Conditional stability for backward parabolic operators with Osgood continuous coefficients

Daniele Casagrande[1], Daniele Del Santo[2], and Martino Prizzi[2]

[1]Dipartimento Politecnico di Ingegneria e Architettura, Università degli Studi di Udine, Via delle Scienze, 206 - 33100 Udine, Italy,
daniele.casagrande@uniud.it
[2]Dipartimento di Matematica e Geoscienze, Università degli Studi di Trieste, Via A. Valerio, 10 - 34100 Trieste, Italy,
delsanto@units.it,mprizzi@units.it

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Abstract

We prove continuous dependence on initial data for a backward parabolic operator whose leading coefficients are Osgood continuous in time. This result fills the gap between uniqueness and continuity results obtained so far.

1 Introduction

Backward parabolic equations are known to generate ill-posed Cauchy problems (in the sense of Hadamard [6, 7]). Due to the smoothing effects of the parabolic operator, in fact, it is not possible, in general, to guarantee existence of the solution for initial data which are not suitably regular. In addition, even when solutions possibly exist, uniqueness cannot be ensured without additional assumptions on the operator. Nevertheless, even for problems which are not well-posed the study of the conditional stability of the solution – the surrogate of the notion of "continuous dependence" when existence of a solution is not guaranteed – is of some interest. Such kind of study is related to the notion of *well-behaved problem* introduced by John [10]: a problem is *well-behaved* if "only a fixed percentage of the significant digits need be lost in determining the solution from the data". More precisely, a problem is well behaved if its solutions in a space $\mathscr{H}$ depend Hölder continuously on the data belonging to a space $\mathscr{K}$, provided they satisfy a prescribed bound in a space $\mathscr{H}'$ (possibly different from $\mathscr{H}$). If the dependence of solutions on data is only continuous, one says that the problem is *conditionally stable*. It is an important task to give a quantification of the continuous dependence on data, because it measures the illness of the problem from the computational point of view.

In this paper we give a contribution to the study of the conditional stability of the Cauchy problem associated with a backward parabolic operator. In particular, we consider the operator $\mathscr{L}$ defined, on the strip $[0,T] \times \mathbb{R}^n$, by

$$\mathscr{L}u = \partial_t u + \sum_{i,j=1}^{n} \partial_{x_i} \left(a_{i,j}(t) \partial_{x_j} u \right) + \sum_{j=1}^{n} b_j(t) \partial_{x_j} u + c(t) u \,, \tag{1}$$

where all the coefficients are bounded. We suppose that $a_{i,j}(t) = a_{j,i}(t)$ for all $i,j = 1,\dots,n$ and for all $t \in [0,T]$. We also suppose that $\mathscr{L}$ is backward parabolic, i.e. there exists $k_A \in]0,1[$ such that, for all $(t,\xi) \in [0,T] \times \mathbb{R}^n$,

$$k_A |\xi|^2 \le \sum_{i,j=1}^{n} a_{i,j}(t) \xi_i \xi_j \le k_A^{-1} |\xi|^2 \,. \tag{2}$$

We shall show that if the coefficients of the principal part of $\mathscr{L}$ are at least Osgood continuous (i.e. their modulus of continuity ω satisfies the condition $\int_0^1 1/\omega(s)\, ds = +\infty$), then there exists a function space in which the associated Cauchy problem

$$\begin{cases} \mathscr{L}u = f \,, & \text{in } (0,T) \times \mathbb{R}^n \,, \\ u|_{t=0} = u_0 \,, & \text{in } \mathbb{R}^n \,, \end{cases} \tag{3}$$

is conditionally stable.

To collocate the new result in the framework of the existing literature, the contents of some publications on the subject are preliminarily recalled. They show that, as one could expect, the strongness of the stability property is related to the degree of regularity of the coefficients of $\mathscr{L}$. Weaker requirements on the regularity of the coefficients give rise to weaker stability estimates, and possibly require stronger *a priori* bounds of the solution.

The overview on available works leads the reader to the new result, concerning operators with Osgood-continuous coefficients. This kind of regularity is critical since it is the minimum required regularity to have uniqueness of the solution and can therefore be considered as a sort of lower limit. Although the proof of the claim is based on the theoretical scheme followed to achieve previous results [4], the modifications needed to obtain an analogous proof in the case of Osgood coefficients are not trivial.

The paper is organised as follows. In Section 2 we give an overview on uniqueness and non-uniqueness results for (3). Moreover, we introduce the notion of modulus of continuity and define the Osgood condition. In Section 3 we recall the notion of conditional stability and we review some known results. In Section 4 we state and prove the main result of the paper (Theorem 4.2). In Section 5 we prove the main results. In Section 6 we consider the particular case of Log-Log-Lipschitz coefficients, where the dependence on initial data can be explicitly determined.

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2 Uniqueness and non-uniqueness results

This section recalls some results on the uniqueness and non-uniqueness of the solution of the problem (3) for an operator like (1) with coefficients depending also on x, namely

$$\mathscr{L}u = \partial_t u + \sum_{i,j=1}^{n} \partial_{x_i}\left(a_{i,j}(t,x)\partial_{x_j}u\right) + \sum_{j=1}^{n} b_j(t,x)\partial_{x_j}u + c(t,x)u\,. \tag{4}$$

Consider the space

$$\mathscr{H}_0 \triangleq C([0,T],L^2)\cap C([0,T[,H^1)\cap C^1([0,T[,L^2)\,. \tag{5}$$

One of the first results concerning uniqueness is due to Lions and Malgrange [11] who consider an equation associated to a sesquilinear operator defined in a Hilbert space. In our context, this result can be read as follows.

Theorem 2.1 *If the coefficients of the principal part of $\mathscr{L}$ are Lipschitz continuous with respect to t and x, $u \in \mathscr{H}_0$ and $u_0 = 0$, then $\mathscr{L}u = 0$ implies $u \equiv 0$.* □

The Lipschitz continuity of the coefficients is a crucial requirement for the claim, as shown some years later by Pliś [12] who proved the following theorem.

Theorem 2.2 *There exist u, b_1, b_2 and $c \in C^\infty(\mathbb{R}^3)$, bounded with bounded derivatives and periodic in the space variables and there exist $l : [0,T] \to \mathbb{R}$, Hölder-continuous of order δ for all $\delta < 1$ but not Lipschitz-continuous, such that $1/2 \le l(t) \le 3/2$ for all t, the support of u is the set $\{t \ge 0\} \times \mathbb{R}^2$, and*

$$\begin{aligned}\partial_t^2 u(t,x_1,x_2) + \partial_{x_1}^2 u(t,x_1,x_2) + l(t)\partial_{x_2}^2 u(t,x_1,x_2) + \\ + b_1(t,x_1,x_2)\partial_{x_1}u(t,x_1,x_2) + b_2(t,x_1,x_2)\partial_{x_2}u(t,x_1,x_2) + \\ + c(t,x_1,x_2)u(t,x_1,x_2) = 0 \qquad \text{in } \mathbb{R}^3\,.\end{aligned} \tag{6}$$

□

Note that the differential operator in (6) is elliptic. However, as explained in [3], the counterexample of Pliś can bemodified in order to obtain a counterexample for the backward parabolic operator

$$\mathscr{L}_P \triangleq \partial_t + \partial_{x_1}^2 + l(t)\partial_{x_2}^2 + b_1(t,x_1,x_2)\partial_{x_1} + b_2(t,x_1,x_2)\partial_{x_2} + c(t,x_1,x_2)\,.$$

Moreover, the result can be extended to the operator $\mathscr{L}$ acting in the space $\mathscr{H}_0$, by considering the problem solved by $u(t,x_1,x_2)e^{-x_1^2-x_2^2}$, thus obtaining the following theorem.

Theorem 2.3 *There exist coefficients $a_{i,j}$, depending only on t, which are Hölder continuous of every order but not Lipschitz continuous and there exist $u \in \mathscr{H}_0$ such that the solution of problem (3) with $u_0 = 0$ and $f = 0$ is not identically zero.* □

In view of the previous results, a question naturally arises: which is the *minimal* regularity (between Lipschitz continuity and Hölder continuity) of the coefficients of the principal part of $\mathscr{L}$ guaranteeing uniqueness of the solution of (3)? To answer to this question, the definition of *modulus of continuity*, that can be exploited to measure the degree of regularity of a function, is useful.

Definition 2.4 *A* modulus of continuity *is a function $\mu : [0,1] \to [0,1]$ which is continuous, strictly increasing, concave and such that $\mu(0) = 0$. A function $f : \mathbb{R} \to \mathbb{R}$ has* regularity μ *if*

$$\sup_{0<|t-s|<1} \frac{|f(t)-f(s)|}{\mu(|t-s|)} < +\infty .$$

The set of all functions having regularity μ is denoted by C^μ.

As particular cases, the Lipschitz continuity, the τ-Hölder continuity ($\tau \in]0,1[$) and the *logarithmic Lipschitz* (in short *Log-Lipschitz*) continuity are obtained for $\mu(s) = s$, $\mu(s) = s^\tau$ and $\mu(s) = s\log(1+1/s)$, respectively.

A further characterization of the modulus of continuity is the so called *Osgood condition* which is crucial in most of the results on uniqueness and stability that are described in the rest of the article. A modulus of continuity μ satisfies the Osgood condition if

$$\int_0^1 \frac{1}{\mu(s)} ds = +\infty .$$

This characterization is used, for instance, in [3] to obtain the following result concerning an operator whose coefficients in the principal part depend also on x.

Theorem 2.5 *Let μ be a modulus of continuity that satisfies the Osgood condition. Let*

$$\mathscr{H}_1 \triangleq H^1([0,T], L^2(\mathbb{R}^n)) \cap L^2([0,T], H^2(\mathbb{R}^n)) \tag{7}$$

and let the coefficients $a_{i,j}$ be such that, for all $i, j = 1, \ldots, n$,

$$a_{i,j} \in C^\mu([0,T], \mathscr{C}_b(\mathbb{R}^n)) \cap \mathscr{C}([0,T], \mathscr{C}_b^2(\mathbb{R}^n)) ,$$

where $\mathscr{C}_b$ is the space of bounded functions and $\mathscr{C}_b^2$ is the space of the bounded functions whose first and second derivatives are bounded. If $u \in \mathscr{H}_1$, $\mathscr{L}u = 0$ on $[0,T] \times \mathbb{R}^n$ and $u(0,x) = 0$ on $\mathbb{R}^n$, then $u \equiv 0$ on $[0,T] \times \mathbb{R}^n$.

More recently, by using Bony's para-multiplication, the result has been improved as far as the regularity with respect to x is concerned, i.e. replacing $\mathscr{C}^2$ regularity with Lipschitz regularity [5].

Note that the claim of Theorem 2.5 refers to the function space defined by (7), however, it is not difficult to extend it to the function space $\mathscr{H}_0$ defined by (5).

3 Conditional stability results with Lipschitz or Log-Lipschitz continuous coefficients

As we explained in the Introduction, the Cauchy problem (3) for the backward parabolic differential operator (4) is in general not well posed. Therefore, the notion of continuous dependence from initial data needs to be replaced with the notion of conditional stability. The question about the conditional stability can be stated as follows: suppose that two functions u and v, defined in $[0,T]\times\mathbb{R}^n$, are solutions of the same equation $\mathscr{L}w = f$; suppose, in addition, that u and v satisfy a fixed bound in a space $\mathscr{K}$ and that $\|u(0,\cdot)-v(0,\cdot)\|_{\mathscr{H}}$ is small (e.g. smaller than some $\varepsilon > 0$). Given these assumptions, what can we say about the quantity $\sup_{t\in[0,T']}\|u(t,\cdot)-v(t,\cdot)\|_{\mathscr{K}}$, where $T' < T$? Does it remains small as well, e.g. smaller than a value related to ε, and possibly tending to 0 as ε tends to 0? In this section some results that give an answer to the above questions are reported.

One of the first results on conditional stability has been proven by Hurd [9] in the same theoretical framework considered by Lions and Malgrange.

Theorem 3.1 *Suppose that the coefficients $a_{i,j}$ in (4) are Lipschitz continuous both in t and in x. For every $T' \in]0,T[$ and for every $D > 0$ there exist $\rho > 0$, $\delta \in]0,1[$ and $M > 0$ such that if $u \in \mathscr{H}_0$ is a solution of $\mathscr{L}u = 0$ on $[0,T]\times\mathbb{R}^n$ with $\|u(t,\cdot)\|_{L^2} \le D$ on $[0,T]$ and $\|u(0,\cdot)\|_{L^2} \le \rho$, then*

$$\sup_{t\in[0,T']}\|u(t,\cdot)\|_{L^2} \le M\|u(0,\cdot)\|_{L^2}^{\delta}. \tag{8}$$

The constants ρ, δ and M depend only on T' and D, on the ellipticity constant of $\mathscr{L}$, on the L^∞ norms of the coefficients $a_{i,j}$, b_j, c, on the L^∞ norms of their spatial derivatives, and on the Lipschitz constant of the coefficients $a_{i,j}$ with respect to time. □

Estimate (8) means that, under the hypotheses of Theorem 3.1, the Cauchy problem (3) is well behaved. Notice that the result expressed by (8) implies uniqueness of the solution for (3). It is therefore apparent that a necessary condition for (3) to be well behaved, is that the coefficients $a_{i,j}$ fulfil the Osgood condition with respect to time. Hence a natural question arises: is Osgood condition also a sufficient condition for (8) to hold? Del Santo and Prizzi [4] gave a negative answer to this question. In particular, mimicking Pliś counterexample, they have shown that if the coefficients $a_{i,j}$ are not Lipschitz continuous but only Log-Lipschitz continuous, then Hurd's result does not hold. Notwithstanding, they proved that if the coefficients are Log-Lipschitz continuous with respect to time and sufficiently regular with respect to space, then a conditional stability property, weaker than (8), does hold. The counterexample in [4] relies on the fact that it is possible to construct

- a sequence $\{\mathscr{L}_k\}_{k\in\mathbb{N}}$ of backward uniformly parabolic operators with uniformly Log-Lipschitz-continuous coefficients (not depending on the space variables) in the principal part and space-periodic uniformly bounded smooth coefficients in the lower order terms,

- a sequence $\{u_k\}_{k\in\mathbb{N}}$ of space-periodic smooth uniformly bounded solutions of $\mathscr{L}_k u_k = 0$ on $[0,1]\times\mathbb{R}^2$,
- a sequence $\{t_k\}_{k\in\mathbb{N}}$ of real numbers, with $t_k \to 0$,

such that

$$\lim_{k\to\infty} \|u_k(0,\cdot,\cdot)\|_{L^2([0,2\pi]\times[0,2\pi])} = 0$$

and

$$\lim_{k\to\infty} \frac{\|u_k(t_k,\cdot,\cdot)\|_{L^2([0,2\pi]\times[0,2\pi])}}{\|u_k(0,\cdot,\cdot)\|^{\delta}_{L^2([0,2\pi]\times[0,2\pi])}} = +\infty$$

for every $\delta > 0$.

