The structure of useful topologies

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ABSTRACT

Let X be an arbitrary set. A topology t on X is said to be useful if every complete and continuous preorder on X is representable by a continuous real-valued order preserving function. It will be shown, in a first step, that there exists a natural one-to-one correspondence between continuous and complete preorders and complete separable systems on X. This result allows us to present a simple characterization of useful topologies t on X. According to such a characterization, a topology t on X is useful if and only if for every complete separable system $\mathcal{E}$ on (X, t) the topology $t_{\mathcal{E}}$ generated by $\mathcal{E}$ and by all the sets $X \setminus \overline{E}$ is second countable. Finally, we provide a simple proof of the fact that the countable weak separability condition (*cwsc*), which is closely related to the countable chain condition (*ccc*), is necessary for the usefulness of a topology.

1. Introduction

Let X be an arbitrarily but fixed chosen set. From Herden, 1991, a topology t on X is said to be *useful* if every *continuous complete (total, linear) preorder* $\precsim$ on X has a continuous utility representation, i.e. can be represented by a continuous real-valued order preserving function (utility function) (see e.g. Herden, 1989a,b, 1991). Continuity of $\precsim$ means that the *order topology* $t^{\precsim}$ induced by $\precsim$ is coarser than t (i.e., the sets $l(x) = \{z \in X | z \prec x\}$ and $r(x) = \{z \in X | x \prec z\}$ are open subsets of X for every $x \in X$).

Other authors call *continuously representable* the topologies satisfying the aforementioned property (see e.g. Candeal et al., 1998 and Campión et al., 2006, 2007, 2009, 2012). In this paper we prefer the original terminology of a useful topology, inherited from the seminal paper Herden, 1991, who first explicitly started a systematic study of this concept.

In some sense the problem of characterizing all useful topologies on X is the most fundamental problem in utility theory. Indeed, the classical theorems by Eilenberg, 1941 (**ET**) and Debreu, 1954, 1964 (**DT**)), that only recently have been proved again by Rébillé, 2018 by very elementary methods, present sufficient conditions for a topology t on X to be useful. Other sufficient conditions, which are based upon familiar topological properties, that ensure usefulness for a topology t on X are provided in Campión et al., 2012, Theorem 4.3.

In Estévez and Hervés, 1995, it was shown that in any non-separable metric space there is a continuous complete preorder that does not admits a utility function. This result, in combination with **DT**, can be used to state that a metric topology t on X is useful if and only if it is second countable. We will refer to this latter result as Estévez–Hervés' theorem (**EHT**) (see also Candeal et al., 1998, Theorem 1).

With help of the concept of a useful topology t on X, the fundamental theorems above can be restated as follows.

ET: *Every connected and separable topology t on X is useful.*
DT: *Every second countable topology t on X is useful.*
EHT: *A metrizable topology t on X is useful if and only if t is second countable.*

On the other hand, it is well known that second countability or separability, in general, is not necessary for t to being useful (cf., for instance, the Niemitzki plane that is extensively discussed in Steen and Seebach, 1970). It is important to observe that, according to Campión et al., 2006, Theorem 3.1, a very important example of a useful topology is represented by the weak topology of a Banach space.

Different characterizations of useful (or representable) topologies appear in the literature. In particular, Campión et al., 2009, Theorem 5.1, proved that a topology t on X is useful if and only if all its preorderable subtopologies are second countable, where a topology t' on X is preorderable if it is the order topology of some continuous complete preorder on (X, t'). It is remarkable that our main result (Theorem 3.1) is somewhat analogous to this characterization.

In this paper we contribute to clarify the structure of useful topologies on X by using the concept of a complete separable system. In particular, we prove that there exists a natural one-to-one correspondence between continuous and complete preorders and complete separable systems on X (cf. Proposition 3.2). This

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one-to-one correspondence can be applied, in particular, in order to simplify and somewhat generalize the characterization of useful topologies that has been presented in Herden, 1991, Herden and Pallack, 2000 and Campión et al., 2009. Indeed, we prove in Theorem 3.1 that a topology t on X is useful if and only if, for every complete separable system $\mathcal{E}$ on X, the topology $t_{\mathcal{E}}$ generated by $\mathcal{E}$ and $\{X \setminus \overline{E} | E \in \mathcal{E}\}$ is second countable. We notice that for every complete and continuous preorder $\precsim$ on (X,t) which is continuously representable, if we take $\mathcal{E} := \{l(x) = \{z \in X | z \prec x\} | x \in X\}$, then the topology $t_{\mathcal{E}}$ is precisely the order topology $t^{\precsim}$ on X.

