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Nonresonance conditions for radial solutions of nonlinear Neumann elliptic problems on annuli

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Abstract. *An existence result to some nonlinear Neumann elliptic problems defined on balls has been provided recently by the author in [21]. We investigate, in this paper, the possibility of extending such a result to annuli.*

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1. Introduction

In a previous paper [21], in order to obtain an existence result, the author introduced a *liminf-limsup* type of nonresonance condition *below* the first positive eigenvalue for Neumann problems defined on the ball $B_R = \{x \in \mathbb{R}^N\,,\ |x| < R\}$. As an example, the following problem

$$\begin{cases} -\Delta u = g(u) + e(|x|) & \text{in } B_R \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial B_R\,, \end{cases}$$

where the functions $g : \mathbb{R} \to \mathbb{R}$ and $e : [0, R] \to \mathbb{R}$ are continuous, has a radial solution if

$$\liminf_{u\to+\infty} \frac{2G(u)}{u^2} < \left(\frac{\pi}{2R}\right)^2 \quad \text{and} \quad \limsup_{u\to-\infty} \frac{g(u)}{u} < \left(\frac{\pi}{2R}\right)^2$$

(here G is a primitive of g), and assuming the existence of a positive d such that

$$\big(g(u) + \bar{e}\big) \operatorname{sgn} u > 0 \qquad \text{when } |u| \ge d\,,$$

where $\bar{e} = \dfrac{N}{R^N} \displaystyle\int_0^R s^{N-1} e(s)\, ds$.

In this paper, we treat Neumann problems defined on annuli and, in the spirit of the above quoted paper, we will provide in Theorem 2.2 some sufficient condition for the existence of radial solutions. Moreover, we will provide with Theorem 2.6 a different result in presence of *not well-ordered* constant upper and lower solutions.

Let us briefly introduce some notations. We denote by $\mathcal{A}^N(R_1,R_2) \subset \mathbb{R}^N$, with $R_2 > R_1 > 0$, the open annulus of internal radius R_1 and external radius R_2:

$$\mathcal{A}^N(R_1,R_2) = B_{R_2} \setminus \overline{B_{R_1}}\,,$$

where $B_r \subset \mathbb{R}^N$ is the open ball of radius r centered at the origin. As usual, we denote the boundary of $\mathcal{A}^N(R_1,R_2)$ with $\partial\mathcal{A}^N(R_1,R_2)$, and the euclidean norm with $|\cdot|$. The problem we are going to study is of the type

$$\begin{cases} -\Delta u = g(|x|,u) + e(|x|) & \text{in } \mathcal{A}^N(R_1,R_2) \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\mathcal{A}^N(R_1,R_2)\,, \end{cases}$$

where $g : [R_1,R_2] \times \mathbb{R} \to \mathbb{R}$ and $e : [R_1,R_2] \to \mathbb{R}$ are continuous functions.

The spirit of this paper follows the idea presented by Fonda, Gossez and Zanolin in [9]. In that paper, the authors deal with a Dirichlet problem defined in a smooth domain $\Omega \subset \mathbb{R}^N$ contained in a ball B_ρ of a certain radius ρ:

$$\begin{cases} -\Delta u = g(u) + h(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega\,. \end{cases}$$

They replace the classical *limsup* nonresonance condition with respect to the first eigenvalue λ_1 provided by Hammerstein in [15],

$$\limsup_{|u|\to\infty} \frac{2G(u)}{u^2} < \lambda_1\,,$$

with a *double liminf* condition like the following one

$$\liminf_{u\to-\infty} \frac{2G(u)}{u^2} < \frac{\pi^2}{4\rho^2}, \quad \text{and} \quad \liminf_{u\to+\infty} \frac{2G(u)}{u^2} < \frac{\pi^2}{4\rho^2} \tag{1}$$

(here, again, G is a primitive of g). Notice that, one has $\pi^2/4\rho^2 < \lambda_1$, except to the case $\Omega = (-2\rho, 2\rho) \subset \mathbb{R}$ where the equality holds. Condition (1) has been first introduced in the frame of the one-dimensional Dirichlet problem in $(-2\rho, 2\rho)$ in [7].

In the case of a Neumann problem, a condition of *liminf* type was studied by Gossez and Omari in [12, 13]. In Neumann problems, a nonresonance condition with respect to the *zero* eigenvalue must be introduced, so that the *liminf* condition will be related to the *first positive* eigenvalue (see also [3, 16] for related problems). Such a situation occurs also when dealing with periodic problems (see for example [8]).

The paper is organized as follows: in Section 2 we will state all the results, the proofs of them are postponed to Section 3.

2. Main results

In this paper we are concerned with the following class of problems defined on an annulus $\mathcal{A}^N(R_1, R_2)$ with $R_2 > R_1 > 0$:

$$\begin{cases} -\Delta u = g(|x|, u) + e(|x|) & \text{in } \mathcal{A}^N(R_1, R_2) \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\mathcal{A}^N(R_1, R_2)\,, \end{cases} \tag{2}$$

where $g : [R_1, R_2] \times \mathbb{R} \to \mathbb{R}$ and $e : [R_1, R_2] \to \mathbb{R}$ are continuous functions. In particular, consider a radial solution $u(x) = v(|x|)$ to (2). Setting $r = |x|$, and denoting with $'$ the derivative with respect to r, we have the equivalent system

$$\begin{cases} -v'' - \dfrac{N-1}{r}\, v' = g(r, v) + e(r) & r \in [R_1, R_2] \\ v'(R_1) = 0 = v'(R_2)\,. \end{cases} \tag{3}$$

Notice that the differential equation in (3) does not present a singularity, being $R_1 > 0$. The case $R_1 = 0$ has been treated by the author in [21]. It will be useful to consider the mean value of the function e

$$\bar{e} = \frac{N}{R_2^N - R_1^N} \int_{R_1}^{R_2} s^{N-1} e(s)\, ds$$

and $\tilde{e}(t) = e(t) - \bar{e}$, so that $\int_{R_1}^{R_2} s^{N-1} \tilde{e}(s)\, ds = 0$.

