Radial solutions of the Dirichlet problem for a class of quasilinear elliptic equations arising in optometry *

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Dedicated to Enzo Mitidieri on the occasion of his 65th birthday

Abstract

This paper deals with the quasilinear elliptic problem

$$-\text{div}\left(\nabla u/\sqrt{1+|\nabla u|^2}\right) + a(x)u = b(x)/\sqrt{1+|\nabla u|^2} \text{ in } B, \;\; u = 0 \text{ on } \partial B,$$

where B is an open ball in $\mathbb{R}^N$, with $N \geq 2$, and $a, b \in C^1(\overline{B})$ are given radially symmetric functions, with $a(x) \geq 0$ in B. This class of anisotropic prescribed mean curvature equations appears in the description of the geometry of the human cornea, as well as in the modeling theory of capillarity phenomena for compressible fluids. Unlike all previous works published on these subjects, existence and uniqueness of solutions of the above problem are here analyzed in the case where the coefficients a, b are not necessarily constant and no sign condition is assumed on b.

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1 Introduction

In this paper we aim to analyze existence and uniqueness of classical solutions of the anisotropic prescribed mean curvature problem

$$\begin{cases} -\mathrm{div}\Big(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) + a(x)u = \dfrac{b(x)}{\sqrt{1+|\nabla u|^2}}, & \text{in } B, \\ u = 0, & \text{on } \partial B, \end{cases} \tag{1.1}$$

where $B = B(0,R)$ is the open ball of center 0 and radius R in $\mathbb{R}^N$, with $N \geq 2$, and $a : \overline{B} \to [0,+\infty[$ and $b : \overline{B} \to \mathbb{R}$ are continuously differentiable radially symmetric functions, i.e., $a(x) = a(|x|, 0, \dots, 0)$ and $b(x) = b(|x|, 0, \dots, 0)$ for all $x \in \overline{B}$. For a radially symmetric function $w : \overline{B} \to \mathbb{R}$, with a slight abuse of notation, we will sometimes write, for $x \in \overline{B}$, $w(|x|)$ in place of $w(x)$.

The equation in (1.1), where a, b are positive constants, has been introduced for modeling capillarity phenomena for compressible fluids, or for describing the geometry of the human cornea, when they are respectively supplemented with non-homogeneous conormal boundary conditions [13, 14, 4, 15, 3], or with homogeneous Dirichlet boundary conditions [20, 21, 22, 9, 24, 25, 23, 26, 10, 11]. We refer to these papers for the derivation of the models, further discussions on the subject, and an additional bibliography.

Concerning the Dirichlet problem (1.1), we recall that, according to [20], the surface of the human cornea is modeled as a membrane, whose shape is described by the graph of the function u and is determined by balancing all forces acting over, that is, surface tension, elasticity, and intra-ocular pressure. The relevant physical parameters are incorporated into the coefficients a and b, which respectively measure the relative importance of the elasticity and of the intra-ocular pressure versus the surface tension.

In [22] it has been pointed out the interest of studying also the case where the coefficients are non-constant functions, in order to provide a better fitting of the model with the experimental data. This actually appears more relevant for the coefficient b, rather than for a; however, in this work we allow both a and b to be non-constant functions. We further stress that in our results we let a to vanish and we impose no sign condition on b.

We notice that confining the consideration of problem (1.1) to spherical domains is justified by the fact that the surface of the cornea may be approximately, although not exactly, modeled as a surface of revolution.

Our existence and uniqueness result for problem (1.1) then reads as follows.

Theorem 1.1. *Let $B = B(0,R)$ be the open ball of center 0 and radius R in $\mathbb{R}^N$, with $N \geq 2$. Suppose that $a, b \in C^1(\overline{B})$ are radially symmetric functions, with $a(x) \geq 0$ in B. Then, there exists a unique solution $u \in C^2(\overline{B})$ of (1.1), which is radially symmetric. In addition, if $b(x) \geq 0$ in B and $b \not\equiv 0$, then*

$$u(x) > 0 \quad \text{in } B \qquad \text{and} \qquad \nabla u(x) \cdot x < 0 \quad \text{on } \partial B.$$

The proof of Theorem 1.1 consists of two parts. First, in Section 2, we prove a uniqueness result for the more general problem

$$\begin{cases} -\operatorname{div}\Big(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) + a(x)u = \dfrac{b(x)}{\sqrt{1+|\nabla u|^2}}, & \text{in } \Omega, \\ \qquad\qquad\qquad\qquad u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where Ω is any bounded domain Ω in $\mathbb{R}^N$, with a Lipschitz boundary $\partial\Omega$, and $a : \overline{\Omega} \to [0, +\infty[$ and $b : \overline{\Omega} \to \mathbb{R}$ are arbitrary continuous functions. This result for (1.2) relies on a rather general comparison principle for the solutions of (1.2), which has an independent interest.

Next, in Section 3, by further requiring that the coefficients a, b are continuously differentiable and radially symmetric, we establish the existence of a classical radial solution of (1.1). This is achieved by solving the one-dimensional singular problem

$$\begin{cases} -\Big(\dfrac{r^{N-1}\, v'}{\sqrt{1+v'^2}}\Big)' + r^{N-1}a(r)v = \dfrac{r^{N-1}b(r)}{\sqrt{1+v'^2}}, & \text{in }]0,R[, \\ \qquad v'(0) = 0, \;\; v(R) = 0, \end{cases} \tag{1.3}$$

where $r = |x|$. The radial solution we find is therefore the unique solution of (1.1).

In general, the study of mean curvature problems requires much care because it is fraught with a number of technical difficulties, the main one being the possible occurrence of gradient blow-up phenomena, even in simple one-dimensional situations (see, e.g., [7, 19] and the references therein). However, for problem (1.1), we are able to show that an a priori bound for the gradient of its possible (radially symmetric) solutions can be obtained by a direct, but delicate, argument which exploits the special structure of the equation, the regularity of the coefficients and the geometry of the domain. These estimates eventually enable us to use a simple continuation method based on the implicit function theorem to prove the solvability of (1.3) and hence of (1.1).