Finally, by introducing the concept of a well-separated family of separable systems on X, we provide a necessary condition for a topology to be useful. This condition, that was introduced by Herden and Pallack, 2000, is referred to as the countable weak separability condition (*cwsc*). It is inspired by two well-known topological concepts: the countable chain condition (*ccc*) and the concept of a locally finite family of subsets of X (cf., Definitions 3.1 and 3.2, and Proposition 3.3).

2. Notation and preliminary results

A *preorder* $\precsim$ on a nonempty set X is a *reflexive* and *transitive* binary relation on X. A preorder is said to be *complete* (*linear*, *total*) if, for all $x, y \in X$, either $x \precsim y$ or $y \precsim x$. The *strict part* (or *asymmetric part*) of a preorder $\precsim$ on X is defined as follows for all $x, y \in X$: $x \prec y$ if and only if $(x \precsim y)$ *and not*$(y \precsim x)$. Further, the *symmetric part* $\sim$ of a preorder $\precsim$ on X is defined as follows for all $x, y \in X$: $x \sim y$ if and only if $(x \precsim y)$ *and* $(y \precsim x)$. We have that $\sim$ is an *equivalence* on X, and we denote by $X_{|\sim}$ the *quotient set*, made up by the equivalence classes $[x] = \{z \in X | z \sim x\}$ $(x \in X)$.

An *order* $\preceq$ on X is a preorder which in addition is *antisymmetric* (i.e., for all $x, y \in X$, $(x \preceq y)$ *and* $(y \preceq x)$ implies that $x = y$).

If t is a *topology* on X, then a family $\mathcal{B}' \subset t$ is said to be a *subbasis* of t if the family $\mathcal{B}$ consisting of all possible intersections of finitely many elements of $\mathcal{B}'$ is a *basis* of t (i.e., every set $O \in t$ is the union of some sets of $\mathcal{B}$).

A topology t on X is said to be *second countable* if there is a countable basis $\mathcal{B} = \{B_n | n \in \mathbb{N}^+\}$ for t.

Let us denote, for every subset A of X, by $\overline{A}$ its topological closure. We recall that a family $\mathcal{A} = \{A_i\}_{i \in I}$ of subsets of (X,t) is said to be *locally finite* if for every $x \in X$ there exists a *neighborhood* U_x of x which intersects finitely many elements of $\mathcal{A}$ (i.e., the set $\{i \in I | U_x \cap A_i \neq \emptyset\}$ is finite). A well known result in general topology states that if $\mathcal{A} = \{A_i\}_{i \in I}$ is a locally finite family of subsets of X, then $\overline{\bigcup_{i \in I} A_i} = \bigcup_{i \in I} \overline{A_i}$ (see e.g. Engelking, 1989, Theorem 1.1.11).

A complete preorder $\precsim$ on the topological space (X,t) is said to be *continuous* if the sets $l(x) = \{z \in X | z \prec x\}$ and $r(x) = \{z \in X | x \prec z\}$ are open subsets of X for every $x \in X$. Equivalently, this is the case when t is *finer* than the *order topology* $t^{\precsim}$ on X associated to $\precsim$, which is precisely the topology generated by the family $\{l(x) | x \in X\} \cup \{r(x) | x \in X\}$ (i.e., $\{l(x) | x \in X\} \cup \{r(x) | x \in X\}$ is a subbasis of t).

A topology t on X is said to be *useful* if every continuous complete preorder on the topological space (X,t) has a *continuous utility representation* (*order preserving function*) u, i.e., there exists a continuous real-valued function u such that $x \precsim y$ if and only if $u(x) \leq u(y)$ for all $x, y \in X$. The Debreu Open Gap lemma (see e.g. Bridges and Mehta, 1995, Lemma 3.3) guarantees that if there exists a utility representation u' for a complete preorder $\precsim$ on a set X, then there exists a utility representation u for $\precsim$ which is continuous in the order topology $t^{\precsim}$ on X. Therefore, if $\precsim$ is a continuous complete preorder on a topological space (X,t), then the existence of a utility representation actually implies the existence of a continuous one.

