CHARACTERIZING THE FORMATION OF SINGULARITIES IN A SUPERLINEAR INDEFINITE PROBLEM RELATED TO THE MEAN CURVATURE OPERATOR

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Abstract. The aim of this paper is characterizing the development of singularities by the positive solutions of the quasilinear indefinite Neumann problem

$$-(u'/\sqrt{1+(u')^2})' = \lambda a(x) f(u) \ \text{ in } (0,1), \quad u'(0)=0,\ u'(1)=0,$$

where $\lambda \in \mathbb{R}$ is a parameter, $a \in L^\infty(0,1)$ changes sign once in $(0,1)$ at the point $z \in (0,1)$, and $f \in \mathcal{C}(\mathbb{R}) \cap \mathcal{C}^1[0,+\infty)$ is positive and increasing in $(0,+\infty)$ with a potential, $\int_0^s f(t)\,dt$, superlinear at $+\infty$. In this paper, by providing a precise description of the asymptotic profile of the derivatives of the solutions of the problem as $\lambda \to 0^+$, we can characterize the existence of singular bounded variation solutions solutions of the problem in terms of the integrability of this limiting profile, which is in turn equivalent to the condition

$$\big(\textstyle\int_x^z a(t)\,dt\big)^{-\frac12} \in L^1(0,z) \quad \text{and} \quad \big(\textstyle\int_x^z a(t)\,dt\big)^{-\frac12} \in L^1(z,1).$$

No previous result of this nature is known in the context of the theory of superlinear indefinite problems.

1. Introduction

This paper analyzes the quasilinear indefinite Neumann problem

$$\begin{cases} -\left(\dfrac{u'}{\sqrt{1+(u')^2}}\right)' = \lambda a(x) f(u) & \text{in } (0,1), \\ u'(0)=u'(1)=0. \end{cases} \tag{1.1}$$

Here, $\lambda \in \mathbb{R}$ is regarded as a parameter and

- (a_1) the function $a \in L^\infty(0,1)$ satisfies, for some $z \in (0,1)$, $a(x) > 0$ a.e. in $(0,z)$ and $a(x) < 0$ a.e. in $(z,1)$, as well as $\int_0^1 a(x)\,dx < 0$;
- (f_1) the function $f \in \mathcal{C}(\mathbb{R}) \cap \mathcal{C}^1[0,+\infty)$ satisfies $f(s) > 0$ and $f'(s) \geq 0$ for all $s > 0$, and there exist four constants, $h > 0$, $k > 0$, $q > 1$ and $p \geq 2$, such that

$$\lim_{s\to+\infty} \frac{f(s)}{s^{q-1}} = qh, \qquad \lim_{s\to 0^+} \frac{f(s)}{s^{p-1}} = pk.$$

Condition (f_1) implies that the potential F of f, defined by $F(s) = \int_0^s f(t)\,dt$, satisfies

$$\lim_{s\to+\infty} \frac{F(s)}{s^q} = h, \qquad \lim_{s\to 0^+} \frac{F(s)}{s^p} = k$$

and, thus, F must be superlinear at $+\infty$. and either quadratic or superquadratic at 0. We also introduce the following condition on the weight function a at the nodal point z, which is going to play a pivotal role in the mathematical analysis carried out in this work

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(a$_2$) $\left(\int_x^z a(t)\,dt\right)^{-\frac{1}{2}} \in L^1(0,z)$ and $\left(\int_x^z a(t)\,dt\right)^{-\frac{1}{2}} \in L^1(z,1)$.

Throughout this paper, we are going to use the following notions of solution.

- A couple (λ, u) is said to be a *regular solution* of (1.1) if $u \in W^{2,1}(0,1)$ and it satisfies the differential equation a.e. in $(0,1)$, as well as the boundary conditions.
- A couple (λ, u) is said to be a *bounded variation solution* of (1.1) if $u \in BV(0,1)$ and it satisfies
$$\int_0^1 \frac{D^a u\, D^a \phi}{\sqrt{1+|D^a u|^2}}\,dx + \int_0^1 \frac{D^s u}{|D^s u|} D^s \phi = \int_0^1 \lambda a f(u)\phi\,dx$$
for all $\phi \in BV(0,1)$ such that $|D^s\phi|$ is absolutely continuous with respect to $|D^s u|$ (cf. [2]).
- A couple (λ, u) is said to be a *singular solution* of (1.1) whenever it is a non-regular bounded variation solution; that is, $u \in BV(0,1) \setminus W^{2,1}(0,1)$.
- When the couple (λ, u) solves (1.1) in any of the previous senses, it is said that (λ, u) is a *positive solution* if, in addition,
$$\lambda > 0, \qquad \operatorname{ess\,inf} u > 0.$$

As usual, for any function $v \in BV(0,1)$,
$$Dv = D^a v\,dx + D^s v$$
stands for the Lebesgue decomposition of the Radon measure Dv and $\frac{D^s v}{|D^s v|}$ denotes the density function of the measure $D^s v$ with respect to its total variation $|D^s v|$ (see [1]). By [22, Prop. 3.6], any positive singular solution, (λ, u), of (1.1) actually satisfies
$$\begin{aligned} u|_{[0,z)} &\in W^{2,1}_{\rm loc}[0,z) \cap W^{1,1}(0,z) \text{ and is concave,}\\ u|_{(z,1]} &\in W^{2,1}_{\rm loc}(z,1] \cap W^{1,1}(z,1) \text{ and is convex;} \end{aligned} \tag{1.2}$$
moreover, $u'(x) < 0$ for every $x \in (0,1) \setminus \{z\}$, $u'(0) = u'(1) = 0$ and
$$u'(z^-) = u'(z^+) = -\infty, \tag{1.3}$$
where $u'(z^-)$ and $u'(z^+)$ are the left and the right Dini derivatives of u at z. In full agreement with (1.3), throughout this paper, for any singular solution (λ, u) of (1.1), it is intended that
$$\frac{-u'(z)}{\sqrt{1+(u'(z))^2}} = 1.$$
The same argument used in [22, Lem. 2.1] shows that $\lambda > 0$ is necessary for the existence of positive non-constant, either regular or singular, solutions.

Problem (1.1) is a one-dimensional prototype model of
$$\begin{cases} -\mathrm{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = g(x,u) & \text{in } \Omega,\\ -\dfrac{\nabla u\cdot \nu}{\sqrt{1+|\nabla u|^2}} = \sigma & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$
where Ω is a bounded regular domain in $\mathbb{R}^N$, with outward pointing normal ν, and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ and $\sigma : \partial\Omega \to \mathbb{R}$ are given functions. Problem (1.4) plays a central role in the mathematical analysis of a number of geometrical and physical issues, such as prescribed mean curvature problems for cartesian surfaces in the Euclidean space [24, 3, 19, 25, 9, 14, 12, 15, 13], capillarity phenomena for incompressible fluids [6, 11, 10, 16, 17, 7], and reaction-diffusion processes where the flux features saturation at high regimes [23, 18, 5, 4, 8].

The model (1.1) has been recently investigated by the authors in [21], [22] and [20]. In [21] the existence of bounded variation solutions was analyzed by using variational methods and in [22] the existence of regular solutions was dealt with by means of classical phase plane and bifurcation techniques. The main result of [20] established the existence of a component of bounded variation solutions bifurcating from the trivial state $(\lambda, 0)$ in the special, but significant, case where $p = 2$. According to the results of these papers, it is already known that, under conditions (a_1) and (f_1), problem (1.1) cannot admit a positive solutions if $\lambda < 0$ and that it possesses at least one positive bounded variation solution for sufficiently small $\lambda > 0$.

Quite strikingly, whether or not these bounded variation solutions are singular depends on whether or not condition (a_2) holds true: this is the main result of this paper, which can be stated as follows.

Theorem 1.1. *Assume* (a_1) *and* (f_1). *Then, the following conclusions hold for sufficiently small* $\lambda > 0$*:*

(i) *any positive solution of* (1.1) *is singular if* (a_2) *holds;*
(ii) *any positive solution of* (1.1) *is regular if* (a_2) *fails.*

In other words, condition (a_2) completely characterizes, under (a_1) and (f_1), the development of singularities by the positive solutions of (1.1) for sufficiently small $\lambda > 0$.

