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Research Article

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Subharmonic solutions of Hamiltonian systems displaying some kind of sublinear growth

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Abstract: We prove the existence and multiplicity of subharmonic solutions for Hamiltonian systems obtained as perturbations of N planar uncoupled systems which, e.g., model some type of asymmetric oscillators. The nonlinearities are assumed to satisfy Landesman–Lazer conditions at the zero eigenvalue, and to have some kind of sublinear behavior at infinity. The proof is carried out by the use of a generalized version of the Poincaré–Birkhoff Theorem. Different situations, including Lotka–Volterra systems, or systems with singularities, are also illustrated.

Keywords: Hamiltonian systems, subharmonic solutions, Poincaré–Birkhoff, Lotka–Volterra

MSC 2010: 34C25

Dedicated to Jean Mawhin on the occasion of his 75th birthday

1 Introduction and main result

We are interested in finding periodic solutions of a nonautonomous Hamiltonian system in $\mathbb{R}^{2N}$. Writing $\mathbf{x} = (x_1, \dots, x_N)$ and $\mathbf{y} = (y_1, \dots, y_N)$, we consider the system

$$\begin{cases} x_k' = f_k(t, y_k) + \dfrac{\partial \mathcal{U}}{\partial y_k}(t, \mathbf{x}, \mathbf{y}; \varepsilon), \\ -y_k' = g_k(t, x_k) + \dfrac{\partial \mathcal{U}}{\partial x_k}(t, \mathbf{x}, \mathbf{y}; \varepsilon), \end{cases} \qquad k = 1, \dots, N. \tag{1.1}$$

All functions $f_k, g_k : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are assumed to be continuous, T-periodic in their first variable and locally Lipschitz continuous in their second variable. The function $\mathcal{U} : \mathbb{R} \times \mathbb{R}^{2N} \times \mathbb{R} \to \mathbb{R}$ is continuous, T-periodic in t, continuously differentiable in $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{2N}$, and

$$\mathcal{U}(t, \mathbf{x}, \mathbf{y}; 0) = 0 \quad \text{for every } (t, \mathbf{x}, \mathbf{y}) \in [0, T] \times \mathbb{R}^{2N}.$$

In addition, for every $k = 1, \dots, N$, the following four assumptions on the functions f_k, g_k are made.

Assumption (A1). There exists a constant $C > 0$ such that

$$|f_k(t, \eta)| \le C(1 + |\eta|), \quad |g_k(t, \xi)| \le C(1 + |\xi|)$$

for every $t \in [0, T]$ and $\eta, \xi \in \mathbb{R}$.

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Assumption (A2). The functions $f_k(t,\eta)$, $g_k(t,\xi)$ are bounded from above for negative ξ, η, bounded from below for positive ξ, η, and

$$\int_0^T \limsup_{\eta\to-\infty} f_k(t,\eta)\,dt < 0 < \int_0^T \liminf_{\eta\to+\infty} f_k(t,\eta)\,dt,$$
$$\int_0^T \limsup_{\xi\to-\infty} g_k(t,\xi)\,dt < 0 < \int_0^T \liminf_{\xi\to+\infty} g_k(t,\xi)\,dt.$$

Assumption (A3). For every $\sigma > 0$ there are $\mathcal{R}_k > 0$ and a planar sector

$$\Theta_k = \{\rho(\cos\theta, \sin\theta) : \rho \geq 0,\ \hat\theta_k \leq \theta \leq \check\theta_k\},$$

with $\hat\theta_k < \check\theta_k \leq \hat\theta_k + 2\pi$, for which

$$\sup\Big\{\frac{g_k(t,\xi)\xi + f_k(t,\eta)\eta}{\xi^2+\eta^2} : (\xi,\eta)\in\Theta_k,\ \xi^2+\eta^2\geq\mathcal{R}_k^2\Big\} \leq \sigma(\check\theta_k - \hat\theta_k).$$

Assumption (A4). Either f_k or g_k is strictly increasing in its second variable.

We now need to recall the notion of rotation number around the origin for a planar curve. For $\tau_1 < \tau_2$, let $\zeta : [\tau_1, \tau_2] \to \mathbb{R}^2$ be continuously differentiable and such that $\zeta(t) = (\xi(t), \eta(t)) \neq (0,0)$ for every $t \in [\tau_1, \tau_2]$. The rotation number of ζ around the origin is defined as

$$\mathrm{Rot}(\zeta; [\tau_1, \tau_2]) = \frac{1}{2\pi}\int_{\tau_1}^{\tau_2} \frac{\xi'(t)\eta(t) - \xi(t)\eta'(t)}{\xi(t)^2 + \eta(t)^2}\,dt.$$

In other terms, writing $\zeta(t) = \rho(t)(\cos\theta(t), \sin\theta(t))$, one has

$$\mathrm{Rot}(\zeta; [\tau_1, \tau_2]) = -\frac{\theta(\tau_2) - \theta(\tau_1)}{2\pi}.$$

We are mainly interested in proving the existence and multiplicity of subharmonic solutions, i.e., periodic solutions of period ℓT for some positive integer ℓ. Writing $z_k = (x_k, y_k)$ for $k = 1, \dots, N$ and $\mathbf{z} = (z_1, \dots, z_N)$, we will find solutions $\mathbf{z}(t)$ whose planar components $z_k(t)$ rotate around the origin a prescribed number of times in their period time. The following is our main result.

Theorem 1.1. *Let assumptions* (A1)–(A4) *hold, let $\bar R$ be a positive real number and let $M_1, \dots, M_N$ be some positive integers. Then there is a positive integer $\bar\ell$ with the following property: for every integer $\ell \geq \bar\ell$, there exists $\varepsilon_\ell > 0$ such that if $|\varepsilon| \leq \varepsilon_\ell$, system* (1.1) *has at least $N+1$ distinct ℓT-periodic solutions $\mathbf{z}(t) = (z_1(t), \dots, z_N(t))$, with $z_k(t) = (x_k(t), y_k(t))$, which satisfy*

$$\min\{|z_k(t)| : t\in[0,\ell T]\} \geq \bar R \quad \textit{and} \quad \mathrm{Rot}(z_k; [0,\ell T]) = M_k \tag{1.2}$$

for every $k = 1, \dots, N$.

Therefore, roughly speaking, when ε is small enough, there are large amplitude subharmonic solutions whose planar components perform a prescribed number of rotations around the origin in its period time ℓT. Hence, if at least one of these components makes exactly one rotation, the solution $\mathbf{z}(t)$ necessarily has *minimal period* equal to ℓT. As a consequence, if $N \geq 2$, there will be a myriad of periodic solutions having minimal period ℓT: when one of the components performs exactly one rotation, the others rotate an arbitrary number of times. We thus have the following direct consequence of Theorem 1.1.

Corollary 1.2. *Let $N \geq 2$ and fix an arbitrary positive integer K. Then, under the assumptions of Theorem 1.1, there is a positive integer $\bar\ell$ with the following property: for every integer $\ell \geq \bar\ell$, there exists $\varepsilon_\ell > 0$ such that if $|\varepsilon| \leq \varepsilon_\ell$, system* (1.1) *has at least K periodic solutions with minimal period ℓT.*

Let us clarify what we mean by *distinct* subharmonic solutions. With the nonlinearities being T-periodic in t, once an ℓT-periodic solution $z(t)$ has been found, many others appear by just making a shift in time, thus giving rise to the periodicity class

$$z(t), z(t+T), z(t+2T), \dots, z(t+(\ell-1)T).$$

We say that two ℓT-periodic solutions are *distinct* if they are not related to each other in this way, i.e., if they do not belong to the same periodicity class.

Some remarks on our hypotheses are now in order. Assumptions (A1)–(A4) involve only the functions f_k, g_k, and are meant to govern the behavior of the solutions of (1.1) when $\varepsilon = 0$. Assumption (A1) is the usual linear growth condition. In assumption (A2) we have the well-known Landesman–Lazer conditions: they will force the large-amplitude solutions of the uncoupled planar systems to rotate around the origin. This property, which might have an independent interest, has already been exploited in [3, 4, 9, 23], and is stated in Lemma 2.5 below. Assumption (A3), first proposed in [5], is needed in order to have a control on the angular velocity of the large-amplitude solutions, while crossing the planar sector Θ_k: it implies that the large-amplitude solutions will not be able to complete an entire rotation in a given period time $[0, \ell T]$. Finally, assumption (A4) will be used, after a change of variables, to forbid counterclockwise rotations in the phase planes.

A particular case of (1.1) is the system

$$\begin{cases} (\phi_1(x_1'))' + g_1(t, x_1) = \dfrac{\partial \mathcal{V}}{\partial x_1}(t, x_1, \dots, x_N; \varepsilon), \\ \qquad\vdots \\ (\phi_N(x_N'))' + g_N(t, x_N) = \dfrac{\partial \mathcal{V}}{\partial x_N}(t, x_1, \dots, x_N; \varepsilon). \end{cases} \tag{1.3}$$

Here, the functions $\phi_k : I_k \to \mathbb{R}$ are strictly increasing diffeomorphisms defined on some open intervals I_k, containing the origin, with $\phi_k(0) = 0$; the functions $g_k : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous, T-periodic in their first variable and locally Lipschitz continuous in their second variable; the function $\mathcal{V} : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is continuous, T-periodic in t, continuously differentiable in $x_1, \dots, x_N$, and

$$\mathcal{V}(t, x_1, \dots, x_N; 0) = 0 \quad \text{for every } (t, x_1, \dots, x_N) \in [0, T] \times \mathbb{R}^N.$$

System (1.3) can be viewed as a mathematical model of N coupled oscillators, with small coupling forces. It can be translated into the form of system (1.1) by setting $f_k(t, y) = \phi_k^{-1}(y)$. Concerning our functions ϕ_k, typically we have in mind either the case $\phi_k(s) = s$, leading to classical second-order differential equations, or the case $\phi_k(s) = s/\sqrt{1-s^2}$, when dealing with a relativistic type of operator. When $N = 1$, the study of the case when the function ϕ is defined on the whole real line was started by García-Huidobro, Manásevich and Zanolin in [15], while, in recent years, following Bereanu and Mawhin [1], a lot of effort has also been devoted to the singular case. See the review paper [20] and the references therein.