As we mentioned above, in the case of Log-Lipschitz coefficients a result weaker that (8) is valid. Consider the equation $\mathscr{L}u = 0$ on $[0,T]\times\mathbb{R}^n$, with $\mathscr{L}$ defined in (4) and suppose that for all $i,j = 1,\ldots,n$, $a_{i,j} \in \mathrm{LogLip}([0,T], L^\infty(\mathbb{R}^n))$, that is

$$\sup\left\{ \frac{|a_{i,j}(t,x) - a_{i,j}(s,x)|}{|t-s|\,(1+\log|t-s|)} \mid t,s \in []0,T], < |t-s| < 1, x \in \mathbb{R}^n \right\} < +\infty;$$

moreover, assume that a_{ij}, b_j and c belong to $L^\infty([0,T], C^2(\mathbb{R}^n))$.

Theorem 3.2 **[4]** *Suppose that the above hypotheses hold. For all $T' \in]0,T[$ and for all $D > 0$ there exist $\rho > 0$, $M > 0$, $N > 0$ and $\delta \in]0,1[$ such that, if $u \in \mathscr{H}_0$ is a solution of $\mathscr{L}u = 0$ on $[0,T]\times\mathbb{R}^n$ with $\|u(t,\cdot)\|_{L^2} \le D$ on $[0,T]$ and $\|u(0,\cdot)\|_{L^2} \le \rho$, then*

$$\sup_{t\in[0,T']} \|u(t,\cdot)\|_{L^2} \le M e^{-N|\log\|u(0,\cdot)\|_{L^2}|^{\delta}}, \tag{9}$$

where the constants ρ, δ, M and N depend only on T', on D, on the ellipticity constant of $\mathscr{L}$, on the L^∞ norms of the coefficients $a_{i,j}$, on the L^∞ norms of their spatial derivatives, and on the Log-Lipschitz constant of the coefficients $a_{i,j}$ with respect to time.

The proof of Theorem 3.2 relies on a weighted energy estimate, with loss of space regularity as time goes on. Such estimate is obtained exploiting Fourier transform of solutions when the coefficients in $\mathscr{L}$ are independent of space. To deal with the case of space dependent coefficients one needs to use Paley-Littlewood dyadic decomposition of solutions. Using Bony's para-product the result can be further improved, lowering the requirements on space regularity of coefficients and allowing them to be just Lipschitz continuous with respect to x [2].

4 Stability with Osgood continuous (with respect to time) coefficients

Let us finally come to the new result contained in this paper. As in the previous section we first present a counterexample to the stability estimate (9) and then a new weaker stability result.

4.1 Counterexample to stability estimate (9) in the LogLog-Lipschitz case

Consider the modulus of continuity ω defined, near 0, by

$$\omega(s) = s \log\left(1 + \frac{1}{s}\right) \log\left(\log\left(1 + \frac{1}{s}\right)\right)$$

and note that ω satisfies the Osgood condition but $\mathscr{C}^\omega$ functions are not Log-Lipschitz continuous. As in the previous section, it is possible [1] to construct

- a sequence $\{\mathscr{L}_k\}_{k\in\mathbb{N}}$ of backward uniformly parabolic operators with uniformly $\mathscr{C}^\omega$-continuous coefficients (independent of the spatial variable) in the principal part and space-periodic uniformly bounded smooth coefficients in the lower order terms,
- a sequence $\{u_k\}_{k\in\mathbb{N}}$ of space-periodic smooth uniformly bounded solutions of $\mathscr{L}_k u_k = 0$ on $[0,1]\times\mathbb{R}^2$,
- a sequence $\{t_k\}_{k\in\mathbb{N}}$ of real numbers, with $t_k \to 0$,

such that

$$\lim_{k\to\infty} \|u_k(0,\cdot,\cdot)\|_{L^2([0,2\pi]\times[0,2\pi])} = 0$$

but (9) does not hold for all δ; more precisely

$$\lim_{k\to\infty} \frac{\|u_k(t_k,\cdot,\cdot)\|_{L^2([0,2\pi]\times[0,2\pi])}}{e^{-N|\log\|u_k(0,\cdot,\cdot)\|_{L^2([0,2\pi]\times[0,2\pi])}|^\delta}} = +\infty$$

for every $\delta > 0$ and $N > 0$.

4.2 Stability result in the Osgood-continuous case

From now on, the following conditions are assumed to hold.

Assumption 4.1 *The operator $\mathscr{L}$ defined in (1) is such that*

- *for all $t \in [0,T]$ and for all $i,j = 1,\dots,n$,*

$$a_{i,j}(t) = a_{j,i}(t);$$

- *there exists $k_A > 0$ such that, for all $(t,\xi) \in [0,T]\times\mathbb{R}^n$,*

$$k_A|\xi|^2 \le \sum_{i,j=1}^n a_{i,j}(t)\xi_i\xi_j \le k_A^{-1}|\xi|^2;$$

- *there exists $k_B > 0$ such that, for all $t \in [0,T]$ and for all $i = 1,\dots,n$, $|b_i(t)| \le k_B$;*
- *there exists $k_C > 0$ such that, for all $t \in [0,T]$, $|c(t)| \le k_C$;*

- *for all $i,j=1,\dots,n$, $a_{i,j}\in C^{\omega}([0,T])$, where ω is a modulus of continuity that satisfies the Osgood condition.*

We can now state our main result.

Theorem 4.2 *For all $T'\in]0,T[$ and for all $D>0$ there exist $\rho'>0$, and an increasing continuous function $G:[0,+\infty[\to[0,+\infty[$, with $G(0)=0$, such that, if $u\in\mathscr{H}_0$ is a solution of $\mathscr{L}u=0$ on $[0,T]$ with $\|u(t,\cdot)\|_{L^2}\le D$ on $[0,T]$ and $\|u(0,\cdot)\|_{L^2}\le\rho'$, then*

$$\sup_{t\in[0,T']}\|u(t,\cdot)\|_{L^2}^2\le G(\|u(0,\cdot)\|_{L^2}^2)\,. \tag{10}$$

The constant ρ' and the function G depend on $k_A,k_B,k_C,\omega,n,T,T'$ and D. □

In Theorem 4.2 all coefficients of the operator $\mathscr{L}$ are assumed to be independent of the space variable x. Compared with the assumptions in [4, 2] this is certainly a strong restriction. In order to deal with the case of space dependent coefficients one could exploit dyadic Littlewood-Paley decomposition of solutions. As it will be apparent in the computations in Section 5 below, although estimate (10) involves only L^2 norms, it is obtained through a massive use of Gevrey-Sobolev norms (see [8] for a theoretical framing). In the case of space dependent coefficients, this would lead to the use of Bony paraproducts (even when only the lower order coefficients are space-dependent) and would require strong regularity assumptions on the x-dependence of coefficients: namely, it would require that the coefficients themselves are Gevrey-Sobolev (hence C^∞) in x (see Theorem 4.1 in [8]). The use of paraproducts would introduce a certain amount of technical difficulties in the management of various commutators, and it is not clear to what extent the result we expose in the present paper can be generalized to the case of an operator $\mathscr{L}$ with x dependent coefficients.

Definition 4.3 *[8] Given $a\ge 0$, $d\in\mathbb{R}$ and $\varepsilon>1$, the* Gevrey-Sobolev *function space $H^d_{a,\varepsilon}$ is the space of the functions $u:\mathbb{R}^n\to\mathbb{R}$ such that*

$$\|u\|_{H^d_{a,\varepsilon}}^2\triangleq\int_{\mathbb{R}^n}\left(1+|\xi|^2\right)^d e^{2a|\xi|^{1/\varepsilon}}\,|\hat{u}(\xi)|^2\,d\xi<+\infty\,,$$

where $\hat{u}$ is the Fourier transform of u.

Besides Gevrey-Sobolev spaces, we need to introduce a new particular class of spaces, taylored on the modulus of continuity ω.

Definition 4.4 *Let $a>0$, $d\in\mathbb{R}$ and ω a modulus of continuity satisfying the Osgood condition. We denote by $H^d_{a,\omega}$ the set of the functions $u:\mathbb{R}^n\to\mathbb{R}$ such that*

$$\|u\|_{H^d_{a,\omega}}^2\triangleq\int_{\mathbb{R}^n}\left(1+|\xi|^2\right)^d e^{a|\xi|^2\omega\left(\frac{1}{|\xi|^2+1}\right)}|\hat{u}(\xi)|^2 d\xi<+\infty\,.$$

We call it Osgood-Sobolev *function space.*

Remark 4.5 *From Definitions 4.3 and 4.4 it is easy to see that, for any Osgood moduls of continuity* ω*, for all* $\varepsilon > 1$*, for all* $a > 0$ *and for all* $d \in \mathbb{R}$*,*

$$H^d_{a,\varepsilon} \subset H^d_{a,\omega}\,.$$

Theorem 4.2 is a consequence of the following local result.

Theorem 4.6 *There exists* $\alpha_1 > 0$ *and, for any* $T'' : 0 < T'' < T$*, there exist constants* $\rho > 0$*,* $C > 0$ *and a function* $g : [0, k_A] \to [0, +\infty[$*, such that, if* $u \in \mathscr{H}_0$ *is a solution of*

$$\mathscr{L} u = 0\,, \tag{11}$$

with $\mathscr{L}$ *fulfilling Assumption 4.1 and* $\|u(0,\cdot)\|^2_{H^0_{\nu,\varepsilon}} < \rho$ *for some* $\nu > 0$ *and some* $\varepsilon > 1$*, then*

$$\sup_{z\in[0,\bar\sigma]} \|u(z,\cdot)\|^2_{H^1} \le C e^{-\sigma g\left(\|u(0,\cdot)\|^2_{H^0_{\nu,\varepsilon}}\right)} \left[1 + \|u(\sigma,\cdot)\|^2_{H^1}\right]\,, \tag{12}$$

where $\sigma = \min\{T'', 1/\alpha_1\}$ *and* $\bar\sigma = \sigma/8$*. The constant* α_1 *depends only on* k_A, k_B, k_C, ω *and* n *while the constants* ρ *and* C *depend also on* T *and* T''*. The function* g *is a strictly decreasing function; it depends on* $k_A, k_B, k_C, \omega, n, T, T'', \varepsilon$ *and* ν *and satisfies* $\lim_{y\to 0} g(y) = +\infty$. □

We end this section with a comment on the functions g and G in estimates (12) and (10) respectively. The function g, as it will be clear in the proof of Theorem 4.6, can be explicitly expressed in terms of the modulus of continuity ω (the precise formula for g is given by (50) below). Being ω "generic", the expression of g is of course only *theoretically* explicit. Nevertheless, when a concrete ω is given, it is possible to compute the concrete explicit expression of g. As an example, in Section 6 we compute the explicit expression of the continuity estimate (12) for an operator $\mathscr{L}$ with LogLog-Lipschitz continuous coefficients, i.e. with $\omega(s) = s(1-\log s)\log(1-\log s)$. In such case the stability estimate becomes

$$\sup_{z\in[0,\bar\sigma/2]} \|u(z,\cdot)\|^2_{L^2} \le \exp\left\{-\frac{\sigma k_A}{2e}\exp\left[\left(\log\left(\frac{1}{\tau}\left|\log\|u(0,\cdot)\|_{L^2}\right|\right)\right)^{\delta}\right]\right\}, \tag{13}$$

where σ, $\bar\sigma$, τ and δ are suitable constants. Concerning the function G in the global estimate (10), it is obtained by iterating a finite number of times estimate (12): therefore, explicit knowledge of g yields explicit knowledge of G. In the LogLog-Lipschitz case we shall see that estimate (13) reproduces itself at each iteration step, so in the corresponding global estimate the function G has the same form as in the local estimate.

Next section is devoted to the proofs of Theorems 4.6 and 4.2.

5 Proofs of the main reults

Theorem 4.6 will be proven with the help of partial results expressed in terms of estimates of some integral quantities. Lemma 5.2 below guarantees that all the integral quantities that will be introduced are finite, so that the obtained estimates make sense.