By having a glance at condition (a_2) it becomes apparent that it fails whenever the function a is differentiable at the nodal point z, whereas a very simple example where (a_2) holds occurs when the function a is discontinuous at z, like, for instance, in the special case when a is assumed to be a positive constant, $A > 0$, in $[z - \eta_1, z)$ and a negative constant, $-B < 0$, in $(z, z + \eta_2]$, for some $\eta_1, \eta_2 > 0$. The huge contrast on the nature of the positive solutions of the problem with respect to the integrability properties of the function a near the node z can also be realized by considering any weight function a satisfying the requirements of (a_1) except for the fact that $a = 0$ in $[z - \eta, z + \eta]$ for some $\eta > 0$. In such case, thanks to the convexity and concavity properties of the positive bounded variation solutions of (1.1) guaranteed by [22, Prop. 3.6], any positive solution u must be linear in the interval $[z - \eta, z + \eta]$ and hence, due to (1.2), it cannot develop singularities.

As a consequence of Theorem 1.1, when $p = 2$, the global structure of the component of the positive solutions of (1.1), $\mathscr{C}_+$, whose existence is guaranteed by the main theorem of [20], drastically changes according to whether or not the condition (a_2) holds as illustrated in Figure 1, where λ_0 stands for the principal positive eigenvalue of the linear weighted problem

$$\begin{cases} -\varphi'' = \lambda a(x)\varphi & \text{in } (0,1), \\ \varphi'(0) = \varphi'(0) = 0. \end{cases}$$

The non-existence of positive regular solutions of (1.1) in the very special cases when $p = 2$ and the

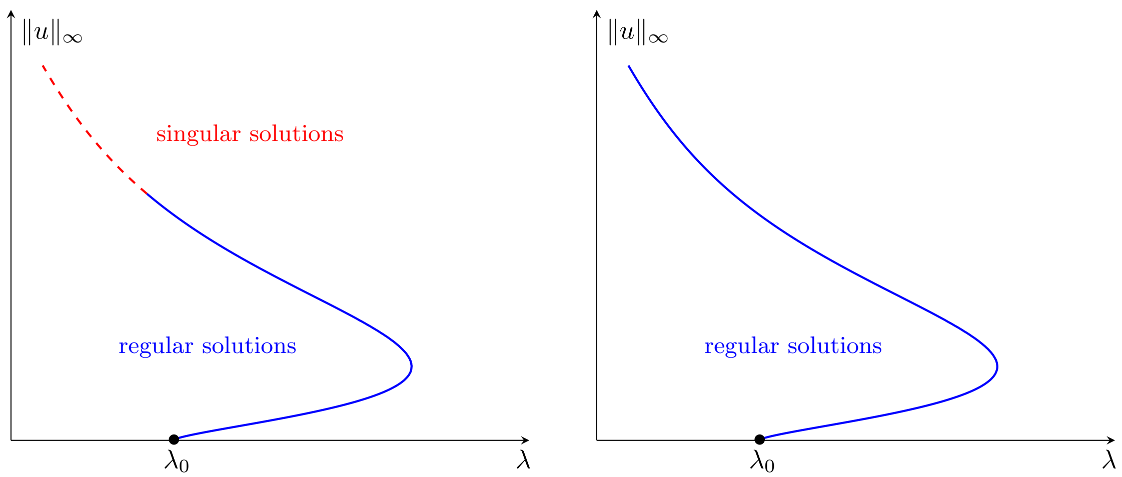

FIGURE 1. Global components emanating from the positive principal eigenvalue λ_0 in case $p = 2$ when (a_2) holds (on the left), or (a_2) fails (on the right).

weight a is constant in $[0, z)$ and in $(z, 1]$ has been recently established in Section 8 of [22] by using some classical, but sophisticated, phase portrait techniques. This induced the authors to presume that an analogous non-existence result should also be valid for general weight functions a, without imposing the integrability condition (a_2). So, they formulated [22, Th. 7.1]. Theorem 1.1 in particular shows that [22, Th. 7.1] has to be complemented with condition (a_2).

Similarly as for $p = 2$, also in the case $p > 2$ the global structure of the set of positive solutions of (1.1), $\mathscr{C}_+$, whose existence is now guaranteed by [21, Th. 1.1] and [22, Th. 10.1], changes for sufficiently small $\lambda > 0$ according to whether or not condition (a_2) holds, as illustrated by Figure 2.

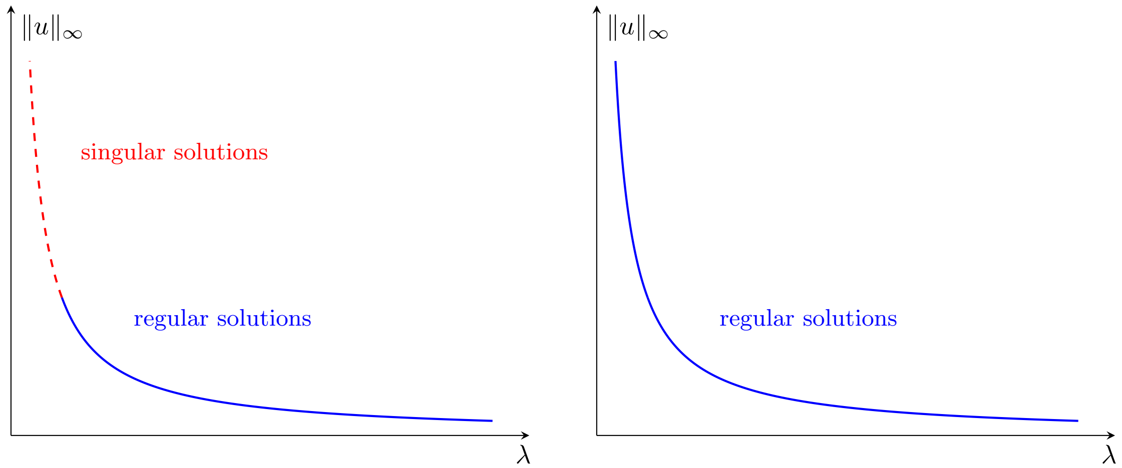

FIGURE 2. Global bifurcation diagrams in case $p > 2$ when (a_2) holds (on the left), or (a_2) fails (on the right).

Our proof of Theorem 1.1 is based upon the characterization of the exact limiting profiles of the positive solutions of (1.1), both regular and singular, as the parameter λ approximates zero. These profiles are provided by the next theorem, regardless their particular nature.

Theorem 1.2. *Assume* (a_1) *and* (f_1), *and let* $((\lambda_n, u_n))_n$ *be an arbitrary sequence of positive solutions of* (1.1) *with* $\lim_{n\to\infty} \lambda_n = 0$. *Then, for sufficiently small* $\eta > 0$, *the following assertions hold:*

$$\lim_{n\to+\infty} \frac{u_n(x)}{u_n(0)} = 1 \quad \text{uniformly in } x \in [0, z-\eta], \tag{1.5}$$

$$\lim_{n\to+\infty} \frac{u_n(x)}{u_n(0)} = \left(\frac{\int_0^z a(t)\,dt}{-\int_z^1 a(t)\,dt} \right)^{\frac{1}{q-1}} \quad \text{uniformly in } x \in [z+\eta, 1], \tag{1.6}$$

$$\lim_{n\to+\infty} (\lambda_n f(u_n(x))) = \frac{1}{\int_0^z a(t)\,dt} \quad \text{uniformly in } x \in [0, z-\eta], \tag{1.7}$$

$$\lim_{n\to+\infty} (\lambda_n f(u_n(x))) = \frac{1}{-\int_z^1 a(t)\,dt} \quad \text{uniformly in } x \in [z+\eta, 1], \tag{1.8}$$

$$\lim_{n\to+\infty} u_n'(x) = \frac{-\int_0^x a(t)\,dt}{\sqrt{\left(\int_0^z a(t)\,dt\right)^2 - \left(\int_0^x a(t)\,dt\right)^2}} \quad \text{uniformly in } x \in [0, z-\eta], \tag{1.9}$$

and

$$\lim_{n\to+\infty} u_n'(x) = \frac{\int_x^1 a(t)\,dt}{\sqrt{\left(\int_z^1 a(t)\,dt\right)^2 - \left(\int_x^1 a(t)\,dt\right)^2}} \quad \text{uniformly in } x \in [z+\eta, 1]. \tag{1.10}$$

Note that condition (a_2) is equivalent to requiring the integrability in both intervals, $(0,z)$ and $(z,1)$, of the asymptotic profile of the derivatives of the positive solutions of (1.1) as $\lambda\to 0+$, which is equivalent to impose that the "limiting derivative" (represented in Figure 3)

$$u'_\omega(x)=\begin{cases}\dfrac{-\int_0^x a(t)\,dt}{\sqrt{\left(\int_0^z a(t)\,dt\right)^2-\left(\int_0^x a(t)\,dt\right)^2}} & \text{for } x\in[0,z),\\[2ex] \dfrac{\int_x^1 a(t)\,dt}{\sqrt{\left(\int_z^1 a(t)\,dt\right)^2-\left(\int_x^1 a(t)\,dt\right)^2}} & \text{for } x\in(z,1],\end{cases}$$

belongs to both $L^1(0,z)$ and $L^1(z,1)$.

