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DYNAMICS AND OPTIMAL CONTROL FOR A SPATIALLY-STRUCTURED ENVIRONMENTAL-ECONOMIC MODEL

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Abstract. A deterministic model of economic growth and pollution accumulation, in the form of a system of partial differential equations, is designed and analyzed. The model assumes pollution as a by-product of production. The stock of pollution has a negative impact on production. The accumulation of pollution is dampened by a share of the investments, in the form of an environmental tax. We consider a linear region where both capital and pollution can diffuse. This economic-environmental model is described by a pair of partial differential equations whose dynamics and steady state characteristics with respect to time and space are studied. Then we look at this ambient environment from the point of view of a social planner who can act on the consumption and taxation, also functions of time and space, considering the dynamics of capital and pollution as constraints.

1. Introduction

In this paper we connect two recent strands of economic literature. The first strand considers the joint evolution of economic growth and pollution. In the recent decades a growing interest both in the preservation of the natural environment and in the long run sustainability of the economic growth has compelled researchers to devise models that could investigate the econo-environment interactions, make predictions, and contrive recommendations for the formulation of optimal policies. Early works in this regard can be traced back to the 70s and early 80s, see for example [4, 9, 15, 18, 19, 22, 25]. Environmental pollution enters neoclassical growth models both as a joint product and as a source of disutility. There are many different approaches in the literature to mathematically model the interconnections between the economy and the environment. Nevertheless they can be grouped in two clusters. It is possible to consider pollution as an input of production, assuming that the more the pollution is allowed, the less costly are the techniques of production; alternatively, a damage function can represent the negative effect the pollution can have on production. See [7, 28, 31, 33] for examples of the former approach and [1, 2, 3, 13, 23] for an example of the latter. Our model formulation is in line with those presented in [1, 3]; with respect to them, in our formulation, the

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level of taxation is affecting the level of pollution (not the level of physical capital) and we have an extra term in the pollution equation that justifies the positive effect of taxation on abatement activities. This makes our model more complicated, due to the presence of quasimonotone functions. Furthermore, the present model does not have an integral term and the objective function only includes a combination of consumption of physical capital and the level of pollution.

The second strand of the literature is represented by the recent attempts into extending some models of economic growth into the spatial dimension, taking into account both the temporal and the spatial dynamics of capital. The idea of this spatio-temporal approach has been mutated from the so-called New Economic Geography, whose founding father is the nobel awarded economist Paul Krugman [20, 21].

In a nutshell, we present a model that combines the evolution of the economy and its effects on the environment in the natural spatio-temporal ambience. The vector that connects the economy and the environment is pollution. The production of capital is negatively affected by the stock of pollution, which in turn is reduced by a tax proportional to the level of production. The environmental tax can be an answer to the pressing objective of ensuring a reasonable and sustainable level of pollution. This is the rationale for the inclusion of such 'green tax' in our model, in order to allow the policy makers to employ a portion of the investment to the reduction of the quantity of pollution per unit of production (see [36] for an overview on applied principles of environmental policies). For models involving two connected modules, applying the same ideas to a slightly different context, see [24, 26, 27].

This article is organized as follows: The model is described in Section 2. The dynamic analysis of the model is presented in Section 3. An associated optimal control problem is formulated and analyzed in Section 4. Section 5 presents the numerical simulations, and the results are discussed in Section 6.

2. The Model

In the first part of the paper we stick to the hypothesis that the decisions about the consumption share of production (c) and the choice of the environmental tax share (τ) are not available to the policy makers (c and τ are given parameters). We drop this limitation in a following section, where it will be possible for the policy maker to choose the consumption and taxation following the path of an optimal control approach. Two partial differential equations are the building blocks of our model, given by:

$$\begin{aligned} \frac{\partial k}{\partial t}(x,t) &= d_k \Delta k + \frac{g(k)(1-\tau-c)}{1+\theta_p p} - \delta_k k, \\ \frac{\partial p}{\partial t}(x,t) &= d_p \Delta p + \frac{\sigma g(k)}{1+\theta_\tau \tau} - \delta_p p, \end{aligned} \tag{2.1}$$

where k and p designate capital and pollution, respectively. The reaction-diffusion system (2.1) describes the mutual interaction between the economy and the environment. The first equation of System (2.1) takes into account the evolution of the capital k. The dynamics of the capital at position x depend on the production function acting at x and the contribution of the diffusive term, Δk. The capital can diffuse in space, which means that the producers can decide to move their plants to positions where they anticipate better returns. The higher the diffusion coefficient