Lemma 5.1 *Let $u:[0,T]\to\mathbb{R}$ a C^1 function. If $u'(t)\geq Mu(t)$, then $u(t)\leq e^{M(t-T)}u(T)$.*

Proof. If is sufficient to note that:

$$u'(t)\geq Mu(t) \;\Rightarrow\; u'(t)e^{-M(t-T)}-Mu(t)e^{-M(t-T)}\geq 0 \;\Rightarrow$$
$$\Rightarrow\; \frac{d}{dt}\left(u(t)e^{-M(t-T)}\right)\geq 0 \;\Rightarrow\; u(t)e^{-M(t-T)}\leq u(T) \;\Rightarrow$$
$$\Rightarrow\; u(t)\leq e^{M(t-T)}u(T)\,.$$

■

Lemma 5.2 *Let $M>0$ and let $u\in\mathscr{H}_0$ be a solution of*

$$\partial_t u+\sum_{i,j=1}^{n}a_{i,j}(t)\partial_{x_i}\partial_{x_j}u+\sum_{i=1}^{n}b_i(t)\partial_{x_i}u+c(t)u=0\,, \tag{14}$$

on $[0,T]$, such that $\|u(t,\cdot)\|_{L^2}\leq M$, for all $t\in[0,T]$. Let $l>0$ and extend the coefficients $a_{i,j}$, b_i and c to $[-l,T]$ by setting $a_{i,j}(t)=a_{i,j}(0)$, $b_i(t)=b_i(0)$ and $c(t)=c(0)$ for all $t\in[-l,0[$. Then u can be extended to a solution of (14) on $[-l,T]$, and there exists $\hat{M}$ such that $\|u(t,\cdot)\|_{L^2}\leq\hat{M}$ on $[-l,T]$. The constant $\hat{M}$ depends only on n, k_A, k_B, K_C, T, l and M. Moreover,

1. *$u\in C^0([-l,T[,H^d_{a,\varepsilon})$ for all $a\geq 0$, $\varepsilon>1$ and $d\in\mathbb{R}$;*
2. *$u\in C^0([-l,T[,H^1)$ and there exists C, which depends on n, k_A, k_B, k_C, T and l, such that*
   $$\|u(t,\cdot)\|_{H^1}\leq C(T-t)^{-1/2}\|u(T,\cdot)\|_{L^2}$$
   for all $t\in[-l,T[$;
3. *there exists $\hat{C}$, which depends on n, k_A, k_B, k_C, l, a and ε and which tends to $+\infty$ when l tends to zero, such that*
   $$\|u(-l,\cdot)\|_{H^0_{a,\varepsilon}}\leq\hat{C}\|u(0,\cdot)\|_{L^2}\,.$$

□

Proof. It is easy to see that for all $t\in[0,T]$ and for almost all $\xi\in\mathbb{R}^n$,

$$\partial_t\hat{u}(t,\xi)-\sum_{i,j=1}^{n}a_{i,j}(t)\xi_i\xi_j\hat{u}(t,\xi)+\imath\sum_{i=1}^{n}b_i(t)\xi_i\hat{u}(t,\xi)+c(t)\hat{u}(t,\xi)=0\,. \tag{15}$$

Multiplying both terms of (15) by $\bar{\hat{u}}$ yields

$$\partial_t\hat{u}(t,\xi)\bar{\hat{u}}(t,\xi)=\sum_{i,j=1}^{n}a_{i,j}(t)\xi_i\xi_j|\hat{u}(t,\xi)|^2-\imath\sum_{i=1}^{n}b_i(t)\xi_i|\hat{u}(t,\xi)|^2-c(t)|\hat{u}(t,\xi)|^2\,. \tag{16}$$

By adding to (16) its complex conjugate, we obtain

$$\partial_t |\hat{u}(t,\xi)|^2 = 2\sum_{i,j=1}^{n} a_{i,j}(t)\xi_i\xi_j|\hat{u}(t,\xi)|^2 + 2\sum_{i=1}^{n}\Im\{b_i(t)\}\xi_i|\hat{u}(t,\xi)|^2 + \\ -2\Re\{c(t)\}|\hat{u}(t,\xi)|^2, \quad (17)$$

hence, recalling the bounds for the coefficients of $\mathscr{L}$ (see Assumption 4.1),

$$\partial_t |\hat{u}(t,\xi)|^2 \geq 2k_A|\xi|^2|\hat{u}(t,\xi)|^2 - 2nk_B|\xi||\hat{u}(t,\xi)|^2 - 2k_C|\hat{u}(t,\xi)|^2,$$

i.e.

$$\partial_t |\hat{u}(t,\xi)|^2 \geq (2k_A|\xi|^2 - 2nk_B|\xi| - 2k_C)|\hat{u}(t,\xi)|^2.$$

Lemma 5.1 allows one to write

$$|\hat{u}(t,\xi)|^2 \leq e^{(2k_A|\xi|^2-2nk_B|\xi|-2k_C)(t-T)}|\hat{u}(T,\xi)|^2. \quad (18)$$

Therefore, for a fixed $t \in [-l,T[$,

$$\int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^d e^{2a|\xi|^{\frac{1}{\varepsilon}}}|\hat{u}(t,\xi)|^2 d\xi \leq \\ \leq \int_{\mathbb{R}^n} \left(1+|\xi|^2\right)^d e^{2a|\xi|^{\frac{1}{\varepsilon}}+(2k_A|\xi|^2-2nk_B|\xi|-2k_C)(t-T)}|\hat{u}(T,\xi)|^2 d\xi < +\infty,$$

where the last inequality comes from the fact that $u \in \mathscr{H}_0$ and therefore, in particular, $u \in \mathscr{C}^0([0,T],L^2(\mathbb{R}^n))$, and, since $t < T$,

$$\lim_{|\xi|\to\infty} \left(1+|\xi|^2\right)^d e^{2a|\xi|^{\frac{1}{\varepsilon}}+(2k_A|\xi|^2-2nk_B|\xi|-2k_C)(t-T)} = 0$$

for all $a > 0$ and all $\varepsilon > 1$. The first claim is then proven. The second claim is proven easily by choosing $d = 1$ and $a = 0$. To prove the third claim it is sufficient to rewrite equation (18) replacing T with 0. ■

5.1 Preliminary results and defintions

In this section some functions that are used in the rest of the article are defined. Let ω be a modulus of continuity satisfying Osgood condition. For a given $\rho > 1$ define the function $\theta : [1,+\infty[\to [0,+\infty]$ as

$$\theta(\rho) = \int_{1/\rho}^{1} \frac{1}{\omega(s)} ds. \quad (19)$$

It is easy to see that θ is bijective and strictly increasing. As a consequence, it can be inverted. For $y \in (0,1]$, for $q > 0$ and for $\lambda > 0$, let $\psi_{\lambda,q} :]0,1] \to [1,+\infty[$ be defined by

$$\psi_{\lambda,q}(y) \triangleq \theta^{-1}(-\lambda q \log y).$$

The relation

$$\theta\left(\psi_{\lambda,q}(y)\right) = -\lambda q \log y$$

immediately follows from the definitions; hence

$$\theta'\left(\psi_{\lambda,q}(y)\right)\psi'_{\lambda,q}(y) = -\frac{\lambda q}{y}.$$

Now, let the function $\phi_{\lambda,q} : (0,1] \to (-\infty, 0]$ be defined as

$$\phi_{\lambda,q}(y) \triangleq q\int_1^y \psi_{\lambda,q}(z)dz. \tag{20}$$

The function $\phi_{\lambda,q}$ is bijective and strictly increasing; moreover,

$$\phi''_{\lambda,q}(y) = q\psi'_{\lambda,q}(y) = \frac{q}{\theta'\left(\psi_{\lambda,q}(y)\right)}\left(-\frac{\lambda q}{y}\right). \tag{21}$$

On the other hand, equation (19), with the change of variable $\eta = 1/s$, becomes

$$\theta(\rho) = -\int_\rho^1 \frac{1}{\omega\left(\frac{1}{\eta}\right)}\frac{1}{\eta^2}d\eta = \int_1^\rho \frac{1}{\eta^2\omega\left(\frac{1}{\eta}\right)}d\eta$$

from which

$$\frac{1}{\theta'\left(\psi_{\lambda,q}(y)\right)} = \psi_{\lambda,q}(y)^2\omega\left(\frac{1}{\psi_{\lambda,q}(y)}\right). \tag{22}$$

Substituting (22) into (21) and recalling that $\psi_{\lambda,q}(y) = \phi'_{\lambda,q}(y)/q$, it is easy to see that $\phi_{\lambda,q}$ satisfies the equation

$$y\phi''_{\lambda,q}(y) = -\lambda\left(\phi'_{\lambda,q}(y)\right)^2\omega\left(\frac{q}{\phi'_{\lambda,q}(y)}\right). \tag{23}$$

Note that for all $\lambda > 0$, for all $q > 0$ and for all $y \in (0,1]$, $\psi_{\lambda,q} \in (1, +\infty)$ and, consequently,

$$\frac{q}{\phi'_{\lambda,q}(y)} \in (0,1).$$

5.2 A pointwise estimate

The first result shows that, once fixed ξ, namely the value of the frequence argument of $\hat{u}$, it is possible to find a bound for a particular time-integral, in an interval $[0,\sigma]$, of a function of $|\hat{u}(t,\xi)|$. This bound consists in the sum of two terms depending on $\hat{u}(0,\xi)$ and $\hat{u}(\sigma,\xi)$, respectively.

Proposition 5.3 *Let $T'' \in]0,T[$. There exist $\alpha_1 > 0$, $\bar{\lambda}$ and $\bar{\gamma} > 0$ such that, setting $\alpha \triangleq \max\{\alpha_1, 1/T''\}$, defining $\sigma = 1/\alpha$, fixing $\tau \in]0, \sigma/4]$, and letting $\beta \geq \sigma + \tau$, whenever $u \in \mathscr{H}_0$ is a solution of (11), one has*

$$\frac{1}{4}\left(k_A|\xi|^2+\gamma\right)\int_0^\sigma e^{(1-\alpha t)|\xi|^2\omega\left(\frac{1}{|\xi|^2+1}\right)} e^{2\gamma t} e^{-2\beta\phi_\lambda\left(\frac{t+\tau}{\beta}\right)}|\hat{u}(t,\xi)|^2 dt \leq$$
$$\leq \phi_\lambda'\left(\frac{\tau}{\beta}\right)\tau e^{|\xi|^2\omega\left(\frac{1}{|\xi|^2+1}\right)} e^{-2\beta\phi_\lambda\left(\frac{\tau}{\beta}\right)}|\hat{u}(0,\xi)|^2+$$
$$+(\sigma+\tau)(\gamma+k_A^{-1}|\xi|^2)e^{2\gamma\sigma}e^{-2\beta\phi_\lambda\left(\frac{\sigma+\tau}{\beta}\right)}|\hat{u}(\sigma,\xi)|^2, \quad (24)$$

for all $\lambda \geq \bar{\lambda}$ and all $\gamma \geq \bar{\gamma}$, where $\phi_\lambda \triangleq \phi_{\lambda,k_A}$ (see (20)). The constant α_1 depends only on n, k_A, k_B, k_C and ω, while $\bar{\gamma}$ and $\bar{\lambda}$ depend on n, k_A, k_B, k_C, ω, T and T''. □

Proof. Let $T'' \in]0,T[$ and let $\alpha \geq 1/T''$, $\gamma > 0$, $\lambda > 0$, $\tau \in]0,T''[$, $\sigma = 1/\alpha$ and $\beta \geq \tau + \sigma$. Consider the function $\hat{v} : [0,\sigma] \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$\hat{v}(t,\xi) = e^{\left(\frac{1-\alpha t}{2}\right)|\xi|^2\omega\left(\frac{1}{|\xi|^2+1}\right)} e^{\gamma t} e^{-\beta\phi_\lambda\left(\frac{t+\tau}{\beta}\right)}\hat{u}(t,\xi). \quad (25)$$

The time-derivative of $\hat{v}$ is

$$\partial_t\hat{v}(t,\xi) = -\frac{\alpha}{2}|\xi|^2\omega\left(\frac{1}{|\xi|^2+1}\right)e^{\left(\frac{1-\alpha t}{2}\right)|\xi|^2\omega\left(\frac{1}{|\xi^2|+1}\right)} e^{\gamma t} e^{-\beta\phi_\lambda\left(\frac{t+\tau}{\beta}\right)}\hat{u}(t,\xi)+$$
$$+\gamma e^{\left(\frac{1-\alpha t}{2}\right)|\xi|^2\omega\left(\frac{1}{|\xi|^2+1}\right)} e^{\gamma t} e^{-\beta\phi_\lambda\left(\frac{t+\tau}{\beta}\right)}\hat{u}(t,\xi)+$$
$$-\phi_\lambda'\left(\frac{t+\tau}{\beta}\right)e^{\left(\frac{1-\alpha t}{2}\right)|\xi|^2\omega\left(\frac{1}{|\xi|^2+1}\right)} e^{\gamma t} e^{-\beta\phi_\lambda\left(\frac{t+\tau}{\beta}\right)}\hat{u}(t,\xi)+$$
$$+e^{\left(\frac{1-\alpha t}{2}\right)|\xi|^2\omega\left(\frac{1}{|\xi|^2+1}\right)} e^{\gamma t} e^{-\beta\phi_\lambda\left(\frac{t+\tau}{\beta}\right)}\partial_t\hat{u}(t,\xi)$$

which may be rewritten as

$$\partial_t\hat{v}+\frac{\alpha}{2}|\xi|^2\omega\left(\frac{1}{|\xi|^2+1}\right)\hat{v}-\gamma\hat{v}+\phi_\lambda'\left(\frac{t+\tau}{\beta}\right)\hat{v}-\sum_{i,j=1}^n a_{i,j}(t)\xi_i\xi_j\hat{v}+$$
$$+\iota\sum_{i=1}^n b_i(t)\xi_i\hat{v}+c(t)\hat{v}=0, \quad (26)$$

where the dependency of $\hat{v}$ and $\partial_t\hat{v}$ on t and on ξ has been neglected for the sake of a simple notation and where the equation (15) has been exploited. The complex conjugate equation of (26) is

$$\partial_t\bar{\hat{v}}+\frac{\alpha}{2}|\xi|^2\omega\left(\frac{1}{|\xi|^2+1}\right)\bar{\hat{v}}-\gamma\bar{\hat{v}}+\phi_\lambda'\left(\frac{t+\tau}{\beta}\right)\bar{\hat{v}}-\sum_{i,j=1}^n a_{i,j}(t)\xi_i\xi_j\bar{\hat{v}}+$$
$$-\iota\sum_{i=1}^n \bar{b}_i(t)\xi_i\bar{\hat{v}}+\bar{c}(t)\bar{\hat{v}}=0. \quad (27)$$