We finally mention that extending Theorem 1.1 to the general setting of problem (1.2) remains an open question. Indeed, in the light of the conclusions achieved in [11], we know that, for non-convex domains, singular solutions possibly not attaining the Dirichlet boundary conditions may occur even in the case of constant coefficients. On the other hand, the method, developed in [10] to deal with (1.2) in case the coefficients a, b are positive constants, fails when b is not constant. In this respect a brief discussion is produced in Section 4.

2 Uniqueness of solutions

The proof of the uniqueness of the solution of problem (1.2) is based on the following comparison lemma, whose proof is partially inspired from [8, 2].

Lemma 2.1. *Let Ω be a bounded domain in $\mathbb{R}^N$, with $N \geq 2$, having a Lipschitz boundary $\partial\Omega$, and let $a, b \in L^\infty(\Omega)$ be given functions, with $\operatorname{ess\,inf} a \geq 0$. Assume that $u_1, u_2 \in W^{1,\infty}(\Omega)$ are such that*

$$\int_\Omega \frac{\nabla u_1 \cdot \nabla \phi}{\sqrt{1+|\nabla u_1|^2}}\,dx + \int_\Omega a u_1 \phi\,dx - \int_\Omega \frac{b\phi}{\sqrt{1+|\nabla u_1|^2}}\,dx$$
$$\leq \int_\Omega \frac{\nabla u_2 \cdot \nabla \phi}{\sqrt{1+|\nabla u_2|^2}}\,dx + \int_\Omega a u_2 \phi\,dx - \int_\Omega \frac{b\phi}{\sqrt{1+|\nabla u_2|^2}}\,dx \quad (2.1)$$

for all $\phi \in W_0^{1,\infty}(\Omega)$, with $\phi(x) \geq 0$ in Ω, and

$$u_1(x) \leq u_2(x) \quad \text{on } \partial\Omega. \quad (2.2)$$

Then, u_1, u_2 satisfy

$$u_1(x) \leq u_2(x) \quad \text{in } \Omega. \quad (2.3)$$

Proof. Set $u = u_1 - u_2$. We want to prove that $u(x) \leq 0$ in Ω. For any given $c \in \mathbb{R}$, define

$$\Omega_c = \{x \in \Omega \mid u(x) = c\}$$

and

$$I = \{c \in \mathbb{R} \mid \operatorname{meas}(\Omega_c) > 0\}.$$

Setting

$$I_1 = \{c \in \mathbb{R} \mid \operatorname{meas}(\Omega_c) > 1\}$$

and, for $n \geq 2$,

$$I_n = \{c \in \mathbb{R} \mid \frac{1}{n} < \operatorname{meas}(\Omega_c) \leq \frac{1}{n-1}\},$$

we can write

$$I = \bigcup_{n=1}^{+\infty} I_n.$$

As meas(Ω) is finite, each I_n is finite and, thus, I is at most countable. The Stampacchia theorem implies that, for any $c \in \mathbb{R}$, $\nabla u(x) = 0$ a.e. in Ω_c and, hence,

$$\nabla u(x) = 0 \quad \text{ a.e. in } \bigcup_{c \in I} \Omega_c.$$

We also set

$$\mathcal{O} = \Omega \setminus \bigcup_{c \in I} \Omega_c.$$

For every $k \geq 0$, define

$$\phi_k = (u-k)^+ \in W_0^{1,\infty}(\Omega)$$

and

$$E_k = \{x \in \mathcal{O} \mid u(x) \geq k\}.$$

Then, pick $\phi = \phi_k$ as a test function in (2.1).

Let us define a function $f : [0,1] \to \mathbb{R}$, by

$$f(s) = \int_\Omega \frac{\nabla(u_2 + su) \cdot \nabla \phi_k}{\sqrt{1 + |\nabla(u_2 + su)|^2}}\, dx,$$

where, to simplify the notation, we omit the indication of the dependence of f on k. Observe that, by the definition of ϕ_k, we have $\nabla u \cdot \nabla \phi_k = |\nabla \phi_k|^2$. Then, the mean value theorem yields the existence of $\theta \in]0,1[$ such that

$$\begin{aligned}
&\int_\Omega \left(\frac{\nabla u_1}{\sqrt{1+|\nabla u_1|^2}} - \frac{\nabla u_2}{\sqrt{1+|\nabla u_2|^2}} \right) \cdot \nabla \phi_k \, dx = f(1) - f(0) = f'(\theta) \\
&\quad = \int_\Omega \frac{\nabla u \cdot \nabla \phi_k}{\sqrt{1+|\nabla(u_2+\theta u)|^2}}\, dx - \int_\Omega \frac{(\nabla(u_2+\theta u)\cdot \nabla u)\,(\nabla(u_2+\theta u)\cdot\nabla\phi_k)}{(1+|\nabla(u_2+\theta u)|^2)^{\frac{3}{2}}}\, dx \\
&\quad = \int_\Omega \frac{|\nabla \phi_k|^2}{\sqrt{1+|\nabla(u_2+\theta u)|^2}}\, dx - \int_\Omega \frac{(\nabla(u_2+\theta u)\cdot\nabla\phi_k)^2}{(1+|\nabla(u_2+\theta u)|^2)^{\frac{3}{2}}}\, dx \\
&\quad \geq \int_\Omega \frac{|\nabla \phi_k|^2}{\sqrt{1+|\nabla(u_2+\theta u)|^2}}\, dx - \int_\Omega \frac{|\nabla(u_2+\theta u)|^2|\nabla\phi_k|^2}{(1+|\nabla(u_2+\theta u)|^2)^{\frac{3}{2}}}\, dx \\
&\quad \geq \int_\Omega \frac{|\nabla \phi_k|^2}{(1+|\nabla(u_2+\theta u)|^2)^{\frac{3}{2}}}\, dx \\
&\quad \geq \frac{1}{(1+(\|\nabla u_1\|_\infty + \|\nabla u_2\|_\infty)^2)^{\frac{3}{2}}} \int_\Omega |\nabla \phi_k|^2 \, dx. \qquad (2.4)
\end{aligned}$$

Incidentally, we notice that this estimate might alternatively be deduced from the strong convexity, on all compact convex subsets of $\mathbb{R}^N$, of the function $g : \mathbb{R}^N \to \mathbb{R}$ defined by $g(\xi) = \sqrt{1+|\xi|^2}$.

