

# Simplified recursive $^3P_0$ model for the fragmentation of polarized quarks

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We revisit the recursive model for the fragmentation of polarized quarks based on the string  $+^3P_0$  mechanism of  $q\bar{q}$  pair creation. We make a different choice for one input function of the model that simplifies the implementation in a Monte Carlo program. No new parameters are introduced, and the relevant results are the same apart from the suppression of the spin-independent correlations between successive quarks. The advantage of the present version is that it allows us to study analytically the preservation of positivity bounds and it is more suitable for an interface with external event generators like PYTHIA. The theoretical aspects and the simulation results obtained with a stand alone program are discussed in detail and compared with those of the previous version of the model.

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## I. INTRODUCTION

The theoretical description of high energy collisions like  $e^+e^-$  annihilations, lepton-nucleon deep inelastic scattering (DIS) and inelastic  $p\bar{p}$  scattering involves factorization theorems which separate the subprocesses calculable in perturbative QCD from the nonperturbative ones. For the semi-inclusive processes where at least one hadron is detected in the final state, the knowledge of fragmentation functions (FFs) is needed. They are universal functions which describe how the colored quarks and gluons transform into observable hadrons and cannot be calculated perturbatively. This issue has been tackled through models, for instance, inspired from field theory or of multiproduction type [1–3].

Within the latter class of models, the most successful one is the symmetric Lund model (SLM) [3], where the interaction among color charges is treated as a relativistic string which decays by a tunneling process into smaller string pieces through the creation of  $q\bar{q}$  pairs in the string world sheet. Such a chain is depicted in Fig. 1 for an initial quark-antiquark pair  $q_A\bar{q}_B$  that hadronizes into mesons. Tunneling of diquarks can account for baryon production. The SLM is symmetric under the reversal of the quark line, namely the hadronization process can be viewed to occur

from the  $q_A$  side to the  $\bar{q}_B$  side or from  $\bar{q}_B$  to  $q_A$  with the same probability. This symmetry will be referred to as the LR symmetry, “LR” standing for *left-right* according to [3] or less subjectively for “line reversal.” This requirement is a strong and important constraint on the form of the splitting function of the SLM. The SLM has been implemented in Monte Carlo event generators like PYTHIA [4] which is successful in the description of experimental data from unpolarized reactions. However it does not incorporate polarization effects.

By now it is well established that the quark polarization produces important effects in  $e^+e^-$  annihilation [5–7] and in polarized reactions like semi-inclusive DIS (SIDIS) [8–10] where large transverse spin asymmetries have been observed for single hadrons and hadron pairs in

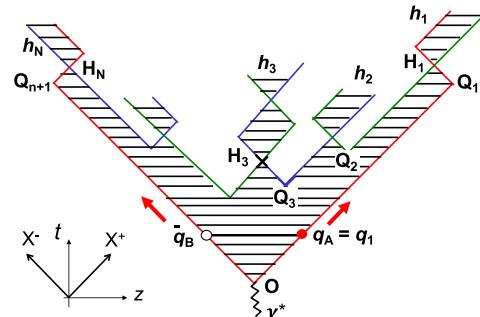


FIG. 1. Space-time history of the hadronization process of a  $q_A\bar{q}_B$  system without gluons produced in  $e^+e^-$  annihilation.  $Q_2, Q_3, \dots$  are the string breaking points whereas  $H_1, H_2, \dots$  are the hadron emission points.

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the same jet [11,12]. Particularly relevant is the Collins effect [13], an asymmetry in the azimuthal spectrum of hadrons produced in the fragmentation of a transversely polarized quark. It is described by the Collins transverse momentum dependent fragmentation function (TMD FF), a nonperturbative and universal function which in SIDIS is coupled to the quark transversity distribution resulting in an observed azimuthal modulation of the hadron in  $\sin(\phi_h + \phi_S)$  in the  $\gamma^*$ -nucleon frame, where  $\phi_S$  is the azimuthal angle of the nucleons transverse polarization about the  $\gamma^*$ -nucleon collision axis. This asymmetry is then used as an observable to access transversity [14–16].

Attempts for the inclusion of the quark spin in the fragmentation process have been made in the past. In particular the model of Refs. [17–19] is an extension of the SLM where the  $q\bar{q}$  pairs at string breaking are produced in the  ${}^3P_0$  state. An alternative model based on a field theoretical approach has been presented in Ref. [20].

In the string  $+{}^3P_0$  model the quark spin is encoded in  $2 \times 2$  density matrices and treated with rules that preserve the LR symmetry. We have recently implemented the general string  $+{}^3P_0$  model in a stand alone MC program [21] which simulates the fragmentation of a quark (or antiquark) with arbitrary polarization into pseudoscalar mesons. The comparison of the resulting Collins and dihadron asymmetries with experimental data from SIDIS and  $e^+e^-$  are very promising [21].

The present work is based on the previous study of Ref. [21], and a simpler choice of one input function of the string  $+{}^3P_0$  model is done. The model is completely LR symmetric and is characterized by a splitting function without dynamical spin-independent correlations between the transverse momenta of two successive quarks [22]. It leads to simpler simulation codes and many analytical calculations can be done. From the practical point of view, it demands much less computer resources and is more suitable for an interface with external event generators [23]. It is as rich as the model in Ref. [21], depends on the same free parameters and, after retuning the latter, gives the same results.

The article is organized as follows. In Secs. II and III the basis of the recursive polarized quark fragmentation model and the splitting matrix of the string  $+{}^3P_0$  model are shortly described. In Sec. IV the possible choices for an input function of the model are discussed. The simplified version of the model is presented in Sec. V and the comparison with Ref. [21] is discussed in Sec. VI. In Sec. VII the positivity conditions are analyzed in the context of the present version of the model.

## II. POLARIZED RECURSIVE QUARK FRAGMENTATION

The hadronization process  $q_A\bar{q}_B \rightarrow h_1\dots h_N$  of the  $q_A\bar{q}_B$  color neutral system can be thought to occur by the chain of splittings

$$q_A \rightarrow h_1 + q_2, \dots, q_r \rightarrow h_r + q_{r+1}, \dots, q_{N-1} \rightarrow h_{N-1} + q_N, \quad (1)$$

$r$  being the “rank” of the hadron  $h_r$ . The chain terminates with  $q_N + \bar{q}_B \rightarrow h_N$ . The nonperturbative interaction between the initial quark and antiquark is treated as a relativistic string with massless endpoints  $q_A$  and  $\bar{q}_B$ . The decay of the string represents the hadronization of the  $q_A\bar{q}_B$  system. In the center of mass frame of the  $q_A\bar{q}_B$  system we orient the  $\hat{\mathbf{z}}$  axis along the momentum of  $q_A$ , which is also the jet or “string” axis.

