On the metric structure of time in classical and quantum mechanics

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Abstract

In this paper we show that, via an extension of time, some metric structures naturally appear in both classical and quantum mechanics when both are formulated via path integrals. We calculate the various Ricci scalar and curvatures associated to these metrics and prove that they can be choosen to be zero in classical mechanics while this is not possible in quantum mechanics.

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1. Introduction

Quantum gravity is for sure one of the most outstanding open problem in theoretical physics. The usual approach is to take a geometric classical theory (like Einstein gravity, string or similar) and apply quantum mechanics to it. People have never reversed the problem, that means first try to understand if there is some hidden universal geometry in quantum mechanics and, second, see if this geometry is compatible with the geometry of the classical model that we want to quantize. In this paper we will concentrate on the first of the two issues above that means try to understand if there is some hidden geometry in quantum mechanics. In doing this we will discover some nice things which may have some application. In a future paper we hope to come back to the second issue mentioned above.

It was shown in [1] that classical mechanics (CM) and quantum mechanics (QM) could have a very similar formulation via path-integrals. The generating function of the first, which we will indicate with the acronym CPI (for *classical path integral*) has the form:

$$\mathcal{Z}_{CPI}\,[\mathbb{J}] = \int \mathcal{D}\Phi^a \exp\left[i\int idtd\theta d\bar{\theta}\,(1)\,(L[\Phi]+\mathbb{J}\,\Phi)\right] \tag{1}$$

where $\theta, \bar{\theta}$ are two grassmanian partners of t, Φ^a are extensions of the phase space coordinates $\varphi^a \equiv (q^1, \ldots, q^n; p^1, \ldots, p^n)$, $a = 1, \ldots, n$ and L is the usual lagrangian. The generating functional for quantum mechanics, which we will indicate with $\mathcal{Z}_{QPI}$ (where QPI stands for *quantum path integral*), has the form

$$\mathcal{Z}_{QPI}\,[\mathbb{J}] = \int \mathcal{D}\Phi^a \exp\left[i\int idtd\theta d\bar{\theta}\left(\frac{\theta\bar{\theta}}{\hbar}\right)(L[\Phi]+\mathbb{J}\,\Phi)\right] \tag{2}$$

which is very similar to Eq. (1) except that the (1) in Eq. (1) is replaced by $\left(\frac{\theta\bar{\theta}}{\hbar}\right)$ in Eqs. (2). As these quantities multiply the measure of integration $\int idtd\theta d\bar{\theta}$, it comes natural to do the following: let us introduce a general dreibein E_A^M in the space $(t,\, \theta, \bar{\theta})$ and let us build the following path-integral

$$\mathcal{Z}_{GPI}\,[\mathbb{J}] = \int \mathcal{D}\Phi^a \exp\left[i\int idtd\theta d\bar{\theta}\,(E)\,(L[\Phi]+\mathbb{J}\,\Phi)\right] \tag{3}$$

(GPI stands for *General Path Integral*) and where E is the determinant (or superdeterminant) of E_A^M. Immediately we notice, comparing Eq. (3) with Eq. (1) and Eq. (2) that the CPI can be considered a GPI with $E = 1$ and the QPI a GPI with $E = \frac{\theta\bar{\theta}}{\hbar}$. As the GPI has a general covariance in the $(t, \theta, \bar{\theta})$ space we could consider the CPI and QPI as two "gauge fixed" versions of Eq. (3)[1]. The reader could object to this by saying that Eq. (1) and Eq. (2) should contain the Fadeev-Popov determinant if considered as gauge fixed versions of Eq. (3). We will return to this issue at the end of Section (2). Somehow we can consider

[1]The Lagrangian of the GPI will have its usual derivatives replaced by general covariant ones. The usual derivatives contanied in Eq. (1) and Eq. (2 will be forced to be the same as the covariant ones associated to their gauge-fixing.

the formulation in Eq. (1), Eq. (2), Eq. (3) similar to that of a field theory Φ in a background gravitational field E_B^A. The reader may also object that if both Eq. (1) and Eq. (2) are different gauge of Eq. (3) then we could turn classical mechanics Eq. (1) into quantum mechanics Eq. (2) via a general covariant transformation in the extension of time. We will show later on why it is not possible to turn CM into QM. Next we will prove that there are various families of E_B^A which give the same CPI and the same for the QPI. These families are parametrized by 4 parameters for the CPI and by 5 for the QPI. From the E_B^A with the help of *Wolfram Mathematica* we will build the metric, the Christoffel symbols, the Ricci curvature tensor $R_{\alpha\beta}$ and the Ricci scalar R. All of these depend on the same parameters as the E_B^A. For the CPI we shall prove that there is a point in parameter space for which the Ricci scalar and tensors are zero. The same does not happen for the QPI. This fact may indicate something very profound but we have not been able to get to it so far.

We leave to the reader the task to explore this last issue. The paper is organized as follows. In Section (2) for completness we briefly review [1, 2, 3]. In Section (3) we introduce the vierbein E_B^A[2] and indicate the general strategy. In Section (4) we show how to obtain the vierbein for both the CPI and the QPI and do the counting of the free parameters. In Section (5) we calculate the metric for both the CPI and the QPI. In Section (6) we proceed to calculate the Ricci scalar and curvature for both theories. In Section (7) we search for the points in parameter space where the Ricci scalar and tensor are zero in the CPI. We also prove that a similar point does not exist for the QPI. In Section (8) we summarize what we had done and the prospects for the future. In few appendices we confine some detailed and long calculations.

[2]What, from now on , we shall call a "vierbein" is actually a dreibein associated to $(t, \theta, \bar{\theta})$.

2. Review

In the thirties Koopman and Von Neumann (KVN) proposed [2, 3] an operatorial and Hilbert space formulation of classical mechanics (CM) on the lines of what had been done few years before for quantum mechanics (QM). It was then natural to give [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14] a path-integral version of the KVN formalism like Feynman had done for the operatorial version of QM [15]. Actually, the path-integral version of classical mechanics (CPI) provided, in a natural way, a generalization of the KVN formalism in the sense that it gave also the classical evolution of differential forms and tensors on phase-space [16].

The procedure has been worked out in details in [1] and [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14] and can be summarized as follows. The KVN postulates for the Hilbert space and operatorial version of CM are the following:

1. a state of a classical system, whose phase-space is indicate by $\mathcal{M}$ with coordinates $\varphi^a \equiv (q^1, \ldots, q^n; p^1, \ldots, p^n)$ is represented by an elememt $|\psi\rangle$ of a Hilbert space $\mathcal{H}$.
2. On this Hilbert space the operators $\hat{p}^i$ and $\hat{q}^j$, whose eigenvalues are p^i and q^j, commutes
$$[\hat{p}^i, \hat{q}^j] = 0$$
and their common eigenstates are indicated as $|q, p\rangle$.
3. The staes $\langle q, p|\psi\rangle$ are square-integrable and their modulus squared $|\psi(q, p)|^2$ is the probability density $\rho(q, p)$ of finding the system in (q, p).
4. The evolution of $\psi(q, p)$ is given by the Liouville equation
$$i\frac{\partial\psi}{\partial t} = \widehat{L}\psi$$
where the Liouvillian $\widehat{L}$ is
$$\widehat{L} = i\left(\frac{\partial H}{\partial q}\frac{\partial}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial}{\partial q}\right)$$

and H is the Hamiltonian of the sistem whose associated equation of motion are

$$\dot{\varphi}^a = \omega^{ab}\frac{\partial H}{\partial\varphi^b} \tag{4}$$

where

$$\omega^{ab} = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}$$

is actually a $2n \times 2n$ matrix called symplectic matrix. It is well-known that the evolution between some initial point φ_i and some final point φ_j has the following form on the states

$$\psi(\varphi_f, t_f) = \int \mathcal{K}(\varphi_f, t_f|\varphi_i, t_i)\, \psi(\varphi_i, t_i)\, d^{2n}\varphi_i \tag{5}$$

where

$$\mathcal{K}(\varphi_f, t_f|\varphi_i, t_i) = \delta[\varphi_f - \Phi_{cl}(t_f; q_i, t_i)]$$

with Φ_{cl} the solution of Eq. (4) with initial condition φ_i. Slicing the time interval $t_f - t_i$ in N intervals, we can re-write the kernel $K(\varphi_f, t_f|\varphi_i, t_i)$ as follows:

$$K(\varphi_f, t_f|\varphi_i, t_i) = \lim_{N\to\infty} \left\{ \int \prod_{J=1}^{N-1} d\varphi_J\, \delta[\varphi_J - \Phi_{cl}(t_J; \varphi_i, t_i]\right\} \delta[\varphi_f - \Phi_{cl}(t_f; q_i, t_i)] \tag{6}$$

where φ_J are the intermediate points between φ_i and φ_f over which we integrate. The Dirac deltas which appear in Eq. (5) can be written as

$$\delta[\varphi_J - \Phi_{cl}(t_J; \varphi_i, t_i)] = \delta\left[\dot{\varphi}^a - \omega^{ab}\frac{\partial H}{\partial \varphi^b}\right]\bigg|_{t_J} \det\left[\delta^a_b \partial_t - \omega^{ac}\frac{\partial^2 H}{\partial\varphi^c\partial\varphi^b}\right]\bigg|_{t_J}. \tag{7}$$

Let us now introduce some auxiliary variables λ_a and let us rewrite the first term on the RHS of Eq. (7) as

$$\delta\left[\dot{\varphi}^a - \omega^{ab}\frac{\partial H}{\partial\varphi^b}\right] = \int d\lambda_a\, \exp\left[i\,\lambda_a\left(\dot{\varphi}^a - \omega^{ab}\frac{\partial H}{\partial\varphi^b}\right)\right] \tag{8}$$

modulo a normalization factor. Let us also introduce $4n$ grassmanian variables [17] $c^a, \bar{c}_a, a = 1, \ldots, 2n$ so that we can rewrite the det on the RHS of Eq. (7) as:

$$\det\left[\delta^a_b\partial_t - \omega^{ac}\frac{\partial^2 H}{\partial\varphi^c\partial\varphi^b}\right] = \int dc^a d\bar{c}_a\, \exp\left[-\bar{c}_a\left(\delta^a_b\partial_t - \omega^{ac}\frac{\partial^2 H}{\partial\varphi^c\partial\varphi^b}\right)c^b\right]. \tag{9}$$

Using Eq. (7), Eq. (8) and Eq. (9) in Eq. (6) we get

$$K(\varphi_f, t_f|\varphi_i, t_i) = \int_{\varphi_i}^{\varphi_f} \mathcal{D}''\varphi\mathcal{D}\lambda\mathcal{D}c\mathcal{D}\bar{c}\, \exp\left[i\int dt\widetilde{\mathcal{L}}\right] \tag{10}$$

where

$$\widetilde{\mathcal{L}} = \lambda_a \left[\dot{\varphi}^a - \omega^{ab}\frac{\partial H}{\partial \varphi^b}\right] + i\,\bar{c}_a\left(\delta^a_b\partial_t - \omega^{ac}\frac{\partial^2 H}{\partial\varphi^c\partial\varphi^b}\right)c^b \tag{11}$$

and $\mathcal{D}''\varphi$ indicates that the integration is done over all its intermediate points and not on the end points φ_i and φ_f. Eq. (10) is basically the path-integral counter-part of the KVN formalism. Remembering how commutators are obtained from the path-integral [15] we get

$$\begin{aligned}[\hat{\varphi}^a, \hat{\varphi}^b] &= 0\\ [\hat{\varphi}^a, \hat{\lambda}_b] &= i\,\delta^a_b\\ [\hat{\bar{c}}_a, \hat{c}^b] &= \delta^a_b\end{aligned} \tag{12}$$

where the last are anticommutators or graded commutators [17]. All the other commutators are zero. From the second commutators of Eq. (12) we can realize operatorially the $\hat{\lambda}_a$ as

$$\hat{\lambda}_a = -i\frac{\partial}{\partial\varphi^a}. \tag{13}$$

Let us now see how the Liouville operator emerges from Eq. (11) from the non-grassmanian part of $\widetilde{\mathcal{L}}$ which we indicate with $\tilde{\mathcal{L}}_B$ the following quantitity:

$$\widetilde{\mathcal{L}}_B = \lambda_a\,\dot{\varphi}^a - \widetilde{\mathcal{H}}_B$$

where

$$\widetilde{\mathcal{H}}_B = \lambda_a\,\omega^{ab}\frac{\partial H}{\partial\varphi^b}. \tag{14}$$

It is then clear that at the operatorial level we have:

$$\int \mathcal{D}\varphi\mathcal{D}\lambda\,\exp\left[i\int dt\widetilde{\mathcal{L}}_B\right] \longrightarrow \exp\left[-\widehat{\widetilde{\mathcal{H}}}_B\,t\right] \tag{15}$$

where $\widehat{\widetilde{\mathcal{H}}}_B$ is the operator associated to Eq. (14) obtained using Eq. (13)

$$\begin{aligned}\widetilde{\mathcal{H}}_B &= -i\frac{\partial}{\partial\varphi^a}\omega^{ab}\frac{\partial H}{\partial\varphi^b} =\\ &= -i\frac{\partial H}{\partial p}\frac{\partial}{\partial q} + i\frac{\partial H}{\partial q}\frac{\partial}{\partial p} = \widehat{L}\end{aligned} \tag{16}$$

and this $\widehat{L}$ is the Liouville operator. The reader may ask now which operator we would get if we had kept also the grassmanian variables. It was shown in [4] that the c^a can be identified with the differential operator $d\varphi^a$. Via these we can build generic differential forms [16]

$$\psi(\varphi, d\varphi) \tag{17}$$

and we know that their evolution is given by an operator [16] called the *Lie derivative of the Hamiltonian flow* which is simbolically written as $\mathcal{L}_{(dH)^\#}$. So

$$\partial_t \psi(\varphi, d\varphi) = \mathcal{L}_{(dH)^\#} \psi(\varphi, d\varphi). \tag{18}$$

This operator is a generalization of the Liouville operator which makes only the evolution of $\psi(\varphi)$ that are called zero forms. As we said in [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14] the c^a of our path-integral can be identified with the $d\varphi^a$ and so the differential form Eq. (17) can be turned into a $\psi(\varphi, c)$

$$\psi(\varphi, d\varphi) \longrightarrow \psi(\varphi, c). \tag{19}$$

From the path-integral Eq. (10) it is clear that the evolution of $\psi(\varphi, c)$ is given by

$$\partial_t \phi(\varphi, c) = \widehat{\widetilde{H}} \psi(\varphi, c), \tag{20}$$

where $\widehat{\widetilde{H}}$ is the operatorial Hamiltonian associated to the Lagrangian of Eq. (11). Comparing Eq. (20) with Eq. (18) we can say that the Hamiltonian operator of our path-integral is a well-known object [16] in differential geometry, i.e. it is the *Lie derivative of the Hamiltonian flow.* The identification with objects of differential geometry can be also extended to the exterior derivatives, the inner contractions, the Lie brackets and the whole Cartan calculus [16]. The details of this important correspondence have been worked out in [4, 5]. So the auxiliary variables that we introduced $c^a, \bar{c}_a, \lambda_a$ are not just tricks to rewrite the path integral in a simpler form but crucial geometrical objects. Let us now go back to the commutation relations i.e Eq. (12). We said before that we can realize the $\hat{\lambda}_a$ as a derivative operator and obviously the φ^a as a multiplicative one:

$$\hat{\varphi}^a |\varphi\rangle = \varphi^a \, |\varphi\rangle. \tag{21}$$

The same can be done for the operators $\hat{c}^a, \hat{\bar{c}}_a$. As they commute with the $\hat{\varphi}^a$ and $\hat{\lambda}_a$, we can generalize the states of Eq. (21) to the following ones:

$$\begin{cases} \hat{\varphi}^a|\varphi, c\rangle &= \varphi^a\,|\varphi, c\rangle \\ \hat{c}^a|\varphi, c\rangle &= c^a\,|\varphi, c\rangle \end{cases} \tag{22}$$

and implement $\hat{\bar{c}}_a$ as a derivative operator

$$\hat{\bar{c}}_a = \frac{\partial}{\partial c^a}.$$

There is another manner to realize the commutation relations of Eq. (12). Noting that $\hat{q}^i$ and $\hat{\lambda}_{p_i}$ commutes and the same $\hat{c}^q$ and $\hat{c}_p$ we can diagonalize these operators [3] and obtain the states

$$\begin{cases} \hat{q}\,|q, \lambda_p, c^q, \bar{c}_p\rangle &= q\,|q, \lambda_p, c^q, \bar{c}_p\rangle \\ \hat{\lambda}_p\,|q, \lambda_p, c^q, \bar{c}_p\rangle &= \lambda_p\,|q, \lambda_p, c^q, \bar{c}_p\rangle \\ \hat{c}^q\,|q, \lambda_p, c^q, \bar{c}_p\rangle &= c^q\,|q, \lambda_p, c^q, \bar{c}_p\rangle \\ \hat{\bar{c}}_p\,|q, \lambda_p, c^q, \bar{c}_p\rangle &= \bar{c}_p\,|q, \lambda_p, c^q, \bar{c}_p\rangle \quad . \end{cases} \tag{23}$$