Let us state a corollary of our main result in the case when $\phi_k(s) = s$.

Corollary 1.3. *Assume $\phi_k(s) = s$, and let the functions g_k satisfy the linear growth assumption* (A1) *and the Landesman–Lazer condition* (A2). *If, moreover,*

$$\lim_{\xi \to +\infty} \frac{g_k(t, \xi)}{\xi} = 0 \quad \textit{uniformly in } t \in [0, T], \tag{1.4}$$

then the same conclusion of Theorem 1.1 holds for system (1.3), *with $z_k(t) = (x_k(t), x_k'(t))$.*

Notice that assumption (1.4) could be replaced by the analogous one at $-\infty$.

On the other hand, in the case when $\phi_k(s) = s/\sqrt{1-s^2}$ we have the following result.

Corollary 1.4. *Assume $\phi_k(s) = s/\sqrt{1-s^2}$, and let the functions g_k satisfy the linear growth assumption* (A1) *and the Landesman–Lazer condition* (A2). *Then the same conclusion of Theorem 1.1 holds for system* (1.3), *with $z_k(t) = (x_k(t), \phi_k(x_k'(t)))$.*

We thus generalize to higher-order systems some of the results obtained in [3, 4, 8, 9, 21, 23] for planar systems and, in particular, for scalar second-order differential equations. We will use phase plane analysis methods, combined with a generalized version of the Poincaré–Birkhoff Theorem for Hamiltonian time maps recently proved by the first author and Ureña in [14]. This last theorem has already been used in [2, 5, 11, 12, 14] to prove the multiplicity of periodic solutions for different kinds of systems.

Let us remark that, when $N \geq 2$, there are few results in the literature concerning the existence of subharmonic solutions for systems in a situation like the one described above. Among those we know, let us mention [7, 25–27], where variational methods have been used. When compared to these results, we can see that our theorem gives more information on the behavior of the solutions, even though it applies only to systems involving small coupling terms. However, let us emphasize that we are not dealing with a standard perturbation problem: the periodic solutions we are looking for do not bifurcate from some particular solutions of the uncoupled system corresponding to $\varepsilon = 0$.

The paper is organized as follows: In Section 2, we provide the proof of Theorem 1.1, which is divided into several steps. First, in Section 2.1, we prove the existence of a T-periodic solution for each of the N uncoupled planar systems corresponding to $\varepsilon = 0$. Then, in Section 2.2, we use this solution to perform a change of variables, which leads to some equivalent planar systems, each of which has the constant solution $(0, 0)$. In Section 2.3, we need a delicate analysis of the rotating behavior of the solutions in the phase plane. Finally, in Section 2.4, we prove our main result by the use of the above mentioned generalized Poincaré–Birkhoff Theorem. In Section 3, besides providing the proofs of Corollaries 1.3 and 1.4, we argue on some variants of our main result, which can be obtained by the same methods. Different situations are illustrated, including Lotka–Volterra systems and systems with singularities.

2 Proof of Theorem 1.1

The proof will be divided into several steps. In order to fix the ideas, we assume in (A4) that

$$f_k(t, \cdot) \text{ is strictly increasing for every } t \in \mathbb{R}.$$

First of all, we recall that the Landesman–Lazer conditions in assumption (A2) can be written in a different form. Following, e.g., [13, Lemma 1], we can find two constants $d_1 > 0$, $\delta > 0$ and four L^1-functions $\varphi_k^\pm, \psi_k^\pm : [0, T] \to \mathbb{R}$, such that the following conditions hold:

$$\begin{aligned}
&[\eta \leq -d_1 \implies f_k(t, \eta) \leq \varphi_k^-(t)] \quad \text{and} \quad \int_0^T \varphi_k^-(t)\,dt \leq -\delta, \\
&[\eta \geq d_1 \implies f_k(t, \eta) \geq \varphi_k^+(t)] \quad \text{and} \quad \int_0^T \varphi_k^+(t)\,dt \geq \delta, \\
&[\xi \leq -d_1 \implies g_k(t, \xi) \leq \psi_k^-(t)] \quad \text{and} \quad \int_0^T \psi_k^-(t)\,dt \leq -\delta, \\
&[\xi \geq d_1 \implies g_k(t, \xi) \geq \psi_k^+(t)] \quad \text{and} \quad \int_0^T \psi_k^+(t)\,dt \geq \delta.
\end{aligned} \tag{2.1}$$

Next, we will find a T-periodic solution of (1.1) with $\varepsilon = 0$, which will be used in a change of variables, in order to have the origin as a constant solution. This will enable us to compute the rotation number on each planar subsystem, so to finally apply a generalized version of the Poincaré–Birkhoff Theorem recently obtained in [14].

2.1 Existence of a T-periodic solution when $\varepsilon = 0$

We consider system (1.1) with $\varepsilon = 0$. We thus have N uncoupled subsystems

$$x_k' = f_k(t, y_k), \quad -y_k' = g_k(t, x_k), \qquad k = 1, \dots, N, \tag{2.2}$$

and we will study each of them separately.

We first prove that system (2.2) has a T-periodic solution. For simplicity in the notation, we write the subsystem corresponding to a given $k \in \{1, \dots, N\}$ as

$$x' = f_k(t, y), \quad -y' = g_k(t, x). \tag{2.3}$$

Since $f_k(t, \cdot)$ is strictly increasing, it is easy to see that the Landesman–Lazer condition in (A2) implies the existence of a constant $\overline{\eta} \in \mathbb{R}$ for which

$$\int_0^T f_k(t, \overline{\eta})\, dt = 0.$$

The change of variables

$$u(t) = x(t) - \int_0^t f_k(\tau, \overline{\eta})\, d\tau, \quad v(t) = y(t) - \overline{\eta}$$

leads to the system

$$u' = \tilde{f}_k(t, v), \quad -v' = \tilde{g}_k(t, u),$$

where

$$\tilde{f}_k(t, v) = f_k(t, v + \overline{\eta}) - f_k(t, \overline{\eta}), \quad \tilde{g}_k(t, u) = g_k\Big(t, u + \int_0^t f_k(\tau, \overline{\eta})\, d\tau\Big).$$

We notice that $\tilde{f}_k(t, 0) = 0$ for every $t \in [0, T]$.

Proposition 2.1. *The assumptions* (A1)–(A4) *hold for the functions* $\tilde{f}_k$ *and* $\tilde{g}_k$ *as well.*

Proof. Conditions (A1), (A2) and (A4) are readily verified. Concerning condition (A3), let us fix $\sigma > 0$. Then there are $\mathcal{R}_k > 0$ and a planar sector

$$\Theta_k = \{\rho(\cos\theta, \sin\theta) : \rho \geq 0,\ \hat{\theta}_k \leq \theta \leq \check{\theta}_k\},$$

with $\hat{\theta}_k < \check{\theta}_k \leq \hat{\theta}_k + 2\pi$, for which

$$g_k(t, \xi)\xi + f_k(t, \eta)\eta \leq \frac{1}{2}\sigma(\check{\theta}_k - \hat{\theta}_k)(\xi^2 + \eta^2)$$

whenever $(\xi, \eta) \in \Theta_k \setminus B((0,0), \mathcal{R}_k)$. Let us choose $\hat{\theta}_k', \check{\theta}_k'$ such that

$$\hat{\theta}_k < \hat{\theta}_k' < \check{\theta}_k' < \check{\theta}_k, \quad \check{\theta}_k' - \hat{\theta}_k' \geq \frac{3}{4}(\check{\theta}_k - \hat{\theta}_k),$$

and consider the planar sector

$$\Theta_k' = \{\rho(\cos\theta, \sin\theta) : \rho \geq 0,\ \hat{\theta}_k' \leq \theta \leq \check{\theta}_k'\}.$$

Taking $\mathcal{R}_k' \geq \mathcal{R}_k$ large enough, if $(u, v) \in \Theta_k'$ and $u^2 + v^2 \geq {\mathcal{R}_k'}^2$, then

$$\Big(u + \int_0^t f_k(\tau, \overline{\eta})\, d\tau, v + \overline{\eta}\Big) \in \Theta_k \setminus B((0,0), \mathcal{R}_k) \quad \text{for every } t \in [0, T].$$

Then, using assumptions (A1) and (A3), we can find a $\mathcal{R}_k'' \geq \mathcal{R}_k'$ such that if $(u, v) \in \Theta_k' \setminus B((0,0), \mathcal{R}_k'')$, then

$$\tilde{g}_k(t, u)u + \tilde{f}_k(t, v)v \leq \sigma(\check{\theta}_k' - \hat{\theta}_k')(u^2 + v^2),$$

thus ending the proof. □

Hence, for the sake of proving the existence of a T-periodic solution to system (2.2) we may assume without loss of generality that $f_k(\cdot, 0)$ is identically equal to zero. Then, by the monotonicity of $f_k(t, \cdot)$,

$$f_k(t,\eta) > 0 \quad \text{for } \eta > 0 \qquad \text{and} \qquad f_k(t,\eta) < 0 \quad \text{for } \eta < 0. \tag{2.4}$$

We will now use the following result due to Mawhin [18, 19].