Multiplying (26) by $(t+\tau)\partial_t\bar{\hat{v}}$ and (27) by $(t+\tau)\partial_t\hat{v}$ and summing the two terms yields

$$2(t+\tau)|\partial_t\hat{v}|^2+\frac{\alpha}{2}(t+\tau)|\xi|^2\omega\left(\frac{1}{|\xi|^2+1}\right)(\hat{v}\partial_t\bar{\hat{v}}+\bar{\hat{v}}\partial_t\hat{v})-\gamma(t+\tau)(\hat{v}\partial_t\bar{\hat{v}}+\bar{\hat{v}}\partial_t\hat{v})+$$
$$+(t+\tau)\phi'_\lambda\left(\frac{t+\tau}{\beta}\right)(\hat{v}\partial_t\bar{\hat{v}}+\bar{\hat{v}}\partial_t\hat{v})-(t+\tau)\sum_{i,j=1}^{n}a_{i,j}(t)\xi_i\xi_j(\hat{v}\partial_t\bar{\hat{v}}+\bar{\hat{v}}\partial_t\hat{v})+$$
$$-2(t+\tau)\sum_{i=1}^{n}\xi_i\Im\{b_i(t)\hat{v}\partial_t\bar{\hat{v}}\}+2(t+\tau)\Re\left\{c(t)\hat{v}\partial_t\bar{\hat{v}}\right\}=0. \quad (28)$$

Substituting in the second term the explicit expressions of $\partial_t\hat{v}$ and $\partial_t\bar{\hat{v}}$, that may be obtained from (26) and (27), one obtains

$$2(t+\tau)|\partial_t\hat{v}|^2-\frac{\alpha^2}{2}(t+\tau)|\xi|^4\left[\omega\left(\frac{1}{|\xi|^2+1}\right)\right]^2|\hat{v}|^2+$$
$$+\alpha\gamma(t+\tau)|\xi|^2\omega\left(\frac{1}{|\xi|^2+1}\right)|\hat{v}|^2-\alpha(t+\tau)|\xi|^2\omega\left(\frac{1}{|\xi|^2+1}\right)\phi'_\lambda\left(\frac{t+\tau}{\beta}\right)|\hat{v}|^2+$$
$$+\alpha(t+\tau)|\xi|^2\omega\left(\frac{1}{|\xi|^2+1}\right)|\hat{v}|^2\left(\sum_{i,j=1}^{n}a_{i,j}(t)\xi_i\xi_j-c(t)\right)+$$
$$-\gamma(t+\tau)(\hat{v}\partial_t\bar{\hat{v}}+\bar{\hat{v}}\partial_t\hat{v})+(t+\tau)\phi'_\lambda\left(\frac{t+\tau}{\beta}\right)(\hat{v}\partial_t\bar{\hat{v}}+\bar{\hat{v}}\partial_t\hat{v})+$$
$$-(t+\tau)(\hat{v}\partial_t\bar{\hat{v}}+\bar{\hat{v}}\partial_t\hat{v})\sum_{i,j=1}^{n}a_{i,j}(t)\xi_i\xi_j+$$
$$-2(t+\tau)\sum_{i=1}^{n}\xi_i\Im\{b_i(t)\hat{v}\partial_t\bar{\hat{v}}\}+2(t+\tau)\Re\left\{c(t)\hat{v}\partial_t\bar{\hat{v}}\right\}=0. \quad (29)$$

Integrating (29) between 0 and s, with $s\le\sigma=1/\alpha$, yields

$$
\begin{aligned}
2\int_0^s (t+\tau)|\partial_t \hat{v}(t,\xi)|^2 dt - \frac{\alpha^2}{2}|\xi|^4 \left[\omega\left(\frac{1}{|\xi|^2+1}\right)\right]^2 \int_0^s (t+\tau)|\hat{v}(t,\xi)|^2 dt +& \\
+\underbrace{\alpha\gamma|\xi|^2\omega\left(\frac{1}{|\xi|^2+1}\right)\int_0^s (t+\tau)|\hat{v}(t,\xi)|^2 dt}_{(A)} +& \\
-\alpha|\xi|^2\omega\left(\frac{1}{|\xi|^2+1}\right)\int_0^s (t+\tau)\phi_\lambda'\left(\frac{t+\tau}{\beta}\right)|\hat{v}(t,\xi)|^2 dt +& \\
+\alpha|\xi|^2\omega\left(\frac{1}{|\xi|^2+1}\right)\int_0^s (t+\tau)\sum_{i,j=1}^n a_{i,j}(t)\xi_i\xi_j|\hat{v}(t,\xi)|^2 dt +& \\
-\alpha|\xi|^2\omega\left(\frac{1}{|\xi|^2+1}\right)\int_0^s (t+\tau)c(t)|\hat{v}(t,\xi)|^2 dt +& \\
+\gamma\int_0^s |\hat{v}(t,\xi)|^2 dt - \gamma(s+\tau)|\hat{v}(s,\xi)|^2 +& \\
+\underbrace{\gamma\tau|\hat{v}(0,\xi)|^2}_{(B)} + \int_0^s \left[-\phi_\lambda''\left(\frac{t+\tau}{\beta}\right)\left(\frac{t+\tau}{\beta}\right) - \phi_\lambda'\left(\frac{t+\tau}{\beta}\right)\right]|\hat{v}(t,\xi)|^2 dt +& \\
+\underbrace{\phi_\lambda'\left(\frac{s+\tau}{\beta}\right)(s+\tau)|\hat{v}(s,\xi)|^2}_{(C)} - \phi_\lambda'\left(\frac{\tau}{\beta}\right)\tau|\hat{v}(0,\xi)|^2 +& \\
\underbrace{-\int_0^s (t+\tau)[\hat{v}(t,\xi)\partial_t\bar{\hat{v}}(t,\xi) + \bar{\hat{v}}(t,\xi)\partial_t\hat{v}(t,\xi)]\sum_{i,j=1}^n a_{i,j}(t)\xi_i\xi_j dt}_{(D)} +& \\
-2\sum_{i=1}^n \xi_i \int_0^s (t+\tau)\Im\{b_i(t)\hat{v}(t,\xi)\partial_t\bar{\hat{v}}(t,\xi)\}dt +& \\
+2\int_0^s (t+\tau)\Re\{c(t)\hat{v}(t,\xi)\partial_t\bar{\hat{v}}(t,\xi)\}dt = 0,& \quad (30)
\end{aligned}
$$

where, to ease the following reasoning, some terms have been identified with capital letters from A to D. Terms (A) and (B) are positive and, since ϕ is strictly increasing, also (C) is positive. To obtain the final estimate, equation (30) needs to be slightly modified. In particular, extend functions $a_{i,j}$ to the whole real axis by setting $a_{i,j}(t) = a_{i,j}(0)$ for $t < 0$ and $a_{i,j}(t) = a_{i,j}(T)$ if $t > T$ and define

$$
a_{i,j}^\varepsilon(t) \triangleq (\rho_\varepsilon * a_{i,j})(t) = \int_{\mathbb{R}^n} \rho_\varepsilon(t-s)a_{i,j}(s)ds
$$

where ρ_ε is a $\mathscr{C}^\infty$ mollifier.

From (30), replacing $a_{i,j}(t)$ with $a_{i,j}(t)+a^{\varepsilon}_{i,j}(t)-a^{\varepsilon}_{i,j}(t)$ in (D), and integrating by parts, we get

$$
\begin{aligned}
&\underbrace{2\int_0^s (t+\tau)|\partial_t \hat{v}(t,\xi)|^2 dt}_{(E)} - \underbrace{\frac{\alpha^2}{2}|\xi|^4\left[\omega\left(\frac{1}{|\xi|^2+1}\right)\right]^2\int_0^s (t+\tau)|\hat{v}(t,\xi)|^2 dt}_{(F)} + \\
&\underbrace{-\alpha|\xi|^2\omega\left(\frac{1}{|\xi|^2+1}\right)\int_0^s (t+\tau)\phi'_\lambda\left(\frac{t+\tau}{\beta}\right)|\hat{v}(t,\xi)|^2 dt}_{(G)} + \\
&\underbrace{+\alpha|\xi|^2\omega\left(\frac{1}{|\xi|^2+1}\right)\sum_{i,j=1}^n \xi_i\xi_j\int_0^s (t+\tau)a_{i,j}(t)|\hat{v}(t,\xi)|^2 dt}_{(H)} + \\
&\underbrace{-\alpha|\xi|^2\omega\left(\frac{1}{|\xi|^2+1}\right)\int_0^s (t+\tau)c(t)|\hat{v}(t,\xi)|^2 dt}_{(I)} + \underbrace{\gamma\int_0^s |\hat{v}(t,\xi)|^2 dt}_{(L)} + \\
&\underbrace{-\gamma(s+\tau)|\hat{v}(s,\xi)|^2}_{(M)} + \underbrace{\int_0^s\left[-\phi''_\lambda\left(\frac{t+\tau}{\beta}\right)\left(\frac{t+\tau}{\beta}\right)-\phi'_\lambda\left(\frac{t+\tau}{\beta}\right)\right]|\hat{v}(t,\xi)|^2 dt}_{(N)} + \\
&\underbrace{-\phi'_\lambda\left(\frac{\tau}{\beta}\right)\tau|\hat{v}(0,\xi)|^2}_{(O)} + \underbrace{2\sum_{i,j=1}^n \xi_i\xi_j\int_0^s (t+\tau)\Re\{\hat{v}(t,\xi)\partial_t\bar{\hat{v}}(t,\xi)\}\widetilde{a}^{\varepsilon}_{i,j}(t)dt}_{(P)} + \\
&\underbrace{+\sum_{i,j=1}^n \xi_i\xi_j\int_0^s |\hat{v}(t,\xi)|^2\frac{\partial}{\partial t}[(t+\tau)a^{\varepsilon}_{i,j}(t)]dt}_{(Q)} + \underbrace{\tau\sum_{i,j=1}^n a^{\varepsilon}_{i,j}(0)\xi_i\xi_j|\hat{v}(0,\xi)|^2}_{(R)} + \\
&\underbrace{-(s+\tau)\sum_{i,j=1}^n a^{\varepsilon}_{i,j}(s)\xi_i\xi_j|\hat{v}(s,\xi)|^2}_{(S)} - \underbrace{2\sum_{i=1}^n \xi_i\int_0^s (t+\tau)\Im\{b_i(t)\hat{v}(t,\xi)\partial_t\bar{\hat{v}}(t,\xi)\}dt}_{(T)} \\
&\qquad + \underbrace{2\int_0^s (t+\tau)\Re\{c(t)\hat{v}(t,\xi)\partial_t\bar{\hat{v}}(t,\xi)\}dt}_{(U)} \le 0\,, \qquad (31)
\end{aligned}
$$

where $\widetilde{a}^{\varepsilon}_{i,j} = a^{\varepsilon}_{i,j} - a_{i,j}$ for all $i,j = 1,\dots,n$.

In the following each term is considered individually, beginning with (P). The properties of the modulus of continuity ω guarantee that there exists a constant C_0 such that

$$|a^{\varepsilon}_{i,j}(t) - a_{i,j}(t)| \le C_0\omega(\varepsilon)\,,$$

for all ε, for all i, for all j and for all t. Hence

$$\left|\sum_{i,j=1}^{n}[a_{i,j}^{\varepsilon}(t)-a_{i,j}(t)]\xi_i\xi_j\right| \leq \sum_{i,j=1}^{n}|a_{i,j}^{\varepsilon}(t)-a_{i,j}(t)||\xi_i\xi_j| \leq C_0 n^2\omega(\varepsilon)|\xi|^2\,,$$

where the property that, for all i, $|\xi_i|\leq|\xi|$ has been exploited. As a consequence, if

$$\varepsilon=\frac{1}{|\xi|^2+1}\,,$$

then

$$|(P)|\leq 2C_0n^2|\xi|^2\omega\left(\frac{1}{|\xi|^2+1}\right)\int_0^s(t+\tau)|\hat{v}(t,\xi)\partial_t\bar{\hat{v}}(t,\xi)|dt\,.$$

Young's inequality yields

$$|(P)|\leq\int_0^s(t+\tau)|\partial_t\hat{v}(t,\xi)|^2dt+C_0^2n^4|\xi|^4\left[\omega\left(\frac{1}{|\xi|^2+1}\right)\right]^2\int_0^s(t+\tau)|\hat{v}(t,\xi)|^2dt$$

and, consequently, since $\omega(s)\in[0,1]$ for all $s\in[0,1]$ and, in turn, $-\omega(s)^2>-\omega(s)$ for all $s\in[0,1]$,

$$(P)\geq-\underbrace{\int_0^s(t+\tau)|\partial_t\hat{v}(t,\xi)|^2dt}_{(P_1)}-\underbrace{C_0^2n^4|\xi|^4\omega\left(\frac{1}{|\xi|^2+1}\right)\int_0^s(t+\tau)|\hat{v}(t,\xi)|^2dt}_{(P_2)}\,.$$