The operators $\hat{p}$ and $\hat{\lambda}_q$ are realized as derivatives operators

$$\hat{p} = i\,\frac{\partial}{\partial\lambda_p}, \quad \hat{\lambda}_q = -i\,\frac{\partial}{\partial q}.$$

The two basis Eq. (22) and Eq. (23) are related by a Fourier transformation [1]. The transition amplitudes in the basis of Eq. (23) is a generalization of Eq. (10) and it has the following path-integral expression

$$\begin{aligned} &\langle q_f, \lambda_f^p, c_f^q, \bar{c}_f^p | q_f, \lambda_i^p, c_i^q, \bar{c}_i^p\rangle = \\ = &\int \mathcal{D}''q\mathcal{D}p\mathcal{D}''\lambda^p\mathcal{D}\lambda^q\mathcal{D}''c^q\mathcal{D}c^p\mathcal{D}\bar{c}^q\mathcal{D}''\bar{c}^p\,\exp\left[i\int dt\widetilde{\widetilde{\mathcal{L}}}\right] \end{aligned} \tag{24}$$

where $\widetilde{\widetilde{\mathcal{L}}}$ is a Lagrangian which differ from $\widetilde{\mathcal{L}}$ of Eq. (11) by surface terms. More details can be found in ref.[1]. At this point we have to introduce two crucial ingredients which are familiar from the supersymmetry formalism [18]. Let us

[3] The index q and p on $\hat{c}, \hat{\bar{c}}, \hat{\lambda}$ indicates respectively the first and the last n-indices on $\hat{c}^a, \hat{\bar{c}}^a, \hat{\lambda}^a$

extend the variable t via two grassmanian partners $\theta, \bar{\theta}$. The triplet $(t, \theta, \bar{\theta})$ is often called "supertime". If we extend t to the 4-dim x^μ then there is an analog extension of super-time called "superspace" [18]. With this tool we can group-together the various variables $(\varphi^a, \lambda_a, c^a, \bar{c}_a)$ into a function of $(t, \theta, \bar{\theta})$ called superfield and defined as follows:

$$\Phi^a(t, \theta, \bar{\theta}) \equiv \varphi^a + \theta\, c^a + \bar{\theta}\, \omega^{ab}\, \bar{c}_b + i\, \bar{\theta}\theta\, \omega^{ab} \lambda_b. \tag{25}$$

We can separate off the q and p part of this superfields as follows:

$$\Phi^a = \begin{pmatrix} Q_i \\ P_i \end{pmatrix} \equiv \begin{pmatrix} q_i \\ p_i \end{pmatrix} + \theta \begin{pmatrix} c^{q_i} \\ c^{p_i} \end{pmatrix} + \bar{\theta} \begin{pmatrix} \bar{c}_{q_i} \\ -\bar{c}_{p_i} \end{pmatrix} + i\bar{\theta}\theta \begin{pmatrix} \lambda_{p_i} \\ -\lambda_{q_i} \end{pmatrix}. \tag{26}$$

Using the superfield there are some nice identities which we will need later on. Let us build the Lagrangian associated to $H(\varphi)$ of the original equations of motion Eq. (4) and let us call it $L(\varphi)$ where we replaced $\dot{q}$ with p. Let us now replace in $L(\varphi)$ the φ with the superfield Φ^a and expand in $\theta, \bar{\theta}$. We get following expression:

$$L[\Phi] = L(\varphi) + \theta\, \mathcal{M} + \bar{\mathcal{M}}\bar{\theta} - i\bar{\theta}\theta\, \tilde{\tilde{\mathcal{L}}} \tag{27}$$

where $\tilde{\tilde{\mathcal{L}}}$ is the Lagrangian which enters in Eq. (24). We will need these identities later on. Let us drop the indices in Eq. (26):

$$\begin{cases} Q(\theta, \bar{\theta}) &= q + \theta\, c^q + \bar{\theta}\, c_p + i\, \bar{\theta}\theta\, \lambda_p \\ P(\theta, \bar{\theta}) &= p + \theta\, c^p - \bar{\theta}\, c_q - i\, \bar{\theta}\theta\, \lambda_q \end{cases}. \tag{28}$$

The variables which enter Q all commute once they are turned into operators so we could define the following states

$$\widehat{Q}|Q\rangle = Q(t, \theta, \bar{\theta})|Q\rangle \tag{29}$$

which clearly satisfy

$$\begin{cases} \hat{q}|Q\rangle &= q\,|Q\rangle \\ \hat{\lambda}_p|Q\rangle &= \lambda_p\,|Q\rangle \\ \hat{c}^q|Q\rangle &= c^q\,|Q\rangle \\ \hat{\bar{c}}_p|Q\rangle &= \bar{c}_p\,|Q\rangle. \end{cases}$$

So we can identify the states $|Q\rangle$ with those of the basis Eq. (29). We can now use this fact and Eq. (27) to rewrite Eq. (24) as follows

$$\langle Q_f, t_f | Q_i, t_i \rangle = \int \mathcal{D}'' Q \mathcal{D} P \, \exp \left[i \int_{t_0}^{t} i \, dt' d\theta d\bar{\theta} \, L[\Phi] \right] \tag{30}$$

where we have used the standard rule of grassmanian integration

$$\int d\theta d\bar{\theta} \, \bar{\theta}\theta = 1.$$

all the details above are carefully explained in ref.[1]. Let us go back to the quantum mechanical path-integral [15] which gives the following expression for the transition amplitude

$$\langle q_f, t_f | q_i, t_i \rangle = \int \mathcal{D}'' q \mathcal{D} p \, \exp \left[\frac{i}{\hbar} \int dt' \, L[\varphi] \right]. \tag{31}$$

Note the great analogy between the classical path-integrac (CPI) Eq. (30) and the quantum path-integral (QPI) Eq. (31). We pass from one to the other by the following steps:

$$\begin{cases} Q & \longrightarrow & q \\ P & \longrightarrow & p \\ i \int dt d\theta d\bar{\theta} & \longrightarrow & 1/\hbar \end{cases} . \tag{32}$$

This is a sort of dimensional reduction which in [1] we proved to be equivalent to geometric quantization [19]. More details can be found in ref.[1]. Differently than in [1] in this paper we still exploit the relation between Eq. (31) and Eq. (30) but following a different route. Let us write Eq. (31) using the superfield and Eq. (27):

$$\int dt \, L(\varphi) = \int dt d\theta d\bar{\theta} \, \bar{\theta}\theta \, L[\Phi] = \int i \, dt d\theta d\bar{\theta} \, (-i \, \bar{\theta}\theta) \, L[\Phi]$$

so

$$\frac{i}{\hbar} \int dt \, L(\varphi) = i \int i \, dt d\theta d\bar{\theta} \left(-\frac{i}{\hbar} \, \bar{\theta}\theta \right) L[\Phi].$$

We can then rewrite Eq. (31) as

$$\langle q_f, t_f | q_i, t_i \rangle = \mathcal{N} \int \mathcal{D}'' Q \mathcal{D} P \, \exp \left[i \int i \, dt' d\theta d\bar{\theta} \left(-\frac{i}{\hbar} \, \bar{\theta}\theta \right) L[\Phi] \right]. \tag{33}$$

where $\mathcal{N}$ is a normalizing factor. On the right hand side of Eq. (33) the integration over $c, \bar{c}, \lambda$ drops off the path-integration because these variables do not enter the weight. The normalizing factor $\mathcal{N}$ is there to get 1 out of those extra intergrations. We could avoid introducing this normalizing factor if we write the L.H.S. of Eq. (33) as $\langle Q_f, t_f | Q_i, t_i\rangle$ The pieces $\langle \lambda_f, t_f | \lambda_i, t_i\rangle$ $\langle c_f, t_f | c_i, t_i\rangle$ $\langle \bar{c}_f, t_f | \bar{c}_i, t_i\rangle$ turn out to be products of "1" at each slice in time exactly as on the path-integral on the R.H.S. So we can summarize the Eq. (33) and Eq. (30) as

$$\langle Q_f, t_f | Q_i, t_i\rangle_{CPI} = \int \mathcal{D}''Q\mathcal{D}P \,\exp\left[i \int_{t_0}^{t} i\, dt' d\theta d\bar{\theta} \,\boxed{\mathbb{I}}\, L[\Phi]\right] \tag{34}$$

$$\langle Q_f, t_f | Q_i, t_i\rangle_{QPI} = \int \mathcal{D}''Q\mathcal{D}P \,\exp\left[i \int_{t_0}^{t} i\, dt' d\theta d\bar{\theta} \,\boxed{-i\frac{\bar{\theta}\theta}{\hbar}}\, L[\Phi]\right]. \tag{35}$$

We have encircled the quantities $\boxed{\mathbb{I}}$ and $\boxed{-i\frac{\bar{\theta}\theta}{\hbar}}$ because they seems to be the only quantities which are different in QM and CM. They somehow modify the *measure of integration* over the superspace $\int dt d\theta d\bar{\theta}$. We can extend the formalism also to the generating functionals as we have indicated in the introduction.

3. General Strategy

The presence of a factor in the measure, both in Eq. (34) and Eq. (35), is reminiscent of another factor which appears in the measure of integration. This happens in Riemannian geometry. There we have distances defined via a metric $g_{\mu\nu}$ as

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$$

and we require that this distance is invariant under general coordinate transformations

$$x'^\mu = x'^\mu(x^\nu).$$

We also require that the volume of integration is invariant and this happens only if we multiply the volume by a factor E:

$$\int \prod_{\nu=1}^{4} dx^{v}\, E. \tag{36}$$

The factor E is a determinant which is built in this way. Let us introduce a tensor called vierbein $e^{a}_{\ \mu}$ which carries an index a transforming under Lorentz transformations and a second index μ transforming under general coordinate transformations. It is possible to show that the metric $g_{\mu\nu}$ can be written in terms of the vierbein as follows

$$g_{\mu\nu} = \eta_{ab}\, e^{a}_{\ \mu}\, e^{b}_{\ \nu}, \tag{37}$$

where η_{ab} is a flat-Lorentz metric. For a review the interested reader can look into ref.[20]. The factor E making the measure invariant is defined as

$$E = \det e^{a}_{\ \mu}. \tag{38}$$

In our case the space on which we would like to introduce the factor E is not the 4-dim. space time but the 3-dim. space $z^A = (t, \theta, \bar{\theta})$. Riemannian spaces with grassmannian coordinates have been studied in [21, 22]. We can define flat supertime in many ways but we choose the following one:

$$dz^A\, \eta_{AB}\, dz^B = dt^2 - d\theta d\bar{\theta} + d\bar{\theta} d\theta, \tag{39}$$

where η_{AB} is

$$\eta_{AB} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{40}$$

The analog of the Lorentz transformation in this case is given by the group $Osp(1|2)$, which is the set of transformations leaving invariant the quantity

$$s = t^2 + \theta\bar{\theta} - \bar{\theta}\theta. \tag{41}$$

The *non-flat* infinitesimal distance is defined as

$$dz^A\, g_{AB}\, dz^B, \tag{42}$$

where g_{AB} is the analog of the metric, which due to the grassmannian character of some of its elements, is called supermetric. Under a general superdiffeomorphisms of our coordinates, which we will indicate as:

$$z^{A'} = z^A + \xi^A(z), \tag{43}$$

the g_{AB}, in order for Eq. (42) to be invariant, must transform as [21, 22]:

$$g'_{AB} = g_{AB}(z) + \frac{\overrightarrow{\partial}\xi^C}{\partial z^A}\, g_{CB} + g_{AC}\,\frac{\xi^C\overleftarrow{\partial}}{\partial z^B} + g_{AB,C}\,\xi^C, \tag{44}$$

where the right and left derivatives above are due to the grassmannian character of some of the z and the fact that in Eq. (42) some infinitesimals are to the left and some to the right. Like in normal Riemannian geometry also in super-Riemannian one [21, 22] we can define the super-vierbein which we will indicate with E^A_Λ, where A is the Lorentz analog (Osp(1,2)) index and Λ the general covariant (in supertime) one. The relation between supermetric and super-vierbein is [21, 22]:

$$g_{\Lambda\Pi} = E^A_{\ \Lambda}\,\eta_{AB}\,(-1)^{(1+B)\Pi} E^B_{\ \Pi}(z). \tag{45}$$

The numbers which are in the exponent of (-1) are 0 for t and 1 for θ and $\bar{\theta}$. For more details about grassmannian number, matrices and super-determinant (which are often indicated by sdet$(\dots)$) the reader can consult [17] or the Appendix A of this paper. From now on we will replace the greek letters on the vierbein with latin letters taken from the end of the latin alphabet like $M, N, P, \dots$. The analog of Eq. (36) for the superspace made of $t, \theta, \bar{\theta}$ will be

$$\int i\, dt\, d\theta\, d\bar{\theta}\, E \tag{46}$$

where $E = \text{sdet}(E^A_M)$. If we compare this with Eq. (34) and Eq. (35) we can say that the CPI is like a "gauge" fixed version of a "super-general covariant" formalism in supertime and the "gauge fixing" is such that

$$E = \mathbb{I} \tag{47}$$

while for the QPI the “gauge fixing” is such that

$$E = -i\frac{\bar{\theta}\theta}{\hbar}. \tag{48}$$

Before going on further we should remember that Eq. (47) and Eq. (48) are not the only conditions we have to impose in order to obtain respectively the CPI and the QPI. We should in fact remember that in a “general covariant” formalism also in the kinetic piece of the Lagrangian there is the presence of the vierbeins. Let us first suppose we integrate out in Eq. (34) and Eq. (35) the P so that the kinetic piece in both of them is reduced to

$$\partial_t Q \partial_t Q, \tag{49}$$

where we omit the indices on Q. An analog “general covariant” piece would be

$$D_t Q \, D_t Q \tag{50}$$

where the general covariant derivative D_t would be

$$D_t = E_t^M \partial_M, \tag{51}$$

with E_t^M components of the *inverse* of the vierbein matrix appearing in Eq. (45) and Eq. (46). The E_t^M should be chosen to be real because the expression in Eq. (50) is real. If instead Eq. (50) were of the form

$$(D_t Q)(D_t Q)^*,$$

then we could choose E_t^M to be complex. We will extend the reality condition of the E_t^M to all the components of the vierbein in order to simplify the treatment. For the expression Eq. (50) to be the same as Eq. (49), we will see later on that we have to make a particular choice for the vierbein. This choice, beside the one of Eq. (47) for the CPI and the Eq. (48) for the QPI, is something like a gauge fixing that we need to impose on the ”general covariant” formalism where the vierbein are free. The reader may object that we should also insert a Faddev-Popov (F.P.) determinant in the functional measure. As we already said earlier, we thought this was not necessary in our two cases because the

F.P. would depend only on E_M^A in our two gauge-fixings and we do not have the integration over E_M^A in the path-integral. But again the reader could object that a gauge-fixing could depend also on the matter Φ, like in the t'Hooft gauge, and over Φ we integrate. As a consequence what we get is not a "gauge fixing" independent formalism and so we cannot pass from the CPI to the QPI via a "gauge transformation". So we should be careful in saying that the CPI and the QPI are something like a "gauge fixing" of a general covariant formalism. In fact we have used the expression "something like". Nevertheless we think that is worth to pursue this analogy and see if it helps us better undertand the interplay between CM and QM

4. Vierbeins

In this section we shall build the vierbein E_A^M, which gives the CPI and the one which gives the QPI. We will bother the reader with several details which are crucial in order to get the precise form of the vierbeins. We will show that there is a whole family of E_M^A, which reproduce the same CPI and the same for the QPI. The vierbein is a 3×3 super matrix

$$E_A^M = \begin{pmatrix} a & \alpha & \beta \\ \gamma & b & c \\ \delta & d & e \end{pmatrix}, \tag{52}$$

where the greek letter indicate an odd element while the latin one indicates an even element. It is easy to see why the elements of E_A^M have the features indicated above by considering how the supermetric g_{AB} is built out of the vierbein in Eq. (45) and from the odd/even characters of the elements of the g_{AB}. The two conditions that we have to satisfy to get the CPI are:

$$\begin{cases} E = 1 \Longrightarrow \text{sdet}(E_A^M) = 1 \\ \\ D_t Q \, D_f Q = \partial_t Q \, \partial_t Q \quad . \end{cases} \tag{53}$$

About the first relation above we already talked a lot, while the second one is there in order to obtain the usual kinetic piece of C.M. from the covariant one.