Theorem 2.2 (Mawhin, 1969). *Let $\mathcal{F} : [0, T] \times R^m \to R^m$ be a Carathéodory vector field, and assume that there exists an open bounded set $\Omega \subseteq \mathbb{R}^m$ such that, for every $\lambda \in \,]0, 1]$, all possible solutions of the problems*

$$\begin{cases} z' = \lambda \mathcal{F}(t, z), \\ z(0) = z(T) \end{cases}$$

satisfy

$$z(t) \in \Omega \quad \textit{for every } t \in [0, T]. \tag{2.5}$$

If the averaged map $\mathcal{F}^\sharp : \mathbb{R}^m \to \mathbb{R}^m$, defined as

$$\mathcal{F}^\sharp(\zeta) = \frac{1}{T}\int_0^T \mathcal{F}(t, \zeta)\, dt,$$

has no zeros on $\partial\Omega$ and the Brouwer degree $d(\mathcal{F}^\sharp, \Omega)$ is different from zero, then the problem

$$\begin{cases} z' = \mathcal{F}(t, z), \\ z(0) = z(T) \end{cases}$$

has a solution satisfying (2.5).

We thus need to find an a priori bound for the T-periodic solutions of the system

$$x' = \lambda f_k(t, y), \quad -y' = \lambda g_k(t, x), \tag{2.6}$$

with $\lambda \in \,]0, 1]$. Integrating in (2.6), we have

$$\int_0^T f_k(t, y(t))\, dt = 0 = \int_0^T g_k(t, x(t))\, dt.$$

Using assumption (A2), we see that the solutions have to cross both the horizontal and the vertical strips of width $2d_1$ around the coordinate axes, where $d_1 > 0$ is the constant introduced in conditions (2.1): there exist $t_1, t_2 \in [0, T]$ such that $|x(t_1)| < d_1$ and $|y(t_2)| < d_1$.

Let us prove that there exists $r > 0$ such that, for every T-periodic solution of (2.6),

$$\min\{x(t)^2 + y(t)^2 : t \in [0, T]\} < r^2. \tag{2.7}$$

By taking $r > \sqrt{2}d_1$, if (2.7) were not true, the fact that $|x(t_1)| < d_1$ and $|y(t_2)| < d_1$, together with (2.4) would imply that the solution has to rotate at least once around the origin as t varies in $[0, T]$. Passing to polar coordinates

$$x(t) = \rho(t)\cos\theta(t), \quad y(t) = \rho(t)\sin\theta(t),$$

we have

$$-\theta'(t) = \frac{\lambda g_k(t, x(t))x(t) + \lambda f_k(t, y(t))y(t)}{x(t)^2 + y(t)^2}.$$

Hence, by taking $\sigma \in \,]0, \frac{1}{T}[$, assumption (A3) tells us that if r is large enough and $\tilde{t}_1 < \tilde{t}_2$ are such that $\theta(\tilde{t}_1) = \check{\theta}_k$ and $\theta(\tilde{t}_2) = \hat{\theta}_k$, with $\theta(t) \in \,]\hat{\theta}_k, \check{\theta}_k[$ for every $t \in \,]\tilde{t}_1, \tilde{t}_2[$, then $\tilde{t}_2 - \tilde{t}_1 > T$, which is a contradiction.

Using assumption (A1), as long as $(x(t), y(t)) \neq (0, 0)$, we have

$$\begin{aligned}
|\rho'(t)| &= \Big|\frac{\lambda f_k(t, y(t))x(t) + \lambda g_k(t, x(t))y(t)}{\sqrt{x(t)^2 + y(t)^2}}\Big| \\
&\leq C\frac{(1 + |y(t)|)|x(t)| + (1 + |x(t)|)|y(t)|}{\sqrt{x(t)^2 + y(t)^2}} \\
&\leq 2C(1 + \rho(t)).
\end{aligned}$$

By the use of Gronwall's lemma, we can then find a constant $R > r$ such that $\rho(t) < R$ for every $t \in [0, T]$. In particular, setting $\Omega =]-R, R[\times]-R, R[$, we have $(x(t), y(t)) \in \Omega$ for every $t \in [0, T]$. The a priori bound is thus established.

Let us consider the averaged functions

$$f_k^\sharp(y) = \frac{1}{T}\int_0^T f_k(t, y)\,dt \quad \text{and} \quad g_k^\sharp(x) = \frac{1}{T}\int_0^T g_k(t, x)\,dt,$$

and define $\mathcal{F}_k^\sharp : \mathbb{R}^2 \to \mathbb{R}^2$ as $\mathcal{F}_k^\sharp(x, y) = (f_k^\sharp(y), g_k^\sharp(x))$. By enlarging R if necessary, in order to have $R > d_1$, conditions (2.1) allow to apply the Poincaré–Miranda Theorem (cf. [6]), and we have that the Brouwer degree $d(\mathcal{F}_k^\sharp, \Omega)$ is different from 0. Then Theorem 2.2 applies, and we conclude that there exists a T-periodic solution of (2.3) for every fixed $k \in \{1, \dots, N\}$.

We have thus proved that system (2.2) has a T-periodic solution.

2.2 A change of variables

Let $(\overline{\mathbf{x}}(t), \overline{\mathbf{y}}(t))$ be a T-periodic solution of system (2.2), with

$$\overline{\mathbf{x}}(t) = (\bar{x}(t), \dots, \bar{x}_N(t)), \quad \overline{\mathbf{y}}(t) = (\bar{y}(t), \dots, \bar{y}_N(t)),$$

whose existence has been proved in the previous section. Going back to system (1.1), we make the change of variables

$$\mathbf{u}(t) = \mathbf{x}(t) - \overline{\mathbf{x}}(t), \quad \mathbf{v}(t) = \mathbf{y}(t) - \overline{\mathbf{y}}(t),$$

thus obtaining a new system

$$\begin{cases} u_k' = \widehat{f}_k(t, v_k) + \dfrac{\partial \widehat{\mathcal{U}}}{\partial v_k}(t, \mathbf{u}, \mathbf{v}; \varepsilon), \\ -v_k' = \widehat{g}_k(t, u_k) + \dfrac{\partial \widehat{\mathcal{U}}}{\partial u_k}(t, \mathbf{u}, \mathbf{v}; \varepsilon), \end{cases} \qquad k = 1, \dots, N, \tag{2.8}$$

where

$$\begin{aligned}
\widehat{f}_k(t, v) &= f_k(t, v + \bar{y}_k(t)) - f_k(t, \bar{y}_k(t)), \\
\widehat{g}_k(t, u) &= g_k(t, u + \bar{x}_k(t)) - g_k(t, \bar{x}_k(t)),
\end{aligned}$$

and

$$\widehat{\mathcal{U}}(t, \mathbf{u}, \mathbf{v}; \varepsilon) = \mathcal{U}(t, \mathbf{u} + \overline{\mathbf{x}}(t), \mathbf{v} + \overline{\mathbf{y}}(t); \varepsilon).$$

All functions $\widehat{f}_k, \widehat{g}_k : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous, T-periodic in their first variable and locally Lipschitz continuous in their second variable. The function $\widehat{\mathcal{U}} : \mathbb{R} \times \mathbb{R}^{2N} \times \mathbb{R} \to \mathbb{R}$ is continuous, T-periodic in t, continuously differentiable in $(\mathbf{u}, \mathbf{v}) \in \mathbb{R}^{2N}$, and

$$\widehat{\mathcal{U}}(t, \mathbf{u}, \mathbf{v}; 0) = 0 \quad \text{for every } (t, \mathbf{u}, \mathbf{v}) \in [0, T] \times \mathbb{R}^{2N}.$$

We write $w_k = (u_k, v_k)$, for $k = 1, \dots, N$, and $\mathbf{w} = (w_1, \dots, w_N)$. Notice that

$$\widehat{f}_k(t, 0) = 0 \quad \text{and} \quad \widehat{g}_k(t, 0) = 0 \qquad \text{for every } t \in [0, T].$$

Proposition 2.3. *Assumptions* (A1)–(A4) *hold for the functions* $\widehat{f}_k$ *and* $\widehat{g}_k$ *as well.*

Proof. The linear growth condition (A1) follows immediately from the boundedness of $\bar{\mathbf{x}}(t)$, $\bar{\mathbf{y}}(t)$ and the continuity of f_k, g_k. Condition (A2) is readily verified after noticing that

$$\int_0^T f_k(t, \bar{y}_k(t))\,dt = \int_0^T \bar{x}_k'(t)\,dt = 0$$

and

$$\int_0^T g_k(t, \bar{x}_k(t))\,dt = -\int_0^T \bar{y}_k'(t)\,dt = 0.$$

The proof of condition (A3) is practically the same as in Proposition 2.1. Finally, if $f_k(t,\cdot)$ is strictly increasing, then also $\widehat{f}_k(t,\cdot)$ is such. Hence, condition (A4) holds as well. □

Let $D > 0$ be such that

$$\bar{x}_k^2(t) + \bar{y}_k^2(t) \le D^2 \quad \text{for every } t \in [0,T] \text{ and } k = 1,\dots,N.$$

Proposition 2.4. *For the sake of proving Theorem 1.1, we may assume without loss of generality that*

$$f_k(t,0) = 0 \quad \textit{and} \quad g_k(t,0) = 0 \qquad \textit{for every } t \in [0,T]. \tag{2.9}$$

Proof. Assume that Theorem 1.1 holds for the new system (2.8). Taking $\widehat{R} > \bar{R} + D$, we will find $N+1$ distinct ℓT-periodic solutions of the new system (2.8) satisfying

$$\min\{|w_k(t)| : t \in [0,\ell T]\} \ge \widehat{R} \quad \text{and} \quad \mathrm{Rot}(w_k;[0,\ell T]) = M_k$$

for every $k = 1,\dots,N$. Then, by the Rouché property, the opposite change of variables

$$\mathbf{x}(t) = \mathbf{u}(t) + \overline{\mathbf{x}}(t), \quad \mathbf{y}(t) = \mathbf{v}(t) + \overline{\mathbf{y}}(t)$$

gives us $N+1$ distinct periodic solutions of the original system (1.1), satisfying both conditions in (1.2). □

Notice that (2.9), together with the fact that $f_k(t,\cdot)$ is strictly increasing, yields that

$$f_k(t,\eta) > 0 \quad \text{for } \eta > 0 \qquad \text{and} \qquad f_k(t,\eta) < 0 \quad \text{for } \eta < 0. \tag{2.10}$$

In the following, we will assume without any further mention that (2.9) and (2.10) hold for every $k = 1,\dots,N$.