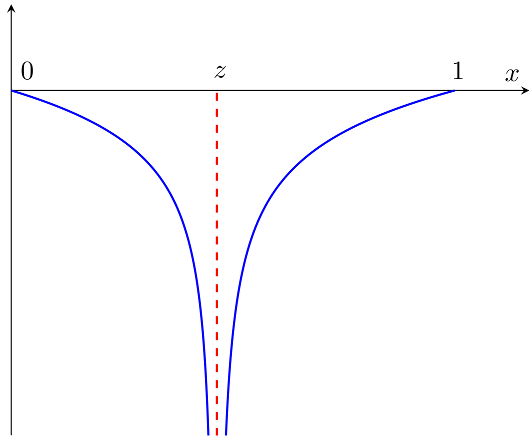

FIGURE 3. Profile of the limiting derivative u'_ω.

The distribution of this paper is as follows. Section 2 contains some preliminary technical lemmas of interest on their own, Section 3 delivers the proof of Theorem 1.2, Section 4 derives another technical result from Theorem 1.2 and, finally, Section 5 consists of the proof of Theorem 1.1. Section 6 collects some additional remarks.

2. Preliminary results of a technical nature

The first result collects some identities that will be used systematically in the sequel.

Lemma 2.1. *Assume* (a_1) *and* (f_1). *Let* (λ,u) *be a positive, regular or singular, solution of* (1.1). *Then, the following identities hold:*

$$-u'(x)=\frac{\frac{-u'(z)}{\sqrt{1+(u'(z))^2}}-\int_x^z\lambda a(t)f(u(t))\,dt}{\sqrt{1-\frac{u'(z)}{\sqrt{1+(u'(z))^2}}-\int_x^z\lambda a(t)f(u(t))\,dt}\sqrt{1+\frac{u'(z)}{\sqrt{1+(u'(z))^2}}+\int_x^z\lambda a(t)f(u(t))\,dt}} \tag{2.1}$$

for all $x\in[0,1]\setminus\{z\}$;

$$\lambda\int_0^x a(t)\,dt=\frac{1}{f(u(x))}\frac{-u'(x)}{\sqrt{1+(u'(x))^2}}+\int_0^x\frac{d}{dt}\left(\frac{1}{f(u(t))}\right)\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\,dt \tag{2.2}$$

for all $x\in[0,z)$;

$$\lambda\int_x^1 a(t)\,dt=\frac{1}{f(u(x))}\frac{u'(x)}{\sqrt{1+(u'(x))^2}}+\int_x^1\frac{d}{dt}\left(\frac{1}{f(u(t))}\right)\frac{u'(t)}{\sqrt{1+(u'(t))^2}}\,dt \tag{2.3}$$

for all $x\in(z,1]$;

$$\frac{-u'(x)}{\sqrt{1+(u'(x))^2}}=\lambda f(u(x))\int_0^x a(t)\,dt-\lambda\int_0^x\left(\int_0^t a(s)\,ds\right)\frac{d}{dt}f(u(t))\,dt \tag{2.4}$$

for all $x \in [0, z)$*;*

$$\frac{-u'(x)}{\sqrt{1+(u'(x))^2}} = \lambda f(u(x)) \int_x^1 -a(t)\,dt + \lambda \int_x^1 \left(\int_t^1 -a(s)\,ds \right) \frac{d}{dt} f(u(t))\,dt \tag{2.5}$$

for all $x \in (z, 1]$.

Proof. Integrating the differential equation of (1.1) in the interval (x, z) yields

$$\frac{u'(x)}{\sqrt{1+(u'(x))^2}} = \frac{u'(z)}{\sqrt{1+(u'(z)^2}} + \int_x^z \lambda a(t) f(u(t))\,dt$$

for all $x \in [0, 1]$, $x \neq z$. Thus, (2.1) follows easily by inverting the function $\varphi : \mathbb{R} \to (-1, 1)$, defined by

$$\varphi(s) = \frac{s}{\sqrt{1+s^2}}.$$

The identities (2.2) and (2.3) can be derived by writing down the differential equation of (1.1) in the form

$$\lambda a(t) = \frac{d}{dt} \left(\frac{1}{f(u(t))} \frac{-u'(t)}{\sqrt{1+(u'(t))^2}} \right) + \frac{d}{dt} \left(\frac{1}{f(u(t))} \right) \frac{u'(t)}{\sqrt{1+(u'(t))^2}}$$

and integrating it in $(0, x)$ and $(x, 1)$, respectively.

The identity (2.4) follows by expressing the differential equation of (1.1) as

$$\frac{-u'(x)}{\sqrt{1+(u'(x))^2}} = \lambda \int_0^x f(u(t))a(t)\,dt = \lambda \int_0^x f(u(t)) \frac{d}{dt} \left(\int_0^t a(s)\,ds \right) dt$$

and integrating by parts the last term. Finally, (2.5) follows by writing down the differential equation of (1.1) as

$$\frac{-u'(x)}{\sqrt{1+(u'(x))^2}} = \lambda \int_x^1 f(u(t))(-a(t))\,dt = \lambda \int_x^1 f(u(t)) \frac{d}{dt} \left(\int_t^1 a(s)\,ds \right) dt$$

and integrating by parts the last term. □

Throughout the rest of this paper, $((\lambda_n, u_n))_n$ stands for a sequence of positive, regular or singular, solutions of (1.1) such that

$$\lim_{n\to\infty} \lambda_n = 0. \tag{2.6}$$

The next series of technical lemmas provides us with some important features of these sequences.

Lemma 2.2. *Assume* (a_1) *and* (f_1)*. Then,*

$$\lim_{n\to+\infty} u_n(x) = +\infty \quad \textit{uniformly in } x \in [0, z].$$

Proof. First, we will prove that

$$\lim_{n\to+\infty} u_n(0) = +\infty. \tag{2.7}$$

Arguing by contradiction, assume that $(u_n(0))_n$ possesses some bounded subsequence. Then, passing to a further subsequence, that we still label with n, one can suppose that there exists a constant $C \geq 0$ such that

$$\lim_{n\to+\infty} u_n(0) = C \tag{2.8}$$

and, since u_n is a decreasing function, for sufficiently large n we have that

$$u_n(x) \leq u_n(0) \leq C + 1 \quad \text{for all } \ x \in [0, 1].$$

Thus, by integrating the differential equation of (1.1) in $[0, z)$ and in $(z, 1]$, we find that, for sufficiently large n,

$$\frac{-u_n'(x)}{\sqrt{1+(u_n'(x))^2}} = \int_0^x \lambda_n f(u_n(t))a(t)\,dt \leq \lambda_n f(C+1) \int_0^z a(t)\,dt \quad \text{for all } \ x \in [0, z)$$

and that

$$\frac{-u_n'(x)}{\sqrt{1+(u_n'(x))^2}} = \int_1^x \lambda_n f(u_n(t))a(t)\,dt \le \lambda_n f(C+1)\int_z^1 -a(t)\,dt \quad \text{for all } \ x\in(z,1],$$

respectively, because f is non-decreasing. By (2.6), these estimates imply that

$$\lim_{n\to+\infty} \frac{u_n'(x)}{\sqrt{1+(u_n'(x))^2}} = 0 \quad \text{uniformly in } \ x\in[0,1]$$

and hence

$$\lim_{n\to+\infty} u_n'(x) = 0 \quad \text{uniformly in } \ x\in[0,1].$$

So, for sufficiently large n, the solutions u_n are regular and, owing to (2.8),

$$u_n(x) = u_n(0) + \int_0^x u_n'(t)\,dt \to C \quad \text{uniformly in } \ x\in[0,1].$$

Thus, integrating the differential equation of (1.1) in $[0,1]$ yields

$$0 = \int_0^1 f(u_n(t))a(t)\,dt \to C\int_0^1 a(t)\,dt \quad \text{as } \ n\to+\infty. \tag{2.9}$$