It is a long calculation, reported in Appendix B, to prove that the vierbein for the CPI, satisfying the constraints (53) is given by

$$E_A^M(CPI) = \begin{pmatrix} \pm 1 & 0 & 0 \\ \gamma & b & c \\ \delta & d & e \end{pmatrix}.$$

where the variables b, c, d, e have to satisfy two constraints reported in Appendix B. A similar but much longer calculation, reported in Appendix C, gives the form of the vierbein for the QPI

$$E_A^M(QPI) = \begin{pmatrix} 1 + a_S\theta\bar{\theta} & \alpha & \beta \\ \gamma & b & c \\ \delta & d & e \end{pmatrix},$$

where a_S is the "soul" (see Appendix A or ref.[17] for the definition of soul). Also the elements of $E_A^M(QPI)$ are not free but must satisfy two constraints presented in Appendix C. Of course for the QPI also Eq. (53) is different and it is reported in details in Appendix C.

5. Metrics

In this section we will calculate the metric from the vierbeins and show that they depend on a lower number of free parameters than the vierbeins. All details of the calculations are in Appendix D and we will often refer to the equations contained in that appendix and in the previous ones. We will skip the similar details in later sections for the curvatures because most of those calculations were done using *Wolfram Mathematica* and using symbols already defined in this and the previous section.

Let us start from the CPI. The vierbein, before implementing the constraint Eq. (B.3), has the form given by Eq. (B.2), i.e.

$$E_A^M = \begin{pmatrix} \pm 1 & 0 & 0 \\ \gamma & b & c \\ \delta & d & e \end{pmatrix}, \tag{54}$$

and the “super-metric” has the following form as a function of the vierbein:

$$g^{MN} = E_A^M\, \eta^{AB}\, (-1)^{(1+B)N} E_B^N. \tag{55}$$

A long but easy calculation gives

$$g^{MN} = \begin{pmatrix} 1-2\,\gamma\,\delta & \gamma\,d-\delta\,b & \gamma\,e-\delta\,c \\ \gamma\,d-\delta\,b & 0 & b\,e-c\,d \\ \gamma\,e-\delta\,c & -(b\,e-c\,d) & 0 \end{pmatrix} \tag{56}$$

and implementing the constraint (B.3) we get:

$$g^{MN} = \begin{pmatrix} 1-2\,\gamma\,\delta & \gamma\,d-\delta\,b & \gamma\,e-\delta\,c \\ \gamma\,d-\delta\,b & 0 & \pm 1 \\ \gamma\,e-\delta\,c & \mp 1 & 0 \end{pmatrix}. \tag{57}$$

Apparently this metric depends on $\gamma, \delta, b, c, d, e$ which means 12 parameters minus the two constraints on b, c, d, e so only on 10 parameters. Actually the combinations of parameters which enter the g^{MN} is less. In fact let us define the following variables:

$$\begin{cases} \pi_1 & \equiv & \gamma_\theta\, e_B - \delta_\theta\, c_B \\ \pi_2 & \equiv & \gamma_{\bar\theta}\, e_B - \delta_{\bar\theta}\, c_B \\ \pi_3 & \equiv & \delta_\theta\, b_B - \gamma_\theta\, d_B \\ \pi_4 & \equiv & \delta_{\bar\theta}\, b_B - \delta_{\bar\theta}\, d_B \end{cases} \tag{58}$$

and

$$\pi_5 \equiv \gamma_{\bar\theta}\, \delta_\theta - \gamma_\theta\, \delta_{\bar\theta}. \tag{59}$$

This π_5 is actually dependent on the other four π_i of Eq. (58), in fact:

$$\pi_5 = \pm(\pi_2\, \pi_3 - \pi_1\, \pi_4).$$

It is easy to see that the metric Eq. (56) can be written in term of the π_i as follows

$$g^{MN} = \begin{pmatrix} 1 \pm 2\bar\theta\theta(\pi_2\pi_3 - \pi_1\pi_4) & -\pi_3\theta - \pi_4\bar\theta & \pi_1\theta + \pi_2\,\bar\theta \\ -\pi_3\theta - \pi_4\bar\theta & 0 & \pm 1 \\ \pi_1\theta + \pi_2\,\bar\theta & \mp 1 & 0 \end{pmatrix}. \tag{60}$$

Later on, in order to build the Christoffel symbols and the various curvatures tensor, we shall need also the inverse of g^{MN} which turns out to have the following expression:

$$g_{MN} = \begin{pmatrix} 1 & \mp(\theta\pi_1 + \bar{\theta}\pi_2) & \mp(\theta\pi_3 + \bar{\theta}\pi_4) \\ \pm(\theta\pi_1 + \bar{\theta}\pi_2) & 0 & \mp(1 + \bar{\theta}\theta(\pi_2\,\pi_3 - \pi_1\,\pi_4)) \\ \pm(\theta\pi_3 + \bar{\theta}\pi_4) & \pm(1 + \bar{\theta}\theta(\pi_2\,\pi_3 - \pi_1\,\pi_4)) & 0 \end{pmatrix}. \tag{61}$$

In both metrics above we have made the choice $a = \pm 1$ which is consistent with the CPI. The reader may wonder why the metric has less free parameters than the vierbein. We feel the reason is because of the particular combination of vierbeins which enters the metric (see Eq. (45)). Moreover the vierbein is a more general object than the metric; in fact it enters the dynamics of particles of any spin. One last question the reader may have is if the π_i are really free parameters or not. We feel they are free because they are made (see Eq. (58)) of combinations of odd variables γ, δ and even ones b, c, d, e and only these last ones are constrained (see Eq. (B.3), while the first ones are totally free. Let us now build the metric for the quantum case (QPI) or better for the "regularized" quantum case. The very long details of the calculations are confined in Appendix D. The result is anyhow the following

$$g^{MN} = \begin{pmatrix} 1 - 2\,\pi_5^Q\,\bar{\theta}\theta & -\pi_3^Q\,\theta - \pi_4^Q\,\bar{\theta} & \pi_1^Q\,\theta + \pi_2^Q\,\bar{\theta} \\ -\pi_3^Q\,\theta - \pi_4^Q\,\bar{\theta} & 0 & \pi_7^Q + \pi_6^Q\bar{\theta}\theta \\ \pi_1^Q\theta + \pi_2^Q\bar{\theta} & -\pi_7^Q - \pi_6^Q\,\bar{\theta}\theta & 0 \end{pmatrix}.$$

where the variables $\pi_1^Q, \ldots, \pi_7^Q$ are properly defined in Appendix D.

6. Curvatures

In this section we will build the curvatures from the metric presented in the previous section. As the calculations are very long we have made used of a package of *Wolfram Mathematica* dedicated to calculations containing grassmannian variables [23]. The same package has been used also for calculating the metric of the previous section and other calculations presented all through the paper. The first thing we have calculated has been the Christoffel symbols associated to our various metrics. If we work in a space with odd and even variables the Christoffel symbols Γ_{AB}^{C} have the following expression [21, 22]:

$$\Gamma_{AB}^{C} = (-)^{BC}\frac{1}{2}\left[(-)^{BD}\,g_{AD,B}+(-)^{A+B+AB+AD}\,g_{BD,A}-g_{AB,D}\right]\,g^{DC} \tag{62}$$

where the comma on the metric like $g_{AD,B}$ means the derivative of g_{AD} respect to the variable B . As usual the exponent on the $(-)$ indicate the even (0) or odd (1) nature of the associated variables. The results of the calculations of Eq. (62) for both the CPI metric Eq. (60) and Eq. (61) and for the QPI one Eq. (D.18) and Eq. (D.25) are confined in Appendix E. Once we have the Christoffel symbols we can calculate the various curvatures. The definition we will use is the one of ref.[21, 22] for SuperRiemannian space:

$$\begin{aligned} R_{ABC}^{D} &= -\Gamma_{AC,B}^{D}+(-)^{BC}\,\Gamma_{AB,C}^{D}-(-)^{C(D+E)}\,\Gamma_{AC}^{E}\,\Gamma_{EB}^{D}+\\ &\quad (-)^{B(C+D+E)}\,\Gamma_{AB}^{E}\,\Gamma_{EC}^{D}. \end{aligned} \tag{63}$$

From this we can build the Ricci curvature tensor defined as

$$R_{AB}\equiv(-)^{C}\,R_{ABC}^{C}. \tag{64}$$

Its expression in term of Christoffel symbols is:

$$\begin{aligned} R_{AB} &= (-)^{C+1}\,\Gamma_{AC,B}^{C}+(-)^{C(B+1)}\,\Gamma_{AB,C}^{C}+\\ &- (-)^{C(C+E-1)}\,\Gamma_{AC}^{E}\,\Gamma_{EB}^{C}+(-)^{BE+C}\,\Gamma_{AB}^{E}\,\Gamma_{EC}^{C}. \end{aligned} \tag{65}$$

From the Ricci curvature tensor we can calculate the so-called Ricci scalar defined in [21, 22] as

$$R=(-)^{B}\,g^{BA}\,R_{AB}. \tag{66}$$

The explicit expression of all components of the Ricci tensor and of the Ricci scalar has been confined to Appendix F and Appendix G. Its calculation, again, has been made possible by the use of *Wolfram Mathematica* [23]. The things to notice for the curvatures of the QPI (see Eq. (F.3) and (G.3)) is they are singular for $\epsilon \to 0$. Infact in many of its component we have the π_7^Q in the denominator and π_7^Q is proportional to ϵ. Only the $R_{\theta\theta}$ and $R_{\bar{\theta}\bar{\theta}}$ are not singular because they are equal to zero. What we should take care of is the Ricci scalar, Eq. (F.4) and (G.4)), where the singularity cannot be a coordinate artifact because it is a scalar independent of the coordinates. The way out could came from the fact that in the true quantum case ($\epsilon \to 0$) we also have that θ and $\bar{\theta}$ have to be sent to zero [21, 22]. So for example in Eq. (F.4) we have that the fourth contribution is proportional to $\bar{\theta}\theta/\pi_7$ that would give a $0/0$, which is an undefined term. But being ϵ and $\bar{\theta}\theta$ totally independent, we could choose that this undefined form is a finite grassman number. The next term is proportional to $\theta\bar{\theta}$ and it would go to zero. In this manner the Ricci scalar would not blow up in QM, which seems a natural thing to require as nothing goes to infinity in QM.

7. Zeros of the curvature

In this section we will check if there are value of the π_i for which the various curvatures turn out to be zero or at least some of them. As the π_i are *arbitrary* we can choose the system to sit on those values and so conclude that those curvatures are zero. This is what happens in the CPI as we will check. Surprisingly this does not happen in the QPI. There is no point where the curvatures vanish. This "may" indicate that there is some sort of "hidden matter" in QM. Of course we don't identify this with the so called "hidden variables" of Einstein [24].

7.1. Zeros of the curvature in the CPI

Let us start with the CPI in the case where the π_i are indipendent of time. The Ricci scalar was given in Eq. (F.2) and it had the following expression:

$$\begin{aligned} R^{CPI} &= -\frac{1}{2}\left(\pi_2^2 - 22\,\pi_2\,\pi_3 + \pi_3^2 + 20\,\pi_1\,\pi_4\right) + \\ &+ 8\,\frac{\bar{\theta}\theta}{a}\,\left(\pi_2\,\pi_3 - \pi_1\,\pi_4\right)^2 . \end{aligned} \tag{67}$$

In order to have $R^{CPI} = 0$ we need to have zero both its soul and body, i.e.:

$$\begin{cases} \pi_2\,\pi_3 - \pi_1\,\pi_4 = 0 \\ \pi_2^2 - 22\,\pi_2\,\pi_3 + \pi_3^2 + 20\,\pi_1\,\pi_4 = 0. \end{cases} \tag{68}$$

From the first equation in (68) we get

$$\pi_1 = \frac{\pi_2\,\pi_3}{\pi_4} \tag{69}$$

and putting this into the second of Eq. (68) we get

$$\pi_2 = \pi_3. \tag{70}$$

In the space described by the four parameters $\pi_1, \pi_2, \pi_3, \pi_4$ the Ricci scalar is zero on a 2-dim surface descibed by Eq. (69) and Eq. (70). Let us now check if on this surface, or at least on some points, also the Ricci tensor is zero. Let us look at the various components of the Ricci tensor presented in Eq. (F.1). Let us start with

$$R_{tt} = \frac{1}{2}\,(\pi_2 + \pi_3)^2 - 2\,\pi_1\,\pi_4.$$

Using Eq. (69) and Eq. (70) it is straightforward to show that $R_{tt} = 0$. Let us now move on to

$$R_{t\theta} = -R_{\theta t} = \frac{(\theta\,\pi_1 + \bar{\theta}\,\pi_2)\,(-(\pi_2 + \pi_3)^2 + 4\,\pi_1\,\pi_4)}{2\,a}.$$

The second factor on the right is zero on Eq. (68) so $R_{t\theta} = R_{\theta t} = 0$. Let us now check

$$R_{t\bar{\theta}} = -R_{\bar{\theta}t} = -\frac{(\theta\,\pi_3 + \bar{\theta}\,\pi_4)\,((\pi_2 + \pi_3)^2 - 4\,\pi_1\,\pi_4)}{2\,a}.$$

Again the second factor on the right is zero on Eq. (68) so $R_{t\bar{\theta}} = R_{\bar{\theta}t} = 0$. Next let us check

$$\begin{aligned} R_{\theta\bar{\theta}} &= \frac{1}{2}\Bigg[\bar{\theta}\theta\,\underbrace{(\pi_1\,\pi_4 - \pi_2\,\pi_3)}_{A}\,\big(\pi_2^2 - 6\,\pi_2\,\pi_3 + \pi_3^2 + 4\,\pi_1\,\pi_4\big) \\ &- \; a\,\underbrace{\big(\pi_2^2 - 10\,\pi_2\,\pi_3 + \pi_3^2 + 8\,\pi_1\,\pi_4\big)}_{B}\Bigg]. \end{aligned}$$

The term $A = 0$ because of Eq. (68) while B, using Eq. (68), can be transformed as follows:

$$B = \pi_2^2 - 10\,\pi_2\,\pi_3 + \pi_3^2 + 8\,\pi_1\,\pi_4 = \pi_2^2 - 10\,\pi_2^2 + \pi_2^2 + 8\,\pi_2^2 = 0.$$

So $R_{\theta\bar{\theta}} = -R_{\bar{\theta}\theta} = 0$.

The reason why we can choose the values of the π_i on which our curvature is zero is because the π_i do not enter the lagrangian of the CPI and we can change them without the Lagrangian getting modified. So far in the CPI we have proved that both the Ricci scalar and the Ricci curvature can be brought to zero. This is a situation very similar to the Schwarzschild case where, for points outside the mass region we have both R and R_{ab} equal to zero. What is not zero there is another scalar built up from the curvature tensor:

$$R_{abcd}R^{abcd} \neq 0.$$

This quantity in the Schwarzschild case is proportional to G/r^6 and it is the indicator of the presence of matter somewhere. We should calculate the analog quantity for the CPI. In this case the quantity to calculate is:

$$R_{abc}R^{abc}.$$

Instead of doing this rather complicated calculation, we should remind ourselves that our analog of space-time is $(t, \theta, \bar{\theta})$ so we should just check if our Ricci scalar and tensor are zero for any value of $(t, \theta, \bar{\theta})$. This would not happen in the Schwarzschild case in the area where there is matter. The calculation we have done is without t and with $\theta, \bar{\theta} \neq 0$ and it gives zero everywhere. Let us now generalize it to the case with t present that means when the π_i depend on t and see if we get zero everywhere in $(t, \theta, \bar{\theta})$. Let us start by finding the points on which the Ricci scalar is zero for π_i depending on time. The R^{CPI} is given by Eq. (G.2)

$$\begin{aligned} R^{CPI} &= -\frac{1}{2}\left[\pi_2^2 - 22\,\pi_2\pi_3 + \pi_3^2 + 20\,\pi_1\pi_4 + 4(\pi_3' - \pi_2')\right] \\ &+ \frac{\bar{\theta}\theta}{a}\left[8\,(\pi_2\pi_3 - \pi_1\pi_4)^2 + 4\,a\,(\pi_3' - \pi_2')\,\pi_5 + 7\,a\,\pi_5'(\pi_3 - \pi_2) + 2\,a\,\pi_5''\right]. \end{aligned} \tag{71}$$

Both the soul and the body must be zero, i.e.

$$\begin{cases} \pi_2^2 - 22\,\pi_2\pi_3 + \pi_3^2 + 20\,\pi_1\pi_4 + 4(\pi_3' - \pi_2') = 0 \\ 8\,(\pi_2\pi_3 - \pi_1\pi_4)^2 + 4\,a\,(\pi_3' - \pi_2')\,\pi_5 + +7\,a\,\pi_5'(\pi_3 - \pi_2) + 2\,a\,\pi_5'' = 0. \end{cases} \tag{72}$$

If in the time-independent case Eq. (69) and Eq. (70) are choosen, starting from the relation between π_5 and $\pi_1, \pi_2, \pi_3, \pi_4$

$$\pi_5 = \frac{\pi_2\,\pi_3 - \pi_1\,\pi_4}{a},$$

it can be easily seen that

$$\pi_5 = 0, \tag{73}$$

from which it follows that $\pi_5' = \pi_5'' = 0$. The system of Eq. (72) consequently reduces to

$$\begin{cases} \pi_2^2 - 22\,\pi_2\pi_3 + \pi_3^2 + 20\,\pi_1\pi_4 = 0 \\ 8\,(\pi_2\pi_3 - \pi_1\pi_4)^2 = 0. \end{cases} \tag{74}$$

which is equivalent to the system of Eq. (68), whose solutions Eq. (69) and Eq. (70) are the one we started with in the time independent case. It follows that the constraints on which R^{CPI} is zero, even in the time dependent case, are Eq. (69) and Eq. (70) like in the case of π_i independent on time. Of course there may be other set of points on which it is zero (because $\pi_2 = \pi_3$ was our choice) but what is important is that we have found points in which it is zero. Next we should check that also the Ricci tensor is zero on the same set of points. Their expression is given in (G.1) of Appendix G and it is easy to check they are all zero on the points where the following condition are satisfied (69),(70),(73):

$$\begin{cases} \pi_2 = \pi_3 \\ \pi_1 = \dfrac{\pi_2\,\pi_3}{\pi_4} \\ \pi_5 = 0. \end{cases} \tag{75}$$

Let us consider for example R_{tt} of (G.1):

$$\begin{aligned} R_{tt} = R_{tt}^{CPI}(\pi_i) + (\pi_3' - \pi_2') + \bar{\theta}\theta\,&\big[\pi_3^2\pi_2' - \pi_2^2\pi_3' + (\pi_2 - \pi_3)(\pi_4\pi_1' + \pi_4'\pi_1) + \\ &+ (\pi_3' - \pi_2')(a\,\pi_5 - 2\,\pi_2\pi_3 + 3\,\pi_1\pi_4) + 2\,a\pi_5'\big]\,. \end{aligned} \tag{76}$$

It turns out that the first term is equal to zero because it has the same expression as the time independent R^{CPI}_{tt}, which was zero on the contraints (69),(70). As for the other contributions they are trivially zero because the conditions $\pi_2 = \pi_3$ and $\pi_5 = 0$ imply $\pi_2' = \pi_3'$ and $\pi_5' = 0$. In the same way it can be shown that all the other contributions are identically equal to zero. As we explainied in the time independent case, we do not calculate

$$R_{abc}R^{abc},$$

bacause we proved that the Ricci scalar and the tensors are zero over the whole $(t, \theta, \bar{\theta})$ space. As a consequence we have that there is no matter anywhere differently than in the Schwarzschild case.

