Upper Semicontinuous Representability of Maximal Elements for Nontransitive Preferences

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Abstract
We characterize the possibility of determining all the maximal elements for a preorder on a topological space by maximizing all the functions in a suitable family of upper semicontinuous order-preserving functions. We discuss the possibility of extending this result to the case of a quasi-preorder (i.e., a reflexive and Suzumura consistent binary relation) on a topological space.

Keywords Order-preserving function · Weak utility · Maximal element · Suzumura consistency · Quasi-preorder · Upper semicontinuous function

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1 Introduction

While it is not frequently cited in the literature, *White's theorem* [1] is important since, for every maximal element relative to a preorder on any nonempty set, it guarantees the existence of an order-preserving function attaining its maximum precisely at that maximal element, provided that an order-preserving function exists. So, at least theoretically, every maximal element is obtained by maximizing a real-valued order-preserving function. When this happens, clearly, every maximal element is *potentially optimal* according to the definition presented by [2], i.e., for every maximal element, there exists a total preorder extending the original preorder with respect to which such maximal element is best preferred.

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On the other hand, we notice that the mere existence of an order-preserving function for a (reflexive) binary relation does not imply transitivity, but the much more general property of *Suzumura consistency* (see [3,4] and the arguments and results in the more recent paper [5]). [6, Theorem 3.1] generalized the aforementioned White's theorem, by proving that, for every maximal element relative to a preorder on any nonempty set, there exists a *weak utility* for the strict part of the preorder attaining its maximum at that maximal element, provided that a weak utility for the strict part exists.

In this paper, we first generalize White's theorem to the *upper semicontinuous case*. This means that we present necessary and sufficient conditions on a preorder on a topological space under which, for every maximal element relative to the preorder, there exists an upper semicontinuous order-preserving function attaining its maximum at such maximal element, provided that an upper semicontinuous order-preserving function exists. Such a problem has been already tackled by [7]. It should be noted that [6] already characterized the property, according to which every maximal element relative to a preorder on a compact topological space can be obtained by maximizing a *transfer weakly upper continuous* weak utility for its strict part (see the generalization of Weierstrass theorem presented in [8]).

Everyone may agree on the importance of such results, due to the well-known fact that every upper semicontinuous (more generally, transfer weakly upper continuous) function attains its maximum on a compact topological space, and the nearly obvious consideration that a point at which an order-preserving function for a preorder (or, more generally, a *weak utility* for its strict part) attains its maximum is also a maximal element for the preorder.

We show that these results are not generalizable to the case of a *quasi-preorder* (i.e., a *reflexive* and *Suzumura consistent* binary relation). In particular, we show that an order-preserving function for a quasi-preorder cannot attain its maximum at any maximal element which is not at the same time a maximal element for its *transitive closure*. Therefore, the best we can do is to apply our results to the transitive closure of a quasi-preorder on a topological space.

2 Notation and Preliminary Results

Let X be a nonempty set (*decision space*). A binary relation $\precsim$ on X is interpreted as *a weak preference relation*, and therefore, for any two elements $x, y \in X$, the scripture "$x \precsim y$" has to be thought of as "the alternative $y \in X$ is at least as preferable as $x \in X$". As usual, $\prec$ denotes the *strict part* of a binary relation $\precsim$ [i.e., for all $x, y \in X$, $x \prec y$ if and only if $(x \precsim y)$ *and not*$(y \precsim x)$]. The *transitive closure* $\precsim^{\mathrm{T}}$ of a binary relation $\precsim$ on a set X (see, e.g., [3,4]) is defined as follows for every $x, y \in X$:

$$x \precsim^{\mathrm{T}} y \Leftrightarrow \exists n \in \mathbb{N}^{+},\ \exists x_0, x_1, \ldots, x_n \in X : \ [x = x_0 \precsim x_1 \precsim \ldots \precsim x_{n-1} \precsim x_n = y].$$