The process in Eq. (1) is the recursive application of the elementary splitting

$$q \rightarrow h + q' \quad (2)$$

where  $q$  is the current fragmenting quark,  $h$  is the emitted hadron, with quark content  $q\bar{q}'$ , and  $q'$  is the leftover quark.  $h$  is restricted here to be a pseudoscalar meson. For a baryon  $q'$  is replaced for instance by an antidiquark. We denote by  $k$  ( $k'$ ) the four-momenta of  $q$  ( $q'$ ) and by  $p$  the four-momentum of  $h$ . They are related by momentum conservation  $k = p + k'$ .

The process in Eq. (2) is described by the splitting function  $F_{q',h,q}(Z, \mathbf{p}_T; \mathbf{k}_T, \mathbf{S}_q)$  which gives the probability

$$dP_{q \rightarrow h + q'} = F_{q',h,q}(Z, \mathbf{p}_T; \mathbf{k}_T, \mathbf{S}_q) \frac{dZ}{Z} d^2 \mathbf{p}_T \quad (3)$$

that the hadron  $h$  is emitted with forward light-cone momentum fraction  $Z = p^+/k^+$  and with transverse momentum  $\mathbf{p}_T = \mathbf{k}_T - \mathbf{k}'_T$ , and is normalized according to

$$\sum_h \int_0^1 \frac{dZ}{Z} \int d^2 \mathbf{p}_T F_{q',h,q}(Z, \mathbf{p}_T; \mathbf{k}_T, \mathbf{S}_q) = 1. \quad (4)$$

The light-cone momenta are defined as  $p^\pm = p^0 \pm p^3$ .  $\mathbf{k}_T$  and  $\mathbf{k}'_T$  are the transverse momenta of  $q$  and  $q'$  with respect to the string axis.  $p^-$  is not an independent variable but fixed by the mass-shell condition  $p^- = \epsilon_h^2/p^+$  where  $\epsilon_h^2 = m_h^2 + \mathbf{p}_T^2$  is the hadrons transverse energy squared and  $m_h$  is its mass. We describe the quark spin states with Pauli spinors and encode the information on the quark polarization in the  $2 \times 2$  spin density matrix  $\rho(q) = (1 + \boldsymbol{\sigma} \cdot \mathbf{S}_q)/2$ . The resulting “polarized splitting function” depends therefore on the polarization vector  $\mathbf{S}_q$ . In Eq. (3) the spin states of  $q'$  are summed over.

The polarized splitting function can be calculated starting from the expression

$$F_{q',h,q} = \text{tr}[T_{q',h,q}\rho(q)T_{q',h,q}^\dagger], \quad (5)$$

where  $T_{q',h,q}$  is a quantum mechanical “splitting matrix” acting on the quark flavor  $\otimes$  momentum  $\otimes$  spin space.

Its elements are defined between the spin states of  $q$  and of  $q'$ . For practical applications, the splitting function in Eq. (5) is used for the generation of the hadron type  $h$  and of its momentum, namely  $Z$  and  $\mathbf{p}_T$ , at the given momentum and polarization state of the quark  $q$ . The spin density matrix of the leftover quark  $q'$  is given by

$$\rho(q') = \frac{T_{q',h,q}\rho(q)T_{q',h,q}^\dagger}{\text{tr}[T_{q',h,q}\rho(q)T_{q',h,q}^\dagger]}. \quad (6)$$

The recursive application of Eq. (5) and of Eq. (6) in the Monte Carlo simulation allows us to generate the hadron jets produced in the hadronization of polarized quarks [21].

### III. POLARIZED SPLITTING FUNCTION FROM THE GENERAL STRING $+^3P_0$ MODEL

The string axis defines a privileged direction in space, thus the splitting matrix has not to be invariant under the full Lorentz group but only under the subgroup generated by rotations about the string axis (here  $\hat{\mathbf{z}}$ ), Lorentz boosts along the same axis and reflections about any plane containing it.

The splitting matrix, defined as [21]

$$T_{q',h,q} = C_{q',h,q}\check{g}(\varepsilon_h^2)\Delta_{q'}(\mathbf{k}'_T)\Gamma_{h,s_h}\hat{u}_q^{-1/2}(\mathbf{k}_T) \\ \times [(1-Z)/\varepsilon_h^2]^{a/2} \exp[-b_L\varepsilon_h^2/(2Z)], \quad (7)$$

respects these symmetries. The  $Z$  dependence as required by LR symmetry is given in the second line and the parameters  $a$  and  $b_L$  have the same meaning as  $a$  and  $b$  in the SLM [3].

The factor  $C_{q',h,q}$  describes the splitting of Eq. (2) in flavor space and is symmetric under the exchange of  $q$  with  $q'$ , more precisely  $C_{q',h,q} = C_{q,\bar{h},q'}$ . It is proportional to the meson wave function  $\langle q\bar{q}'|h\rangle$  in flavor space and also takes into account the suppression of strange mesons and the suppression of  $\eta$  with respect to  $\pi^0$ .

The complex  $2 \times 2$  matrix in quark spin space

$$\Delta_q(\mathbf{k}_T) = (\mu_q + \sigma_z \boldsymbol{\sigma} \cdot \mathbf{k}_T) f_T(\mathbf{k}_T^2) \quad (8)$$

gives the  $\mathbf{k}_T$ -dependent part of the quark propagator inspired to the  $^3P_0$  mechanism. It depends on the complex mass parameter  $\mu_q$  which is responsible for the single spin effects. In the following, we take the same complex parameter for all quark flavors, i.e.,  $\mu_q \equiv \mu$ . The function  $f_T$  is a fast decreasing function of the quark transverse momentum at the string breaking. As in Ref. [21] we take

$$f_T(\mathbf{k}_T^2) = \sqrt{\frac{b_T}{\pi}} \exp(-b_T \mathbf{k}_T^2/2), \quad (9)$$

where the parameter  $b_T$  acts upon the width of the quark (and antiquark) transverse momentum at each string

breaking. This choice of  $f_T$  leads to an exponential decay of the hadrons  $p_T^2$  spectrum. A different choice is discussed in Sec. V.

$\Gamma_{h,s_h}$  is the  $2 \times 2$  vertex matrix which describes the  $q - h - q'$  coupling. It depends on the hadron spin state  $s_h$  and possibly on  $\mathbf{k}_T$  and  $\mathbf{k}'_T$ , at most as a polynomial. Neglecting the latter possibility and considering only pseudoscalar meson emission, it is

$$\Gamma_h = \sigma_z, \quad (10)$$

analogous to the Dirac  $\gamma_5$  coupling.