2.3 The rotational lemma

The following lemma tells us that the Landesman–Lazer conditions force all components $z_k(t) = (x_k(t), y_k(t))$ of the solutions to rotate around the origin, provided that they start sufficiently far away from the origin itself.

Lemma 2.5. *Let $M \ge 1$ be an integer and $R_1 > 0$ a real number. If assumption* (A2) *holds, then there are an $R_2 > R_1$ and an increasing function $\tau : [R_2, +\infty[\to]0,+\infty[$, satisfying the following property: for every $R \ge R_2$, if $\mathbf{z}(t) = (z_1(t),\dots,z_N(t))$, with $z_k(t) = (x_k(t), y_k(t))$, is a solution of* (2.2) *such that, for some index k and some $t_0 \in \mathbb{R}$, one has that $|z_k(t_0)| = R$, then there is a $t_1 \in]t_0, t_0 + \tau(R)]$ such that z_k is defined on $[t_0,t_1]$,*

$$|z_k(t)| > R_1 \quad \textit{for every } t \in [t_0,t_1], \qquad \mathrm{Rot}(z_k;[t_0,t_1]) > M.$$

Proof. It will be sufficient to analyze the behavior of each component $z_k(t) = (x_k(t), y_k(t))$ of the solution $\mathbf{z}(t)$. Hence, we fix $k \in \{1,\dots,N\}$ and, to simplify the notation, we consider system (2.3) and denote by $z(t) = (x(t), y(t))$ its solutions. Set

$$D_k := \max\{\|\psi_k^+\|_{L^1}, \|\psi_k^-\|_{L^1}, \|\varphi_k^+\|_{L^1}, \|\varphi_k^-\|_{L^1}\},$$

where $\psi_k^\pm$ and $\varphi_k^\pm$ are the functions introduced in condition (2.1). The key information for the argument of the proof is contained in the following two propositions.

Proposition 2.6 (Eastern region of the plane). *For any fixed $\alpha < \beta$ and $\gamma \geq d_1$, by setting*

$$\gamma^* = \gamma + \Big(\frac{\beta + D_k - \alpha}{\delta} + 1\Big) T \big\| f_k|_{[0,T]\times[\alpha,\beta+D_k]} \big\|_\infty, \tag{2.11}$$

if $z(t_0) \in]\gamma^*, +\infty[\times]\alpha, \beta[$, then there is a $t_1 > t_0$ with the following three properties:
(i) $y(t_1) = \alpha$;
(ii) *for every $t \in [t_0, t_1[$, one has $\alpha < y(t) < \beta + D_k$ and*

$$\gamma < x(t) \leq x(t_0) + (t_1 - t_0) \big\| f_k|_{[0,T]\times[\alpha,\beta+D_k]} \big\|_\infty;$$

(iii) $t_1 - t_0 \leq (\frac{\beta+D_k-\alpha}{\delta} + 1)T$.

Proof. We assume that $z(t_0) \in]\gamma^*, +\infty[\times]\alpha, \beta[$, and we define $t_1 > t_0$ as the maximal time for which $z(t) \in]\gamma, +\infty[\times]\alpha, +\infty[$ for every $t \in]t_0, t_1[$. So,

$$\begin{aligned} y(t) &= y(t_0) - \int_{t_0}^{t} g_k(t, x(t))\, dt \\ &\leq y(t_0) - \Big\lfloor \frac{t - t_0}{T} \Big\rfloor \int_0^T \psi_k^+(t)\, dt + \int_{t_0 + \lfloor \frac{t-t_0}{T} \rfloor T}^{t} \psi_k^+(t)\, dt \\ &\leq y(t_0) - \delta \Big\lfloor \frac{t - t_0}{T} \Big\rfloor + \|\psi_k^+\|_{L^1}. \end{aligned}$$

(Here and below, we denote by $\lfloor \alpha \rfloor$ the integer part of a real number α, that is, the integer $n(\alpha)$ such that $n(\alpha) \leq \alpha < n(\alpha) + 1$.) In particular, $\alpha < y(t) < \beta + \|\psi_k^+\|_{L^1}$ for every $t \in [t_0, t_1[$, and

$$\Big\lfloor \frac{t - t_0}{T} \Big\rfloor \leq \frac{\beta + \|\psi_k^+\|_{L^1} - \alpha}{\delta},$$

whence, since

$$\frac{t - t_0}{T} \leq \Big\lfloor \frac{t - t_0}{T} \Big\rfloor + 1,$$

we have that (iii) holds. Moreover, since

$$|x'(t)| \leq |f_k(t, y(t))| \leq \big\| f_k|_{[0,T]\times[\alpha,\beta+\|\psi_k^+\|_{L^1}]} \big\|_\infty$$

for every $t \in [t_0, t_1[$, we see that there is no blow-up in finite time, and

$$|x(t) - x(t_0)| \leq \int_{t_0}^{t} |x'(t)|\, dt \leq (t - t_0) \big\| f_k|_{[0,T]\times[\alpha,\beta+\|\psi_k^+\|_{L^1}]} \big\|_\infty.$$

So, (ii) follows by the choice of γ^*, and hence necessarily $y(t_1) = \alpha$. □

Proposition 2.7 (North-eastern region of the plane). *For any fixed $\mu \geq d_1$ and $\nu \geq d_1$, by setting*

$$\nu^* = \nu + D_k, \tag{2.12}$$

if $z(t_0) \in]\nu^*, +\infty] \times]\mu, +\infty[$, then there is a $t_1 > t_0$ with the following three properties:
(i) $y(t_1) = \mu$;
(ii) *for every $t \in [t_0, t_1[$, one has $\mu < y(t) < y(t_0) + D_k$ and*

$$\nu < x(t) \leq x(t_0) + (t_1 - t_0) \big\| f_k|_{[0,T]\times[\mu,y(t_0)+D_k]} \big\|_\infty;$$

(iii) $t_1 - t_0 \leq (\frac{y(t_0)+D_k-\mu}{\delta} + 1)T$.

Proof. We assume that $z(t_0) \in\,]\nu^*, +\infty[\, \times\,]\mu, +\infty[$, and we define $t_1 > t_0$ as the maximal time for which $z(t) \in\,]\nu, +\infty[\, \times\,]\mu, +\infty[$ for every $t \in\,]t_0, t_1[$. As in the proof of Proposition 2.6, we have

$$y(t) \leq y(t_0) - \delta\Big\lfloor \frac{t - t_0}{T} \Big\rfloor + \|\psi_k^+\|_{L^1}$$

for every $t \in [t_0, t_1[$. In particular, $\mu < y(t) < y(t_0) + \|\psi_k^+\|_{L^1}$ and

$$\Big\lfloor \frac{t - t_0}{T} \Big\rfloor \leq \frac{y(t_0) + \|\psi_k^+\|_{L^1} - \mu}{\delta},$$

whence (iii) holds. On the other hand,

$$\begin{aligned} x(t) &= x(t_0) + \int_{t_0}^{t} f_k(t, y(t))\, dt \\ &\geq x(t_0) + \Big\lfloor \frac{t - t_0}{T} \Big\rfloor \int_0^T \varphi_k^+(t)\, dt + \int_{t_0 + \lfloor \frac{t-t_0}{T} \rfloor T}^{t} \varphi_k^+(t)\, dt \\ &\geq x(t_0) - \|\varphi_k^+\|_{L^1} \end{aligned}$$

for every $t \in [t_0, t_1[$. Moreover, since

$$|x'(t)| \leq |f_k(t, y(t))| \leq \big\|f_k|_{[0,T]\times[\mu, y(t_0) + \|\psi_k^+\|_{L^1}]}\big\|_\infty$$

for every $t \in [t_0, t_1[$, we see that there is no blow-up in finite time, and

$$|x(t) - x(t_0)| \leq (t_1 - t_0)\big\|f_k|_{[0,T]\times[\mu, y(t_0) + \|\psi_k^+\|_{L^1}]}\big\|_\infty.$$

So, (ii) follows by the choice of ν^*, and hence necessarily $y(t_1) = \mu$. □

By the symmetry of our assumption (A2), we can write the analogous of Proposition 2.6 in the northern, western and southern regions, and the analogous of Proposition 2.7 in the north-western, south-western and south-eastern regions. For briefness, we leave this easy but tedious charge to the patient reader.

Let us now proceed with the proof of Lemma 2.5. Let M be a positive integer and let $R_1 > 0$ be fixed: we can assume, without loss of generality, that $R_1 \geq 2d_1$. We will define two polygonal curves Γ_1^k and Γ_2^k, represented in Figure 1, which will guide the components of the solutions: they are spiral-like curves, rotating counterclockwise around the origin infinitely many times as their distance from the origin goes to infinity. (For a similar approach, see also [10].)

We start by fixing three constants $\beta_1 \geq R_1$, $\alpha_1 = -\beta_1$ and $\gamma_1 \geq R_1$.

First part of Γ_1^k. This is simply the segment $\{\gamma_1\} \times [-\beta_1, \beta_1]$.