Therefore, since we are assuming that $\int_0^1 a(t)\,dt<0$, it follows that $C=0$ and hence

$$\lim_{n\to+\infty} u_n = 0 \quad \text{in } \ \mathcal{C}^1[0,1]. \tag{2.10}$$

Let us set, for each n,

$$v_n(x) = \frac{u_n(x)}{u_n(0)} \quad \text{for all } x\in[0,1].$$

It is apparent, from (1.1), that each v_n satisfies

$$\begin{cases} -v_n'' = \lambda_n a\dfrac{f(u_n)}{u_n(0)}(1+(u_n')^2)^{\frac{3}{2}} & \text{in } \ (0,1), \\ v_n'(0)=v_n'(1)=0. \end{cases} \tag{2.11}$$

As we assumed $p\ge 2$ in (f$_1$), we can find a constant $L>0$ such that

$$0\le f(s)\le Ls^{p-1}\le Ls \quad \text{for all } 0\le s\le 1.$$

and thus, by (2.10), we have that, for sufficiently large n,

$$0\le f(u_n(x))\le u_n(x)\le Lu_n(0) \quad \text{for all } x\in[0,1].$$

Therefore, using (2.10) and $\lim_{n\to+\infty}\lambda_n=0$, we get

$$\lim_{n\to+\infty} \lambda_n a\frac{f(u_n)}{u_n(0)}(1+(u_n')^2)^{\frac{3}{2}} = 0 \quad \text{in } \ L^\infty(0,1).$$

Accordingly, we infer from (2.11) that the sequence $(v_n'')_n$ is bounded in $L^\infty(0,1)$ and hence $(v_n')_n$ is bounded in $L^\infty(0,1)$. Since, for all $n\ge 1$, $\|v_n\|_\infty=1$, the sequence $(v_n)_n$ is bounded in $W^{2,\infty}(0,1)$. Therefore, there exist a subsequence of $(v_n)_n$, still labeled by n, and a function $v\in\mathcal{C}^1[0,1]$ such that

$$\lim_{n\to+\infty} v_n = v \quad \text{in } \ \mathcal{C}^1[0,1].$$

As by (2.11)

$$\lim_{n\to+\infty} v_n'' = 0 \quad \text{in } \ L^\infty(0,1),$$

we derive that $v'=0$ and $v=1$. This entails, in particular, that

$$\lim_{n\to+\infty} \frac{u_n(x)}{u_n(0)} = 1 \quad \text{uniformly in } \ x\in[0,1].$$

Consequently, by (f_1), it follows from (2.9) that

$$0=\lim_{n\to+\infty}\int_0^1 a(t)\frac{f(u_n(t))}{u_n^{p-1}(t)}\left(\frac{u_n(t)}{u_n(0)}\right)^{p-1}dt=p\,k\int_0^1 a(t)\,dt<0,$$

which is impossible. This contradiction yields (2.7).

Since u_n is decreasing for all $n\geq 1$, to conclude the proof of the lemma it suffices to show that

$$\lim_{n\to+\infty}u_n(z)=+\infty.$$

Assume by contradiction that there exists a constant C such that, along some subsequence relabeled by n, we have that

$$u_n(x)\leq u_n(z)\leq C$$

for all $n\geq 1$ and $x\in[z,1]$. Then, arguing as above, we see that

$$\lim_{n\to+\infty}u_n'(x)=0\quad\text{uniformly in }\ x\in[z,1].$$

Hence u_n is a regular solution of (1.1) and, as $\|u_n'\|_\infty=-u_n'(z)$,

$$\lim_{n\to+\infty}\|u'\|_\infty=\lim_{n\to+\infty}-u_n'(z)=0.$$

Therefore, for sufficiently large n, we find that

$$u_n(0)=u_n(z)-\int_0^z u_n'(t)\,dt\leq C+1,$$

which contradicts (2.7) and completes the proof. □

Lemma 2.3. *Assume* (a_1) *and* (f_1). *Then,*

$$\lim_{n\to+\infty}u_n'(z)=-\infty.$$

Proof. It suffices to prove the conclusion for regular solutions, for as we already know that any singular solution satisfies (1.3). Arguing by contradiction, assume that there exist a constant $C>0$ and a subsequence of $(u_n'(z))_n$, still labeled by n, such that, for all $n\geq 1$,

$$|u_n'(z)|=\|u_n'\|_\infty\leq C.$$

Hence, by Lemma 2.2, we infer that

$$\frac{u_n(x)}{u_n(0)}=1+\frac{\int_0^x u_n'(t)\,dt}{u_n(0)}\to 1\quad\text{uniformly in }\ x\in[0,1].$$

In particular, this entails that

$$\lim_{n\to+\infty}u_n=+\infty\quad\text{uniformly in}\quad x\in[0,1].$$

Thus, according to condition (f_1), we find that

$$\lim_{n\to+\infty}\frac{f(u_n(x))}{u_n^{q-1}(0)}=\lim_{n\to+\infty}\left[\frac{f(u_n(x))}{u_n^{q-1}(x)}\left(\frac{u_n(x)}{u_n(0)}\right)^{q-1}\right]=qh,$$

uniformly in $x\in[0,1]$. Hence, we get

$$\lim_{n\to+\infty}\int_0^z\frac{f(u_n(t))}{u_n^{q-1}(0)}a(t)\,dt=qh\int_0^z a(t)\,dt.$$

and

$$\lim_{n\to+\infty}\int_z^1\frac{f(u_n(t))}{u_n^{q-1}(0)}a(t)\,dt=qh\int_z^1 a(t)\,dt.$$

On the other hand, integrating the differential equation of (1.1) in $(0,1)$ yields

$$\int_0^z a(t)f(u_n(t))\,dt=-\int_z^1 a(t)f(u_n(t))\,dt. \tag{2.12}$$

Consequently, dividing (2.12) by $u_n^{q-1}(0)$ and letting $n \to +\infty$ yields

$$qh \int_0^z a(t)\,dt = -qh \int_z^1 a(t)\,dt,$$

which implies $\int_0^1 a(t)\,dt = 0$. As this identity contradicts (a_1), the proof is complete. □

Lemma 2.4. *Assume* (a_1) *and* (f_1). *Then, the following estimates hold:*

$$\begin{aligned} \lambda_n f(u_n(x)) &\leq \frac{-u_n'(x)}{\sqrt{1+(u_n'(x))^2}} \left(\int_0^x a(t)\,dt\right)^{-1} \quad \text{for all } x \in (0,z], \\ \lambda_n f(u_n(x)) &\leq \left(\int_0^z a(t)\,dt\right)^{-1} \quad \text{for all } x \in [z,1], \end{aligned} \tag{2.13}$$

$$\begin{aligned} \lambda_n f(u_n(x)) &\geq \frac{-u_n'(z)}{\sqrt{1+(u_n'(z))^2}} \left(\int_z^1 -a(t)\,dt\right)^{-1} \quad \text{for all } x \in [0,z], \\ \lambda_n f(u_n(x)) &\geq \frac{-u_n'(x)}{\sqrt{1+(u_n'(x))^2}} \left(\int_x^1 -a(t)\,dt\right)^{-1} \quad \text{for all } x \in [z,1], \end{aligned} \tag{2.14}$$

and, moreover,

$$\begin{aligned} \lambda_n f(u_n(0)) &\geq \frac{-u_n'(z)}{\sqrt{1+(u_n'(z))^2}} \left(\int_0^z a(t)\,dt\right)^{-1}, \\ \lambda_n f(u_n(1)) &\leq \frac{-u_n'(z)}{\sqrt{1+(u_n'(z))^2}} \left(\int_z^1 -a(t)\,dt\right)^{-1}. \end{aligned} \tag{2.15}$$

Proof. It should be remembered that $f(u_n)$ is non-increasing in $[0,1]$ for all $n \geq 1$, because f is non-decreasing and u_n is decreasing. Thus, the second term on the right hand side of (2.2) is non-positive. Hence it follows from (2.2) that

$$\lambda_n \int_0^x a(t)\,dt \leq \frac{1}{f(u_n(x))} \frac{-u_n'(x)}{\sqrt{1+(u_n'(x))^2}}$$

for all $x \in [0,z]$. Thus, the first estimate of (2.13) holds. Similarly, from (2.3) we infer that, for every $x \in [z,1]$,

$$\lambda_n \int_x^1 a(t)\,dt \leq \frac{1}{f(u_n(x))} \frac{u_n'(x)}{\sqrt{1+(u_n'(x))^2}},$$

which implies the second estimate of (2.14).