On the other hand, we have

$$\int_\Omega au\phi_k \, dx \geq \int_\Omega a|\phi_k|^2 \, dx \geq 0 \tag{2.5}$$

and

$$\begin{aligned}
&\int_\Omega b\Big(\frac{1}{\sqrt{1+|\nabla u_1|^2}} - \frac{1}{\sqrt{1+|\nabla u_2|^2}}\Big)\phi_k \, dx \\
&= \int_\Omega \frac{b\,\left(|\nabla u_2|^2 - |\nabla u_1|^2\right)\,\phi_k}{\sqrt{1+|\nabla u_1|^2}\sqrt{1+|\nabla u_2|^2}\left(\sqrt{1+|\nabla u_1|^2}+\sqrt{1+|\nabla u_2|^2}\right)} \, dx \\
&\leq \|b\|_\infty \int_\Omega \big||\nabla u_2| - |\nabla u_1|\big|\,\phi_k \, dx \\
&\leq \|b\|_\infty \int_\Omega \big|\nabla u\big|\,\phi_k \, dx = \|b\|_\infty \int_\Omega \big|\nabla \phi_k\big|\,\phi_k \, dx. \qquad (2.6)
\end{aligned}$$

Setting

$$M = \|b\|_\infty^{-1}(1 + (\|\nabla u_1\|_\infty + \|\nabla u_2\|_\infty)^2)^{-\frac{3}{2}},$$

from (2.1), (2.4), (2.5) and (2.6), we get

$$\begin{aligned}
M\int_\Omega |\nabla\phi_k|^2 \, dx &\leq \int_\Omega \big|\nabla\phi_k\big|\,\phi_k \, dx \\
&= \int_{\mathcal{O}} \big|\nabla\phi_k\big|\,\phi_k \, dx = \int_{\mathcal{O}} \chi_{E_k}\big|\nabla\phi_k\big|\,\phi_k \, dx. \qquad (2.7)
\end{aligned}$$

We first suppose $N \geq 3$. Using the Hölder inequality and the Sobolev imbedding theorem, we obtain from (2.7)

$$\begin{aligned}
M\|\nabla\phi_k\|^2_{L^2(\Omega)} &\leq \int_{\mathcal{O}} \chi_{E_k}\big|\nabla\phi_k\big|\,\phi_k \, dx \\
&\leq \|\chi_{E_k}\|_{L^N(\mathcal{O})}\|\nabla\phi_k\|_{L^2(\Omega)}\|\phi_k\|_{L^{\frac{2N}{N-2}}(\Omega)} \\
&\leq C\,\|\chi_{E_k}\|_{L^N(\mathcal{O})}\|\nabla\phi_k\|^2_{L^2(\Omega)},
\end{aligned}$$

where $C = C(N,\Omega)$ is the imbedding constant of $H_0^1(\Omega)$ into $L^{\frac{2N}{N-2}}(\Omega)$. If

$N=2$, we fix $p>2$ and, arguing as above, we find

$$\begin{aligned} M\|\nabla\phi_k\|^2_{L^2(\Omega)} &\le \int_{\mathcal{O}} \chi_{E_k}\big|\nabla\phi_k\big|\,\phi_k\,dx \\ &\le \|\chi_{E_k}\|_{L^p(\mathcal{O})}\|\nabla\phi_k\|_{L^2(\Omega)}\|\phi_k\|_{L^{\frac{2p}{p-2}}(\Omega)} \\ &\le C\,\|\chi_{E_k}\|_{L^p(\mathcal{O})}\|\nabla\phi_k\|^2_{L^2(\Omega)}, \end{aligned}$$

where $C=C(p,N,\Omega)$, with $N=2$, is the imbedding constant of $H^1_0(\Omega)$ into $L^{\frac{2p}{p-2}}(\Omega)$. Thus, we conclude that, for each $N\ge 2$, there is $p=p_N\ge N$ such that

$$M\|\nabla\phi_k\|_{L^2(\Omega)} \le C\,\|\chi_{E_k}\|_{L^p(\mathcal{O})}\|\nabla\phi_k\|_{L^2(\Omega)} \tag{2.8}$$

holds for all k.

In order to show that $u(x)\le 0$ in Ω, we assume by contradiction that $u^+\ne 0$. The definition of the set $\mathcal{O}$ implies that

$$\begin{aligned} 0=\operatorname{meas}(E_{\|u^+\|_\infty}) &= \operatorname{meas}(\{x\in\mathcal{O}\mid u(x)\ge\|u^+\|_\infty\}) \\ &= \operatorname{meas}(\{x\in\mathcal{O}\mid u(x)\ge k \text{ for all } k<\|u^+\|_\infty\}) \\ &= \operatorname{meas}\Bigg(\bigcap_{k<\|u^+\|_\infty} E_k\Bigg) \\ &= \lim_{k\to\|u^+\|_\infty^-}\operatorname{meas}(E_k). \end{aligned}$$

Therefore, we can pick $k_0\in]0,\|u^+\|_\infty[$ such that

$$\|\chi_{E_{k_0}}\|_{L^p(\mathcal{O})}=\operatorname{meas}(E_{k_0})^{\frac{1}{p}}\le\frac{1}{2}\frac{M}{C}.$$