d_k, the easier it is to move the capital from place to place (d_k describes the level of free circulation of the capital that the policy maker allows between positions x and $x + dx$). In the neoclassical economic theory better returns realize where less capital has been accumulated, thanks to the law of decreasing returns, an immediate consequence of the convexity of the production function. This is still true for a convex-concave production function, provided that the level of accumulated capital is above the poverty trap threshold. Not all the outcome of the process of production is invested in the accumulation of new capital. A fraction of this outcome, c, is dedicated to consumption. As a pollution counteracting measure, another share of production, τ, is devoted to an environmental tax whose amount would be invested to reduce the impact of pollution on the environment. This is a version of the Solow Model of Growth Theory [34, 35] and of its recent extension to the spatial dimension [38], where a constant fraction of production, $1 - c - \tau$, is invested in the accumulation of the capital. It is important to underline how the parameter A, the total factor productivity, reads in our model. Usually this parameter conveys information about the effect of technology on the production function, and it can be either exogenously or endogenously determined. In our model, $A = \frac{1}{1+\theta_p p}$; that is, we allow the pollution to negatively impact the production through a term that can be interpreted as a factor inhibiting the performance of the economy, or a damage function. This formulation says that if pollution is zero, then there is no externality on production. Otherwise the production decreases proportionally with the increase in pollution. One can imagine, for example, that high levels of pollution, by destroying environmental amenities, make disconsolate and/or less productive human beings (see [37] for a study of the effect of the environmental pollution on the productivity of labor). The depreciation of the capital is taken into account by the term $\delta_k k$, a rather standard assumption.

The second equation in (2.1) describes the accumulation and diffusion of pollution p. Pollution is a by-product of the production, as is clear from the source term $\frac{\sigma g(k)}{1+\theta_\tau \tau}$. The level of pollution at position x is also given by the amount of pollution reaching x through the process of diffusion. The current model considers a composite pollutant, whose diffusion properties are summed up by the diffusion coefficient d_p. One can think about a combination of greenhouse gases. Moreover we observe that the flow of pollution per unit of production, $\frac{\sigma}{1+\theta_\tau \tau}$, depends on the level of the environmental tax share. The model assumes that the resources collected through the environmental taxation are employed to develop cleaner industrial processes or abatement activities that facilitate the reduction of pollution level, given the same amount of production. The term $\delta_p p$ describes the self-cleaning capacity of the environment. In a more realistic approach the exponential decay of pollution must be accompanied by a nonlinear feedback term that recounts the possible irreversibilities and hysteresis connected to the environmental degradation. Indeed, it is not always possible to restore the initial conditions of the environment by stopping the economic activity that deteriorated the environment in the first place.

The general form of the production function $g(k)$ is

$$g(k) = \frac{\alpha_1 k^n}{1 + \alpha_2 k^n}, \quad n \in \mathbb{Z},\ \alpha_1, \alpha_2 \geq 0. \tag{2.2}$$

We consider two different shapes of the production function:

- $S-$shaped production function with $n = 2$ and $\alpha_2 \neq 0$;

- Concave production function with $n = 1$ and $\alpha_2 \neq 0$.

The standard neoclassical (concave) production function, with $n = 1$ and $\alpha_2 \neq 0$, has convenient properties from the point of view of the neoclassical economic theory: Positivity and decreasing return to capital. Yet there are circumstances that are not properly modeled if some departures from pure concavity are not allowed for. Hence, following the famous idea of Skiba [32], we allow for the existence of non-concavity. With $n = 2$ and $\alpha_2 \neq 0$ the function $g(k)$ is a convex-concave ($S-$shaped) production function, meaning that for values of k up to a certain threshold the function exhibits convexity, and then concavity (it has the so-called $S-$shaped form). In terms of the economic literature, the function exhibits increasing and then decreasing return to capital. The $S-$shaped curve is not a pure neoclassical production function because it does not respect the law of diminishing return for all the values of k (the second derivative is not always negative), but it gives rise to richer dynamics.

3. Dynamic Analysis

Let $D = \Omega \times [0, T]$, where Ω is a bounded domain. Consider the problem

$$\begin{aligned} \frac{\partial k}{\partial t}(x,t) &= d_k \Delta k + f_1(k,p) \quad \text{in } D, \\ \frac{\partial p}{\partial t}(x,t) &= d_p \Delta p + f_2(k,p) \quad \text{in } D, \end{aligned} \tag{3.1}$$

where

$$f_1(k,p) = \frac{g(k)(1-\tau-c)}{1+\theta_p p} - \delta_k k, \quad f_2(k,p) = \frac{\sigma g(k)}{1+\theta_\tau \tau} - \delta_p p,$$

with g given by (2.2).