First part of Γ_2^k. This is made up of three joined segments, which will now be defined. Using Proposition 2.6 (eastern region), with $\alpha = \alpha_1$, $\beta = \beta_1$ and $\gamma = \gamma_1$, we find a $\gamma_1^* > \gamma_1$, defined as in (2.11), i.e.,

$$\gamma_1^* = \gamma_1 + \Big(\frac{2\beta_1 + \|\psi_k^+\|_{L^1}}{\delta} + 1\Big) T \big\|f_k|_{[0,T]\times[-\beta_1, \beta_1 + \|\psi_k^+\|_{L^1}]}\big\|_\infty.$$

The first of the three segments is $\{\gamma_1^*\} \times [-\beta_1, \beta_1]$. We now use Proposition 2.7 (north-eastern region), with $\mu = \beta_1$ and $\nu = \gamma_1^*$, and we find a $\nu_1^* > \gamma_1^*$, defined as in (2.12), i.e.,

$$\nu_1^* = \gamma_1^* + \|\varphi_k^+\|_{L^1}.$$

The second of the three segments is $[\gamma_1^*, \nu_1^*] \times \{\beta_1\}$. We now use the northern version of Proposition 2.6, with $\alpha = -\nu_1^*$, $\beta = \nu_1^*$ and $\gamma = \beta_1$, and we find a $\gamma_2^* > \beta_1$, defined similarly to (2.11), precisely

$$\gamma_2^* = \beta_1 + \Big(\frac{2\nu_1^* + \|\varphi_k^+\|_{L^1}}{\delta} + 1\Big) T \big\|g_k|_{[0,T]\times[-\nu_1^* - \|\varphi_k^+\|_{L^1}, \nu_1^*]}\big\|_\infty.$$

The third of the three segments is then $\{\nu_1^*\} \times [\beta_1, \gamma_2^*]$.

We now iterate such a procedure in the other regions, as briefly explained below.

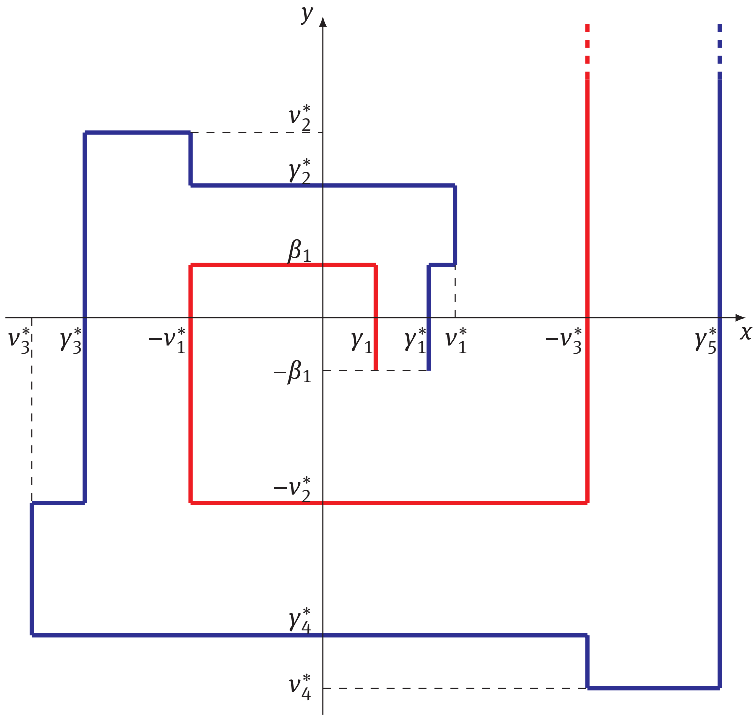

Figure 1: The curves Γ_1^k and Γ_2^k.

Second part of Γ_1^k. This is the segment $[-v_1^*, \gamma_1] \times \{\beta_1\}$.

Second part of Γ_2^k. As before, this is made up of three segments. The first one is $[-v_1^*, v_1^*] \times \{\gamma_2^*\}$. We now use the north-western version of Proposition 2.7, with $\mu = -v_1^*$ and $v = \gamma_2^*$, and we find a $v_2^* > \gamma_2^*$, similarly to (2.12), precisely

$$v_2^* = \gamma_2^* + \|\psi_k^+\|_{L^1}.$$

The second of the three segments is $\{-v_1^*\} \times [\gamma_2^*, v_2^*]$. We then use the western version of Proposition 2.6, with $\alpha = -v_2^*$, $\beta = v_2^*$ and $\gamma = -v_1^*$, and we find a $\gamma_3^* > v_1^*$ (we prefer writing γ_3^* instead of $-\gamma_3^*$ in order to deal with a positive constant, even if $x(t)$ is negative in this region), similarly to (2.11), precisely

$$\gamma_3^* = v_1^* + \Big(\frac{2v_2^* + \|\psi_k^-\|_{L^1}}{\delta} + 1\Big) T \big\| f_k|_{[0,T]\times[-v_2^* - \|\psi_k^-\|_{L^1}, v_2^*]} \big\|_\infty .$$

The third of the three segments is then $[-\gamma_3^*, -v_1^*] \times \{v_2^*\}$.

Third part of Γ_1^k. This is the segment $\{-v_1^*\} \times [-v_2^*, \beta_1]$.

Third part of Γ_2^k. This time, the first segment is $\{-\gamma_3^*\} \times [-v_2^*, v_2^*]$. Using the south-western version of Proposition 2.7, with $\mu = -v_2^*$ and $v = -\gamma_3^*$, we find a $v_3^* > \gamma_3^*$ (again we prefer dealing with positive constants), precisely

$$v_3^* = \gamma_3^* + \|\varphi_k^-\|_{L^1}.$$

The second segment is $[-v_3^*, -\gamma_3^*] \times \{-v_2^*\}$. Using the southern version of Proposition 2.6, with $\alpha = -v_3^*$, $\beta = v_3^*$ and $\gamma = -v_2^*$, and we find a $\gamma_4^* > v_2^*$ (again a positive constant), precisely

$$\gamma_4^* = v_2^* + \Big(\frac{2v_3^* + \|\varphi_k^-\|_{L^1}}{\delta} + 1\Big) T \big\| g_k|_{[0,T]\times[-v_3^*, v_3^* + \|\varphi_k^-\|_{L^1}]} \big\|_\infty .$$

The third segment is then $\{v_3^*\} \times [-\gamma_4^*, -v_2^*]$.

Fourth part of Γ_1^k. This is simply the segment $[-v_1^*, v_3^*] \times \{-v_2^*\}$.

Fourth part of Γ_2^k. As usual, this is made up of three segments. The first one is $[-\nu_3^*, \nu_3^*] \times \{-\gamma_4^*\}$. We now use the south-eastern version of Proposition 2.7, with $\mu = \nu_3^*$ and $\nu = -\gamma_4^*$, and we find a $\nu_4^* > \gamma_4^*$ (again positive), precisely

$$\nu_4^* = \gamma_4^* + \|\psi_k^-\|_{L^1}.$$

The second segment is $\{\nu_3^*\} \times [-\nu_4^*, -\gamma_4^*]$. We now use the eastern version of Proposition 2.6, with $\alpha = -\nu_4^*$, $\beta = \nu_4^*$ and $\gamma = \nu_3^*$, and we find a $\gamma_5^* > \nu_3^*$, precisely

$$\gamma_5^* = \nu_3^* + \Big(\frac{2\nu_4^* + \|\psi_k^+\|_{L^1}}{\delta} + 1\Big)T\big\|f_k|_{[0,T]\times[-\nu_4^*, \nu_4^* + \|\psi_k^+\|_{L^1}]}\big\|_\infty.$$

The third segment is then $[\nu_3^*, \gamma_5^*] \times \{-\nu_4^*\}$.

Fifth part of Γ_1^k. This is simply the segment $\{\nu_3^*\} \times [-\nu_2^*, \nu_4^*]$.

Fifth part of Γ_2^k. This is constructed exactly as the first part, starting with the segment $\{\gamma_5^*\} \times [-\nu_4^*, \nu_4^*]$, and then continuing analogously.

After having completed the first lap, we can now proceed recursively, until the curves Γ_1^k and Γ_2^k have completed $M + 1$ rotations around the origin.

Fix $R_2 > 0$ so that the curves Γ_1^k and Γ_2^k are contained in the ball centered at the origin, with radius R_2. Choose $R \ge R_2$, and let $z(t) = (x(t), y(t))$ be a solution of (2.3) such that, for some $t_0 \in \mathbb{R}$, one has that $|z(t_0)| = R$. We will analyze the behavior of $z(t)$ showing that its orbit is controlled, and in some sense guided, by the curves Γ_1^k and Γ_2^k. Indeed, the curve Γ_1^k keeps $z(t)$ from getting too close to the origin, while Γ_2^k provides some reference lines which must be crossed by the orbit of $z(t)$, forcing it to rotate around the origin. Moreover, the estimates given in Propositions 2.6 and 2.7, and their analogues in the other regions of the plane, show that the amplitudes of the orbit and the times needed by the orbit to cross the different regions of the plane are all controlled by some constants which can be chosen to depend only on R.

More precisely, let $z(t)$ be a solution with $|z(t_0)| = R \ge R_2$. It is possible to determine the region where $z(t_0)$ is located with respect to the last lap of Γ_2^k. Assume, for instance, that it is in the “northern region”, by which we mean that $\alpha \le x(t_0) \le \beta$ and $y(t_0) \ge \gamma$, where $\alpha = -\beta$ and γ are as shown in Figure 2.

Then, by the analogue of Proposition 2.6, there is a first time $t_1 \ge t_0$ at which the orbit reaches a point $z(t_1) = (x(t_1), y(t_1))$, with $x(t_1) = \beta$, and

$$\alpha - D_k \le x(t) \le \beta, \quad \mu \le y(t) \le \kappa_1(R) \qquad \text{for every } t \in [t_0, t_1],$$

where $\mu > 0$ is determined by the inner curve Γ_1^k, and

$$\kappa_1(R) = R + \Big(\frac{2R + D_k}{\delta} + 1\Big)T\big\|g_k|_{[0,T]\times[-R-D_k, R]}\big\|_\infty.$$

Moreover, by the analogue of Proposition 2.6, the time interval $t_1 - t_0$ is controlled from above by a constant which may be chosen to depend only on R, since the starting point lies on a compact set.