In the proof of the theorem, we will use the so-called *time-map* function. Let us spend a few words about it. Consider the scalar second order differential equation $x'' + \psi(x) = 0$. It is possible to write the associated system in the plane

$$x' = y\,, \qquad -y' = \psi(x)\,. \tag{4}$$

Suppose $\psi(x)x > 0$ for every $x \neq 0$, and consider the primitive $\Psi(x) = \int_0^x \psi(\xi)\, d\xi$. The function

$$\tau_\psi(x) = \operatorname{sgn}(x)\sqrt{2} \int_0^x \frac{d\xi}{\sqrt{\Psi(x) - \Psi(\xi)}}\,,$$

is defined as the time-map associated to the planar system (4), and gives an estimate of the time between two subsequent zeroes t_1 and t_2 of the function

$x = x(t)$. In particular, if the function x reaches its maximum $x(t_0) = x_M$ at $t_0 \in (t_1, t_2)$, then $t_2 - t_1 = \tau_\psi(x_M)$ and $t_2 - t_0 = t_0 - t_1 = \tau_\psi(x_M)/2$. See, e.g., [6, 11, 18, 19] for details and their applications to periodic scalar problems.

In view of this, let us define the *half-valued* time-map

$$\mathcal{T}_\psi(x) = \mathrm{sgn}(x) \frac{1}{\sqrt{2}} \int_0^x \frac{d\xi}{\sqrt{\Psi(x) - \Psi(\xi)}} \,,$$

and the following limits

$$\mathcal{T}_\psi^\pm = \limsup_{x \to \pm\infty} \mathcal{T}_\psi(x) \,, \qquad \mathcal{T}_\pm^\psi = \liminf_{x \to \pm\infty} \mathcal{T}_\psi(x) \,.$$

In [11], Fonda and Zanolin provided some estimates on these values, some of which we collect in the following proposition.

PROPOSITION 2.1 ([11]). *Assume that ψ is a continuous function, with primitive Ψ, and ℓ_+, ℓ_- are positive constants. If ψ satisfies at $+\infty$ or $-\infty$ some of the following limits on the left, then the correspondent estimate on the right holds.*

$$\liminf_{x \to \pm\infty} \frac{2\Psi(x)}{x^2} \le \ell_\pm \quad \Rightarrow \quad \mathcal{T}_\psi^\pm \ge \frac{\pi}{2\sqrt{\ell_\pm}},$$

$$\limsup_{x \to \pm\infty} \frac{\psi(x)}{x} \le \ell_\pm \quad \Rightarrow \quad \mathcal{T}_\pm^\psi \ge \frac{\pi}{2\sqrt{\ell_\pm}},$$

$$\exists \lim_{x \to \pm\infty} \frac{2\Psi(x)}{x^2} \le \ell_\pm \quad \Rightarrow \quad \mathcal{T}_\pm^\psi \ge \frac{\pi}{2\sqrt{\ell_\pm}}.$$

We can now state our main result.

THEOREM 2.2. *Assume the existence of a continuous function $\phi : \mathbb{R} \to \mathbb{R}$, and of a constant $d > 0$ such that*

$$-\bar{e} < g(r, v) \le \phi(v) \qquad \text{for every } r \in [R_1, R_2] \text{ and every } v \ge d \,, \tag{5}$$

$$\phi(v) \le g(r, v) < -\bar{e} \qquad \text{for every } r \in [R_1, R_2] \text{ and every } v \le -d \,. \tag{6}$$

Moreover assume the existence of a constant $\eta > 0$ such that

$$\phi(v)v \ge \eta\, v^2 \qquad \text{for every } |v| \ge d \,. \tag{7}$$

Suppose that the function $\mathcal{T}_\phi$ is well-defined for $|v| > d$ and its limits satisfy either

$$\mathcal{T}_\phi^+ > R_2 - R_1 \quad \text{and} \quad \mathcal{T}_-^\phi > R_2 - R_1 \,, \tag{8}$$

or

$$\mathcal{T}_+^\phi > R_2 - R_1 \quad \text{and} \quad \mathcal{T}_\phi^- > R_2 - R_1 \,. \tag{9}$$

Then (2) has at least one radial solution.

Remark 2.3. Assuming (5) and (6) we implicitly have that the function $\mathcal{T}_\phi = \mathcal{T}_\phi(v)$ is well-defined for v large enough. The assumptions of the theorem require only that the value d is chosen large enough to guarantee that the domain of $\mathcal{T}_\phi$ contains the set $(-\infty, -d) \cup (d, +\infty)$.

We will prove this theorem in Section 3.1. We will give now, as an example of application, some possible corollaries to Theorem 2.2 using the estimates in Proposition 2.1. In order to simplify the statement, in the setting of problem (2), we assume g not depending by x.