Let us consider now the term (Q). For the properties of the modulus of continuity, there exists C_1 such that

$$|(a_{i,j}^{\varepsilon})'(t)|\leq C_1\frac{\omega(\varepsilon)}{\varepsilon}\,,$$

for all ε, for all i, for all j and for all t. As a consequence, if

$$\varepsilon=\frac{1}{|\xi|^2+1}\,,$$

then

$$\begin{aligned}(Q)=&\sum_{i,j=1}^{n}\xi_i\xi_j\int_0^s|\hat{v}(t,\xi)|^2(t+\tau)(a_{i,j}^{\varepsilon})'(t)dt+\\&+\sum_{i,j=1}^{n}\xi_i\xi_j\int_0^s|\hat{v}(t,\xi)|^2a_{i,j}^{\varepsilon}(t)dt\geq\\&\geq-\underbrace{C_1n^2|\xi|^2(|\xi|^2+1)\omega\left(\frac{1}{|\xi|^2+1}\right)\int_0^s(t+\tau)|\hat{v}(t,\xi)|^2dt}_{(Q_1)}+\\&+\underbrace{\sum_{i,j=1}^{n}\xi_i\xi_j\int_0^s a_{i,j}^{\varepsilon}(t)|\hat{v}(t,\xi)|^2dt}_{(Q_2)}\,.\end{aligned}$$

As far as the terms (T) and (U) are concerned,

$$(U)-(T) \geq -(U_1)-(U_2)-(T_1)-(T_2)\,,$$

where

$$(U_1)=2k_C^2\int_0^s (t+\tau)|\hat{v}(t,\xi)|^2dt\,,\quad (U_2)=\frac{1}{2}\int_0^s (t+\tau)|\partial_t\hat{v}(t,\xi)|^2dt\,,$$

$$(T_1)=2n^2k_B^2|\xi|^2\int_0^s (t+\tau)|\hat{v}(t,\xi)|^2dt\,,\quad (T_2)=\frac{1}{2}\int_0^s (t+\tau)|\partial_t\hat{v}(t,\xi)|^2dt\,.$$

Note, moreover, that

$$(H)\geq \alpha k_A|\xi|^4\omega\left(\frac{1}{|\xi|^2+1}\right)\int_0^s (t+\tau)|\hat{v}(t,\xi)|^2dt\,,$$

and

$$(Q_2)\geq k_A|\xi|^2\int_0^s |\hat{v}(t,\xi)|^2dt\,.$$

We claim now that there exist two positive constants α_1 and γ_1 such that, for all $\xi\in\mathbb{R}^n$,

$$\begin{aligned}\frac{\gamma_1}{4T}+\frac{\alpha_1}{2}k_A|\xi|^4\omega\left(\frac{1}{|\xi|^2+1}\right)-C_0^2n^4|\xi|^4\omega\left(\frac{1}{|\xi|^2+1}\right)+\\ -C_1n^2|\xi|^2\left(|\xi|^2+1\right)\omega\left(\frac{1}{|\xi|^2+1}\right)-2n^2k_B^2|\xi|^2-2k_C^2+\\ -\frac{\alpha_1^2}{2}|\xi|^4\left(\omega\left(\frac{1}{|\xi|^2+1}\right)\right)^2-\alpha_1|\xi|^2\omega\left(\frac{1}{|\xi|^2+1}\right)k_C\geq 0\,.\end{aligned}\tag{32}$$

In fact, from the properties of the modulus of continuity, we know that, for $|\xi|\geq 1$, the function

$$\xi\to|\xi|^2\omega\left(\frac{1}{|\xi|^2+1}\right)$$

is bounded from below by a positive constant. Consequently there exists $\alpha_1>0$ and $\xi_0\in\mathbb{R}^n$ such that, if $|\xi|>|\xi_0|$, then

$$\begin{aligned}\frac{\alpha_1}{4}k_A|\xi|^4\omega\left(\frac{1}{|\xi|^2+1}\right)-C_0^2n^4|\xi|^4\omega\left(\frac{1}{|\xi|^2+1}\right)+\\ -C_1n^2|\xi|^2\left(|\xi|^2+1\right)\omega\left(\frac{1}{|\xi|^2+1}\right)-2n^2k_B^2|\xi|^2-2k_C^2+\\ -\alpha_1|\xi|^2\omega\left(\frac{1}{|\xi|^2+1}\right)k_C\geq 0\,.\end{aligned}$$

We use now the fact that

$$\lim_{|\xi|\to+\infty}\omega\left(\frac{1}{|\xi|^2+1}\right)=0\,,$$

and, for the above chosen α_1, possibly taking a larger $|\xi_0|$, we have that, for all $|\xi| > |\xi_0|$,

$$\frac{\alpha_1}{4} k_A |\xi|^4 \omega\left(\frac{1}{|\xi|^2+1}\right) - \frac{\alpha_1^2}{2}|\xi|^4\left(\omega\left(\frac{1}{|\xi|^2+1}\right)\right)^2 \geq 0.$$

Finally, to obtain (32) for all $\xi \in \mathbb{R}^n$, it is sufficient to choose a suitable γ_1. We remark also that the inequality (32) remains true with α at the place of α_1, provided the choice of a possibly larger γ_1.

As a consequence, if $\alpha = \max\{\alpha_1, 1/T''\}$ and $\gamma \geq \gamma_1$, then

$$\frac{1}{2}(L) + \frac{1}{2}(H) - (P_2) - (Q_1) - (T_1) - (U_1) - (F) - (I) \geq 0\,. \tag{33}$$

By using (33) into (31) and taking into account that $(E) = (T_2) + (U_2) = (P_1)$ and that $(R) \geq 0$, we obtain

$$\begin{aligned}
&\frac{1}{2}(H) + (Q_2) - \underbrace{\alpha|\xi|^2\omega\left(\frac{1}{|\xi|^2+1}\right)\int_0^s (t+\tau)\phi_\lambda'\left(\frac{t+\tau}{\beta}\right)|\hat{v}(t,\xi)|^2 dt}_{(G)} + \frac{1}{2}(L) + \\
&- \underbrace{\gamma(s+\tau)|\hat{v}(s,\xi)|^2}_{(M)} + \underbrace{\int_0^s \left[-\phi_\lambda''\left(\frac{t+\tau}{\beta}\right)\left(\frac{t+\tau}{\beta}\right) - \phi_\lambda'\left(\frac{t+\tau}{\beta}\right)\right]|\hat{v}(t,\xi)|^2 dt}_{(N)} + \\
&- \underbrace{\phi_\lambda'\left(\frac{\tau}{\beta}\right)\tau|\hat{v}(0,\xi)|^2}_{(O)} - \underbrace{(s+\tau)\sum_{i,j=1}^n a_{i,j}^\varepsilon(s)\xi_i\xi_j|\hat{v}(s,\xi)|^2}_{(S)} \leq 0\,.
\end{aligned} \tag{34}$$

Recall, now, that ϕ_λ is a solution of equation (23) with $q = k_A$. Since $\omega(z)/z > 1$ for all $z \in (0,1)$, equation (23) implies

$$-\frac{1}{2}y\phi_\lambda''(y) > \frac{\lambda k_A}{2}\phi_\lambda'(y)\,, \qquad \text{for all } y \in (0,1)\,. \tag{35}$$

Hence, if ϕ_λ is solution of (23) with $\lambda > 2/k_A$,

$$(N) \geq -\frac{1}{2}\int_0^s \phi_\lambda''\left(\frac{t+\tau}{\beta}\right)\left(\frac{t+\tau}{\beta}\right)|\hat{v}(t,\xi)|^2 dt\,,$$

provided that $(t+\tau)/\beta \in (0,1)$ for all $t \in (0,s)$. Consider, now, the following two cases.

1. If

$$\phi_\lambda'\left(\frac{t+\tau}{\beta}\right) \leq \frac{(|\xi|^2+1)k_A}{4}\,,$$

then

$$(G) \leq \frac{1}{4}\alpha k_A|\xi|^2(|\xi|^2+1)\omega\left(\frac{1}{|\xi|^2+1}\right)\int_0^s (t+\tau)|\hat{v}(t,\xi)|^2 dt$$

and hence, if

$$\gamma > \bar{\gamma} \triangleq \max\left\{\gamma_1, 8T\alpha k_A \omega\left(\frac{1}{2}\right)\right\}, \tag{36}$$

then

$$\frac{1}{2}(H) + \frac{1}{4}(L) \geq (G).$$

In fact if $|\xi| > 1$, then

$$\frac{1}{4}\alpha k_A |\xi|^2(|\xi|^2+1)\omega\left(\frac{1}{|\xi|^2+1}\right)\int_0^s (t+\tau)|\hat{v}(t,\xi)|^2 dt \leq \frac{1}{2}(H).$$

If $|\xi| \leq 1$, then

$$(|\xi|^2+1)|\xi|^2\omega\left(\frac{1}{|\xi|^2+1}\right) \leq 2\omega\left(\frac{1}{2}\right)$$

and choosing γ according to (36) guarantees $(G) \leq (L)/4$.

2. On the contrary, if

$$\phi'_\lambda\left(\frac{t+\tau}{\beta}\right) > \frac{(|\xi|^2+1)k_A}{4},$$

then, since the function $h : (0,1) \to \mathbb{R}$ defined by $h(y) = \omega(y)/y$ is decreasing,

$$(|\xi|^2+1)\omega\left(\frac{1}{|\xi|^2+1}\right) = \frac{\omega\left(\frac{1}{|\xi|^2+1}\right)}{\frac{1}{|\xi|^2+1}} \leq$$

$$\leq \frac{\omega\left(\frac{k_A}{4\phi'_\lambda\left(\frac{t+\tau}{\beta}\right)}\right)}{\frac{k_A}{4\phi'_\lambda\left(\frac{t+\tau}{\beta}\right)}} = \frac{4}{k_A}\phi'_\lambda\left(\frac{t+\tau}{\beta}\right)\omega\left(\frac{k_A}{4\phi'_\lambda\left(\frac{t+\tau}{\beta}\right)}\right)$$

and, since ω is increasing,

$$(|\xi|^2+1)\omega\left(\frac{1}{|\xi|^2+1}\right) \leq \frac{4}{k_A}\phi'_\lambda\left(\frac{t+\tau}{\beta}\right)\omega\left(\frac{k_A}{\phi'_\lambda\left(\frac{t+\tau}{\beta}\right)}\right).$$

As a consequence, if ϕ_λ is solution of (23) with $\lambda > 4/k_A$, then

$$(N) \geq -\frac{1}{2}\int_0^s \phi''_\lambda\left(\frac{t+\tau}{\beta}\right)\left(\frac{t+\tau}{\beta}\right)|\hat{v}(t,\xi)|^2 dt =$$

$$= \frac{\lambda}{2}\int_0^s \phi'_\lambda\left(\frac{t+\tau}{\beta}\right)\left(\phi'_\lambda\left(\frac{t+\tau}{\beta}\right)\omega\left(\frac{k_A}{\phi'_\lambda\left(\frac{t+\tau}{\beta}\right)}\right)\right)|\hat{v}(t,\xi)|^2 dt \geq$$

$$\geq \frac{\lambda k_A}{8}(|\xi|^2+1)\omega\left(\frac{1}{|\xi|^2+1}\right)\int_0^s \phi'_\lambda\left(\frac{t+\tau}{\beta}\right)|\hat{v}(t,\xi)|^2 dt. \tag{37}$$

Moreover, if

$$\lambda > \bar{\lambda} \triangleq \max\left(\frac{4}{k_A}, \frac{16T\alpha}{k_A}\right),$$

then

$$(N) \geq \alpha(|\xi|^2+1)\omega\left(\frac{1}{|\xi|^2+1}\right)\int_0^s (t+\tau)\phi_\lambda'\left(\frac{t+\tau}{\beta}\right)|\hat{v}(t,\xi)|^2 dt \geq (G).$$

In conclusion, taking into account that $(N) \geq 0$, $(H) \geq 0$, $(L) \geq 0$ and $(G) \geq 0$, leads to the inequality

$$\frac{1}{2}(H) + \frac{1}{4}(L) + (N) - (G) \geq 0. \tag{38}$$

Furthermore, using (38) into (34) and taking into account that

$$\frac{1}{2}(Q_2) \geq \frac{1}{2}k_A|\xi|^2\int_0^s |\hat{v}(t,\xi)|^2 dt,$$

yields

$$\left(\frac{k_A|\xi|^2}{2} + \frac{\gamma}{4}\right)\int_0^s |\hat{v}(t,\xi)|^2 dt \leq$$
$$\leq \phi_\lambda'\left(\frac{\tau}{\beta}\right)\tau|\hat{v}(0,\xi)|^2 + (s+\tau)(\gamma + k_A^{-1}|\xi|^2)|\hat{v}(s,\xi)|^2. \tag{39}$$

Finally, substituting (25) into (39) yields

$$\frac{1}{4}\left(k_A|\xi|^2+\gamma\right)\int_0^s e^{(1-\alpha t)|\xi|^2\omega\left(\frac{1}{|\xi|^2+1}\right)} e^{2\gamma t} e^{-2\beta\phi_\lambda\left(\frac{t+\tau}{\beta}\right)}|\hat{u}(t,\xi)|^2 dt \leq$$
$$\leq \phi_\lambda'\left(\frac{\tau}{\beta}\right)\tau e^{|\xi|^2\omega\left(\frac{1}{|\xi|^2+1}\right)} e^{-2\beta\phi_\lambda\left(\frac{\tau}{\beta}\right)}|\hat{u}(0,\xi)|^2 +$$
$$+ (s+\tau)(\gamma + k_A^{-1}|\xi|^2)e^{(1-\alpha s)|\xi|^2\omega\left(\frac{1}{|\xi|^2+1}\right)} e^{2\gamma s} e^{-2\beta\phi_\lambda\left(\frac{s+\tau}{\beta}\right)}|\hat{u}(s,\xi)|^2. \tag{40}$$

Equation (40) holds for all $s \in (0,\sigma]$; choosing $s = \sigma$ one obtains (24). ■

5.3 An integral estimate

Proposition 5.3 provides a punctual estimate of the Fourier transform of u which will allow us to obtain, by integration, an analogously estimate on the norm of u. To obtain this result the following lemma and Definition 4.4 are accessory.