Therefore, we have that $z(t_1) \in \{\beta\} \times [\mu, \kappa_1(R)]$. The solution now enters the “north-eastern region” depicted in Figure 3 and, by Proposition 2.7, there is a first time $t_2 \ge t_1$ at which the orbit reaches a point $z(t_2) = (x(t_2), y(t_2))$ with $y(t_2) = \mu$, and

$$\nu \le x(t) \le \kappa_2(R), \quad \mu \le y(t) \le \kappa_1(R) + D_k \qquad \text{for every } t \in [t_1, t_2],$$

where $\nu = \beta - D_k$ and

$$\kappa_2(R) = \kappa_1(R) + \Big(\frac{\kappa_1(R) + D_k}{\delta} + 1\Big)T\big\|f_k|_{[0,T]\times[0,\kappa_1(R)+D_k]}\big\|_\infty.$$

By Proposition 2.7, the time interval $t_2 - t_1$ is controlled from above by a constant which may be chosen to depend only on R since we started from a compact set.

Now the solution has arrived at $z(t_2) \in [\beta - D_k, \kappa_2(R)] \times \{\mu\}$, and it enters the “eastern region”, where it behaves similarly to the northern region: we will find a first time $t_3 \ge t_2$ at which the orbit reaches a point $z(t_3) = (x(t_3), y(t_3))$, with $y(t_3) = -\mu$, and

$$\rho \le x(t) \le \kappa_3(R), \quad -\mu \le y(t) \le \mu + D_k \qquad \text{for every } t \in [t_2, t_3],$$

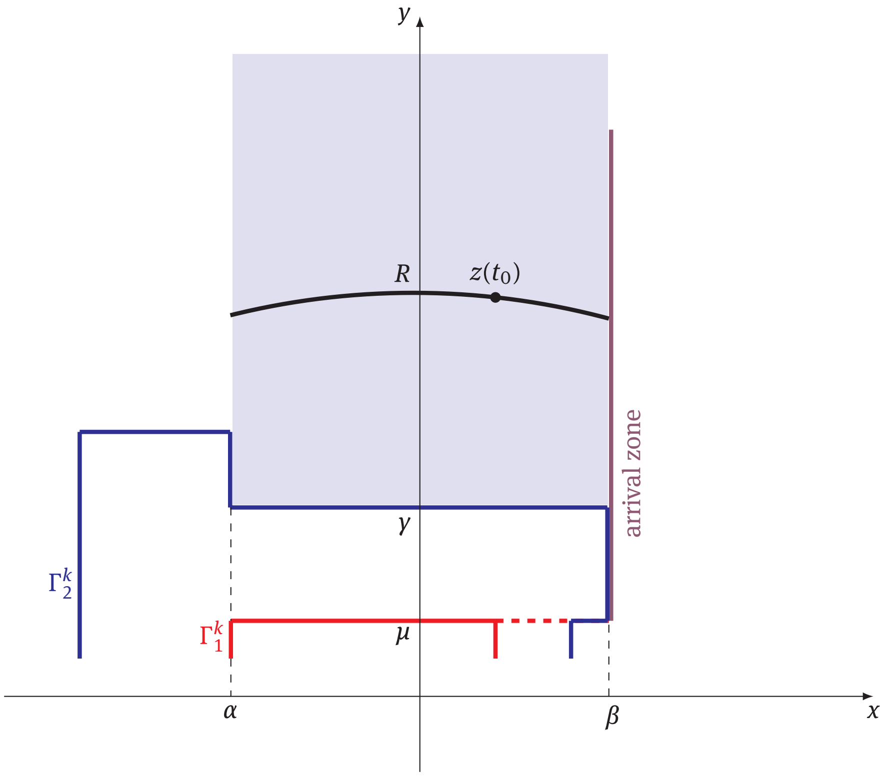

Figure 2: The northern region.

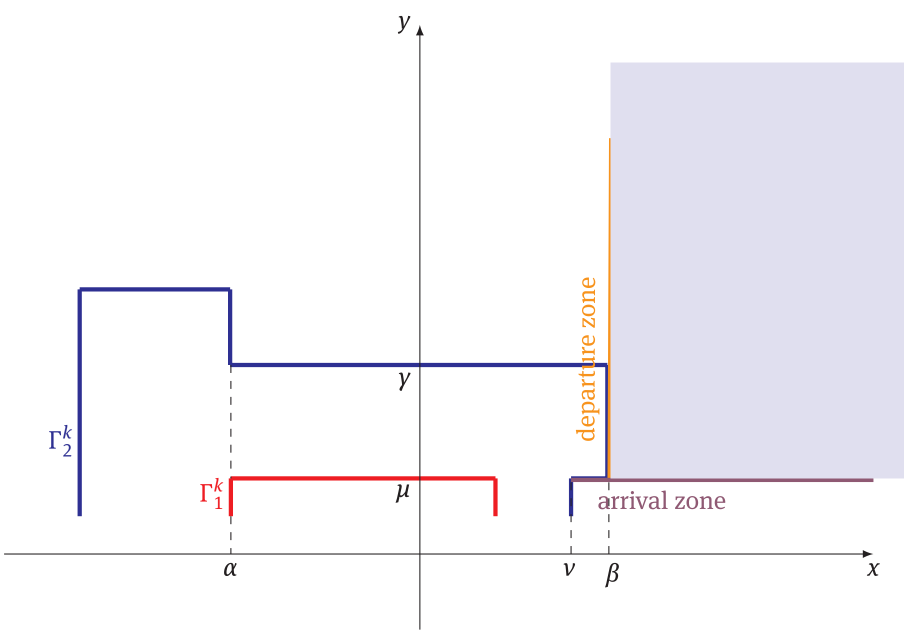

Figure 3: The north-eastern region.

where $\rho > 0$ is determined by Γ_1^k, and $\kappa_3(R)$ is a constant depending only on R (see Figure 4). Again, the time interval $t_3 - t_2$ is controlled from above by a constant which only depends on R.

And this can be repeated on and on, until the solution has completed one rotation around the origin. Observe that, while crossing the different regions, the orbit of $z(t)$ is always "controlled from below" by the

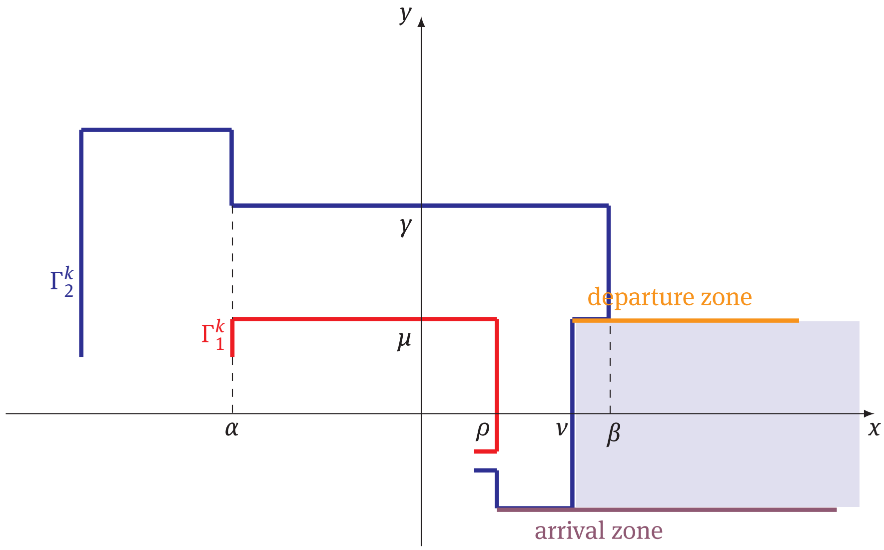

Figure 4: The eastern region.

inner curve Γ_1^k, which will guarantee that, during all the time needed to perform a complete rotation, the distance from the origin will remain greater than R_1.

Clearly enough, the same type of reasoning applies when $z(t_0)$, instead of being in the "northern region", belongs to the "north-eastern region". The estimates will still depend only on R by continuity and compactness. When $z(t_0)$ belongs to any of the other regions, the situation is perfectly symmetrical with the above, as can be seen by rotating Figure 1 by a multiple of 90 degrees.

After the solution has completed one rotation around the origin, it could have approached the origin, but not too much, due to the fact that it cannot intersect the curve Γ_1^k. Hence, we can repeat the same argument, taking this time as reference regions those determined by the inner lap of Γ_2^k, until the solution has completed the second rotation around the origin. And all this can be repeated until the solution has performed $M+1$ rotations around the origin, thus completing the proof. □

Remark 2.8. Assumptions (A2) and (A3) alone imply that the solutions of system (2.2) are globally defined.

Indeed, assume by contradiction that for some $k \in \{1, \dots, N\}$ there is a solution z_k of (2.3) and a strictly increasing bounded sequence $(t_n)_n$ along which

$$|z_k(t_n)| \to +\infty.$$

By (A3), there is an $R_1 > 0$ and a sector Θ_k such that, as long as $|z_k(t)|$ remains greater than R_1, the time needed to cross this sector is greater than 1. Using Lemma 2.5 with $M = 1$, we determine $R_2 > R_1$. Take n such that $|z_k(t_n)| \geq R_2$. Then there is a $\hat{t}_n > t_n$ such that z_k is defined on $[t_n, \hat{t}_n]$,

$$|z_k(t)| > R_1 \quad \text{for every } t \in [t_n, \hat{t}_n], \qquad \operatorname{Rot}(z_k; [t_n, \hat{t}_n]) > 1.$$

It follows that $\hat{t}_n - t_n \geq 1$. Now let us take an $n_1 > n$ such that $t_{n_1} \geq \hat{t}_n$ and $|z_k(t_{n_1})| \geq R_2$. Repeating the same argument, we find a $\hat{t}_{n_1} > t_{n_1}$ such that z_k is defined on $[t_{n_1}, \hat{t}_{n_1}]$,

$$|z_k(t)| > R_1 \quad \text{for every } t \in [t_{n_1}, \hat{t}_{n_1}], \qquad \operatorname{Rot}(z_k; [t_{n_1}, \hat{t}_{n_1}]) > 1.$$

It follows that $\hat{t}_{n_1} - t_{n_1} \geq 1$. Iterating this process, we find a subsequence $(t_{n_j})_j$ such that $t_{n_{j+1}} - t_{n_j} \geq 1$, thus contradicting the boundedness of $(t_n)_n$.