Using (2.8), we have

$$\|\nabla\phi_{k_0}\|_{L^2(\Omega)}\le\tfrac{1}{2}\|\nabla\phi_{k_0}\|_{L^2(\Omega)}$$

and hence

$$\|\nabla(u-k_0)^+\|_{L^2(\Omega)}=0.$$

This implies that

$$u(x)\le k_0<\|u^+\|_\infty \quad \text{in } \Omega,$$

which is impossible. □

Remark 2.1 From the proof of Lemma 2.1 it follows that a more general version of the above stated comparison principle holds true. Namely, suppose that $h:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ satisfies the L^1-Carathéodory conditions and, for any given $r > 0$, there is $c_r \in L^p_{\rm loc}(\Omega)$, with $p \geq N$, if $N \geq 3$, and $p > 2$, if $N = 1$ or $N = 2$, such that, for a.e. $x \in \Omega$, for all $s_1, s_2 \in \mathbb{R}$, with $-r \leq s_1 \leq s_2 \leq r$, and all $\xi_1, \xi_2 \in \mathbb{R}^N$, with $|\xi_1|, |\xi_2| \leq r$,

$$h(x, s_1, \xi_1) - h(x, s_2, \xi_2) \leq c_r(x)\,|\xi_1 - \xi_2|.$$

If $u_1, u_2 \in W^{1,\infty}(\Omega)$ are such that

$$\int_\Omega \frac{\nabla u_1 \cdot \nabla \phi}{\sqrt{1 + |\nabla u_1|^2}}\,dx + \int_\Omega h(x, u_1, \nabla u_1)\,\phi\,dx \\ \leq \int_\Omega \frac{\nabla u_2 \cdot \nabla \phi}{\sqrt{1 + |\nabla u_2|^2}}\,dx + \int_\Omega h(x, u_2, \nabla u_2)\,\phi\,dx$$

for all $\phi \in W^{1,\infty}_0(\Omega)$, with $\phi(x) \geq 0$ in Ω, and (2.2) holds, then, u_1, u_2 satisfy (2.3).

To prove this conclusion, we follow that same patterns of the proof of Lemma 2.1. We pick $r = \max\{\|u_i\|_\infty, \|\nabla u_i\|_\infty \mid i = 1, 2\}$ and we first show that there is a constant $M > 0$ such that

$$M \int_\Omega |\nabla \phi_k|^2\,dx \leq \int_{\mathcal{O}} c_r \chi_{E_k} \big|\nabla \phi_k\big|\,\phi_k\,dx$$

and then that there is a constant $C > 0$ such that

$$M \|\nabla \phi_k\|_{L^2(\Omega)} \leq C\,\|c_r \chi_{E_k}\|_{L^p(\mathcal{O})} \|\nabla \phi_k\|_{L^2(\Omega)}.$$

As

$$\lim_{k \to \|u^+\|_\infty^-} \mathrm{meas}\,(E_k) = 0,$$

we get

$$\lim_{k \to \|u^+\|_\infty^-} \|c_r \chi_{E_k}\|_{L^p(\mathcal{O})} = 0.$$

Thus, we can conclude as in the proof of Lemma 2.1.

This statement extends various classical comparison principles previously obtained in the literature (see, e.g., [16, Section 10.1]).

The following general uniqueness result is a direct consequence of Lemma 2.1.

Theorem 2.2. *Let Ω be a bounded domain in $\mathbb{R}^N$, with $N \geq 2$, having a Lipschitz boundary $\partial\Omega$, and let $a, b \in L^\infty(\Omega)$ be given functions, with* ess inf $a \geq 0$*. Then, problem* (1.2) *has at most one weak solution $u \in W^{1,\infty}_0(\Omega)$.*

3 Existence of solutions

In this section we prove the existence of solutions of problem (1.3) and, hence, of problem (1.1). Throughout we assume $N \geq 2$. We begin with an elementary regularity result.

Lemma 3.1. *Let $a, b \in C^0([0,R]) \cap C^1(]0,R[)$ be given. Then, any solution $v \in C^1([0,R]) \cap C^2(]0,R])$ of (1.3) belongs to $C^2([0,R]) \cap C^3(]0,R[)$ and satisfies the equation in (1.3) for all $r \in [0,R]$.*

Proof. Pick $r \in]0,R]$ and integrate the equation in (1.3) between 0 and r. We get

$$\frac{r^{N-1}v'(r)}{\sqrt{1+v'(r)^2}} = \int_0^r s^{N-1}\Big(a(s)v(s) - \frac{b(s)}{\sqrt{1+v'(s)^2}}\Big)\, ds$$

and hence

$$\frac{v'(r)}{r} = \frac{\sqrt{1+v'(r)^2}}{r^N} \int_0^r s^{N-1}\Big(a(s)v(s) - \frac{b(s)}{\sqrt{1+v'(s)^2}}\Big)\, ds.$$

By applying L'Hospital's rule, we easily see that there exists

$$\lim_{r\to 0}\frac{1}{r^N} \int_0^r s^{N-1}\Big(a(s)v(s) - \frac{b(s)}{\sqrt{1+v'(s)^2}}\Big)\, ds = \frac{a(0)\,v(0) - b(0)}{N},$$

that is, $v''(0)$ exists and

$$v''(0) = \frac{a(0)\,v(0) - b(0)}{N}.$$

Since we can write the equation in (1.3) in the form

$$-\frac{v''}{(1+v'^2)^{\frac{3}{2}}} - \frac{N-1}{r}\frac{v'}{\sqrt{1+v'^2}} + a(r)v = \frac{b(r)}{\sqrt{1+v'^2}}, \tag{3.1}$$

or equivalently

$$-v'' - \frac{N-1}{r}v'(1+v'^2) + a(r)v(1+v'^2)^{\frac{3}{2}} = b(r)(1+v'^2), \tag{3.2}$$

we conclude that $v \in C^2([0,R]) \cap C^3(]0,R[)$. □

Remark 3.1 Setting, by convention,

$$\frac{v'(r)}{r} = v''(0), \quad \text{if } r = 0,$$

we can also say that v satisfies (3.1) for all $r \in [0,R]$.