Corollary 2.4. *Let be $g(|x|, u) = g(u)$ with primitive G. Assume*

$$\liminf_{u\to+\infty} \frac{2G(u)}{u^2} < \left(\frac{\pi}{2(R_2 - R_1)}\right)^2 \quad \textit{and} \quad \limsup_{u\to-\infty} \frac{g(u)}{u} < \left(\frac{\pi}{2(R_2 - R_1)}\right)^2$$

and that there exists $d > 0$ such that

$$(g(u) + \bar{e}) \operatorname{sgn} u > 0 \quad \textit{when } |u| > d\,.$$

Then, problem (2) has at least one radial solution.

Corollary 2.5. *Let be $g(|x|, u) = g(u)$ with primitive G. Assume*

$$\liminf_{u\to+\infty} \frac{2G(u)}{u^2} < \left(\frac{\pi}{2(R_2 - R_1)}\right)^2 \quad \textit{and} \quad \exists \lim_{u\to-\infty} \frac{2G(u)}{u^2} < \left(\frac{\pi}{2(R_2 - R_1)}\right)^2$$

and that there exists $d > 0$ such that

$$(g(u) + \bar{e}) \operatorname{sgn} u > 0 \quad \textit{when } |u| > d\,.$$

Then, problem (2) has at least one radial solution.

The proof is obtained by defining, for $\eta > 0$ sufficiently small, the function

$$\phi(v) = \begin{cases} \max\{g(v)\,,\, \eta\, v\} & \text{if } v \geq d \\ \min\{g(v)\,,\, \eta\, v\} & \text{if } v \leq -d\,, \end{cases} \tag{10}$$

enlarging d if necessary, and extending its domain to the whole $\mathbb{R}$.

Another existence result can be obtained assuming the existence of constant lower and upper solutions which are not *well-ordered*, as the following theorem states. For further results on non-well-ordered lower and upper solutions, see, e.g., [1, 5, 10, 14, 17].

Theorem 2.6. *Assume the existence of a continuous function $\phi : \mathbb{R} \to \mathbb{R}$ and of some positive constants d, χ, η such that*

$$-\chi < g(r, v) \leq \phi(v) \qquad \textit{for every } r \in [R_1, R_2] \textit{ and every } v \geq d\,,$$

$$\phi(v) \le g(r,v) < \chi \qquad \textit{for every } r \in [R_1,R_2] \textit{ and every } v \le -d\,,$$
$$\phi(v)v \ge \eta v^2 \qquad \textit{for every } |v| \ge d\,.$$

Moreover, assume that there exist some constants $\beta < \alpha$ such that

$$g(r,\beta) + e(r) < 0 < g(r,\alpha) + e(r)\,, \qquad \textit{for every } r \in [R_1,R_2]\,. \tag{11}$$

Suppose that the function $\mathcal{T}_\phi$ is well-defined for $|v| > d$ and its limits satisfy either

$$\mathcal{T}_\phi^+ > R_2 - R_1 \quad \textit{and} \quad \mathcal{T}_-^\phi > R_2 - R_1\,,$$

or

$$\mathcal{T}_+^\phi > R_2 - R_1 \quad \textit{and} \quad \mathcal{T}_\phi^- > R_2 - R_1\,.$$

Then (2) *has at least one radial solution.*

Such a statement has been inspired by a result obtained by Gossez and Omari in [12], and the following results follow as a direct consequence of the previous theorem. We will refer also to [13] for comparison.

Theorem 2.7. *Assume g to be a continuous function, with primitive G, such that*

$$\liminf_{u\to+\infty} \frac{2G(u)}{u^2} < \left(\frac{\pi}{2(R_2-R_1)}\right)^2 \quad \textit{and} \quad \limsup_{u\to-\infty} \frac{g(u)}{u} < \left(\frac{\pi}{2(R_2-R_1)}\right)^2.$$

Then,

$$\begin{cases} -\Delta u = g(u) + e(|x|) & \textit{in } \mathcal{A}^N(R_1,R_2) \\ \dfrac{\partial u}{\partial \nu} = 0 & \textit{on } \partial\mathcal{A}^N(R_1,R_2)\,, \end{cases}$$

has a solution for every continuous function e if and only if $g(\mathbb{R}) = \mathbb{R}$.

Theorem 2.8. *Assume g to be a continuous function, with primitive G, such that*

$$\liminf_{u\to+\infty} \frac{2G(u)}{u^2} < \left(\frac{\pi}{2(R_2-R_1)}\right)^2 \quad \textit{and} \quad \exists \lim_{u\to-\infty} \frac{2G(u)}{u^2} < \left(\frac{\pi}{2(R_2-R_1)}\right)^2.$$

Then,

$$\begin{cases} -\Delta u = g(u) + e(|x|) & \textit{in } \mathcal{A}^N(R_1,R_2) \\ \dfrac{\partial u}{\partial \nu} = 0 & \textit{on } \partial\mathcal{A}^N(R_1,R_2)\,, \end{cases}$$

has a solution for every continuous function e if and only if $g(\mathbb{R}) = \mathbb{R}$.

For comparison, let us quote here a possible application of [13, Theorem 1.1]. The quoted theorem is a more general application to asymmetric nonlinearities.

THEOREM 2.9 ([13]). *Let λ_2 be the second eigenvalue of $-\Delta$ on $\mathcal{A}^N(R_1, R_2)$ with Neumann boundary condition. Assume g to be a continuous function, with primitive G, such that*

$$\limsup_{u\to+\infty} \frac{g(u)}{u} \le \lambda_2 \quad \text{and} \quad \limsup_{u\to-\infty} \frac{g(u)}{u} \le \lambda_2\,,$$

with moreover $\liminf_{u\to+\infty} \frac{2G(u)}{u^2} < \lambda_2$. *Then,*

$$\begin{cases} -\Delta u = g(u) + e(x) & \text{in } \mathcal{A}^N(R_1, R_2) \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\mathcal{A}^N(R_1, R_2)\,, \end{cases}$$

has a solution for every continuous function e if and only if $g(\mathbb{R}) = \mathbb{R}$.