Definition 2.1 (Herden, 1989a). Let a topology t on X be given. A family $\mathcal{E}$ of open subsets of the topological space (X,t) such that $\bigcup_{E \in \mathcal{E}} E = X$ is said to be a *separable system* on (X,t) if it satisfies the following conditions:

S1 : There exist sets $E_1 \in \mathcal{E}$ and $E_2 \in \mathcal{E}$ such that $\overline{E_1} \subset E_2$.
S2 : For all sets $E_1 \in \mathcal{E}$ and $E_2 \in \mathcal{E}$ such that $\overline{E_1} \subset E_2$ there exists some set $E_3 \in \mathcal{E}$ such that $\overline{E_1} \subset E_3 \subset \overline{E_3} \subset E_2$.

Let us now introduce the fundamental notion of a *complete separable system* on a topological space (X,t).

Definition 2.2. Let a topology t on X be given. A separable system $\mathcal{E}$ on (X,t) is said to be *complete* if for all sets $E \in \mathcal{E}$ and $E' \in \mathcal{E}$ at least one of the following conditions holds: $E = E'$ or $\overline{E} \subset E'$ or $\overline{E'} \subset E$.

Remark 2.1. It should be noted that the concept of a complete separable system is stronger than the notion of a *linear separable system* as it was presented in Herden, 1991 and Herden and Pallack, 2000. Indeed, a linear separable system was defined to be a separable system $\mathcal{E}$ on (X,t) which is linearly ordered by set inclusion (i.e., for all sets $E \in \mathcal{E}$ and $E' \in \mathcal{E}$ either $E \subset E'$ or $E' \subset E$).

Remark 2.2. The consideration of a complete separable systems in connection with useful topologies can be motivated as follows. Let $(X,t,\precsim)$ be a completely preordered topological space and assume that there exists a continuous utility representation u for $\precsim$. Then it is easily seen that the family $\mathcal{E} = \{E_q = u^{-1}(]-\infty, q))\}_{q \in \mathbb{Q}}$ is a complete separable system on (X,t). We just observe that, for all $q \in \mathbb{Q}$, $\overline{E_q} = u^{-1}(]-\infty, q])$ and this fact clearly implies that, for all $q, r \in \mathbb{Q}$ such that $q < r$, $\overline{E_q} \subset E_r$.

The following proposition holds, which illustrates the concept of a complete separable system on X.

Proposition 2.1. *Let t be a topology on X, and let $\mathcal{E}$ be a family of open subsets of the topological space (X,t) such that $\bigcup_{E \in \mathcal{E}} E = X$. In order for $\mathcal{E}$ to be a complete separable system on (X,t) it is necessary and sufficient that $\mathcal{E}$ satisfies the following conditions.*

CS1: *$\mathcal{E}^c := \mathcal{E} \cup \{\overline{E} | E \in \mathcal{E}\}$ is linearly ordered by set inclusion.*
CS2: *$E = \bigcup_{\overline{E'} \subset E,\, E' \in \mathcal{E}} E' = \bigcup_{\overline{E'} \subset E,\, E' \in \mathcal{E}} \overline{E'}$ for every $E \in \mathcal{E}$.*
CS3: *$\overline{E} = \bigcap_{\overline{E} \subset E' \in \mathcal{E}} E' = \bigcap_{\overline{E} \subset E' \in \mathcal{E}} \overline{E'}$ for every $E \in \mathcal{E}$.*

Proof. If $\mathcal{E}$ is a complete separable system on X, then it is clear that the above condition **CS1** holds since $\mathcal{E}$ is in particular linear. In order to show, for example, that also condition **CS3** holds, first consider that, from condition **S2** in Definition 2.1, for every set $E \in \mathcal{E}$ and for every $E' \in \mathcal{E}$ such that $\overline{E} \subset E'$ there exists a set $E'' \in \mathcal{E}$ such that $\overline{E} \subset E''$ and $\overline{E} \subset \overline{E''} \subset E'$, which implies that $\bigcap_{\overline{E} \subset E' \in \mathcal{E}} E' = \bigcap_{\overline{E} \subset E' \in \mathcal{E}} \overline{E'}$. In order to show that $\bigcap_{\overline{E} \subset E' \in \mathcal{E}} E' = \bigcap_{\overline{E} \subset E' \in \mathcal{E}} \overline{E'} \subset \overline{E}$, take any $x \in X \setminus \overline{E}$. Then there exists $E'' \in \mathcal{E}$ such that $x \in E''$, and since the separable system $\mathcal{E}$ is linear, it must be $\overline{E} \subset E''$. Therefore we have that $x \notin \bigcap_{\overline{E} \subset E' \in \mathcal{E}} E'$. Condition **CS2** can be proven in a perfectly analogous way.