Remark 3.2 If $v \in C^2([0,R])$ is a solution of (1.3), then the function $u : \overline{B} \to \mathbb{R}$, defined by $u(x) = v(|x|)$, satisfies, for $i, j = 1, \dots, N$,

$$\partial_i u(x) = v'(|x|)\frac{x_i}{|x|} \quad \text{in } B \setminus \{0\}, \qquad \partial_i u(0) = 0,$$

$$\partial_{ij} u(x) = \Big(v''(|x|) - \frac{v'(|x|)}{|x|}\Big)\frac{x_i x_j}{|x|^2} + \frac{v'(|x|)}{|x|}\delta_{ij} \quad \text{in } B\setminus\{0\}, \quad \partial_{ij}u(0) = \delta_{ij} v''(0),$$

and thus $u \in C^2(\overline{B})$. Further, as

$$\operatorname{div}\Big(\frac{\nabla u(x)}{\sqrt{1+|\nabla u(x)|^2}}\Big) = \frac{v''(|x|)}{(1+v'(|x|)^2)^{\frac{3}{2}}} + \frac{N-1}{|x|}\frac{v'(|x|)}{\sqrt{1+v'(|x|)^2}} \quad \text{in } B \setminus \{0\},$$

and

$$\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big)(0) = N v''(0),$$

we conclude that u is a solution of (1.1).

The proof of the solvability of problem (1.3) is based on a simple continuation method combining the implicit function theorem with the obtention of suitable a priori bounds on the solutions. To this end we imbed problem (1.3) into the one-parameter family

$$\begin{cases} -\Big(\dfrac{r^{N-1}\,v'}{\sqrt{1+v'^2}}\Big)' + r^{N-1}a(r)v = \lambda\dfrac{r^{N-1}b(r)}{\sqrt{1+v'^2}}, & \text{in }]0,R[, \\ v'(0) = 0, \ v(R) = 0, \end{cases} \tag{3.3}$$

where $\lambda \in [0,1]$.

The following result provides the desired a priori bound, uniform in λ, on the possible solutions of (3.3).

Lemma 3.2. *Assume $a, b \in C^1([0,R])$, with $a(r) \geq 0$ in $[0,R]$. Then, there exists a constant $K > 0$ such that any solution $v \in C^2([0,R])$ of (3.3), for some $\lambda \in [0,1]$, satisfies*

$$\|v\|_{C^2} = \max\{\|v\|_\infty, \|v'\|_\infty, \|v''\|_\infty\} \leq K. \tag{3.4}$$

Proof. We prove the following slightly more general conclusion. Given $a \in C^1([0,R])$, with

$$a(r) \geq 0 \qquad \text{in } [0,R], \tag{3.5}$$

and $B_1, B_2 \in [0,+\infty[$, there exists $K \in]0,+\infty[$ such that, for all $b \in C^0([0,R]) \cap C^1(]0,R[)$ satisfying

$$|b(r)| \le B_1 \quad \text{in } [0,R], \tag{3.6}$$

and

$$b'(r) \le B_2 \quad \text{in }]0,R[, \tag{3.7}$$

any solution $v \in C^1([0,R]) \cap C^2(]0,R])$ of (3.3) belongs to $C^2([0,R]) \cap C^3(]0,R])$ and satisfies (3.4). The former assertion follows from Lemma 3.1, while the latter one is a consequence of the next three steps.

Step 1. For all $r \in [0,R]$, we have

$$\frac{B_1}{2(N-1)}(r^2 - R^2) \le v(r) \le \frac{B_1}{2(N-1)}(R^2 - r^2) \tag{3.8}$$

and hence there exists $K_1 \in]0,+\infty[$ such that $|v(r)| \le K_1$ in $[0,R]$. Define a function $\alpha : [0,R] \to \mathbb{R}$ by setting $\alpha(r) = \frac{B_1}{2(N-1)}(r^2 - R^2)$. Let us prove the former inequality in (3.8). For all $r \in [0,R]$, we have

$$\begin{aligned} -r^{N-1}\alpha''(r) - (N-1)r^{N-2}\alpha'(r)(1+\alpha'(r)^2) + r^{N-1}a(r)\alpha(r)(1+\alpha'(r)^2)^{\frac{3}{2}} \\ \le -r^{N-1}B_1\frac{N}{N-1} \\ \le r^{N-1}\frac{b(r)}{\sqrt{1+\alpha'(r)^2}}. \end{aligned}$$

Hence the functions $u_1(x) = \alpha(|x|)$ and $u_2(x) = v(|x|)$ satisfy the assumptions of Lemma 2.1 and therefore we have $v(r) \ge \alpha(r)$ in $[0,R]$. Setting $\beta = -\alpha$, a similar computation shows that also the latter inequality in (3.8) holds true.

Step 2. There exists $K_2 \in]0,+\infty[$ such that $|v'(r)| \le K_2$ in $[0,R]$. Assume by contradiction the existence of sequences $(b_n)_n$, satisfying, for all n, $b_n \in C^0([0,R]) \cap C^1(]0,R[)$, (3.6) and (3.7), and $(v_n)_n$, satisfying, for all n,

$$\begin{cases} -\Big(\dfrac{r^{N-1}\,v_n'}{\sqrt{1+{v_n'}^2}}\Big)' + r^{N-1}a(r)v_n = \dfrac{r^{N-1}b_n(r)}{\sqrt{1+{v_n'}^2}}, & \text{in }]0,R[, \\ v_n'(0) = 0,\ v_n(R) = 0, \end{cases}$$

such that

$$\lim_{n\to+\infty}(\max v_n') = +\infty. \tag{3.9}$$

For each n, let $r_n \in [0,R]$ be such that $v_n'(r_n) = \max v_n'$. Since $v_n'(0) = 0$, and by (3.8), $|v_n'(R)| \le \frac{B_1 R}{N-1}$, we have that $r_n \in]0,R[$, for all large n and hence

$$v_n''(r_n) = 0. \tag{3.10}$$

Passing to a subsequence, still labeled by $(r_n)_n$, we can also suppose the existence of $\overline{r} \in [0, R]$ such that
$$\lim_{n\to+\infty} r_n = \overline{r}.$$