Lemma 5.4 *If $u \in \mathscr{H}_0$ is solution of (1), then there exists $\bar{\gamma}$, not depending on ξ, such that, for all ξ, $e^{2\bar{\gamma}t}|\hat{u}(t,\xi)|^2$ is (weakly) increasing in t.* □

Proof. We want to show that there exists $\bar{\gamma}$ such that

$$\partial_t(e^{2\bar{\gamma}t}\hat{u}(t,\xi)\bar{\hat{u}}(t,\xi)) \geq 0\,.$$

Note that

$$\partial_t(e^{2\bar{\gamma}t}\hat{u}(t,\xi)\bar{\hat{u}}(t,\xi)) = 2\bar{\gamma}e^{2\bar{\gamma}t}|\hat{u}(t,\xi)|^2 + \\ + e^{2\bar{\gamma}t}\partial_t(\hat{u}(t,\xi))\bar{\hat{u}}(t,\xi) + e^{2\bar{\gamma}t}\hat{u}(t,\xi)\partial_t(\bar{\hat{u}}(t,\xi))\,. \quad (41)$$

From (15), multiplying by $\bar{\hat{u}}(t,\xi)$ we obtain

$$\bar{\hat{u}}(t,\xi)\partial_t\hat{u}(t,\xi) = \\ = \sum_{i,j=1}^{n} a_{i,j}(t)\xi_i\xi_j|\hat{u}(t,\xi)|^2 - \imath\sum_{i=1}^{n} b_i(t)\xi_i|\hat{u}(t,\xi)|^2 + c(t)|\hat{u}(t,\xi)|^2$$

and also, taking in both term the complex conjugate values,

$$\hat{u}(t,\xi)\partial_t\bar{\hat{u}}(t,\xi) = \\ = \sum_{i,j=1}^{n} a_{i,j}(t)\xi_i\xi_j|\hat{u}(t,\xi)|^2 + \imath\sum_{i=1}^{n} \bar{b}_i(t)\xi_i|\hat{u}(t,\xi)|^2 + \bar{c}(t)|\hat{u}(t,\xi)|^2$$

and, consequently,

$$\partial_t(e^{2\bar{\gamma}t}\hat{u}(t,\xi)\bar{\hat{u}}(t,\xi)) = 2\bar{\gamma}e^{2\bar{\gamma}t}|\hat{u}(t,\xi)|^2 + 2e^{2\bar{\gamma}t}\sum_{i,j=1}^{n} a_{i,j}(t)\xi_i\xi_j|\hat{u}(t,\xi)|^2 + \\ + 2e^{2\bar{\gamma}t}\sum_{i=1}^{n}\Im\{b_i(t)\}\xi_i|\hat{u}(t,\xi)|^2 + 2e^{2\bar{\gamma}t}\Re\{c(t)\}|\hat{u}(t,\xi)|^2 \geq \\ 2e^{2\bar{\gamma}t}|\hat{u}(t,\xi)|^2(\bar{\gamma} + k_A|\xi|^2 - nk_B|\xi| - k_C)\,. \quad (42)$$

Now, if $|\xi| \geq nk_B/k_A$, then $k_A|\xi|^2 > nk_B|\xi|$ and hence, if $\bar{\gamma} > k_C$, we have

$$\bar{\gamma} + k_A|\xi|^2 - nk_B|\xi| - k_C \geq 0\,.$$

On the other hand, if $|\xi| < nk_B/k_A$, then $-|\xi| > -nk_B/k_A$ and hence $-nk_B|\xi| > -n^2k_B^2/k_A$. In conclusion, the claim holds for any $\bar{\gamma}$ such that $\bar{\gamma} > 2\max\{k_C, n^2k_B^2/k_A\}$. ■

Let us, now, come back to inequality (24). By integrating it with respect to ξ, the following result can be obtained.

Proposition 5.5 *Let σ and τ be as in Proposition 5.3. Set $\bar{\sigma} \triangleq \sigma/8$. There exists $C > 0$ such that, whenever $u \in \mathscr{H}_0$ is a solution of (1), with $\mathscr{L}$ fulfilling Assumption 4.1, one has, for all $\beta \geq \sigma + \tau$,*

$$\sup_{z\in[0,\bar{\sigma}]} \|u(z,\cdot)\|^2_{H^1_{\frac{1}{2},\omega}} \leq \\ \leq Ce^{-\sigma\phi'\left(\frac{\sigma+\tau}{\beta}\right)}\left[\phi'\left(\frac{\tau}{\beta}\right)e^{-2\beta\phi\left(\frac{\tau}{\beta}\right)}\|u(0,\cdot)\|^2_{H^0_{1,\omega}} + \|u(\sigma,\cdot)\|^2_{H^1}\right]\,, \quad (43)$$

where $\phi = \phi_{\bar{\lambda},k_A}$ with $\bar{\lambda}$ given by Proposition 5.3. The constant C depends no n, k_A, k_B, k_C, ω, T and T''. □

Proof. In the hypotheses of the stetement, Proposition 5.3 guarantees the existence of σ, α, γ and ϕ_λ such that (24) holds. The integrand function in (24) is positive and, consequently, the term on the left hand side can be bounded from below by integrating on an interval contained in $[0,\sigma]$. Let $\tau \leq \sigma/4$ and let z be a value such that $0 < z \leq \bar{\sigma}$; we have

$$[z, 2z+\tau] \subset [0, \sigma/2]\,;$$

by integrating with respect to ξ and taking into account that, since $\sigma = 1/\alpha$,

$$1 - \alpha t \geq 1 - \alpha \frac{\sigma}{2} \geq \frac{1}{2}\,,$$

for all $t \in [0, \sigma/2]$, one obtains

$$\begin{aligned} \frac{1}{4}\int_{\mathbb{R}^n} \left(k_A|\xi|^2+\gamma\right) e^{\frac{1}{2}|\xi|^2\omega\left(\frac{1}{|\xi|^2+1}\right)} \int_z^{2z+\tau} e^{2\gamma t} e^{-2\beta\phi_\lambda\left(\frac{t+\tau}{\beta}\right)} |\hat{u}(t,\xi)|^2 dt d\xi \leq \\ \leq \tau\phi_\lambda'\left(\frac{\tau}{\beta}\right) e^{-2\beta\phi_\lambda\left(\frac{\tau}{\beta}\right)} \int_{\mathbb{R}^n} e^{|\xi|^2\omega\left(\frac{1}{|\xi|^2+1}\right)} |\hat{u}(0,\xi)|^2 d\xi + \\ + (\sigma+\tau) e^{2\gamma\sigma} e^{-2\beta\phi_\lambda\left(\frac{\sigma+\tau}{\beta}\right)} \int_{\mathbb{R}^n} (\gamma + k_A^{-1}|\xi|^2)|\hat{u}(\sigma,\xi)|^2 d\xi\,. \end{aligned} \tag{44}$$

Now, let $\bar{\bar{\gamma}}$ be a value of γ fulfilling equation (36), let $\bar{\gamma}$ be the value provided by Lemma 5.4 and let

$$\gamma > \max\{\bar{\bar{\gamma}}, \bar{\gamma}\}\,.$$

Since ϕ_λ is increasing, we have that

$$e^{-2\beta\phi_\lambda\left(\frac{t+\tau}{\beta}\right)} \geq e^{-2\beta\phi_\lambda\left(\frac{2(z+\tau)}{\beta}\right)}$$

for all $t < 2z+\tau$. As a consequence, using also the fact that $e^{2\gamma z} \geq 1$, equation (44) yields

$$\begin{aligned} c_1(z+\tau)\int_{\mathbb{R}^n} (|\xi|^2+1) e^{\frac{1}{2}|\xi|^2\omega\left(\frac{1}{|\xi|^2+1}\right)} |\hat{u}(z,\xi)|^2 d\xi \leq \\ \leq \tau\phi_\lambda'\left(\frac{\tau}{\beta}\right) e^{2\beta\left[\phi_\lambda\left(\frac{2(z+\tau)}{\beta}\right)-\phi_\lambda\left(\frac{\tau}{\beta}\right)\right]} \int_{\mathbb{R}^n} e^{|\xi|^2\omega\left(\frac{1}{|\xi|^2+1}\right)} |\hat{u}(0,\xi)|^2 d\xi + \\ + c_2(\sigma+\tau) e^{2\gamma\sigma} e^{2\beta\left[\phi_\lambda\left(\frac{2(z+\tau)}{\beta}\right)-\phi_\lambda\left(\frac{\sigma+\tau}{\beta}\right)\right]} \int_{\mathbb{R}^n} (1+|\xi|^2)|\hat{u}(\sigma,\xi)|^2 d\xi\,, \end{aligned} \tag{45}$$

where the constant values

$$c_1 \triangleq \frac{1}{4}\min\{k_A, \gamma\}\,, \qquad c_2 \triangleq \max\left\{\gamma, k_A^{-1}\right\}$$

have been introduced. Dividing by τ and taking into account that $(z+\tau)/\tau > 1$ and that ϕ_λ is negative, it is easy to see that (45) implies

$$\begin{aligned} c_1\|u(z,\cdot)\|^2_{H^1_{\frac{1}{2},\omega}} &\le \phi'_\lambda\left(\frac{\tau}{\beta}\right)e^{2\beta\left[\phi_\lambda\left(\frac{2(z+\tau)}{\beta}\right)-\phi_\lambda\left(\frac{\tau}{\beta}\right)\right]}\|u(0,\cdot)\|^2_{H^0_{1,\omega}}+\\ &\quad + c_2\frac{\sigma+\tau}{\tau}e^{2\gamma\sigma}e^{2\beta\left[\phi_\lambda\left(\frac{2(z+\tau)}{\beta}\right)-\phi_\lambda\left(\frac{\sigma+\tau}{\beta}\right)\right]}\|u(\sigma,\cdot)\|^2_{H^1}\le\\ &\le \phi'_\lambda\left(\frac{\tau}{\beta}\right)e^{2\beta\left[\phi_\lambda\left(\frac{2(z+\tau)}{\beta}\right)-\phi_\lambda\left(\frac{\tau}{\beta}\right)-\phi_\lambda\left(\frac{\sigma+\tau}{\beta}\right)\right]}\|u(0,\cdot)\|^2_{H^0_{1,\omega}}+\\ &\quad + c_2\frac{\sigma+\tau}{\tau}e^{2\gamma\sigma}e^{2\beta\left[\phi_\lambda\left(\frac{2(z+\tau)}{\beta}\right)-\phi_\lambda\left(\frac{\sigma+\tau}{\beta}\right)\right]}\|u(\sigma,\cdot)\|^2_{H^1}\,, \end{aligned} \tag{46}$$

Moreover, with respect to ϕ_λ, note that since ϕ_λ is increasing,

$$2(z+\tau)\le\frac{\sigma}{2}+\tau \quad\Rightarrow\quad \phi_\lambda\left(\frac{2(z+\tau)}{\beta}\right)\le\phi_\lambda\left(\frac{\frac{\sigma}{2}+\tau}{\beta}\right).$$

In addition, since ϕ_λ is also concave,

$$\phi_\lambda\left(\frac{\sigma+\tau}{\beta}\right)-\phi_\lambda\left(\frac{\frac{\sigma}{2}+\tau}{\beta}\right)\ge\frac{\sigma}{2\beta}\phi'_\lambda\left(\frac{\sigma+\tau}{\beta}\right).$$

As a consequence, from (46) one obtains

$$\begin{aligned} &c_1\|u(z,\cdot)\|^2_{H^1_{\frac{1}{2},\omega}}\le\\ &\le e^{-\sigma\phi'_\lambda\left(\frac{\sigma+\tau}{\beta}\right)}\left[\phi'_\lambda\left(\frac{\tau}{\beta}\right)e^{-2\beta\phi_\lambda\left(\frac{\tau}{\beta}\right)}\|u(0,\cdot)\|^2_{H^0_{1,\omega}}+c_2\frac{\sigma+\tau}{\tau}e^{2\gamma\sigma}\|u(\sigma,\cdot)\|^2_{H^1}\right], \end{aligned} \tag{47}$$

namely

$$\begin{aligned} &\|u(z,\cdot)\|^2_{H^1_{\frac{1}{2},\omega}}\le\\ &\qquad\le Ce^{-\sigma\phi'_\lambda\left(\frac{\sigma+\tau}{\beta}\right)}\left[\phi'_\lambda\left(\frac{\tau}{\beta}\right)e^{-2\beta\phi_\lambda\left(\frac{\tau}{\beta}\right)}\|u(0,\cdot)\|^2_{H^0_{1,\omega}}+\|u(\sigma,\cdot)\|^2_{H^1}\right], \end{aligned} \tag{48}$$

where

$$C=\max\left\{\frac{1}{c_1},\frac{c_2(\sigma+\tau)e^{2\gamma\sigma}}{c_1\tau}\right\}.$$

Equation (48) holds for all $z\in[0,\bar{\sigma}]$ and hence equation (43) immediately follows. ■

5.4 Proof of Theorem 4.6

Proposition 5.5 states, in particular, that the norm of u in any insatant of the sub-interval $[0,\bar{\sigma}]\subset[0,\sigma]$ is bounded by a quantity depending on the value of the norm in the initial and final instants, i.e. on $\|u(0,\cdot)\|_{H^0_{1,\omega}}$ and $\|u(\sigma,\cdot)\|_{H^1}$. Nevertheless, to obtain a stability result, the right hand side term in equation (48) must tend to zero when $\|u(0,\cdot)\|_{H^0_{1,\omega}}$ tends to zero, which is not immediate to guess. The following lemma allows one to choose β in such a way that (48) can be written in a form from which the stability property can be obtained more easily.

Lemma 5.6 *Let ϕ be a solution of (23) with $\lambda > 0$ and $q > 0$ and let $\tau > 0$. Let $h :]0,1[\to]q,+\infty[$ be defined by*

$$h(z) \triangleq e^{-2\tau\phi(z)/z}\phi'(z)\,.$$

The function h so defined is strictly decreasing with

$$\lim_{z\to 0} h(z) = +\infty\,, \quad \lim_{z\to 1} h(z) = q\,.$$

□

Proof. The claim is easily proven by computing h'. ■

As a consequence of Lemma 5.6, h can be inverted and its inverse $h^{-1} :]q,+\infty[\to]0,1[$ is strictly increasing and

$$\lim_{y\to+\infty} h^{-1}(y) = 0\,.$$

Now the main stability result can be proven.

