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Yang–Lee zeros of the Yang–Lee model

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Abstract

To understand the distribution of the Yang–Lee zeros in quantum integrable field theories we analyse the simplest of these systems given by the 2D Yang–Lee model. The grand-canonical partition function of this quantum field theory, as a function of the fugacity z and the inverse temperature β , can be computed in terms of the thermodynamics Bethe Ansatz based on its exact S -matrix. We extract the Yang–Lee zeros in the complex plane by using a sequence of polynomials of increasing order N in z which converges to the grand-canonical partition function. We show that these zeros are distributed along curves which are approximate circles as it is also the case of the zeros for purely free theories. There is though an important difference between the interactive theory and the free theories, for the radius of the zeros in the interactive theory goes continuously to zero in the high-temperature limit $\beta \rightarrow 0$ while in the free theories it remains close to 1 even for small values of β , jumping to 0 only at $\beta = 0$.

Keywords: integrable quantum field theory, thermodynamics Bethe Ansatz, Yang–Lee zeros

(Some figures may appear in colour only in the online journal)

1. Introduction

Many physical quantities reveal their deeper structure by going to the complex plane. This is the case, for instance, of the analytic properties of the scattering amplitudes where the angular momentum is not longer restricted to be an integer but allowed to take any complex value giving rise in this way to the famous Regge poles [1]. Another famous example is the Yang–Lee theory of equilibrium phase transitions [2, 3] based on the zeros of the grand-canonical partition function in the complex plane of the fugacity: in a nutshell, the main observation of Yang and Lee was that the zeros of the grand-canonical partition functions in the thermodynamic limit usually

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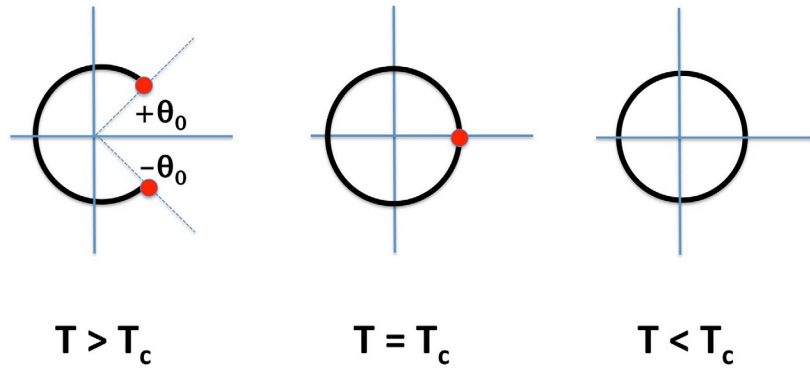


Figure 1. Distribution of the Yang–Lee zeros for the Ising model in the complex plane of the fugacity z .

accumulate in certain regions or curves of the complex plane, with their positive local density $\eta(z)$ which changes by changing the temperature; if at a critical value T_c , the zeros accumulate and pinch a positive value of the real axis, this is what may mark the onset of a phase transition.

As we are going to discuss extensively through the rest of the paper, the pattern of zeros of grand-canonical partition functions can be generally quite interesting and this study alone is a source of many stimulating physical and mathematical questions. If the study of the patterns of Yang–Lee zeros is then the first topic of this paper, the Yang–Lee model (and its zeros!) is our second main topic. In order to introduce such a model and present the work of this paper in its proper perspective, we need to talk about the pattern of zeros of just one particular statistical system: the Ising model. In [3] Yang and Lee showed that for ferromagnetic Ising-like models, independently on the dimensionality and regularity of the lattice and also largely independently on the nature of the couplings, the zeros of the Ising model lie on the unit circle⁵ in the complex plane of the variable $z = e^{-2\beta h}$ (where $\beta = 1/(kT)$ and h is the external magnetic field): posing $z = e^{i\theta}$, they have the following structure (see figure 1):

- for $T > T_c$ the zeros are placed along a ‘C’, namely a symmetric arc around $\theta = \pi$ whose edges are at $\pm\theta_0(T)$;
- at $T = T_c$ these edges move to the real axis and pinch it;
- for $T < T_c$ the zeros densely cover the entire circle.

Kortman and Griffiths [9] were the first to notice that the density of the Yang–Lee zeros of the Ising model nearby the edges $\pm\theta_0(T)$ gives rise to a problem which has its own interest since such a density presents an anomalous behaviour with a scaling law ruled by a critical exponent σ

$$\eta(\theta, T) \sim |\theta - \theta_0(T)|^\sigma, \quad T > T_c. \quad (1)$$

Such a behavior is closely analogous to the usual critical phenomena (although in this case triggered by a purely imaginary magnetic field ih) and therefore Fisher [10] posed the question about its effective quantum field theory and argued that, in sufficiently high dimension d , this consists of a ϕ^3 Landau–Ginzburg theory for the scalar field $\phi(x)$ with euclidean action given by

$$\mathcal{A} = \int d^d x \left[\frac{1}{2} (\partial\phi)^2 + i(h - h_0)\phi + ig \phi^3 \right]. \quad (2)$$

⁵This circle-theorem was later extended by many authors to ferromagnetic Ising model of arbitrarily high spin and with many-body spin interactions [4–8].

This action is what defines the Yang–Lee model, i.e. the quantum field theory studied in this paper, and the imaginary couplings present in such an action is what makes the Yang–Lee model a non-hermitian theory; the theory is however invariant under a CP transformations [11, 13] and therefore its spectrum is real. At criticality, i.e. when $h = h_0$, the corresponding fixed point of the renormalization Group presents only one relevant operator, namely the field $\phi(x)$ itself: in two dimensions Cardy showed [12] that all these properties are encoded into the simplest non-unitarity minimal model of Conformal Field Theories which has central charge $c = -22/5$ and only one relevant operator of conformal dimension $\Delta = -1/5$. Still in two dimensions, Cardy and Mussardo [11] then argued that the action (2), regarded as deformation of this minimal conformal model by means of its relevant operator $\phi(x)$ corresponds to an integrable quantum field theory. This fact has far-reaching consequences: indeed, the infinite number of conservation laws—the fingerprint of any integrable field theory [15–17]—implies that scattering processes of the Yang–Lee model are completely elastic and factorizable in terms of the two-body scattering matrix which was exactly computed in [11]. The spectrum can be easily extracted by the poles of the S -matrix and turns out to consist of only one massive particle which may be regarded as a bound state of itself. Exact Form Factors and two-point correlation functions of this massive model were later computed and discussed in [14].

Based on the S -matrix scattering theory and the thermodynamic Bethe Ansatz introduced by Zamolodchikov [18], one can then in principle determine exactly (although numerically) the free-energy of the integrable models as function of the inverse temperature β and the fugacity z [20, 21]. This is indeed what we have done in this paper for the Yang–Lee model, i.e. one of the simplest representative of integrable quantum field theories, for the purpose of studying what kind of distribution emerges for the zeros of the grand-canonical partition function of these systems. We hope these brief introductory considerations were useful to clarify the general aims of this paper and the ‘recursive’ use of Yang–Lee names, both to denote the zeros and the eponymous model, and the zeros of the model itself!

A comment is in order on the logic of this work. In the Yang–Lee theory of phase transitions one relates thermodynamical properties, such the free energy and the magnetization, to the density of Yang–Lee zeros $\eta(z)$. Usually the focus is on the (often approximate) determination of $\eta(z)$ in order then to extract the equilibrium quantities of interest. However, the point of view we adopt to deal with integrable quantum field theories as, for instance the Yang–Lee model, is rather the opposite, since the free energy of these models are already known in terms of the thermodynamics Bethe Ansatz equations, and this gives the possibility to determine the properties of the zeros. We have applied such a procedure to the Yang–Lee model but it is clear that it can be applied as well to other integrable quantum field theories.

The rest of the paper is organised as follows. Section 2 is devoted to a brief recap of the main points of the Yang–Lee theory of the phase transition and the importance to control the distribution of the zeros of the grand-canonical partition function in order to study the thermodynamics. Section 3 contains a detailed discussion on the nature of the roots of polynomials which may be regarded as grand-canonical partition functions of some appropriate physical system. In section 4 we present the closed formulas of the partition functions of free bosonic and fermionic theories, both in the non-relativistic and relativistic case, and also in the presence of an harmonic trap. Sections 5 and 6 contain a detailed discussion of the zeros of the grand-canonical partition functions of the non-relativistic and relativistic free theories respectively: apart from some peculiar features emerging from this study, the basic purpose of these sections is to set the stage for the analysis of the interacting integrable models discussed in the later sections. In particular, in section 7 we start recalling the basic properties of integrable quantum field theories, namely the S -matrix formulation and the thermodynamics

Bethe Ansatz which allows us to recover the partition function of the integrable models. In section 8 we study in detail the Yang–Lee zeros of the simplest quantum integrable field theories, namely the Yang–Lee model. Our conclusions are finally collected in section 9.

2. Yang–Lee theory of phase transitions

In order to overcome some inadequacies of the Mayer method [22] for dealing with the condensation of a gas and also to understand better the underlying mathematical reasons behind the occurrence of phase transitions, in 1952 Yang and Lee proposed to analytically continue the grand-canonical partition function to the complex plane of its fugacity and determine the pattern of zeros of this function. Although only real values of the fugacity determine the physical value of the pressure, the magnetization or other relevant thermodynamical quantities, the overall analytic behavior of these observables can only be understood by looking at how the zeros move as a function of an external parameter such as the temperature. In this section we are going to simply write down, mainly for future reference, the basic formulas of the Yang–Lee formalism with few extra comments.

Concerning general references on this topic, in addition to the original papers [2, 3] and others previously mentioned [4–7, 9–12], the reader may also benefit of some standard books [23, 24] or reviews such as [25, 26] and references therein. As a matter of fact, the literature on the subject is immense, even ranging across several fields of physics and mathematics. For this reason we cannot definitely do justice to all authors who contributed to the development of the subject but we would like nevertheless to explicitly mention few more references which we have found particularly useful, such as the series of papers by Ikeda [27–29] or by Abe [30, 31], the papers by Katsura on some analytic expressions of the density of zeros [5], the paper by Fonseca and Zamolodchikov [33] on the analytic properties of the free energy of the Ising field theory and some related references on this subject such as [18, 19, 34–39], and finally some references on the experimental observations of the Yang–Lee zeros [40], in particular those based on the coherence of a quantum spin coupled to an Ising-type thermal bath [41, 42].

2.1. Yang–Lee formulation

Consider for simplicity the grand-canonical partition function $\Omega_N(z)$ of a gas made of N particles with hard cores b in a volume V , of activity z and at temperature T is given by

$$\Omega_N(z) = \sum_{k=0}^N \frac{1}{k!} Z_k(V, T) z^k = \prod_{l=1}^N \left(1 - \frac{z}{z_l}\right), \quad (3)$$

where $N = V/b$ is the largest number of particles that can be contained in the volume, the coefficients $Z_k(V, T)$ are the canonical partition functions of a system of k particles and the z is the fugacity. As a polynomial of order N , $\Omega_N(z)$ has z_1, z_2, \dots, z_N zeros in the complex plane. The thermodynamics of the system is recovered by defining, in the limit $V \rightarrow \infty$, the pressure $p(z)$ and the density $\rho(z)$ of the system as

$$\frac{p(z)}{kT} \equiv \hat{f}(z) = \lim_{V \rightarrow \infty} \frac{1}{V} \log \Omega_N(z) = \lim_{V, N \rightarrow \infty} \frac{1}{V} \sum_{l=1}^N \log \left(1 - \frac{z}{z_l}\right), \quad (4)$$

$$\rho(z) \equiv \hat{z}'(z) = \lim_{V \rightarrow \infty} \frac{1}{V} z \frac{d}{dz} \log \Omega_N(z) = \lim_{V, N \rightarrow \infty} \frac{1}{V} \sum_{l=1}^N \frac{z}{z - z_l}. \quad (5)$$

For extended systems the limit $V \rightarrow \infty$ also enforces $N \rightarrow \infty$ and therefore there will be an infinite number of zeros: these may become densely distributed in the complex plane according to their positive density function $\eta(z)$, which can be different from zero either in a region \mathcal{A} of the complex plane (such a situation we will call later *area law* for the zeros) or along a curve, \mathcal{C} (to which we refer to as a *perimeter law*). Apart from these two generic cases, it can also be that the zeros may remain isolated points in the complex plane or, rather pathologically, accumulated instead around single points.

As shown in the original papers by Yang and Lee [2, 3], the entire thermodynamics can be recovered in terms of the density $\eta(z)$ of the zeros of the grand-canonical partition function. Notice that, for the reality of the original $\Omega_N(z)$, this function must satisfy the property

$$\eta(z) = \eta(z^*) , \quad (6)$$

i.e. must be symmetric with respect to the real axis. The region \mathcal{A} or the curve \mathcal{C} depend on the temperature T and they change their shape by changing T . Let us initially assume that the zeros are placed on an extended area \mathcal{A} : in this case, for all points outside this region one can analytically extend the definition of the pressure $p(z)$ as

$$\frac{p(z)}{kT} = \int_{\mathcal{A}} d\xi \eta(\xi) \log \left(1 - \frac{z}{\xi} \right). \quad (7)$$

We can split this function into its real and imaginary part

$$\frac{p(z)}{kT} \equiv P(z) = \varphi(z) + i\psi(z) , \quad (8)$$

where its real part

$$\varphi(z) = \int d\xi \eta(\xi) \log \left| 1 - \frac{z}{\xi} \right| \quad (9)$$

involves $\log |z|$, i.e. the Green function of the 2D Laplacian operator Δ . Therefore $\varphi(z)$ satisfies the Poisson equation

$$\Delta\varphi(z) = 2\pi \eta(z). \quad (10)$$

Drawing an electromagnetic analogy, this equation implies that $\varphi(z)$ can be thought as the electrostatic potential generated by the (positive) distribution of charges with density $\eta(z)$. Posing $z = x + iy$, the corresponding components of the electric field are given by

$$E_1 = -\frac{\partial\varphi}{\partial x}, \quad E_2 = -\frac{\partial\varphi}{\partial y} . \quad (11)$$

Since in any region not occupied by the charges both $\varphi(z)$ and its companion $\psi(z)$ are analytic functions related by the Cauchy–Riemann equations, we have

$$\frac{dP}{dz} = \frac{d\varphi}{dx} + i\frac{d\psi}{dx} = \frac{d\varphi}{dx} - i\frac{d\psi}{dy} = -\bar{E} , \quad (12)$$

where $E = E_1 + iE_2$ is the complex electric field ($\bar{E} = E_1 - iE_2$).

This electrostatic analogy is pretty appealing but it is important to realise that *not* all charge densities are appropriate for the statistical mechanics problem. For this purpose, there are indeed certain requirements to fulfil, such as: its pressure $p(z)$, computed according to equation (4), must be necessarily a continuous and positive function of z , monotonically increasing, along the real axis of the variable z ; at the same time, its density $\rho(z)$, computed according to equation (5), must also be a positive but not necessarily a continuous function, increasing too along the real

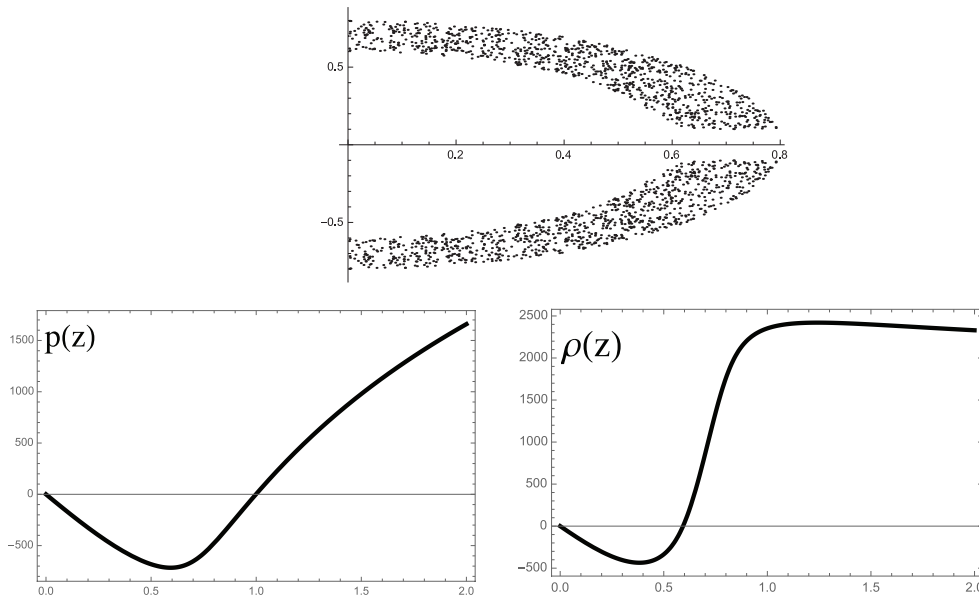


Figure 2. A random distribution of zeros (on top) which gives rise to a non positive and non monotonic behavior both of the pressure $p(z)$ and the density $\rho(z)$ along the positive real values of the fugacity z .

axis of z . It may be stressed that these conditions alone may be not sufficient to define a physical systems but if they are violated the system at hand is surely unphysical. For instance, for the random distributions of zeros shown in the top of figure 2, the corresponding pressure and density, shown on the bottom of the same figure, do not fulfil the physical conditions of positivity and monotonicity: therefore, this set of zeros shown does not correspond to any physical statistical system. We will come back again to this issue later at the end of this section.

2.2. Ising models

The formulas (4) and (5) for the pressure and the density of a gas are also useful to express the parametric equations of state for the Ising model whose spins can assume values ± 1 . This can be easily done in terms of the mapping which exists between the quantities of the Ising model with N spin and the lattice gas in a volume V [3]. In the following N_{\pm} denotes the number of positive/negative spins, M the magnetization of the system $M = (N_+ - N_-)/N$, F the free energy of the Ising model and H an external magnetic field. So, making the substitutions in equations (4) and (5) according to the following dictionary

| Ising model | = | Lattice gas | |
|-------------|---|-------------|------|
| N | = | V | |
| $1 - M$ | = | 2ρ | (13) |
| $F + H$ | = | $-p$ | |

one can easily write down the free energy and the magnetization of the Ising spin in terms of the zeros in the variable $z = e^{-2\beta H}$.

