix $\hat{A} := A/\lambda_A$ may be assumed to be block diagonal. More precisely, the first block is 1×1 and consists in the maximum simple eigenvalue $\lambda_{\hat{A}} = 1$. The second one is a $(d-1) \times (d-1)$-block, upper bidiagonal, whose diagonal entries are the secondary eigenvalues of $\hat{A}$, all with modulus < 1, and the upper diagonal entries are equal to 1 or to 0. Therefore, when we take the kth power of $\hat{J}$, the first block remains unchanged, while the second clearly goes to zero. Hence, we obtain the rank-one limit matrix $\hat{J}^\infty$ with only one nonzero entry equal to 1 in the left upper corner.

Finally, the form of the limit $\hat{A}^\infty$ is easily determined by taking into account that it has the leading eigenvector v_A related to the eigenvalue 1 and that, analogously, its transpose $(\hat{A}^\infty)^T$ has the leading eigenvector h_A. □

The following characterization rephrases Theorem 4.4 in Vandergraft [17].

THEOREM 4.17. *A matrix A is asymptotically rank-one if and only if A or $-A$ admits a strictly invariant proper cone.*

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