The  $2 \times 2$  matrix  $\hat{u}_q(\mathbf{k}_T)$  is related to the single quark density in momentum  $\otimes$  spin space and can be written as [21]

$$\begin{aligned} \hat{u}_q(\mathbf{k}_T) &= \sum_h \hat{u}_{q,h}(\mathbf{k}_T) \\ &= \sum_h |C_{q',h,q}|^2 \int d^2 \mathbf{k}'_T \check{g}^2(\varepsilon_h^2) N_a(\varepsilon_h^2) \\ &\quad \times \sum_{s_h} \Gamma_{h,s_h}^\dagger \Delta_{q'}^\dagger(\mathbf{k}'_T) \Delta_{q'}(\mathbf{k}'_T) \Gamma_{h,s_h} \\ &\equiv \hat{u}_{0q}(\mathbf{k}_T^2) + \hat{u}_{1q}(\mathbf{k}_T^2) \boldsymbol{\sigma} \cdot \tilde{\mathbf{n}}(\mathbf{k}_T), \end{aligned} \quad (11)$$

where  $\tilde{\mathbf{n}}(\mathbf{k}_T) = \hat{\mathbf{z}} \times \mathbf{k}_T / |\mathbf{k}_T|$  and

$$N_a(\varepsilon_h^2) = \int_0^1 dZ Z^{-1} \left( \frac{1-Z}{\varepsilon_h^2} \right)^a \exp \left( -b_L \frac{\varepsilon_h^2}{Z} \right). \quad (12)$$

The matrix  $\hat{u}_q$  is positive definite, with  $\hat{u}_{0q} > |\hat{u}_{1q}|$ , and allows the splitting function to be normalized according to Eq. (4). The presence of  $\hat{u}_q^{-1/2}$  in Eq. (7) is necessary to fulfill the LR symmetry requirement. The partial contribution  $\hat{u}_{q,h}$  to  $\hat{u}_q$  is also positive definite and can be decomposed in the same form as  $\hat{u}_q$ .

The function  $\check{g}(\varepsilon_h^2)$  appearing in Eq. (7) plays an important role as can be seen considering the splitting function of the string  $+^3P_0$  model obtained from the general splitting matrix in Eq. (7),

$$\begin{aligned} F_{q',h,q} &= |C_{q',h,q}|^2 \check{g}^2(\varepsilon_h^2) \left( \frac{1-Z}{\varepsilon_h^2} \right)^a \exp \left( -b_L \frac{\varepsilon_h^2}{Z} \right) \\ &\quad \times \text{tr}[\Delta_{q'}(\mathbf{k}'_T) \hat{u}_q^{-1/2}(\mathbf{k}_T) \rho(q) \hat{u}_q^{-1/2}(\mathbf{k}_T) \Delta_{q'}^\dagger(\mathbf{k}'_T)], \end{aligned} \quad (13)$$

and different choices are allowed.

For a generic dependence of  $\check{g}$  on  $\varepsilon_h^2$ , integrating Eq. (13) on  $Z$  and  $\mathbf{p}_T$  for a fixed hadron type  $h$ , the probability of generating the hadron is

$$P_{q \rightarrow h} = \text{tr}[\hat{u}_{q,h}(\mathbf{k}_T) \hat{u}_q^{-1}(\mathbf{k}_T) \rho(q)]. \quad (14)$$

It depends on the transverse polarization  $\mathbf{S}_{qT}$  and on the transverse momentum  $\mathbf{k}_T$  of the quark  $q$ , due to the matrix  $\hat{u}_{q,h}(\mathbf{k}_T)\hat{u}_q^{-1}(\mathbf{k}_T)$  which is not calculable analytically.

Moreover, a generic  $\check{g}$  precludes a factorization in  $\mathbf{k}_T$  and  $\mathbf{k}'_T$  of the splitting function in Eq. (13) leading to a *spin-independent* correlation between  $\mathbf{k}_T$  and  $\mathbf{k}'_T$  which is also expected for a general spinless model [22]. Such correlation may depend on  $Z$  and adds to the *spin-mediated* ( $Z$ -independent)  $\mathbf{k}_T - \mathbf{k}'_T$  correlation which originates from the  $^3P_0$  mechanism that pushes  $\mathbf{k}_T$  and  $\mathbf{k}'_T$  to be in opposite directions, namely  $\langle \mathbf{k}_T \cdot \mathbf{k}'_T \rangle < 0$ . The factor  $\check{g}^2(\epsilon_h^2)N_a(\epsilon_h^2)$ , as can be seen from the integral over  $Z$  of Eq. (13) for fixed  $\mathbf{p}_T$ , governs the  $Z$ -integrated spin-independent  $\mathbf{k}_T - \mathbf{k}'_T$  correlation.

The simplest choices for  $\check{g}$  already quoted in Ref. [21] are discussed in the next section.

#### IV. POSSIBLE CHOICES FOR THE INPUT FUNCTION $\check{g}$ AND $\mathbf{k}_T - \mathbf{k}'_T$ CORRELATIONS

The choice of  $\check{g}$  made in the model of Ref. [21] is

$$(a) \quad \check{g}^2(\epsilon_h^2) = (\epsilon_h^2)^a. \quad (15)$$

It gives spin-independent  $\mathbf{k}_T - \mathbf{k}'_T$  correlations dependent on  $Z$  which favor the quark transverse momenta to be aligned, namely  $\langle \mathbf{k}_T \cdot \mathbf{k}'_T \rangle > 0$ . The matrix  $\hat{u}_{q,h}(\mathbf{k}_T)\hat{u}_q^{-1}$  is nontrivial and it needs to be tabulated in order to implement the model in a simulation program. This was indeed the case of the previous work [21]. The probability  $P_{q \rightarrow h}$  depends on  $\mathbf{k}_T$  and  $\mathbf{S}_{Tq}$ , and this property was not implemented in the stand alone Monte Carlo program of Ref. [21] introducing a breaking of the LR symmetry.

In the present work we adopt the different choice

$$(c) \quad \check{g}^2(\epsilon_h^2) = 1/N_a(\epsilon_h^2). \quad (16)$$

In this case there is no spin-independent  $\mathbf{k}_T - \mathbf{k}'_T$  correlation, the dependence of  $P_{q \rightarrow h}$  on  $\mathbf{k}_T$  and  $\mathbf{S}_{qT}$  is removed and the generation of the hadron type in the simulation code is greatly simplified without breaking the LR symmetry. The choice in Eq. (16) is in fact implicit in the standard spinless SLM [3] implemented in PYTHIA [4]. Thus it is more suitable in view of the inclusion of spin effects in the hadronization part of this event generator [23]. Also, it allows for a simpler description of the spin transfer mechanism, as will be shown in Sec. V.

Alternative choices consist in introducing to (a) and (c) an exponential factor dependent on  $\epsilon_h^2$  which correlates  $\mathbf{k}_T$  and  $\mathbf{k}'_T$ . They are

$$(b) \quad \check{g}^2(\epsilon_h^2) = (\epsilon_h^2)^a e^{cb_L \epsilon_h^2} \quad (17)$$

and

$$(d) \quad \check{g}^2(\epsilon_h^2) = e^{-b_e \epsilon_h^2} / N_a(\epsilon_h^2). \quad (18)$$

Choice (b) introduces the new phenomenological parameter  $c \leq 1 + b_T/2b_L$  and is a generalization of (a). The spin-independent  $\mathbf{k}_T - \mathbf{k}'_T$  correlation foreseen in (a) is reinforced with respect to (a) for  $c < 0$ , weakened and eventually changing sign for  $c > 0$ . For vanishing  $c$  we recover (a). This choice is characterized by the same complications as (a).