Table 1. Description and values of the model parameters

Parameters	Description	Typical value (range)
$\frac{\alpha_1}{\alpha_2}$	Maximum production level	10
$1/\alpha_2$	Half-saturation constant	$\alpha_2 = 0.1$
δ_k	Capital depreciation rate	0.02
θ_p	Trade-off parameter	0.001
θ_τ	Trade-off parameter	2
σ	Trade-off parameter	2
δ_p	Pollution abatement rate	0.4
τ	Green tax rate	Varies

The boundary conditions are the homogeneous Neumann conditions

$$\begin{aligned} \frac{\partial k}{\partial n} &= 0 \quad \text{for } x \in \partial\Omega,\ t \in [0,T], \\ \frac{\partial p}{\partial n} &= 0 \quad \text{for } x \in \partial\Omega,\ t \in [0,T], \end{aligned} \tag{3.2}$$

where n designates a unit outward normal vector to $\partial\Omega$. The initial conditions are given by

$$\begin{aligned} k(x,0) &= k_0(x) \quad \text{for } x \in \Omega, \\ p(x,0) &= p_0(x) \quad \text{for } x \in \Omega, \end{aligned} \tag{3.3}$$

where functions k_0 and p_0 are smooth. The goal is to prove the existence of a solution of this problem (we will also prove the uniqueness in this case) and to study the steady-state solutions. We start by introducing several definitions and results that will be useful for the study of Problem (3.1)–(3.3). The process we will follow is based on finding what is referred to as lower and upper solutions in order to prove the existence and, under suitable hypotheses, uniqueness of the solution of Problem (3.1)–(3.3). This technique is introduced in details in [30] for the systems of parabolic as well as elliptic partial differential equations. The current model contains mixed quasimonotone functions, which will be elaborated in the following section.

Preliminary results. To prove the existence of a solution of System (3.1), subject to the boundary and initial conditions (3.2) and (3.3), we need to introduce some preliminary results. These results can be found in more details in [30]. Consider the system

$$\begin{gathered}
\frac{\partial u_i}{\partial t} = \mathcal{L}_i u_i + g_i(x,t,u_1,u_2) \quad \text{in } D, \\
\mathcal{B}_i u_i = h_i(x,t) \quad \text{for } x \in \partial\Omega,\ t \in [0,T], \\
u_i(x,0) = u_{i,0}(x) \quad \text{for } x \in \Omega,
\end{gathered} \tag{3.4}$$

for $i = 1, 2$, where $u_{i,0} \in C^0_{L^\infty}(\Omega)$ and the operators $\mathcal{L}_i$ are uniformly elliptic with Hölder continuous coefficients and having the form

$$\mathcal{L}_i \equiv \sum_{j,l=1}^{n} a^{(i)}_{j,l}(x,t)\frac{\partial^2}{\partial x_j \partial x_l} + \sum_{j=1}^{n} b^{(i)}_j(x,t)\frac{\partial}{\partial x_j},$$

and $\mathcal{B}_i$ are defined as

$$\mathcal{B}_i \equiv \beta_i(x,t)\frac{\partial}{\partial n} + \gamma_i(x,t),$$

where β_i and γ_i are continuous for $i = 1, 2$, $\beta_i \geq 0$, $\gamma_i \geq 0$, $\beta_i + \gamma_i > 0$, and n being the unit outward normal vector to $\partial\Omega$.

Definition 3.1. $(g_1, g_2) = (g_1(r,s), g_2(r,s))$ is called mixed quasimonotone when

$$\frac{\partial g_1}{\partial s} \leq 0 \quad \text{and} \quad \frac{\partial g_2}{\partial r} \geq 0,$$

or vice versa.

Definition 3.2. Suppose that (g_1, g_2) is mixed quasimonotone. We call $\overline{u} = (\overline{u_1}, \overline{u_2})$ and $\underline{u} = (\underline{u_1}, \underline{u_2})$ in $\left(C^{2,1}_{L^\infty}(\Omega \times [0,T])\right)^2$ coupled ordered upper and lower solutions of (3.4) if they satisfy the following relations for $i = 1, 2$:

$$\begin{gathered}
\overline{u} \geq \underline{u}, \\
\mathcal{B}_i \overline{u}_i \geq h_i(x,t) \geq \mathcal{B}_i \underline{u}_i, \\
\overline{u}_i(x,0) \geq u_{i,0}(x) \geq \underline{u}_i(x,0), \\
\frac{\partial \overline{u}_1}{\partial t} - \mathcal{L}_1 \overline{u}_1 - g_1(x,t,\overline{u}_1,\underline{u}_2) \geq 0 \geq \frac{\partial \underline{u}_1}{\partial t} - \mathcal{L}_1 \underline{u}_1 - g_1(x,t,\underline{u}_1,\overline{u}_2), \\
\frac{\partial \overline{u}_2}{\partial t} - \mathcal{L}_2 \overline{u}_2 - g_2(x,t,\overline{u}_1,\overline{u}_2) \geq 0 \geq \frac{\partial \underline{u}_2}{\partial t} - \mathcal{L}_2 \underline{u}_2 - g_2(x,t,\underline{u}_1,\underline{u}_2).
\end{gathered}$$

Theorem 3.3. *Let* $(\overline{u}_1, \overline{u}_2)$ *and* $(\underline{u}_1, \underline{u}_2)$ *be coupled upper and lower solutions of* (3.4) *and let* (g_1, g_2) *be mixed quasimonotone in*

$$\langle \underline{u}, \overline{u} \rangle := \big\{(u_1, u_2) \in \big(C^{2,1}_{L^\infty}(\Omega \times [0,T])\big)^2 : (\underline{u}_1, \underline{u}_2) \le (u_1, u_2) \le (\overline{u}_1, \overline{u}_2)\big\}.$$

Then, Problem (3.4) *has a unique solution* u *in* $\langle \underline{u}, \overline{u} \rangle$.