Proof of Theorem 4.6. In (43) of Proposition 5.5 we want to choose $\beta > \sigma + \tau$ in such a way that

$$\phi'\left(\frac{\tau}{\beta}\right) e^{-2\beta\phi\left(\frac{\tau}{\beta}\right)} = \|u(0,\cdot)\|^{-2}_{H^0_{1,\omega}}\,.$$

This goal is achieved by taking

$$\beta = \frac{\tau}{h^{-1}\left(\|u(0,\cdot)\|^2_{H^0_{1,\omega}}\right)}\,,$$

provided that $\|u(0,\cdot)\|_{H^0_{1,\omega}} < q^{-1/2}$ and $\|u(0,\cdot)\|_{H^0_{1,\omega}} < h\left(\frac{\tau}{\sigma+\tau}\right)^{-1/2}$. With this choice of β, one obtains, from (43),

$$\sup_{z\in[0,\bar{\sigma}]} \|u(z,\cdot)\|^2_{H^1_{\frac{1}{2},\omega}} \leq C e^{-\sigma\widehat{g}\left(\|u(0,\cdot)\|^2_{H^0_{1,\omega}}\right)} \left[1 + \|u(\sigma,\cdot)\|^2_{H^1}\right]\,, \tag{49}$$

where $\widehat{g}$ is defined by

$$\widehat{g}(y) = \phi'\left(\frac{\sigma+\tau}{\tau} h^{-1}(y^{-1})\right)\,, \tag{50}$$

so that

$$\lim_{y\to 0} \widehat{g}(y) = +\infty\,.$$

Note, in particular, that taking $\tau = \sigma/4$ the condition $\|u(0,\cdot)\|_{H^0_{1,\omega}} < h(\tau/(\sigma+\tau))^{-1/2}$ yields $\|u(0,\cdot)\|_{H^0_{1,\omega}} < \widehat{\rho}$ where

$$\widehat{\rho} \triangleq \min\{e^{-\tau\frac{4}{5}\phi\left(\frac{5}{4}\right)}\phi'\left(\frac{5}{4}\right)^{1/2}, q^{-1/2}\}.$$

Note, now, that

$$\|u(z,\cdot)\|^2_{H^1} \le \|u(z,\cdot)\|^2_{H^1_{\frac{1}{2},\omega}} \tag{51}$$

and that, for all $\nu > 0$ and all $\varepsilon > 0$, there exists $\widetilde{C}_{\nu,\varepsilon}$ such that

$$\|u(0,\cdot)\|^2_{H^0_{1,\omega}} \le \widetilde{C}_{\nu,\varepsilon}\|u(0,\cdot)\|^2_{H^0_{\nu,\varepsilon}}.$$

It follows that

$$\sup_{z\in[0,\bar{\sigma}]} \|u(z,\cdot)\|^2_{H_1} \le Ce^{-\sigma\widehat{g}\left(\widetilde{C}_{\nu,\varepsilon}\|u(0,\cdot)\|^2_{H^0_{\nu,\varepsilon}}\right)}\left[1+\|u(\sigma,\cdot)\|^2_{H^1}\right], \tag{52}$$

provided that

$$\|u(0,\cdot)\|_{H^0_{\nu,1}} < \frac{\widehat{\rho}}{C^{1/2}_{\nu,\varepsilon}}.$$

By defining $g(y) = \widetilde{g}(\widetilde{C}_{\nu,\varepsilon}y)$, equation (52) allows one to easily obtain (12). ■

The claim of Theorem 4.6 to the whole interval $[0,T]$.

5.5 Proof of Theorem 4.2

Theorem 4.2 is proven iterating a finite number of times the estimate given by the following lemma.

Lemma 5.7 *Under the same hypotheses of Theorem 4.6,*

$$\sup_{z\in[0,\bar{\sigma}/2]} \|u(z,\cdot)\|_{L^2} \le C'Ce^{-\sigma g\left(C''\|u(0,\cdot)\|^2_{L^2}\right)}\left[1+\|u(\sigma,\cdot)\|^2_{L^2}\right].$$

The constants C' and C'' depend on n, k_A, k_B, k_C, ν, ε and σ and tend to $+\infty$ as σ tends to zero. □

Proof. Analogously to Lemma 5.2, extend $a_{i,j}$, b_i and c on $[-\sigma/2,T]$ and u to a solution of $\mathscr{L}$ on $[-\sigma/2,T]$. Then the results of Theorem 4.6 on $[-\bar{\sigma}/2, T-\bar{\sigma}/2]$ gives

$$\sup_{z\in[-\bar{\sigma}/2,\bar{\sigma}/2]} \|u(z,\cdot)\|^2_{H_1} \le Ce^{-\sigma g\left(\|u(-\bar{\sigma}/2,\cdot)\|^2_{H^0_{\nu,\varepsilon}}\right)}\left[1+\|u(\sigma-\bar{\sigma}/2)\|^2_{H_1}\right].$$

By Lemma 5.2 we obtain

$$\sup_{z\in[0,\bar{\sigma}/2]} \|u(z,\cdot)\|_{L^2}^2 \le Ce^{-\sigma g\left(C''\|u(0,\cdot)\|_{L^2}^2\right)} \left[1+\|u(\sigma-\frac{\sigma}{16},\cdot)\|_{H^1}^2\right] \le$$

$$\le C'Ce^{-\sigma g\left(C''\|u(0,\cdot)\|_{L^2}^2\right)} \left[1+\|u(\sigma,\cdot)\|_{L^2}^2\right] . \quad (53)$$

■

Now set $G(y) \triangleq (1+D)C'Ce^{-\sigma g(C''y)}$ and note that $\lim_{y\to 0} G(y) = 0$. We have just proven that

$$\sup_{z\in[0,\bar{\sigma}/2]} \|u(z,\cdot)\|_{L^2}^2 \le G\left(\|u(0,\cdot)\|_{L^2}^2\right) . \quad (54)$$

Finally, let $T' : 0 < T' < T$; take $T'' = (T+T')/2$ (so that $T' < T'' < T$). Note that $\bar{\sigma}/2 = \sigma/16$ and recall that $\sigma = \min\{1/\alpha_1, T''\}$. To complete the proof of Theorem 4.2 it is sufficient to iterate inequality (54) a finite number of times. Indeed, set $T_0 = 0$ and, for $i \ge 0$,

$$T_{i+1} = T_i + \frac{1}{16}\min\left\{\frac{1}{\alpha_1}, T'' - T_i\right\} .$$

For all i inequality (54) provides

$$\sup_{z\in[T_i,T_{i+1}]} \|u(z,\cdot)\|_{L^2}^2 \le G_i\left(\|u(T_i,\cdot)\|_{L^2}^2\right) .$$

The result follows by noting that

$$T_{i+1} - T_i = \frac{1}{16}\min\left\{\frac{1}{\alpha_1}, T'' - T_i\right\} ,$$

and that, for all j

$$T_{j+1} = \sum_{i=0}^{j} \frac{1}{16}\min\left\{\frac{1}{\alpha_1}, T'' - T_i\right\} .$$

The sequence $\left\{T_j\right\}_{j\in\mathbb{N}}$ is increasing and bounded from above by T''; hence it admits a limit. Let this limit be T^*; we want to show that $T^* = T''$. Obviously, $T^* \le T''$; suppose that $T^* < T''$, then $T'' - T_i \ge T'' - T^* > 0$ and, consequently,

$$T_{j+1} \ge \sum_{i=0}^{j} \frac{1}{16}\min\left\{\frac{1}{\alpha_1}, T'' - T^*\right\}$$

for all j, yielding $\lim_{j\to\infty} T_j = +\infty$, which is a contradiction. Therefore it must be $T^* = T''$ which means that $T_j > T'$ for some j. ■

6 A specific case

In this section the explicit expressions of the functions g and G appearing in the statements of 4.6 and 4.2 respectively is computed when the modulus of continuity $\omega :]0, e^{1-e}] \to \mathbb{R}$ is defined by

$$\omega(s) = s(1-\log s)\log(1-\log s) . \quad (55)$$

Note that ω is increasing and fulfils the Osgood condition, but LogLog-Lipschitz continuity is strictly weaker than Log-Lipschitz continuity.

The computations below are not just a straightforward application of theorems 4.6 and 4.2 with ω given by (55). Indeed, the specific form of ω allows a more effective handling of inequality (5.5) and yields a slightly different function g. The explicit form of the "new" g is particularly well suited for the iteration procedure which in turn gives G, since it "reproduces itself" after each iteration step, up to some possible changes in the constants.

To begin with, we need to revisit Section 5.1 in the light of (55). The function $\theta : [e^{e-1}, +\infty[\to [0, +\infty[$ is now defined by

$$\theta(\tau) = \int_{1/\tau}^{e^{1-e}} \frac{1}{\omega(s)} ds = \log(\log(1 + \log\tau))$$

and the function $\psi_{\lambda,q} :]0,1] \to [e^{e-1}, +\infty[$ is defined by

$$\psi_{\lambda,q}(y) = \theta^{-1}(-\lambda q \log y) = \exp(e^{y^{-\lambda q}} - 1)\,. \tag{56}$$

From the definition of $\psi_{\lambda,q}$, one can easily check that it is strictly decreasing and that

$$\psi'_{\lambda,q}(y) = \exp\left(e^{y^{-\lambda q}} - 1\right) e^{y^{-\lambda q}} (-\lambda q) y^{-\lambda q - 1} = -\frac{\lambda q}{y} \left(\psi_{\lambda,q}(y)\right)^2 \omega\left(\frac{1}{\psi_{\lambda,q}(y)}\right), \tag{57}$$

hence the function $\phi_{\lambda,q} :]0,1] \to]-\infty, 0]$ defined by

$$\phi_{\lambda,q}(y) = -q \int_y^1 \psi_{\lambda,q}(z) dz$$

is such that

$$\phi''_{\lambda,q}(y) = -\frac{\lambda}{y} \left(\phi'_{\lambda,q}(y)\right)^2 \omega\left(\frac{q}{\phi'_{\lambda,q}}(y)\right)$$

i.e. $\phi_{\lambda,q}$ is a solution of equation (23). Note, as an accessory result, that

$$\phi'_{\lambda,q}(y) = q\phi_{\lambda,q}(y) \geq q e^{e-1}\,. \tag{58}$$

We need also to introduce the function $\Lambda : [0, +\infty[\to]-\infty, 0]$ defined by

$$\Lambda(y) = y\phi_\lambda\left(\frac{1}{y}\right) \tag{59}$$

which is strictly decreasing and, hence, invertible. Its inverse, $\Lambda^{-1} :]-\infty, 0] \to [1, +\infty[$ is also strictly decreasing. We have the following

Lemma 6.1 *The functions ψ_{λ,k_A} and Λ are such that*

$$\lim_{\zeta \to +\infty} \frac{\psi_{\lambda,k_A}\left(\frac{1}{\zeta}\right)}{|\Lambda(\zeta)|} = +\infty\,.$$

Proof. Note that

$$\lim_{\zeta\to+\infty}\frac{\psi_{\lambda,k_A}\left(\frac{1}{\zeta}\right)}{|\Lambda(\zeta)|}=\lim_{\rho\to 0}-\frac{\rho\,\psi_{\lambda,k_A}(\rho)}{\phi_{\lambda,k_A}(\rho)}=\lim_{\rho\to 0}-\frac{\psi_{\lambda,k_A}(\rho)+\rho\,\psi'_{\lambda,k_A(\rho)}}{k_A\,\psi_{\lambda,k_A}(\rho)}=$$

$$=-\frac{1}{k_A}-\lim_{\rho\to 0}\frac{\rho\,\psi'_{\lambda,k_A}(\rho)}{k_A\,\psi_{\lambda,k_A}(\rho)}=-\frac{1}{k_A}+\lim_{\rho\to 0}\frac{1}{k_A}\lambda\,\psi_{\lambda,k_A}(\rho)\,\omega\left(\frac{1}{\psi_{\lambda,k_A}(\rho)}\right)=$$

$$=-\frac{1}{k_A}+\lim_{q\to 0}\frac{\lambda}{k_A}\frac{\omega(q)}{q}=-1+\lim_{q\to 0}(1-\log q)\log(1-\log q)=+\infty.$$

■

From now on, we choose $q=k_A$ and $\lambda\geq\bar{\lambda}$ as in the proof of Proposition 5.3, and for the sake of a simpler notation, we write ϕ_λ and ψ_λ instead of $\phi_{\lambda,q}$ and $\psi_{\lambda,q}$, respectively. Proposition 5.5 and (58) then, for $\beta\geq\sigma+\tau$, give

$$\sup_{z\in[0,\bar{\sigma}]}\|u(z,\cdot)\|^2_{L^2}\leq Ce^{-\sigma\phi'_\lambda\left(\frac{\sigma+\tau}{\beta}\right)}\phi'_\lambda\left(\frac{\tau}{\beta}\right)\left[e^{-2\beta\phi_\lambda\left(\frac{\tau}{\beta}\right)}\|u(0,\cdot)\|^2_{H^0_{1,\omega}}+\|u(\sigma,\cdot)\|^2_{H^1}\right], \tag{60}$$

where $\bar{\sigma}=\sigma/8$. Arguing as in Lemma 5.7 one obtains

$$\sup_{z\in[0,\bar{\sigma}/2]}\|u(z,\cdot)\|^2_{L^2}\leq Ce^{-\sigma\phi'_\lambda\left(\frac{\sigma+\tau}{\beta}\right)}\phi'_\lambda\left(\frac{\tau}{\beta}\right)\left[e^{-2\beta\phi_\lambda\left(\frac{\tau}{\beta}\right)}\|u(0,\cdot)\|^2_{L^2}+\|u(\sigma,\cdot)\|^2_{L^2}\right] \tag{61}$$

with a possibly larger C.