Choice (d) depends on the parameter  $b_e \geq -b_T/2$  and it is a generalization of (c). It reintroduces the spin-independent (and  $Z$ -independent)  $\mathbf{k}_T - \mathbf{k}'_T$  correlations while maintaining the matrix  $\hat{u}_{q,h}(\mathbf{k}_T)\hat{u}_q^{-1}(\mathbf{k}_T)$  simple. Also, the probability  $P_{q \rightarrow h}$  as function of  $\mathbf{k}_T$  and  $\mathbf{S}_{qT}$  is calculable analytically. For vanishing  $b_e$  we recover (c).

In spite of different possible options for  $\check{g}$ , we find that the bulk predictions of the model do not depend too much on its specific analytical form, provided that the experimental  $\langle \mathbf{p}_T^2 \rangle$  of the hadrons is reproduced. This constraint correlates the parameters of  $\check{g}(\epsilon_h^2)$  and  $f_T(\mathbf{k}_T^2)$ . For instance, in the spinless model with Eq. (9) and choice (d) we have

$$\langle \mathbf{p}_T^2 \rangle = 1/(b_e + b_T/2). \quad (19)$$

This result was also obtained in Ref. [24]. For other choices of  $\check{g}$  and  $f_T$ , Eq. (19) can be generalized by replacing  $b_T$  and  $b_e$  by the derivatives  $d \ln f_T^2/dk_T^2$  and  $d \ln(\check{g}^2 N_a)/d\epsilon_h^2$  evaluated at typical values, e.g.,  $\mathbf{k}_T^2 = \langle \mathbf{p}_T^2 \rangle/2$  and  $\epsilon_h^2 = \langle m_h^2 + \mathbf{p}_T^2 \rangle$ . Equation (19) tells that by retuning the respective parameters of (b) and (d) such that they reproduce the same  $\langle \mathbf{p}_T^2 \rangle$ , the two choices give practically the same results.

The strength of the spin-independent  $\mathbf{k}_T - \mathbf{k}'_T$  correlations is quantified by the parameter  $\beta = \langle \mathbf{k}_T \cdot \mathbf{k}'_T \rangle / \langle \mathbf{k}_T^2 \rangle$ . In the spinless model with Eq. (9) and choice (d) we get

$$\beta = b_e/(b_T + b_e), \quad (20)$$

$$\langle \mathbf{k}'_T \rangle = \beta \mathbf{k}_T \text{(at given } \mathbf{k}_T\text{)}, \quad (21)$$

$$\langle \mathbf{k}_{r,T} \cdot \mathbf{k}_{s,T} \rangle = \beta^{|r-s|} \langle \mathbf{k}_T^2 \rangle, \quad (22)$$

$$\langle \mathbf{p}_{r,T} \cdot \mathbf{p}_{s,T} \rangle = \beta^{|r-s|-1} (\beta - 1) \langle \mathbf{p}_T^2 \rangle / 2 \quad (r \neq s), \quad (23)$$

where  $r$  and  $s$  refer to different ranks. In our model with spin, Eqs. (20)–(23) are expected to still hold approximately for all choices of  $\check{g}$ , as suggested by the recursive nature of the model. Then  $\langle \mathbf{k}_T \cdot \mathbf{k}'_T \rangle$  includes both the spin-independent and spin-mediated  $\mathbf{k}_T - \mathbf{k}'_T$  correlations. Equation (23) tells how a large  $\mathbf{p}_T$  is compensated [25] by the  $\mathbf{p}_T$ 's of nearby ranks. The “compensation length” in rank space is  $1/\ln(1/|\beta|)$ . In principle it could be experimentally estimated if the correlation between rank ordering and rapidity ordering is sufficiently known. However,

that would not be sufficient to tell if spin-independent  $Z$ -integrated  $\mathbf{k}_T - \mathbf{k}'_T$  correlations are necessary, because of the mixing with spin-mediated correlations.

To summarize, up to now there is no compelling reason to introduce spin-independent  $\mathbf{k}_T - \mathbf{k}'_T$  correlations in our model and we consider the choice (c) for  $\check{g}$  as the most suitable for the interface with existing event generators [23] and for the future developments like the introduction of vector mesons.

## V. THE SIMPLIFIED STRING + $^3P_0$ MODEL

Equation (16) introduces a remarkable simplification with respect to Ref. [21], in particular the matrix  $\hat{u}_q$  of Eq. (11) becomes proportional to the unit matrix. With only pseudoscalar mesons and Eq. (10), it is

$$\hat{u}_q(\mathbf{k}_T) = \mathbf{1} \sum_h |C_{q',h,q}|^2 (|\mu|^2 + \langle \mathbf{k}_T^2 \rangle_{f_T}), \quad (24)$$

where we have defined the weighting operation for a generic function  $A(\mathbf{k}_T^2)$  (for normalized  $f_T^2$ ) as

$$\langle A \rangle_{f_T} = \int d^2 \mathbf{k}_T A(\mathbf{k}_T^2) f_T^2(\mathbf{k}_T^2). \quad (25)$$

The matrix  $\hat{u}_{q,h}$  has the same property as  $\hat{u}_q$  in Eq. (24).

Using Eqs. (5), (7)–(9), the splitting function becomes

$$F_{q',h,q}(Z, \mathbf{p}_T; \mathbf{k}_T, \mathbf{S}_q) = \frac{|C_{q',h,q}|^2}{\sum_H |C_{q',H,q}|^2} \frac{|\mu|^2 + \mathbf{k}'_T^2}{|\mu|^2 + \langle \mathbf{k}_T^2 \rangle_{f_T}} f_T^2(\mathbf{k}'_T^2) \times \left[ 1 - \frac{2\text{Im}(\mu)\mathbf{k}'_T}{|\mu|^2 + \mathbf{k}'_T^2} \mathbf{S}_q \cdot \tilde{\mathbf{n}}(\mathbf{k}'_T) \right] \times \left( \frac{1-Z}{\varepsilon_h^2} \right)^a \frac{\exp(-b_L \varepsilon_h^2 / Z)}{N_a(\varepsilon_h^2)}, \quad (26)$$

where the third line is source of the Collins effect in the model. The splitting function satisfies the normalization condition in Eq. (4) and is much simpler than the one given by Eqs. (52)–(54) of Ref. [21].