Conversely, consider a family $\mathcal{E}$ satisfying conditions **CS1**, **CS2** and **CS3**. Condition **CS1** clearly implies that for any two sets $E_1, E_2 \in \mathcal{E}$ such that $E_1 \neq E_2$ either $\overline{E_1} \subset E_2$ or $\overline{E_2} \subset E_1$. Now consider any two sets $E_1, E_2 \in \mathcal{E}$ such that $\overline{E_1} \subsetneq E_2$. Then there exists $x \in E_2 \setminus \overline{E_1}$, and by conditions **CS2** and **CS3** there exists $E_3 \in \mathcal{E}$ such that $x \in \overline{E_3} \subset E_2$, implying that $\overline{E_1} \subset E_3 \subset \overline{E_3} \subset E_2$. Now the proof is complete. □

3. Characterization of useful topologies

Let $\mathbb{S}_C(X)$ be the set of all complete separable systems on X that contain X. Then we proceed by considering the preorder $\precsim_{\mathbb{S}}$ on $\mathbb{S}_C(X)$ that is defined by setting, for all separable systems $\mathcal{E} \in \mathbb{S}_C(X)$ and $\mathcal{L} \in \mathbb{S}_C(X)$,

$$\mathcal{E} \precsim_{\mathbb{S}} \mathcal{L} \Leftrightarrow \forall E \in \mathcal{E} \left(\left(E = \bigcup_{L \in \mathcal{L}, \overline{L} \subset E} L = \bigcup_{L \in \mathcal{L}, \overline{L} \subset E} \overline{L} \right) \wedge \left(\overline{E} = \bigcap_{\overline{E} \subset L \in \mathcal{L}} L = \bigcap_{\overline{E} \subset L \in \mathcal{L}} \overline{L} \right) \right).$$

In order to better understand $\precsim_{\mathbb{S}}$ we denote, for every separable system $\mathcal{E} \in \mathbb{S}_C(X)$, by $t_{\mathcal{E}}$ the topology on X that is generated by $\mathcal{E} \cup \{X \setminus \overline{E} | E \in \mathcal{E}\}$ (i.e., $\mathcal{E} \cup \{X \setminus \overline{E} | E \in \mathcal{E}\}$ is a subbasis of $t_{\mathcal{E}}$). Indeed, using this notation, the following proposition holds.

Proposition 3.1. *For all separable systems $\mathcal{E} \in \mathbb{S}_C(X)$ and $\mathcal{L} \in \mathbb{S}_C(X)$ it holds that*

$$\mathcal{E} \precsim_{\mathbb{S}} \mathcal{L} \Leftrightarrow t_{\mathcal{E}} \subset t_{\mathcal{L}}.$$

Proof. Assume that, for two complete separable systems $\mathcal{E} \in \mathbb{S}_C(X)$ and $\mathcal{L} \in \mathbb{S}_C(X)$, $\mathcal{E} \precsim_{\mathbb{S}} \mathcal{L}$. In order to show that $t_{\mathcal{E}} \subset t_{\mathcal{L}}$ it suffices to show that the sets E and $X \setminus \overline{E}$ are open in the topology $t_{\mathcal{L}}$ for every set $E \in \mathcal{E}$. These facts are immediate, since in particular, we have that $E = \bigcup_{L \in \mathcal{L}, \overline{L} \subset E} L$ and $X \setminus \overline{E} = \bigcup_{\overline{E} \subset L \in \mathcal{L}} (X \setminus \overline{L})$.