From the first estimate of (2.13), it becomes apparent that, for every $x \in [z,1]$,

$$\lambda_n f(u_n(x)) \leq \lambda_n f(u_n(z)) \leq \frac{-u_n'(z)}{\sqrt{1+(u_n'(z))^2}} \left(\int_0^z a(t)\,dt\right)^{-1} \leq \left(\int_0^z a(t)\,dt\right)^{-1},$$

which provides us with the second estimate of (2.13). Analogously, from the second estimate of (2.14) it can be inferred that, for every $x \in [0,z]$,

$$\lambda_n f(u_n(x)) \geq \lambda_n f(u_n(z)) \geq \frac{-u_n'(z)}{\sqrt{1+(u_n'(z))^2}} \left(\int_z^1 -a(t)\,dt\right)^{-1},$$

which provides us with the first estimate of (2.14).

Our proof of the first estimate of (2.15) is based upon (2.2). Indeed, thanks to (2.2), we have that

$$\begin{aligned}\lambda_n \int_0^z a(t)\,dt =& \frac{1}{f(u_n(z))}\frac{-u_n'(z)}{\sqrt{1+(u_n'(z))^2}} + \int_0^z \frac{d}{dt}\left(\frac{1}{f(u_n(t))}\right)\frac{u_n'(t)}{\sqrt{1+(u_n'(t))^2}}\,dt \\ \geq& \frac{-u_n'(z)}{\sqrt{1+(u_n'(z))^2}}\left[\frac{1}{f(u_n(z))} - \int_0^z \frac{d}{dt}\left(\frac{1}{f(u_n(t))}\right)dt\right] \\ =& \frac{-u_n'(z)}{\sqrt{1+(u_n'(z))^2}}\frac{1}{f(u_n(0))},\end{aligned}$$

which provides us with the desired estimate. Similarly, our proof of the second estimate of (2.13) relies upon (2.3). Indeed, changing of sign (2.3), it is apparent that

$$\begin{aligned}\lambda_n \int_z^1 -a(t)\,dt =& \frac{1}{f(u_n(z))}\frac{-u_n'(z)}{\sqrt{1+(u_n'(z))^2}} + \int_z^1 \frac{d}{dt}\left(\frac{1}{f(u_n(t))}\right)\frac{-u_n'(t)}{\sqrt{1+(u_n'(t))^2}}\,dt \\ \leq& \frac{-u_n'(z)}{\sqrt{1+(u_n'(z))^2}}\left[\frac{1}{f(u_n(z))} + \int_z^1 \frac{d}{dt}\left(\frac{1}{f(u_n(t))}\right)dt\right] \\ =& \frac{-u_n'(z)}{\sqrt{1+(u_n'(z))^2}}\frac{1}{f(u_n(1))},\end{aligned}$$

which provides us with the second estimate of (2.13) and ends the proof. □

Lemma 2.5. *Assume* (a$_1$) *and* (f$_1$). *Then, for sufficiently small $\eta > 0$, there exist constants $C_1 = C_1(\eta) > 0$, $C_2 = C_2(\eta) > 0$ and an integer $n_0 = n_0(\eta)$ such that, for every $n \geq n_0$,*

$$C_1 \leq \lambda_n f(u_n(x)) \leq C_2 \quad \text{for all } x \in [z-\eta, z+\eta]. \tag{2.16}$$

Proof. Pick $\eta \in (0, z)$. Then, owing to the first estimate of (2.13), we find that, for every $n \geq 1$ and $x \in [z-\eta, z]$,

$$\lambda_n f(u_n(x)) \leq \frac{-u_n'(x)}{\sqrt{1+(u_n'(x))^2}}\left(\int_0^x a(t)\,dt\right)^{-1} \leq \left(\int_0^x a(t)\,dt\right)^{-1} \leq \left(\int_0^{z-\eta} a(t)\,dt\right)^{-1}.$$

Thus, since $f(u_n)$ is non-increasing in $[0,1]$, it becomes apparent that

$$\lambda_n f(u_n(x)) \leq \left(\int_0^{z-\eta} a(t)\,dt\right)^{-1} \quad \text{for all } x \in [z-\eta, 1].$$

This yields the upper estimate of (2.16).

The proof of the lower estimate is technically more delicate. It relies on the fact that there exist $\bar{x} \in (z, 1)$ and $\bar{C} > 0$ such that, for sufficiently large n,

$$\frac{-u_n'(x)}{\sqrt{1+(u_n'(x))^2}} \geq \frac{-u_n'(\bar{x})}{\sqrt{1+(u_n'(\bar{x}))^2}} \geq \bar{C} \quad \text{for all } x \in [z, \bar{x}]. \tag{2.17}$$

Our proof of (2.17) follows by contradiction. Suppose that, for every $\bar{x} \in (z, 1)$, there is a subsequence of $(u_n'(\bar{x}))_n$, labeled again by n, such that

$$\lim_{n\to+\infty} \frac{-u_n'(\bar{x})}{\sqrt{1+(u_n'(\bar{x}))^2}} = 0.$$

As u_n' is increasing in $(z, 1]$, this implies that

$$\lim_{n\to+\infty} \frac{-u_n'(x)}{\sqrt{1+(u_n'(x))^2}} = 0 \quad \text{uniformly in } x \in [\bar{x}, 1].$$

Since, integrating the differential equation of (1.1) on $(\bar{x}, 1)$, we have that

$$\frac{-u_n'(\bar{x})}{\sqrt{1+(u_n'(\bar{x}))^2}} = \int_{\bar{x}}^1 \lambda_n a(t) f(u_n(t))\, dt,$$

it follows that

$$\lim_{n\to+\infty} \int_{\bar{x}}^1 \lambda_n a(t) f(u_n(t))\, dt = 0. \tag{2.18}$$

On the other hand, using the second estimate of (2.13), it is easily seen that

$$\begin{aligned}\int_z^1 -a(t)\lambda_n f(u_n(t))\, dt &= \int_z^{\bar{x}} -a(t)\lambda_n f(u_n(t))\, dt + \int_{\bar{x}}^1 -a(t)\lambda_n f(u_n(t))\, dt \\ &\le \frac{\int_z^{\bar{x}} -a(t)\, dt}{\int_0^z a(t)\, dt} + \int_{\bar{x}}^1 -a(t)\lambda_n f(u_n(t))\, dt.\end{aligned}$$

Consequently, owing to (2.18), we infer that

$$\limsup_{n\to+\infty} \int_z^1 -a(t)\lambda_n f(u_n(t))\, dt \le \frac{\int_z^{\bar{x}} -a(t)\, dt}{\int_0^z a(t)\, dt}. \tag{2.19}$$

On the other hand, using Lemma 2.3, it follows from the first estimate of (2.14) that

$$\liminf_{n\to+\infty} \int_0^z a(t)\lambda_n f(u_n(t))\, dt \ge \frac{\int_0^z a(t)\, dt}{\int_z^1 -a(t)\, dt} \lim_{n\to+\infty} \frac{-u_n'(z)}{\sqrt{1+(u_n'(z))^2}} = \frac{\int_0^z a(t)\, dt}{\int_z^1 -a(t)\, dt}. \tag{2.20}$$

Therefore, since

$$\int_0^1 \lambda_n a(t) f(u_n(t))\, dt = 0$$

for all $n \ge 1$, we can conclude from (2.19) and (2.20) that

$$\frac{\int_0^z a(t)\, dt}{\int_z^1 -a(t)\, dt} \le \frac{\int_z^{\bar{x}} -a(t)\, dt}{\int_0^z a(t)\, dt},$$

which is impossible if $\bar{x}$ is sufficiently close to z.

Finally, combining (2.17) with the second estimate of (2.14) shows that, for sufficiently large n,

$$\lambda_n f(u_n(\bar{x})) \ge \left(\int_{\bar{x}}^1 -a(t)\, dt\right)^{-1} \frac{-u_n'(\bar{x})}{\sqrt{1+(u_n'(\bar{x}))^2}} \ge \left(\int_{\bar{x}}^1 -a(t)\, dt\right)^{-1} \bar{C}$$

and hence

$$\lambda_n f(u_n(x)) \ge \left(\int_{\bar{x}}^1 -a(t)\, dt\right)^{-1} \bar{C} \quad \text{for all } x \in [0, \bar{x}].$$

Therefore, for every $\eta \in (0, \bar{x} - z)$, the lower estimate of (2.16) also holds. This completes the proof. □

Finally, the next result provides a uniform a priori bounds for u_n' in $[0, z-\eta] \cup [z+\eta, 1]$ for sufficiently large n.