For the function  $f_T$  we take the exponential in Eq. (9), which was proposed in Ref. [1], but other forms can be used to enhance the tail of the distribution. For instance in the event generator PYTHIA it is  $f_T^2(\mathbf{k}_T^2) = p_0 \exp(-\mathbf{k}_T^2/\sigma_0^2) + p_1 \exp(-\mathbf{k}_T^2/\sigma_1^2)$ . An alternative class of functions is

$$f_T(\mathbf{k}_T^2) \propto \frac{\exp(-b_T \mathbf{k}_T^2/2)}{(|\mu|^2 + \mathbf{k}_T^2)^\alpha}, \quad (27)$$

where the denominator is inspired from the Feynman propagator  $1/(\gamma \cdot k - m_q) = (\gamma \cdot k + m_q)/(k^2 - m_q^2)$ , the analog of  $(k^2 - m_q^2)$  being  $-(\mathbf{k}_T^2 + |\mu|^2)$ . This analogy suggests  $\alpha = 1$ , but since we are not in a QCD perturbation

regime, the choice of  $\alpha$  is phenomenological. The choice  $\alpha = 0$  brings back to Eq. (9) whereas taking  $\alpha \neq 0$  while decreasing  $b_T$  extends the  $\mathbf{k}_T^2$  tail for a given  $\langle \mathbf{k}_T^2 \rangle$ . We have performed simulations using both Eqs. (9) and (27) for different values of  $\alpha$  obtaining predictions only slightly different, allowing the choice  $\alpha = 0$  of Ref. [21].

In this new version of the model it is more convenient to draw the hadron  $h$  generating first its type according to the first line of Eq. (26), then the transverse momentum  $\mathbf{p}_T = \mathbf{k}_T - \mathbf{k}'_T$  according to the second and third lines and then finally the longitudinal momentum fraction  $Z$  according to the last line of Eq. (26). In Ref. [21] the simplest order was the hadron type first, then  $Z$  and finally  $\mathbf{p}_T$ .

As already mentioned, with the choice of Eq. (16), there is no spin-independent correlation between  $\mathbf{k}_T$  and  $\mathbf{k}'_T$  in the  $Z$ -integrated splitting function. The only source of correlation between  $\mathbf{k}_T$  and  $\mathbf{k}'_T$  comes from the  $^3P_0$  mechanism associated to the correlation between the spins of  $q$  and  $\bar{q}'$  in the hadron. For a pseudoscalar hadron it gives  $\langle \mathbf{k}_T \cdot \mathbf{k}'_T \rangle < 0$ , i.e., on the average  $\mathbf{k}_T$  and  $\mathbf{k}'_T$  are antiparallel and  $\langle \mathbf{p}_T^2 \rangle > 2\langle \mathbf{k}_T^2 \rangle$ .

The polarization vector of the leftover quark  $q'$  can then be calculated from Eq. (6). The transverse and the longitudinal components are

$$\begin{aligned} \mathbf{S}_{q'T} = & \frac{1}{N} [ -(|\mu|^2 + \mathbf{k}'_T^2) \mathbf{S}_{qT} + 2(\mathbf{S}_{qT} \cdot \mathbf{k}'_T) \mathbf{k}'_T \\ & - 2\text{Im}(\mu) \mathbf{k}'_T \tilde{\mathbf{n}}(\mathbf{k}'_T) - 2\text{Re}\mu S_{qL} \mathbf{k}'_T ], \end{aligned} \quad (28)$$

$$S_{q'L} = \frac{1}{N} [ (|\mu|^2 - \mathbf{k}'_T^2) S_{qL} - 2\text{Re}\mu \mathbf{S}_{qT} \cdot \mathbf{k}'_T ] \quad (29)$$

where the normalization  $N$  is given by

$$N = |\mu|^2 + \mathbf{k}'_T^2 - 2\text{Im}\mu \mathbf{k}'_T \mathbf{S}_{qT} \cdot \tilde{\mathbf{n}}(\mathbf{k}'_T). \quad (30)$$

From Eq. (28) it is clear that the transverse polarization of  $q'$  has several different types of contributions: it inherits some (depending on  $\mathbf{k}'_T$ ) of the transverse polarization of  $q$  but can also receive contributions from  $\mathbf{k}'_T$  alone. In addition, there can be a transfer from longitudinal to transverse polarization and vice-versa. If the quark  $q$  is in a pure state ( $\mathbf{S}_q^2 = 1$ ), then also  $q'$  will be in a pure state ( $\mathbf{S}_{q'}^2 = 1$ ). This is due to the fact that the emitted meson has spin zero, thus cannot take spin information away.

If the transverse momentum of  $q'$  is integrated over there is a leakage of spin information on  $q'$  ( $\mathbf{k}'_T$  is correlated with  $\mathbf{S}'_q$ ) and the quark polarization decays along the fragmentation chain. Therefore, at each step of the recursive process both the quark transverse and longitudinal polarizations decay.

The polarized decay process is described by the transverse and the longitudinal depolarization factors  $D_{TT}$  and

$D_{LL}$ . They are obtained from Eqs. (28)–(29) integrating over  $\mathbf{k}'_T$  separately the numerator and the denominator. For  $f_T$  in Eq. (9) we obtain

$$\mathbf{S}_{q'T} = -\frac{|\mu|^2}{|\mu|^2 + \langle \mathbf{k}_T^2 \rangle_{f_T}} \mathbf{S}_{qT} \equiv D_{TT} \mathbf{S}_{qT} \quad (31)$$

$$S_{q'L} = \frac{|\mu|^2 - \langle \mathbf{k}_T^2 \rangle_{f_T}}{|\mu|^2 + \langle \mathbf{k}_T^2 \rangle_{f_T}} S_{qL} \equiv D_{LL} S_{qL}. \quad (32)$$

The depolarization factors depend on the complex mass and on the width of quark transverse momentum  $\mathbf{k}'_T^2$ , i.e., on the choice of the function  $f_T$ . For  $f_T$  of Eq. (9) it is  $D_{TT} = -b_T |\mu|^2 / (b_T |\mu|^2 + 1)$  and  $D_{LL} = (b_T |\mu|^2 - 1) / (b_T |\mu|^2 + 1)$  as in Ref. [17]. We note that  $D_{TT} < 0$  as expected for the production of a pseudoscalar meson in the string +  ${}^3P_0$  model. This gives Collins effects of opposite sign for even and odd rank mesons.

## VI. COMPARISON WITH THE PREVIOUS RESULTS

As in its previous version [21], we have implemented the model in a recursive stand alone Monte Carlo using now the choice of Eq. (16) for the function  $\check{g}$ . The code is the same except for the routines used for the generation of  $Z$  and  $\mathbf{k}'_T$  which have been changed according to Eq. (26). The free parameters are the same and have the same values as in Ref. [21] except for  $b_T$ . In particular  $a = 0.9$ ,  $b_L = 0.5 \text{ GeV}^{-2}$ ,  $\mu = (0.42 + i0.76) \text{ GeV}$  and  $b_T = 8.43 \text{ GeV}^{-2}$  which is 1.63 times larger than the value used in Ref. [21] in order to have similar  $p_T^2$  distributions in spite of the different choices for  $\check{g}$ . The increase in  $b_T$  is necessary to compensate the exponential growth, at large  $\mathbf{p}_T^2$ , of  $\check{g}(\epsilon_h^2)$  given by Eq. (16).