Conversely, assume that for two complete separable systems $\mathcal{E} \in \mathbb{S}_C(X)$ and $\mathcal{L} \in \mathbb{S}_C(X)$, we have that $t_{\mathcal{E}} \subset t_{\mathcal{L}}$. Then any set $E \in \mathcal{E}$ must be expressed as union of sets which are open in the topology $t_{\mathcal{L}}$, and therefore it must be the case that $E = \bigcup_{L \in \mathcal{L}, \overline{L} \subset E} L$. Also, $X \setminus \overline{E}$ must be open in the topology $t_{\mathcal{L}}$, and therefore it must be the case that $\overline{E} = \bigcap_{\overline{E} \subset L \in \mathcal{L}} \overline{L}$. This actually means that $\mathcal{E} \precsim_{\mathbb{S}} \mathcal{L}$, and the proof is complete. □

Denote by $\sim_{\mathbb{S}}$ the symmetric part of the above defined preorder $\precsim_{\mathbb{S}}$ on $\mathbb{S}_C(X)$ (i.e., for all separable systems $\mathcal{E} \in \mathbb{S}_C(X)$ and $\mathcal{L} \in \mathbb{S}_C(X)$, $\mathcal{E} \sim_{\mathbb{S}} \mathcal{L}$ if and only if $(\mathcal{E} \precsim_{\mathbb{S}} \mathcal{L})$ *and* $(\mathcal{L} \precsim_{\mathbb{S}} \mathcal{E})$). From Proposition 3.1, we have that, for all separable systems $\mathcal{E} \in \mathbb{S}_C(X)$ and $\mathcal{L} \in \mathbb{S}_C(X)$, $\mathcal{E} \sim_{\mathbb{S}} \mathcal{L}$ if and only if $t_{\mathcal{E}} = t_{\mathcal{L}}$.

We are now fully prepared in order to prove the following proposition that is fundamental in the theory of complete and continuous preorders on X.

Proposition 3.2. *Let $\mathbb{P}(\trianglelefteq)$ be the set of all continuous and complete preorders on X. Then there exists a one-to-one correspondence between $\mathbb{P}(\trianglelefteq)$ and $\mathbb{S}_C(X)_{|\sim_{\mathbb{S}}}$.*

Proof. Let, in a first step, some continuous and complete preorder $\precsim \in \mathbb{P}(\trianglelefteq)$ be arbitrarily chosen. Then we set $l(x) := \{y \in X | y \prec x\}$ for every $x \in X$ in order to then define a function $\Phi : \mathbb{P}(\trianglelefteq) \rightarrow \mathbb{S}_C(X)_{|\sim_{\mathbb{S}}}$ by setting $\Phi(\precsim) := [\{l(x)\}_{x \in X}]$ for every continuous and complete preorder $\precsim \in \mathbb{P}(\trianglelefteq)$. Let, conversely, some equivalence class $[\mathcal{E}] \in \mathbb{S}_C(X)_{|\sim_{\mathbb{S}}}$ be chosen. Then we define a function $\Psi : \mathbb{S}_C(X)_{|\sim_{\mathbb{S}}} \rightarrow \mathbb{P}(\trianglelefteq)$ by choosing some representant $\mathcal{E} \in [\mathcal{E}]$ in order to then consider the complete preorder $\precsim_{[\mathcal{E}]} := \Psi([\mathcal{E}])$ that is defined, for all $x \in X$ and $y \in X$, by setting

$$x \prec_{[\mathcal{E}]} y \Leftrightarrow \exists E \in \mathcal{E}\, \exists E' \in \mathcal{E}\, ((E \subset \overline{E} \subset E') \wedge (x \in E) \wedge (y \in X \setminus E')),$$

$$x \sim_{[\mathcal{E}]} y \Leftrightarrow \neg(x \prec_{[\mathcal{E}]} y) \wedge \neg(y \prec_{[\mathcal{E}]} x).$$

The reader may compare the previous definition with assertion (iii) of Theorem 3.1 in Herden, 1989b. It follows that

$$\precsim_{[\mathcal{E}]} = \{(x, y) \in X \times X | \forall E \in \mathcal{E}\, (y \in E \Rightarrow x \in E)\}.$$

In addition, one immediately verifies that Ψ is well-defined, i.e. independent of the particular chosen representative $\mathcal{E} \in [\mathcal{E}]$.