Notice that Theorem 2.7 does not require any *limsup* type of condition at $+\infty$, but it requires that all the limits are *below* the second eigenvalue.

REMARK 2.10. It is known that the Fučik spectrum for the radial elliptic Neumann problem on an annulus presents two monotone curves departing from the point (λ_2, λ_2), where λ_2 is the first positive eigenvalue (cf. [2, 20]). Such curves have two different asymptotes, call $a > 0$ the smaller one. A natural question arises about the order of a and of the constants $k = \frac{\pi^2}{4(R_2-R_1)^2}$ involved in the previous theorems. The value a is strictly related to the zeroes of Bessel functions of index ν and $\nu + 1 = N/2$ and also, in particular, to the choice of R_1 and R_2. It is possible to find suitable values for them thus obtaining both the cases $a < k$ and $a > k$, so that the *liminf* condition in Corollary 2.4 and Theorem 2.7 is sometimes not necessary, but it is hard to verify if this situation occurs when R_1 and R_2 are arbitrarily fixed.

REMARK 2.11. Similar results can be obtained by assuming the existence of non-constant lower and upper solutions which are not well-ordered, following the main ideas of the paper by Alif and Omari [1]. We do not enter in such details for briefness.

REMARK 2.12. Several other theorems can be formulated using the estimates in Proposition 2.1 and the other ones contained in [11]. As a trivial example, the asymptotic behaviour of the nonlinearities at $+\infty$ and $-\infty$ can be switched in all the previous theorems. For briefness we do not enter in such details.

3. Proofs

3.1. Proof of Theorem 2.2

We will prove the theorem under assumption (8). The proof of the other case is specular.

Define the function $T : (-\infty, -d] \cup [d, +\infty) \to \mathbb{R}$ as

$$T(v) = \frac{1}{\sqrt{2}} \int_d^v \frac{d\xi}{\sqrt{\Phi(v) - \Phi(\xi) + \|e\|_\infty (v - \xi)}}\,, \qquad \text{for } v \geq d\,,$$

$$T(v) = \frac{1}{\sqrt{2}} \int_v^{-d} \frac{d\xi}{\sqrt{\Phi(v) - \Phi(\xi) - \|e\|_\infty (v - \xi)}}\,, \qquad \text{for } v \leq -d\,.$$

The following proposition was proved in [21, Lemma 3.1].

PROPOSITION 3.1. *For every $\epsilon > 0$ there exists $v_\epsilon > d$ such that the following inequalities hold*

$$T(v) \leq \mathcal{T}_\phi(v) \leq (1+\epsilon)\, T(v) + \epsilon\,,$$

for every v with $|v| > v_\epsilon$.

By (8), it is possible to find a sufficiently small $\epsilon > 0$ such that there exist an increasing sequence of positive real values $(\omega_n)_n$, with $\lim_n \omega_n = +\infty$, and $\bar\omega > 0$ with the following property:

$$\mathcal{T}_\phi(\omega_n) > (R_2 - R_1)(1+\epsilon) + \epsilon \qquad \text{for every } n \in \mathbb{N}\,,$$

$$\mathcal{T}_\phi(v) > (R_2 - R_1)(1+\epsilon) + \epsilon \qquad \text{for every } v < -\bar\omega\,.$$

We can assume $\bar\omega$ and ω_0 to be greater than $d+1$ and v_ϵ, where v_ϵ is given by Proposition 3.1, thus permitting to have the following estimates

$$T(\omega_n) \geq \frac{\mathcal{T}_\phi(\omega_n) - \epsilon}{1+\epsilon} > R_2 - R_1 \qquad \text{for every } n \in \mathbb{N}\,, \tag{12}$$

$$T(v) \geq \frac{\mathcal{T}_\phi(v) - \epsilon}{1+\epsilon} > R_2 - R_1 \qquad \text{for every } v < -\bar\omega\,. \tag{13}$$

We introduce the following family of problems, for $\lambda \in [0,1]$,

$$\begin{cases} -v'' - \dfrac{N-1}{r}\, v' = \lambda\big(g(r,v) + e(r)\big) + (1-\lambda)\eta v\,, \quad r \in [R_1, R_2]\,, \\ v'(R_1) = 0 = v'(R_2)\,, \end{cases} \tag{14}$$

where η was introduced in (7). We define the following sets

$$C_N^k = \{v \in C^k([R_1, R_2]) \,:\, v'(R_1) = 0 = v'(R_2)\}\,, \quad k = 1, 2\,.$$

It is not restrictive to assume that the constant η introduced in (7) is smaller than the first positive eigenvalue, so to have the existence of a unique solution of the problem

$$\begin{cases} -v'' - \dfrac{N-1}{r}\, v' = \eta v + f(r)\,, \quad r \in [R_1, R_2]\,, \\ v'(R_1) = 0 = v'(R_2)\,, \end{cases}$$

for every continuous function f. Call $T : C^0 \to C^1_N$ the operator that sends the continuous function f to the unique solution v. Then (14) is equivalent to the fixed point problem

$$v = \mathcal{G}_\lambda(v) := \lambda\, T\big(- \eta v + g(r,v) + e(r)\big)\,.$$

where $\mathcal{G}_\lambda : C^1_N \to C^1_N$ is a completely continuous operator. Moreover, any fixed point of $\mathcal{G}_\lambda$ is a function belonging to C^2_N and $d_{LS}(I - \mathcal{G}_0, \Omega, 0) = 1$ for every open bounded set $\Omega \subset C^1_N$ such that $0 \in \Omega$. Hence, by Leray-Schauder degree theory, it will be sufficient to find an open bounded set $\Omega \subset C^1_N$, containing 0, such that there are no solutions of (14) on $\partial\Omega$, for every $\lambda \in [0,1]$, in order to prove the existence of a solution of (3).