2.3. Singular behavior

It may happen that in the thermodynamic limit the zeros of the grand-canonical partition function do not spread on an area but lie instead along a particular curve \mathcal{C} thus satisfying a *perimeter law*. Such a curve has not to be necessarily a circle but in any case the previous formulas for the pressure and the density become in this case

$$\frac{p(z)}{kT} \equiv \hat{f}(z) = \int_{\mathcal{C}} \eta(s) \log \left(1 - \frac{z}{z(s)} \right) ds , \quad (14)$$

$$\rho(z) = z\hat{f}'(z) = z \int_{\mathcal{C}} \frac{\eta(s)}{z - z(s)} ds . \quad (15)$$

These equations are particularly useful to characterise the nature of the phase transition which occurs when the zeros pinch the real axis at same point z_0 . Let us indeed suppose, as it is often the case, that nearby this point the zeros lie on a smooth curve \mathcal{C} : using the electrostatic analogy, in this case we are in presence of a line distribution of the charges and therefore there will be a discontinuity in the electric field, given by the gradient of $\varphi(z)$, across the line charge distribution. Denoting by $P_1(z)$ and $P_2(z)$ the pressure (up to kT factor) across the line of the zeros and using the parameter s to move along this line, applying the Gauss law to the rectangle R shown in figure 3, one sees that [25, 26]

$$(\nabla\varphi_2 - \nabla\varphi_1) \cdot n|_{\mathcal{C}} = 2\pi\eta(s). \quad (16)$$

Using the Cauchy–Riemann equation, this relation can be also expressed as

$$\frac{d}{ds}(\psi_2 - \psi_1)|_{\mathcal{C}} = 2\pi\eta(s). \quad (17)$$

The nature of the phase transition at $z = z_0$ is then determined by the behavior of the density of the zeros nearby this point. Particularly important are two cases:

1. When $\eta(0) \neq 0$ we have a jump in the derivative of the pressure at $z = z_0$, therefore we are in presence of a first order phase transition.
2. When $\eta \sim |s|$, the density of zeros vanishes at $z = z_0$ and therefore the first derivative of the pressure is continuous at $z = z_0$ while there is a discontinuity in its second derivative. In this case we are in presence of a second order phase transition.

2.4. Zeros on a circle

Notice that the equations (14) and (15) further simplify when the curve \mathcal{C} is a circle, say of radius equal to 1: in this case the density $\eta(z)$ becomes an even function of the angle θ , normalised as

$$\int_{-\pi}^{\pi} \eta(\theta) d\theta = 1 , \quad (18)$$

and the pressure and density are expressed as

$$\frac{p(z)}{kT} = \int_0^{\pi} \eta(\theta) \log(z^2 - 2z \cos \theta + 1) d\theta , \quad (19)$$

$$\rho(z) = 2z \int_0^{\pi} \eta(\theta) \frac{z - \cos \theta}{z^2 - 2z \cos \theta + 1} d\theta . \quad (20)$$

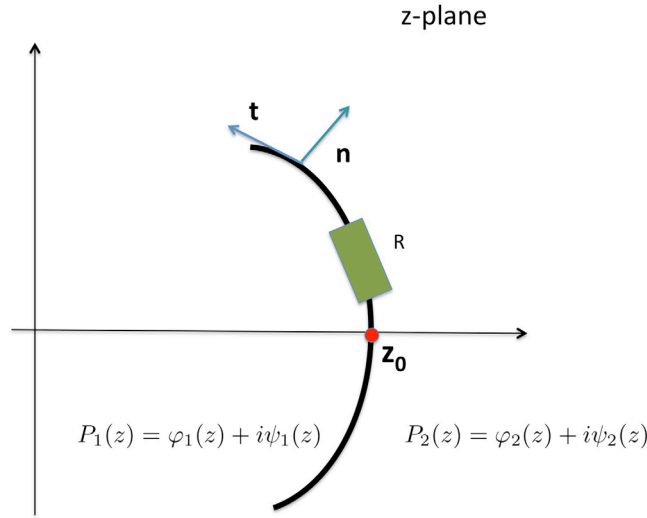


Figure 3. Linear distribution of the Yang–Lee zeros nearby the real zero z_0 and the Gauss law for the rectangle R across the line of zeros. \mathbf{n} and \mathbf{t} are respectively the normal and the tangential unit vectors of the the curve \mathcal{C} .

For a circle distribution of the zeros, it is easy to find a condition on the density $\eta(\theta)$ which ensures that both the pressure and the density are positive monotonic functions of z : as shown in [27], it is in fact sufficient that the density $\eta(\theta)$ is bounded and continuous, while its derivative $\eta'(\theta)$ is a bounded, continuous and positive function. Indeed, taking the derivative of $\rho(z)$ with respect to z we have

$$\frac{d\rho}{dz} = 2 \int_0^\pi \eta(\theta) \frac{2z - (1 + z^2) \cos \theta}{(z^2 - 2z \cos \theta + 1)^2} d\theta, \tag{21}$$

which, with the change of variable $\xi = \tan(\theta/2)$ and an integration by part, can be expressed as

$$\frac{d\rho}{dz} = -4 \left[\frac{\xi \eta(2 \arctan \xi)}{(z + 1)^2 \xi^2 + (z - 1)^2} \right]_0^\infty + 8 \int_0^\pi \frac{\xi \eta'(2 \arctan \xi)}{(1 + \xi)^2 (z + 1)^2 \xi^2 + (z - 1)^2} d\xi. \tag{22}$$

With the hypothesis that $\eta(\theta)$ is a bounded function, the first term of this expression vanishes while the second term, as far as $\eta'(\theta) > 0$, is positive. Once established that the function $\rho(z)$ is then an increasing monotonic function, to show that it is always positive is sufficient to calculate its value at the origin and, if not negative, the function $\rho(z)$ will be indeed always positive. Since at $z = 0$ we have $\rho(0) = 0$, this is sufficient to show the positivity of $\rho(z)$.

Under the same hypothesis for $\eta(\theta)$, we can also conclude that the pressure $p(z)$ is a positive and increasing function of z : since

$$\rho(z) = z \frac{dP}{dz} = \frac{dP}{d \log z}, \tag{23}$$

the positivity of $\rho(z)$ implies that $p(z)$ is an increasing function of $\log z$, i.e. of z itself since the logarithmic function is a monotonic function. So, also for $p(z)$ to prove that this is always a positive function it is sufficient to compute its value at the origin and check that it is not negative. Since $p(0) = 0$, this concludes the argument.

The importance of these considerations becomes more clear once one realises that even assuming the most favourable distribution of the zeros, i.e. along a circle, the analytic expression of the densities $\eta(\theta)$ is largely unknown. The very few cases where we have such an information include the 1D Ising model with nearest-neighbor interaction [3] and various versions of the mean field solutions of the same model [32]. As a further intriguing remark, it seems that if one is able to point out what are the conditions in order to have a proper physical density $\eta(\theta)$, there is a plenty of room to define new statistical model by inverting somehow the theory of Yang and Lee. We will further comment on this point in the conclusions of the paper.

2.5. Yang–Lee edge singularities

We can take advantage of the known expression of the density of zeros of the 1D ferromagnetic Ising model for presenting the simplest example of Yang–Lee edge singularities. The absence of a phase transition in this 1D model of course implies that $\eta(\theta)$ must vanish in an interval around the origin. This is indeed the case and the exact expression of the distribution is given by [3]

$$\eta(\theta) = \frac{1}{2\pi} \frac{\sin \frac{\theta}{2}}{\sqrt{\sin^2 \frac{\theta}{2} - \sin^2 \frac{\theta_0}{2}}}, \quad \text{if } |\theta| > \theta_0 \quad (24)$$

otherwise 0, where $\theta_0 = \arccos(1 - 2e^{-2\beta J})$ and J is the coupling constant between two next-neighbor spins. The zeros take then the C -shape of the plot on the left in figure 1. In this example θ_0 plays the role of an edge singularity and this value pinches the origin only when $\beta J \rightarrow \infty$, namely at $T = 0$. Nearby θ_0 the density of zeros behaves anomalously as

$$g(\theta) \sim \frac{C}{\sqrt{\theta - \theta_0}}, \quad (25)$$

($C = 1/(2\pi)\sqrt{\tan \frac{\theta_0}{2}}$), therefore for the 1D Ising model the Yang–Lee edge singularity exponent defined in equation (1) is equal to $\sigma = -1/2$. Consider now the expression of the magnetization in terms of the density of the zeros using the dictionary previously established

$$M(z) = 1 - 4z \int_0^\pi \eta(\theta) \frac{z - \cos \theta}{z^2 - 2z \cos \theta + 1} d\theta. \quad (26)$$

Performing the integral we have

$$M(z) = \frac{z - 1}{\sqrt{(z - e^{i\theta_0})(z - e^{-i\theta_0})}}. \quad (27)$$

Posing $z = e^{-2\beta h}$ and $\theta_0 = -2\beta h_c$, and expanding this formula around $h = ih_c$, we have

$$M(h) \sim \frac{M_0}{(h - ih_c)^{1/2}}, \quad (28)$$

i.e. the value ih_c can be considered as a singular point.

2.6. Polynomials versus series

The discussion made so far concerned with the singular behavior which emerges by increasing the order N of a sequence of genuine polynomials, in particular when $N \rightarrow \infty$. But what about

is the partition function $\Omega(z)$ is *ab-initio* given in terms not of a polynomial but an infinite series? Let us say $\Omega(z)$ in a neighbourhood of $z = 0$ is given by the infinite series

$$\mathcal{S}(z) = \sum_{k=0}^{\infty} \alpha_n z^n . \quad (29)$$

In this case one must be aware that analysing the Yang–Lee zeros of an expression such as in equation (29) there may be a condensation of zeros along some positive value z_0 of the fugacity z which however does not necessarily signal a phase transition of the physical system but rather the finite radius of convergence of the series itself! Imagine in fact that the series (29) could be analytically continued and that the corresponding function $\Omega(z)$ has the closest singularity nearby the origin at a negative value $z = -R$. This automatically fixes the radius of convergence of the series (29) to be R and therefore if we would use such a series to define the partition function, the corresponding Yang–Lee zeros may also condensate at the *positive* value $z = R$, even though this point is not associated to a phase transition of the actual function $\Omega(z)$. A simple example of this phenomena is worked out in detail in appendix A. We will see that similar cases also emerge in discussing the Yang–Lee zero distributions of fermionic theories whose corresponding series $\mathcal{S}(z)$ has alternating sign and therefore a singularity at a negative value of z : their zeros however also condensate at a positive real value of z .

3. Playing with polynomials

In this section we are going to deal extensively with the properties of the main mathematical object of this paper, namely the class of *real* polynomials $\Omega_N(z)$ of order N in the variable z

$$\Omega_N(z) = \gamma_0 + \gamma_1 z + \gamma_2 z^2 + \dots + \gamma_N z^N . \quad (30)$$

The coefficients γ_n of these polynomials are real and we choose hereafter $\gamma_0 = 1$. Since we are going to interpret $\Omega_N(z)$ as a generalised grand-canonical partition function of a statistical model, either classical or quantum, we pose

$$\Omega_N(z) \equiv e^{\mathcal{F}_N(z)} , \quad (31)$$

where we have defined the so-called free-energy $\mathcal{F}_N(z)$ of the system, directly related to the pressure of the system, see equation (4). In light of this statistical interpretation of $\Omega_N(z)$ in the following we will also express it with a different normalization of the coefficients

$$\Omega_N(z) = \sum_{k=0}^N \frac{a_k}{k!} z^k , \quad (32)$$

with $\gamma_k = a_k/k!$ and $a_0 = 1$ while the higher coefficients a_k 's assume the familiar meaning of canonical partition functions of k particles. Formulas which we derive below are sometimes more elegantly expressed in terms of the a_k 's although we will switch often between the two equivalent expressions (30) and (32), hoping that this will not confuse the reader.

3.1. Sign of the coefficients

For models coming from classical statistical physics, it is easy to argue that the coefficients a_k of the relative polynomial $\Omega_N(z)$ are generically all positive, $a_k > 0$. In this respect, consider for instance two significant examples:

- **Classical gas.** A classical model of a gas in d -dimension, made of N particles of mass m and Hamiltonian of the form

$$H = \sum_{i=1}^N \frac{p_i^2}{2m} + \sum_{ij} u(r_{ij}) , \quad (33)$$

where r_{ij} is the distance between the i th and j th particle. In this case the grand-canonical partition function of the system assumes the form (32), where the variable z expressed by

$$z = e^{\beta\mu} (2\pi m\beta/h^2)^{\frac{d}{2}} , \quad (34)$$

where μ is the fugacity, $\beta = 1/kT$ where T is the temperature and h is the Planck constant here introduced to normalise the phase-space integral. Therefore in this example the coefficients a_k are given the positive integrals

$$a_k = \int \cdots \int \mathbf{d}r_1 \cdots \mathbf{d}r_k \exp \left[-\beta \sum_{ij} u(r_{ij}) \right] > 0 . \quad (35)$$

- **Ising in a magnetic field.** As a second example of classical statistical mechanics, consider the ferromagnetic Ising model in a magnetic field H studied originally by Lee and Yang in [3]: in this case, with $\sigma_i = \{\pm 1\}$ and a general two-body ferromagnetic Hamiltonian of N spins in a regular lattice in arbitrary d -dimensional lattice of the form

$$H = - \sum_{ij} J_{ij} \sigma_i \sigma_j - H \sum_i \sigma_i, \quad J_{ij} > 0 \quad (36)$$

posing

$$z = e^{-2\beta H} , \quad (37)$$

the partition function of the model is expressed by a palindrome polynomial in this variable, i.e. a polynomial of the form (30) where the coefficients satisfy the additional condition

$$\gamma_k = \gamma_{N-k} > 0 . \quad (38)$$

This because γ_k is the contribution to the partition function of the Ising model in zero magnetic field coming from configurations in which the number N_- of spins $\sigma_i = -1$ is equal to k ; this contribution is evidently the same also for $N_- = (N - k)$.

It is however important to stress that for situations which come from quantum statistics and which involve, in particular, fermionic degrees of freedom, the sign of the coefficients γ_k 's of the polynomial may be not necessarily all positive but rather alternating. However, despite the presence of negative coefficients, in the *physical domain* of the variable z (alias the positive real axis, $z > 0$), the partition function $\Omega_N(z)$ of these systems assumes only positive values and is a monotonic function of z . The request of positivity of $\Omega_N(z)$ in its physical domain will play an important role in our future analysis when we are going to truncate our expressions of the partition function.

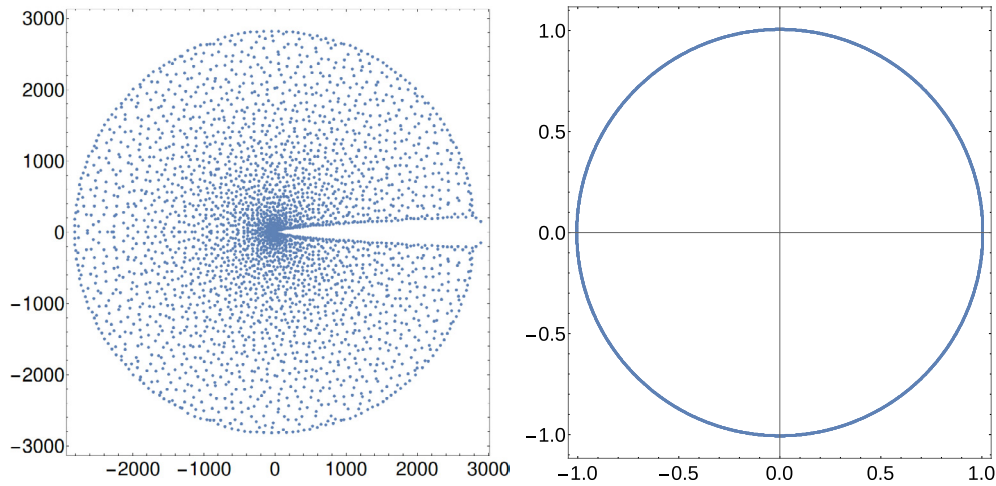


Figure 4. (a) Distribution of zeros of the partition function where the coefficients a_n are given by prime numbers. (b) Distribution of zeros of the partition function where the coefficients are given by $\gamma_n = (n-1)!/n$. Notice the very different scales of the two figures.

3.2. Patterns of the zeros

Let us now make a preliminar discussion about the nature of the zeros of a polynomial such as $\Omega_N(z)$ before focusing our attention on a series of examples. Being $\Omega_N(z)$ a real polynomial, its N zeros are either real or grouped in pairs of complex conjugated values in the complex z plane. Hence, any distribution of the zeros has to be symmetric with respect the real axis. Concerning the real zeros, when all coefficients γ_k of $\Omega_N(z)$ are positive, they cannot be obviously positive. Hence, for any *finite* order N of the polynomial, it should exist a region R which contains the whole positive real axis which is free of zeros: a positive real zero z_c can emerge only in the limit $N \rightarrow \infty$, i.e. as an accumulation point of complex zeros and, in this case, it corresponds to a point of singularity of the grand-canonical partition function [2, 3].

Clearly, the overall distribution of zeros and its details depend upon the choice of the coefficients γ_k and therefore, in general, there may be a wide range of situations. We find however important to try to identify at least two important patterns of zeros, hereafter called *area* and *perimeter* distributions respectively, for which in section 3.6 we will find some criteria for their realisation. At the level of terminology, we say:

- we are in presence of an *area distribution* when the the N zeros are distributed in an extended area of the complex plane. This is the case, for instance, of the distribution shown in the first plot in figure 4, obtained by taking as coefficients p_k of Ω_N just the k th prime number. We will comment more on this example later.
- we are in presence of a *perimeter distribution* when the N zeros are distributed along particular lines. A particular interesting case of perimeter distributions is when all the zeros are along a circle. This is the case, for instance, of the distribution of the zeros shown in the second plot in figure 4, obtained by considering a grand-canonical partition function $\Omega_N(z)$ with coefficients a_k given the sequence $\gamma_k = 1/k^2$. This and other similar cases will be discussed in more details below.

3.3. Bounds on the absolute module of the zeros

Given a polynomial of the form (30), there are several bounds on the magnitudes $|z_i|$ of its roots. Here we simply state some of these bounds without any proof for them (it may be useful to consult the book by Prasolov [43] as a general reference on polynomials and bounds on their roots). Some of these bounds are more stringent than others (and we list them in the order of increasing their level of refinement) but altogether they may help in getting an idea about the distribution of the roots.

- **Cauchy bound.** The Cauchy bound states that the roots z_i have an absolute value less than R_c , $|z_i| < R_c$, where

$$R_c = 1 + \frac{1}{|\gamma_N|} \max \{ |\gamma_0|, |\gamma_1|, \dots, |\gamma_{N-1}| \} . \tag{39}$$

This is usually the less stringent bound on the module on the roots.

- **Sun-Hsieh bound.** The Cauchy bound can be refined in terms of the Sun-Hsieh bound, $|z_i| < R_{SH}$, where

$$R_{SH} = 1 + \frac{1}{2} \left((|\gamma_{N-1}/\gamma_N| - 1) + \sqrt{(|\gamma_{N-1}/\gamma_N| - 1)^2 + 4a} \right) , \quad a = \max \{ |\gamma_k/\gamma_N| \} . \tag{40}$$

- **Fujiwara bound.** A further refinement comes from the Fujiwara bound where all roots are within the disc of radius R_F , $|z_i| < R_F$, where

$$R_K = 2 \max \left\{ \left| \frac{\gamma_{N-1}}{\gamma_N} \right| , \left| \frac{\gamma_{N-2}}{\gamma_N} \right|^{\frac{1}{2}} , \dots , \left| \frac{\gamma_1}{\gamma_N} \right|^{\frac{1}{N-1}} , \frac{1}{2} \left| \frac{\gamma_0}{\gamma_N} \right|^{\frac{1}{N}} \right\} . \tag{41}$$

- **Eneström bounds.** When all $\gamma_k > 0$, we can have a lower and an upper bound of the magnitude of the roots given by

$$R_d \leq |z_i| \leq R_u , \tag{42}$$

where

$$R_d = \min \left\{ \frac{\gamma_k}{\gamma_{k+1}} \right\} , \quad R_u = \max \left\{ \frac{\gamma_k}{\gamma_{k+1}} \right\} , \quad k = 0, 1, \dots, N-1 . \tag{43}$$

The Eneström bounds are usually the best estimate of the annulus in the complex plane where the roots are located.