Since the verification for $\Phi(\precsim) := \{l(x)\}_{x \in X}$ to be, for every continuous and complete preorder $\precsim \in \mathbb{P}(\trianglelefteq)$, a complete separable system on X as well as the verification $\Psi([\mathcal{E}])$ to be, for every equivalence class $[\mathcal{E}] \in \mathbb{S}_C(X)_{|\sim_{\mathbb{S}}}$, a continuous and complete preorder on X is lengthy but immediate, the proposition will follow if we are able to show that $\Psi \circ \Phi = id_{\mathbb{P}(\trianglelefteq)}$ and that, conversely, $\Phi \circ \Psi = id_{\mathbb{S}_C(X)_{|\sim_{\mathbb{S}}}}$.

Let us, therefore, start with arbitrarily choosing some continuous and complete preorder $\precsim \in \mathbb{P}(\trianglelefteq)$. Then $\Phi(\precsim) := [\{l(z)\}_{z \in X}]$ and $\Psi([\{l(z)\}_{z \in X}]) := \{(x, y) \in X \times X | \forall z \in X\, (y \prec z \Rightarrow x \prec z)\} = \{(x, y) \in X \times X | x \precsim y\} = \precsim$, which means that $\Psi \circ \Phi = id_{\mathbb{P}(\trianglelefteq)}$.

Let us, conversely, choose some equivalence class $[\mathcal{E}] \in \mathbb{S}_C(X)_{|\sim_{\mathbb{S}}}$. Then $\Psi([\mathcal{E}]) := \precsim_{[\mathcal{E}]} := \{(x, y) \in X \times X \mid \forall E \in \mathcal{E} \quad (y \in E \Rightarrow x \in E)\}$. The definition of $\precsim_{[\mathcal{E}]}$ implies that $\Phi(\precsim_{[\mathcal{E}]}) = [\{l(z)\}_{z \in X}] = [\mathcal{E}]$. This means that $\Phi \circ \Psi = id_{\mathbb{S}_C(X)_{|\sim_{\mathbb{S}}}}$ and, thus, finishes the proof of the proposition. □

Now a simple solution to the problem of characterizing all useful topologies on X can be presented.

Theorem 3.1. *Let t be a topology on a set X. The following assertions are equivalent:*

(i) t is useful.
(ii) For every separable system $\mathcal{E} \in \mathbb{S}_C(X)$, the topology $t^l_{\mathcal{E}}$ generated by $\mathcal{E}$ is second countable.
(iii) For every separable system $\mathcal{E} \in \mathbb{S}_C(X)$, the topology $t_{\mathcal{E}}$ generated by $\mathcal{E} \cup \{X \setminus \overline{E} | E \in \mathcal{E}\}$ is second countable.

Proof. (i) $\Rightarrow$ (ii). The proof of this implication is found in Herden and Pallack, 2000, Proposition 5.1, assertion (iv).

(ii) $\Rightarrow$ (iii). Consider any separable system $\mathcal{E} \in \mathbb{S}_C(X)$ on (X, t), and assume that the subtopology $t^l_{\mathcal{E}}$ of t is second countable. Let $\mathcal{B}^l = \{B_n\}_{n \in \mathbb{N}^+}$ be a countable basis for $t^l_{\mathcal{E}}$. Since $t^l_{\mathcal{E}}$ is linearly ordered by set inclusion, we can assume without loss of generality that every element B_n of the basis $\mathcal{B}^l$ belongs to $\mathcal{E}$, i.e. $\mathcal{B}^l = \{E_n\}_{n \in \mathbb{N}^+} \subset \mathcal{E}$. From condition **CS3** in Proposition 2.1 we have that, for every set $E \in \mathcal{E}$, $X \setminus \overline{E} = \bigcup_{\overline{E} \subset E' \in \mathcal{E}} (X \setminus \overline{E'}) = \bigcup_{\overline{E} \subset E_n \in \mathcal{B}^l} (X \setminus \overline{E_n})$. Therefore, we have that $\mathcal{B}^u = \{X \setminus \overline{E_n}\}_{n \in \mathbb{N}^+}$ is a basis for the linearly ordered subtopology $t^u_{\mathcal{E}}$ of t which is generated by $\{X \setminus \overline{E} | E \in \mathcal{E}\}$, and $\mathcal{B} = \mathcal{B}^l \cup \mathcal{B}^u$ is a countable subbasis of $t_{\mathcal{E}}$. Hence, $t_{\mathcal{E}}$ is second countable.