Existence and uniqueness of a solution.

Theorem 3.4. *Problem* (3.1) *subject to the conditions* (3.2) *and* (3.3) *has a solution and this solution is unique.*

Proof. We first observe that $f = (f_1(k,p), f_2(k,p))$ is mixed quasimonotone. Indeed

$$\frac{\partial f_1}{\partial p} = g(k)(1-\tau-c)\frac{(-\theta_p)}{(1+\theta_p p)^2} \le 0,$$
$$\frac{\partial f_2}{\partial k} = \frac{\sigma}{1+\theta_\tau \tau} g'(k) = \frac{\sigma}{1+\theta_\tau \tau}\frac{n\alpha_1 k^{n-1}}{(1+\alpha_2 k^n)^2} \ge 0.$$

We now prove the existence of lower and upper solutions of (3.1), denoted by $(\underline{k}, \underline{p})$ and $(\overline{k}, \overline{p})$ respectively, satisfying the following conditions:

$$\overline{k} \ge \underline{k}, \quad \overline{p} \ge \underline{p},$$
$$\frac{\partial \overline{k}}{\partial n} \ge 0 \ge \frac{\partial \underline{k}}{\partial n},$$
$$\frac{\partial \overline{p}}{\partial n} \ge 0 \ge \frac{\partial \underline{p}}{\partial n},$$
$$\overline{k}(x,0) \ge k_0(x) \ge \underline{k}(x,0),$$
$$\overline{p}(x,0) \ge p_0(x) \ge \underline{p}(x,0),$$
$$\frac{\partial \overline{k}}{\partial t} - d_k \Delta \overline{k} - f_1(\overline{k}, \underline{p}) \ge 0 \ge \frac{\partial \underline{k}}{\partial t} - d_k \Delta \underline{k} - f_1(\underline{k}, \overline{p}),$$
$$\frac{\partial \overline{p}}{\partial t} - d_p \Delta \overline{p} - f_2(\overline{k}, \overline{p}) \ge 0 \ge \frac{\partial \underline{p}}{\partial t} - d_p \Delta \underline{p} - f_2(\underline{k}, \underline{p}).$$

Let $\underline{k} = 0$ and note that it satisfies $\frac{\partial \underline{k}}{\partial t} \le d_k \Delta \underline{k} + f_1(\underline{k}, \overline{p})$. Using $\underline{k} = 0$, $\underline{p}$ satisfies

$$\frac{\partial \underline{p}}{\partial t} \le d_p \Delta \underline{p} - \delta_p \underline{p},$$
$$\frac{\partial \underline{p}}{\partial n} \le 0,$$
$$\underline{p}(x,0) \le p_0.$$

It is reasonable to choose $\underline{p} = \min\{0, \inf_\Omega p_0\}$, which leads to $\underline{p} = 0$ (since $p_0(x,t) \ge 0$ for all valid (x,t), by definition). Considering $\overline{k}$ independent of x and noticing that $\underline{p} = 0$ the upper solution $\overline{k}$ satisfies

$$\frac{\partial \overline{k}}{\partial t} \ge g(\overline{k})(1-\tau-c) - \delta_k \overline{k},$$
$$\frac{\partial \overline{k}}{\partial n} \ge 0,$$
$$\overline{k}(x,0) \ge k_0.$$

Denoting $a := \alpha_1(1-\tau-c)$,

$$\frac{\partial \overline{k}}{\partial t} \geq h(\overline{k}) - \delta_k \overline{k},$$

where $h(x) = \frac{ax^n}{1+\alpha_2 x^n}$. Since h is increasing for $x > 0$ and $\lim_{x\to\infty} h(x) = \frac{a}{\alpha_2}$, it is sufficient to look for a $\overline{k}$ that satisfies

$$\frac{\partial \overline{k}}{\partial t} \geq \frac{a}{\alpha_2} - \delta_k \overline{k}.$$

Therefore, we consider the initial value problem

$$\begin{gathered} y'(t) = \frac{a}{\alpha_2} - \delta_k y, \\ y(0) = \sup_{\Omega} k_0. \end{gathered} \tag{3.5}$$

The solution of (3.5) is

$$y(t) = \frac{a}{\delta_k \alpha_2} + \Big(\sup_{\Omega} k_0 - \frac{a}{\delta_k \alpha_2}\Big) e^{-\delta_k t}.$$