It should be noted that the strict part $\prec^{\mathrm{T}}$ of the transitive closure $\precsim^{\mathrm{T}}$ of a binary relation $\precsim$ on a set X is defined as follows for every $x, y \in X$:

$$x \prec^{\mathrm{T}} y \Leftrightarrow (x \precsim^{\mathrm{T}} y)\ \mathit{and}([x = x_0 \precsim x_1 \precsim \ldots \precsim x_{n-1} \precsim x_n = y] \Rightarrow \\ x_i \prec x_{i+1} \text{ for some } i \in \{0, 1, \ldots, n-1\}).$$

We say that a binary relation $\precsim$ on X is a *quasi-preorder* because, following the definition presented by [5], $\precsim$ is *reflexive*, i.e., $x \precsim x$ for every $x \in X$, and *Suzumura consistent* [3], i.e., for all $x, y \in X$, $x \precsim^{\mathrm{T}} y$ implies *not*$(y \prec x)$. [4, Lemma 3] proved the following result.

Proposition 2.1 *A binary relation $\precsim$ on X is Suzumura consistent if and only if $\precsim \subseteq \precsim^{\mathrm{T}}$ and $\prec \subseteq \prec^{\mathrm{T}}$.*

Therefore, a binary relation is Suzumura consistent if and only if it is "extended" by its transitive closure. Denote by $\sim$ the *indifference relation* associated with a quasi-preorder $\precsim$ on set X [i.e., for all $x, y \in X$, $x \sim y$ if and only if $(x \precsim y)$ *and* $(y \precsim x)$]. Clearly, $\sim$ is not transitive in general.

A *preorder* (i.e., a reflexive and *transitive* binary relation) is in particular a quasi-preorder. An *antisymmetric* preorder $\precsim$ is referred to as an *order*. It is well known that $\precsim^{\mathrm{T}}$ is the smallest transitive binary relation containing $\precsim$.

We have that $\sim$ is an *equivalence relation* on X whenever $\precsim$ is a preorder on X. In this case, we denote by $\precsim_{|\sim}$ the *quotient order* on the *quotient set* $X_{|\sim}$ (i.e., for all $x, y \in X$, $[x] \precsim_{|\sim} [y]$ if and only if $x \precsim y$, where $[x] = \{z \in X : z \sim x\}$ is the *indifference class* associated with $x \in X$).

In the general case of a quasi-preorder $\precsim$ on a set X, we set, for every $x \in X$,

$$l_{\precsim}(x) = \{z \in X : z \prec x\}, \quad i_{\precsim}(x) = \{z \in X : x \precsim z\}.$$

Therefore, $l_{\precsim}(x)$ is the *strict lower section*, and $i_{\precsim}(x)$ the *weak upper section* associated with the element $x \in X$.

Given a *quasi-preordered set* $(X, \precsim)$, a point $x_0 \in X$ is said to be a *maximal element of* $(X, \precsim)$ if for no $z \in X$ it occurs that $x_0 \prec z$. In the sequel, we shall denote by $X_M^{\precsim}$ the set of all the maximal elements of a quasi-preordered set $(X, \precsim)$. Please observe that $X_M^{\precsim}$ can be empty. The following proposition is an immediate consequence of Proposition 2.1.

Proposition 2.2 *If $\precsim$ is a Suzumura consistent binary relation on a set X, then $X_M^{\precsim^{\mathrm{T}}} \subset X_M^{\precsim}$.*

Denote by $\bowtie$ the *incomparability relation* associated with a quasi-preorder $\precsim$ on a set X [i.e., for all $x, y \in X$, $x \bowtie y$ if and only if *not*$(x \precsim y)$ *and not*$(y \precsim x)$].

We recall that a function $u : (X, \precsim) \longrightarrow (\mathbb{R}, \leq)$ is said to be

1. $\precsim$-*isotonic* or $\precsim$-*increasing* if, for all $x, y \in X$, $x \precsim y \Rightarrow u(x) \leq u(y)$;
2. a *weak utility for* $\prec$ if, for all $x, y \in X$, $x \prec y \Rightarrow u(x) < u(y)$;
3. $\precsim$-*strictly isotonic* or $\precsim$-*order-preserving* if it is both $\precsim$-isotonic and a weak utility for $\prec$.