(iii) $\Rightarrow$ (i). Consider any complete and continuous preorder $\precsim$ on (X, t). In the proof of Proposition 3.2 we noticed that $\mathcal{E} = \{l(x)\}_{x \in X}$ is a complete separable system on (X, t). It is not difficult to show that the order topology $t^{\precsim}$ corresponding to $\precsim$ is contained in the topology $t_{\mathcal{E}}$ generated by $\mathcal{E}$ and $\{X \setminus \overline{E} | E \in \mathcal{E}\}$ (see Herden, 1989b, Lemma 2.1). Since this latter topology is second countable, we have that also $t^{\precsim}$ is second countable, which implies that $\precsim$ is representable by a utility function (see Herden, 1989b, Lemma 3.1, assertion (v)), and therefore it is continuously representable by the Debreu Open Gap Lemma (see Debreu, 1954, 1964). This consideration completes the proof. □

The usual proofs of **ET** and **DT** that can be found in the literature do not even touch any result that at least is somewhat related to Proposition 3.2. On the other hand, however, **ET** as well as **DT** are implicit in Theorem 3.1. Indeed, one immediately verifies that Theorem 3.1 is a common generalization of **ET** and **DT**, since every linearly ordered subtopology t' of a second countable or connected and separable topology t is itself second countable (and this is the case of the subtopology topology $t^l_{\mathcal{E}}$ of t generated by a complete separable system $\mathcal{E}$). Because of the naturalness of Proposition 3.2, this observation is surely remarkable.

EHT, however, cannot be deduced from Theorem 3.1. It is of quite different nature than any of the theorems **ET** and **DT**, respectively. Indeed, a generalization of **EHT** is based upon the already announced condition *cwsc*. In order to prepare the fundamental definition of *countable weak separability condition* (*cwsc*), let us first recall the definition of *well-separated family* of separable systems as it is found in Herden and Pallack, 2000.

Definition 3.1 (*Herden and Pallack, 2000*). Let $\Theta = \{\mathcal{E}_i\}_{i\in I}$ be a family of separable systems on X. Then Θ is said to be *well-separated* if it satisfies the following conditions.

WS1: $\bigcup_{E\in\mathcal{E}_i} E \cap \bigcup_{L\in\mathcal{E}_j} L = \emptyset$ for all $i \in I$ and all $j \in I$ such that $i \neq j$.

WS2: Let, for every $i \in I$, some fixed set $E_i \in \mathcal{E}_i$ be arbitrarily chosen. Then

$$\bigcup_{i\in I} \overline{E_i} = \overline{\bigcup_{i\in I} E_i}.$$

The reader may recall that condition **WS2** is satisfied if the family $\left\{\bigcup_{E\in\mathcal{E}_i} E\right\}_{i\in I}$ of open subsets of X is locally finite, i.e. each point $x \in X$ has a neighborhood U_x such that $U_x \cap \bigcup_{E\in\mathcal{E}_i} E \neq \emptyset$ for at most finitely many $i \in I$. This means that condition **WS2** may be considered as a slight generalization of the locally finiteness condition in topological spaces.

Definition 3.2. A topology t on a set X is said to satisfy the *countable weak separability condition* (*cwsc*) if every well-separated family $\Theta = \{\mathcal{E}_i\}_{i\in I}$ of separable systems on X is countable.

In metric spaces *cwsc* is equivalent to second countability. This is the contents of **EHT**. Following the spirit of the proof that has been presented by Estévez and Hervés, 1995, a proof of the following proposition already has been given in Herden and Pallack, 2000, Lemma 6.1. Here we want to present a somewhat modified and simpler proof that is based upon Proposition 3.2 and Theorem 3.1, respectively.