The results shown in the next sections are obtained from simulations of the fragmentation of fully transversely polarized  $u$  quarks whose momentum is determined using the same sample of  $x_B$  and  $Q^2$  values of SIDIS events as in Ref. [21].

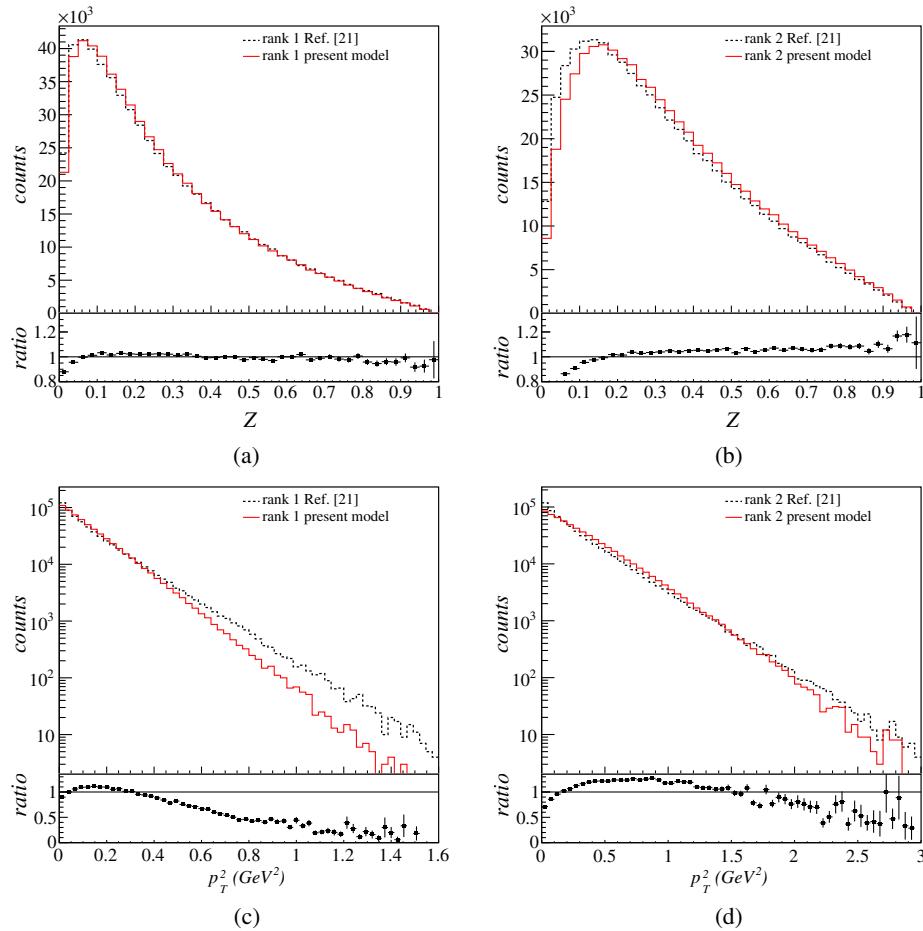


FIG. 2. Comparison between the model of Ref. [21] (dotted histogram) and the simplified  ${}^3P_0$  (continuous histogram) for (a)  $Z$  distribution for rank 1 hadrons, (b)  $Z$  distribution for rank 2 hadrons, (c)  $p_T^2$  distribution for rank 1 hadrons and (d)  $p_T^2$  distribution for rank 2 hadrons. Their ratios are shown in the bottom plots. Note the different horizontal scales in plots (c) and (d).

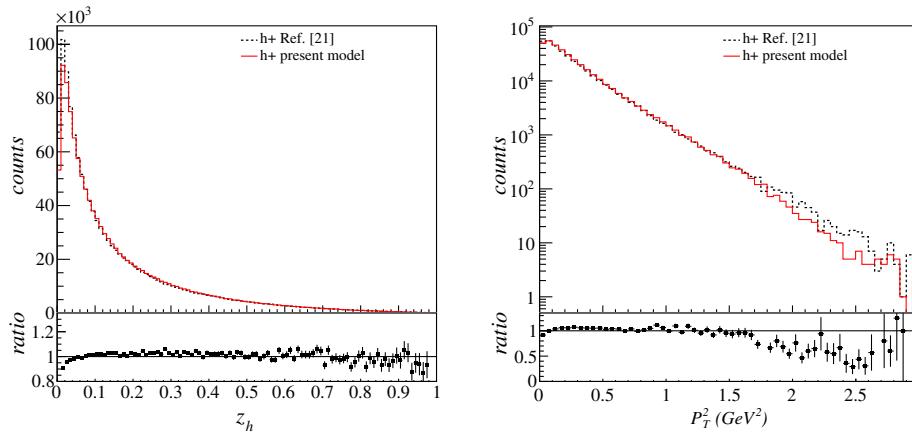


FIG. 3. Comparison between  $z_h$  (left plot) and  $p_T^2$  (right plot) distributions of positively charged hadrons as obtained with the model of Ref. [21] (dotted histogram) and with the simplified  $^3P_0$  (continuous histogram). Their ratio is also displayed in the respective bottom panels. We have applied the cuts  $z_h > 0.2$  and  $p_T > 0.1$  GeV.

### A. Kinematical distributions

The rank dependence of the kinematical distributions is a characteristic of a recursive model and is about the same as that in Ref. [21]. In particular, the  $Z$  and  $\mathbf{p}_T^2$  distributions do not depend on the rank for  $r \geq 2$ .

In Fig. 2 we compare the  $Z$  and  $\mathbf{p}_T^2$  distributions for the  $r = 1$  (left plots) and  $r = 2$  (right plots) hadrons as obtained with the present model (continuous histograms) and with the model of Ref. [21] (dotted histograms). Their ratio is shown in the bottom plot of each panel. The two models produce almost the same  $Z$  distribution for rank 1 [plot (a)] as expected because the initial quark has no  $\mathbf{k}_T$ . For rank 2 [plot (b)] the  $Z$  distribution in this model is slightly shifted towards greater values of  $Z$ . This is correlated to a somewhat larger  $\langle \mathbf{p}_T^2 \rangle$ , as can be seen from plot (d).

From plot (c) it is also clear that the  $\mathbf{p}_T^2$  distribution for rank 1 of Ref. [21] has two slopes on the contrary to this model. In fact the  $\mathbf{p}_T^2$  distribution of Ref. [21] is a sum of contributions of different slopes, one for each  $Z$ , due to the factor  $\exp(-b_L \epsilon_h^2/Z)$ . In the present model also there is a different slope for each  $Z$ , but the factor  $1/N_a(\epsilon_h^2)$  “rectifies” the slope of the  $Z$ -integrated  $\mathbf{p}_T^2$  spectrum.