Strictly isotonic functions on $(X, \precsim)$ are also called *Richter–Peleg representations* of $\precsim$ in the economic literature (see, e.g., [9,10]). [5, Proposition 1] proved the following proposition.

Proposition 2.3 *If there exists a* $\precsim$*-order-preserving function* $u : (X, \precsim) \longrightarrow (\mathbb{R}, \leq)$ *for a reflexive binary relation* $\precsim$ *on* X*, then* $\precsim$ *is a quasi-preorder.*

By putting together Proposition 2.1 and Proposition 2.3, we immediately get the following proposition.

Proposition 2.4 *If there exists a* $\precsim$*-order-preserving function* $u' : (X, \precsim) \longrightarrow (\mathbb{R}, \leq)$ *for a (reflexive) binary relation* $\precsim$ *on* X*, then every* $\precsim^{\mathrm{T}}$*-order-preserving function* $u : (X, \precsim^{\mathrm{T}}) \longrightarrow (\mathbb{R}, \leq)$ *(respectively, weak utility for* $\prec^{\mathrm{T}}$*) is also a* $\precsim$*-order-preserving function (respectively, weak utility for* $\prec$*).*

The following proposition, whose straightforward proof is omitted for the sake of brevity, will be used in the sequel.

Proposition 2.5 *Let* $\precsim$ *be a quasi-preorder on a set* X*. If* $u : (X, \precsim) \longrightarrow (\mathbb{R}, \leq)$ *is a* $\precsim$*-order-preserving function, then* u *does not attain its maximum at any point* $x_0 \in X_M^{\precsim} \setminus X_M^{\precsim^{\mathrm{T}}}$.

As usual, for a real-valued function u on X, we denote by arg max u the set of all the points $x \in X$ such that u attains its maximum at x (i.e., $\arg\max u = \{x \in X : u(z) \leq u(x) \text{ for all } z \in X\}$).

The aim of the following example is to illustrate the fact that the consideration of quasi-preorders instead of preorders leads to important complications, loosely speaking.

Example 2.1 Consider the binary relation $\precsim$ on the real interval $X = [0, 1]$ defined as follows:

$$x \precsim y \Leftrightarrow \begin{cases} x, y \in [0, \frac{1}{2}] \\ \text{or} \\ (x \leq y) \text{ and } (x, y \in [\frac{1}{2}, 1]) \\ \text{or} \\ (x \leq \frac{1}{4}) \text{ and } (y \geq \frac{3}{4}). \end{cases}$$

Clearly, $\precsim$ is reflexive. Notice that $x \bowtie y$ for all $x, y \in [0, 1]$ such that $x \leq \frac{1}{4}$ and $y \in]\frac{1}{2}, \frac{3}{4}[$, and respectively $x \in]\frac{1}{4}, \frac{1}{2}[$ and $y \in]\frac{1}{2}, 1]$. We have that $\precsim$ is not transitive since, for example, $\frac{1}{3} \sim \frac{1}{2} \precsim \frac{3}{5}$ but $\frac{1}{3} \bowtie \frac{3}{5}$. On the other hand, it is easily seen that the following function $u : [0, 1] \longrightarrow [0, \frac{1}{2}]$ is a $\precsim$-order-preserving function, so that $\precsim$ is Suzumura consistent (actually a quasi-preorder) on $[0, 1]$ by Proposition 2.3:

$$u(x) = \begin{cases} 0, & \text{if } x \in [0, \frac{1}{2}[, \\ x - \frac{1}{2}, & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$$

Notice that $x \sim y$ for all $x, y \in [0, \frac{1}{2}]$, and that $X_M^{\precsim} =]\frac{1}{4}, \frac{1}{2}[\cup \{1\}$ and $X_M^{\precsim^{\mathrm{T}}} = \{1\}$. Clearly, the above function u attains its maximum at $1 \in X_M^{\precsim^{\mathrm{T}}}$. On the other hand, there cannot exist any $\precsim$-order-preserving function on $(X, \precsim)$ such that $\arg\max u =]\frac{1}{4}, \frac{1}{2}[= X_M^{\precsim} \setminus X_M^{\precsim^{\mathrm{T}}}$, since $x \sim \frac{1}{2} \prec 1$ for every $x \in]\frac{1}{4}, \frac{1}{2}[$, implying that $u(x) < u(1)$.