- **Eneström–Kakeya bound.** Let us also mention the Eneström–Kakeya bound which refers to polynomials where the coefficients γ_k satisfy the condition

$$\gamma_0 \leq \gamma_1 \leq \gamma_2 \dots \leq \gamma_N . \tag{44}$$

In this case we have that $|z_i| \leq (|\gamma_N| - \gamma_0 + |\gamma_0|)/|\gamma_N|$. In particular, if all coefficients γ_k are positive, the roots are within the unit disc, $|z_i| \leq 1$. Of course, when the coefficients satisfy instead the condition

$$\gamma_0 \geq \gamma_1 \geq \gamma_2 \dots \geq \gamma_N , \tag{45}$$

when $|z_i| \geq 1$.

Let us make a final comment on the information provided by the bounds on the modules of the zeros: these bounds refer to *all* roots, so if knew for instance that $|z_i| < 100$, it may happen that just one or few zeros has module $|z| \sim 100$ while all the rest may have a very small module. In other words, bounds usually refer to the *largest* of the zeros rather than their overall distribution.

3.4. Motion of the zeros: perturbative analysis

We can study how the distribution of the zeros of a polynomial is modified by the addition of a new root as the order of the polynomial is increased by 1. To this aim, consider a polynomial of the form

$$\Omega_N(x) = \Omega_{N-1}(x) + \gamma_N x^N. \quad (46)$$

The addition of the new term $\gamma_N x^N$ to $\Omega_{N-1}(x)$ has two effects:

1. it creates a new zero;
2. it moves the previous ones.

Let us address the first issue by considering the equation for the zeros of the new polynomial

$$\Omega_{N-1}(x) + \gamma_N x^N = 0. \quad (47)$$

Writing it as

$$x = -\frac{\Omega_{N-1}(x)}{\gamma_N x^{N-1}} = -\frac{\gamma_{N-1}}{\gamma_N} - \frac{\gamma_{N-2}}{\gamma_{N-1} x} + \dots, \quad (48)$$

we see that, perturbatively in a_N , the new root x^* is roughly placed at

$$x^* \simeq -\frac{\gamma_{N-1}}{\gamma_N}. \quad (49)$$

If γ_N is infinitesimally small with respect to the other previous coefficients, the value x^* determined from this equation is large and therefore, self-consistently, it is justified to neglect its corrections coming from the inverse powers $(1/x^*)^l$ present in the left hand side of (48).

Let us now address the second issue, i.e. the motion of the other zeros, once again perturbatively in the quantity a_N . Let x_i ($i = 1, \dots, N-1$) be one of the zeros of the polynomial $P_{N-1}(x)$. Let us now write

$$x_i + \delta x_i \quad (50)$$

as the new position of the root, once the new term $a_N x^N$ has been added

$$\begin{aligned} \Omega_N(x_i + \delta x_i) &= \Omega_{N-1}(x_i + \delta x_i) + \gamma_N (x_i + \delta x_i)^N = 0 \\ &= \Omega_{N-1}(x_i) + \delta x_i \frac{d\Omega_{N-1}}{dx_i} + \gamma_N x_i^N + N\gamma_N x_i^{N-1} \delta x_i \\ &= \delta x_i \left[\frac{d\Omega_{N-1}}{dx_i} + N\gamma_N x_i^{N-1} \right] + \gamma_N x_i^N = 0, \end{aligned} \quad (51)$$

hence the displacement of the i th root due to the addition on the term $a_N x^N$ is given by

$$\delta x_i = -\frac{\gamma_N x_i^N}{\left(\frac{d\Omega_{N-1}}{dx_i} + \gamma_N N x_i^{N-1} \right)}. \quad (52)$$

3.5. Statistical approach

In order to understand better the pattern of the zeros and, in particular, to see whether we are able to qualitatively predict if an area or a perimeter law occurs, it is worth setting up a statistical analysis of the zeros. The obvious quantities to look at are the statistical moments s_k of the zeros defined as

$$s_k \equiv \sum_{l=1}^N z_l^k, \quad (53)$$

where k can be either a positive or a negative integer. Notice that, being the polynomial real, all s_k are real quantities.

3.5.1. Negative moments. Let us see how to relate the negative moments of the zeros

$$s_{-m} \equiv \hat{s}_m = \sum_{l=1}^N \left(\frac{1}{z_l}\right)^m, \quad (54)$$

to the coefficients of the polynomial $\Omega_N(z)$. In order to do so, let us factorise the partition functions in terms of its zeros as

$$\Omega_N = \prod_{l=1}^N \left(1 - \frac{z}{z_l}\right). \quad (55)$$

Taking the logarithm of both sides we arrive to the familiar cluster expansion of the free-energy $\mathcal{F}_N(z)$

$$\begin{aligned} \mathcal{F}_N(z) = \log \Omega_N(z) &= \sum_{l=1}^N \log \left(1 - \frac{z}{z_l}\right) = - \sum_{l=1}^N \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{z}{z_l}\right)^m \\ &\equiv \sum_{m=1}^{\infty} b_m z^m. \end{aligned}$$

The negative moments \hat{s}_m are then related to the cluster coefficients b_m as [2]

$$b_m = -\frac{1}{m} \sum_{l=1}^N \left(\frac{1}{z_l}\right)^m = -\frac{1}{m} \hat{s}_m. \quad (56)$$

It is also custom to expand the free-energy $\mathcal{F}_N(z)$ in terms of the cumulants c_k defined by

$$\mathcal{F}_N(z) = \ln \Omega_N \equiv \sum_{k=1}^{\infty} \frac{c_k}{k!} z^k. \quad (57)$$

If we use for $\Omega(z)$ the expression (32) and the coefficients a_k , the cumulants are given by

$$\begin{aligned} c_1 &= a_1 \\ c_2 &= a_2 - a_1^2 \\ c_3 &= a_3 - 3a_2a_1 + 2a_1^3 \\ &\vdots \end{aligned} \quad (58)$$

As a matter of fact there exists a closed formula which relates the two sets of coefficients c_k and a_k . To this aim let us introduce the determinant of a $(k \times k)$ matrix $M(k)$ whose entries involve the first k coefficients a_l ($l = 1, 2, \dots, k$)

$$M(k) = \begin{pmatrix} a_1 & 1 & 0 & \cdots & 0 \\ a_2 & a_1 & 1 & & \vdots \\ a_3 & a_2 & \binom{2}{1} a_1 & & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_k & a_{k-1} & \binom{k}{1} a_{k-2} & \cdots & \binom{k-1}{k-2} a_1 \end{pmatrix}. \quad (59)$$

The final formula is then

$$c_k = (-1)^{k-1} \det M(k). \quad (60)$$

The relevant thing to note is that the k th cluster coefficient c_k , which is of course related to the k th moment of the inverse roots as

$$c_k = -(k-1)! \hat{s}_k, \quad (61)$$

is entirely determined only by the first k coefficients of the original polynomial Ω_N . This implies that, if we increase the order $N \rightarrow N + \tilde{N}$ of the polynomial by adding to Ω_N the new coefficients $a_{N+1}, \dots, a_{N+\tilde{N}}$ but keeping fixed all the previous ones, there will be an increasing number of the zeros but their overall positions are constrained by the condition that the first N negative moments

$$s_{-k} \equiv \hat{s}_k = \sum_{l=1}^N \left(\frac{1}{z_l}\right)^k = \sum_{l=1}^{\tilde{N}} \left(\frac{1}{z_l}\right)^k, \quad k = 1, \dots, N \quad (62)$$

before and after adding the new coefficients, remain constant.

3.5.2. Positive moments. Let us now discuss the positive moments of the zeros

$$s_k = \sum_{l=1}^N (z_l)^k, \quad k > 0. \quad (63)$$

We can relate them to the coefficients of the polynomial $\Omega_N(z)$ as follows. Let us first introduce the elementary symmetric polynomials $\sigma_k(x_1, \dots, x_N)$ defined as

$$\sigma_k(x_1, \dots, x_N) = \sum_{1 \leq j_1 \leq j_2 \leq \dots \leq j_k \leq N} x_{j_1} \dots x_{j_k}, \quad k = 0, \dots, N. \quad (64)$$

Notice that $\sigma_0 = 1$ and $\sigma_N = x_1 x_2 \dots x_N$. Since

$$G(z) \equiv \prod_{l=1}^N (z - z_l) = \sum_{k=0}^N (-1)^k \sigma_k(z_1, \dots, z_N) z^{N-k}, \quad (65)$$

we can express the partition function $\Omega_N(z)$ as

$$\begin{aligned} \Omega_N(z) &= \sum_{m=0}^N \gamma_m z^m \equiv \gamma_N G(z) \\ &= \gamma_N \sum_{k=0}^N (-1)^k \sigma_k(z_1, \dots, z_N) z^{N-k}. \end{aligned} \tag{66}$$

Therefore

$$\gamma_m = (-1)^{N-m} \gamma_N \sigma_{N-m}, \tag{67}$$

and, sending $m \rightarrow N - m$, we have the final relation between the symmetric polynomials and the coefficients γ_k (or a_k) of the partition function $\Omega_N(z)$

$$\sigma_m = (-1)^m \frac{\gamma_{N-m}}{\gamma_N} = (-1)^m \frac{N!}{(N-m)!} \frac{a_{N-m}}{a_N}. \tag{68}$$

Moreover, we can relate the moments s_k to the elementary symmetric polynomials thanks to the *Newton–Girard formula*

$$(-1)^m m \sigma_m(z_1, \dots, z_N) + \sum_{k=1}^m (-1)^{k+m} s_k(z_1, \dots, z_N) \sigma_{m-k} = 0, \tag{69}$$

so that

$$\begin{aligned} s_1 - \sigma_1 &= 0, \\ s_2 - s_1 \sigma_1 + 2\sigma_2 &= 0, \\ s_3 - s_2 \sigma_1 + s_1 \sigma_2 - 3\sigma_3 &= 0, \\ &\vdots \\ s_N - s_{N-1} \sigma_1 + s_{N-2} \sigma_2 - \dots + (-1)^N \sigma_N &= 0. \end{aligned} \tag{70}$$

A closed solution of these relations can be given in terms of a formula which employs the following determinant

$$s_p = \begin{vmatrix} \sigma_1 & 1 & 0 & \dots & 0 \\ 2\sigma_2 & \sigma_1 & 1 & \dots & 0 \\ 3\sigma_3 & \sigma_2 & \sigma_1 & & \\ \vdots & \vdots & \vdots & & \\ p\sigma_p & \sigma_{p-1} & \sigma_{p-2} & \dots & \sigma_1 \end{vmatrix}. \tag{71}$$

The relevant thing to notice in this case is that the positive moment s_p is determined by the first p elementary symmetric polynomials which, on the other hand, are fully determined by the last $(N - p)$ coefficients of the polynomial Ω_N (see equation (68)).

3.5.3. Moments: summary. So far we have seen that the negative moments \hat{s}_k of the zeros are determined by the first k coefficients of the polynomial $\Omega_N(z)$ while the positive moments s_l are determined instead by the last l coefficients of $\Omega_N(z)$

$$\Omega_N = \underbrace{1 + \gamma_1 z + \gamma_2 z^2 + \dots + \gamma_k z^k}_{\hat{s}_k = s_{-k}} + \dots + \underbrace{\gamma_{N-l} z^{N-l} + \gamma_{N-l+1} z^{N-l+1} + \dots + \gamma_N z^N}_{s_l}. \tag{72}$$

Obviously all the negative and positive moments higher or equal than N are linearly dependent from the previous ones. To show this, let us interpret the polynomial $\Omega_N(z)$ as the characteristic polynomial of a $(N \times N)$ matrix \mathbf{M} which satisfies the same equation satisfied by the zeros themselves of $\Omega_N(z)$

$$\mathbf{1} + \gamma_1 \mathbf{M} + \gamma_2 \mathbf{M}^2 + \cdots + \gamma_N \mathbf{M}^N = 0, \tag{73}$$

where $\mathbf{1}$ is the $(N \times N)$ identity matrix. Since $s_k = \text{Tr} \mathbf{M}^k$, taking now the trace of the equation above, it is easy to see that the N th positive moment is linearly dependent from the previous $N - 1$ moments

$$s_N = -\frac{1}{\gamma_N} (N + \gamma_1 s_1 + \gamma_2 s_2 + \cdots + \gamma_{N-1} s_{N-1}). \tag{74}$$

Moreover, to get the linear equation which links the higher positive moment s_{N+m} to the N previous ones, it is sufficient to multiply by \mathbf{M}^m equation (73) and then to take the trace of the resulting expression.

Concerning instead the linear combinations which involve the negative moments higher or equal to N , it is sufficient to multiply equation (73) by \mathbf{M}^{-N} and then repeating the steps described above. For instance, the negative moment \hat{s}_N depends linearly from the previous ones as

$$\hat{s}_N = -\gamma_1 \hat{s}_{N-1} - \gamma_2 \hat{s}_{N-2} - \cdots - \gamma_N N. \tag{75}$$

3.6. Area and perimeter laws

Hereafter we are going to set a certain number of ‘rules of thumb’ that allow us to have a reasonable guess whether the zeros satisfy the area or the perimeter laws. The criteria make use of the bounds on the modules of the zeros, the geometrical and the arithmetic means of the zeros and also of their variance. All these quantities are easy to compute in terms of the coefficients of the partition function and therefore they provide a very economical way for trying to anticipate their distribution. In other words, we must subscribe to a reasonable compromise between the reliability of the prediction and the effort to compute the indicators on the zeros: of course, would one increase the number of computed moments of the zeros, then he/she would narrow better and better the prediction but at the cost of course to engage into the full analysis of the problem! This is precisely the origin of the compromise. One must be aware, however, of the heuristic nature of the arguments we are going to discuss below, which have not at all the status of a theorem, and therefore they must be taken with a grain of salt. Moreover, one must also be aware that not all distributions are either area or perimeter laws, since there exist cases where the roots have simultaneously both distributions or they are made by isolate points.

3.6.1. Geometrical mean of the zeros. From equation (68), taking $m = N$ we have

$$\sigma_N \equiv \prod_{k=1}^N z_k = (-1)^N \frac{N!}{a_N} = (-1)^N \frac{1}{\gamma_N}. \tag{76}$$

Let us put $q_N \equiv \frac{N!}{p_N} = \frac{1}{\gamma_N}$ and take the N th root of both terms in equation (76): for the left hand side, we have the geometrical mean of the roots

$$\langle z \rangle_{\text{geom}} = \left(\prod_{k=1}^N z_k \right)^{1/N}, \tag{77}$$

while for the right-hand side we have:

$$(-1)(q_N)^{1/N}. \tag{78}$$

For the nature of the zeros of a real polynomial—which either pair in complex conjugate values $z_k = \rho_k e^{i\theta_k}$ and $z_k^* = \rho_k e^{-i\theta_k}$ or are real $z_a = \rho_a$ (where ρ_a can be also negative)—the product of all zeros is a real quantity which depends upon only the product of the *modules* ρ_k of all roots. When the module of the geometrical mean is finite, say $|\langle z \rangle_{\text{geom}}| \sim \xi$, the zeros basically must be symmetric under the mapping $z \rightarrow \xi^2/z$. Notice that the easiest way to implement such a symmetry is that the zeros are placed along a circle of radius ξ .

3.6.2. Arithmetic mean of the zeros. Given the N zeros of a polynomial we can define their arithmetic mean: this is simply the first positive moment of the zeros divided by N

$$\langle z \rangle_{\text{arith}} = \frac{1}{N} \sum_{k=1}^N z_k = \frac{1}{N} s_1 = \frac{1}{N} \sigma_1, \tag{79}$$

and therefore, using equations (68) and (70), it is easily related to the ratio of the last two coefficients

$$\langle z \rangle_{\text{arith}} = -\frac{a_{N-1}}{a_N} = -\frac{1}{N} \frac{\gamma_{N-1}}{\gamma_N}. \tag{80}$$

For the reality of the polynomials considered in this paper it is obvious that the arithmetic mean of the zeros just depends on their real part and therefore it is a real number.

3.6.3. Variance of the zeros. The variance⁶ of the zeros is defined as

$$\mu^2 = \frac{1}{N} \sum_{k=1}^N (z_k - \bar{z}_{\text{arith}})^2. \tag{81}$$

Putting $\langle z \rangle_{\text{arith}} \equiv a \in \mathbb{R}$ and expressing the zeros in terms of their real and imaginary parts, $z_k = x_k + iy_k$, we have

$$\mu^2 = \frac{1}{N} \sum_{k=1}^N [(x_k - a)^2 - y_k^2], \tag{82}$$

while the imaginary part of μ^2 vanishes for the symmetry $z_k \leftrightarrow z_k^*$ of the zeros. Hence, the variance μ^2 essentially measures the unbalance between the imaginary and the real components of the zeros (the latter component measured with respect to the arithmetic mean of the zeros). The distributions of the zeros which have small values of μ^2 are those which are almost symmetric for the interchange $(x_k - a) \leftrightarrow y_k$, as for instance is the case for zeros placed uniformly on a circle of center at $z = a$. However, notice that another and quite different way to have small values of μ^2 is that the zeros are real and grouped very close to their arithmetic mean.

Using equations (68) and (70), one can see that the variance of the zeros can be expressed in terms of the coefficients of the polynomial $\Omega_N(z)$ as

⁶Notice that what we call here the *variance* μ^2 is *not* the familiar variance of real random variables, since our definition employs the square of the differences of complex numbers and therefore μ^2 is not necessarily positive.

Table 1. Cases relative to the various behavior of the arithmetic mean, the geometrical mean and the variance.

| | $\langle z \rangle_{\text{geom}}$ | $\langle z \rangle_{\text{arith}}$ | μ^2 |
|---|-----------------------------------|------------------------------------|------------|
| A | ξ | c | ϵ |
| B | ξ | c | N |
| C | ξ | N | N |
| D | ξ | N | ϵ |
| E | N | c | ϵ |
| F | N | c | N |
| G | N | N | ϵ |
| H | N | N | N |

$$\mu^2 = \frac{1}{N} \left[\left(1 - \frac{1}{N} \right) \left(\frac{\gamma_{N-1}}{\gamma_N} \right)^2 - 2 \frac{\gamma_{N-2}}{\gamma_N} \right]. \tag{83}$$

3.6.4. Rules of thumb. Let us now see how the three indicators introduced above, used for instance together with the Eneström bounds (42), can help us in discriminating between different types of zero distributions. We are of course interested in the behaviour of the zeros for large N . To this aim let us consider in particular the cases⁷ gathered together in table 1 in which the symbol N means that the relative quantity scales as N or even with higher power of N , while ξ and c denotes finite quantities, independent of N , and ϵ finally denotes an infinitesimal quantity. So, for example, the case *B* corresponds to a situation in which both the arithmetic and geometrical means of the zeros are finite while the variance μ^2 diverges when N goes to infinity. In the following we will analyse and illustrate the various cases of table 1 by means of some explicit polynomials⁸ whose means and variance satisfy the values in the table: the role of these polynomials is to guide us in understanding, if possible, more general situations. Notice that, from a statistical physics point of view, some of the polynomials shown below may be considered rather pathological.