Hence we can choose

$$\overline{k} = \max\big\{\frac{\alpha_1(1-\tau-c)}{\delta_k \alpha_2}, \sup_{\Omega} k_0\big\}.$$

Finally, suppose $\overline{p}$ is also space-independent. Then $\overline{p}$ must satisfy

$$\frac{\partial \overline{p}}{\partial t} \geq \frac{\sigma g(\overline{k})}{1+\theta_\tau \tau} - \delta_p \overline{p},$$

along with the corresponding boundary and initial conditions. To identify $\overline{p}$, we similarly look for a solution of

$$\begin{gathered} z'(t) = \mu - \delta_p z, \\ z(0) = \sup_{\Omega} p_0, \end{gathered} \tag{3.6}$$

where

$$\mu = \frac{\sigma g(\overline{k})}{1+\theta_\tau \tau}.$$

The solution of (3.6) is

$$z(t) = \frac{\mu}{\delta_p} + \Big(\sup_{\Omega} p_0 - \frac{\mu}{\delta_p}\Big) e^{-\delta t}.$$

One can therefore choose

$$\overline{p} = \max\Big\{\frac{\sigma g(\overline{k})}{\delta_p(1+\theta_\tau \tau)}, \sup_{\Omega} p_0\Big\}.$$

We can now apply Theorem 3.3 to deduce the existence of a unique solution u to (3.1)-(3.3), with

$$u \in \Big\langle (0,0), \Big(\max\big\{\frac{\alpha_1(1-\tau-c)}{\delta_k \alpha_2}, \sup_{\Omega} k_0\big\}, \max\big\{\frac{\sigma g(\overline{k})}{\delta_p(1+\theta_\tau \tau)}, \sup_{\Omega} p_0\big\}\Big)\Big\rangle.$$

$\square$

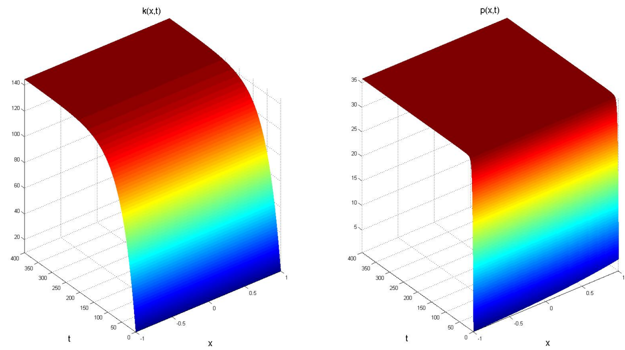

FIGURE 1. Plot of $k(x,t)$ and $p(x,t)$ using the concave production function

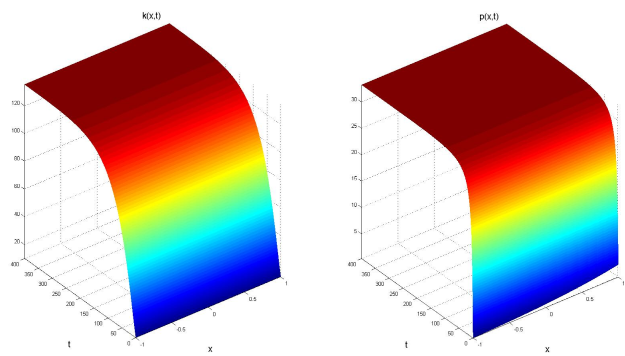

FIGURE 2. Plot of $k(x,t)$ and $p(x,t)$ using the $S-$shaped production function

A solution profile of $k(x,t)$ and $p(x,t)$ is provided in Figures 1 and 2.

The steady state solution of (3.1)-(3.3) is a pair of smooth functions $(k^\star, p^\star)$ satisfying

$$
\begin{gathered}
d_k \Delta k^\star + f_1(k^\star, p^\star) = 0 \quad \text{in } \Omega, \\
d_p \Delta p^\star + f_2(k^\star, p^\star) = 0 \quad \text{in } \Omega, \\
\frac{\partial k^\star}{\partial n} = \frac{\partial p^\star}{\partial n} = 0 \quad \text{on } \partial\Omega.
\end{gathered}
$$

This problem, although involving elliptic partial differential equations, can be addressed in a similar way as (3.1)-(3.3) by looking for upper and lower solutions in order to deduce the existence of solutions of the problem; that is, equilibrium solutions of (3.1)-(3.3). The tools needed for treating this problem are similar to the ones used in the previous section and can be found in [30]. The steady state solution profiles of k^* and p^* are provided in Figures 3 and 4.