A preorder $\precsim$ on a topological space (X, τ) is said to be *upper semicontinuous* if $i_{\precsim}(x) = \{z \in X : x \precsim z\}$ is a closed subset of X for every element $x \in X$. [11, Theorem 1] proved the classical and very frequently cited result according to which an upper semicontinuous preorder on a compact topological space admits a maximal element. From [12], a preorder $\precsim$ on a topological space (X, τ) is said to be *quasi upper semicontinuous* if there exists an upper semicontinuous preorder $\lesssim$ on (X, τ) such that $\prec \subset <$. The concept of quasi upper semicontinuity can be naturally extended to quasi-preorders.

Denote by τ_{nat} the *natural* (*interval*) topology on the real line $\mathbb{R}$. We recall that, if (X, τ) is a topological space, then a real-valued function $u : (X, \tau) \longrightarrow (\mathbb{R}, \tau_{nat})$ is said to be upper semicontinuous if one of the following equivalent conditions holds:

1. For every real number α, the set $u^{-1}(]-\infty, \alpha[) = \{x \in X : u(x) < \alpha\}$ is an open subset of X;
2. For every point $x \in X$, and every net $\{x_d\}_{d \in D}$ such that $u(x) \leq u(x_d)$ for all $d \in D$, the following implication holds:

$$\lim_{d \in D} x_d = x \Rightarrow \lim_{d \in D} u(x_d) \leq u(x);$$

3. For every point $x_0 \in X$, the following property holds:

$$u(x_0) \geq \limsup_{x \to x_0} u(x).$$

A very well-known theorem guarantees that every upper semicontinuous function on a compact topological space attains its maximum.

3 Maximal Elements of Quasi-Preorders from Maximization of Upper Semicontinuous Functions

The following theorem was proved by [1]. Given any maximal element x_0 relative to a preorder $\precsim$ on a set X, it guarantees the existence of some $\precsim$-order-preserving function u attaining its maximum at x_0, provided that a $\precsim$-order-preserving function u' on $(X, \precsim)$ exists. Therefore, in order to determine all the maximal elements of a preorder $\precsim$ on a set X, the agent maximizes all the functions u in a family $\mathcal{U}$ of bounded order-preserving functions for $\precsim$. Needless to say, this is a very important fact from the point of view of computational aspects.

Theorem 3.1 ([1]) *Let $(X, \precsim)$ be a preordered set and assume that there exists an order-preserving function $u' : (X, \precsim) \longrightarrow (\mathbb{R}, \leq)$. If $X_M^{\precsim}$ is nonempty, then for every $x_0 \in X_M^{\precsim}$ there exists an order-preserving function $u : (X, \precsim) \longrightarrow (\mathbb{R}, \leq)$ such that arg max $u = [x_0] = \{z \in X : z \sim x_0\}$.*

Clearly, the previous theorem can be applied to the transitive closure $\precsim^{\mathrm{T}}$ of any quasi-preorder $\precsim$. It is important to notice that, by Proposition 2.5, the previous theorem cannot be extended to the case of a quasi-preorder (see also Example 2.1). Indeed, no $\precsim$-order-preserving function can attain its maximum at any maximal element of $(X, \precsim)$ which is not, at the same time, a maximal element of $(X, \precsim^{\mathrm{T}})$.

We now present a generalization of the above theorem to the "upper semicontinuous transitive case."