Lemma 2.6. *Assume* (a_1) *and* (f_1). *Then, for any* $\eta \in (0, \min\{z, 1-z\})$, *there exists a constant* $C = C(\eta)$ *and an integer* $n_0 = n_0(\eta)$ *such that, for every* $n \ge n_0$,

$$|u_n'(x)| \le C \quad \text{for all } x \in [0, z-\eta] \cup [z+\eta, 1].$$

Proof. Fix $\eta \in (0, \min\{z, 1-z\})$. We claim that

$$\sup_{n\ge 1} |u_n'(z-\eta)| < +\infty. \tag{2.21}$$

Assume, on the contrary, that there is a subsequence of $(u_n'(z-\eta))_n$, relabeled by n, such that

$$\lim_{n\to+\infty} u_n'(z-\eta) = -\infty.$$

Then, integrating the differential equation of (1.1) on $(z-\eta, z)$, it follows from the first estimate of (2.14) that

$$\begin{aligned}\frac{u_n'(z-\eta)}{\sqrt{1+(u_n'(z-\eta))^2}} - \frac{u_n'(z)}{\sqrt{1+(u_n'(z))^2}} &= \int_{z-\eta}^{z} a(t)\lambda_n f(u_n(t))\,dt \\ &\geq \frac{-u_n'(z)}{\sqrt{1+(u_n'(z))^2}} \frac{\int_{z-\eta}^{z} a(t)\,dt}{\int_z^1 -a(t)\,dt}.\end{aligned}$$

Thus, letting $n \to +\infty$ in this inequality and using Lemma 2.3 yields

$$0 \geq \frac{\int_{z-\eta}^{z} a(t)\,dt}{\int_z^1 -a(t)\,dt} > 0,$$

which is impossible. Therefore, (2.21) holds.

Analogously, to prove that

$$\sup_{n\geq 1} |u_n'(z+\eta)| < +\infty, \tag{2.22}$$

we will argue by contradiction assuming that, along some subsequence $(u_n'(z+\eta))_n$ labeled again by n,

$$\lim_{n\to+\infty} u_n'(z+\eta) = -\infty.$$

As above, integrating the differential equation of (1.1) on $(z, z+\eta)$, we find from the second estimate of (2.14) that

$$\begin{aligned}\frac{-u_n'(z)}{\sqrt{1+(u_n'(z))^2}} + \frac{u_n'(z+\eta)}{\sqrt{1+(u_n'(z+\eta))^2}} &= -\int_z^{z+\eta} a(t)\lambda_n f(u_n(t))\,dt \\ &\geq \int_z^{z+\eta} -a(t)\left(\int_t^1 -a(t)dt\right)^{-1} \frac{-u_n'(t)}{\sqrt{1+(u_n'(t))^2}}\,dt \\ &\geq \frac{\int_z^{z+\eta} -a(t)\,dt}{\int_z^1 -a(t)\,dt} \frac{-u_n'(z+\eta)}{\sqrt{1+(u_n'(z+\eta))^2}}.\end{aligned}$$

Thus, letting $n \to +\infty$, again from Lemma 2.3, it follows that

$$0 \geq \frac{\int_z^{z+\eta} -a(t)\,dt}{\int_z^1 -a(t)\,dt} > 0,$$

which is impossible. Therefore, (2.22) holds true and the proof is complete. □

3. Proof of Theorem 1.2

From Lemmas 2.2 and 2.6 we infer that, for any given $\eta \in (0, \min\{z, 1-z\})$,

$$\lim_{n\to+\infty} \frac{u_n(x)}{u_n(0)} = \lim_{n\to+\infty} \left(1 + \frac{1}{u_n(0)} \int_0^x u_n'(t)\,dt\right) = 1 \quad \text{uniformly in } x \in [0, z-\eta],$$

which provides us with (1.5). In particular, we have that

$$\lim_{n\to+\infty} \frac{u_n(x)}{u_n(0)} = 1 \quad \text{for all } \ x \in [0, z).$$

Thus, condition (f_1) implies that

$$\lim_{n\to+\infty} \frac{f(u_n(x))}{u_n^{q-1}(0)} = \lim_{n\to+\infty} \left[\frac{f(u_n(x))}{u_n^{q-1}(x)} \left(\frac{u_n(x)}{u_n(0)}\right)^{q-1}\right] = qh \quad \text{uniformly in } \ x \in [0, z-\eta] \tag{3.1}$$

and, in particular,

$$\lim_{n\to+\infty} \frac{f(u_n(x))}{u_n^{q-1}(0)} = qh \quad \text{for all } \ x \in [0, z). \tag{3.2}$$

Moreover, by the monotonicity properties of u_n and f, there exists an integer n_0 such that, for every $n \geq n_0$ and $x \in [0,1]$,

$$0 \leq \frac{f(u_n(x))}{u_n^{q-1}(0)} \leq \frac{f(u_n(0))}{u_n^{q-1}(0)} \leq qh + 1. \tag{3.3}$$

Consequently, integrating the differential equation of (1.1) on $(0,z)$ and using (3.2) and (3.3), it follows from the dominated convergence theorem that

$$\lim_{n\to+\infty} \left(\frac{1}{\lambda_n u_n^{q-1}(0)} \frac{-u_n'(z)}{\sqrt{1+(u_n'(z))^2}} \right) = \lim_{n\to+\infty} \int_0^z a(x) \frac{f(u_n(x))}{u_n^{q-1}(0)}\, dx = qh \int_0^z a(x)\, dx \tag{3.4}$$

and hence

$$\lim_{n\to+\infty} \left(\lambda_n u_n^{q-1}(0)\right) = \frac{1}{qh \int_0^z a(x)\, dx}. \tag{3.5}$$

Therefore, it follows from (3.1) that

$$\begin{aligned} \lim_{n\to+\infty} \left(\lambda_n f(u_n(x))\right) &= \lim_{n\to+\infty} \left(\frac{f(u_n(x))}{u_n^{q-1}(0)} \lambda_n u_n^{q-1}(0) \right) \\ &= \frac{1}{\int_0^z a(x)\, dx} \quad \text{uniformly in } \ x \in [0, z-\eta], \end{aligned}$$

which provides us with (1.7).

According to Lemma 2.5, for sufficiently small $\eta > 0$, we have that

$$\lim_{n\to+\infty} f(u_n(z+\eta)) \geq \lim_{n\to+\infty} \frac{C_1}{\lambda_n} = +\infty.$$

Thus, according to (f_1), we get

$$\lim_{n\to+\infty} u_n(z+\eta) = +\infty$$

and hence, thanks to Lemma 2.6,

$$\lim_{n\to+\infty} \frac{u_n(1)}{u_n(z+\eta)} = \lim_{n\to+\infty} \left(1 + \frac{1}{u_n(z+\eta)} \int_{z+\eta}^1 u_n'(t)\, dt\right) = 1.$$

Consequently, we also have that

$$\lim_{n\to+\infty} u_n(1) = +\infty.$$

Thus, by Lemma 2.6, we infer

$$\lim_{n\to+\infty} \frac{u_n(x)}{u_n(1)} = \lim_{n\to+\infty} \left(1 + \frac{1}{u_n(1)} \int_1^x u_n'(t)\, dt\right) = 1 \quad \text{uniformly in } \ x \in [z+\eta, 1],$$

or, equivalently,

$$\lim_{n\to\infty} \frac{u_n(x)}{u_n(z+\eta)} = 1 \quad \text{uniformly in } \ x \in [z+\eta, 1]. \tag{3.6}$$

Integrating the differential equation of (1.1) in $(0,1)$ yields

$$\int_0^1 f(u_n(x)) a(x)\, dx = 0$$

and hence

$$\int_0^z \frac{f(u_n(x))}{u_n^{q-1}(0)} a(x)\, dx = -\int_z^{z+\eta} \frac{f(u_n(x))}{u_n^{q-1}(0)} a(x)\, dx - \int_{z+\eta}^1 \frac{f(u_n(x))}{u_n^{q-1}(0)} a(x)\, dx. \tag{3.7}$$