The differences are even smaller when looking at measurable quantities. The distributions of the fraction  $z_h$  of the fragmenting quark energy taken by a positive hadron in the two models are shown in the left plot of Fig. 3. The region of very small  $z_h$  is less populated in the simplified  $^3P_0$  model. The  $\mathbf{p}_T^2$  distribution for positive hadrons is almost the same in both models as shown in the right plot of Fig. 3.

Figure 4 compares the  $z_h$  dependence of the transverse momentum width  $\langle p_T^2 \rangle$  of charged hadrons in the two models. The present model gives a larger difference between the  $\langle p_T^2 \rangle$  for positive hadrons and the  $\langle p_T^2 \rangle$  for negative hadrons than the model of Ref. [21], which

already was not in agreement with experiments. Indeed, due to the pure spin correlations which gives  $\langle \mathbf{k}_T \cdot \mathbf{k}'_T \rangle < 0$ , now at ranks larger than 1 we have  $\langle \mathbf{p}_T^2 \rangle > 2\langle \mathbf{k}_T^2 \rangle$ . In Ref. [21], on the other hand, the spin-independent correlation, if taken alone, would give the opposite correlation  $\langle \mathbf{k}_T \cdot \mathbf{k}'_T \rangle > 0$ , therefore  $\langle \mathbf{p}_T^2 \rangle < 2\langle \mathbf{k}_T^2 \rangle$ .

### B. Single hadron transverse spin asymmetries

Hadrons in the fragmentation of transversely polarized quarks exhibit a left-right asymmetry with respect to the plane defined by the transverse spin and the momentum of the quark, according to the azimuthal distribution

$$\frac{dN_h}{dz_h d\mathbf{p}_T} \propto 1 + a^{q_A \uparrow \rightarrow h+X} S_{AT} \sin \phi_C \quad (33)$$

where  $a^{q_A \uparrow \rightarrow h+X}$  is the Collins analyzing power for hadron  $h$ ,  $S_{AT}$  is the transverse polarization of the fragmenting

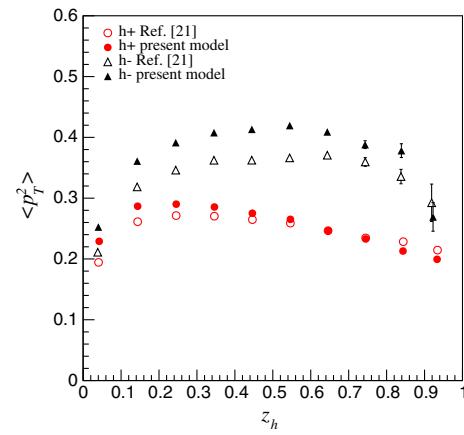


FIG. 4. Comparison between the  $z_h$  dependence of  $\langle p_T^2 \rangle$  in the present model (full points) and in the model of Ref. [21] (open points).

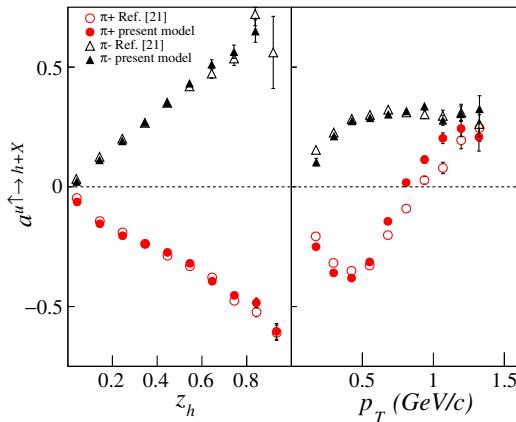


FIG. 5. Collins analyzing power for charged pions as function of  $z_h$  (left panel) and  $p_T$  (right panel) as obtained with the present model (full points) and with the model of Ref. [21] (open points). The cuts  $z_h > 0.2$  and  $p_T > 0.1$  GeV have been applied.

quark  $q_A$  and  $\phi_C = \phi_h - \phi_{S_{AT}}$  is the Collins azimuthal angle. Being formulated at the amplitude level, this model produces a pure  $\sin \phi_C$  modulation.

Figure 5 shows the Collins analyzing power for charged pions produced in jets of transversely polarized  $u$  quarks estimated as  $2\langle \sin \phi_C \rangle$  (full points). They are compared with the results of Ref. [21] (open points). The analyzing power is shown as function of  $z_h$  in the left panel and as function of  $p_T$  in the right panel of Fig. 5. The cuts  $z_h > 0.2$  and  $p_T > 0.1$  GeV have been applied. Both models produce the same features for the analyzing power. Some slight differences can be seen for the analyzing power as function of  $p_T$  for  $\pi^+$  (right panel) which are due to the different  $k_T^2$  dependencies of the respective splitting functions.

The absolute value of the Collins analyzing power as function of the rank is shown in Fig. 6 for the present model (full points) and for the model of Ref. [21] (open points).

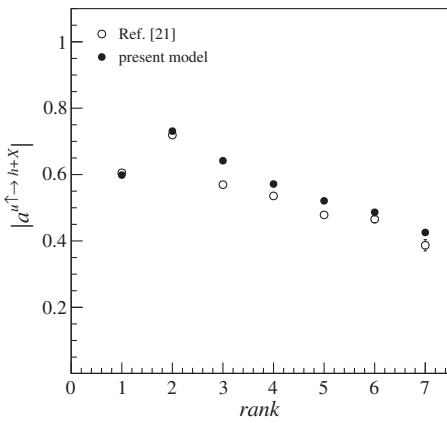


FIG. 6. Comparison of the absolute value of the Collins analyzing power as function of rank as obtained with the present model (full points) and with the model of Ref. [21] (open points). The cuts  $z_h > 0.2$  and  $p_T > 0.1$  GeV have been applied.

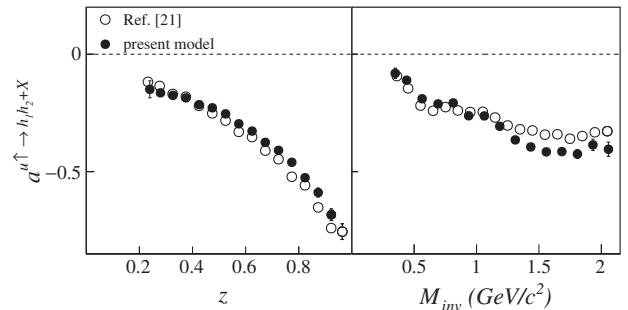


FIG. 7. Comparison between the dihadron transverse spin asymmetry as function of  $z = z_{h_1} + z_{h_2}$  (left panel) and of  $M_{\text{inv}}$  (right panel), as obtained for unidentified hadrons with the present model (full points) and with the model of Ref. [21] (open points).

In the present model the analyzing power decays slower because of the triviality of the  $\hat{u}_q$  matrix.