We are going to prove the existence of such a set, looking for some positive constants A, B, M defining Ω as follows:

$$\Omega = \{v \in C^1_N : -A < v(r) < B \text{ and } |v'(r)| < M\,,\ \forall r \in [R_1, R_2]\}\,. \tag{15}$$

First of all, we show now that all the solutions of (14) cannot *remain large* because of assumptions (5) and (6). We will prove the following claim.

Claim. Every solution v of (14) satisfies $|v(r)| < d$ for some $r \in [R_1, R_2]$.

Consider a solution v of (14) such that $v(r) > d$ for every $r \in [R_1, R_2]$. It satisfies also the following differential equation for every $r \in [R_1, R_2]$:

$$\frac{d}{dr}\big(r^{N-1}v'(r)\big) = -r^{N-1}\left[\lambda\big(g(r,v(r)) + e(r)\big) + (1-\lambda)\eta v(r)\right]\,. \tag{16}$$

Integrating it in the interval $[R_1, R_2]$, we get

$$0 = -\int_{R_1}^{R_2} r^{N-1}\left[\lambda\big(g(r,v(r)) + \bar{e}\big) + (1-\lambda)\eta v(r)\right]\, dr\,.$$

Notice that, by (5), the integral must be negative, providing a contradiction. A similar computation proves also the impossibility of having a solution v of (14) such that $v(r) < -d$ for every $r \in [R_1, R_2]$. We have so proved the claim.

The proof of Theorem 2.2 consists of three steps: in each one we provide one of the needed constants A, B, M which appear in (15).

▶ *Step 1 (Find the constant B).* The positive constant B can be chosen in the set of the values of the previously introduced sequence $(\omega_n)_n$, taking n sufficiently large. In fact, suppose by contradiction that there exist a sequence $(\lambda_n)_n$, with $\lambda_n \in [0,1]$ for every n, a subsequence of $(\omega_n)_n$, still denoted $(\omega_n)_n$, and a sequence of solutions v_n to (14), with $\lambda = \lambda_n$, such that $\max_{[R_1,R_2]} v_n = \omega_n$. The maximum is reached at the instant

$$r^n_M = \max\{r \in [R_1, R_2] : v_n(r) = \omega_n\}\,.$$

The Claim permits us to define also

$$r_d^n = \max\{r \in [R_1, R_2] : v_n(r) = d\}\,.$$

We consider two different situations, up to subsequences.

$\diamond$ *Case 1:* $r_M^n < r_d^n$.
In this situation, the solution reaches its maximum at r_M^n and then *becomes small* reaching the value d at

$$\tilde r_n = \min\{r \in [r_M^n, R_2] : v_n(r) = d\}\,.$$

For every $r \in [r_M^n, \tilde r_n]$ such that $v_n'(r) < 0$, it is possible to find a value $s(r) \in [r_M^n, r)$ such that $v_n'(s(r)) = 0$ and $v_n'(r) < 0$ for every $s \in (s(r), r]$. Consider the differential equation in (14) with $v = v_n$ and $\lambda = \lambda_n$. Using (5) and (7), we have

$$-v_n''(s) \le \phi(v_n(s)) + \|e\|_\infty \qquad \text{for every } s \in [s(r), r]\,,$$

and multiplying by $v_n'(s) \le 0$ and integrating in the interval $[s(r), r]$, we obtain

$$-\frac{1}{2} v_n'(r)^2 \ge \Phi(v_n(r)) - \Phi(v_n(s(r))) + \|e\|_\infty \big(v_n(r) - v_n(s(r))\big)\,.$$

Using the monotonicity of Φ in the interval $[d, +\infty)$, we get

$$1 \ge \frac{1}{\sqrt{2}} \frac{-v_n'(r)}{\sqrt{\Phi(\omega_n) - \Phi(v_n(r)) + \|e\|_\infty (\omega_n - v_n(r))}}\,,$$

for every $r \in [r_M^n, \tilde r_n]$ such that $v_n'(r) < 0$. The previous inequality holds also when $v_n' > 0$, so we can obtain the following contradiction using (12):

$$\begin{aligned} \tilde r_n - r_M^n \ge & \frac{1}{\sqrt{2}} \int_{r_M^n}^{\tilde r_n} \frac{-v_n'(r)}{\sqrt{\Phi(\omega_n) - \Phi(v_n(r)) + \|e\|_\infty (\omega_n - v_n(r))}}\, dr \\ = & T(\omega_n) > R_2 - R_1\,. \end{aligned}$$