7.2. Lack of zeros in the QPI curvature

Let us now turn to the QPI case starting with the Ricci scalar in the case independent on time. Its expression was given in Eq. (F.4):

$$\begin{aligned} R^{QPI} &= R^{CPI}(\pi_i^Q) + (2\,\sigma_1 - 3\,\pi_6^Q) + \\ &+ \frac{\bar{\theta}\theta}{\pi_7^Q}\left[-(\pi_2^{Q^2} + 6\,\pi_2^Q\pi_3^Q + \pi_3^{Q^2} - 8\,\pi_1^Q\pi_4^Q)\pi_6^Q + 4\,\pi_6^{Q^2} + \right. \\ &- \left. 4\,\sigma_1(\pi_2^{Q^2} + 3\,\pi_2^Q\pi_3^Q + \pi_3^{Q^2} - 5\,\pi_1^Q\pi_4^Q - \pi_6^Q + \sigma_1)\right]. \end{aligned} \tag{77}$$

The σ_1 was defined in (E.2) of Appendix E and the parameters of (77) are not four but six. The various π_i^Q were introduced in (D.20). Note that π_5^Q does not appear because it is related to the others. Let us suppose we stay on the surface where

$$R^{CPI}(\pi_i^Q) = 0. \tag{78}$$

and this happens when the constraints Eq. (69) and Eq. (70) are satisfied but with the π_i replaced by the π_i^Q. Next, for the body of Eq. (77) to be zero we need that

$$2\,\sigma_1 - 3\,\pi_6^Q = 0$$

i.e.

$$\sigma_1 = \frac{3}{2}\,\pi_6^Q. \tag{79}$$

Like in the CPI, we keep π_4^Q and π_3^Q free and link the other vabiables to these two via the $R^{CPI}(\pi_i^Q) = 0$. Also π_6^Q seems to be free and the same π_7^Q, which does not make its appearance in Eq. (77) but was present in the formalism. So Eqs. (79) and (78) bring to zero the body of R^{QPI}; now we have to make the soul zero. Using the constrain Eq. (78), which leads to $\pi_2^Q = \pi_3^Q$, and Eq. (79), after straightforward calculations we get that the soul of R^{QPI} is given by

$$\mathrm{soul}(R^{QPI}) = \frac{\pi_6^{Q^2}}{\pi_7^Q}.$$

So to be zero we have to set

$$\pi_6^Q = 0. \tag{80}$$

We can summarize the set of constraints which make $R^{QPI} = 0$ as:

$$\begin{cases} \pi_2^Q = \pi_3^Q \\ \pi_1^Q = \dfrac{\pi_2^Q\,\pi_3^Q}{\pi_4^Q} \\ \pi_6^Q = 0 \\ \sigma_1 = \dfrac{3}{2}\pi_6^Q \end{cases} \tag{81}$$

Let us now see if also the Ricci tensor for the QPI, given by Eq. (F.3) is zero on the points of Eq. (81). Let us start from R_{tt}^{QPI}:

$$\begin{aligned} R_{tt}^{QPI} &= \underbrace{R_{tt}^{CPI}(\pi_i^Q)}_{A} + 2\,\sigma_1 + \underbrace{\frac{\bar{\theta}\theta}{\pi_7^Q}\Big[((\pi_2^Q+\pi_3^Q)^2 - 4\,\pi_1^Q\pi_4^Q)\pi_6^Q}_{B} \\ &+ \underbrace{2\,(\pi_2^{Q^2} + 4\,\pi_2^Q\pi_3^Q + \pi_3^{Q^2} - 6\,\pi_1^Q\pi_6^Q - \pi_6^Q)\,\sigma_1 + 6\sigma_1^2}_{C}\Big]. \end{aligned}$$

The A-term calculated using Eq. (81) gives

$$\begin{aligned} A &= \left[\frac{1}{2}(\pi_2^Q + \pi_3^Q)^2 - 2\,\pi_1^Q\pi_4^Q\right] \\ &= \left[\frac{1}{2}(2\,\pi_2^Q)^2 - 2\,\pi_2^Q\pi_3^Q\right] = 2\,\pi_2^{Q^2} - 2\,\pi_2^{Q^2} = 0. \end{aligned}$$

The term B is zero because is multiplied by π_6^Q which is zero by Eq. (81). The term C is zero because $\sigma_1 = 0$. After the A piece there is a σ_1 which is zero. So $R_{tt}^{QPI} = 0$. Let us now analyze the $R_{t\theta}^{QPI}$ which is

$$\begin{aligned} R_{t\theta}^{QPI} &= -R_{\theta t}^{QPI} \\ &= R_{t\theta}^{CPI}(\pi_i^Q) - \frac{\pi_6^Q + 3\,\sigma_1}{\pi_7^Q}\left[\theta\pi_1^Q + \bar{\theta}(\pi_2^Q + \pi_3^Q)\right]. \end{aligned} \tag{82}$$

The $R_{t\theta}^{CPI} = 0$ on the constraints Eq. (81). The second piece in Eq. (82) is zero because $\pi_6^Q = \sigma_1 = 0$, so $R_{t\theta}^{QPI} = 0$. Next let us analyze $R_{\theta\bar{\theta}}$ which is

$$\begin{aligned} R_{\theta\bar{\theta}}^{QPI} &= -R_{\bar{\theta}\theta} = R_{\theta\bar{\theta}}^{CPI}(\pi_i^Q) + \frac{\sigma_1 - 3\,\pi_6^Q}{\pi_7^Q} \\ &+ \frac{\bar{\theta}\theta}{\pi_7^{Q^2}}\left[2\,\pi_6^{Q^2} + 8\,\pi_1^Q\pi_4^Q\pi_6^Q - 8\,\pi_2^Q\pi_3^Q\pi_6^Q\sigma_1\left((\pi_2^Q - \pi_3^Q\right)^2 + 4\,\pi_6^Q + 2\,\sigma_1)\right]. \end{aligned} \tag{83}$$

Again $R_{\theta\bar{\theta}}^{CPI} = 0$ is zero on the constraints Eq. (81) and all the rest is zero because $\pi_6^Q = \sigma_1 = 0$.

Now let us check if the constraint Eq. (81) leads to any contradiction. Let us go back to the definition of π_6^Q given in Eq. (D.23)

$$\pi_6^Q = \epsilon\, a_S - \frac{i}{\hbar} a_B\,(1-\epsilon) + a_B(\alpha_\theta\,\pi_2 - \alpha_{\bar{\theta}}\,\pi_1 + \beta_\theta\,\pi_4 - \beta_{\bar{\theta}}\,\pi_3) + \alpha_{\bar{\theta}}\,\beta_\theta - \alpha_\theta\,\beta_{\bar{\theta}}.$$

Going to the true-quantum case $\epsilon = 0$ we would get

$$\pi_6^Q = -\frac{i}{\hbar} a_B + a_B \underbrace{(\alpha_\theta\,\pi_2 - \alpha_{\bar{\theta}}\,\pi_1 + \beta_\theta\,\pi_4 - \beta_{\bar{\theta}}\,\pi_3)}_{A} + \underbrace{\alpha_{\bar{\theta}}\,\beta_\theta - \alpha_\theta\,\beta_{\bar{\theta}}}_{B}. \tag{84}$$

If, as we did before in our calculations, we choose $\alpha = \beta = 0$, we get in the true quantum case:

$$\pi_6^Q = -\frac{i}{\hbar} a_B$$

as $a_B = \pm 1$ we get

$$\pi_6^Q = \mp\frac{i}{\hbar}\,. \tag{85}$$

So it is never zero and this contradicts Eq. (81) or, saying it better, because of Eq. (85) the constraint Eq. (81) is not satisfied. The Ricci scalar curvature and the associated tensor are never zero in the true quantum case. Let us suppose we do not make the choice $\alpha = \beta = 0$, then in Eq. (D.23) we would have three terms: one which is a complex number $\epsilon\, a_S - \frac{i}{\hbar} a_B\,(1-\epsilon)$ and the A, B of Eq. (84) which are the product of couple of grassmannian odd number like π_2 and α_θ and similar. These A and B are grassmann even and they will never be equal to a complex number. So A and B cannot cancel the above mentioned complex number in order to put π_6^Q equal to zero. As the A and B contain the parameterm π_i^Q which are free they could be put to zero, but then also the complex number has to be put to zero. This is possible in the regularized QPI and it would mean

$$\epsilon\, a_S - \frac{i}{\hbar} a_B (1-\epsilon) = 0$$

which is equivalent to:

$$a_S = \frac{i}{\hbar}\, a_B\, \frac{1-\epsilon}{\epsilon}. \tag{86}$$

But in the true quantum case $\epsilon \to 0$ we would get $a_S \to \infty$, which does not make sense. Let us do the last attempt and see if we can put $R^{QPI} = 0$ without putting $\pi_6^Q = 0$. Let us go back to formula Eq. (77) and let us see if there is a different method to get $R^{QPI} = 0$. If in Eq. (77) we put first to zero the body and then the soul we get

$$R^{CPI}(\pi_i^Q) + 2\sigma_1 - 3\pi_6^Q = 0$$

which leads to

$$\pi_6^Q = \frac{1}{3}\,(R^{CPI}(\pi_i^Q) + 2\sigma_1). \tag{87}$$

From this formula it seems that we do not have to put $\pi_6^Q = 0$. But let us analize Eq. (87) in detail. On the R.H.S. we have only terms which are product of grassmann variables like

$$\sigma_1 = \pi_2^Q \pi_3^Q - \pi_1^Q \pi_4^Q - \pi_5^Q \pi_7^Q.$$

The same for R^{CPI} which is

$$\begin{aligned} R^{CPI}(\pi_i^Q) &= -\frac{1}{2}\left(\pi_2^{Q^2} - 22\,\pi_2^Q \pi_4^Q + \pi_3^{Q^2} + 20\,\pi_1^{Q^2}\pi_4^Q\right). \\ &+ \bar{\theta}\theta \frac{8}{a}(\pi_2^Q \pi_3^Q - \pi_1^Q \pi_4^Q). \end{aligned}$$

So on the R.H.S. of Eq. (87) we have products of 2 grassmannian numbers or higher terms (like those with $\bar{\theta}\theta$ which anyhow goes to zero in the true quantum case $\theta, \bar{\theta} \to 0$), while on the L.H.S. we have π_6^Q which (see Eq. (84)) contains both product of grassmannian numbers but also complex number which must be put to zero separately and we go back to the case described by Eq. (86). So we can conlude that in the true quantum case we cannot bring the curvature to zero. Of course we should do the same calculations in the case dependent on t, but we feel that the result will be the same.

8. Conclusions

In this paper we have shown that an "intrinsic" vierbein, present in the CPI version of CM, gives zero curvature (at least the Ricci one). This seems natural because there is no external mass generating a curvature in the space on which our test particle of the CPI would move. In the quantum case instead there is an intrinsic curvature. One could immediately ask what is the matter which produce this curvature. We could speculate saying that there are some non-local hidden variables of the type Bell [24] proposed long ago or it is some sort of dark matter or dark energy so fashionable these days. We do not know and we prefer not to speculate. Our goal at the beginning was to see if there was some "intrinsic" geometry in Q.M and we feel we have found some hints of it. We also would like to notice that this intrinsic geometry appear when we do not look only at the usual bosonic variables φ^a of Q.M but also at their differential forms c^a, which we would like to call "*quantum forms*". This is a topic which has not been studied deeply except by few mathematicians and in a language difficult for physicists. We feel a more intense study should be done of this sector of mathematics.

Appendix A.

Appendix A.1. Grassmannian algebras

Given a set of N-elements $\xi^a, a = 1, \ldots, N$ obeying the following properties

$$\xi^a \xi^b = -\xi^b \xi^a,\ (\xi^a)^2 = 0, \text{for all } a, b,$$

they are called generators of a grassmannian algebra Λ_N.
The elements $1, \xi^a, \xi^a \xi^b, \xi^a \xi^b \xi^c, \ldots$ form a set of 2^N objects, called the basis of the algebra. An addition in this basis and a multiplication by complex numbers is defined among its elements and so they form a linear vector space of dimension 2^N.

Appendix A.2. Super-numbers

Every element z of the vector space above can be written as

$$z = z_B + z_S$$

where z_B is an ordinary complex number and it is called the "body" of z and z_S, called the "soul", is:

$$z_S = \sum_{n=1}^{2^N} \frac{1}{n!} c_{a_1 \cdots a_n} \xi^{a_n} \cdots \xi^{a_1}, \tag{A.1}$$

where the $c_{a_1 \cdots a_n}$ are also complex numbers. The $c_{a_1 \cdots a_n}$ are antisymmetric in the exchange of their indices. It is easy to prove that

$$z_S^{N+1} = 0.$$

Appendix A.3. Inverse of a super-number

The inverse z^{-1} of a super-number, defined by $z\, z^{-1} = 1$, turns out to be

$$z^{-1} = z_B^{-1} \sum_{n=0}^{2^N} \left(-z_B^{-1}\, z_S \right),$$

so if $z_B = 0$ the inverse does not exist.

Appendix A.4. C-number and A-number

Any super-number can be split into its even "e" and odd "o" part as

$$\begin{aligned} z &= e + o \\ e &\equiv z_B + \sum_n \frac{1}{2n!} c_{a_2 \cdots a_{2n}} \xi^{a_{2n}} \cdots \xi^{a_2} \\ o &\equiv \sum_n \frac{1}{2n+1!} c_{a_1 \cdots a_{2n+1}} \xi^{a_{2n+1}} \cdots \xi^{a_1}. \end{aligned}$$

If a super-number has only an "e" part is called an even super-number while if it has only an "o" part it is called an odd super-number. The grassmann index of even or odd numbers is the number $2n$ or $2n+1$ modulo 2 so for even number is zero and for odd number is one. Usually it is put as exponent of (-1) and is indicated with square brackets: $[e], [o]$.

Appendix A.5. Super-vectors and super-matrices

Super-vectors are defined regorously in [17], but basically they are rows or columns of super-numbers. The elements in the basis of these vectors are arranged in such a manner that the even elements "e" come above the odd ""o"" one, like

$$\begin{pmatrix} e \\ o \end{pmatrix}. \tag{A.2}$$

This feature can always be realized because super-vectors can be multiplied by super-numbers. If the basis has the form of Eq. (A.2) then a super-matrix $\mathcal{K}$ can always be arranged in the form

$$K = \begin{pmatrix} A & C \\ D & B \end{pmatrix}, \tag{A.3}$$

where the elements of the super-matrices A and B are made of even super-numbers while C and D are made of odd numbers. More details are given in [17].