### C. Dihadron transverse spin asymmetry

The azimuthal distribution of hadron pairs of opposite charge in the same jet produced in the fragmentation of a transversely polarized quark is described by the equation

$$\frac{dN_{h_1 h_2}}{dz dM_{\text{inv}} d\phi_R} \propto 1 + a^{q_A \uparrow \rightarrow h_1 h_2 + X} S_{AT} \sin(\phi_R - \phi_{S_{AT}}) \quad (34)$$

where  $z = z_{h_1} + z_{h_2}$  is the sum of the fractional energies of the positive ( $h_1$ ) and negative ( $h_2$ ) hadrons and  $M_{\text{inv}}$  is the invariant mass of the pair. The angle  $\phi_R$  is the azimuthal angle of the transverse vector  $\mathbf{R}_T = (z_{h_2} \mathbf{p}_{1T} - z_{h_1} \mathbf{p}_{2T})/z$ .  $\mathbf{p}_{1T} (\mathbf{p}_{2T})$  is the transverse momentum of the positively (negatively) charged hadron of the pair.

Figure 7 compares the dihadron  $h^+ h^-$  analyzing power as function of  $z$  (left panel) and  $M_{\text{inv}}$  (right panel) as obtained with the present model (full points) and with the model of Ref. [21] (open points). The cuts  $z_{h_i} > 0.1$ ,  $R_T > 0.07$  GeV and  $|\mathbf{p}_i| > 3$  GeV ( $i = 1, 2$ ) have been applied. The overall trends are the same in both models and only some slight differences can be seen. In particular as function of the invariant mass the present model saturates to somewhat larger absolute values of the analyzing power at large  $M_{\text{inv}}$ . All in all, the main features of the results obtained from the two implementations are the same.

## VII. POSITIVITY BOUNDS

The present simplified model allows for explicit calculations of the spin transfer coefficients between the quark  $q$  and  $q'$  and the positivity bounds can be checked easily.

In general an all-polarized splitting function can be defined as measured by a *gedanken* quark detector accepting only quarks of spins  $+\hat{S}_{q'}/2$ . Then Eq. (5) is generalized to

$$F_{q',h,q} = \text{tr}[T_{q',h,q}\rho(q)T_{q',h,q}^\dagger\check{\rho}(q')]. \quad (35)$$

Here the vector  $\check{\mathbf{S}}_{q'}$  encoded in  $\check{\rho}(q')$  is a property of the detector and is imposed. At variance with the vector  $\mathbf{S}_{q'}$  in Eqs. (28)–(29), it does depend either on  $\mathbf{S}_q$  or on the involved momenta. The all-polarized splitting function of Eq. (35) can be written as

$$\begin{aligned} F_{q',h,q}(Z, \mathbf{p}_T, \check{\mathbf{S}}_{q'}; \mathbf{k}_T, \mathbf{S}_q) \\ = \frac{|C_{q',h,q}|^2}{\sum_H |C_{q',H,q}|^2} \left(\frac{1-Z}{\varepsilon_h^2}\right)^a \frac{\exp(-b_L \varepsilon_h^2/Z)}{N_a(\varepsilon_h^2)} \\ \times \frac{|\mu|^2 + \mathbf{k}_T'^2}{|\mu|^2 + \langle \mathbf{k}_T'^2 \rangle_{f_T}} f_T^2(\mathbf{k}_T'^2) \times \frac{1}{2} C(\mathbf{S}_q, \check{\mathbf{S}}_{q'}). \end{aligned} \quad (36)$$

Projecting the quark spins on the axes  $M = \hat{\mathbf{k}}'_T$ ,  $N = \hat{\mathbf{z}} \times \hat{\mathbf{k}}'_T$  and  $L = \hat{\mathbf{z}}$ , the correlation function  $C(\mathbf{S}_q, \check{\mathbf{S}}_{q'})$  is decomposed as

$$\begin{aligned} C(\mathbf{S}_q, \check{\mathbf{S}}_{q'}) = 1 + C_{N0} S_{qN} + C_{0N} \check{S}_{q'N} \\ + C_{NN} S_{qN} \check{S}_{q'N} + C_{MM} S_{qM} \check{S}_{q'M} \\ + C_{ML} S_{qM} \check{S}_{q'L} + C_{LM} S_{qL} \check{S}_{q'M} \\ + C_{LL} S_{qL} \check{S}_{q'L}, \end{aligned} \quad (37)$$

with  $|C_{ij}| < 1$ , where  $i, j$  take the values M, N, L or 0 in the unpolarized case. Only the coefficients appearing in Eq. (37) are allowed by parity conservation and are given by

$$C_{N0} = -\frac{2\text{Im}\mu k'_T}{|\mu|^2 + \mathbf{k}_T'^2} = -C_{0N} \quad (38)$$

$$C_{NN} = -1 \quad (39)$$

$$C_{MM} = -\frac{|\mu|^2 + \mathbf{k}_T'^2}{|\mu|^2 + \mathbf{k}_T'^2} = -C_{LL} \quad (40)$$

$$C_{ML} = -\frac{2\text{Re}\mu k'_T}{|\mu|^2 + \mathbf{k}_T'^2} = C_{LM}. \quad (41)$$

These coefficients describe the dynamics of the transfer of polarization from  $q$  to  $q'$  in the elementary splitting and are

connected to the actual polarization  $\mathbf{S}_{q'}$  of  $q'$ , given in Eqs. (28)–(29), through the relation

$$\mathbf{S}_{q'} = \frac{\nabla_{\check{\mathbf{S}}_{q'}} C(\mathbf{S}_q, \check{\mathbf{S}}_{q'})}{C(\mathbf{S}_q, \mathbf{0})}. \quad (42)$$

In addition they must obey the positivity conditions [26]

$$\begin{aligned} (1 \pm C_{NN})^2 \geq (C_{0N} \pm C_{N0})^2 + (C_{LL} \pm C_{MM})^2 \\ + (C_{LM} \mp C_{ML})^2. \end{aligned} \quad (43)$$

In the present model they saturate these inequalities as expected for a quantum mechanical model of the fragmentation process formulated at the amplitude level. The saturation comes from the fact that the spin-0 mesons do not carry spin information. Our Monte Carlo simulation program takes into account this property. The same is true also for the model of Ref. [21], the correlation coefficients of which have more complicated expressions due to the nonvanishing  $\hat{u}_{1q}(\mathbf{k}_T^2)$  function.

## VIII. CONCLUSIONS

We have presented a new version of the string +  $^3P_0$  model of Ref. [21] where the dynamical correlations between the quark transverse momenta are suppressed. In fact, it is the same model with the same parameters but with a different choice for the input function  $\check{g}$ . It gives nearly the same results. This work shows that the model is stable against changes of the function  $\check{g}$  and does not lose predictive power. The advantage of the present choice of  $\check{g}$  is that it allows us to take more simply into account the exact left-right symmetry and simplifies analytical calculations. As for the dynamical  $\mathbf{k}_T - \mathbf{k}'_T$  correlations, there is no compelling reason at present to reintroduce them. The model as presented here is also more suitable to be interfaced with external event generators and in particular with PYTHIA [23] and will also be extended to include the production of vector mesons.

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