$\diamond$ *Case 2:* $r_M^n > r_d^n$.
We want to show that, in this situation, the solutions v_n must reach a negative minimum $m_n = \min_{[R_1,R_2]} v_n$ which is large in absolute value, in particular we will prove that

$$\lim_n m_n = -\infty\,. \tag{17}$$

We consider the last point of minimum

$$r_m^n = \max\{r \in [R_1, R_2] : v_n(r) = m_n\} < r_M^n.$$

Arguing by contradiction, we suppose that there exists a constant $C > 0$ such that, up to a subsequence, $v_n(r) \geq -C$ for every $r \in [R_1, R_2]$ and every $n \in \mathbb{N}$. Defining

$$\tilde{g}_n(r) = -r^{N-1}\left[\lambda_n\big(g(r, v_n(r)) + \bar{e}\big) + (1-\lambda_n)\eta v_n(r)\right], \tag{18}$$

we verify, using (16), that

$$\int_{R_1}^{R_2} \tilde{g}_n(r)dr = 0\,.$$

Hence, being $\tilde{g}_n$ negative when $v_n > d$,

$$\begin{aligned}\int_{R_1}^{R_2} |\tilde{g}_n(r)|dr &= \int\limits_{v_n>d} -\tilde{g}_n(r)dr + \int\limits_{-C\leq v_n\leq d} |\tilde{g}_n(r)|dr\\ &\leq 2\int\limits_{-C\leq v_n\leq d} |\tilde{g}_n(r)|dr\,,\end{aligned}$$

which is bounded. Form (16) and the previous computation, we obtain

$$\left\|\frac{d}{dr}(r^{N-1}v_n')\right\|_{L^1} \leq D\,, \tag{19}$$

for a suitable constant D, independent of n. Thus, for every $r \in (r_m^n, R_2]$,

$$r^{N-1}v_n'(r) = (r_m^n)^{N-1}v_n'(r_m^n) + \int_{r_m^n}^{r} \big(s^{N-1}v_n'(s)\big)'ds \leq D\,.$$

Hence, $v_n'(r) < D/R_1^{N-1}$ for every $r > r_m^n$, for every n. The following computation gives us a contradiction with the assumption $\omega_n \to +\infty$ giving us the proof of the limit in (17):

$$\omega_n = v_n(r_M^n) = v_n(r_m^n) + \int_{r_m^n}^{r_M^n} v_n'(s)\,ds \leq d + \frac{D}{R_1^{N-1}}(R_2 - R_1)\,.$$

Being (17) valid, we can assume $m_n < -d$ for every n. Consider

$$\hat{r}_n = \min\{r \in (r_m^n, r_d^n) : v_n(r) = -d\}\,.$$

Arguing as in Case 1, we can find, for every $r \in [r_m^n, \hat{r}_n]$ such that $v_n'(r) > 0$, a value $s(r) \in [r_m^n, r)$ such that $v_n'(s(r)) = 0$ and $v_n'(s) > 0$ for every $s \in (s(r), r]$. Considering the differential equation in (14) with $v = v_n$ and $\lambda = \lambda_n$, we can write, using (6) and (7),

$$-v_n''(s) \geq \phi(v_n(s)) - \|e\|_\infty \qquad \text{for every } s \in [s(r), r]\,.$$

Multiplying it by $v_n'(s) \geq 0$ and integrating in the interval $[s(r), r]$, using the monotonicity of Φ in $(-\infty, -d]$, we obtain, arguing as above,

$$1 \geq \frac{1}{\sqrt{2}} \frac{v_n'(r)}{\sqrt{\Phi(m_n) - \Phi(v_n(r)) - \|e\|_\infty (m_n - v_n(r))}} \,.$$

for every $r \in [r_m^n, \hat{r}_n]$, thus giving us the following contradiction when n is large enough, using (13):

$$\begin{aligned} \hat{r}_n - r_m^n \geq & \frac{1}{\sqrt{2}} \int_{r_m^n}^{\hat{r}_n} \frac{v_n'(r)}{\sqrt{\Phi(m_n) - \Phi(v_n(r)) - \|e\|_\infty (m_n - v_n(r))}} \, dr \\ = & T(m_n) > R_2 - R_1 \,. \end{aligned}$$

We have just proved that there cannot exist solutions to (14) such that $\max_{[R_1,R_2]} v_n = \omega_n$ if n is large enough. So, we can choose B among such values.

► *Step 2 (Find the constant A).* When B is fixed, it is possible to prove that there cannot exist solutions to (14), for a certain λ, having $\max_{[R_1,R_2]} v < B$ with a large (in absolute value) negative minimum.

Suppose by contradiction that, for every $m \in \mathbb{N}$, there exists a solution v_m to (14), for a certain λ, with $\max_{[R_1,R_2]} v_m < B$, such that $\min_{[R_1,R_2]} v_m < -m$. By the Claim, if $-m < -d$ then $\max_{[R_1,R_2]} v_m > -d$.

Arguing as above, we can define the function $\tilde{g}_m$ as in (18) and, being $\tilde{g}_m$ positive when $v_m < -d$, with a similar procedure, we can find a constant D' (independent of m) such that $v_m' < D'/R_1^{N-1}$ so to obtain

$$\begin{aligned} -d < \max_{[R_1,R_2]} v_m \leq & \min_{[R_1,R_2]} v_m + \frac{D'}{R_1^{N-1}} (R_2 - R_1) \\ < & -m + \frac{D'}{R_1^{N-1}} (R_2 - R_1) \,, \end{aligned}$$

which gives us a contradiction when m is large enough. Hence, we can find a positive constant A, such that every solution v to (14) satisfying $\max_{[R_1,R_2]} v < B$ must also satisfy $\min_{[R_1,R_2]} v > -A$.

► *Step 3 (Find the constant M).* Consider a solution v of (14) with $-A < v < B$, then by (16) it is easy to see that

$$r^{N-1} |v'(r)| \leq \int_{R_1}^{R_2} \left| \frac{d}{ds} (s^{N-1} v'(s)) \right| ds \leq K r^N \,,$$

for a suitable positive constant K. So, we get $|v'| \leq K R_2$ and setting, for example, $M = K R_2 + 1$ also the third step of the proof is completed.