From (3.4), we already know that

$$\lim_{n\to+\infty} \int_0^z \frac{f(u_n(x))}{u_n^{q-1}(0)} a(x)\, dx = qh \int_0^z a(x)\, dx. \tag{3.8}$$

On the other hand, from (3.3), we infer that

$$\limsup_{n\to+\infty}\int_z^{z+\eta} -a(x)\frac{f(u_n(x))}{u_n^{q-1}(0)}\,dx \le \|a\|_\infty (qh+1)\eta$$

and thus

$$\limsup_{n\to+\infty}\int_z^{z+\eta} -a(x)\frac{f(u_n(x))}{u_n^{q-1}(0)}\,dx = O(\eta) \quad \text{as } \eta\to 0^+. \tag{3.9}$$

In order to estimate the second term on the right hand side of (3.8) we proceed as follows. Since the sequence $\left(\frac{u_n(z+\eta)}{u_n(0)}\right)_n$ is bounded, from each of its subsequences we can extract a further subsequence, labeled by n_k, such that

$$\lim_{k\to+\infty}\frac{u_{n_k}(z+\eta)}{u_{n_k}(0)} = c(\eta)\in[0,1]. \tag{3.10}$$

Hence, thanks to (f_1), it follows from (3.6) that

$$\lim_{k\to+\infty}\frac{f(u_{n_k}(x))}{u_{n_k}^{q-1}(z+\eta)} = \lim_{k\to+\infty}\left(\frac{f(u_{n_k}(x))}{u_{n_k}^{q-1}(x)}\,\frac{u_{n_k}^{q-1}(x)}{u_{n_k}^{q-1}(z+\eta)}\right) = qh \quad \text{uniformly in } x\in[z+\eta,1].$$

So, due to (3.10), we find that

$$\begin{aligned}\lim_{k\to+\infty}\int_{z+\eta}^1 a(x)\frac{f(u_{n_k}(x))}{u_{n_k}^{q-1}(0)}\,dx &= \lim_{k\to+\infty}\int_{z+\eta}^1 a(x)\frac{f(u_{n_k}(x))}{u_{n_k}^{q-1}(z+\eta)}\,\frac{u_{n_k}^{q-1}(z+\eta)}{u_{n_k}^{q-1}(0)}\,dx\\ &= qh(c(\eta))^{q-1}\int_{z+\eta}^1 a(x)\,dx.\end{aligned} \tag{3.11}$$

Consequently, particularizing (3.7) at $n=n_k$, $k\ge 1$, letting $k\to+\infty$, using (3.8), (3.9) and (3.11), and dividing by qh, we are driven to the identity

$$\int_0^z a(x)\,dx = (c(\eta))^{q-1}\int_{z+\eta}^1 -a(x)\,dx + O(\eta). \tag{3.12}$$

Therefore, from (3.6) and (3.10) one can also infer that

$$\lim_{k\to+\infty}\frac{u_{n_k}(x)}{u_{n_k}(0)} = \lim_{k\to\infty}\left(\frac{u_{n_k}(x)}{u_{n_k}(z+\eta)}\,\frac{u_{n_k}(z+\eta)}{u_{n_k}(0)}\right) = c(\eta) \quad \text{uniformly in } x\in[z+\eta,1]. \tag{3.13}$$

Subsequently, we pick any $\eta_1\in(0,\eta)$ and fix it. As the sequence $\left(\frac{u_{n_k}(z+\eta_1)}{u_{n_k}(0)}\right)_k$ is bounded, we can extract from it a further subsequence, relabeled by n_k, such that

$$\lim_{k\to+\infty}\frac{u_{n_k}(z+\eta_1)}{u_{n_k}(0)} = c(\eta_1)\in[0,1]$$

and, arguing as above, we also have that

$$\int_0^z a(x)\,dx = (c(\eta_1))^{q-1}\int_{z+\eta_1}^1 -a(x)\,dx + O(\eta_1),$$

as well as

$$\lim_{k\to+\infty}\frac{u_{n_k}(x)}{u_{n_k}(0)} = c(\eta_1) \quad \text{uniformly in } x\in[z+\eta_1,1]. \tag{3.14}$$

Since $\eta_1<\eta$, we can conclude from (3.13) and (3.14) that $c(\eta)=c(\eta_1)$. This shows that $c(\eta)$ is constant in a right neighborhood of z, say $c(\eta)=c$. Consequently, (3.12) becomes into

$$\int_0^z a(x)\,dx = c^{q-1}\int_{z+\eta}^1 a(x)\,dx + O(\eta).$$

Therefore, letting $\eta \to 0^+$, it turns out that

$$c = \left(\frac{\int_0^z a(x)\,dx}{\int_z^1 -a(x)\,dx} \right)^{\frac{1}{q-1}}.$$

As this constant is independent of the particular subsequence chosen, one can conclude that, for the whole sequence,

$$\lim_{n\to+\infty} \frac{u_n(x)}{u_n(0)} = \left(\frac{\int_0^z a(x)\,dx}{\int_z^1 -a(x)\,dx} \right)^{\frac{1}{q-1}} \quad \text{uniformly in } \; x \in [z+\eta, 1], \tag{3.15}$$

which ends the proof of (1.6). Moreover, by (f_1), (3.5) and (3.15), we also find that

$$\begin{aligned}\lim_{n\to+\infty} (\lambda_n f(u_n(x))) &= \lim_{n\to+\infty} \left(\lambda_n u_n^{q-1}(0) \frac{f(u_n(x))}{u_n^{q-1}(x)} \frac{u_n^{q-1}(x)}{u_n^{q-1}(0)} \right) \\ &= \frac{1}{qh \int_0^z a(x)\,dx} qh \frac{\int_0^z a(x)\,dx}{\int_z^1 -a(x)\,dx} \\ &= \frac{1}{\int_z^1 -a(x)\,dx} \quad \text{uniformly in } \; x \in [z+\eta, 1],\end{aligned}$$

which provides us with (1.8).

Finally, integrating the differential equation of (1.1) and using the identities (1.7) and (1.8), we conclude that

$$\lim_{n\to+\infty} \frac{-u_n'(x)}{\sqrt{1+(u_n'(x))^2}} = \lim_{n\to+\infty} \int_0^x \lambda_n f(u_n(x))\,dx = \frac{\int_0^x a(t)\,dt}{\int_0^z a(t)\,dt} \quad \text{uniformly in} \quad x \in [0, z-\eta]$$

and

$$\lim_{n\to+\infty} \frac{-u_n'(x)}{\sqrt{1+(u_n'(x))^2}} = \lim_{n\to+\infty} \int_x^1 \lambda_n f(u_n(x))\,dx = \frac{\int_x^1 -a(t)\,dt}{\int_z^1 -a(t)\,dt} \quad \text{uniformly in} \quad x \in [z+\eta, 1].$$

From these relations, (1.9) and (1.10) can be easily obtained. This ends the proof of Theorem 1.2. □

4. A technical lemma derived from the proof of Theorem 1.2

As a direct consequence of the proof of Theorem 1.2, the next result holds.

Lemma 4.1. *Assume* (a_1) *and* (f_1)*. Then,*

$$\lim_{n\to+\infty} (u_n(0) - u_n(1)) = +\infty.$$

Proof. From (3.15) it follows that

$$\lim_{n\to+\infty} \frac{u_n(1)}{u_n(0)} = \left(\frac{\int_0^z a(x)\,dx}{\int_z^1 -a(x)\,dx} \right)^{\frac{1}{q-1}} \in (0,1),$$

because $\int_0^1 a(x)\,dx < 0$. Consequently, by Lemma 2.2, we get

$$\lim_{n\to+\infty} (u_n(0) - u_n(1)) = \lim_{n\to+\infty} \left[u_n(0) \left(1 - \frac{u_n(1)}{u_n(0)} \right) \right] = +\infty,$$

which ends the proof. □

The proof of this result can be substantially simplified if one further assumes that f is globally Lipschitz in $[0,+\infty)$. Indeed, in such case, (3.15) is far from necessary in the proof of Lemma 4.1. Indeed, the estimates (2.15) in Lemma 2.4 yield

$$f(u_n(0)) - f(u_n(1)) \geq \frac{1}{\lambda_n}\frac{-u_n'(z)}{\sqrt{1+(u_n'(z))^2}}\left[\left(\int_0^z a(t)\,dt\right)^{-1} - \left(\int_z^1 -a(t)\,dt\right)^{-1}\right]$$

for all $n \geq 1$. Thus, letting $n \to +\infty$ in the previous estimate, it follows from Lemma 2.3 that

$$\lim_{n\to+\infty} \left(f(u_n(0)) - f(u_n(1))\right) = +\infty,$$

because $\int_0^1 a(x)\,dx < 0$. Moreover, as f is globally Lipschitz continuity, there is a constant $L > 0$ such that

$$f(u_n(0)) - f(u_n(1)) \leq L\,(u_n(0) - u_n(0))$$

and then

$$\lim_{n\to+\infty} (u_n(0) - u_n(0)) = +\infty.$$

It is clear that this situation is compatible only with assuming $q \leq 2$ in (f_1).