We have thus proved global existence in the future. Concerning the past, this can be obtained by reversing the time and arguing similarly.

2.4 End of the proof

The proof will follow from a generalized version of the Poincaré–Birkhoff Theorem recently proposed in [14]. We now recall this result, which is stated for a general Hamiltonian system of the type

$$\begin{cases} x_k' = \dfrac{\partial \mathcal{H}}{\partial y_k}(t,\mathbf{x},\mathbf{y}), \\ -y_k' = \dfrac{\partial \mathcal{H}}{\partial x_k}(t,\mathbf{x},\mathbf{y}), \end{cases} \qquad k = 1,\dots,N. \tag{2.13}$$

Here, $\mathcal{H} : \mathbb{R}\times\mathbb{R}^{2N} \to \mathbb{R}$ is continuous, T-periodic in t and continuously differentiable in $(\mathbf{x},\mathbf{y}) \in \mathbb{R}^{2N}$.

Assume that for each $k = 1,\dots,N$ we have two strictly star-shaped Jordan curves around the origin $\mathcal{C}_1^k, \mathcal{C}_2^k \subseteq \mathbb{R}^2$ such that, by denoting by $\mathcal{D}(\Gamma)$ the open bounded region delimited by the Jordan curve Γ,

$$0 \in \mathcal{D}(\mathcal{C}_1^k) \subseteq \overline{\mathcal{D}(\mathcal{C}_1^k)} \subseteq \mathcal{D}(\mathcal{C}_2^k).$$

We consider the annular regions $\mathcal{A}_k = \overline{\mathcal{D}(\mathcal{C}_2^k)} \setminus \mathcal{D}(\mathcal{C}_1^k)$ for $k = 1,\dots,N$, and set

$$\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_N.$$

We will write $z_k = (x_k, y_k)$, for $k = 1,\dots,N$, and $\mathbf{z} = (z_1,\dots,z_N)$. Let us state the result in [14, Theorem 1.2].

Theorem 2.9. *Assume that every solution of the Hamiltonian system* (2.13)*, departing with* $\mathbf{z}(0) \in \mathcal{A}$*, is defined on* $[0, \ell T]$*, where* ℓ *is a positive integer, and satisfies*

$$z_k(t) \neq (0,0) \quad \textit{for every } t \in [0,\ell T] \textit{ and } k = 1,\dots,N.$$

Assume moreover that, for each $k = 1,\dots,N$*, there is an integer* M_k *such that*

$$\mathrm{Rot}(z_k; [0,\ell T]) \begin{cases} > M_k & \textit{if } z_k(0) \in \Gamma_1^k, \\ < M_k & \textit{if } z_k(0) \in \Gamma_2^k. \end{cases}$$

Then the Hamiltonian system (2.13) *has at least* $N+1$ *distinct* ℓT*-periodic solutions* $\mathbf{z}^0(t),\dots,\mathbf{z}^N(t)$*, with* $\mathbf{z}^0(0),\dots,\mathbf{z}^N(0) \in \mathcal{A}$*, such that*

$$\mathrm{Rot}(z_k^j; [0,\ell T]) = M_k \quad \textit{for every } j = 0,\dots,N \textit{ and } k = 1,\dots,N.$$

Why do we say that these solutions $\mathbf{z}^0(t),\dots,\mathbf{z}^N(t)$ are distinct? Could they not belong to the same periodicity class? Well, the fact that these $N+1$ solutions are distinct is a consequence of the proof of [14, Theorem 1.2], which is carried out by a variational method. Indeed, these solutions are obtained as critical points of a suitable functional $\varphi : \mathbb{T}^N \times \mathcal{H} \to \mathbb{R}$, using a generalized Lusternik–Schnirelmann theorem. Here, $\mathbb{T}^N$ is the N-dimensional torus, and $\mathcal{H}$ is a Hilbert space. The theory says that either all the corresponding critical levels are different or the set of critical points is not contractible. The claim then follows since the solutions belonging to the same periodicity class are critical points on which the functional has the same value.

Let us go back to the proof of Theorem 1.1. We first consider system (1.1) with $\varepsilon = 0$, which is split in the N uncoupled subsystems, as in (2.2). Notice that, by (2.9), the subsystem (2.3) has the solution $z_k = (x_k, y_k)$ with x_k and y_k identically equal to 0. Hence, as a consequence of the uniqueness of solutions to Cauchy problems, if $z_k(t)$ is a solution of (2.3) with $z_k(0) \neq (0,0)$, then $z_k(t) \neq (0,0)$ for every $t \geq 0$.

We now use Lemma 2.5: by taking $M = \max\{M_1,\dots,M_N\} + 1$ and $R_1 = \bar{R}$, there is an $r_k > \bar{R}$ such that if z_k is a solution of (2.3) satisfying $|z_k(t_0)| = r_k$ for some $t_0 \in \mathbb{R}$, then there is a $t_1^k \in \,]t_0, t_0 + \tau(r_k)]$ such that $\mathrm{Rot}(z_k, [t_0, t_1^k]) > M$. Fix an integer $\bar{\ell}$ such that

$$\bar{\ell} T \geq \max\{\tau(r_k) : k = 1,\dots,N\},$$

and take an integer $\ell \geq \bar{\ell}$. By (2.10), the solutions can never rotate counterclockwise more than half a turn. Hence,

$$|z_k(0)| = r_k \implies \mathrm{Rot}(z_k, [0,\ell T]) > M + \frac{1}{2}. \tag{2.14}$$

We now estimate the rotation of large amplitude solutions. By assumption (A3), taking $\sigma = 1/(2\ell T)$, we can find an $\mathcal{R}_k > r_k$ such that if $z_k(t)$ is a solution of (2.3) such that $|z_k(t)| \geq \mathcal{R}_k$ for every $t \in [0, \ell T]$, then the time needed to cross the angular sector Θ_k is greater than $2\ell T$. Hence, there is a $\gamma \in]0, 1[$ such that $\mathrm{Rot}(z_k, [0, \ell T]) < 1 - \gamma$. On the other hand, assumption (A1) implies that there is an $R_k \geq \mathcal{R}_k$ such that if $|z_k(0)| \geq R_k$, then $|z_k(t)| \geq \mathcal{R}_k$ for every $t \in [0, \ell T]$. Hence,

$$|z_k(0)| = R_k \implies \mathrm{Rot}(z_k, [0, \ell T]) < 1 - \gamma. \tag{2.15}$$

We define the planar annulus

$$\mathcal{A}_k = \overline{B}((0,0), R_k) \setminus B((0,0), r_k),$$

so that, by taking $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_N$, the assumptions of Theorem 2.9 are satisfied by system (1.1) when $\varepsilon = 0$, i.e., by system (2.2). Since $\mathcal{A}$ is a compact set, the solutions of this system, starting with $\mathbf{z}(0) \in \mathcal{A}$, will remain, for every $t \in [0, \ell T]$, in the interior of a larger set $\widetilde{\mathcal{A}} = \widetilde{\mathcal{A}}_1 \times \cdots \times \widetilde{\mathcal{A}}_N$, where

$$\widetilde{\mathcal{A}}_k = \overline{B}((0,0), \widetilde{R}_k) \setminus B((0,0), \tilde{r}_k)$$

for some positive $\tilde{r}_k < r_k$ and $\widetilde{R}_k > R_k$. Since the partial derivatives of $\mathcal{U}$ with respect to x_k and y_k are continuous, they are bounded on the compact set $[0, T] \times \widetilde{\mathcal{A}}_k \times [-1, 1]$. Therefore, if $\varepsilon \in [-1, 1]$ is taken with $|\varepsilon|$ small enough, the solutions $\mathbf{z}(t)$ of (1.1) starting with $\mathbf{z}(0) \in \mathcal{A}$ will also remain in $\widetilde{\mathcal{A}}$ for every $t \in [0, \ell T]$. Moreover, by (2.14) and (2.15), if $|\varepsilon|$ is sufficiently small, then

$$|z_k(0)| = r_k \implies \mathrm{Rot}(z_k, [0, \ell T]) > M_k$$

and

$$|z_k(0)| = R_k \implies \mathrm{Rot}(z_k, [0, \ell T]) < 1 \leq M_k.$$

Hence, Theorem 2.9 applies, providing the existence of $N + 1$ distinct ℓT-periodic solutions $\mathbf{z}^0(t), \dots, \mathbf{z}^N(t)$ of (1.1), with $\mathbf{z}^0(0), \dots, \mathbf{z}^N(0) \in \mathcal{A}$, such that

$$\mathrm{Rot}(z_k^j; [0, \ell T]) = M_k \quad \text{for every } j = 0, \dots, N \text{ and } k = 1, \dots, N.$$

Moreover, it has to be $|z_k^j(t)| > r_k$, for every $t \in [0, \ell T]$, since otherwise Lemma 2.5 would imply the rotation number to be greater than $M_k + 1$.

The proof of Theorem 1.1 is thus completed.

3 Proof of the corollaries and final remarks

Let us first prove the two corollaries stated in Section 1.