As $v_n \in C^3(]0, R[)$, from (3.2), we can compute, for all $r \in]0, R[$,

$$\begin{aligned} v_n'''(r) = {} & a'(r)v_n(r)(1 + v_n'(r)^2)^{\frac{3}{2}} + a(r)v_n'(r)(1 + v_n'(r)^2)^{\frac{3}{2}} \\ & + 3a(r)v_n(r)v_n'(r)v_n''(r)\sqrt{1 + v_n'(r)^2} \\ & - b_n'(r)(1 + v_n'(r)^2) - 2b_n(r)v_n'(r)v_n''(r) \\ & + \frac{N-1}{r^2}v_n'(r)(1 + v_n'(r)^2) - \frac{N-1}{r}v_n''(r)(1 + 3v_n'(r)^2) \end{aligned}$$

and, hence, letting $r = r_n$,

$$\begin{aligned} \frac{v_n'''(r_n)}{(1 + v_n'(r_n)^2)^{\frac{3}{2}}} = {} & a'(r_n)v_n(r_n) + a(r_n)v_n'(r_n) \\ & - \frac{b_n'(r_n)}{\sqrt{1 + v_n'(r_n)^2}} + \frac{N-1}{r_n^2}\frac{v_n'(r_n)}{\sqrt{1 + v_n'(r_n)^2}}. \end{aligned} \tag{3.11}$$

Suppose that $\overline{r} = 0$. We first observe that

$$\lim_{n\to+\infty} \frac{N-1}{r_n^2}\frac{v_n'(r_n)}{\sqrt{1 + v_n'(r_n)^2}} = +\infty.$$

From (3.11), by using the continuity of a and a' at 0, (3.8), (3.7) and (3.9), we find that

$$\lim_{n\to+\infty} \frac{v_n'''(r_n)}{(1 + v_n'(r_n)^2)^{\frac{3}{2}}} = +\infty. \tag{3.12}$$

Thus, we conclude that, for all large n,

$$v_n'''(r_n) > 0, \tag{3.13}$$

which is impossible, as r_n is an interior maximum point of v_n'.

Suppose that $\overline{r} \in]0, R[$. Two cases may occur: either $a(\overline{r}) > 0$, or else $a(\overline{r}) = 0$ and then, due to (3.5), $a'(\overline{r}) = 0$. In the former case, by using the continuity of a at $\overline{r}$ and (3.9), we get

$$\lim_{n\to+\infty} a(r_n)v_n'(r_n) = +\infty.$$

Thus, by the continuity of a' at $\bar{r}$, (3.8) and (3.7), we find again that (3.12) and (3.13) hold and then a contradiction follows. In the latter case, by using the continuity of a' at $\bar{r}$, (3.8), (3.7) and (3.9), we infer

$$\liminf_{n\to+\infty} \frac{v_n'''(r_n)}{(1+v_n'(r_n)^2)^{\frac{3}{2}}} \geq \frac{N-1}{\bar{r}^2}.$$

Hence, (3.13) holds, for all large n, yielding a contradiction as before.

Assume that $\bar{r} = R$. Again we distinguish two cases: either $a(R) > 0$, or $a(R) = 0$. In the former case, by using the continuity of a and a' at R, (3.8), (3.7) and (3.9), we find as above that (3.12) and (3.13) hold and then the desired contradiction follows. In the latter case, from (3.1), by using (3.10), (3.8) and (3.6), we get the contradiction

$$\begin{aligned}\frac{N-1}{R} &= \lim_{n\to+\infty} \frac{N-1}{r_n}\frac{v_n'(r_n)}{\sqrt{1+v_n'(r_n)^2}}\\ &= \lim_{n\to+\infty}\left(a(r_n)v_n(r_n) - \frac{b_n(r_n)}{\sqrt{1+v_n'(r_n)^2}}\right) = 0.\end{aligned}$$

Therefore, we have shown that there exists $K' \in]0,+\infty[$ such that $v'(r) \leq K'$ in $[0,R]$. In a totally symmetric way we can prove the existence of $K'' \in]0,+\infty[$ such that $v'(r) \geq -K''$ in $[0,R]$. Thus, by setting $K_2 = \max\{K', K''\}$, the conclusion follows.

Step 3. There exists $K_3 \in]0,+\infty[$ such that $|v''(x)| \leq K_3$ in $[0,R]$. As in the proof of Lemma 3.1, for all $r \in]0,R]$, we have

$$\frac{v'(r)}{r} = \frac{\sqrt{1+v'(r)^2}}{r^N}\int_0^r s^{N-1}\left(a(s)v(s) - \frac{b(s)}{\sqrt{1+v'(s)^2}}\right) ds,$$

and, therefore, from Step 1 and Step 2,

$$\begin{aligned}\frac{|v'(r)|}{r} &\leq \frac{\sqrt{1+K_2^2}}{r^N}\int_0^r s^{N-1}\left(\|a\|_\infty K_1 + \|b\|_\infty\right) ds\\ &= \frac{1}{N}\sqrt{1+K_2^2}\left(\|a\|_\infty K_1 + \|b\|_\infty\right).\end{aligned} \tag{3.14}$$

The desired bound on v'' is finally deduced from (3.2), by using (3.6), (3.14) and the conclusions achieved in the previous steps. □

An application of the implicit function theorem yields the existence of solutions of (3.3) close to a known solution, in particular, the existence of solutions for all small values of $\lambda \in [0,1]$.

Lemma 3.3. *Let $a : [0, R] \to [0, +\infty[$ and $b : [0, R] \to \mathbb{R}$ be continuous functions. Assume that problem* (3.3) *has a solution v_0 for some $\lambda_0 \in [0, 1[$. Then, there exists $\delta_0 > 0$ such that, for all $\lambda \in [\lambda_0, \lambda_0 + \delta_0]$, problem* (3.3) *has a unique solution $v \in C^2([0, R])$, which satisfies $\|v - v_0\|_{C^2} < \delta_0$.*

Proof. Define the Banach space

$$C^2_0([0, R]) = \{v \in C^2([0, R]) \mid v'(0) = 0,\ v(R) = 0\},$$

endowed with the norm of $C^2([0, R])$, and the operator $\mathcal{F} : C^2_0([0, R]) \times \mathbb{R} \to C^0([0, R])$ by setting

$$\begin{aligned}\mathcal{F}(v, \lambda) = r^{N-1}v'' + (N-1)r^{N-2}v'(1 + v'^2)\\ - r^{N-1}av(1 + v'^2)^{\frac{3}{2}} + \lambda r^{N-1}b(1 + v'^2).\end{aligned}$$

It is clear that $v \in C^2_0([0, R])$ is a solution of (3.3), for some $\lambda \in \mathbb{R}$, if and only if

$$\mathcal{F}(v, \lambda) = 0.$$

Further, it is a standard matter to verify that $\mathcal{F}$ is of class C^∞, with partial derivative

$$\begin{aligned}\partial_v\mathcal{F}(v, \lambda)[w] = r^{N-1}w'' + (N-1)r^{N-2}(1 + v'^2)w' + 2(N-1)r^{N-2}v'^2w'\\ - 3r^{N-1}av\sqrt{1 + v'^2}v'w' + 2\lambda r^{N-1}bv'w' - r^{N-1}a(1 + v'^2)^{\frac{3}{2}}w,\end{aligned}$$

for all $w \in C^2_0([0, R])$.