Appendix A.6. Super-trace

The super-trace of the matrix $\mathcal{K}$ is defined as

$$\mathrm{str}\,\mathcal{K} = (-1)^{[i]}\mathcal{K}^i_i,$$

where $[i]$ is the grassmann index of the "i" elements and we sum over "i".

Appendix A.7. Super-determinant and its inverse

For a standard matrix X we know that the following relation holds between the variation "δ" of parameters entering the determinant and the ones entering the trace:

$$\delta[\ln\,\mathrm{det}X] = \mathrm{tr}[X^{-1}\delta X].$$

We use this relation to define the super-determinant in case of a super-matrix $\mathcal{K}$ which has the form $\mathcal{K}$ of Eq. (A.3). The result [17] is:

$$\mathrm{sdet}\begin{pmatrix} A & C \\ D & B \end{pmatrix} = \mathrm{det}(A - CB^{-1}D)(\mathrm{det}B)^{-1} \tag{A.4}$$

where the symbol "det" has the same meaning as if the entries were complex numbers. It is also possible to define the inverse of the supermatrix X as:

$$X^{-1} = \begin{pmatrix} \widetilde{A} & \widetilde{C} \\ \widetilde{D} & \widetilde{B} \end{pmatrix} \tag{A.5}$$

where

$$\begin{aligned}
\widetilde{A} &= (\mathbb{I} - A^{-1}CB^{-1}D)^{-1}A^{-1} \\
\widetilde{C} &= -(\mathbb{I} - A^{-1}CB^{-1}D)^{-1}A^{-1}CB^{-1} \\
\widetilde{D} &= -(\mathbb{I} - B^{-1}DA^{-1}C)^{-1}B^{-1}DA^{-1} \\
\widetilde{B} &= (\mathbb{I} - B^{-1}DA^{-1}C)^{-1}B^{-1}.
\end{aligned}$$

Note that the inverse above exists if only A and B are not singular. It is also easy to calculate the determinant of the inverse:

$$\mathrm{sdet}\begin{pmatrix} A & C \\ D & B \end{pmatrix}^{-1} = (\mathrm{det}A)^{-1}\mathrm{det}(B - DA^{-1}C).$$

Appendix A.8. Left and right derivatives

The rigourous definition for these operations are given in [17]. Here we will give only an example. Let us give a function $f(\xi_1, \xi_2)$ of two grassmannian odd variables ξ_1, ξ_2 of the form

$$f(\xi_1, \xi_2) = \xi_1 \, \xi_2.$$

Let us define the right or left derivative of f with respect to ξ_1:

$$\begin{aligned} \frac{\overrightarrow{\partial} f}{\partial \xi_1} &= \xi_2 \qquad \text{left derivative} \\ \frac{\overleftarrow{\partial} f}{\partial \xi_1} &= -\xi_2 \qquad \text{right derivative} \end{aligned}$$

"Somehow" roughly speaking in the right derivative it looks like if we had put $\overleftarrow{\partial}/\partial\xi_1$ to the right of the function so that $\overleftarrow{\partial}/\partial\xi_1$ has to pass through ξ_2 in order to act on ξ_1. In going through ξ_2 it acquires a minus sign because ξ_1 and ξ_2 anticommute. On grassmannian spaces we can also define the concept of integration. All the details are given in [17]. The few things we need in this paper were already indicated in the body of the paper and will not be repeated in this appendix.

Appendix B.

In this appendix we will give details of the calculations of the vierbein for the CPI case.
Using the expression Eq. (52) for the E_A^M we get

$$D_t = \partial_M E_t^M = a\,\partial_t + \alpha\,\partial_\theta + \beta\,\partial_{\bar\theta},$$

so

$$(D_t Q)(D_t Q) = a^2\,\partial_t Q\,\partial_t Q + 2\,a\,\alpha\,\partial_t Q\,\partial_\theta Q + 2\,a\,\beta, \partial_t Q\,\partial_{\bar\theta} Q + \alpha\,\beta\,\partial_\theta Q\,\partial_{\bar\theta} Q.$$

Using the expression above, it is easy to prove that a choice of parameters for which the second of Eq. (53) holds is the following one:

$$a = \pm 1, \quad \alpha = \beta = 0. \tag{B.1}$$

So the supervierbein in Eq. (52) takes the form

$$E_A^M = \begin{pmatrix} \pm 1 & 0 & 0 \\ \gamma & b & c \\ \delta & d & e \end{pmatrix}. \tag{B.2}$$

Next we have to impose the first of the condition Eq. (53) using the definition of superdeterminant given in [17] or Appendix A of this paper and applied to Eq. (B.2):

$$\text{sdet} E_A^M = 1 \Longrightarrow \det \begin{pmatrix} b & c \\ d & c \end{pmatrix} \Longrightarrow b\,e - c\,d = \pm 1. \tag{B.3}$$

The quantity b, c, d, e are even so they have the form

$$\begin{aligned} b &\equiv b_B + b_S\,\bar\theta\theta \\ c &\equiv c_B + c_S\,\bar\theta\theta \\ d &\equiv d_B + d_S\,\bar\theta\theta \\ e &\equiv e_B + e_S\,\bar\theta\theta, \end{aligned} \tag{B.4}$$

where b_B, c_B, d_B, e_B are the "bodies" of the numbers while b_S, c_S, d_S, e_S are called the "souls" of the numbers. Again for details about these numbers see

Appendix A of this paper and consult ref.[17]. Using Eq. (B.4) the relation (B.3) gives the two equations

$$\begin{cases} b_B\, e_B - c_B\, d_B = \pm 1 \\ b_S\, e_B + b_B\, e_S - c_S\, d_B - c_B\, d_S = 0 \end{cases}. \tag{B.5}$$

A set of two solutions has the form

$$\begin{aligned} &(1) \quad e_B = 0,\, c_B = \mp\frac{1}{d_B},\, c_S = \frac{\mp d_S + d_B\, b_B\, e_S}{d_B^2} \\ &(2) \quad b_B = \frac{\pm 1 + c_B\, d_B}{e_B},\, b_S = \frac{\mp e_S - c_B\, d_B\, e_S}{e_B^2} + \frac{c_B\, d_S\, e_B + c_S\, d_B\, e_B}{e_B^2} \end{aligned} \tag{B.6}$$

It is a long but easy calculation to build the inverse [17] of the matrix E_A^M, the result is

$$\begin{pmatrix} \pm 1 & 0 & 0 \\ \pm c\,\delta \mp e\,\gamma & \pm e & \mp c \\ \pm d\,\gamma \mp b\,\delta & \mp d & \pm b \end{pmatrix}. \tag{B.7}$$

So now we have all the matrix elements to build the kinetic term and the superdeterminat. The number of free elements that we know in Eq. (B.2) is 12 because each $b, c, d, e, \gamma, \delta$ is made of two numbers either the body and the soul of the even ones or for the odd elements, like $\gamma = \gamma_\theta\,\theta + \gamma_{\bar\theta}\,\bar\theta$, they are the coefficient of θ and of $\bar\theta$. We have 2 constraints in Eq. (B.5), so the number of free variables is 10 and we choose them to be real. Considering the gauge freedom the carefull reader may envision the following problem. We said that building the CPI or the QPI is "like a gauge fixing" of a more general theory. For the CPI this "gauge fixing " is given by the constraints of Eq. (53). One important thing to check is that the " gauge fixing" Eq. (53) does not fix more parameters than those allowed by the gauge freedom. This is not so and we prove it below. Our diffeomorphism, Eq. (43), can be explicitly written as:

$$\begin{aligned} \delta t &= A(t) + \widetilde{\alpha}(t)\theta + \widetilde{\beta}(t)\bar\theta + \beta(t)\theta\bar\theta \\ \delta\theta &= \widetilde{\gamma}(t) + C(t)\theta + D(t)\bar\theta + \epsilon(t)\theta\bar\theta \\ \delta\bar\theta &= \widetilde{\delta}(t) + F(t)\theta + G(t)\bar\theta + \xi(t)\theta\bar\theta, \end{aligned} \tag{B.8}$$

where the latin symbols are real even numbers and the greek ones are odd number functions only of t and not of θ and $\bar{\theta}$. So in (B.8) we have 12 parameters. The general vierbein has the form

$$E_A^M(z) = \begin{pmatrix} a & \alpha & \beta \\ \gamma & b & c \\ \delta & d & e \end{pmatrix} \tag{B.9}$$

and it contains 18 variables because each $a, \alpha, \ldots, e$ is made of two entries. This vierbein transforms in the following manner under (B.8) or (43):

$$E_A'^M(z) = \frac{\vec{\partial} z_B'}{\partial z_A} E_B^M(z'). \tag{B.10}$$

If we were able to fully exploit the 12-parameter gauge freedom of (B.8), we could reduce the 18-variables of E_A^M to just six. In the CPI the vierbein that we use is:

$$E_A^M(CPI) = \begin{pmatrix} \pm 1 & 0 & 0 \\ \gamma & b & c \\ \delta & d & e \end{pmatrix} \tag{B.11}$$

so we have 12 parameters minus the 2 constraints (B.5) and this brings down to 10 parameters that are more than 6. So we have done only a partial gauge fixing.

Appendix C.

In this appendix we will give details of the calculation of the vierbein for the QPI. We will start with a vierbein of the form

$$E_A^M(QPI) = \begin{pmatrix} 1 + a_s\theta\bar{\theta} & \alpha & \beta \\ \gamma & b & c \\ \delta & d & e \end{pmatrix}, \tag{C.1}$$

which has 17 parameters minus 2 constraints (that we will see later on) bringing the total free parameters down to 15. Moreover we will choose $\alpha = \beta = 0$ like in the CPI, so we will come down to 11 parameters, which again is more than 6. This is consistent with considering our procedure as a partial gauge fixing. This would not be so if the procedure would bring the number of free parameters to less than 6 both in the CPI and in the QPI.

Let us now build the vierbein for the quantum case that is the QPI of Eq. (2). In this case the determinant of the vierbein $E = \mathrm{sdet}(E_M^A)$ has to be

$$E = -i\frac{\bar{\theta}\theta}{\hbar}. \tag{C.2}$$

This number has a body equal to zero and as explained as explained in [17] and in Appendix A, it does not admit an inverse E^{-1}. The way out is is to add a small "regulating" body ϵ to Eq. (C.2) so that the determinant can be inverted. This "regularized" determinant is

$$E^{\mathrm{reg}} = \epsilon - i\frac{\bar{\theta}\theta}{\hbar}. \tag{C.3}$$

The inverse can now be built [17] and it is

$$E^{-1} = \frac{1}{\epsilon} + \frac{1}{\epsilon^2}\frac{\bar{\theta}\theta}{\hbar}. \tag{C.4}$$

We will now go on to find for the QPI the analog of the two constraints of Eq. (53). Let us insert the regularized E^{reg} of Eq. (C.3) into the action of the QPI written in Eq. (35) once we have integrated out the P. Moreover let us keep only the kinetic piece:

$$S_{QPI}^{\mathrm{reg}} = i\int dt d\theta d\bar{\theta}\left(\epsilon - i\frac{\bar{\theta}\theta}{\hbar}\right)\left(\frac{1}{2}D_tQ\, D_tQ\right). \tag{C.5}$$

Performing the products above,we get:

$$S_{QPI}^{\rm reg} = i\,\epsilon \int dt d\theta d\bar{\theta} \left(\frac{1}{2} D_t Q\, D_t Q \right) + \frac{i}{\hbar} \int dt d\theta d\bar{\theta}\, \bar{\theta}\theta \left(\frac{1}{2} D_t Q\, D_t Q \right). \qquad \text{(C.6)}$$

The first term goes to zero in the true quantum-case because in this case $\epsilon \to 0$. So we will work out only the second term in Eq. (C.6) using the general form of the vierbein written in Eq. (52) and using the expression Eq. (51) for the covariant derivative. The second term in Eq. (C.6) turns out to be

$$\frac{1}{\hbar} \int dt\, d\theta\, d\bar{\theta}\, \frac{1}{2} \left(a\, \partial_t Q + d\, \partial_t Q + \beta\, \partial_{\bar{\theta}} Q \right)^2 \bar{\theta}\theta. \qquad \text{(C.7)}$$

As the α and β are odd and get multiplied by $\bar{\theta}\theta$ the only term which survives is

$$\frac{1}{\hbar} \int dt\, d\theta\, d\bar{\theta}\, \frac{1}{2}\, a^2\, (\partial_t Q)^2\, \bar{\theta}\theta. \qquad \text{(C.8)}$$

Differently than in the classical case of Eq. (B.1), note that in the quantum case a is an even element made of a body a_B and a soul a_S. So Eq. (C.8) turns out to be

$$\frac{1}{\hbar} \int dt\, d\theta\, d\bar{\theta} \frac{1}{2} \left(a_B^2 + 2\, a_B\, a_S\, \bar{\theta}\theta \right) \bar{\theta}\theta \partial_t Q \partial_t Q = \frac{1}{2\,\hbar} \int a_B^2\, (\partial_t q)(\partial_t q). \qquad \text{(C.9)}$$

where q is the first component of Q like in Eq. (26) and we have omitted the indices for simplicity. In order to get the usual kinetic piece of quantum mechanics in Eq. (C.9) we need to have:

$$a_B = \pm 1, \qquad \text{(C.10)}$$

while a_S is free. Next we have to impose the conditions Eq. (C.4) and Eq. (C.10) on the determinant, i.e.:

$$\text{sdet} \begin{pmatrix} \pm 1 + a_S\, \bar{\theta}\theta & \alpha & \beta \\ \gamma & b & c \\ \delta & d & e \end{pmatrix} = \frac{1}{\epsilon} + \frac{i}{\epsilon} \frac{\bar{\theta}\theta}{\hbar}. \qquad \text{(C.11)}$$

Working out the sdet on the L.H.S. of Eq. (C.11) using the usual rules given in

[17], we get

$$\text{sdet}\begin{pmatrix}\pm 1 + a_S\,\bar\theta\theta & \alpha & \beta\\ \gamma & b & c\\ \delta & d & e\end{pmatrix} \tag{C.12}$$

$$= \left[\pm 1 + a_S\,\bar\theta\theta - \begin{pmatrix}\alpha & \beta\end{pmatrix}\begin{pmatrix}b & c\\ d & e\end{pmatrix}^{-1}\begin{pmatrix}\gamma\\ \delta\end{pmatrix}\right]\cdot \det{}^{-1}\begin{pmatrix}b & c\\ d & e\end{pmatrix}.$$

Let us now simplify things by introducing some new symbols p, q, r defined as:

$$\begin{aligned} p\,\bar\theta\theta &\equiv \begin{pmatrix}\alpha & \beta\end{pmatrix}\begin{pmatrix}b & c\\ d & e\end{pmatrix}^{-1}\begin{pmatrix}\gamma\\ \delta\end{pmatrix}\\ q + r\,\bar\theta\theta &\equiv \det{}^{-1}\begin{pmatrix}b & c\\ d & e\end{pmatrix}. \end{aligned} \tag{C.13}$$

The powers of $\theta, \bar\theta$ present on the L.H.S.of Eq. (C.13) can be easily understood by remembering the powers of $\theta, \bar\theta$ present in the even and odd elements. Using Eq. (C.13) the relation Eq. (C.12) can be written as

$$\text{sdet}\begin{pmatrix}\pm 1 + a_S\,\bar\theta\theta & \alpha & \beta\\ \gamma & b & c\\ \delta & d & e\end{pmatrix} = \pm q + (a_S\,q - p\,q \pm r)\bar\theta\theta. \tag{C.14}$$

Combining Eq. (C.14) with Eq. (C.11) we get

$$\begin{cases} q = \pm\dfrac{1}{\epsilon} \\ a_s q - p\,q \pm r = \dfrac{i}{\epsilon^2\,\hbar} \end{cases}. \tag{C.15}$$

which can be combined to give

$$\pm\left(\frac{a_S}{\epsilon} - \frac{p}{\epsilon} + r\right) = \frac{1}{\epsilon^2\,\hbar}. \tag{C.16}$$

From the second equation in (C.13) we get that the matrix

$$D \equiv \begin{pmatrix}b & c\\ d & e\end{pmatrix} \tag{C.17}$$

must be invertible and the determinant of the inverse must be equal to the L.H.S. of the following equation:

$$q + r\,\bar{\theta}\theta = \pm\frac{1}{\epsilon} + r\,\bar{\theta}\theta. \tag{C.18}$$