Proof of Corollary 1.3. Since $f_k(t, \eta) = \eta$, we have that (A1), (A2) and (A4) certainly hold, hence we just have to verify (A3). Let $\sigma \in]0, \frac{\pi}{3}[$ be fixed. We consider the planar sector Θ_k with $\hat{\theta}_k = -\sigma$ and $\check{\theta}_k = \sigma$. If $(\xi, \eta) \in \Theta_k$, by writing $\xi = \rho\cos\theta$ and $\eta = \rho\sin\theta$, since $\cos\theta \geq \frac{1}{2}$, there is an $\mathcal{R}_k > 0$ such that if $\rho \geq \mathcal{R}_k$, then

$$\begin{aligned}\frac{g_k(t,\xi)\xi + f_k(t,\eta)\eta}{\xi^2 + \eta^2} &\leq \sin^2\theta + \left|\frac{g(\rho\cos\theta)}{\rho\cos\theta}\right|\cos^2\theta \\ &\leq \theta^2 + \left|\frac{g(\rho\cos\theta)}{\rho\cos\theta}\right| \\ &\leq 2\sigma^2 = \sigma(\check{\theta}_k - \hat{\theta}_k),\end{aligned}$$

thus completing the proof. □

Proof of Corollary 1.4. Since $f_k(t,\eta) = \eta/\sqrt{1+\eta^2}$, also in this case (A1), (A2) and (A4) hold, and we need to verify only (A3). Recall that, by (A1), $|g(t,\xi)| \le C(1+|\xi|)$. Let $\sigma \in \,]0, \frac{C\pi}{3}[$ be fixed. We consider the planar sector Θ_k with

$$\hat\theta_k = \frac{\pi}{2} - \frac{\sigma}{C}, \qquad \check\theta_k = \frac{\pi}{2} + \frac{\sigma}{C}.$$

If $(\xi,\eta) \in \Theta_k$, by writing $\xi = \rho\cos\theta$ and $\eta = \rho\sin\theta$, since $\sin\theta \ge \frac{1}{2}$ and $|\cos\theta| \le \frac{\sigma}{C}$, there is an $\mathcal{R}_k > 0$ such that if $\rho \ge \mathcal{R}_k$, then

$$\begin{aligned}\frac{g_k(t,\xi)\xi + f_k(t,\eta)\eta}{\xi^2+\eta^2} &\le C\cos^2\theta + \frac{C|\cos\theta|}{\rho} + \frac{\sin^2\theta}{\sqrt{1+\rho^2\sin^2\theta}}\\ &\le C\frac{\sigma^2}{C^2} + \frac{C}{\rho} + \frac{2}{\sqrt{4+\rho^2}}\\ &\le 2\frac{\sigma^2}{C} = \sigma(\check\theta_k - \hat\theta_k),\end{aligned}$$

and the proof is thus completed. □

We conclude with some final remarks.

1. Our results still hold if the continuity assumptions are replaced by some L^p-Carathéodory conditions, with $p > 1$. Indeed, [14, Theorem 1.2] still holds in this case, as noticed in [14, Section 8].

2. Instead of having a single parameter ε, in our $2N$ equations we could have several of them. The statements of our theorems can be easily modified, in this case.

3. Our results hold for weakly coupled systems, but we think that they should not be included in what is usually called *perturbation theory* [22]. (For the use of the Poincaré–Birkhoff Theorem to the study of periodic perturbations of Hamiltonian systems, see [5].) Indeed, we do not have some known solutions of the uncoupled system with $\varepsilon = 0$, which give rise to the periodic solutions we are looking for. This fact suggests that there should be some generalizations of our Theorem 1.1 to systems which do not necessarily explicitly depend on one or more parameters but satisfy some assumptions guaranteeing the main qualitative properties of the solutions, which have been emphasized in this paper.

4. When $N = 1$, a scalar second-order equation has been proposed as a simple model for the vertical oscillations of suspension bridges by Lazer and McKenna in [16]. For such a model, subharmonic solutions have been found in [8, 9]. A more realistic model would involve the partial differential equation of an elastic beam, cf. [17]. However, one could try to discretize this equation in space, thus obtaining a system of second-order differential equations, coupled by a symmetric matrix. It would be interesting to generalize the results obtained in this paper, showing that large amplitude subharmonic vertical oscillations also arise for this type of suspension bridge models.

5. Another model which can be reduced to our setting is the Lotka–Volterra predator-prey system

$$\begin{cases} u_k' = \alpha_k u_k - \Big(\beta_k + \dfrac{\partial\mathcal{W}}{\partial v_k}(t,\mathbf{u},\mathbf{v};\varepsilon)\Big)u_k v_k, \\ -v_k' = \gamma_k v_k - \Big(\delta_k + \dfrac{\partial\mathcal{W}}{\partial u_k}(t,\mathbf{u},\mathbf{v};\varepsilon)\Big)u_k v_k, \end{cases} \qquad k = 1,\dots,N, \tag{3.1}$$

where $\alpha_k, \beta_k, \gamma_k, \delta_k$ are positive constants, $\mathbf{u} = (u_1,\dots,u_N)$, $\mathbf{v} = (v_1,\dots,v_N)$, and $\mathcal{W}$ is T-periodic in t and identically zero when $\varepsilon = 0$. Notice that the point $(\gamma_k/\delta_k, \alpha_k/\beta_k)$ is an equilibrium for the corresponding planar subsystem with $\varepsilon = 0$. We look for solutions having all components u_k, v_k positive. By following [3], the change of variables $(x_k, y_k) = (\ln u_k, \ln v_k)$ can be performed to translate the system into the form (1.1). We thus get the following result.

Theorem 3.1. *Let $\bar R$ be a positive real number and let $M_1,\dots,M_N$ be some positive integers. Then there is a positive integer $\bar\ell$ with the following property: for every integer $\ell \ge \bar\ell$, there exists $\varepsilon_\ell > 0$ such that if $|\varepsilon| \le \varepsilon_\ell$, system* (3.1) *has at least $N+1$ distinct ℓT-periodic solutions $\mathbf{w}(t) = (w_1(t),\dots,w_N(t))$, with $w_k(t) = (u_k(t), u_k(t))$ having positive components, which satisfy*

$$\min\{|\ln u_k(t)| + |\ln v_k(t)| : t \in [0,\ell T]\} \ge \bar R$$

and

$$\operatorname{Rot}\Big(\Big(u_k - \frac{\gamma_k}{\delta_k}, v_k - \frac{\alpha_k}{\beta_k}\Big); [0, \ell T]\Big) = M_k$$

for every $k = 1, \dots, N$.

The proof uses the same ideas as before, but is simplified by the fact that the system with $\varepsilon = 0$ is autonomous. The boundaries of the planar annuli needed for the application of the Poincaré–Birkhoff Theorem can indeed be chosen as the orbits of this autonomous system.

A more general system where α_k, β_k, γ_k, δ_k are replaced by T-periodic positive continuous functions could be considered as well. The same result still holds, and the proof can be carried out similarly, using the estimates in [3].

As an example of application, we could have four species involved, the first species predating only the second, and the third species predating only the fourth. A weak interaction among all of them then preserves the existence of periodic solutions.

6. Some nonlinearities with a singularity can also be treated with the same approach. For example, let us consider the system

$$\begin{cases} x_1'' + g_1(x_1) = \dfrac{\partial \mathcal{V}}{\partial x_1}(t, x_1, \dots, x_N; \varepsilon), \\ \qquad \vdots \\ x_N'' + g_N(x_N) = \dfrac{\partial \mathcal{V}}{\partial x_N}(t, x_1, \dots, x_N; \varepsilon), \end{cases}$$

and let us assume $g_k : \mathbb{R}_+ \to \mathbb{R}$, with $\mathbb{R}_+ =]0, +\infty[$, to be locally Lipschitz continuous functions, while $\mathcal{V} : \mathbb{R} \times \mathbb{R}_+^N \times \mathbb{R} \to \mathbb{R}$ satisfies the assumptions stated for system (1.3). Let us define the primitive functions

$$G_k(\xi) = \int_1^\xi g_k(s)\, ds. \tag{3.2}$$

The following is an illustrative example of the corresponding existence result.

Theorem 3.2. *Assume the following conditions:*
(i) $\lim_{\xi \to +\infty} G_k(\xi)/\xi^2 = 0$;
(ii) $g_k(\xi)(\xi - 1) > 0$ *for every* $\xi \neq 1$;
(iii) $\lim_{\xi \to 0^+} G_k(\xi) = \lim_{\xi \to +\infty} G_k(\xi) = +\infty$.

Then the same conclusion of Theorem 1.1 holds, with $z_k(t) = (x_k(t), x_k'(t))$ *and* (1.2) *replaced by*

$$\min\Big\{\Big(x_k(t)^2 + \frac{1}{x_k(t)^2} + x_k'(t)^2\Big)^{1/2} : t \in [0, \ell T]\Big\} \geq \bar{R},$$

and

$$\operatorname{Rot}((x_k - 1, x_k'); [0, \ell T]) = M_k$$

for every $k = 1, \dots, N$.

The proof is indeed easier in this case, since each planar annulus is determined by choosing two level curves of the corresponding Hamiltonian function $H_k(\xi, \eta) = \frac{1}{2}\eta^2 + G_k(\xi)$. Condition (i) then implies that the time map has an infinite limit, i.e., the large amplitude solutions rotate very slowly. A similar argument as in the proof of Theorem 1.1 then leads to the conclusion. It should be clear that the choice of the point $(1, 0)$ around which the solutions rotate is not significant.

7. We can also adapt our approach to a system like (1.1), with f_k and g_k defined on $\mathbb{R}\times]0, +\infty[$ and both having a singularity at 0, a situation which has already been considered in [24]. For instance, in the case when f_k and g_k do not depend on t, define G_k as in (3.2) and F_k similarly, and assume for both that conditions (ii) and (iii) of Theorem 3.2 hold true. In this setting, the orbits of the unperturbed planar subsystems are the level lines of the function $H_k(\xi, \eta) = G_k(\xi) + F_k(\eta)$, which are star-shaped closed curves surrounding the point $(1, 1)$. If in addition assumptions (A3) and (A4) hold, with $0 < \hat{\theta}_k < \check{\theta}_k < \frac{\pi}{2}$, we are able to conclude

similarly: there are large-amplitude subharmonic solutions performing a given number of rotations around the point $(1, 1)$ in their period time.

8. As observed in Corollary 1.2, when $N \geq 2$, in all the above examples we get a myriad of subharmonic solutions with *minimal period* ℓT, with one planar component performing exactly one rotation, while the other components rotate an arbitrary number of times.

9. Clearly enough, the different types of equations considered above could be mixed up in the same system.

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