5. Proof of Theorem 1.1

Proof of Part (i). Let us prove that if

$$\text{both} \quad \left(\int_x^z a(t)\,dt\right)^{-\frac{1}{2}} \in L^1(0,z) \quad \text{and} \quad \left(\int_x^z a(t)\,dt\right)^{-\frac{1}{2}} \in L^1(z,1)$$

then, for sufficiently small $\lambda > 0$, any positive solution of (1.1) is singular. Our proof of this feature proceeds by contradiction. Assume that there exists a sequence of positive regular solutions, $((\lambda_n, u_n))_n$, with

$$\lim_{n\to+\infty} \lambda_n = 0. \tag{5.1}$$

Then, integrating the differential equation of (1.1), it is easily seen that, for every $n \geq 1$,

$$0 \leq \int_x^z \lambda_n a(t) f(u_n(t))\,dt = \frac{u_n'(x)}{\sqrt{1+(u_n'(x))^2}} - \frac{u_n'(z)}{\sqrt{1+(u_n'(z))^2}} \leq 1 \quad \text{for all } x \in [0,1]. \tag{5.2}$$

According to Lemma 2.3, we also have that, for sufficiently large n,

$$\frac{1}{2} < \frac{-u_n'(z)}{\sqrt{1+(u_n'(z))^2}} < 1. \tag{5.3}$$

Moreover, if we pick a sufficiently small $\eta > 0$ so that Lemma 2.5 holds, then there exist constants C_1, C_2, with $0 < C_1 < C_2$, such that, for sufficiently large n,

$$C_1 \leq \lambda_n f(u_n(x)) \leq C_2 \quad \text{for all } x \in [z-\eta, z+\eta]. \tag{5.4}$$

Incorporating (5.2), (5.3) and (5.4) into the estimate (2.1) of Lemma 2.1, it follows that there exists a constant $D > 0$ such that, for sufficiently large n,

$$|u_n'(x)| \leq D\left(\int_x^z a(t)\,dt\right)^{-\frac{1}{2}} \quad \text{for all } x \in [z-\eta, z+\eta] \setminus \{z\}.$$

Therefore, according to Lemma 2.6, we have that

$$\begin{aligned} u_n(0) - u_n(1) &= \int_0^{z-\eta} |u_n'(t)|\,dt + \int_{z-\eta}^z |u_n'(t)|\,dt + \int_z^{z+\eta} |u_n'(t)|\,dt + \int_{z+\eta}^1 |u_n'(t)|\,dt \\ &\leq C(1-2\eta) + D\left[\int_{z-\eta}^z \left(\int_x^z a(t)\,dt\right)^{-\frac{1}{2}} dx + \int_z^{z+\eta}\left(\int_x^z a(t)\,dt\right)^{-\frac{1}{2}} dx\right] \\ &< +\infty. \end{aligned}$$

As this contradicts the thesis of Lemma 4.1, the proof of Part (i) is complete. □

Proof of Part (ii). Let us prove that if

$$\text{either} \quad \left(\int_x^z a(t)\,dt\right)^{-\frac{1}{2}} \notin L^1(0,z) \quad \text{or} \quad \left(\int_x^z a(t)\,dt\right)^{-\frac{1}{2}} \notin L^1(z,1),$$

then, for sufficiently small $\lambda > 0$, any positive solution of (1.1) is regular. Indeed, suppose, by contradiction, that there is a sequence $((\lambda_n, u_n))_n$ of positive singular solutions of (1.1) satisfying (5.1). As in the proof of Part (i), integrating the differential equation also provides us with the identity (5.2). Similarly, by Lemma 2.5, $\eta > 0$ can be chosen so that (5.4) holds for sufficiently large n. Shortening the size of η, we can also assume that

$$C_2 \int_{z-\eta}^{z+\eta} |a(t)|\,dt < 1. \tag{5.5}$$

Thus, substituting (5.2), (5.4) and (5.5) into the estimate (2.1) of Lemma 2.1 and taking into account that now

$$\frac{-u_n'(z)}{\sqrt{1+(u_n'(z))^2}} = 1,$$

we find that, for sufficiently large n and every $x \in [z-\eta, z+\eta] \setminus \{z\}$,

$$-u_n'(x) = \frac{1 - \int_x^z \lambda_n a(t) f(u_n(t))\,dt}{\sqrt{2 - \int_x^z \lambda_n a(t) f(u_n(t))\,dt}\sqrt{\int_x^z \lambda_n a(t) f(u_n(t))\,dt}} \geq \frac{1 - C_2 \int_{z-\eta}^{z+\eta} |a(t)|\,dt}{\sqrt{2}\sqrt{C_2 \int_x^z a(t)\,dt}}.$$

Consequently, there exists a constant, $C_3 > 0$, such that, for sufficiently large n,

$$|u_n'(x)| \geq C_3 \left(\int_x^z a(t)\,dt\right)^{-\frac{1}{2}} \quad \text{if } 0 < |x - z| < \eta.$$

Therefore, either $u_n \notin L^1(z-\eta, z)$, or $u_n' \notin L^1(z, z+\eta)$, which contradicts the fact that both $u_n \in W^{1,1}(0,z)$ and $u_n \in W^{1,1}(z,1)$, as it was already established by (1.2). This ends the proof of the theorem. □

6. Final remarks

The restriction $p \geq 2$ that we have imposed in assumption (f_1) has been used only in the proof of Lemma 2.2 to guarantee that any sequence $((\lambda_n, u_n))_n$ of positive bounded variation solutions of problem (1.1), with

$$\lim_{n\to\infty} \lambda_n = 0, \tag{6.1}$$

satisfies

$$\lim_{n\to+\infty} u_n(0) = +\infty. \tag{6.2}$$

Regardless the assumptions that one might impose to f at 0, condition (6.2) implies that

$$\lim_{n\to+\infty} u_n(x) = +\infty \quad \text{uniformly in } x \in [0,z]. \tag{6.3}$$

Therefore, whenever (6.1) and (6.2) are satisfied, we can replace (f_1) with

(f_2) the function $f \in \mathcal{C}(\mathbb{R}) \cap \mathcal{C}^1(0,+\infty)$ satisfies $f(s) > 0$ and $f'(s) \geq 0$ for all $s > 0$, and there exist four constants, $h > 0$, $k > 0$, $q > 1$ and $p > 1$, such that

$$\lim_{s\to+\infty} \frac{f(s)}{s^{q-1}} = qh, \qquad \lim_{s\to 0^+} \frac{f(s)}{s^{p-1}} = pk,$$

in order to conclude that, under (a_1) and (f_2), the condition (a_2) characterizes the development of singularities by the positive solutions of (1.1) having large L^∞-norms.

Theorem 6.1. *Assume* (a_1) *and* (f_2). *Let* $((\lambda_n, u_n))_n$ *be an arbitrary sequence of positive solutions of* (1.1) *satisfying* (6.1) *and* (6.2). *Then, for all large* n, *the following assertions hold:*

(i) (λ_n, u_n) *is singular if* (a_2) *holds;*
(ii) (λ_n, u_n) *is regular if* (a_2) *fails.*

It should be further observed that, under (a_1) and (f_2), condition (6.1) is a consequence, via (6.3), of (6.2). Indeed, as we are assuming in particular that $a(x) > 0$ for a.e. $x \in [0, z]$ and

$$\lim_{s\to+\infty} f(s) = +\infty,$$

integrating the equation of (1.1) on $[0, z]$ and using (6.3) yield

$$\liminf_{n\to+\infty} \frac{1}{\lambda_n} \geq \liminf_{n\to+\infty} \left(\frac{1}{\lambda_n} \frac{-u_n'(z)}{\sqrt{1+(u_n'(z))^2}} \right) = \lim_{n\to+\infty} \int_0^z a(x) f(u_n(x))\, dx = +\infty.$$

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