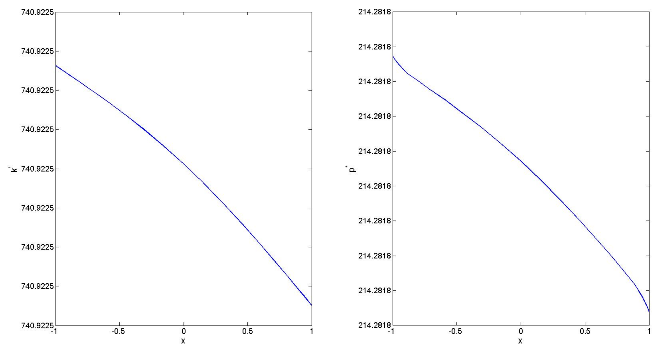

FIGURE 3. Plot of k^* and p^* using the concave production function

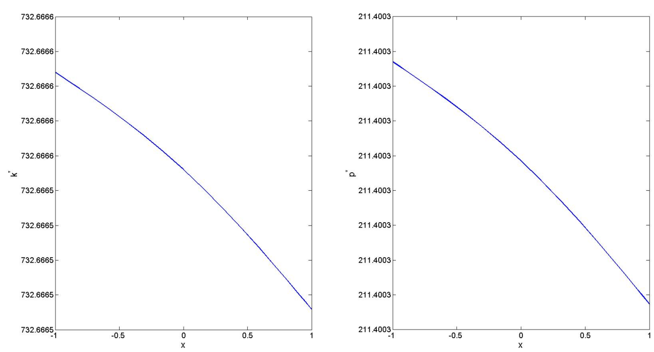

FIGURE 4. Plot of k^* and p^* using the $S-$shaped production function

4. Optimal control

So far the consumption share c and the taxation share τ have been considered exogenous, meaning that their values have been treated as the result of exogenous choices. Now we want to allow a social planner to choose these parameters in such a way to optimize an objective function. The objective function takes into account consumption as a source of utility and pollution as a source of disutility. The planner's problem reads

$$\max_{\{c(x,t),\tau(x,t)\}} \int_0^T \int_a^b [\theta c(x,t) g(k(x,t)) - \gamma p(x,t)]\, dx\, dt,$$

subject to

$$\begin{gathered}
\frac{\partial}{\partial t} k(x,t) = d_k \frac{\partial^2 k(x,t)}{\partial x^2} + \frac{g(k(x,t))[1-\tau(x,t)-c(x,t)]}{1+\theta_p p(x,t)} - \delta_k k(x,t), \\
\frac{\partial}{\partial t} p(x,t) = d_p \frac{\partial^2 p(x,t)}{\partial x^2} + \frac{\sigma g(k(x,t))}{1+\theta_\tau \tau(x,t)} - \delta_p p(x,t).
\end{gathered} \tag{4.1}$$

The planner can choose whatever values for c and τ (s)he considers optimal, provided that these two control variables are picked in a reasonable control set. c and τ are shares, so they need to belong to the set $[0,1]$. Moreover it is plausible to think that there is a bottom limit for the investment in new capital that cannot be crossed. In other words, the control set has been built in such a way that consumption and tax shares do not exhaust the investment; that is, $c+\tau \leq \theta_{c\tau}$. We chose $\theta_{c\tau}=0.8$, but other values do not affect our qualitative results. The Hamiltonian function $\mathcal{H}$ is

$$\begin{aligned}\mathcal{H} &= \theta c(x,t)g(k(x,t)) - \gamma p(x,t)\\ &\quad + \lambda_k(x,t)\Big[d_k\frac{\partial^2 k(x,t)}{\partial x^2} + \frac{g(k(x,t))[1-\tau(x,t)-c(x,t)]}{1+\theta_p p(x,t)} - \delta_k k(x,t)\Big]\\ &\quad + \lambda_p(x,t)\Big[d_p\frac{\partial^2 p(x,t)}{\partial x^2} + \frac{\sigma g(k(x,t))}{1+\theta_\tau \tau(x,t)} - \delta_p p(x,t)\Big].\end{aligned} \tag{4.2}$$

The first order conditions to the previous problem are

$$\begin{aligned}\frac{\partial}{\partial t}\lambda_k(x,t) &= -d_k\frac{\partial^2 \lambda_k(x,t)}{\partial x^2} - \theta c(x,t)g_k(k(x,t))\\ &\quad - \lambda_k(x,t)g_k(k(x,t))\frac{[1-\tau(x,t)-c(x,t)]}{1+\theta_p p(x,t)}\\ &\quad - \lambda_p(x,t)g_k(k(x,t))\frac{\sigma}{1+\theta_\tau\tau(x,t)} + \delta_k\lambda_k(x,t),\\ \frac{\partial}{\partial t}\lambda_p(x,t) &= -d_p\frac{\partial^2\lambda_p(x,t)}{\partial x^2} + \gamma\\ &\quad + \lambda_k(x,t)g(k(x,t))\frac{\theta_p[1-\tau(x,t)-c(x,t)]}{[1+\theta_p p(x,t)]^2}\\ &\quad + \delta_p\lambda_p(x,t),\end{aligned} \tag{4.3}$$

subject to homogeneous Neumann boundary conditions and the final conditions

$$\lambda_k(x,T) = \lambda_p(x,T) = 0.$$

5. Numerical Simulations

Description of the algorithm. We implemented the forward-backward sweep method for System (4.1), (4.2) and (4.3) as follows:

(1) Choose an initial guess: $(c^{(0)},\tau^{(0)}) = (c^{(0)}(t),\tau^{(0)}(t))$.