We can now state and prove the refinement of the local stability estimate of Theorem 4.6:

Theorem 6.2 *Let ω be as in (55) and let the operator $\mathscr{L}$ fulfil Assumption 4.1. Let $D>0$. There exists $\alpha_1>0$ and, for any T'', $0<T''<T$, there exist constants $\rho>0$ and $0<\delta<1$, such that, if $u\in\mathscr{H}_0$ is a solution of*

$$\mathscr{L}u=0, \tag{62}$$

with $\sup_{t\in[0,T]}\|u(t,\cdot)\|^2_{L^2}\leq D$ and $\|u(0,\cdot)\|^2_{L^2}<\rho$, then

$$\sup_{z\in[0,\bar{\sigma}/2]}\|u(z,\cdot)\|^2_{L^2}\leq\exp\left\{-\frac{\sigma k_A}{2e}\exp\left[\left(\log\left(\frac{1}{\tau}|\log\|u(0,\cdot)\|_{L^2}|\right)\right)^{\delta}\right]\right\} \tag{63}$$

where $\sigma=\min\{T'',1/\alpha_1\}$, $0<\tau\leq\sigma/4$ and $\bar{\sigma}=\sigma/8$. The constant α_1 depends only on k_A,k_B,k_C,ω and n, while the constant δ depends also on T and T'' and ρ depends also on T and T'' and D.

Proof. Set

$$\beta=\tau\Lambda^{-1}\left(\frac{1}{\tau}\log\|u(0,\cdot)\|_{L^2}\right) \tag{64}$$

so that

$$e^{-2\tau\Lambda\left(\frac{\beta}{\tau}\right)} = \|u(0,\cdot)\|_{L^2}^{-2}.$$

Notice that the value of β is larger than $\sigma+\tau$ if and only if

$$\|u(0,\cdot)\|_{L^2} < e^{\tau\Lambda\left(\frac{\sigma+\tau}{\tau}\right)} \triangleq \rho .$$

In particular, if $\tau = \sigma/4$ then $\rho = e^{\tau\Lambda(5/4)}$ (however, we show below that a smaller value of τ performs better). Note, now, that for $\zeta > 1$ and $y < 1/\zeta$

$$\log\left(\psi_{\lambda,q}(\zeta y)\right) = \left(\log\left(\psi_{\lambda,q}(y)\right)+1\right)^{\zeta^{-\lambda q}} - 1 ;$$

therefore

$$\phi_\lambda'\left(\frac{\sigma+\tau}{\beta}\right) = \frac{k_A}{e}\exp\left[\left(\log\left(\psi_{\lambda,k_A}\left(\frac{\tau}{\beta}\right)\right)+1\right)^{\delta}\right], \tag{65}$$

where $\delta = ((\sigma+\tau)/\tau)^{-\lambda k_A}$. From (61), (64) and (65) one obtains

$$\begin{aligned}\sup_{z\in[0,\bar\sigma/2]} \|u(z,\cdot)\|_{L^2}^2 \le C k_A \psi_{\lambda,k_A}\left(\frac{1}{\Lambda\left(\frac{1}{\tau}\log\|u(0,\cdot)\|_{L^2}\right)}\right)\times \\ \times\exp\left\{-\frac{\sigma k_A}{e}\exp\left[\left(\log\left(\psi_{\lambda,k_A}\left(\frac{1}{\Lambda^{-1}\left(\frac{1}{\tau}\log\|u(0,\cdot)\|_{L^2}\right)}\right)\right)+1\right)^{\delta}\right]\right\}\times \\ \times\left(1+\|u(\sigma,\cdot)\|_{L^2}^2\right).\end{aligned} \tag{66}$$

Consider, now, the function F defined by

$$F(\zeta) \triangleq (1+D)Ck_A\zeta\exp\left\{-\frac{\sigma k_A}{2e}\exp\left[(\log\zeta+1)^{\delta}\right]\right\}$$

and note that

$$\lim_{\zeta\to+\infty} F(\zeta) = 0 .$$

Indeed, let $\varepsilon > 0$. It is easy to check that

$$\begin{aligned}F(\zeta) < \varepsilon \Leftrightarrow \exp\left\{-\frac{\sigma k_A}{2e}\exp\left[(\log\zeta+1)^{\delta}\right]\right\} \le \frac{\varepsilon\zeta^{-1}}{Ck_A(1+D)} \Leftrightarrow \\ \Leftrightarrow -\frac{\sigma k_A}{2e}\exp\left[(\log\zeta+1)^{\delta}\right] \le -\log\zeta + \log\frac{\varepsilon}{Ck_A(1+D)} \Leftrightarrow \\ \Leftrightarrow \frac{\sigma k_A}{2e}\exp\left[(\log\zeta+1)^{\delta}\right] \ge \log\zeta - \log\frac{\varepsilon}{Ck_A(1+D)} \Leftrightarrow \\ \Leftrightarrow \exp\left[(\log\zeta+1)^{\delta}\right] \ge \frac{2e}{\sigma k_A}\log\zeta - \frac{2e}{\sigma k_A}\log\frac{\varepsilon}{Ck_A(1+D)} \Leftrightarrow \\ \Leftrightarrow (\log\zeta+1)^{\delta} \ge \log\left(\frac{2e}{\sigma k_A}\log\zeta - \frac{2e}{\sigma k_A}\log\frac{\varepsilon}{Ck_A(1+D)}\right),\end{aligned}$$

which is true for sufficiently large ζ. It follows that, for sufficiently small $\|u(0,\cdot)\|_{L^2}$, one has

$$(1+D)Ck_A\,\psi_{\lambda,k_A}\left(\frac{1}{\Lambda^{-1}\left(\frac{1}{\tau}\log\|u(0,\cdot)\|_{L^2}\right)}\right)\times$$

$$\times\exp\left\{-\frac{\sigma k_A}{2e}\exp\left[\left(\log\left(\psi_{\lambda,k_A}\left(\frac{1}{\Lambda^{-1}\left(\frac{1}{\tau}\log\|u(0,\cdot)\|_{L^2}\right)}\right)\right)+1\right)^{\delta}\right]\right\}\leq 1\,.$$

So, if $\|u(0,\cdot)\|_{L^2}\leq\widetilde{\rho}$ for a suitable $\widetilde{\rho}$, one has

$$\sup_{z\in[0,\bar{\sigma}/2]}\|u(z,\cdot)\|^2_{L^2}\leq\exp\left\{-\frac{\sigma k_A}{2e}\exp\left[\left(\log\left(\psi_{\lambda,k_A}\left(\frac{1}{\Lambda^{-1}\left(\frac{1}{\tau}\log\|u(0,\cdot)\|\right)}\right)\right)+1\right)^{\delta}\right]\right\}. \tag{67}$$

Now thanks to Lemma 6.1 we have

$$\lim_{\zeta\to+\infty}\frac{\psi_{\lambda,k_A}\left(\frac{1}{\zeta}\right)}{|\Lambda(\zeta)|}=+\infty$$

and therefore for $\|u(0,\cdot)\|_{L^2}$ sufficiently small we get

$$\psi_{\lambda,k_A}\left(\frac{1}{\Lambda^{-1}\left(\frac{1}{\tau}\left(\log\|u(0,\cdot)\|_{L^2}\right)\right)}\right)\geq\frac{1}{\tau}\left|\log\|u(0,\cdot)\|_{L^2}\right|\,.$$

As a consequence, (67) yields

$$\sup_{z\in[0,\bar{\sigma}/2]}\|u(z,\cdot)\|^2_{L^2}\leq\exp\left\{-\frac{\sigma k_A}{2e}\exp\left[\left(\log\left(\frac{1}{\tau}\left|\log\|u(0,\cdot)\|_{L^2}\right|\right)\right)^{\delta}\right]\right\}. \tag{68}$$

The proof is complete. ■

Finally, we state and prove the refinement of the global stability estimate of Theorem 4.2:

Theorem 6.3 *Let ω be as in (55) and let the operator $\mathscr{L}$ fulfil Assumption 4.1. Then for any $0<T'<T$ and $D>0$ there exist constants $\widetilde{\sigma},\widetilde{\tau},\widetilde{\rho}>0$ and $0<\widetilde{\delta}<1$, such that, if $u\in\mathscr{H}_0$ is a solution of*

$$\mathscr{L}u=0\,, \tag{69}$$

with $\sup_{t\in[0,T]}\|u(t,\cdot)\|^2_{L^2}\leq D$ and $\|u(0,\cdot)\|^2_{L^2}<\widetilde{\rho}$, then

$$\sup_{z\in[0,T']}\|u(z,\cdot)\|^2_{L^2}\leq\exp\left\{-\frac{\widetilde{\sigma}k_A}{2e}\exp\left[\left(\log\left(\frac{1}{2\widetilde{\tau}}\left|\log\|u(0,\cdot)\|^2_{L^2}\right|\right)\right)^{\widetilde{\delta}}\right]\right\}. \tag{70}$$

The constants $\widetilde{\sigma}$, $\widetilde{\tau}$, $\widetilde{\delta}$ and $\widetilde{\rho}$ depend only on k_A,k_B,k_C,ω,n, T,T' and D.

Proof. Fix $T' < T'' < T$. Setting $\sigma_1 := \sigma = \min\{T'', 1/\alpha_1\}$, $\bar{\sigma}_1 := \bar{\sigma}/2 = \sigma_1/16$, $\delta_1 := \delta$, and choosing

$$\tau_1 = \min\left\{\frac{\sigma_1}{4}, \frac{\sigma_1 k_A}{4e}\right\}$$

we can rewrite (63) as

$$\sup_{z\in[0,\bar{\sigma}_1]} \|u(z,\cdot)\|_{L^2}^2 \leq \exp\left\{-\frac{\sigma_1 k_A}{2e}\exp\left[\left(\log\left(\frac{1}{2\tau_1}\left|\log\|u(0,\cdot)\|_{L^2}^2\right|\right)\right)^{\delta_1}\right]\right\}. \quad (71)$$

Repeating the above arguments on the interval $[\bar{\sigma}_1, T]$, we find

$$\sup_{z\in[\bar{\sigma}_1,\bar{\sigma}_2]} \|u(z,\cdot)\|_{L^2}^2 \leq \exp\left\{-\frac{\sigma_2 k_A}{2e}\exp\left[\left(\log\left(\frac{1}{2\tau_2}\left|\log\|u(\bar{\sigma}_1,\cdot)\|_{L^2}^2\right|\right)\right)^{\delta_2}\right]\right\}, \quad (72)$$

where $\sigma_2 = \min\{1/\alpha_1, T'' - \sigma_1\}$, $\bar{\sigma}_2 = \sigma_2/16$ and $\tau_2 = \min\{\sigma_2/4, \sigma_2 k_A/4e\}$. Notice that $\sigma_2 \leq \sigma_1$ and $\tau_2 \leq \tau_1$. As a consequence,

$$\sup_{z\in[\bar{\sigma}_1,\bar{\sigma}_2]} \|u(z,\cdot)\|_{L^2}^2 \leq$$

$$\exp\left\{-\frac{\sigma_2 k_A}{2e}\exp\left[\left(\log\left|\frac{1}{2\tau_2}\log\left(\exp\left\{-\frac{\sigma_1 k_A}{2e}\exp\left[\left(\log\frac{\left|\log\|u(0,\cdot)\|_{L^2}^2\right|}{2\tau_1}\right)^{\delta_1}\right]\right\}\right)\right|\right)^{\delta_2}\right]\right\} =$$

$$= \exp\left\{-\frac{\sigma_2 k_A}{2e}\exp\left[\left(\log\left|-\frac{\sigma_1 k_A}{4e\tau_2}\exp\left[\left(\log\frac{\left|\log\|u(0,\cdot)\|_{L^2}^2\right|}{2\tau_1}\right)^{\delta_1}\right]\right|\right)^{\delta_2}\right]\right\} =$$

$$= \exp\left\{-\frac{\sigma_2 k_A}{2e}\exp\left[\left(\log\frac{\sigma_1 k_A}{4e\tau_2} + \left(\log\frac{\left|\log\|u(0,\cdot)\|_{L^2}^2\right|}{2\tau_1}\right)^{\delta_1}\right)^{\delta_2}\right]\right\} \leq$$

$$\leq \exp\left\{-\frac{\sigma_2 k_A}{2e}\exp\left[\left(\log\frac{1}{2\tau_1}\left|\log\|u(0,\cdot)\|_{L^2}^2\right|\right)^{\delta_1\delta_2}\right]\right\},$$

where the last inequality holds since $\sigma_1 k_A \geq 4e\tau_2$. Merging the estimates obtained for the two intervals, yields

$$\sup_{[0,\bar{\sigma}_2]} \|u(z,\cdot)\|_{L^2}^2 \leq \exp\left\{-\frac{\sigma_2 k_A}{2e}\exp\left[\left(\log\frac{1}{2\tau_1}\left|\log\|u(0,\cdot)\|_{L^2}^2\right|\right)^{\delta_1\delta_2}\right]\right\},$$

which has the same form of the inequality obtained in $[0, \bar{\sigma}_1]$. Hence, iterating the arguments above a finite number of times one obtains an estimate on $[0, T']$ of the form

$$\sup_{[0,T']} \|u(z,\cdot)\|_{L^2}^2 \leq \exp\left\{-\frac{\widetilde{\sigma}_2 k_A}{2e}\exp\left[\left(\log\frac{1}{2\tilde{\tau}}\left|\log\|u(0,\cdot)\|_{L^2}^2\right|\right)^{\tilde{\delta}}\right]\right\}.$$

The proof is complete. ∎

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