The R.H.S. of Eq. (C.18) is otained from the first of Eq. (C.15). From the determinant of the inverse we can get the determinant of D which from Eq. (C.18) turns out to be

$$\mathrm{det}D = \pm\epsilon - \epsilon^2\, r\,\bar{\theta}\theta. \tag{C.19}$$

The determinant of D is equal to $(b\,e - c\,d)$ so Eq. (C.19) becomes

$$b\,e - c\,d = \pm\epsilon - \epsilon^2\, r\,\bar{\theta}\theta, \tag{C.20}$$

which is equal to

$$(b_B + b_S\,\bar{\theta}\theta)(e_B + e_S\,\bar{\theta}\theta) - (c_B + c_S\,\bar{\theta}\theta)(d_B + d_S\,\bar{\theta}\theta) = \pm\epsilon - \epsilon^2\, r\,\bar{\theta}\theta \tag{C.21}$$

and comparing equal powers of θ and $\bar{\theta}$ we get that Eq. (C.21) is equivalent to the following two equations

$$\begin{cases} b_B\,e_B - c_B\,d_B = \pm\epsilon \\ b_S\,e_B + b_B\,e_S - c_S\,d_B - c_B\,d_S = -\epsilon^2\, r. \end{cases} \tag{C.22}$$

The first equation is a true constraint equation while the second one relates the parameter r to the variables b, c, d, e. From Eq. (C.13) we can also obtain the detail expression of p in terms of the entries of the vierbein. A long calculation leads to the following equation

$$\begin{aligned} p = \pm\frac{1}{\epsilon}\,(&\alpha_{\bar{\theta}}\,\gamma_\theta\,e_B - \gamma_{\bar{\theta}}\,\alpha_\theta\,e_B - \alpha_{\bar{\theta}}\,\delta_\theta\,c_B + \alpha_\theta\,\delta_{\bar{\theta}}\,c_B \\ &+\beta_\theta\,\gamma_{\bar{\theta}}\,d_B - \beta_{\bar{\theta}}\,\gamma_\theta\,d_B + \beta_{\bar{\theta}}\,\delta_\theta\,b_B - \beta_\theta\,\delta_{\bar{\theta}}\,b_B)\,. \end{aligned} \tag{C.23}$$

Inserting Eq. (C.23) and the second of Eq. (C.22) into the second of Eq. (C.15) we get a constraint among the $a, b, c, d, \alpha, \beta, \gamma, \delta$. This constraint together with the first of Eq. (C.22) provides the two QPI constraints analog to the two of the CPI of Eq. (B.5) but much more complicated. In order to simplify things let us choose $\alpha = \beta = 0$ like in the CPI case. This choice, once inserted in Eq. (C.23), gives $p = 0$. Using this inside Eq. (C.16) we get

$$\frac{a_S}{\epsilon} + r = \pm \frac{i}{\epsilon^2\,\hbar} \tag{C.24}$$

and using for r the expression in the second equation of Eq. (C.22) we get from Eq. (C.24) the following constraint

$$b_S\,e_B + b_B\,e_S - c_S\,d_B - c_B\,d_S = \mp\frac{i}{\hbar} + \epsilon\,a_S. \tag{C.25}$$

This together with the first relation in Eq. (C.22) are the two constraints for the QPI analog to the two for the CPI in Eq. (B.5). Let us write together those of the QPI

$$\begin{cases} b_B\,e_B - c_B\,d_B = \pm\epsilon \\ \\ b_S\,e_B + b_B\,e_S - c_S\,d_B - c_B\,d_S = \mp\dfrac{i}{\hbar} + \epsilon\,a_S. \end{cases} \tag{C.26}$$

We can find some solutions of these equations like for example

$$\begin{aligned} &(1) \qquad e_B = 0, \quad c_B = \mp\frac{1}{d_B}, \quad c_S = \frac{\pm\epsilon\,d_S + d_B\,b_B\,a_S \pm \frac{i}{\hbar}a_S\,d_B - \epsilon\,a_S\,d_B}{d_B^2} \\ &(2) \quad b_B = \frac{\pm\epsilon + c_B\,d_B}{e_B}, \quad b_S = \frac{\mp\epsilon\,e_S - c_B\,d_B\,b_S + c_B\,d_S\,e_B}{e_B^2} \\ &\qquad + \frac{c_B\,d_B\,e_B \mp \frac{i}{\hbar}e_B + \epsilon\,a_S\,a_B}{e_B^2}. \end{aligned} \tag{C.27}$$

Appendix D.

Here we will give details for the construction of the metric in the QPI case. Let us start with the most general vierbein:

$$E_A^M = \begin{pmatrix} a & \alpha & \beta \\ \gamma & b & c \\ \delta & d & e \end{pmatrix} \tag{D.1}$$

and later on we will insert the quantum constraints Eq. (C.22) in the associated matrix. Using the relation Eq. (55) between vierbein and metric we get that the metric associated to the general vierbein in Eq. (D.1) has the form:

$$\begin{pmatrix} 1-2\gamma\,\delta+2\,\bar{\theta}\theta\,a_B a_S & d\,\gamma-b\delta+\alpha a_B & e\,\gamma-c\,\delta+\beta\,a_B \\ d\,\gamma-b\,\delta+\alpha\,a_B & 0 & be-cd+\alpha\,\beta \\ e\,\gamma-c\,\delta+\beta\,a_B & cd-be-\alpha\,\beta & 0 \end{pmatrix}. \tag{D.2}$$

Let us now rewrite the metric using the π_i introduced in Eq. (58) and Eq. (59). From the definition of π_5 in Eq. (59) it is easy to prove that

$$\gamma\,\delta = \pi_5\,\bar{\theta}\theta$$

so the element g^{11} of Eq. (D.2) can be written as

$$\begin{aligned} 1-2\gamma\,\delta+2\,\bar{\theta}\theta\,a_B a_S &= 1-2(\pi_5-a_B a_S)\bar{\theta}\theta \\ &\equiv 1-\pi_5^Q\,\bar{\theta}\theta \end{aligned} \tag{D.3}$$

where we have defined a new quantity π_5^Q as

$$\pi_5^Q \equiv \pi_5 - a_B a_S. \tag{D.4}$$

The index "Q" is to indicate that these are objects related to the QPI. The element g^{12} of Eq. (D.2) can be written as

$$\begin{aligned} d\,\gamma-b\,\delta+\alpha\,a_B &= -\pi_3\,\theta-\pi_4\,\bar{\theta}+\alpha_\theta\,a_B\,\theta+\alpha_{\bar{\theta}}\,a_B\,\bar{\theta} \\ &= -(\pi_3-\alpha_\theta a_B)\theta-(\pi_4-\alpha_{\bar{\theta}}a_B)\bar{\theta} \\ &\equiv -\pi_3^Q\,\theta-\pi_4^Q\,\bar{\theta} \end{aligned} \tag{D.5}$$

where

$$\begin{aligned} \pi_3^Q &\equiv \pi_3 - \alpha_\theta a_B \\ \pi_4^Q &\equiv \pi_4 - \alpha_{\bar\theta} a_B. \end{aligned} \tag{D.6}$$

The element g^{13} of Eq. (D.2) can be written as

$$\begin{aligned} e\,\gamma - c\,\delta + \beta\,a_B &= \pi_1\,\theta + \pi_2\,\bar\theta + \beta\,a_B \\ &= \pi_1\,\theta + \pi_2\,\bar\theta + \beta_\theta\,\theta\,a_B + \beta_{\bar\theta}\,\bar\theta\,a_B \\ &= (\pi_1 + \beta_\theta a_B)\theta + (\pi_2 + \alpha_{\bar\theta} a_B)\bar\theta \\ &\equiv \pi_1^Q\,\theta + \pi_2^Q\,\bar\theta \end{aligned} \tag{D.7}$$

where

$$\begin{aligned} \pi_1^Q &\equiv \pi_1 + \beta_\theta a_B \\ \pi_2^Q &\equiv \pi_2 + \beta_{\bar\theta} a_B. \end{aligned} \tag{D.8}$$

Next let now examine the term g^{23} of Eq. (D.2)

$$\begin{aligned} be - cd + \alpha\,\beta &= (b_B\,e_B - c_B\,d_B) + \\ &+ (b_S\,e_B + b_B\,e_S - c_S\,d_B - c_B\,d_S + \alpha_{\bar\theta}\,\beta_\theta - \beta_{\bar\theta}\,\alpha_\theta)\,\bar\theta\theta. \end{aligned} \tag{D.9}$$

When $a_B \neq \pm 1$ the first of relation (C.22) and (C.16) will turn into the following two relations:

$$\begin{cases} b_B\,e_B - e_B\,d_B = a_B\,\epsilon \\ \\ \dfrac{a_B a_S}{\epsilon} - \dfrac{a_B p}{\epsilon} + a_B r = \dfrac{i}{\epsilon^2\,\hbar} \end{cases}. \tag{D.10}$$

The first of relations (C.15) will turn into

$$q = \frac{a_B}{\epsilon}$$

while the relation Eq. (C.23) for p becomes

$$\begin{aligned} p &= \frac{a_B}{\epsilon}\,[\alpha_{\bar\theta}(\gamma_\theta\,e_B - \delta_\theta\,c_B) + \alpha_\theta(\delta_{\bar\theta}\,c_B - \gamma_{\bar\theta}\,e_B) + \\ &\quad + \beta_\theta(\gamma_{\bar\theta}\,d_B - \delta_{\bar\theta}\,b_B) + \beta_{\bar\theta}(\delta_\theta\,b_B - \gamma_\theta\,d_B)] \\ &= \frac{a_B}{\epsilon}\,(\alpha_{\bar\theta}\,\pi_1 - \alpha_\theta\,\pi_2 - \beta_\theta\,\pi_4 + \beta_{\bar\theta}\,\pi_3) \end{aligned} \tag{D.11}$$

Multiplying the second equation of (D.10) by ϵ^2/a_B we get

$$\epsilon\, a_S - \epsilon\, p + \epsilon^2\, r = \frac{i}{\hbar\, a_B}$$

which, using Eq. (D.11), implies

$$\begin{aligned} -\epsilon^2\, r &= \epsilon\, a_S - \epsilon\, p - \frac{i}{\hbar\, a_B} = \\ &= \epsilon\, a_S - \frac{i}{\hbar\, a_B} - a_B(\alpha_{\bar\theta}\,\pi_1 - \alpha_\theta\,\pi_2 - \beta_\theta\,\pi_4 + \beta_{\bar\theta}\,\pi_3). \end{aligned} \tag{D.12}$$

Let us now remember the second relation of Eq. (C.22)

$$b_S\, e_B + b_B\, e_S - c_S\, d_B - c_B\, d_S = -\epsilon^2\, r. \tag{D.13}$$

Note that the L.H.S. of this equation are exactly the first four terms of the soul of $b\,e - c\,d + \alpha\,\beta$ in Eq. (D.9). Replacing them with the expression of $-\epsilon^2\, r$, which appear on the L.H.S. of Eq. (D.13), we get that the soul of $b\,e - c\,d + \alpha\,\beta$ is equal to

$$\begin{aligned} \epsilon\, a_S &- \frac{i}{\hbar\, a_B} + a_B(\alpha_\theta\,\pi_2 - \alpha_{\bar\theta}\,\pi_1 + \beta_\theta\,\pi_4 - \beta_{\bar\theta}\,\pi_3) + \\ &+ \alpha_{\bar\theta}\,\beta_\theta - \alpha_\theta\,\beta_{\bar\theta} \equiv \pi_6^Q. \end{aligned} \tag{D.14}$$

In the equation above the soul of $b\,e - c\,d + \alpha\,\beta$ has been set equal to π_6^Q.

Going now back to Eq. (D.9) and using for its body the first constraint of Eq. (D.10) we get

$$b\,e - c\,d + \alpha\,\beta = a_B\,\epsilon + \bar\theta\theta\,\pi_6^Q. \tag{D.15}$$

We have now all the elements to write down the metric with all constraints implemented

$$g^{MN} = \begin{pmatrix} 1 - 2\,\pi_5^Q\,\bar\theta\theta & -\pi_3^Q\,\theta - \pi_4^Q\,\bar\theta & \pi_1^Q\,\theta + \pi_2^Q\,\bar\theta \\ -\pi_3^Q\,\theta - \pi_4^Q\,\bar\theta & 0 & a_B\,\epsilon + \pi_6^Q\bar\theta\theta \\ \pi_1^Q\theta + \pi_2^Q\bar\theta & -a_B\,\epsilon - \pi_6^Q\,\bar\theta\theta & 0 \end{pmatrix}.. \tag{D.16}$$

If we define a new variable π_7^Q as

$$\pi_7^Q = a_B\,\epsilon, \tag{D.17}$$

the metric Eq. (D.16) can be written using only π_i^Q variables as follows:

$$g^{MN} = \begin{pmatrix} 1 - 2\,\pi_5^Q\,\bar{\theta}\theta & -\pi_3^Q\,\theta - \pi_4^Q\,\bar{\theta} & \pi_1^Q\,\theta + \pi_2^Q\,\bar{\theta} \\ -\pi_3^Q\,\theta - \pi_4^Q\,\bar{\theta} & 0 & \pi_7^Q + \pi_6^Q\bar{\theta}\theta \\ \pi_1^Q\theta + \pi_2^Q\bar{\theta} & -\pi_7^Q - \pi_6^Q\,\bar{\theta}\theta & 0 \end{pmatrix}. \tag{D.18}$$

Let us summarize the various quantities we have introduced:

$$\begin{cases} \pi_1^Q \equiv \pi_1 + \beta_\theta\, a_B \\ \pi_2^Q \equiv \pi_2 + \beta_\theta\, a_B \\ \pi_3^Q \equiv \pi_3 - \alpha_\theta\, a_B \\ \pi_4^Q \equiv \pi_4 - \alpha_{\bar{\theta}}\, a_B \\ \pi_5^Q \equiv \pi_5 - a_B\, a_S \\ \pi_6^Q \equiv \epsilon\, a_S - \dfrac{i}{\hbar\, a_B} + a_B(\alpha_\theta\,\pi_2 - \alpha_{\bar{\theta}}\pi_1 + \beta_\theta\pi_4 - \beta_{\bar{\theta}}\pi_3) + \alpha_{\bar{\theta}}\beta_\theta - \alpha_\theta\beta_{\bar{\theta}} \\ \pi_7^Q \equiv a_B\epsilon \end{cases} \tag{D.19}$$

Let us now count the number of free parameters. α and β do not enter any of the constraints in Eq. (C.22) so they are free. In order to simplify things we can put them by hand equal to zero like in the CPI and we suggested this already after Eq. (C.23). Moreover we should remember that $a_B = \pm 1$ as proved in Eq. (C.10) but differently than the classical case a_S is a free parameter. So with this choice Eq. (D.19), becomes

$$\begin{cases} \pi_1^Q = \pi_1 \\ \pi_2^Q = \pi_2 \\ \pi_3^Q = \pi_3 \\ \pi_4^Q = \pi_4 \\ \pi_5^Q = \pi_5 \mp a_S \\ \pi_6^Q = \epsilon\, a_S \mp \dfrac{i}{\hbar} \\ \pi_7^Q = \pm\epsilon \end{cases} \tag{D.20}$$

So the "quantum" metric depend on 5 parameters $\pi_1^Q, \pi_2^Q, \pi_3^Q, \pi_4^Q$ and a_S while the classical one only on 4. The reader may object that also α and β were free and should be counted. He is right. Anyhow we put them equal to zero both in the CPI and the QPI and so the difference in the numbers of free parameters remains one between QM and CM Let us look at the vierbein. For the CPI we have 10 free parameters, while in the QPI will be 11 because we have a_S as extra variable. If we had not put $\alpha = \beta = 0$ we would have 14 parameters for the vierbein of the CPI and 15 for the QPI. For the metrics instead, as the α and β get incorporated into the π_i^Q (see Eq. (D.19)) the number of free parameters is 6 for the QPI, while for the CPI we do not know because we should re-derive the metric keeping the α and β different from zero.To finish this section let us explore the issue of wether we can recover the classical case from the "regulated quantum" one without setting $\hbar \to 0$ but manipolating the ϵ parameter and the others. For sure we have to require that $\alpha = \beta = a_S = 0$ which are the values set previously in the CPI. Moreover in the CPI we had the constraint

$$b\,e - c\,d = \pm 1 \tag{D.21}$$

while in the QPI we had (with $\alpha = \beta = 0$) Eq. (D.15):

$$b\,e - c\,d = a_B\,\epsilon + \bar{\theta}\theta\,\pi_6^Q. \tag{D.22}$$

For Eq. (D.22) to be equal to Eq. (D.21), as we know that $a_B = \pm 1$, we have to require that $\epsilon \to 1$ and $\pi_6^Q = 0$. Actually, remembering the form of π_6^Q present in Eq. (D.19), we see that the following other form of π_6^Q

$$\begin{aligned} \pi_6^Q &= \epsilon\, a_S - \frac{i}{\hbar} a_B\,(1-\epsilon) + a_B(\alpha_\theta\,\pi_2 - \alpha_{\bar{\theta}}\,\pi_1 + \beta_\theta\,\pi_4 + \\ &- \beta_{\bar{\theta}}\,\pi_3) + \alpha_{\bar{\theta}}\,\beta_\theta - \alpha_\theta\,\beta_{\bar{\theta}} \end{aligned} \tag{D.23}$$

has the same quantum limit ($\epsilon \to 0$) as the one conained in Eq. (D.19). So we can use Eq. (D.23) in order to reproduce Q.M. This new π_6^Q has the feature that it goes to zero for $\epsilon = 1$ (of course this has to be combined with the other things we require for CM: $a_S = \alpha = \beta = 0$). So in the limit $\epsilon \to 0$ we would get QM and in the limit $\epsilon \to 1$ we would get CM This is equivalent of having required that the determinant of the vierbein had the form

$$E = \epsilon - i\,(1-\epsilon)\,\frac{\bar{\theta}\theta}{\hbar}. \tag{D.24}$$

For $\epsilon \to 1$ we would get from Eq. (D.24)

$$E = \mathbb{I}$$

which is the $C.P.I$ and for $\epsilon \to 0$ we would get

$$E = -i\,\frac{\bar{\theta}\theta}{\hbar},$$

which is QM For ϵ in between 0 and 1 we would get a family of models which are betwwen CM and QM and could interpolate all the mesoscopic physics. Before concluding this section let us provide the inverse of g^{MN} of Eq. (D.18). This quantity will be useful for the calculations provided in the next section:

$$g_{MN} = \begin{pmatrix} 1 - \frac{2\,\bar{\theta}\theta(\varphi^Q - \pi_5^Q \pi_7^Q)}{\pi_7^Q} & -\frac{\theta\,\pi_1^Q + \bar{\theta}\,\pi_2^Q}{\pi_7^Q} & -\frac{\theta\,\pi_3^Q + \bar{\theta}\,\pi_4^Q}{\pi_7^Q} \\ \frac{\theta\,\pi_1^Q + \bar{\theta}\,\pi_2^Q}{\pi_7^Q} & 0 & -\frac{\pi_7^Q + (\varphi^Q - \pi_6^Q)}{\pi_7^{Q^2}} \\ \frac{\theta\,\pi_3^Q + \bar{\theta}\,\pi_4^Q}{\pi_7^Q} & \frac{\pi_7^Q + \bar{\theta}\theta(\varphi^Q - \pi_6^Q)}{\pi_7^{Q^2}} & 0 \end{pmatrix}. \tag{D.25}$$

In the expression above to be compact we have defined a new quantity:

$$\varphi^Q \equiv \pi_2^Q\,\pi_3^Q - \pi_1^Q\,\pi_4^Q.$$

Note that in the true quantum limit $\epsilon \to 0$ this metric is singular because $\pi_7^Q \to 0$.