Proposition 3.3 (*Herden and Pallack, 2000, Lemma 6.1*). *In order for a topology t on X to be useful, it is necessary that t satisfies cwsc.*

Proof. Let t be a useful topology on X. Then we consider a family $\Theta = \{\mathcal{E}_i\}_{i\in I}$ of separable systems on X that satisfies the conditions **WS1** and **WS2** in Definition 3.1 in order to assume, in contrast, I to be uncountable. In this case the Well-ordering Theorem of Zermelo allows us to assume without loss of generality that $I := [0, \Omega)$, where Ω is the first uncountable ordinal number. Let now some ordinal number $\alpha < \Omega$ be fixed given. Since $\mathcal{E}_\alpha$ is a separable system on X we may choose for every ordinal number $\alpha' \leq \alpha$ and every rational number $p \in \mathbb{Q}$ some set $E^\alpha_{\alpha' p} \in \mathcal{E}_\alpha$ in such a way that, for all pairs of ordinal numbers (α', α'') with $\alpha' \leq \alpha$ and $\alpha'' \leq \alpha$, and for all pairs (p, q) of rational numbers such that $(\alpha', p) \neq (\alpha'', q)$,

$$\overline{E^\alpha_{\alpha' p}} \subset E^\alpha_{\alpha'' q} \Leftrightarrow (\alpha', p) <_{lex} (\alpha'', q) \Leftrightarrow (\alpha' < \alpha'') \textit{ or } ((\alpha' = \alpha'') \textit{ and } (p < q)).$$

We proceed by setting $E^\alpha_p := \bigcup_{\alpha'\leq\alpha} \bigcup_{\alpha''\leq\alpha'} E^{\alpha'}_{\alpha'' p} = \bigcup_{\alpha'\leq\alpha} \mathbf{E}^{\alpha'}_{\alpha' p}$ for every ordinal number $\alpha < \Omega$ and every $p \in \mathbb{Q}$. The construction of the (open) sets E^α_p ($\alpha < \Omega$, $p \in \mathbb{Q}$) implies, with help of the conditions **WS1** and **WS2** of Definition 3.1, that the collection $\mathcal{E} = \{E^\alpha_p\}_{\alpha<\Omega,\, p\in\mathbb{Q}}$ is a complete separable system on X. Since it is clear that $\mathcal{E} = \{E^\alpha_p\}_{\alpha<\Omega,\, p\in\mathbb{Q}}$ is a separable system on X, we limit ourselves to show that completeness is verified (see Definition 2.2), and to this aim we notice that, whenever $(\alpha, p) <_{lex} (\beta, q)$,

$$\overline{E^\alpha_p} = \bigcup_{\alpha'\leq\alpha} \overline{\mathbf{E}^{\alpha'}_{\alpha' p}} = \bigcup_{\alpha'\leq\alpha} \bigcup_{\alpha''\leq\alpha'} \overline{E^{\alpha'}_{\alpha'' p}} \subset \bigcup_{\beta'\leq\beta} \bigcup_{\beta''\leq\beta'} E^{\beta'}_{\beta'' q} = \bigcup_{\beta'\leq\beta} \mathbf{E}^{\beta'}_{\beta' q} = E^\beta_q.$$

In addition, condition **WS1** of Definition 3.1 implies that, for any two sets E^α_p and E^β_q, the proper inclusion $E^\alpha_p \subsetneq E^\beta_q$ holds whenever $\alpha < \beta$ and for all pairs (p, q) of rational numbers. Hence, the not countability of Ω implies that the topology $t^l_{\mathcal{E}}$ generated by the family $\mathcal{E} = \{E^\alpha_p\}_{\alpha<\Omega,\, p\in\mathbb{Q}}$ cannot be second countable. Assertion (ii) of Theorem 3.1, thus, allows us to conclude that t cannot be useful. This contradiction finishes the proof of the proposition. □

4. Conclusions

In this paper, based on the concept of a *complete separable system* on a topological space, we have presented a simple characterization of a useful topology t on a set X (i.e., a topology t such that every continuous complete preorder on the topological space (X, t) admits a continuous utility representation). Indeed, we have shown that a topology t on X is useful if and only if, for every complete separable system $\mathcal{E}$ on (X, t), the linearly ordered subtopology $t^l_{\mathcal{E}}$ which is generated by the family $\mathcal{E}$ is second countable. This is peculiar of the consideration of a complete separable system as defined in the present work. While there are many other characterizations of useful (or representable) topologies in the literature (see the various papers appearing in our section of references), it seems to us that our approach is enough general in order to be considered as interesting and widely applicable.

We are confident that our considerations may be used in order to characterize useful topologies under the *Souslin Hypothesis*. If this is possible, then the corresponding material will be presented in a future paper.

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