We have just found the three constants A, B, M describing a set Ω suitable to apply the Leray-Schauder degree theory, completing the proof of Theorem 2.2.

3.2. Proof of Theorem 2.6

The proof is rather similar to the one of Theorem 2.2. Proposition 3.1 remains valid also under the assumptions of Theorem 2.6.

It is not restrictive to assume $d > \max\{-\beta, \alpha\}$. Let us consider first the case $\beta < 0 < \alpha$. As above, we can find a sequence of values $(\omega_n)_n$ and a constant $\bar{\omega}$ satisfying (12) and (13). We can introduce problem (14) and the operator $\mathcal{G}_\lambda$, but we are now going to look for a different kind of set Ω. In particular it will be of the form

$$\Omega = \{v \in C^1_N : -A < v(r) < B\,,\, |v'(r)| < M \text{ for every } r \in [R_1, R_2] \\ \text{and } \exists r_0 \in [R_1, R_2] \,:\, \beta < v(r_0) < \alpha\}\,.$$

The impossibility of having solutions v to (14) satisfying $\max_{[R_1,R_2]} v = \beta$ or $\min_{[R_1,R_2]} v = \alpha$ is given by assumption (11). The proof of this theorem follows the main procedure of the one of Theorem 2.2 and consists of three steps, too.

► *Step 1 (Find the constant B).* It is possible to find the constant B in the set of the values of the sequence $(\omega_n)_n$. If n is chosen large enough, one can prove that any solution v to (14), with $\beta < v(r_0) < \alpha$ for a certain $r_0 \in [R_1, R_2]$, must satisfy $\max_{[R_1,R_2]} v \neq \omega_n$.

► *Step 2 (Find the constant A).* It is possible to find the constant A choosing it sufficiently large in order to obtain that any solution v to (14), with $\beta < v(r_0) < \alpha$ for a certain $r_0 \in [R_1, R_2]$ and satisfying $\max_{[R_1,R_2]} v < B$, must also satisfy $\min_{[R_1,R_2]} v > -A$.

► *Step 3 (Find the constant M).* It is possible to find a constant M, sufficiently large, in order to guarantee that any solution v to (14), with $-A < v(r) < B$ for every $r \in [R_1, R_2]$, must satisfy $\max_{[R_1,R_2]} |v'| < M$.

We emphasize that, in order to adapt the proof of Theorem 2.2 to the assumptions of Theorem 2.6, we have to rewrite the part involving the estimate in (19). In this part, in fact, we have used the property $\operatorname{sgn}(v)(g(r,v) + \bar{e}) > 0$ which does not hold necessarily under the assumptions of Theorem 2.6. So, we are going to rewrite this part.

We can assume $\chi > \bar{e}$ and rename the function $\tilde{g}_n$, appearing in (18), as

$$\tilde{g}_n(r) = -r^{N-1}\left[\lambda_n\big(g(r, v_n(r)) + \chi\big) + (1-\lambda_n)\eta v_n(r)\right],$$

so that

$$\frac{d}{dr}(r^{N-1}v_n'(r))' = \tilde{g}_n(r) + \lambda_n r^{N-1}(\chi - e(r))\,.$$

Integrating the equation in the interval $[R_1, R_2]$, we obtain

$$\int_{R_1}^{R_2} \tilde{g}_n(s)ds \geq -\frac{\chi - \bar{e}}{N}(R_2^N - R_1^N)\,.$$

Let us define $H = (\chi - \bar{e})(R_2^N - R_1^N)/N$. Being $\tilde{g}_n < 0$ when $v_n > d$, we have

$$\begin{aligned}\int_{R_1}^{R_2} |\tilde{g}_n(s)|ds &= \int_{v_n > d} -\tilde{g}_n(s)ds + \int_{-C \le v_n \le d} |\tilde{g}_n(s)|ds \\ &\le H + \int_{-C \le v_n \le d} \tilde{g}_n(s)ds + \int_{-C \le v_n \le d} |\tilde{g}_n(s)|ds \\ &\le H + \quad 2 \int_{-C \le v_n \le d} |\tilde{g}_n(s)|ds\,,\end{aligned}$$

which is bounded, independently of n. Then,

$$\begin{aligned}\int_{R_1}^{R_2} \left| \frac{d}{ds}(s^{N-1} v_n'(s)) \right| ds \le \int_{R_1}^{R_2} |\tilde{g}_n(s)|ds + \frac{\chi}{N}(R_2^N - R_1^N) \\ + \int_{R_1}^{R_2} s^{N-1}|e(s)|\, ds \le D'\,,\end{aligned}$$

for a suitable constant D', independent of n. We thus obtain

$$\left\| \frac{d}{dr}(r^{N-1} v_n') \right\|_{L^1} \le D'\,,$$

for every n. Then the proof works as the one of Theorem 2.2.

The proof of Theorem 2.6, is now completed only in the case $\beta < 0 < \alpha$. Suppose now that this condition is not fulfilled. Choosing $\xi \in (\beta, \alpha)$ and defining $h(r, v) = g(r, v+\xi)$, we can verify that the assumptions of Theorem 2.6 are also satisfied for $\beta_1 = \beta - \xi < 0 < \alpha - \xi = \alpha_1$ and $\phi_1 = \phi(\,\cdot\, + \xi)$, even slightly modifying the other values. Thus, we can find a solution z of

$$\begin{cases} -\Delta z = h(|x|, z) + e(|x|) & \text{in } \mathcal{A}^N(R_1, R_2) \\ \dfrac{\partial z}{\partial \nu} = 0 & \text{on } \partial\mathcal{A}^N(R_1, R_2)\,, \end{cases}$$

so that the function $u = z + \xi$ is a solution to (2), thus completing the proof.