(2) Iterate for $j\geq 0$: Using the spectral method, we solved

$$\begin{aligned}\frac{\partial k^{(j+1)}(x,t)}{\partial t} &= d_k\frac{\partial^2 k^{(j+1)}(x,t)}{\partial x^2} + \frac{g(k^{(j+1)}(x,t))[1-\tau^{(j)}(x,t)-c^{(j)}(x,t)]}{1+\theta_p p^{(j+1)}(x,t)}\\ &\quad - \delta_k k^{(j+1)}(x,t),\\ \frac{\partial p^{(j+1)}(x,t)}{\partial t} &= d_p\frac{\partial^2 p^{(j+1)}(x,t)}{\partial x^2} + \frac{\sigma g(k^{(j+1)}(x,t))}{1+\theta_\tau\tau^{(j)}(x,t)} - \delta_p p^{(j+1)}(x,t),\end{aligned}$$

subject to

$$\begin{aligned}k^{(j+1)}(x,0) &= k_0(x) \quad \text{in } \Omega,\\ p^{(j+1)}(x,0) &= p_0(x) \quad \text{in } \Omega,\end{aligned}$$

$$
\begin{gathered}
\frac{\partial k^{(j+1)}(x,t)}{\partial n} = 0 \quad \text{for } x \in \partial\Omega, \\
\frac{\partial p^{(j+1)}(x,t)}{\partial n} = 0 \quad \text{for } x \in \partial\Omega,
\end{gathered}
$$

from $t = 0$ to $t = T$. We reversed the equations (4.3) in time, via the change of variable $\bar{t} = T - t$, turning the problem into a forward problem with zero initial conditions. Then, we solved

$$
\begin{aligned}
\frac{\partial \lambda_k^{(j+1)}(x,\bar{t})}{\partial \bar{t}} &= d_k \frac{\partial^2 \lambda_k^{(j+1)}(x,\bar{t})}{\partial x^2} + \theta c^{(j)}(x,\bar{t}) g_k(k^{(j+1)}(x,\bar{t})) \\
&\quad + \lambda_k^{(j+1)}(x,\bar{t}) g_k(k^{(j+1)}(x,\bar{t})) \frac{[1 - \tau^{(j)}(x,\bar{t}) - c^{(j)}(x,\bar{t})]}{1 + \theta_p p^{(j+1)}(x,\bar{t})} \\
&\quad + \lambda_p^{(j+1)}(x,\bar{t}) g_k(k^{(j+1)}(x,\bar{t})) \frac{\sigma}{1 + \theta_\tau \tau^{(j)}(x,\bar{t})} - \delta_k \lambda_k^{(j+1)}(x,\bar{t}),
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \lambda_p^{(j+1)}(x,\bar{t})}{\partial \bar{t}} &= d_p \frac{\partial^2 \lambda_p^{(j+1)}(x,\bar{t})}{\partial x^2} - \gamma \\
&\quad - \lambda_k^{(j+1)}(x,\bar{t}) g(k^{(j+1)}(x,\bar{t})) \frac{\theta_p [1 - \tau^{(j)}(x,\bar{t}) - c^{(j)}(x,\bar{t})]}{[1 + \theta_p p^{(j+1)}(x,\bar{t})]^2} \\
&\quad - \delta_p \lambda_p^{(j+1)}(x,\bar{t}).
\end{aligned}
$$

subject to

$$
\begin{gathered}
\lambda_k^{(j+1)}(x,0) = 0 \quad \text{in } \Omega, \\
\lambda_k^{(j+1)}(x,0) = 0 \quad \text{in } \Omega, \\
\frac{\partial \lambda_k^{(j+1)}(x,\bar{t})}{\partial n} = 0 \quad \text{for } x \in \partial\Omega, \\
\frac{\partial \lambda_k^{(j+1)}(x,\bar{t})}{\partial n} = 0 \quad \text{for } x \in \partial\Omega,
\end{gathered}
$$

from $\bar{t} = 0$ to $\bar{t} = T$. We used the 'fmincon' function of MATLAB (dedicated to finding the minimum of a constrained nonlinear multivariable function) defined below

$$
\min_x f(x) \text{ such that } \begin{cases} c(x) \leq 0, \\ ceq(x) = 0, \\ A \cdot x \leq b, \\ Aeq \cdot x = beq, \\ lb \leq x \leq ub, \end{cases}
$$

to determine the values of $c(x,t)$ and $\tau(x,t)$ that maximize $\mathcal{H}$. We achieved this by finding the values of $c(x,t)$ and $\tau(x,t)$ that minimized $-\mathcal{H}$. Here, we define

$$
\begin{gathered}
A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ -1 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 1 \\ 0.8 \\ 0 \end{pmatrix}, \\
lb = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad ub = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad x = \begin{pmatrix} c^{(j+1)}(x,t) \\ \tau^{(j+1)}(x,t) \end{pmatrix}.
\end{gathered}
$$