Case A. Without losing generality, if the arithmetic mean is finite we can assume to be zero, since we can always shift the variable of the relative polynomial as $z \rightarrow z - \langle z \rangle_{\text{arith}}$. So this case concerns with

$$|\bar{z}_{\text{geom}}| \simeq \xi, \quad |\bar{z}_{\text{arith}}| = 0, \quad \mu^2 \simeq \epsilon. \tag{84}$$

The fact that the arithmetic mean vanishes obviously implies that the barycenter of the zeros is the origin of the complex plane; since the geometrical mean of this case goes instead as a constant for large N , this can be interpreted as the fact that the module of all the zeros (except probably few) cannot grow with N . Finally, since the variance vanishes for $N \rightarrow \infty$ this seems to imply a certain localization of the zeros, so most probable the zeros of this case satisfy a perimeter law. Notice all these features are realised by zeros placed along a circle of radius ξ , as for instance

⁷This table is not at all exhaustive of all possible zero distributions but it rather points out few significant cases.

⁸For convenience we often choose the zeros of the polynomials to be along the positive real axis but of course they can be placed along the negative axis if one has to comply the condition of dealing with polynomials with real positive coefficients.

$$A(z) = z^N + \xi^N . \tag{85}$$

In this case, in fact $\langle z \rangle_{\text{geom}} = \xi$, $\langle z \rangle_{\text{arith}} = 0$ and $\mu^2 = 0$. The distribution of the zeros of this polynomial satisfies indeed the perimeter law. Notice that for this polynomial the value ξ sets the upper bound of the module of zeros, as seen applying the Fujiwara bound (41). More generally, all polynomials whose coefficients γ_k scale as a power of k , i.e. $\gamma_k \simeq k^\delta$, where δ is either positive or negative real number, have asymptotically as Eneström bounds $R_d = R_u = 1$ and for $N \rightarrow \infty$

$$\begin{aligned} \langle z \rangle_{\text{geom}} &= (N^\delta)^{1/N} \rightarrow 1 , \\ \langle z \rangle_{\text{arith}} &= -\frac{1}{N} \left(\frac{N-1}{N} \right)^k \rightarrow 0 , \\ \mu^2 &= \frac{1}{N} \left[\left(\frac{N-1}{N} \right)^{2k} - 2 \left(\frac{N-2}{N} \right)^k \right] \rightarrow 0 , \end{aligned} \tag{86}$$

namely all of them belong to the class A. Therefore we expect that the polynomials of this class will satisfy the perimeter law, apart eventually few zeros displaced somewhere else. Few experiments, in addition to the one shown in figure 4, are reported in figure 5, and indeed they show that the zeros of these polynomials satisfy the perimeter law.

Case B. This case concerns with the values

$$|\bar{z}_{\text{geom}}| \simeq \xi, \quad |\bar{z}_{\text{arith}}| = c, \quad \mu^2 \simeq N. \tag{87}$$

A polynomial which has such values is

$$B(z) = z^N + Nz^{N-1} + 1 , \tag{88}$$

(plus all other lower coefficients which however scale with lower powers in N) since

$$\begin{aligned} \langle z \rangle_{\text{geom}} &= (1)^{1/N} = 1 , \\ \langle z \rangle_{\text{arith}} &= -1 , \\ \mu^2 &= \frac{1}{N} \left(1 - \frac{1}{N} \right) N^2 \rightarrow N . \end{aligned} \tag{89}$$

Notice that, in general, to have a finite geometrical mean not all zeros can grow as N , the same is also true in order to have a finite value for the arithmetic mean. However, to have a variance which grows as N it is enough that some of the zeros are far off from $\langle z \rangle_{\text{arith}}$, order N . This is indeed the case for the roots of the polynomial (88), where one of them is far away from the arithmetic mean, $z_* \sim -N$, while all the others are essentially along a circle of radius 1 and center at $\langle z \rangle_{\text{arith}} = -1$. Therefore for polynomials of this class we expect that, apart few zeros, the others satisfy a perimeter law. Notice that in this case a bound such as the Fujiwara bound, taken alone, is pretty loose since it states that all roots satisfy $|z_i| < 2N$, a condition which is indeed true but in this case only one root is order N .

Case C. This case concerns with the values

$$|\bar{z}_{\text{geom}}| \simeq \xi, \quad |\bar{z}_{\text{arith}}| \sim N, \quad \mu^2 \simeq N. \tag{90}$$

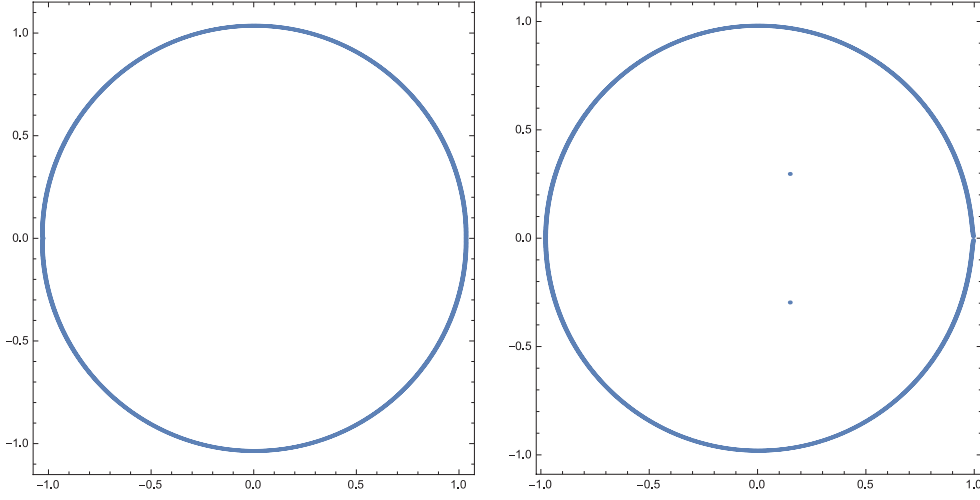


Figure 5. (a) Distribution of zeros of the partition function whose coefficients γ_k go as $1/k^5$. (b) Distribution of zeros of the partition function whose coefficients γ_k go as k^3 . Notice that in both cases, all zeros, apart from a few, are placed along a circle, i.e. they satisfy the perimeter law.

A polynomial with these properties is given by

$$C(z) = (z - N)^{\frac{N}{2}} \left(z - \frac{1}{N} \right)^{\frac{N}{2}}. \tag{91}$$

Assume N to be an even number. This polynomial has $N/2$ zeros at $z = N$ and $N/2$ zeros at $z = 1/N$. Therefore the geometrical mean is equal to 1 while the arithmetic mean is equal to $\bar{z}_{arith} = (N + 1/N)/2$ and its variance $\mu^2 = (N - 1/N)^2$. In this case we are in presence of a bunch concentration of zeros (in this case placed at reciprocal positions) which essentially do not satisfy neither area or perimeter laws.

Case D. This case concerns with

$$|\bar{z}_{geom}| \simeq \xi, \quad |\bar{z}_{arith}| = N, \quad \mu^2 \simeq \epsilon. \tag{92}$$

A corresponding polynomial with these features is given by

$$D(z) = 1 + (N^4 - N^3 + N) z^{N-2} + 2N z^{N-1} + 2z^N \tag{93}$$

plus lowest order coefficients which scale with lower powers in N . Such a polynomial, apart few of its zeros which have large modules (order N) so that their arithmetic average goes with N , has the rest of the zeros with bounded modules. Since their spread is small, most probably we are in presence of a perimeter law. In the case of the polynomial (93), its bounded zeros are indeed all around a circle.

Case E. This case concerns with

$$|\bar{z}_{geom}| \simeq N, \quad |\bar{z}_{arith}| \simeq c, \quad \mu^2 \simeq \epsilon. \tag{94}$$

A representative polynomial with these features is

$$E(z) = 1 + \frac{N(N+1)}{2N!} z^{N-2} + \frac{1}{(N-1)!} z^{N-1} + \frac{1}{N!} z^N. \tag{95}$$

In order to compute the various statistical quantities we need the Stirling formula for the factorial

$$N! \simeq \sqrt{2\pi N} \left(\frac{N}{e}\right)^N. \quad (96)$$

The divergence with N of the geometrical mean implies that almost all zeros has a module which increases with N although their barycenter remains at a finite distance. The infinitesimal value of the variance once again suggests that we may be in presence of a perimeter law. As a matter of fact, almost all roots of the polynomial (95), varying N , are along a circle of increasing radius.

Case F. This case concerns with

$$|\bar{z}_{\text{geom}}| \simeq N, \quad |\bar{z}_{\text{arith}}| \simeq c, \quad \mu^2 \simeq N. \quad (97)$$

This class may have as a significant representative a polynomial which employs the prime numbers P_k

$$F(z) = 1 + P_1 z + \frac{1}{2!} P_2 z^2 + \dots + \frac{1}{N!} P_N z^N. \quad (98)$$

In order to compute the various statistical quantities, in addition to the Stirling formula for the factorial, we also need the approximate formula for the N -the prime P_N

$$P_N \simeq N \log N. \quad (99)$$

Therefore

$$\begin{aligned} |\bar{z}_{\text{geom}}| &= \left(\frac{1}{\gamma_N}\right)^{1/N} \sim N, \\ \bar{z}_{\text{arith}} &= -\frac{1}{N} \frac{\gamma_{N-1}}{\gamma_N} \sim -1 \\ \mu^2 &\sim N. \end{aligned} \quad (100)$$

Applying either the Fujiwara or the Eneström bound, one sees that the radius of the disc which includes all zeros increases linearly with N . All these behaviors suggest that the zeros of the polynomial (98) satisfy the area law, which is indeed the case, as shown in figure 4.

Case G. This case concerns with

$$|\bar{z}_{\text{geom}}| \simeq N, \quad |\bar{z}_{\text{arith}}| \simeq N, \quad \mu^2 \simeq \epsilon. \quad (101)$$

A polynomial with these features is given by

$$G(z) = (z - N)^N. \quad (102)$$

The zeros of this polynomial are obviously all at $z_* = N$. Both their arithmetic and geometric means are equal to N and therefore diverge when $N \rightarrow \infty$, however their variance is identically zero. When we are in presence of these values of the indicators it is quite probable that the distribution of zero is sharply peaked around a value z_* that grows with N . Also in this case the zeros do not satisfy neither area or perimeter laws.

Case H. This case concerns with

$$|\bar{z}_{\text{geom}}| \simeq N, \quad |\bar{z}_{\text{arith}}| \simeq N, \quad \mu^2 \simeq N. \quad (103)$$

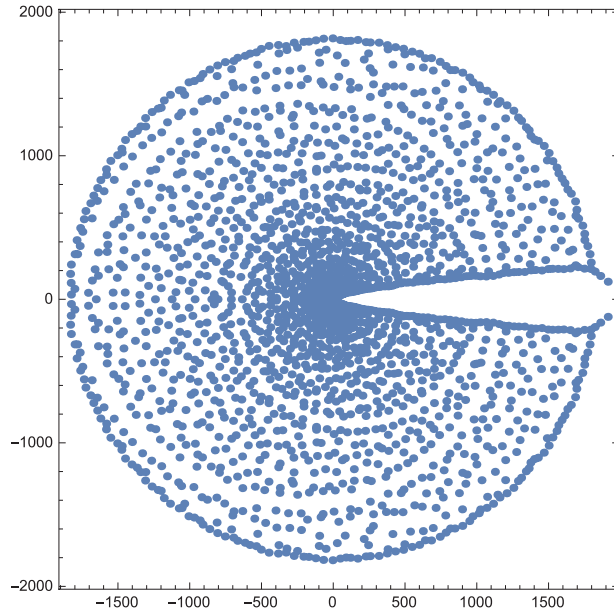


Figure 6. An example of area law: distribution of zeros of the partition function where the coefficients are given by $a_n = [n \log n]$.

A representative polynomial may be

$$H(z) = (z - N)(z - N + 1)(z - N + 2) \cdots (z - 2N) . \tag{104}$$

The zeros of this polynomial are $z = N, N + 1, \dots, 2N$. Their arithmetic mean is $\bar{z}_{\text{arith}} = 3(N + 1)/2$, its geometrical mean goes as $\bar{z}_{\text{geom}} \sim N$ and its variance goes as $\mu^2 \sim N^2/12$. The geometrical shape of these zeros consists of a cluster of N isolated zeros, all of order N , which move away from the origin enlarging their spreading. This kind of zeros do not satisfy neither area nor perimeter law but are rather discrete isolated points which move at $z = \infty$ by increasing N .

Let us close our discussion on the patterns of zeros by commenting on the information one can extract from the statistical indicators. To this aim let us consider another example of zeros which satisfy the area law, as the case F analysed above: this example consists of a polynomial whose coefficients p_m are given not by the prime numbers but by their approximation $p_m = [m \log m]$, where $[x]$ denotes the integer part of the real number x . In this case all the three statistical indicators (as well as the bounds) coincide with those of the polynomial based exactly on the prime numbers and therefore also for this example we expect that its zeros satisfy the area law. They do indeed but their actual distribution has a finer structure, with arcs and other curves along which the zeros are placed, as evident from their plot in the complex plane shown in figure 6. This example shows explicitly that the arithmetic and geometrical means of the zeros and their variance may help in capturing the gross structure of the zeros but of course not the detailed features of their actual distribution.

4. Free energy of the free bosonic and fermionic theories

In this section we consider quantum theories in which there is no interactions among the particles. For these theories we can study the distribution of the zeros and learn some interesting facts about these zeros. The theories that we consider are:

1. non-relativistic free bosonic and fermionic theories in d dimensions;
2. non-relativistic bosonic and fermionic theories in a (spherical symmetrical) harmonic trap in d -dimensions;
3. relativistic free bosonic and fermion theories in d dimensions.

As shown below, all these cases can be treated altogether. The starting point is the grand-canonical partition function Ω for bosons and fermions [23]

$$\Omega(z) = \prod_p (1 \pm z e^{-\beta \epsilon_p})^{\pm 1}, \quad (105)$$

where $+$ refers to fermion and $-$ to boson, with the infinite product extended to all values p of the momentum which parameterises the energies of the excitation. Taking the logarithm of both sides we end up in

$$\log \Omega(z) = \pm \sum_p \log (1 \pm z e^{-\beta \epsilon_p}) \equiv F_{\pm}(z, \beta), \quad (106)$$

where

$$F_{\pm}(z, \beta) = \pm \int d\epsilon g(\epsilon) \log (1 \pm z e^{-\beta \epsilon}). \quad (107)$$

This universal way to express the free energy simply employs the *density of states* $g(\epsilon)$ of each system, whose definition is

$$g(\epsilon) = \int \frac{d\mathbf{r} d\mathbf{p}}{(2\pi\hbar)^d} \delta(\epsilon - H(r, p)). \quad (108)$$

We report hereafter the expression of the various densities of states (got by a straightforward calculation) and the expression of the relative free-energies for the various cases enumerated above.

1. **Non-relativistic free bosonic and fermionic theories in d dimensions.** Both these theories have the Hamiltonian

$$H = \frac{p^2}{2m}, \quad (109)$$

and their density of states is given by

$$g_{nr}(\epsilon) = V \left(\frac{m}{2\pi\hbar^2} \right)^{\frac{d}{2}} \frac{1}{\Gamma(\frac{d}{2})} \epsilon^{\frac{d}{2}-1}. \quad (110)$$

Substituting this expression in equation (107) and expanding the logarithm for $|z| < 1$, we have

$$\begin{aligned}
 F_{\pm}(z, \beta) &= \sum_{k=1}^{\infty} \int_0^{\infty} d\epsilon g(\epsilon) (\mp 1)^{k+1} \frac{z^k}{k} e^{-k\beta\epsilon} \\
 &= \frac{V}{\lambda_T^d} f_{\pm}(z; d),
 \end{aligned}
 \tag{111}$$

where λ_T is the thermal wave-length

$$\lambda_T = \left(\frac{2\pi\hbar^2}{m k T} \right)^{1/2},
 \tag{112}$$

while the functions $f_{\pm}(z; d)$ depend upon *only* z and the dimensionality d , but not from the temperature T

$$f_{\pm}(z; d) = \sum_{k=1}^{\infty} (\mp 1)^{k+1} \frac{z^k}{k^{\frac{d}{2}+1}}.
 \tag{113}$$

These functions can be expressed in terms of poly-logarithmic functions, defined by

$$L_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}.
 \tag{114}$$

More precisely, for the boson, we have

$$f_-(z; d) = L_{\frac{d}{2}+1}(z),
 \tag{115}$$

while for the fermion

$$f_+(z; d) = -L_{\frac{d}{2}+1}(-z).
 \tag{116}$$

2. Non-relativistic bosonic and fermion theories in a (spherical symmetrical) harmonic trap in d -dimensions. The Hamiltonian in this case is given by

$$H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 r^2,
 \tag{117}$$

and the density of states is

$$g_h(\epsilon) = \left(\frac{1}{\hbar\omega} \right)^d \frac{1}{\Gamma(d)} \epsilon^{d-1}.
 \tag{118}$$

Notice that, apart from the prefactors, the density of states for the harmonic trap in d -dimension is the *same* of the one of the free theories but in a dimension twice as large! Therefore, repeating the same computations as before, the expression of the free energies is

$$F_{\pm}(z, \beta) = \left(\frac{1}{\hbar\omega} \right)^d f_{\pm}(z; 2d).
 \tag{119}$$

3. Relativistic free bosonic and fermionic theories in d dimensions. The Hamiltonian of these cases is

$$H = \sqrt{p^2 + m^2},
 \tag{120}$$

and for the density of states we have

$$g_r(\epsilon) = 2V \left(\frac{1}{4\pi\hbar^2} \right)^{\frac{d}{2}} \epsilon (\epsilon^2 - m^2)^{\frac{d}{2}-1} . \quad (121)$$

Repeating the same steps as before, we end up in the following expression of the free energies

$$F_{\pm}(z, \beta) = \frac{V}{\lambda_T^d} \sqrt{\frac{2m\beta}{\pi}} H_{\pm}(z; d; \beta) , \quad (122)$$

where

$$H_{\pm}(z; d; \beta) = \sum_{n=1}^{\infty} \frac{(\mp 1)^{n+1}}{n^{\frac{d+1}{2}}} K_{\frac{d+1}{2}}(n\beta m) z^n , \quad (123)$$

and $K_{\nu}(z)$ is the modified Bessel function. In these relativistic cases the dependence on the temperature T no longer factorises as in the non-relativistic cases, but also enters explicitly the cluster coefficients through the Bessel function. These distributions correspond to free bosonic (−subscript) and fermionic fields (+ subscript). In particular, for $d = 1$ the fermionic distribution is relevant for the 2D classical Ising model, regarded as a 1D quantum model of free Majorana fermions (see, for instance [17, 20, 21, 46, 47]).