Claim: $\partial_v\mathcal{F}(v_0, \lambda_0)[w] = 0$ if and only if $w = 0$.

Let us define the functions

$$p_0 = (N-1)\frac{{v'_0}^2}{r} + 2(N-1)\frac{{v'_0}^2}{r} - 3av_0\sqrt{1 + {v'_0}^2}v'_0 + 2\lambda_0 bv'_0$$

and

$$q_0 = a(1 + {v'_0}^2)^{\frac{3}{2}}.$$

Observe that p_0, $q_0 \in C^0([0, R])$ and, for $w \in C^2_0([0, R])$,

$$\partial_v\mathcal{F}(v_0, \lambda_0)[w] = 0$$

is equivalent to

$$w'' + \frac{N-1}{r}w' + p_0(r)w' - q_0(r)w = 0. \tag{3.15}$$

For all $r \in]0, R]$, define

$$P(r) = \int_R^r \frac{N-1}{s}\, ds + \int_R^r p_0(s)\, ds = \ln\left(\frac{r}{R}\right)^{N-1} + P_0(r),$$

with

$$P_0(r) = \int_R^r p_0(s)\, ds.$$

Hence, for $w \in C_0^2([0, R])$, (3.15) can be written as

$$(e^{P(r)} w')' = e^{P(r)} q_0(r) w. \tag{3.16}$$

Multiplying (3.16) by w and integrating, we get, for all $r \in]0, R]$,

$$-e^{P(r)} w'(r) w(r) - \int_r^R e^{P(s)} w'^2(s)\, ds = \int_r^R e^{P(s)} q_0(s) w^2(s)\, ds, \tag{3.17}$$

where

$$e^{P(r)} = \left(\frac{r}{R}\right)^{N-1} e^{P_0(r)}$$

is continuous in $[0, R]$. Hence, passing to the limit in (3.17), we obtain

$$0 = -\lim_{r \to 0} e^{P(r)} w'(r) w(r) = \int_0^R e^{P(r)} w'^2(r)\, dr + \int_0^R e^{P(r)} q_0(r) w^2(r)\, dr,$$

which implies $w' = 0$ and hence $w = 0$.

Let us define the compact operator $K : C^0([0, R]) \to C^0([0, R])$ by

$$(Kw)(r) = \int_R^r \Big(e^{-P(s)} \int_0^s e^{P(t)} q_0(t) w(t)\, dt \Big)\, ds,$$

and observe that $w \in C_0^2([0, R])$ is a solution of (3.15) if and only if it is a fixed point of K. Then, we can apply the Fredholm alternative [12, Theorem D.5] and conclude that

$$\partial_v \mathcal{F}(v_0, \lambda_0) : C_0^2([0, R]) \to C^0([0, R])$$

is a linear homeomorphism. Hence, the implicit function theorem [1, p. 38] yields the existence of a constant $\delta_0 > 0$ and a map $V :]\lambda_0 - \delta_0, \lambda_0 + \delta_0[\to C_0^2([0, R])$ of class C^∞ such that, for all $(v, \lambda) \in C_0^2([0, R]) \times \mathbb{R}$, with $\|v - v_0\|_{C^2} < \delta_0$ and $|\lambda - \lambda_0| < \delta_0$,

$$\mathcal{F}(v, \lambda) = 0 \quad \text{if and only if} \quad v = V(\lambda).$$

The conclusions then follow by also using Theorem 2.2, as far as uniqueness is concerned. $\square$

Now we are in position of proving our main existence result for (1.3).

Theorem 3.4. *Assume $a, b \in C^1([0, R])$, with $a(r) \geq 0$ in $[0, R]$. Then, there exists a solution $v \in C^2([0, R]) \cap C^3(]0, R])$ of (1.3).*

Proof. Define

$$\lambda^* = \sup\{\hat{\lambda} \in]0, 1[\mid (3.3) \text{ has a solution for each } \lambda \in [0, \hat{\lambda}]\}.$$

As Lemma 3.3 applies for $v_0 = 0$ and $\lambda_0 = 0$, we infer that $\lambda^* > 0$. Let us show that (3.3) has a solution for $\lambda = \lambda^*$ too. Let $(\lambda_n)_n$ be such that $\lambda_n \in]0, \lambda^*]$, for all n, and

$$\lambda_n \to \lambda^*, \quad \text{as } n \to +\infty,$$

and let $(v_n)_n$ be the corresponding sequence of solutions of (3.3). Lemma 3.2 implies that there is a constant $K > 0$ such that, for all n,

$$\|v_n\|_{C^2} \leq K.$$

The Ascoli-Arzelà theorem yields the existence of a subsequence of $(v_n)_n$, still labeled as $(v_n)_n$, and a function $v^* \in C^1([0, R])$ such that

$$v_n \to v^*, \quad v_n' \to v^{*\prime}, \quad \text{uniformly in } [0, R], \quad \text{as } n \to +\infty. \tag{3.18}$$