(3) We checked convergence by computing the difference between the respective values of $c(x,t)$, $\tau(x,t)$, $k(x,t)$, $p(x,t)$, $\lambda_k(x,t)$ and $\lambda_p(x,t)$ in two consecutive iterations. If the maximum of the L^2-norm of the difference was negligibly small, we output the current function as a solution, otherwise we continued iterating.

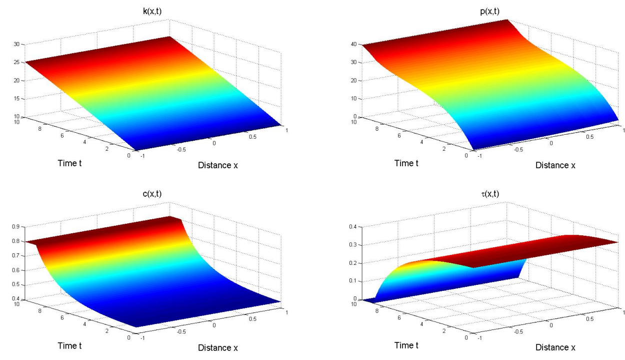

FIGURE 5. Plot of $k(x,t), p(x,t), c(x,t)$ and $\tau(x,t)$ using the concave production function

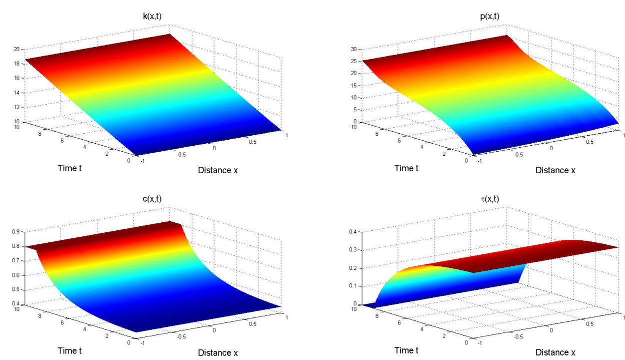

FIGURE 6. Plot of $k(x,t), p(x,t), c(x,t)$ and $\tau(x,t)$ using the S-shaped production function

6. Conclusions

In this article we have studied the dynamics in space and time of a coupled environment-growth model. In the first part the analysis focused on the existence and uniqueness of solutions. This first part can be considered a short-run analysis of the economy with fixed values for the control variables. Then we allowed for these variables to be chosen by a social planner. The social planner wants to optimize the objective function given the partial differential equations describing the coupled environmental economic system. We used a generalized version of Pontryagin's maximum principle as in [36], but our analysis went through the complete dynamical study of the solutions. In the spatial environmental economic literature, see [20, 21, 16, 17, 38, 8, 11, 12, 6, 14, 5], only the study of the solutions around a space-homogeneous steady state was performed. By applying the Sweep Algorithm to our framework, we have been able to perform numerical simulations of all the spatio-temporal path.

As we can see from Figures 5 and 6, both in the case of an concave production function and in the case of a S-shaped function, the diffusion creates spatial homogeneity: The spatial heterogeneous initial time profiles of capital and pollution are smoothed out. As for the temporal dynamics, we see that both capital and pollution grow along a sustainable path. The social planner finds it optimum to let consumption share grow over time, while the green taxation share decreases simultaneously. In other words, it is optimum to first dedicate the major part of the available investment resources to abatement activities, and then progressively let consumption increase while the taxation share is being reduced. This model proposes a precise suggestion for the consumption and taxation policy to be followed when the time horizon is set to T: Consumption is supposed to increase slowly, giving the time to the abatement activities financed by the taxation share to do its job, namely driving the growth path of capital and pollution toward a sustainable outcome (this is in-line with the definition of green taxation that is mainly devoted to pollution abatement) and with the recommendations of the Organization for Economic Co-operation and Development (OECD) on green growth and taxation. As pointed out in [29]: “Environmentally related taxes are increasingly being used in OECD economies and can provide significant incentives for innovation, as firms and consumers seek new, cleaner solutions in response to the price put on pollution. These incentives also make it commercially attractive to invest in R&D activities to develop technologies and consumer products with a lighter environmental footprint”. According to the economic conclusion of our model, there exist optimal paths for consumption and taxation that allow to reach a sustainable level of pollution, together with a hopefully satisfactory level of capital. This is an interesting result from the economic perspective and extends similar results in the literature (see [10]).

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