Theorem 3.2 *Let $(X, \tau, \precsim)$ be a topological preordered space, and assume that $X_M^{\precsim}$ is nonempty. Consider an element $x_0 \in X_M^{\precsim}$. Then, the following conditions are equivalent:*

(i) *There exists an upper semicontinuous $\precsim$-order-preserving function $u' : (X, \tau, \precsim) \longrightarrow (\mathbb{R}, \tau_{nat}, \leq)$ and $[x_0] = \{z \in X : z \sim x_0\}$ is a closed subset of X;*
(ii) *There exists an upper semicontinuous $\precsim$-order-preserving function $u : (X, \tau, \precsim) \longrightarrow (\mathbb{R}, \tau_{nat}, \leq)$ such that*

$$arg\ max\, u = [x_0] = \{z \in X : z \sim x_0\}.$$

Proof Consider a topological preordered space $(X, \tau, \precsim)$.
(i) $\Rightarrow$ (ii). Let u' be an upper semicontinuous $\precsim$-order-preserving function on $(X, \tau, \precsim)$. Without loss of generality, we can assume u' to be bounded. Consider a point $x_0 \in X_M^{\precsim}$ and define the function u as follows for any choice of a positive real number δ:

$$u(x) = \begin{cases} u'(x), & \text{if } not(x \sim x_0), \\ \sup\, u'(X) + \delta, & \text{if } x \sim x_0. \end{cases} \quad (1)$$

[1, Theorem 1] already proved that the above function u is order-preserving for $\precsim$ as soon as u' is order-preserving for $\precsim$. For the sake of completeness, let us recall here the arguments supporting this consideration. Clearly, we have that $u'(x) \leq u(x)$ for all $x \in X$.

In order to show that u is increasing with respect to $\precsim$, consider any two points $x, y \in X$ such that $x \precsim y$. If $y \sim x_0$, then, clearly, $u(x) \leq u(y)$ from the definition of u. On the other hand, if $not(y \sim x_0)$, then it must be also $not(x \sim x_0)$, since $x_0 \sim x \precsim y$ would imply $x_0 \precsim y$, and in turn $x_0 \sim y$ due to the fact that x_0 is a maximal element relative to $\precsim$. Hence, since neither $x \sim x_0$ nor $y \sim x_0$, we have that $u(x) = u'(x) \leq u'(y) = u(y)$ from the definition of u and the fact that u' is increasing with respect to $\precsim$.

In order to show that u is a weak utility for $\prec$, consider any two points $x, y \in X$ such that $x \prec y$. Then, we have that $not(x \sim x_0)$, since $x_0 \sim x \prec y$ implies that $x_0 \prec y$ (a contradiction, since x_0 is assumed to be a maximal element for $\precsim$). Therefore, from the definition of u and the fact that u' is a weak utility for $\prec$, we have that $u(x) = u'(x) < u'(y) \leq u(y)$, which obviously implies that $u(x) < u(y)$.

Further, u attains its maximum at x_0 and actually, since δ is a positive real number, arg max $u = [x_0] = \{z \in X : z \sim x_0\}$. Therefore, it only remains to show that under our assumptions u is upper semicontinuous.

Clearly, u is upper semicontinuous at every point $x \in X$ such that $x \sim x_0$. Therefore, consider any point $x \in X$ such that $not(x \sim x_0)$, and $\alpha \in \mathbb{R}$ such that $u(x) < \alpha$. We can limit our considerations to the case when $\alpha \leq \sup\, u'(X) + \delta$. Since in this case $u(x) = u'(x)$, from upper semicontinuity of u' there exists an open set U_x' containing x

such that $u'(z) < \alpha$ for every $z \in U'_x$. Since $[x_0]$ is closed, we have that $U_x = U'_x \setminus [x_0]$ is an open set containing x such that $u'(z) = u(z) < \alpha$ for every $z \in U'_x$. Hence, u is an upper semicontinuous function.

(ii) $\Rightarrow$ (i). Assume that there exists an upper semicontinuous order-preserving function $u : (X, \tau, \precsim) \longrightarrow (\mathbb{R}, \tau_{nat}, \leq)$ such that $\arg\max u = [x_0] = \{z \in X : z \sim x_0\}$. If $[x_0]$ is not closed, then there exists an element $z \in X \setminus [x_0]$ such that $U_z \cap [x_0] \neq \emptyset$ for every neighborhood U_z of the element z. But this contradicts the fact that u is upper semicontinuous, since in this case $u^{-1}(]-\infty, u(x_0)[)$, an open set containing z, should contain an element $z' \in [x_0]$, for which $u(z') = u(x_0)$. This consideration completes the proof. □

Remark 3.1 It is clear that Theorem 3.2 generalizes White's theorem, due to the fact that these two results precisely coincide when we consider the discrete topology τ on X.