Passing to the limit, as $n \to +\infty$, on both sides of the integral reformulation of the equation in (3.3)

$$\frac{s^{N-1}\, v_n'(s)}{\sqrt{1 + v_n'(s)^2}} = \int_0^s r^{N-1}\Big(a(r)v_n(r) - \lambda_n \frac{b(r)}{\sqrt{1 + v_n'(r)^2}}\Big)\, dr \quad \text{in }]0, R[,$$

we get

$$\frac{s^{N-1}\, v^{*\prime}(s)}{\sqrt{1 + v^{*\prime}(s)^2}} = \int_0^s r^{N-1}\Big(a(r)v^*(r) - \lambda^* \frac{b(r)}{\sqrt{1 + v^{*\prime}(r)^2}}\Big)\, dr \quad \text{in }]0, R[.$$

As, by (3.18), $v^{*\prime}(0) = 0$ and $v^*(R) = 0$, we conclude that v^* is the solution of (3.3) for $\lambda = \lambda^*$. Lemma 3.3 would contradict the definition of λ^*, unless $\lambda^* = 1$. This implies the existence of a solution v of (1.3). □

Remark 3.3 If, in addition to the assumptions of Theorem 3.4, we suppose that $b(r) \geq 0$ in $[0, R]$, then the solution v of (1.3) satisfies either $v(r) > 0$ in $[0, R[$ and $v'(R) < 0$, or else $v = 0$ and, in this case, $b = 0$. This assertion follows by setting $u(x) = v(|x|)$ in B and applying the strong maximum principle and the Hopf boundary point lemma [16, Section 3.2] to problem (1.1).

If we further assume that $a'(r) \geq 0$ and $b'(r) \leq 0$ in $]0, R[$, then the solution v of (1.3) is also decreasing in $[0, R]$. Indeed, assume, by contradiction, that $\max v' > 0$ and let $r_0 \in]0, R]$ be such that $v'(r_0) = \max v'$. As $v'(R) < 0$ we must have $r_0 \in]0, R[$ and then $v''(r_0) = 0$. The same calculations performed along Step 2 in the proof of Lemma 3.2 and, in particular, the identity

$$\begin{aligned} \frac{v'''(r_0)}{(1 + v'(r_0)^2)^{\frac{3}{2}}} &= a'(r_0)v(r_0) + a(r_0)v'(r_0) \\ &\quad - \frac{b'(r_0)}{\sqrt{1 + v'(r_0)^2}} + \frac{N-1}{r_0^2} \frac{v'(r_0)}{\sqrt{1 + v'(r_0)^2}} \end{aligned}$$

imply that $v'''(r_0) > 0$, which is impossible at an interior maximum point of v'.

4 Concluding remarks

As already noticed in the Introduction, extending Theorem 1.1 to the general setting of problem (1.2) is still an open question. Indeed, it is a well established fact that the existence of classical solutions of the, possibly non-homogeneous, Dirichlet problem for the prescribed mean curvature equation

$$-\mathrm{div}\Big(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\Big) = NH(x) \quad \text{in } \Omega, \tag{4.1}$$

as well as for the capillarity equation

$$-\mathrm{div}\Big(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\Big) = -au \quad \text{in } \Omega, \tag{4.2}$$

with a a positive coefficient, is intimately related to the geometric properties of $\partial\Omega$. In this respect J. Serrin established in [27] a fundamental criterion for the solvability of the Dirichlet problem associated with (4.1) and (4.2). This relies on a mean convexity assumption on $\partial\Omega$, introduced in [17, 27], which was proven to be sufficient, and in suitable sense even necessary, for the existence of classical solutions. Yet, in [27, p. 480] J. Serrin also emphasized "the delicacy of the situation when any but the simplest equations are treated". Notwithstanding, basically applying Serrin's method, the solvability of problem (1.2) can be proved under a smallness assumption on the size of the coefficient b and a version of the mean convexity condition on $\partial\Omega$. Namely, from [18], one can infer the following result.

Proposition 4.1. *Let Ω be a bounded domain in $\mathbb{R}^N$, with $N \geq 2$, having a boundary $\partial\Omega$ of class $C^{2,\alpha}$, for some $\alpha \in]0,1[$, with non-negative mean curvature. Suppose that $a, b \in C^{1,\alpha}(\overline{\Omega})$ satisfy*

$$a(x) \geq 0 \qquad \text{and} \qquad C_\Omega |b(x)| < 1 \quad \text{in } \overline{\Omega},$$

where $C_\Omega > 0$ is the embedding constant of $W_0^{1,1}(\Omega)$ into $L^1(\Omega)$. Then, problem (1.2) has a unique solution $u \in C^{2,\alpha}(\overline{\Omega})$. In addition, if $b(x) \geq 0$ in Ω and $b \neq 0$, then

$$u(x) > 0 \quad \text{in } \Omega \qquad \text{and} \qquad \nabla u(x) \cdot \nu(x) < 0 \quad \text{on } \partial\Omega,$$

$\nu(x)$ being the unit outer normal to Ω at $x \in \partial\Omega$.

In [6, Remark (a), p. 342] it was further claimed, yet without an explicit proof, that using the methods of [5] the mean convexity assumption might be suitably relaxed, allowing boundary points with negative mean curvature, but at the expense of requiring severe conditions on the size both of the coefficients and of the domain.

In view of Theorem 1.1, these results however do not appear satisfactory, due to the assumed, rather unnatural, smallness restrictions.

On the other hand, it was shown in [11] that singular solutions, which do not attain the homogeneous Dirichlet conditions at boundary points having negative mean curvature, may occur, even in the case of constant coefficients. In this specific frame the problem was rather exhaustively investigated in [10], where an existence and uniqueness result was proven within a suitable class of generalized solutions, without placing any additional condition either on the (positive constant) coefficients, or on the (Lipschitz) boundary of the domain. Since one cannot expect to find classical solutions, in [10] an explicit quantitative condition was introduced, which relates the size of the ratio of the coefficients of the equation with the geometry of the domain and guarantees that the solution previously obtained attains the homogeneous Dirichlet condition, even at those boundary points where the Serrin's mean convexity assumption fails. The approach, developed in [10] to deal with (1.2), was based on converting, by a change of variable, the problem into a variational one. Yet, unfortunately, such method does not work when the coefficient b is not constant, thus leaving open the question of the solvability of problem (1.2) in a general setting.

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