5. Distribution of the zeros for non-relativistic free theories

In this section we study the distribution of the zeros relative to the non-relativistic free theories. Surprisingly enough, there are two alternative ways to address the problem that end up in two different distributions of the zeros. We denote the first method as *truncated series approach* (TSA) while the second method as *infinite product approach* (IPA). With the TSA, we will get an expression $F_1(z)$ of the free energy of the systems which is valid in the disk $|z| < 1$ while, with the IPA, we will get instead an expression $F_2(z)$ which extends to the entire complex plane, providing in particular the analytic continuation of $F_1(z)$ outside its disk of convergence. It is important to notice, though, that both distributions of zeros share the same value of all the moments of the zeros. Let us discuss these two methods in more detail.

5.1. Truncated series approach

As shown in section 4, there is a closed expression for the free energies $F_{\pm}(z, \beta)$ for the non-relativistic free theories of bosons and fermions, see equation (111). However, to determine the Yang–Lee zeros we need to consider the zeros of the partition function itself, alias of a sequence of polynomials $\hat{\Omega}_{\pm}^{(N)}(z, \beta)$ in the limit $N \rightarrow \infty$. This sequence of polynomials can be constructed by truncating the series relative to $F_{\pm}(z, \beta)$ up to a given order N and then expanding the exponential exactly up to that order. In more detail:

1. Let us define the truncated expression of the free energies up to the order N as

$$F_{\pm}^{(N)}(z, \beta) \equiv \frac{V}{\lambda_T^d} \sum_{k=1}^N (\mp 1)^{k+1} \frac{z^k}{k^{\frac{d}{2}+1}} . \quad (124)$$

2. Let us also define

$$\hat{\Omega}_{\pm}^{(N)}(z, \beta) \equiv e^{F_{\pm}^{(N)}(z, \beta)} , \quad (125)$$

where $\hat{\Omega}_{\pm}^{(N)}(x, \beta)$ is given by the Taylor series

$$\hat{\Omega}_{\pm}^{(N)}(x, \beta) \equiv \sum_{k=0}^N \frac{p_k}{k!} z^k = \sum_{k=0}^N \frac{z^k}{k!} \frac{d^k \hat{\Omega}_{\pm}^{(N)}}{dz^k}(0, \beta), \quad (126)$$

truncated at the order N . With the coefficients of the truncated polynomial $\hat{\Omega}_{\pm}^{(N)}(x, \beta)$ determined as in equation (126), it is easy to see (compare with section 3) that—by construction—the negative moments of the zeros of this polynomial coincide exactly with those given in equation (56)

$$\hat{s}_k = -\frac{V}{\lambda_T^d} (\mp 1)^{k+1} \frac{1}{k^{\frac{d}{2}}}. \quad (127)$$

Let us underline that in equation (126) the coefficients p_k relative to the bosonic case are all positive, while those of the fermionic case have alternating sign. In order to have always a positive and increasing value of $F(z)$ by increasing z , in the case of fermions it is convenient to always truncate at an even power of N . Let us observe, moreover, that in all the expressions given above of the free energy the dependence upon the volume V is always factorised. So, it is convenient to divide by V and consider the intensive part of all quantities. Equivalently we can just work putting $V = 1$, which is what we have done in the rest of the paper for the distributions of the zeros associated to the actual partition functions. The numerical determination of the zeros of the polynomials (126) are shown in figure 7, where the module of the roots are expressed in unit of V/λ_T^d .

There are many interesting features of the distribution of these zeros:

- for a finite value of N , the zeros are placed on a curve which looks to be a circle but as a matter of fact is not! A fit of their location with a circle (i.e. with fit parameters given by the mean radius R_N and the center $z_N^{(0)}$) gives values of the mean radius R_N typically larger than 1 and a non-zero value of z_0 , therefore the circle determined by the fit is anyhow displaced from the origin. However these are just finite-size effects. In fact increasing the values of N and extrapolating the asymptotic values of R_N and $z_N^{(0)}$ we have⁹ (see figure 8)

$$\lim_{N \rightarrow \infty} R_N \rightarrow 1, \quad \lim_{N \rightarrow \infty} z_N^{(0)} \rightarrow 0. \quad (128)$$

Notice that the ratio $r_n = \gamma_n/\gamma_{n-1}$ of two consecutive coefficients of $\Omega_N(z)$ is not a flat curve and the minimum R_d and the maximum R_u of this sequence do not coincide, see figure 9. According to the Eneström bounds, the roots then generally lay in the annulus

$$R_d \leq |z_i| \leq R_u. \quad (129)$$

- However, despite the fact that the asymptotic value of the mean radius being $R_{\infty} = 1$ and the apparent circular shape of the zeros as shown in figure 7, the asymptotic distribution of the zeros is not on a circle! This can be proved in two different ways. The first consists in parameterising the various zeros z_k as $z_k = \rho_k e^{i\theta_k}$ and plotting the dependence of ρ_k from the angle θ_k : if the distribution was a circle, we shall find a flat function for the radius $\rho(\theta)$ while the actual functions for boson and fermion are the ones shown in figure 10. This figure shows that the radius of the zeros is not constant and this feature persists also in the limit $N \rightarrow \infty$. However, in addition to this numerical evidence, there is a simple

⁹The result $\lim_{N \rightarrow \infty} R_N \rightarrow 1$ holds by taking $V/\lambda_T^d = 1$. Restoring this factor, the radius as a function of the inverse temperature β goes as $\beta^{d/2}$.

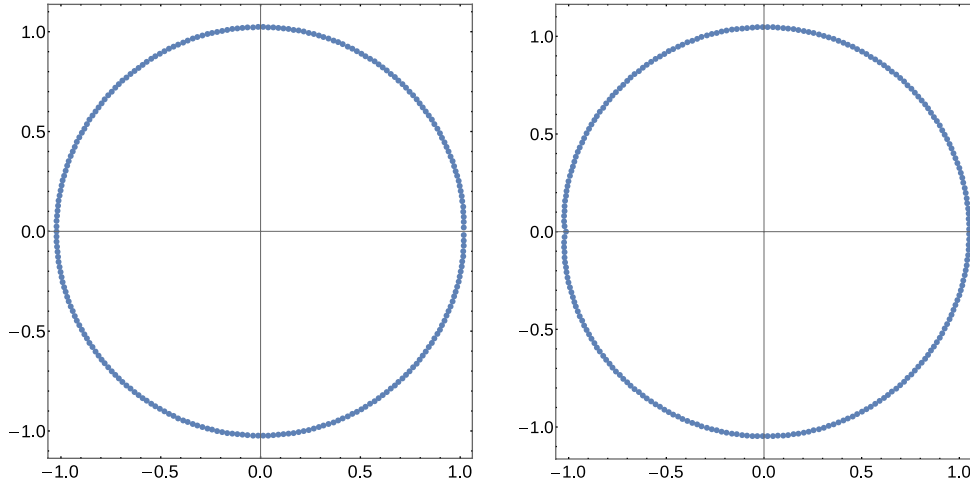


Figure 7. (a) Distribution of zeros of the partition function of the boson (left) and fermion (right) for $d = 2$ and with the partition function of equation (126) truncated to the first 300 terms. The module of the roots are in units of V/λ_T^d .

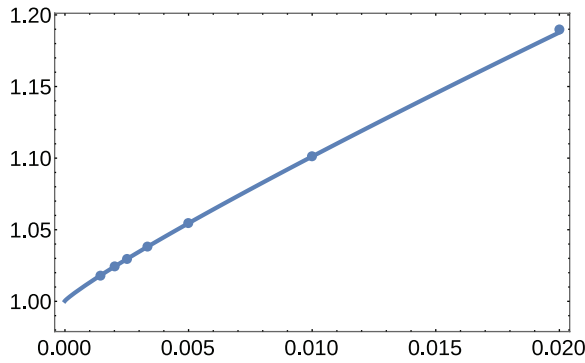


Figure 8. Scaling of the radius of the distribution of zeros of the non-relativistic fermionic free theory for increasing number of terms N of the series at $\beta = 1$. The values of the radius R_N are plotted together a function of $1/N^\alpha$, where the best fit of the parameter α in this case is given by $\alpha = 0.91 \pm 0.04$. For $1/N \rightarrow 0$ the average radius R_N goes to 1.

but incisive analytic argument to show that in general, for the free non relativistic theories (and for the harmonic trap as well) the zeros are *not* placed on a circle. Indeed, if they were on a circle, the coefficients of the expansion of the functions $f_{\pm}(z)$ would be simply related to the Fourier coefficients of the distribution function $\eta(\theta)$ of the zeros along the unitary circle, parameterised by the angle θ [3]. Namely, if

$$\left(\frac{\lambda_T}{V}\right) \log \Omega = \sum_{n=1}^{\infty} b_n z^n, \tag{130}$$

then

$$b_n = -\frac{1}{n} \sum_{l=1}^N \left(\frac{1}{z_l}\right)^n = -\frac{1}{n} \int_{-\pi}^{\pi} \eta(\theta) \cos(n\theta) d\theta. \tag{131}$$

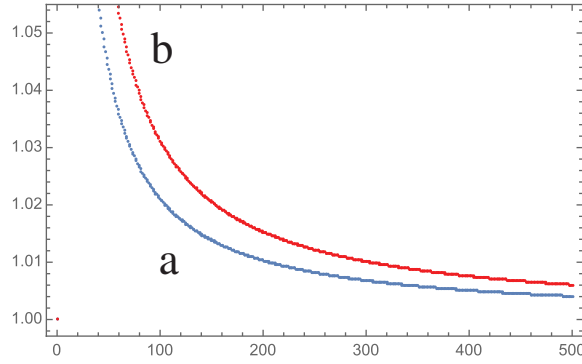


Figure 9. Ratio $r_n = \gamma_n/\gamma_{n-1}$ versus n of the coefficients of $\Omega_N(z)$ for bosonic system with $d = 2$ (curve a) and $d = 4$ (curve b). The minimum and the maximum values of these curve do not coincide: while the minimum tends to 1, the maximum of these sequence is always larger than 1.

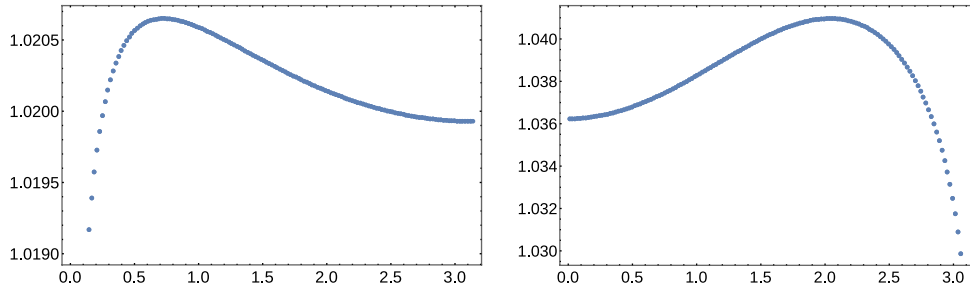


Figure 10. Radius ρ versus the angle θ for the 300 zeros shown in figure 7. The boson case is on the left, the fermion case on the right.

Hence, the distribution function $p(\theta)$ of the zeros would be given by the Fourier series

$$\eta(\theta) = \frac{1}{2\pi} - \frac{1}{\pi} \sum_{n=1}^{\infty} n b_n \cos(n\theta) . \tag{132}$$

The interesting point is now that, for various values of d and both for boson and fermion, one knows how to sum the Fourier series! For instance

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos n\theta = \frac{1}{4} \left(\frac{\pi^2}{3} - \theta^2 \right) , \tag{133}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \cos n\theta = \frac{1}{4} (\pi - |\theta|)^2 - \frac{\pi^2}{12} , \tag{134}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cos n\theta = \log \left(2 \cos \frac{\theta}{2} \right) , \tag{135}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \cos n\theta = -\log \left(2 \sin \frac{\theta}{2} \right) . \tag{136}$$

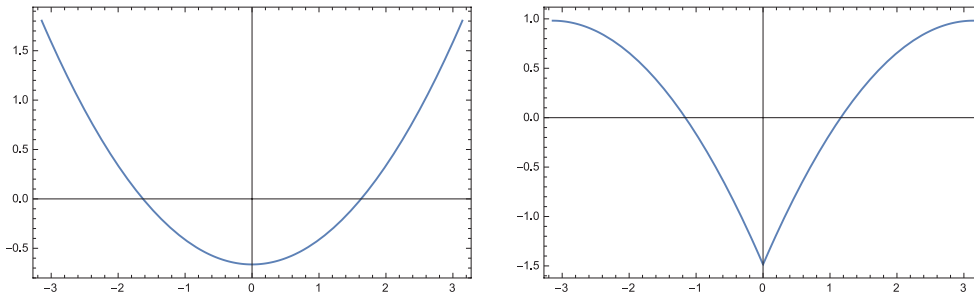


Figure 11. Plot of the function $\eta(\theta)$ defined in equation (132) with the Fourier series given by equation (133) (left-hand side plot) and equation (134) (right-hand side plot). In both cases the function $\eta(\theta)$ has an interval in which it is negative.

For all these cases (which corresponds to fermion or boson in different dimensions), one can easily see that the density function $\eta(\theta)$ of the zeros becomes *negative* in some intervals (see figures 11 and 12). But this is a contradiction, since $\eta(\theta)$ has to be intrinsically a positive function. So, for free non-relativistic theories, this proves that their Yang–Lee zeros cannot be exactly on a circle.

- The distribution of the N zeros, both in the bosonic and in the fermionic case, are dense nearby $z = 1$. While this may signal an actual phase transition in the bosonic case (and this, we know, only happens for a finite density of the bosonic gas for $d > 2$), for the fermionic case we know that there cannot be a phase transition at $z = 1$, therefore this singularity for the fermionic systems simply emerges from the finite radius of convergence of the series (111) employed in the method. The difference between the two cases of course would become evident by employing the analytic continuation of the expression of the pressure given by the original formula (107) in which it will be present the singularity at $z = 1$ for the boson and its absence for the fermion.

5.2. Infinite product approach

Ikeda, in a series of papers [28, 29], has worked out the distribution of the zeros of the grand-canonical partition function of free bosons and fermions starting directly from the factorised formula (105). The particles of these gas do not have hard core and consequently the grand-canonical partition function has always an infinite number of terms, even at a finite volume. Notice that the fermionic grand-canonical partition function has genuine zeros whose physical origin may be traced back to the effective repulsion of particles due to Fermi–Dirac statistics. The bosonic grand-canonical partition function has instead poles rather than zeros: however, for what the free-energy of the system is concerned (where it matters taking the logarithm of the partition function), these poles play essentially the same role as the zeros. In light of this remark, let us initially focus the attention on the computation of the density of the zeros for the fermionic case.

Contrary to the TSA, it is important to stress that in the IPA there is no consistent way of getting a sequence of polynomials of increasing order N . Indeed, there are infinitely many polynomials of order N that could be defined: to do so, simply pick up N arbitrary terms out of the infinite product (105). Therefore in this approach we must necessarily consider at once the totality of all the infinite zeros.

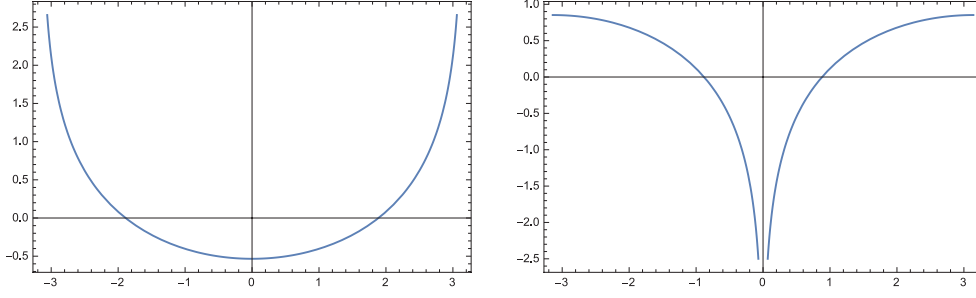


Figure 12. Plot of the function $\eta(\theta)$ defined in equation (132) with the Fourier series given by equation (135) (left-hand side plot) and equation (136) (right-hand side plot). In both cases the function $\eta(\theta)$ has an interval in which it is negative.

5.2.1. *Fermionic case.* From the explicit factorised formula (105), it is obvious that for the fermionic case the zeros are on the negative real axis, from $z = -1$ to ∞ , and localised at the positions

$$z_p = -e^{\beta\epsilon_p} . \tag{137}$$

Their density $\eta(z)$ can be explicitly computed using the density of states $g(\epsilon)$. Indeed, from equation (137), solving for ϵ_p , we have

$$\epsilon_p = kT \log(-z_p) , \tag{138}$$

and therefore

$$\frac{d\epsilon}{dz} = \frac{kT}{z} . \tag{139}$$

Since the number of energy states with energy values in the shell $(\epsilon, \epsilon + d\epsilon)$ is given by $g(\epsilon)d\epsilon$, the density of zeros $\eta(x)$ in the interval between x and $x + dx$ along the negative real line is

$$\eta(x) = g(\epsilon) \left| \frac{d\epsilon}{dx} \right| . \tag{140}$$

Let us first consider the density of zeros for the non-relativistic fermionic system in d dimensions. Employing equation (110) (with $V = 1$), we have

$$\eta(x) = - \left(\frac{m}{2\pi\hbar^2} \right)^{\frac{d}{2}} \frac{1}{\Gamma(\frac{d}{2})} \frac{kT}{x} (kT \log(-x))^{\frac{d}{2}-1}, \quad -\infty < x < -1. \tag{141}$$

This function changes its behavior by varying the dimensionality d of the system, in particular for $d > 2$, it vanishes at $x = -1$, while for $d < 2$ it diverges (see figure 13). Notice that the density of the zeros given in equation (141) is not normalisable

$$\int_{-\infty}^{-1} \eta(x) dx = \infty , \tag{142}$$

since the number of states of the fermionic gas is infinite, independently from the volume V . One can easily compute the negative moments of the zeros and see that they correctly coincide with the cluster coefficients of the free energy of the fermionic system

$$b_m = -\frac{1}{m} \int_{-\infty}^{-1} \eta(x) x^{-m} dx = (-1)^{m-1} m^{-d/2-1} . \tag{143}$$

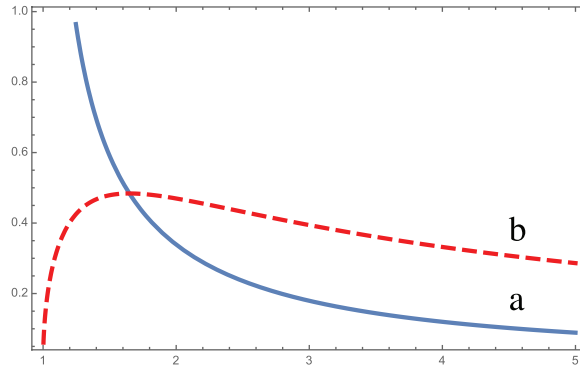


Figure 13. Plot of the density of zeros $\eta(-x)$ defined in equation (141) along the real axis of the fermionic free theories, where the curve (a) refers to $d = 1$ while the curve (b) to $d = 3$.