From the above considerations, the best we can do when dealing with a quasi-preorder $\precsim$ on a topological space (X, τ) is to apply Theorem 3.2 to the transitive closure $\precsim^{\mathrm{T}}$ of $\precsim$.

Corollary 3.1 *Let $(X, \tau, \precsim)$ be a topological quasi-preordered space, and assume that $X_M^{\precsim^{\mathrm{T}}}$ is nonempty. Consider an element $x_0 \in X_M^{\precsim^{\mathrm{T}}}$. Then, the following conditions are equivalent:*

(i) *There exists an upper semicontinuous $\precsim^{\mathrm{T}}$-order-preserving function $u' : (X, \tau, \precsim^{\mathrm{T}}) \longrightarrow (\mathbb{R}, \tau_{nat}, \leq)$ and $[x_0] = \{z \in X : z \sim^{\mathrm{T}} x_0\}$ is a closed subset of X;*
(ii) *There exists an upper semicontinuous $\precsim^{\mathrm{T}}$-order-preserving function $u : (X, \tau, \precsim^{\mathrm{T}}) \longrightarrow (\mathbb{R}, \tau_{nat}, \leq)$ such that*

$$\arg\max u = [x_0] = \{z \in X : z \sim^{\mathrm{T}} x_0\}.$$

Since in order to determine a maximal element relative to a preorder $\precsim$ on a set X it suffices to maximize a weak utility for the strict part $\prec$ of $\precsim$, the following corollary can be considered as useful. Indeed, the reader can easily verify that the implication "(i) $\Rightarrow$ (ii)" in Theorem 3.2 is still valid if one considers weak utilities for $\prec$ instead of order-preserving functions u', u.

Corollary 3.2 *Let $(X, \tau, \precsim)$ be a topological quasi-preordered space with τ a compact topology. If there exists an upper semicontinuous weak utility u' for $\prec^{\mathrm{T}}$, and $[x] = \{z \in X : z \sim^{\mathrm{T}} x\}$ is a closed set for all $x \in X_M^{\precsim^{\mathrm{T}}}$, then $X_M^{\precsim^{\mathrm{T}}}$ is nonempty and for every $x_0 \in X_M^{\precsim^{\mathrm{T}}}$ there exists an upper semicontinuous weak utility u for $\prec$ such that $\arg\max u = [x_0] = \{z \in X : z \sim^{\mathrm{T}} x_0\}$.*

[12, Theorem 2.11] proved that there exists an upper semicontinuous weak utility for the strict part $\prec$ of a quasi upper semicontinuous preorder $\precsim$ on a *second countable* (i.e., with a *countable base*) topological space (X, τ). Hence, we finally get the following corollary.

Corollary 3.3 *Let $(X, \tau, \precsim)$ be a topological quasi-preordered space. Assume that τ is compact and second countable, and that $\precsim^{\mathrm{T}}$ is quasi upper semicontinuous. If $[x] = \{z \in X : z \sim^{\mathrm{T}} x\}$ is a closed set for all $x \in X_M^{\precsim^{\mathrm{T}}}$, then $X_M^{\precsim^{\mathrm{T}}}$ is nonempty and for every $x_0 \in X_M^{\precsim^{\mathrm{T}}}$ there exists an upper semicontinuous weak utility u for $\prec$ such that $\arg\max u = [x_0] = \{z \in X : z \sim^{\mathrm{T}} x_0\}$.*

4 Conclusions

In this paper, we have presented a characterization of the existence of an upper semicontinuous $\precsim$-order-preserving function u attaining its maximum at any maximal element x_0 of a preorder $\precsim$ on a topological space (X, τ), provided that an upper semicontinuous $\precsim$-order-preserving function u' exists. We have addressed the problems posed by the consideration of reflexive and Suzumura consistent (therefore nontransitive in general) binary relations, when trying to extend this result to the case of nontransitive and nontotal binary relations.

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