3.3. Proof of Theorems 2.7 and 2.8

It is easy to prove that the requirement $g(\mathbb{R}) = \mathbb{R}$ is a necessary condition. So, we will prove only that it is also sufficient. By hypothesis, for every continuous function e, it is possible to find $\alpha, \beta \in \mathbb{R}$ such that $g(\alpha) \ge \|e\|_\infty$ and $g(\beta) \le -\|e\|_\infty$, thus having the property of being respectively a lower and an upper constant solution to (2). The case $\alpha \le \beta$ follows from classical results (see, e.g., [4]), so let us assume $\beta < \alpha$. Suppose that $\liminf_{u\to+\infty} g(u) = -\infty$, then it is possible to find a constant upper solution $\beta_2 > \alpha$, thus concluding. Similarly, if $\limsup_{u\to-\infty} g(u) = +\infty$, we can find $\alpha_2 < \beta$ being a constant lower solution. The interesting case is the remaining one: there exists a positive constant $\chi > 0$ such that $g(v)\,\mathrm{sgn}(v) \geq -\chi$. Defining ϕ as in (10), the assumption of Theorem 2.6 are fulfilled using the estimates in Proposition 2.1. Applying Theorem 2.6 we complete the proof.

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References

[1] M. Alif and P. Omari, *On a p-Laplace Neumann problem with asymptotically asymmetric perturbations*, Nonlinear Anal. **51** (2002), 369–389.

[2] M. Arias and J. Campos, *Radial Fučik spectrum of the Laplace operator*, J. Math. Anal. Appl. **190** (1995), 654–666.

[3] C. Bereanu, P. Jebelean and J. Mawhin, *Radial solutions for Neumann problems involving mean curvature operators in Euclidean and Minkowski spaces*, Math. Nachr. **3** (2010), 379–391.

[4] C. De Coster and P. Habets, *Two-point boundary value problems. Lower and upper solutions*, Elsevier, Amsterdam, 2006.

[5] C. De Coster and M. Henrard, *Existence and localization of solution for elliptic problem in presence of lower and upper solutions without any order*, J. Differential Equations **145** (1998), 420–452.

[6] T. Ding, R. Iannacci and F. Zanolin, *Existence and multiplicity results for periodic solutions of semilinear Duffing equations*, J. Differential Equations **105** (1993), 364–409.

[7] L. Fernandes, P. Omari and F. Zanolin, *On the solvability of a semilinear two-point BVP around the first eigenvalue*, Differential Integral Equations **2** (1989), 63–79.

[8] L. Fernandes and F. Zanolin, *Periodic solutions of a second order differential equation with one-sided growth restriction on the restoring term.* Arch. Math. **51** (1988), 151–163.

[9] A. Fonda, J.P. Gossez and F. Zanolin, *On a nonresonance condition for a semilinear elliptic problem.* Differential Integral Equations **4** (1991), 945–951.

[10] A. Fonda and R. Toader, *Lower and upper solutions to semilinear boundary value problems: an abstract approach*, Top. Meth. Nonlin. Anal. **38** (2011), 59–93.

[11] A. Fonda and F. Zanolin, *On the use of time-maps for the solvability of nonlinear boundary value problems.* Arch. Math. **59** (1992), 245–259.

[12] J.P. Gossez and P. Omari, *A necessary and sufficient condition of nonresonance for a semilinear Neumann problem*, Proc. Amer. Math. Soc. **114** (1992), 433–442.

[13] J.P. Gossez and P. Omari, *On a semilinear elliptic Neumann problem with asymmetric nonlinearities*, Trans. Amer. Math. Soc. **347** (1995), 2553–2562.

[14] P. Habets and P. Omari, *Existence and localization of solutions of second order elliptic problems using lower and upper solutions in the reversed order*, Top. Meth. Nonlin. Anal. **8** (1996), 25–56.

[15] A. Hammerstein, *Nichtlineare Integralgleichungen nebst Anwendungen*, Acta Math. **54** (1930), 117–176.

[16] J. Mawhin, *Radial solutions of Neumann problem for periodic perturbations of the mean extrinsic curvature operator*. Milan J. Math. **79** (2011), 95–112.

[17] P. Omari, *Non-ordered lower and upper solutions and solvability of the periodic problem for the Liénard and the Rayleigh equations*, Rend. Ist. Mat. Univ. Trieste **20** (1988), 54–64.

[18] Z. Opial, *Sur les périodes des solutions de l'équation différentielle* $x''+g(x)=0$, Bull. Acad. Polon. Sci. Sér. Sci. Math., Astronom. Phys. **8** (1960), 151–156.

[19] Z. Opial, *Sur l'existence des solutions périodiques de l'équation différentielle* $x''+f(x,x')x'+g(x)=p(t)$, Ann. Polon. Math. **11** (1961), 149–159.

[20] W. Reichel and W. Walter, *Sturm-Liouville type problems for the p-Laplacian under asymptotic non-resonance conditions*, J. Differential Equations **156** (1999), 50–70.

[21] A. Sfecci, *A nonresonance condition for radial solutions of a nonlinear Neumann elliptic problem*, Nonlinear Anal. **75** (2012), 6191–6202.

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