Finally, let us comment that it is also easy to compute the density of zeros also in the relativistic case, the only thing that changes in this case is the density of states employed by the formula which in this case is given in equation (121). Hence

$$\eta(x) = -2 \left(\frac{1}{4\pi\hbar^2} \right)^{\frac{d}{2}} kT \log(-x) \left((kT \log(-x))^2 - m^2 \right)^{\frac{d}{2}-1} \frac{kT}{x}. \quad (144)$$

This distribution is defined on an interval determined by the mass, $x \in (-\infty, -e^{m/kT})$.

5.2.2. Bosonic case. Let us now briefly discuss the bosonic case and the phase transition which this system may have at $z = 1$. Repeating the same considerations as in the fermionic case, we arrive to the density of zeros given by this distribution

$$\eta(x) = \left(\frac{m}{2\pi\hbar^2} \right)^{\frac{d}{2}} \frac{1}{\Gamma\left(\frac{d}{2}\right)} \frac{kT}{x} (kT \log(x))^{\frac{d}{2}-1}, \quad 1 < x < \infty \quad (145)$$

which extends on the positive interval $(1, \infty)$. Hence, this positive interval is made of singular values for the pressure, the smaller singular value being $z = 1$. This seems to signal a phase transition of the bosonic system for any value of its dimensionality d . Let us compute however the density of the system which, for $|z| < 1$, is given by

$$\rho(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^{\frac{d}{2}}}, \quad (146)$$

and, at $z = 1$, by the Riemann zeta-function $\zeta(s)$ computed at $s = d/2$

$$\rho(1) = \zeta\left(\frac{d}{2}\right). \quad (147)$$

Since for $d \leq 2$, $\rho(1)$ diverges while is finite for $d > 2$, the conclusion is that the bosonic system have a phase transition at a finite density of the gas only for $d > 2$, while for lower values of d the phase transition at $z = 1$ is only realised at the price to have an infinite density of the gas, in other words, correctly it will never happen at a finite density of the gas.

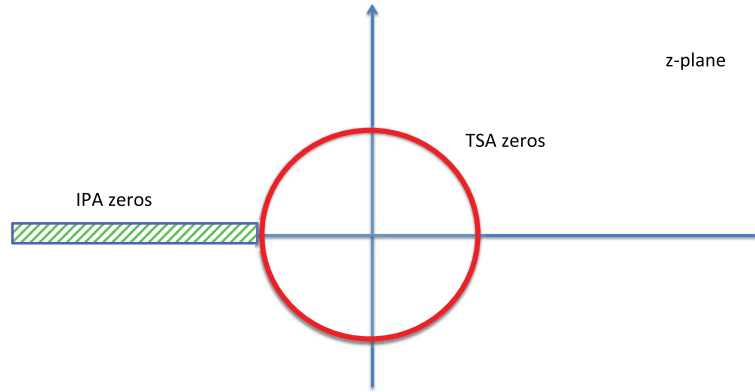


Figure 14. Distribution of the Yang–Lee zeros in the infinite product approach (IPA) and in the truncated series approach (TSA).

5.3. Discussion

For the partition function of the free non-relativistic fermionic theories¹⁰ we have seen that it can be characterised by two different sets of zeros: the first consisting of zeros approximately placed along a circle of radius 1, the second consisting of zeros placed on the negative real axis from $z = -1$ to $-\infty$, see figure 14. Although these sets of zeros are quite different, both give rise to the *same* values of the negative moments of the zeros: in the TSA, this is true by construction, since the coefficients of the truncated polynomials come directly from the cluster coefficients; in the IPA, this was shown by the integrals (143). Imagine now that we know the zeros coming from TSA and imagine we employ equation (4) to compute the pressure of the gas: notice that in this case we can define the pressure $p(z)$ *only* for $|z| < 1$, because this was, from the start, the domain of validity of this approach: indeed, it is based on the expansion of the logarithm present in equation (107), so that all the following series have inherently a finite radius of convergence coming from the radius of convergence of the series expansion of the logarithmic function. This state of matter is similar to the case discussed in appendix A. Concerning the IPA, it is worth to notice that it may be simply interpreted as a change of variable done in such a way to bring the original expression (107) of the free energy in a ‘Yang–Lee’ form (14): hence, the corresponding integral on the set of zeros ranging in the interval $(-1, -\infty)$ provides a function which is also defined for $|z| > 1$, which is of course the *analytic continuation* of free-energy defined by the TSA. In the real interval $0 < z < 1$, the two functions obtained by employing the two different sets of zeros of course coincide. Even though in the free theories the IPA seems to perform better than TSA, in the interactive theories discussed in the next sections we will see however that the the proper Yang–Lee zeros can be identified only using the TCS.

6. Distribution of the zeros for relativistic free theories

Let us now employ the TSA to determine the zeros of the relativistic free theories whose free energy was given in equations (122) and (123). Particularly important for later comparisons is to work out in some detail the bosonic/fermionic case for $d = 1$: for this value of d , for instance, the fermionic case can be directly related to the partition function of the 2D classical

¹⁰The same is also true for the bosonic theories.

Ising model [17, 20, 21, 46, 47] and moreover it provides the proper limit of the interactive integrable model analysed in the next sections when $\beta \rightarrow \infty$.

For $d = 1$ the pressure of boson/fermion is given by

$$F_{\pm}(z, \beta) = \frac{V}{\lambda_T} \sqrt{\frac{2m\beta}{\pi}} \hat{H}_{\pm}(z; \beta), \quad (148)$$

where

$$\hat{H}_{\pm}(z; \beta) = \sum_{n=1}^{\infty} \frac{(\mp 1)^{n+1}}{n} K_1(n\beta m) z^n. \quad (149)$$

Notice that, with the dependence of the thermal length λ_T from the temperature $T = 1/\beta$ given in equation (112), the pre-factor in front of $\hat{H}_{\pm}(z; \beta)$ in equation (148) is completely independent on the temperature and therefore for $d = 1$ the only dependence on temperature comes from the argument of the Bessel functions. From now on we put the value of the mass to 1, $m = 1$.

For any finite value of β , using the TSA we can determine the zeros of the truncated partition function $\hat{\Omega}_{\pm}^{(N)}(z)$ and they essentially lay along circles (see figure 15) whose radius $R(\beta)$ has an exponential dependence on β , $D \simeq e^{\beta}$ as shown in figure 16. For large β this exponential behaviour has an easy explanation. Consider the bosonic case, where all coefficients of $\hat{\Omega}_{+}^{(N)}(z)$ are positive (or equivalently one can consider the fermionic case computed at $-z$) so that we can apply the Eneström bounds (42) and (43) given by the ratio of the consecutive coefficients of $\hat{\Omega}_{+}^{(N)}(z)$: it is easy to see that the maximum of the sequence $\left\{ \frac{\gamma_k}{\gamma_{k+1}} \right\}$ is obtained for $k = 0$ and therefore is

$$\max \left\{ \frac{\gamma_k}{\gamma_{k+1}} \right\} = \frac{\gamma_0}{\gamma_1} = \frac{1}{K_1(\beta)} \sim \sqrt{\frac{2\beta}{\pi}} \exp(\beta). \quad (150)$$

Concerning the minimum of the same sequence, it is also easy to see that scale with the same exponential law: in fact from the exponential expansion of (148) the general expression of the k th term γ_k is given by [45]

$$\gamma_k = \alpha_1 K_1^k(\beta) + \alpha_2 K_1^{k-2}(\beta) K_1(2\beta) + \alpha_3 K_1^{k-3}(\beta) K_1(3\beta) + \alpha_3 K_1^{k-4}(\beta) K_1^2(2\beta) + \dots + \alpha_p K_1(k\beta), \quad (151)$$

where all terms, for large β , scale as $e^{-k\beta}$ and therefore $\gamma_k \simeq A_k e^{-k\beta}$. Hence, taking the ratio of two consecutive coefficients we have that

$$\min \left\{ \frac{\gamma_k}{\gamma_{k+1}} \right\} = \left(\min \left\{ \frac{A_k}{A_{k+1}} \right\} \right) e^{\beta}. \quad (152)$$

Putting $B = \lim_{k \rightarrow \infty} \frac{A_k}{A_{k+1}}$ (and it possible to prove that this number is of order 1), we have that the module of the all zeros satisfy the bounds

$$B e^{\beta} \leq |z_i| \leq \sqrt{\frac{2\beta}{\pi}} e^{\beta}. \quad (153)$$

More subtle is the behavior of the radius $R(\beta)$ of the zeros when $\beta \rightarrow 0$. Indeed, we can prove that this function must have a discontinuous behavior at $\beta = 0$, namely arbitrarily close to this value, we have $R = 1$ while at exactly $\beta = 0$, we have $R(0) = 0$. To show this discontinuity, we follow two different routes: the first consists of computing the series (149) for a

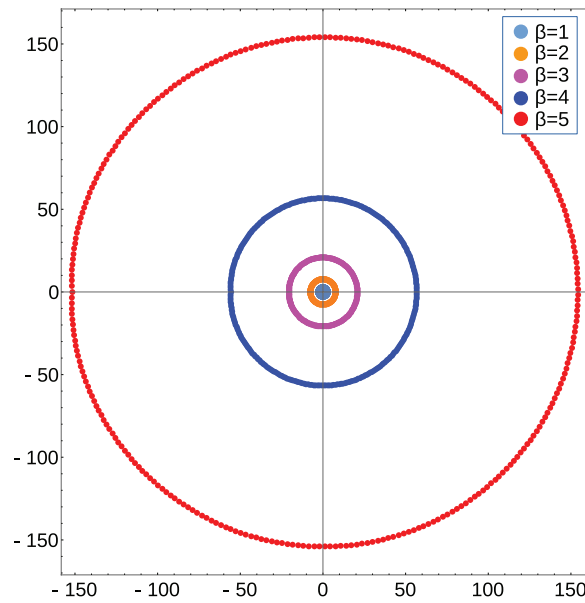


Figure 15. Yang–Lee zeros of the relativistic free fermionic case for $d = 1$ for various values of β .

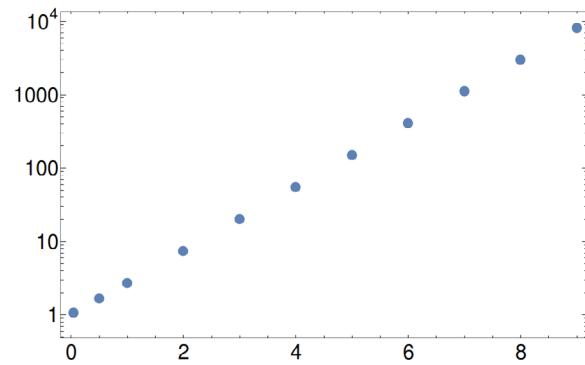


Figure 16. Logarithm of the radius R versus β .

small value of β : it is clear that in this case the first coefficients may be quite large (since they are computed near the origin of the Bessel function where $K_1(x) \simeq \frac{1}{x}$ if $x \rightarrow 0$) but for all indices n larger than a certain index \tilde{n} (with $\tilde{n}\beta > 1$), all the remaining coefficients become once again small since they refer to large value of the Bessel function. Under such circumstances, the sequence $\left\{ \frac{\gamma_k}{\gamma_{k+1}} \right\}$ is no longer monotonically decreasing as it was for large β but nevertheless has a min and a max of order 1, so that the circle of the zeros has also a radius $D \simeq 1$.

The second route consists instead in taking directly the limit $\beta \rightarrow 0$ in the series (149): if we do so, we can substitute each Bessel function with its short distance asymptotic behavior and therefore one has

$$\hat{H}_{\pm}(z; \beta) = \frac{1}{\beta} \sum_{n=1}^{\infty} \frac{(\mp 1)^{n+1}}{n^2} z^n = \mp \frac{1}{\beta} L_2(\pm z) , \tag{154}$$

where $L_2(z)$ is the dilogarithmic function, obtained by equation (114) with $s = 2$ (the interested reader may consult the [44] for the remarkable mathematical properties of this function!). It is now clear why at $\beta = 0$ the radius of the zeros vanishes. Indeed in this case the partition function is given by (consider for simplicity the bosonic case where we can easily apply the Eneström bounds)

$$\Omega_+(z) = e^{\frac{1}{\beta} L_2(z)} . \tag{155}$$

Strictly speaking this result just applies at $\beta = 0$ ¹¹ but if we assume the validity of equation (155) even for infinitesimal values of β , this implies that the ratio of two consecutive coefficients of $\Omega_+(z)$ is always proportional to β and therefore for all roots we have the bounds

$$\mu \beta \leq |z_i| \leq \nu \beta , \tag{156}$$

(for some constant μ and ν) and therefore when $\beta = 0$ the module of the roots vanishes. The discontinuous behavior of $R(\beta)$ at $\beta = 0$ comes because equation (155) in free theories actually holds only at $\beta = 0$.

7. Integrable quantum field theories

7.1. Elastic S-matrix

Relativistic quantum integrable models have the distinguished property that their S -matrix is elastic and factorizable in terms of the two-body S -matrix amplitudes [15–17]. Hereafter, for simplicity, we will only consider quantum integrable models in $(1 + 1)$ dimensions with the spectrum consisting of just one neutral particle A of mass m . Our main example is the Yang–Lee model [11, 14, 18, 47].

In order to write down explicitly the scattering amplitude of such simple integrable theory it is convenient to parameterise the energy E and the momentum p of the particle in terms of the so-called rapidity variable θ as

$$E = m \cosh \theta, \quad p = m \sinh \theta . \tag{157}$$

The two-body S -matrix amplitude is then defined by

$$|A(\theta_1) A(\theta_2)\rangle = S(\theta_1 - \theta_2) |A(\theta_2) A(\theta_1)\rangle . \tag{158}$$

It depends on the difference of the rapidities $\theta \equiv \theta_1 - \theta_2$ and satisfies the unitarity and crossing equations [15–17]

$$S(\theta) S(-\theta) = 1 , \tag{159}$$

$$S(\theta) = S(i\pi - \theta) . \tag{160}$$

As evident from equation (158), the S -matrix also plays the role of braiding matrix, interchanging the rapidities of the two particles. Notice that the unitarity condition (159) forces the value of the S -matrix at $\theta = 0$ to be $S(0) = \pm 1$. Consistent models have been found only for $S = -1$ (see, for instance, [48]): models whose S -matrix satisfies $S(0) = -1$ are called

¹¹ This means that the proper mathematical statement is that $\beta \log \Omega(z)$ has a well-defined value in $\beta = 0$ given by $L_2(z)$.

fermionic models because, looking at equation (158), interchanging two excitations of equal rapidities, their wave-function changes sign. As discussed below, this property is important for the thermodynamics of the integrable models. Let us notice that for such simple quantum integrable systems the S -matrix is a pure phase

$$S(\theta) = e^{i\delta(\theta)} , \quad (161)$$

where $\delta(\theta)$ is the phase shift. For later use it is convenient to introduce the kernel $\varphi(\theta)$, given by the derivative of the phase shift

$$\varphi(\theta) = \frac{d\delta(\theta)}{d\theta} . \quad (162)$$

7.2. Thermodynamic Bethe Ansatz

Zamolodchikov [18] derived the basic equations which allows us to compute the free-energy of a quantum integrable model defined on a cylinder of width β and length L . The radius β of the cylinder is directly associated to the temperature T of the system through the relation $\beta = 1/T$ while L is the volume of the system. The formalism can be generalised to include also the chemical potential [20, 21] and the final equations read as follow: the pressure $p(z)$ is expressed by

$$\beta p(z) = \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \cosh \theta \log \left(1 + z e^{-\epsilon(\theta,z)} \right) , \quad (163)$$

where the function $\epsilon(\theta, z)$ (called pseudo-energy) satisfies the non-linear integrable equation

$$\epsilon(\theta, z) = m\beta \cosh \theta - \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \varphi(\theta - \theta') \log \left(1 + z e^{-\epsilon(\theta',z)} \right) . \quad (164)$$

Therefore the partition function of the system is given by

$$\Omega(z) = \exp \left[mL \int_{-\infty}^{+\infty} \cosh \theta \log \left(1 + z e^{-\epsilon(\theta,z)} \right) \frac{d\theta}{2\pi} \right] . \quad (165)$$

Notice that $\Omega(z)$ refers already to the thermodynamic limit where both L and the number of particles N go to infinity with their ratio given by the density $\rho(z) = z \frac{d\Omega}{dz}$ at a fixed value of z . It is worth noticing that the pressure has formally the same expression as in the free theories, the only difference is that while in free theories the pseudo-energy $\epsilon(\theta)$ coincides with the energy of the particle¹² multiplied by β , in an interactive integrable theory it satisfies instead the non-linear integral equation (164). Observe that the pseudo-energy $\epsilon(\theta, z)$ of course also depends on β but we avoid to write this explicitly to keep the notation simpler. The partition function (165) does not have a singularity for positive value of z , in agreement with the purely massive behavior of the theory and its fermionic nature.

In summary, for a quantum integrable model, it is sufficient to know its S -matrix for computing its partition function (165) at a finite temperature and a chemical potential through the equations (163) and (164). In the next section we will study the zeros of this partition function for one of the simplest integrable quantum field theories, the Yang–Lee model.

¹² If the theory is free the S -matrix is a constant and therefore the kernel $\varphi(\theta)$ vanishes. Therefore, in this case, equation (164) states that $\epsilon(\theta) = m\beta \cosh \theta$.

8. Partition function of the Yang–Lee model

In this section, using the TSA described in section 5.1, we are going to determine the distribution of the Yang–Lee zeros of the Yang–Lee model in $(1 + 1)$ dimensions. To proceed, let us briefly recall the exact S -matrix of this model.

8.1. S -matrix of the Yang–Lee model

An example of the S -matrix which satisfy all the requirements discussed above is

$$S_{YL}(\theta) = \frac{\tanh \frac{1}{2} \left(\theta + \frac{2\pi i}{3} \right)}{\tanh \frac{1}{2} \left(\theta - \frac{2\pi i}{3} \right)}. \quad (166)$$

In [11] it has been identified with the exact scattering amplitude of the massive theory that comes from the deformation of the non-unitarity conformal minimal model with central charge $c = -22/5$ [12] made by its only relevant field ϕ of conformal dimension $\Delta = -1/5$. Apart being a fermionic S -matrix, the amplitude (166) has its own peculiarities: first of all, it has a pole at $\theta = \frac{2\pi i}{3}$ and another at $\theta = i\frac{\pi}{3}$ which correspond to the same particle A that appears as a bound state of itself both in the s and the t -channel. Secondly, the residue of the S -matrix at the pole in the s -channel has the ‘*wrong*’ sign with what usually expected in an hermitian theory. Generally, the residue in the s -channel pole at iu of an S -matrix is associated to the Feynman diagram shown in figure 17

$$S(\theta) = i \frac{g^2}{\theta - iu}, \quad (167)$$

and therefore it corresponds to the square of the *on-shell* three-point coupling constant g . Since this quantity in an hermitian theory is real, the residue is positive. But in Yang–Lee model the coupling constant is purely imaginary, as evident from its action (2) and therefore the residue of this theory at the s -channel pole $\theta = \frac{2\pi i}{3}$ is negative, as confirmed by its explicit computation

$$i g^2 = i \tan \frac{2\pi}{3} = -\sqrt{3} i. \quad (168)$$

The action (2) is not hermitian but nevertheless CP invariant. The way to see this is to introduce the operator CP which changes ϕ in $-\phi$, under which $\mathcal{A}^\dagger = (CP) \mathcal{A} (CP)$. This relation ensures the reality of the spectrum but implies a modification in the completeness relation given by the multi-state sum, for the Yang–Lee model given by

$$\sum_{N=0} (-1)^N |N\rangle \langle N| = 1. \quad (169)$$

8.2. Solution of the Bethe Ansatz equations

The non-linear integral equation for the pseudo-energy of the Yang–Lee model is given by (we put the mass $m = 1$)

$$\epsilon(\theta, z) = \beta \cosh \theta + \int_{-\infty}^{\infty} \frac{\sqrt{3} \cosh(\theta)}{\left(\sinh^2(\theta) + \frac{3}{4} \right)} \ln(1 + z e^{-\epsilon(\theta', z)}) \frac{d\theta'}{2\pi}. \quad (170)$$

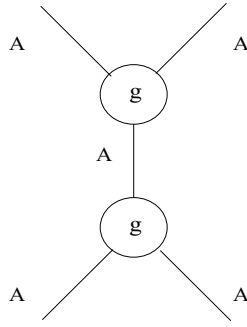


Figure 17. Residue at the pole expressed in terms of the *on-shell* coupling constant.