Appendix E.

We will now calculate the Christofel symbols for the CPI leaving the body "a" of the vierbein undetermined. The results, obtained assuming the π_i independent on t and using *Mathematica*, are the following ones:

$$\begin{aligned}
\Gamma^t_{t\theta} &= \frac{\theta\pi_1(\pi_2-\pi_3)+\bar\theta(\pi_2(\pi_2+\pi_3)-2\pi_1\pi_4)}{2\,a} = -\Gamma^t_{\theta t} \\
\Gamma^\theta_{t\theta} &= \frac{\pi_2+\pi_3}{2} = \Gamma^\theta_{\theta t} = -\Gamma^{\bar\theta}_{t\bar\theta} = -\Gamma^{\bar\theta}_{\bar\theta t} \\
\Gamma^{\bar\theta}_{t\theta} &= -\pi_1 = \Gamma^{\bar\theta}_{\theta t} \\
\Gamma^t_{t\bar\theta} &= \frac{\theta(-\pi_3(\pi_2+\pi_3)+2\pi_1\pi_4)+\bar\theta\,\pi_4(\pi_2-\pi_3)}{2\,a} = -\Gamma^t_{\bar\theta t} \\
\Gamma^\theta_{t\bar\theta} &= \pi_4 = \Gamma^\theta_{\bar\theta t} \\
\Gamma^t_{\theta\bar\theta} &= \frac{\pi_2-\pi_3}{2\,a} - 2\,\bar\theta\theta(\pi_2-\pi_3)(\pi_2\pi_3-\pi_1\pi_4) = -\Gamma^t_{\bar\theta\theta} \\
\Gamma^\theta_{\theta\bar\theta} &= \frac{\theta(\pi_3(3\pi_2-\pi_3)-2\pi_1\pi_4)+\bar\theta\pi_4(\pi_2-\pi_3)}{2\,a} = -\Gamma^\theta_{\bar\theta\theta} \\
\Gamma^{\bar\theta}_{\theta\bar\theta} &= \frac{\theta(\pi_1(\pi_3-\pi_2)+\bar\theta(\pi_2(3\pi_3-\pi_2)-2\pi_1\pi_4))}{2\,a} = -\Gamma^{\bar\theta}_{\bar\theta\theta}.
\end{aligned} \tag{E.1}$$

All the other Christofel symbols are equal to zero. Similarly we can calculate the Christofel symbols for the QPI using the metric (D.18) and (D.25). In order to simplify the expression for the Christofel symbols and curvatures, we need to introduce the following quantity:

$$\sigma_1 \equiv \pi_2^Q\pi_3^Q - \pi_1^Q\pi_4^Q - \pi_5^Q\pi_7^Q. \tag{E.2}$$

Note that in the classical limit $\pi_i^Q \to \pi_i$, $i=1,\ldots,4$ since $\alpha_\theta, \alpha_{\bar\theta}, \beta_\theta, \beta_{\bar\theta} \to 0$ and $\pi_7^Q \to a$ being $\epsilon \to 1$. Therefore $\sigma_1 \to (\pi_2\pi_3 - \pi_1\pi_4 - \pi_5\, a_B)$, that in the classical limit is equal to zero. The result, via Mathematica [23], for parameters

π_i independent on t, turns out to be:

$$
\begin{aligned}
\Gamma^t_{tt} &= \bar\theta\theta\frac{(\pi_3^Q-\pi_2^Q)}{\pi_7^Q}\sigma_1 \\
\Gamma^\theta_{tt} &= -\bar\theta\,\sigma_1 \\
\Gamma^t_{t\theta} &= \Gamma^{t\,CPI}_{t\theta}(\pi_i^Q)+\frac{\sigma_1}{\pi_7^Q} = -\Gamma^t_{\theta t} \\
\Gamma^\theta_{t\theta} &= \Gamma^{\theta\,CPI}_{t\theta}(\pi_i^Q)+\bar\theta\theta\,\frac{\pi_6^Q(\pi_2^Q+\pi_3^Q)+2\pi_3^Q\,\sigma_1}{2\,\pi_7^Q} = \Gamma^\theta_{\theta t} \\
\Gamma^{\bar\theta}_{t\theta} &= \Gamma^{\bar\theta\,CPI}_{t\theta}(\pi_i^Q)-\bar\theta\theta\,\frac{\pi_1^Q\,(\sigma_1+\pi_6^Q)}{\pi_7^Q} = -\Gamma^{\bar\theta}_{\theta t} \\
\Gamma^t_{t\bar\theta} &= \Gamma^{t\,CPI}_{t\bar\theta}(\pi_i^Q)-\frac{\sigma_1}{\pi_7^Q} = -\Gamma^t_{\bar\theta t} \\
\Gamma^\theta_{t\bar\theta} &= \Gamma^{\theta\,CPI}_{t\bar\theta}(\pi_i^Q)+\bar\theta\theta\,\frac{\pi_4^Q\,(\sigma_1+\pi_6^Q)}{\pi_7^Q} = \Gamma^\theta_{\bar\theta t} \qquad \text{(E.3)} \\
\Gamma^{\bar\theta}_{t\bar\theta} &= \Gamma^{\bar\theta\,CPI}_{t\bar\theta}(\pi_i^Q)-\bar\theta\theta\,\frac{\pi_6^Q\,(\pi_2^Q+\pi_3^Q)+2\,\pi_2^Q\,\sigma_1}{2\,\pi_7^Q} = \Gamma^{\bar\theta}_{\bar\theta t} \\
\Gamma^t_{\theta\bar\theta} &= \Gamma^{t\,CPI}_{\theta\bar\theta}(\pi_i^Q)-\bar\theta\theta\,\frac{(\pi_2^Q-\pi_3^Q)(\sigma_1+\pi_6^Q+2\,\pi_5^Q\pi_7^Q)}{2\,\pi_7^{Q^2}} \\
&= -\Gamma^t_{\bar\theta\theta} \\
\Gamma^\theta_{\theta\bar\theta} &= \Gamma^{\theta\,CPI}_{\theta\bar\theta}(\pi_i^Q)-\theta\,\frac{\pi_6^Q}{\pi_7^Q} = -\Gamma^\theta_{\bar\theta\theta} \\
\Gamma^{\bar\theta}_{\theta\bar\theta} &= \Gamma^{\bar\theta\,CPI}_{\theta\bar\theta}(\pi_i^Q)-\bar\theta\,\frac{\pi_6^Q}{\pi_7^Q} = -\Gamma^{\bar\theta}_{\bar\theta\theta},
\end{aligned}
$$

where

$$\Gamma^{C\,CPI}_{AB}(\pi_i^Q)\equiv\Gamma^{C\,CPI}_{AB}(\pi_i\to\pi_i^Q, a\to\pi_7^Q).$$

All the other Christofel symbols are equal to zero.

If we choose the π_i dependent on time for the CPI Christofel symbols we get the following expressions:

$$
\begin{aligned}
\Gamma^{t}_{tt} &= \bar\theta\theta\,\pi_5' \\
\Gamma^{\theta}_{tt} &= \theta\,\pi_3' + \bar\theta\,\pi_4' \\
\Gamma^{\bar\theta}_{tt} &= -\theta\,\pi_1' - \bar\theta\,\pi_2' \\
\Gamma^{t}_{t\theta} &= -\theta\,\frac{\pi_1\,(\pi_2-\pi_3)}{2\,a} + \bar\theta\,\frac{(\pi_2+\pi_3)\pi_2 - 2\,\pi_1\,\pi_4}{2\,a} \\
\Gamma^{\theta}_{t\theta} &= \frac{\pi_2+\pi_3}{2} + \bar\theta\theta\,\frac{\pi_5'}{2} \\
\Gamma^{\bar\theta}_{t\theta} &= -\pi_1 \\
\Gamma^{t}_{t\bar\theta} &= \bar\theta\,\frac{(\pi_2-\pi_3)\pi_4}{2\,a} + \theta\,\frac{2\,\pi_1\,\pi_4 - \pi_3\,(\pi_2+\pi_3)}{2\,a} \\
\Gamma^{\theta}_{t\bar\theta} &= \pi_4 \\
\Gamma^{\bar\theta}_{t\bar\theta} &= -\frac{\pi_2+\pi_3}{2} + \bar\theta\theta\,\frac{\pi_5'}{2} \\
\Gamma^{t}_{\theta t} &= -\Gamma^{t}_{t\theta} \\
\Gamma^{\theta}_{\theta t} &= \Gamma^{\theta}_{t\theta} \\
\Gamma^{\bar\theta}_{\theta t} &= \Gamma^{\bar\theta}_{t\theta} \\
\Gamma^{t}_{\theta\bar\theta} &= \frac{\pi_2-\pi_3}{2\,a} - \bar\theta\theta\,\frac{4\,\pi_2\,\pi_5 - \pi_5'}{2\,a} \\
\Gamma^{\theta}_{\theta\bar\theta} &= \bar\theta\,\frac{(\pi_2-\pi_3)\,\pi_4}{2\,a} + \theta\,\frac{(\pi_2-\pi_3)\,\pi_3 + 2\,a\,\pi_5}{2\,a} \\
\Gamma^{\bar\theta}_{\theta\bar\theta} &= -\bar\theta\,\frac{(\pi_2-\pi_3)\,\pi_1}{2\,a} + \bar\theta\,\frac{2\,a\,\pi_5 - (\pi_2-\pi_3)\,\pi_2}{2\,a} \\
\Gamma^{t}_{\bar\theta t} &= -\Gamma^{t}_{t\bar\theta} \\
\Gamma^{\theta}_{\bar\theta t} &= \Gamma^{\theta}_{t\bar\theta} \\
\Gamma^{\bar\theta}_{\bar\theta t} &= -\frac{\pi_2+\pi_3}{2} + \bar\theta\theta\,\frac{\pi_2'+\pi_3'}{2} \\
\Gamma^{t}_{\bar\theta\theta} &= -\Gamma^{t}_{\theta\bar\theta} \\
\Gamma^{\theta}_{\bar\theta\theta} &= -\Gamma^{\theta}_{\theta\bar\theta}. \\
\Gamma^{\theta}_{t\theta} &= -\frac{\pi_2+\pi_3}{2} + \bar\theta\theta\,\frac{\pi_5'}{2}
\end{aligned}
\tag{E.4}
$$

For the QPI case, when the coefficient π_i depend on time the Christofel symbols turn out to be:

$$
\begin{aligned}
\Gamma^t_{tt} &= \Gamma^{t\,CPI}_{tt} - \bar\theta\theta\,\frac{(\pi_2-\pi_3)\,\sigma_1}{\pi_7}\\
\Gamma^\theta_{tt} &= \Gamma^{\theta\,CPI}_{tt} - \theta\,\sigma_1\\
\Gamma^{\bar\theta}_{tt} &= \Gamma^{\bar\theta\,CPI}_{tt} - \bar\theta\,\sigma_1\\
\Gamma^t_{t\theta} &= \Gamma^{t\,CPI}_{t\theta} + \bar\theta\,\frac{\sigma_1}{\pi_7}\\
\Gamma^\theta_{t\theta} &= \Gamma^{\theta\,CPI}_{t\theta} + \bar\theta\theta\,\frac{\sigma_1' + \pi_6\,(\pi_2+\pi_3) + 2\,\pi_3\,\sigma_1 - \pi_6'}{2\,\pi_7}\\
\Gamma^{\bar\theta}_{t\theta} &= \Gamma^{\bar\theta\,CPI}_{t\theta} - \bar\theta\theta\,\frac{\pi_1\,(\sigma_1+\pi_6)}{\pi_7}\\
\Gamma^t_{t\bar\theta} &= \Gamma^{t\,CPI}_{t\bar\theta} - \theta\,\frac{\sigma_1}{\pi_7}\\
\Gamma^\theta_{t\bar\theta} &= \Gamma^{\theta\,CPI}_{t\bar\theta} + \bar\theta\theta\,\frac{\pi_4\,(\sigma_1+\pi_6)}{\pi_7}\\
\Gamma^{\bar\theta}_{t\bar\theta} &= \Gamma^{\bar\theta\,CPI}_{t\bar\theta} + \bar\theta\theta\,\frac{\sigma_1' - \pi_6\,(\pi_2+\pi_3) - 2\,\pi_2\,\sigma_1 - \pi_6'}{\pi_7}\\
\Gamma^t_{\theta t} &= -\Gamma^t_{t\theta}\\
\Gamma^\theta_{\theta t} &= \Gamma^\theta_{t\theta}\\
\Gamma^{\bar\theta}_{\theta t} &= \Gamma^{\bar\theta}_{t\theta}\\
\Gamma^t_{\theta\bar\theta} &= \Gamma^{t\,CPI}_{\theta\bar\theta} - \bar\theta\theta\,\frac{\pi_6' - \sigma_1' + 2\,(\pi_2-\pi_3)(\sigma_1-\pi_6)}{2\,\pi_7^2}\\
\Gamma^\theta_{\theta\bar\theta} &= \Gamma^{\theta\,CPI}_{\theta\bar\theta} + \theta\,\frac{\sigma_1-\pi_6}{\pi_7}\\
\Gamma^{\bar\theta}_{\theta\bar\theta} &= \Gamma^{\bar\theta\,CPI}_{\theta\bar\theta} + \bar\theta\,\frac{\sigma_1-\pi_6}{\pi_7}\\
\Gamma^t_{\bar\theta t} &= -\Gamma^t_{t\bar\theta}\\
\Gamma^\theta_{\bar\theta t} &= \Gamma^\theta_{t\bar\theta}\\
\Gamma^{\bar\theta}_{\bar\theta t} &= \Gamma^{\bar\theta}_{\bar\theta t}\\
\Gamma^t_{\bar\theta\theta} &= -\Gamma^t_{\theta\bar\theta}\\
\Gamma^\theta_{\bar\theta\theta} &= -\Gamma^\theta_{\theta\bar\theta}\\
\Gamma^{\bar\theta}_{\bar\theta\theta} &= -\Gamma^{\bar\theta}_{\theta\bar\theta}.
\end{aligned}
$$

The other symbols are zero. All π_i appearing above have to be intended as π_i^Q.

Appendix F.