To find the Yang–Lee zeros of this model we have used the TSA implemented as follows. Firstly we express ϵ as a series in the variable z

$$\epsilon(\theta, z) = \sum_{n=0}^{\infty} \epsilon_n(\theta) z^n, \tag{171}$$

where the functions $\epsilon_n(\theta)$ can be obtained by substituting this expression in (170), expanding the logarithm in the integral in powers of z and comparing the various order of this expansion. Therefore

$$\begin{aligned} \epsilon(\theta, z) &= \beta \cosh \theta - \int_{-\infty}^{\infty} \varphi(\theta - \theta') \log \left(1 + ze^{-\epsilon(\theta', z)} \right) \frac{d\theta'}{2\pi} \\ &= \beta \cosh(\theta) - \int_{-\infty}^{\infty} \varphi(\theta - \theta') \sum_{k=0}^{\infty} \frac{z^k}{k!} \frac{\partial^k}{\partial z^k} \left[\ln \left(1 + ze^{-\epsilon(\theta', z)} \right) \right] \Big|_{z=0} \frac{d\theta'}{2\pi}. \end{aligned} \tag{172}$$

Let us now calculate the coefficients of the expansion of the logarithm by evaluating its various derivatives in $z = 0$:

- $k = 0$:

$$\ln \left(1 + ze^{-\epsilon_0(\theta') - \epsilon_1(\theta')z - \epsilon_2(\theta')z^2 - \dots} \right) \Big|_{z=0} = 0 \tag{174}$$

- $k = 1$:

$$\begin{aligned} &\frac{\partial}{\partial z} \ln \left(1 + ze^{-\epsilon_0(\theta') - \epsilon_1(\theta')z - \epsilon_2(\theta')z^2 - \dots} \right) \Big|_{z=0} \\ &= \left[\frac{e^{-\epsilon_0(\theta') - \epsilon_1(\theta')z - \epsilon_2(\theta')z^2 - \dots}}{\left(1 + ze^{-\epsilon_0(\theta') - \epsilon_1(\theta')z - \epsilon_2(\theta')z^2 - \dots} \right)} + \frac{ze^{-\epsilon_0(\theta') - \epsilon_1(\theta')z - \epsilon_2(\theta')z^2 - \dots} (-\epsilon_1(\theta')z - \epsilon_2(\theta')z^2 - \dots)}{\left(1 + ze^{-\epsilon_0(\theta') - \epsilon_1(\theta')z - \epsilon_2(\theta')z^2 - \dots} \right)} \right] \Big|_{z=0} \\ &= e^{-\epsilon_0(\theta')} \end{aligned} \tag{175}$$

- $k = 2$:

$$\begin{aligned}
 & \frac{\partial^2}{\partial z^2} \ln \left(1 + z e^{-\epsilon_0(\theta') - \epsilon_1(\theta')z - \epsilon_2(\theta')z^2 - \dots} \right) \Big|_{z=0} \\
 &= \left[\frac{e^{-\epsilon_0(\theta') - \epsilon_1(\theta')z - \epsilon_2(\theta')z^2 - \dots} [(-\epsilon_1(\theta') - \epsilon_2(\theta')z - \dots - \epsilon_1(\theta') - \epsilon_2(\theta')z - \dots)]}{(1 + z e^{-\epsilon_0(\theta') - \epsilon_1(\theta')z - \epsilon_2(\theta')z^2 - \dots})^2} \right. \\
 & \quad \frac{z(-\epsilon_1(\theta') - \epsilon_2(\theta')z - \dots)^2 (1 + z e^{-\epsilon_0(\theta') - \epsilon_1(\theta')z - \epsilon_2(\theta')z^2 - \dots})}{(1 + z e^{-\epsilon_0(\theta') - \epsilon_1(\theta')z - \epsilon_2(\theta')z^2 - \dots})^2} \\
 & \quad \left. - \frac{[e^{-\epsilon_0(\theta') - \epsilon_1(\theta')z - \epsilon_2(\theta')z^2 - \dots} (1 + z(-\epsilon_1(\theta') - \epsilon_2(\theta')z - \dots))]^2}{(1 + z e^{-\epsilon_0(\theta') - \epsilon_1(\theta')z - \epsilon_2(\theta')z^2 - \dots})^2} \right] \Big|_{z=0} \\
 &= -2\epsilon_1(\theta') e^{-\epsilon_0(\theta')} - e^{-2\epsilon_0(\theta')} \\
 &= -e^{-\epsilon_0(\theta')} \left[2\epsilon_1 + e^{-\epsilon_0(\theta')} \right] \tag{176}
 \end{aligned}$$

and so on.

By comparing now the various terms which appear with the same power in z in equation (170), we can then identify the functions $\epsilon_n(\theta)$ as:

$$\epsilon_0(\theta) = \beta \cosh(\theta) \tag{177}$$

$$\epsilon_1(\theta) = - \int_{-\infty}^{\infty} d\theta' \varphi(\theta - \theta') e^{-\epsilon_0(\theta')} \tag{178}$$

$$\epsilon_2(\theta) = \frac{1}{2!} \int_{-\infty}^{\infty} d\theta' \varphi(\theta - \theta') [2\epsilon_1 e^{-\epsilon_0(\theta')} + e^{-2\epsilon_0(\theta')}] \tag{179}$$

$$\epsilon_3(\theta) = - \frac{1}{3!} \int_{-\infty}^{\infty} d\theta' \varphi(\theta - \theta') \left[2e^{-3\epsilon_0(\theta')} + 6\epsilon_1(\theta') e^{-2\epsilon_0(\theta')} \right. \tag{180}$$

$$\left. + 3\epsilon_1(\theta')^2 e^{-\epsilon_0(\theta')} - 6\epsilon_2(\theta') e^{-\epsilon_0(\theta')} \right] \tag{181}$$

and so on.

Let us observe that, once we fix the inverse temperature β , the coefficients $\epsilon_n(\theta)$ are obtained in terms of convolutions between the kernel $\varphi(\theta)$ and a function G involving the previous coefficients, $G(\epsilon_1(\theta'), \dots, \epsilon_{n-1}(\theta'))$. The actual calculation of these coefficients consists in solving numerically this hierarchical set of integral equations. The plots of the first two coefficients of the expansion of the ϵ as a function of theta are shown in figure 18.

Notice that, except for the term $\epsilon_0(\theta) = \beta \cosh(\theta)$ that is positive, the sign of the other coefficients oscillates, and in particular it is positive for the odd index coefficients, and negative for the even index ones. This will induce an expansion of the free-energy in terms of a series of alternating terms and ultimately a partition function with positive and negative coefficients, as in the case of free fermionic theory.

8.3. Calculation of the free energy

We can now compute the pressure using the same method adopted for the pseudo-energy. In fact we can write it as an expansion:

$$p(\beta, z) \equiv \sum_n f_n(\beta) z^n \tag{182}$$

with the coefficients $f_n(\beta)$ obtained by expanding the integrand of the equation (163)

$$p(\beta, z) = \frac{1}{\beta} \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \sum_k \frac{z^k}{k!} \frac{\partial^k}{\partial z^k} \left[\ln(1 + ze^{-\epsilon(\theta)}) \right] \Big|_{z=0}. \tag{183}$$

The coefficients have the following expression:

$$f_0(\beta) = 0 \tag{184}$$

$$f_1(\beta) = \frac{1}{\beta} \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \cosh(\theta) e^{-\epsilon_0(\theta)} \tag{185}$$

$$f_2(\beta) = -\frac{1}{2!\beta} \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \cosh(\theta) [2\epsilon_1 e^{-\epsilon_0(\theta')} + e^{-2\epsilon_0(\theta')}] \tag{186}$$

$$f_3(\beta) = \frac{1}{3!\beta} \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \cosh(\theta) \left[2e^{-3\epsilon_0(\theta')} + 6\epsilon_1(\theta') e^{-2\epsilon_0(\theta')} \right. \tag{187}$$

$$\left. + 3\epsilon_1(\theta')^2 e^{-\epsilon_0(\theta')} - 6\epsilon_2(\theta') e^{-\epsilon_0(\theta')} \right] \tag{188}$$

and so on. Notice that also in this case we have alternating sign for the coefficients, but the negative ones are those with an even index.

8.4. Calculation of the zeros of the partition function

We can now determine the zeros of the partition function, written as polynomial in the fugacity z using the expansion:

$$\Omega(z) = \exp \left[\beta \sum_i f_i(z) z^i \right] \equiv \sum_k p_k z^k \tag{189}$$

where:

$$p_k = \frac{1}{k!} \frac{\partial^k \exp \left[\beta \sum_i f_i(z, \beta) z^i \right]}{\partial z^k} \Big|_{z=0}. \tag{190}$$

We can verify that also in this case, as for the fermionic free theory, the coefficients of the expansion in z of the partition function, except for the zero order term, have alternating signs, in particular the even index coefficients are negative and the odd index coefficients are positive.

8.5. Results for finite β

Once we obtain the expression of the partition function as a polynomial in the fugacity z , we can study the distribution of its zeros. For each value of β , we find that the zeros lay along an approximate circumference, whose radius increases with β as shown in figure 19. As a matter of fact, given the enormous number of terms generated by the various series expansions

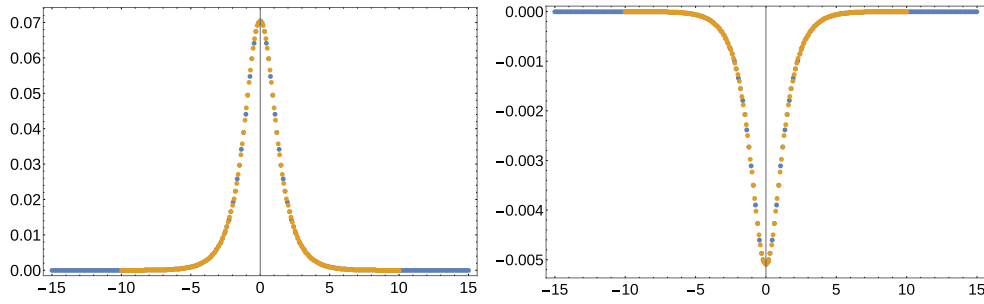


Figure 18. Plots of the functions ϵ_1 and ϵ_2 , evaluated in the grid of the θ (blue dots) and of the θ' (yellow dots), for $\beta = 4$.

described above, the analytic computation can be handle only up to 24 terms. However, we have pushed further the order of the polynomial by using a scaling law satisfied by the coefficients $f_n(\beta)$ in equation (182)

$$\frac{f_n(\beta)}{f_{n-1}(\beta)} \simeq \frac{1}{z_c(\beta)} \left(1 + \frac{\alpha(\beta) - 1}{n} \right), \tag{191}$$

where both the parameters $z_c(\beta)$ and $\alpha(\beta)$ depends on β . This scaling law is shown in figure 20 for a particular value of β . Using the scaling law (191) we study the zeros of polynomials up to order 500. Increasing the order of the polynomial we observe that the average values of the radius of the circle distribution of the zeros shows a convergence to final values which depend on β , as shown in figure 21. Contrary to the free cases, for the interactive Yang–Lee model the radius $R(\beta)$ of its zeros goes continuously to zero when $\beta \rightarrow 0$ with an overall law which a best fit of the data fixes to be

$$R(\beta) = e^\beta - 1. \tag{192}$$

We now show that we can easily predict the behavior of $R(\beta)$ at $\beta \rightarrow \infty$ and at $\beta \rightarrow 0$.

8.6. The $\beta \rightarrow \infty$ limit

In the limit $\beta \rightarrow \infty$, the distribution of zeros of the partition function of the Yang–Lee model simply reduces to that of the free fermionic theory. In fact, for large value of β , the pseudo-energy becomes the one of free theory $\epsilon(\theta) \simeq \beta \cosh \theta$ and also its partition function

$$\Omega(z) \simeq \exp \left[L \int_{-\infty}^{+\infty} \cosh \theta \log \left(1 + z e^{-\beta \cosh \theta} \right) \frac{d\theta}{2\pi} \right]. \tag{193}$$

The zeros of the Yang–Lee model in this limit thus coincide with those of the 1D free fermionic model and therefore the radius $R(\beta)$ of the zeros scales for $\beta \rightarrow \infty$ as $R(\beta) \simeq e^\beta$.

8.7. The limit case $\beta \rightarrow 0$

When $\beta \rightarrow 0$, the thermodynamics Bethe Ansatz solution for interactive theories has a qualitatively different behavior with respect to the behavior present in free theories, namely in an interactive theory such as the Yang–Lee model, the pseudo-energy $\epsilon(\theta, z)$ flattens in the central region $-\Lambda(\beta) \ll \theta \ll \Lambda(2/\beta)$, where $\Lambda(\beta) = \log(2/\beta)$ [18]. This means that the function

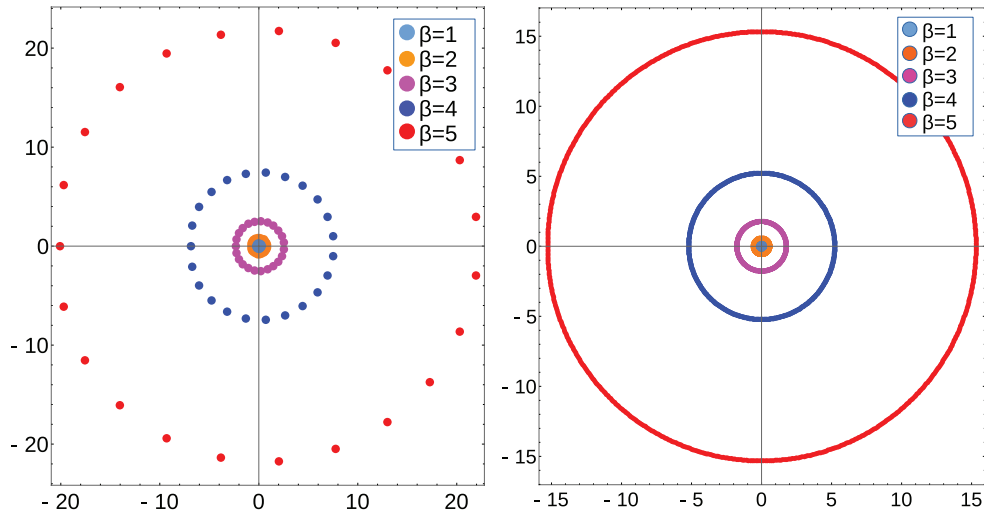


Figure 19. (a) Distribution of zeros of the partition function for fermionic Yang–Lee model for different values of β , obtained with the analytical calculations of the first 24 coefficients of the expansion; (b) distribution of zeros of the partition function for fermionic Yang–Lee model for different values of β , obtained with 500 coefficients obtained by using their scaling law (191).

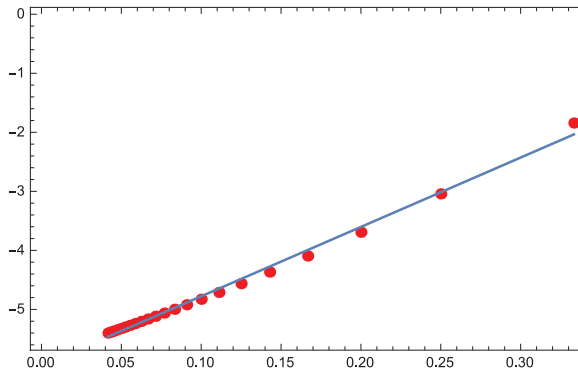


Figure 20. Linear scaling law of the coefficients $f_n(\beta)$ in the variable $x = 1/n$. The plot refers to $\beta = 1$.

$$L(\theta, z) = \log \left(1 + e^{-\epsilon(\theta, z)} \right), \tag{194}$$

shows a plateau in the same region, with a constant value which depends on z and with a double falloff outside (see figure 22). For $\beta \rightarrow 0$ the shape of the right and left edge becomes universal and gives rise to the so-called *kink-solution* of the thermodynamics Bethe Ansatz equation. As shown in the appendix B, this kink-solution allows us to compute the constant value $\epsilon_0(z)$ in the plateau region, solution of the non-linear equation

$$\epsilon_0(z) = \log \left(1 + z e^{-\epsilon_0(z)} \right), \tag{195}$$

and the exact partition function in this limit

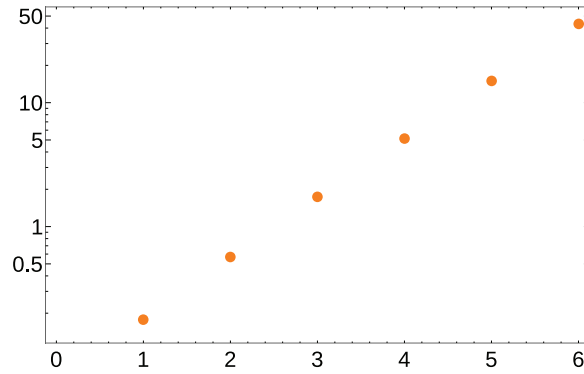


Figure 21. Logarithm of the radius R of the zeros versus the inverse temperature β .

$$\Omega(z) = e^{\frac{1}{\beta}f(z)}, \tag{196}$$

where

$$f(z) = -\frac{1}{\pi} \mathcal{L} \left(\frac{z}{z + \frac{1+\sqrt{1+4z}}{2}} \right) - \frac{1}{2\pi} \ln(z) \ln \left(\frac{\frac{1+\sqrt{1+4z}}{2}}{\frac{1+\sqrt{1+4z}}{2} + z} \right), \tag{197}$$

and $\mathcal{L}(x)$ is the the Rogers Dilogarithmic function

$$\mathcal{L}(x) = -\frac{1}{2} \int_0^x dy \left[\frac{\log y}{1-y} + \frac{\log(1-y)}{y} \right]. \tag{198}$$

As in the free case (see equation (155) and the relative discussion there), equation (196) strictly speaking applied just at $\beta = 0$ but the qualitative difference with respect to the free case is that in an interactive theory as the Yang–Lee model the logarithm of the partition function $\Omega(z)$ presents a $1/\beta$ behavior also at a finite small values of β ! This is a direct consequence of the plateau behavior of the pseudo-energy: indeed, we can first use the parity of the Bethe Ansatz equation and the quantity $\Lambda(\beta)$ that sets the scale of the plateau behavior to split the integral of the free-energy as

$$\int_0^{+\Lambda} \cosh \theta \log \left(1 + z e^{-\epsilon(\theta,z)} \right) \frac{d\theta}{2\pi} + \int_{-\Lambda}^{+\infty} \cosh \theta \log \left(1 + z e^{-\epsilon(\theta,z)} \right) \frac{d\theta}{2\pi}. \tag{199}$$

It is easy now to argue that the first term, for small but not zero value of β , is essentially proportional to $1/\beta$. Indeed, in the interval $(0, \Lambda(\beta))$, the pseudo-energy $\epsilon(\theta, z)$ is essentially constant while a term as $\beta \cosh \theta$ is significantly different from zero in an interval around $\Lambda(\beta)$. So, multiplying and dividing for β , for the first term in equation (199) we have

$$\frac{1}{\beta} \int_0^{+\Lambda} \beta \cosh \theta \log \left(1 + z e^{-\epsilon(\theta,z)} \right) \frac{d\theta}{2\pi} \simeq \frac{1}{\beta} \log \left(1 + z e^{-\epsilon_0(z)} \right) = \frac{\epsilon_0(z)}{\beta}, \tag{200}$$

where in the last step we have used the identity (195). Concerning the second term in (199), for $\beta \rightarrow 0$ it gives essentially a negligible contribution.

Having established a $1/\beta$ behavior at small but finite value of β in the logarithm of the partition function, the presence of this factor implies that the module of the zeros of $\Omega(z)$ satisfies the bounds

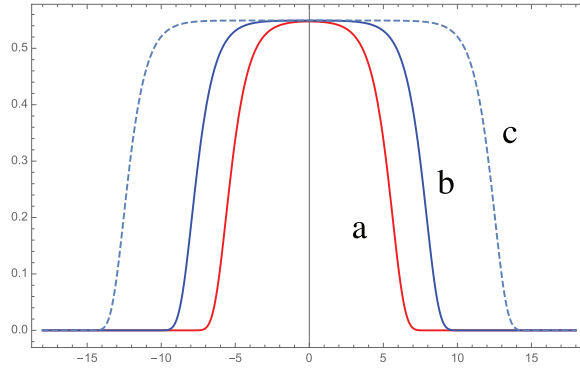


Figure 22. Plot of the function $L(\theta, z)$ versus θ at $z = 0.5$ and for various values of β : the curve **a** refers to $\beta = 0.01$, the curve **b** to $\beta = 0.001$ and the curve **c** to $\beta = 0.0001$. Going to smaller value of β , the function $L(\theta, z)$ develops a larger interval for its plateau value.

$$\tilde{\mu} \beta \leq |z_i| \leq \tilde{\nu} \beta, \tag{201}$$

(for some constants $\tilde{\mu}$ and $\tilde{\nu}$) and therefore for $\beta \rightarrow 0$, it vanishes, in agreement with the scaling law (192).

8.8. Remarks

In the previous sections we have found the Yang–Lee zeros of the Yang–Lee model by using the TSA and these zeros could be in principle used to reconstruct the free energy of the model inside the disk of the complex plane limited by their radius. But could we in principle find another set of zeros which allows us to define an analytic continuation of the free-energy in the entire complex plane of z ? In other words, is there a way to implement the IPA for an interactive theory such as the Yang–Lee model? In practise, to implement the IPA is equivalent to firstly make the change of variable $\beta \rightarrow t$ in (165), where the variable t is solution of the equation

$$t = -e^{\epsilon(\beta, z)}, \tag{202}$$

and secondly to find the corresponding density of the solutions of this equation. If this procedure could be implemented, the logarithm of the partition function (165) could be formally put in the form (14), i.e. an integral on the zeros of the theory. All these steps seem to follow closely the IPA implemented for the free theories (see section 5.2) but for interactive theories there is an important difference: the variable t that solves equation (202) depends itself on z ! This implies that, in the interactive case, in order to implement the IPA one is forced to introduce an infinite number of distributions of zeros, according to the values of z for which we want to evaluate the free-energy. Although nothing wrong with this procedure, of course its implementation is not particularly practical. This essentially seems to leave the TSA as the most convenient method to identify the Yang–Lee zeros in interacting integrable models.

9. Conclusions

In this paper we have discussed the Yang–Lee zeros for a quantum integrable field theory particularly simple, the Yang–Lee model. To compute such zeros, we have relied on the exact

expression of the free-energy of the model expressed by the thermodynamics Bethe Ansatz equations based on the exact S -matrix of the theory. We have used in particular the so-called Truncated Series approach to define a sequence of polynomials, each of them regarded as an approximation of the exact partition function of the model at a given order N in the fugacity z : increasing N , the zeros of these polynomials identify the Yang–Lee zeros of the model. As a matter of fact, they are distributed around approximate circles whose radius depends exponentially on the inverse temperature β . This feature may be also present in other integrable quantum field theories and it would be interesting to confirm or disprove this fact.

In addition to satisfy the curiosity to see, at least in one explicit example, where the Yang–Lee zeros are located for quantum integrable field theories, in our opinion there are other interesting questions that come out from our analysis and which deserve future attention. The most important one concerns the possibility to invert the logic behind the theory of Yang–Lee zeros, namely to see whether it would be possible to use the distributions of zeros for defining well defined statistical models and what properties they would have. Is any distribution of zeros which leads to positive polynomials a good one? For instance, what is (if any) the statistical model behind the partition function (98) based on the prime numbers? Is it a local or non-local model? We denote such questions from the *inverse Yang–Lee problem*.

To make any progress in such a program, one shall be aware however of another important aspect: beside some very special cases, as for instance the one analysed in [31], the knowledge of the Yang–Lee zeros alone is of course not enough to address other important properties of the model, as for instance, the computation of its correlation functions. In light of this remark, as a first important step forward, it would be extremely interesting to identify what is the minimum set of quantities (in addition to the density of the Yang–Lee zeros) which would allow—at least in principle—to successfully deal with the inverse Yang–Lee problem.

Acknowledgments

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Appendix A. Condensation of Yang–Lee zeros for a series expression

In this appendix we analyse the pattern of Yang–Lee zeros and, in particular, their condensation for a fictitious system whose grand-canonical partition function is given by the infinite series of alternating signs

$$\Omega(z) = 1 + \sum_{k=1}^{\infty} (-1)^{k+1} z^k . \quad (\text{A.1})$$

This series converges absolutely for $|z| < 1$ and its unique singularity is at $z = -1$, as can be seen by the exact resummation of the series

$$1 + \sum_{k=1}^{\infty} (-1)^{k+1} z^k = \frac{1 + 2z}{1 + z} , \quad (\text{A.2})$$

which of course provides the analytic continuation of $\Omega(z)$ in all the complex plane of z except at $z = -1$. Hence, the exact expression of the pressure $p(z)$ is given in terms of the function which provides the analytic continuation of the original expression (A.1)

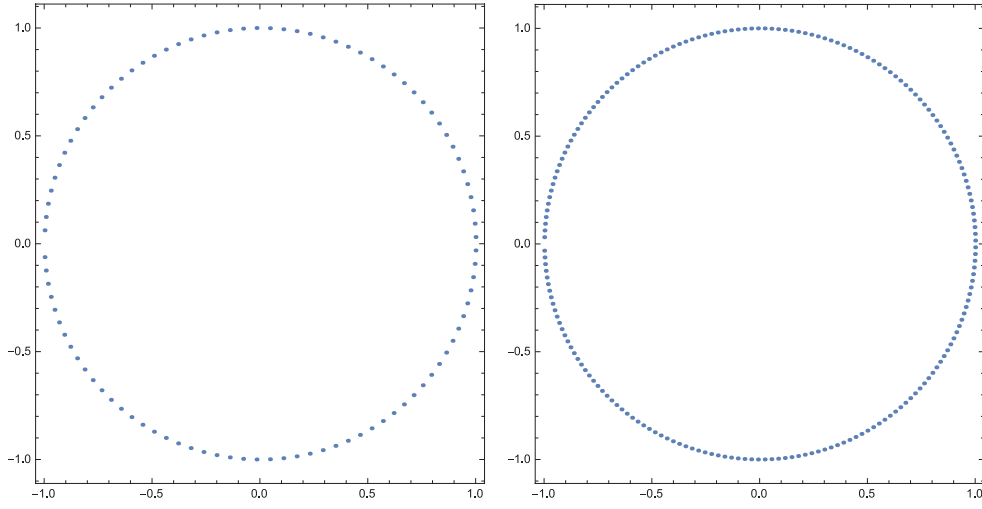


Figure A1. Distribution of zeros of the polynomial $Y_N(z)$ for $N = 100$ (left hand side figure) and for $N = 200$ (right hand side figure). Notice that the zeros are becoming equally placed on the circle except in a small arc nearby -1 .

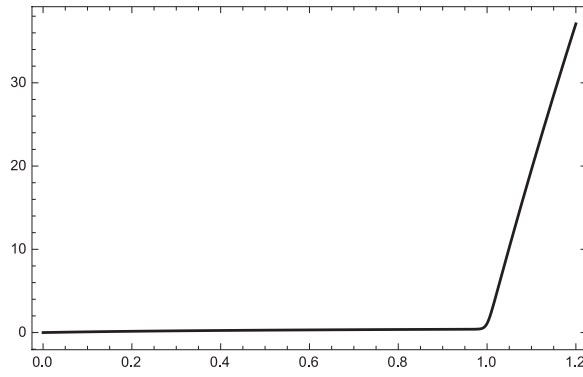


Figure A2. Plot of the pressure $\tilde{p}_N(z)$ versus z computed employing the first $N = 100$ zeros.

$$\frac{p(z)}{kT} = \frac{1}{V} \log \frac{1+2z}{1+z}. \tag{A.3}$$

This expression is obviously not singular at $z = 1$, which is indeed a smooth point of the pressure. However, this is not the conclusion to which one arrives using a different kind of route. To see this, let us instead compute the pressure $p(z)$ in terms of the expression (4) which involves a sum on the Yang–Lee zeros of $\Omega(z)$. To this aim, let us approximate $\Omega(z) \rightarrow \hat{\Omega}_N(z)$ in terms of the sequence of polynomials

$$\hat{\Omega}_N(z) = 1 + \sum_{k=1}^N (-1)^{k+1} z^k = (1+2z) \sum_{k=0}^N (-1)^k z^k \equiv (1+2z)Y_N(z) \tag{A.4}$$

obtained by truncating the series (A.1) to its first N terms. It is now easy to see that, apart of an isolate zero at $z = -1/2$, all the remaining zeros of $\hat{\Omega}_N(z)$, alias of the polynomials $Y_N(z)$, are along the unit circle. Consider in fact the transformation $z \rightarrow -z$ in the polynomial $Y_N(z)$: this produces the polynomial $Y_N(-z)$ whose coefficients are all positive. Applying to this polynomial the Eneström bounds (42), we see that the zeros of $Y_N(-z)$ are on the unit circle, and in particular they become dense near $z = 1$, as evident by their numerical determination shown in figure A1 for two different values of N .

Imagine now to compute the pressure by summing on the isolate zero at $z = -1/2$ and on the N zeros of $Y_N(z)$, according to equation (4). This provides an approximation of the actual pressure and let us call this function $\tilde{p}_N(z)$, defined by

$$\frac{\tilde{p}_N(z)}{kT} = \frac{1}{V} \left[\log(1 + 2x) + \sum_{l=1}^N \log \left(1 - \frac{z}{z_l} \right) \right]. \quad (\text{A.5})$$

Plotted versus z , for positive values of z , this function has a jump in its derivative at $z = 1$ due to a finite density of the zeros nearby $z = 1$, see figure A2. While for $0 < z < 1$ summing on an increasing but finite number of zeros provides better and better approximation of the actual pressure $p(z)$ in this interval, the fact that we employ a finite number of zeros produces the unphysical jump of the approximate function $\tilde{p}_N(z)$. This behavior does not correspond to any phase transition but it is simple due to the finite radius of convergence of the original series (A.1). Proper physical behavior can be obtained only resumming the entire series. Notice that this resummation can be directly done using the density of zeros on the circle: indeed, using the cluster exponents coming from the expansion

$$\log(1 + x) = \sum_{n=1}^{\infty} b_n z^n = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n} z^n, \quad (\text{A.6})$$

we can get the density of zeros $\eta(\theta)$ by using the Fourier series

$$\begin{aligned} \eta(\theta) &= \frac{1}{2\pi} - \frac{1}{\pi} \sum_{n=1}^{\infty} n b_n \cos(n\theta) \\ &= \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^n \cos(n\theta) = \delta(\theta - \pi) \end{aligned} \quad (\text{A.7})$$

where we have used the Fourier transform of the periodic δ -function

$$\delta(x - a) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{\infty} \cos(n(x - a)). \quad (\text{A.8})$$

It is worth underlying that the infinite sum on the zeros has produced a density distribution for them which is peaked at $z = -1$ while is zero everywhere on the circle! Of course, quite the contrary of what happens at any finite N . Using now this exact distribution of the zeros and employing equation (19), one can easily see that the integral on the zeros gives rise to the correct expression (19) of the pressure.

Summarising: when the partition function $\Omega(z)$ is defined *ab-initio* by an infinite series, the truncation of this series at order N produces a set of N zeros which can be employed in the equation (4) to get the correct value of the pressure for values of z *inside* the radius of convergence of the series. Outside the domain of convergence, the pressure may present a jump in its derivative at some positive value of z which is not necessarily associated to a phase transition but it is rather a different manifestation of the radius of convergence of the original

series. In absence of some physical principle which allows us to decide *a priori* on the nature of this jump, to resolve whether or not it is associated to an actual phase transition, one should in principle sum the series to get its analytic continuation and see the location of the actual singularities of the function.

Appendix B. Plateau value of the thermodynamics Bethe Ansatz equations

In this appendix we derive the plateau value $\epsilon_0(z)$ of the pseudo-energy by using the kink-solution of the thermodynamics Bethe Ansatz equation and the value of the free energy at $\beta = 0$. The kink solution emerges when $\beta \rightarrow 0$ and concerns the behavior of $\epsilon(\theta, z)$ nearby the edge $\theta \simeq \Lambda(\beta) = \log(2/\beta)$: it solves the non-linear integral equation

$$\epsilon^{\text{kink}}(\theta) = e^\theta - \frac{1}{2\pi} \int_{-\infty}^{\infty} d\theta' \varphi(\theta - \theta') \ln \left(1 + ze^{-\epsilon^{\text{kink}}(\theta')} \right), \quad (\text{B.1})$$

and it allows to compute the free energy at $\beta = 0$

$$f(z) = -\frac{1}{\pi} \int_{-\infty}^{\infty} e^\theta \ln \left(1 + ze^{-\epsilon^{\text{kink}}(\theta')} \right). \quad (\text{B.2})$$

Using some standard steps (see for instance [20]), this expression can be also written as

$$f(z) = \frac{1}{2\pi} \int_{\epsilon_0}^{\infty} d\epsilon \left[\frac{\epsilon z e^\epsilon}{1 + z e^{-\epsilon}} + \ln \left(1 + z e^{-\epsilon} \right) \right]. \quad (\text{B.3})$$

The lower bound of the integral $\epsilon_0(z)$ is the value of the pseudo-energy at the plateau and for the Yang–Lee model it is solution of the non-linear equation

$$\epsilon_0(z) = \ln \left(1 + z e^{-\epsilon_0(z)} \right). \quad (\text{B.4})$$

If we define $x_0 \equiv e^{\epsilon_0}$, we obtain the equation:

$$\begin{aligned} x_0 &= 1 + \frac{z}{x_0} \\ x_0^2 - x_0 - z &= 0 \\ x_0 &= \frac{1 + \sqrt{1 + 4z}}{2} \end{aligned} \quad (\text{B.5})$$

where we have taken the positive solution of the quadratic equation. We can now make a change of variable in the integral (B.2) in terms of

$$y = \frac{z e^{-\epsilon}}{1 + z e^{-\epsilon}} \quad (\text{B.6})$$

so that the extrema of the integral become:

$$y_- = \frac{z e^{-\epsilon_0}}{1 + z e^{-\epsilon_0}} = \frac{z}{e^{\epsilon_0} + z} = \frac{z}{x_0 + z}, \quad (\text{B.7})$$

$$y_+ = 0. \quad (\text{B.8})$$

With this substitution we also have:

$$\epsilon = \ln(1 - y) - \ln y - \ln z, \quad (\text{B.9})$$

so that

$$d\epsilon = \left(\frac{1}{1-y} - \frac{1}{y} \right) dy = -\frac{dy}{y(1-y)}. \quad (\text{B.10})$$

Therefore

$$\begin{aligned} f(z) &= \frac{1}{2\pi} \int_{y_-}^0 \frac{dy}{y(1-y)} [y(\ln(1-y) - \ln y - \ln z) - \ln(1-y)] \\ &= \frac{1}{2\pi} \int_{y_-}^0 \frac{dy}{y(1-y)} [(y-1)\ln(1-y) - y\ln y - y\ln z] \\ &= \frac{1}{2\pi} \left[-\int_{y_-}^0 \frac{dy}{y} \ln(1-y) - \int_{y_-}^0 dy \frac{\ln y}{(1-y)} \right] - \frac{1}{2\pi} \int_{y_-}^0 \frac{\ln z}{(1-y)} dy \\ &= -\frac{1}{\pi} \left[-\frac{1}{2} \int_0^{y_-} \frac{dy}{y} \ln(1-y) - \int_0^{y_-} dy \frac{\ln y}{(1-y)} \right] - \frac{1}{2\pi} \int_{y_-}^0 \frac{\ln z}{(1-y)} dy. \end{aligned} \quad (\text{B.11})$$

The function within the square parenthesis is the Rogers Dilogarithmic function $\mathcal{L}(y_-) = \mathcal{L}(z/(z+x_0))$ and therefore

$$\begin{aligned} f(z) &= -\frac{1}{\pi} L\left(\frac{z}{z+x_0}\right) - \frac{1}{2\pi} \int_{y_-}^0 \frac{\ln z}{(1-y)} dy \\ &= -\frac{1}{\pi} L\left(\frac{z}{z+x_0}\right) + \frac{1}{2\pi} \ln z \left(\ln(1-y) \Big|_{y_-}^0 \right) \\ &= -\frac{1}{\pi} L\left(\frac{z}{z+e^{\epsilon_0}}\right) - \frac{1}{2\pi} \ln(z) \ln\left(\frac{x_0}{x_0+z}\right) \\ &= -\frac{1}{\pi} \mathcal{L}\left(\frac{z}{z + \frac{1+\sqrt{1+4z}}{2}}\right) - \frac{1}{2\pi} \ln(z) \ln\left(\frac{\frac{1+\sqrt{1+4z}}{2}}{\frac{1+\sqrt{1+4z}}{2} + z}\right). \end{aligned} \quad (\text{B.12})$$

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