Appendix F.1. Time Independent CPI Ricci tensor and scalar

In the case of π_i independent of t the CPI Ricci curvature tensor turns out to be:

$$\begin{aligned}
R_{\theta\theta} &= R_{\bar\theta\bar\theta} = 0 \qquad \text{(F.1)}\\
R_{tt} &= \frac{(\pi_2+\pi_3)^2 - 4\,\pi_1\pi_4}{2}\\
R_{t\theta} &= -\frac{(\theta\,\pi_1 + \bar\theta\,\pi_2)((\pi_2+\pi_3)^2 - 4\,\pi_1\pi_4)}{2\,a} = -R_{\theta t}\\
R_{t\bar\theta} &= -\frac{(\theta\,\pi_3 + \bar\theta\,\pi_4)((\pi_2+\pi_3)^2 - 4\,\pi_1\pi_4)}{2\,a} = -R_{\bar\theta t}\\
R_{\theta\bar\theta} &= \frac{\bar\theta\theta}{2}(\pi_1\pi_4 - \pi_2\pi_3)(\pi_2^2 - 6\,\pi_2\pi_3 + \pi_3^2 + 4\,\pi_1\,\pi_4)\\
&- \frac{a}{2}(\pi_2^2 - 10\,\pi_2\pi_3 + \pi_3^2 + 8\,\pi_1\,\pi_4) = -R_{\bar\theta\theta}
\end{aligned}$$

From the components we can build the Ricci scalar

$$R^{CPI} = -\frac{1}{2}(\pi_2^2 - 22\,\pi_2\pi_3 + \pi_3^2 + 20\,\pi_1\,\pi_4) + 8\,\frac{\bar\theta\theta}{a}(\pi_2\pi_3 - \pi_1\pi_4)^2 \qquad \text{(F.2)}$$

Appendix F.2. Time Independent QPI Ricci tensor and scalar

For the QPI, the components of the Ricci tensor can be written as:

$$
\begin{aligned}
R_{\theta\theta} &= R_{\bar\theta\bar\theta} = 0 \qquad \text{(F.3)}\\
R_{tt} &= R^{CPI}_{tt}(\pi^Q_i) + 2\sigma_1 + \frac{\bar\theta\theta}{\pi^Q_7}\Big[\pi^Q_6((\pi^Q_2+\pi^Q_3)^2 - 4\,\pi^Q_1\pi^Q_4) + \\
&+ 2\,\sigma_1\,(\pi^{Q^2}_2 + 4\,\pi^Q_2\pi^Q_3 + \pi^{Q^2}_3 - 6\,\pi^Q_1\pi^Q_4 - \pi^Q_6) + 6\,\sigma_1^2\Big]\\
R_{t\theta} &= R^{CPI}_{t\theta}(\pi^Q_i) - \frac{\pi^Q_6 + 3\,\sigma_1}{\pi^Q_7}\Big[\theta\pi^Q_1 + \bar\theta(\pi^Q_2+\pi^Q_3)\Big] = -R_{\theta t}\\
R_{t\bar\theta} &= R^{CPI}_{t\bar\theta}(\pi^Q_i) - \frac{\pi^Q_6 + 3\,\sigma_1}{\pi^Q_7}\Big[\bar\theta\pi^Q_4 + \theta(\pi^Q_2+\pi^Q_3)\Big] = -R_{\bar\theta t}\\
R_{\theta\bar\theta} &= R^{CPI}_{\theta\bar\theta}(\pi^Q_i) - \frac{\sigma_1 - 3\,\pi^Q_6}{\pi^Q_7} + \frac{\bar\theta\theta}{2\,\pi^{Q^2}_7}\Big[2\,\pi^{Q^2}_6 + 8\,\pi^Q_1\pi^Q_4\pi^Q_6\\
&- 8\,\pi^Q_2\pi^Q_3\pi^Q_6 - \sigma_1((\pi^Q_2-\pi^Q_3)^2 + 4\,\pi^Q_6 + 2\,\sigma_1)\Big] = -R_{\bar\theta\theta},
\end{aligned}
$$

where

$$R^{CPI}_{AB}(\pi^Q_i) \equiv R^{CPI}_{AB}(\pi_i \to \pi^Q_i, a \to \pi^Q_7).$$

The Ricci scalar turns out to be

$$
\begin{aligned}
R^{QPI} &= R^{CPI}(\pi^Q_i) + 2\,\sigma_1 - 3\,\pi^Q_6 + \qquad \text{(F.4)}\\
&+ \frac{\bar\theta\theta}{\pi^Q_7}\Big[-\pi^Q_6(\pi^{Q^2}_2 + 6\,\pi^Q_2\pi^Q_3 + \pi^{Q^2}_3 - 8\,\pi^Q_1\pi^Q_4) + 4\,\pi^{Q^2}_6\\
&- 4\,\sigma_1(\pi^{Q^2}_2 + 3\,\pi^Q_2\pi^Q_3 + \pi^{Q^2}_3 - 5\,\pi^Q_1\pi^Q_4\pi^Q_6 + \sigma_1)\Big]
\end{aligned}
$$

where

$$R^{CPI}(\pi^Q_i) = R^{CPI}(\pi_i \to \pi^Q_i, a \to \pi^Q_7).$$

Appendix G.

Appendix G.1. Time Dependent CPI Ricci tensor and scalar

In the case of π_i dependent on t we get for the CPI the expressions below where π_i' and π_i'' indicate the first and second derivative of π_i with respect to t.

$$\begin{aligned}
R_{\theta\theta} &= R_{\bar\theta\bar\theta} = 0 \qquad \text{(G.1)}\\
R_{tt} &= R^{CPI}_{tt}(\pi_i) + (\pi_3' - \pi_2') + \bar\theta\theta\left[\pi_3^2\pi_2' - \pi_2^2\pi_3' + \right.\\
&\quad + (\pi_3' - \pi_2')(a\,\pi_5 - 2\,\pi_2\pi_3 + 3\,\pi_1\pi_4) + 2\,a\pi_5' + \\
&\quad \left. + (\pi_2 - \pi_3)(\pi_4\pi_1' + \pi_4'\pi_1)\right]\\
R_{t\theta} &= R^{CPI}_{t\theta}(\pi_i) + \frac{\theta\pi_1 + \bar\theta\pi_2}{2\,a}(\pi_3' - \pi_2') + \frac{\bar\theta\pi_5'}{2}\\
R_{t\bar\theta} &= R^{CPI}_{t\bar\theta}(\pi_i) + \frac{\theta\pi_3 + \bar\theta\pi_4}{2\,a}(\pi_2' - \pi_3') - \frac{\theta\pi_5'}{2}\\
R_{\theta t} &= R^{CPI}_{\theta t}(\pi_i) - \frac{\theta\pi_3 + \bar\theta\pi_4}{2\,a}(\pi_2' - \pi_3') - \frac{3\,\theta\pi_5'}{2}\\
R_{\theta\bar\theta} &= R^{CPI}_{\theta\bar\theta}(\pi_i) + \frac{a\,(\pi_2' - \pi_3')}{2} + \frac{\bar\theta\theta}{2}[5\,a\,\pi_5'(\pi_3 - \pi_2) + \\
&\quad + 4\,a\,\pi_5(\pi_3' - \pi_2') + a\pi_5'']\\
R_{\bar\theta t} &= R^{CPI}_{\bar\theta t}(\pi_i) - \frac{\theta\pi_3 + \bar\theta\pi_4}{2\,a}(\pi_2' - \pi_3') - \frac{3\,\theta\pi_5'}{2}\\
R_{\bar\theta\theta} &= R^{CPI}_{\bar\theta\theta}(\pi_i) - \frac{a\,(\pi_2' - \pi_3')}{2} - \frac{\bar\theta\theta}{2}[5\,a\,\pi_5'(\pi_3 - \pi_2) + \\
&\quad + 4\,a\,\pi_5(\pi_3' - \pi_2') + a\pi_5'']\,.
\end{aligned}$$

$R^{CPI}_{AB}(\pi_i)$ are the expressions previously presented in Appendix C for the time independent CPI Ricci tensor. The other tensor components are all equal to zero. The CPI Ricci scalar, when the parameters are explicit functions of time, turns out to be:

$$\begin{aligned}
R^{CPI} &= R^{CPI}(\pi_i) + 2(\pi_2' - \pi_3') + \qquad \text{(G.2)}\\
&+ \bar\theta\theta\,[4\,\pi_5\,(\pi_3' - \pi_2') + 7\,(\pi_3 - \pi_2)\,\pi_5' + 2\,\pi_5'']\,,
\end{aligned}$$

where, once again, $R^{CPI}(\pi_i)$ is the CPI Ricci scalar given in Appendix C.

Appendix G.2. Time Dependent QPI Ricci Tensor and Curvature

In the case of π_i dependent on t we get for the QPI Ricci tensor:

$$\begin{aligned}
R_{tt} &= R^{QPI}_{tt}(\pi_i) + \pi_2'(t) - \pi_3'(t) + \qquad \text{(G.3)}\\
&- \frac{\bar\theta\theta}{\pi_7}[-\pi_2(t)\left(2\pi_4(t)\pi_1'(t) + 4\pi_3(t)\left(\pi_2'(t) - \pi_3'(t)\right) + 2\pi_1(t)\pi_4'(t) + \right.\\
&+ \pi_7\pi_5'(t) + \pi_3''(t)) + \pi_3(t)\left(2\pi_4(t)\pi_1'(t) + 2\pi_1(t)\pi_4'(t) + \pi_7\pi_5'(t) + \right.\\
&- \left.\pi_2''(t)\right) + 2\pi_2(t)^2\pi_3'(t) - 2\pi_3(t)^2\pi_2'(t) + 6\pi_1(t)\pi_4(t)\pi_2'(t) +\\
&+ 2\pi_7\pi_5(t)\pi_2'(t) + \pi_6(t)\pi_2'(t) - 6\pi_1(t)\pi_4(t)\pi_3'(t) - 2\pi_7\pi_5(t)\pi_3'(t) +\\
&- \pi_6(t)\pi_3'(t) - 2\pi_2'(t)\pi_3'(t) + 2\pi_1'(t)\pi_4'(t) + \pi_4(t)\pi_1''(t) + \pi_1(t)\pi_4''(t) +\\
&+ \pi_6''(t)]\\
R_{t\theta} &= R^{QPI}_{t\theta}(\pi_i) + \frac{\theta}{2\pi_7}\left(\pi_1(t)\pi_2'(t) - \pi_1(t)\pi_3'(t)\right) + \frac{\bar\theta}{2\pi_7}\,[-\pi_4(t)\pi_1'(t) +\\
&+ \left(\pi_2(t) + \pi_3(t)\right)\pi_2'(t) - \pi_1(t)\pi_4'(t) + 3\pi_6'(t)]\\
R_{t\bar\theta} &= R^{QPI}_{t\bar\theta}(\pi_i) + \frac{\theta}{2\pi_7}\,[\pi_4(t)\pi_1'(t) - \left(\pi_2(t) + \pi_3(t)\right)\pi_3'(t) + \pi_1(t)\pi_4'(t) +\\
&+ -3\pi_6'(t)] + \frac{\bar\theta}{2\,\pi_7}\left(\pi_4(t)\left(\pi_2'(t) - \pi_3'(t)\right)\right)\\
R_{\theta t} &= R^{QPI}_{\theta t}(\pi_i) + \frac{\theta}{2\pi_7}\,\pi_1(t)\left(\pi_3'(t) - \pi_2'(t)\right) + \frac{\bar\theta}{2\pi_7}\,[-3\pi_4(t)\pi_1'(t) +\\
&+ -\left(\pi_2(t) - 3\pi_3(t)\right)\pi_2'(t) + 4\pi_2(t)\pi_3'(t) - 3\left(\pi_1(t)\pi_4'(t) + \pi_6'(t)\right)]\\
R_{\theta\bar\theta} &= R^{QPI}_{\theta\bar\theta}(\pi_i) + \frac{\pi_2'(t) - \pi_3'(t)}{2\pi_7} - \frac{\bar\theta\theta}{2\,\pi_7^2}\,[-\pi_2(t)\left(2\pi_4(t)\pi_1'(t) + \right.\\
&+ 4\pi_3(t)\left(\pi_3'(t) - \pi_2'(t)\right) + 2\pi_1(t)\pi_4'(t) - 3\pi_7\pi_5'(t) + 2\pi_6'(t) + \pi_3''(t)) +\\
&+ \pi_3(t)\left(2\pi_4(t)\pi_1'(t) + 2\pi_1(t)\pi_4'(t) - 3\pi_7\pi_5'(t) + 2\pi_6'(t) - \pi_2''(t)\right) +\\
&+ 2\pi_2(t)^2\pi_3'(t) - 2\pi_3(t)^2\pi_2'(t) - 2\pi_1(t)\pi_4(t)\pi_2'(t) + 2\pi_7\pi_5(t)\pi_2'(t) +\\
&- 2\pi_6(t)\pi_2'(t) + 2\pi_1(t)\pi_4(t)\pi_3'(t) - 2\pi_7\pi_5(t)\pi_3'(t) + 2\pi_6(t)\pi_3'(t) +\\
&- 2\pi_2'(t)\pi_3'(t) + 2\pi_1'(t)\pi_4'(t) + \pi_4(t)\pi_1''(t) + \pi_1(t)\pi_4''(t) + \pi_6''(t)]
\end{aligned}$$

$$
\begin{aligned}
R_{\bar\theta t} &= R^{QPI}_{\bar\theta t}(\pi_i) + \frac{\theta}{2\,\pi_7}\left[3\pi_4(t)\pi_1'(t) + \pi_3(t)\left(\pi_3'(t) - 4\pi_2'(t)\right) + 3\left(-\pi_2(t)\pi_3'(t)\right.\right. \\
&+ \left.\left. +\pi_1(t)\pi_4'(t) + \pi_6'(t)\right)\right] + \frac{\bar\theta}{2\pi_7}\pi_4(t)\left[\pi_3'(t) - \pi_2'(t)\right] \\
R_{\bar\theta\theta} &= R^{QPI}_{\bar\theta\theta}(\pi_i) + \frac{\pi_3'(t) - \pi_2'(t)}{2\pi_7} - \frac{\bar\theta\theta}{\pi_7^2}\left[\pi_2(t)\left(2\pi_4(t)\pi_1'(t) + 4\pi_3(t)\left(\pi_3'(t) - \pi_2'(t)\right)\right)\right. \\
&+ 2\pi_1(t)\pi_4'(t) - 3\pi_7\pi_5'(t) + 2\pi_6'(t) + \pi_3''(t)) - \pi_3(t)\left(2\pi_4(t)\pi_1'(t) + \right. \\
&+ 2\pi_1(t)\pi_4'(t) - 3\pi_7\pi_5'(t) + 2\pi_6'(t) - \pi_2''(t)) - 2\pi_2(t)^2\pi_3'(t)2\pi_3(t)^2\pi_2'(t) + \\
&+ 2\pi_1(t)\pi_4(t)\pi_2'(t) - 2\pi_7\pi_5(t)\pi_2'(t) + 2\pi_6(t)\pi_2'(t) - 2\pi_1(t)\pi_4(t)\pi_3'(t) + \\
&+ 2\pi_7\pi_5(t)\pi_3'(t) - 2\pi_6(t)\pi_3'(t) + 2\pi_2'(t)\pi_3'(t) - 2\pi_1'(t)\pi_4'(t) - \pi_4(t)\pi_1''(t) + \\
&- \left.\pi_1(t)\pi_4''(t) - \pi_6''(t)\right]
\end{aligned}
$$

where $R^{CPI}_{AB}(\pi_i)$ are the expression previously presented in Appendix C for the QPI Ricci tensor. The other tensor components are all equal to zero. The QPI Ricci scalar, when the parameters are explicit functions of time, turns out:

$$
\begin{aligned}
R^{QPI} &= R^{QPI}(\pi_i) + 2(\pi_2' - \pi_3') + \\
&+ \frac{\bar\theta\theta}{\pi_7}\left[-\pi_2(t)\left(5\pi_4(t)\pi_1'(t) + 3\pi_3(t)\left(\pi_3'(t) - \pi_2'(t)\right) + 5\pi_1(t)\pi_4'(t)\right.\right. \\
&- 2\pi_7\pi_5'(t) + 5\pi_6'(t) + 2\pi_3''(t)) + \pi_3(t)\left(5\pi_4(t)\pi_1'(t) + 5\pi_1(t)\pi_4'(t)\right. \\
&- 2\pi_7\pi_5'(t) + 5\pi_6'(t) - 2\pi_2''(t)) + 2\left(\pi_1(t)\left(\pi_4(t)\left(\pi_2'(t) - \pi_3'(t)\right) + \pi_4''(t)\right)\right. \\
&+ \left(3\pi_7\pi_5(t) - \pi_6(t)\right)\pi_2'(t) + \left(-2\pi_2'(t) - 3\pi_7\pi_5(t) + \pi_6(t)\right)\pi_3'(t) \\
&+ \left.2\pi_1'(t)\pi_4'(t) + \pi_4(t)\pi_1''(t) + \pi_6''(t)) + 5\pi_2(t)^2\pi_3'(t) - 5\pi_3(t)^2\pi_2'(t)\right]
\end{aligned}
\tag{G.4}
$$

where, once again, $R^{QPI}(\pi_i)$ is the QPI Ricci scalar given in Appendix C.

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