Global components of positive bounded variation solutions of a one-dimensional indefinite quasilinear Neumann problem *

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Abstract

This paper investigates the topological structure of the set of the positive solutions of the one-dimensional quasilinear indefinite Neumann problem

$$-\left(u'/\sqrt{1+u'^2}\right)' = \lambda a(x) f(u) \ \text{ in } (0,1), \quad u'(0)=0,\ u'(1)=0,$$

where $\lambda \in \mathbb{R}$ is a parameter, $a \in L^\infty(0,1)$ changes sign, and $f \in C^1(\mathbb{R})$ is positive in $(0,+\infty)$. The attention is focused on the case $f(0)=0$ and $f'(0)=1$, where we can prove, likely for the first time in the literature, a bifurcation result for this problem in the space of bounded variation functions. Namely, the existence of global connected components of the set of the positive solutions, emanating from the line of the trivial solutions at the two principal eigenvalues of the linearized problem around 0 is established. The solutions in these components are regular, as long as they are small, while they may develop jump singularities at the nodes of the weight function a, as they become larger, thus showing the possible coexistence along the same component of regular and singular solutions.

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1 Introduction

In this paper we study the topological structure of the set of the positive bounded variation solutions of the quasilinear Neumann problem

$$\begin{cases} -\left(\dfrac{u'}{\sqrt{1+u'^2}}\right)' = \lambda a(x) f(u) \quad \text{in } (0,1), \\ u'(0)=0, \ u'(1)=0, \end{cases} \tag{1.1}$$

where $\lambda \in \mathbb{R}$ is a parameter, $a \in L^\infty(0,1)$ changes sign, $f \in C^1(\mathbb{R})$ satisfies $f(s)\, s > 0$ for all $s \neq 0$ and $f'(0) = 1$. Problem (1.1) is a particular version of

$$\begin{cases} -\operatorname{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = g(x,u) \quad \text{in } \Omega, \\ -\dfrac{\nabla u \cdot \nu}{\sqrt{1+|\nabla u|^2}} = \sigma \quad \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where Ω is a bounded regular domain in $\mathbb{R}^N$, with outward pointing normal ν, and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ and $\sigma : \partial\Omega \to \mathbb{R}$ are given functions. This model plays a central role in the mathematical analysis of a number of geometrical and physical issues, such as prescribed mean curvature problems for cartesian surfaces in the Euclidean space [53, 12, 37, 54, 26, 32, 29, 31, 30], capillarity phenomena for incompressible fluids [20, 27, 28, 34, 35], and reaction-diffusion processes where the flux features saturation at high regimes [52, 36, 15].

Although there is a large amount of literature devoted to the existence of positive solutions for semilinear elliptic problems with indefinite nonlinearities [1, 2, 8, 9, 3, 33, 40, 45], no results were available for the problem (1.2), even in the one-dimensional case (1.1), before [43, 44, 42], where we began the analysis of the effects of spatial heterogeneities in the simplest prototype problem (1.1). Even if part of our discussion in this paper has been influenced by some results in the context of semilinear equations, it must be stressed that the specific structure of the mean curvature operator, $u \mapsto -\operatorname{div}\left(\nabla u / \sqrt{1+|\nabla u|^2}\right)$, makes the analysis in this paper much more delicate and sophisticated, as (1.1) may determine spatial patterns which exhibit sharp transitions between adjacent profiles, up to the formation of discontinuities [36, 24, 10, 11, 48, 15, 16, 50, 23, 21, 22]. This special feature explains why the existence intervals of regular positive solutions of [47, 18, 19] are smaller than those given in the former references when dealing with bounded variation solutions. It is a well-agreed fact that the space of bounded variation functions is the most appropriate setting for discussing these topics. The precise notion of bounded variation solution of (1.1) used in this paper has been basically introduced in [5, 6] and, for the sake of completeness, will be shortly revisited in Section 2.

In [43] we discussed the existence and the multiplicity of positive bounded variation solutions of (1.1) under various representative configurations of the behavior at zero and at infinity of the function f. The solutions of [43] can be singular, for as they may exhibit jump discontinuities at the nodal points of the weight function a, while they are regular, at least of class C^1, on each open interval where the weight function a has a constant sign. Instead, in [44, 42] we investigated the existence and the non-existence of positive regular solutions. Some of the most intriguing findings of [43, 44, 42] can be synthesized by saying that the solutions of (1.1) obtained in [43] are regular as long as they are small, in a sense to be precise later, whereas they develop singularities as they become sufficiently large. This is in complete agreement with the peculiar structure of the mean curvature operator, which combines the regularizing features of the 2-laplacian, when ∇u is sufficiently small, with the severe sharpening effects of the 1-laplacian, when ∇u becomes larger.

A natural question arising at the light of these novelties is the problem of ascertaining whether or not these regular and singular solutions can be obtained, simultaneously, by establishing the existence of connected components of bounded variation solutions bifurcating from $(\lambda, u) = (\lambda, 0)$, which stem regular from $(\lambda, 0)$ and develop singularities as their sizes increase; thus establishing the coexistence along the same component of both regular and singular solutions, as synoptically illustrated by the two bifurcation diagrams in Figure 1. Although this phenomenology has been already documented by the special example of [44, Section 8], by means of a rather sophisticated phase plane analysis, solving this problem in our general setting still was a challenge.

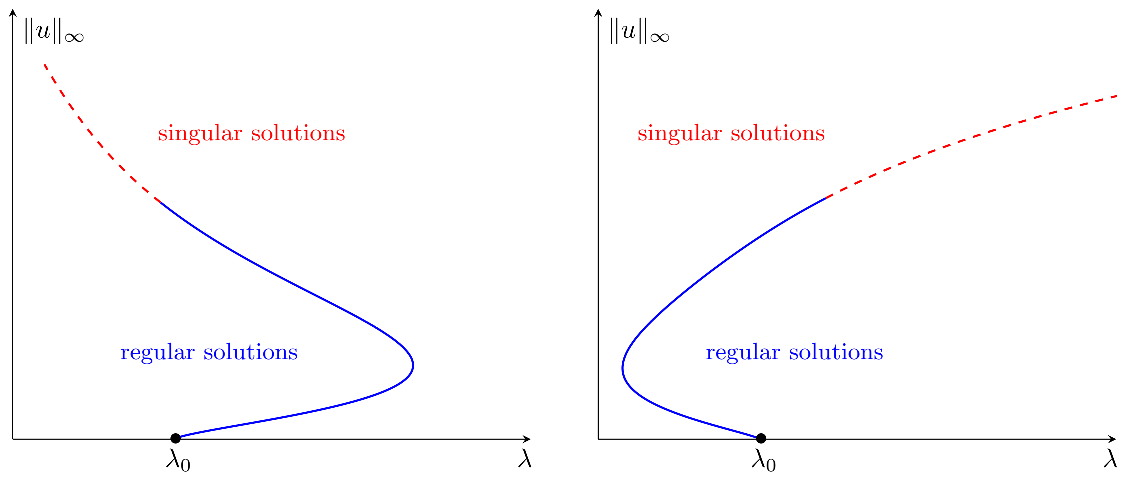

Figure 1: Global bifurcation diagrams emanating from the positive principal eigenvalue λ_0, according to the nature of the potential $\int_0^s f(t)\,dt$ of f: superlinear at infinity (on the left), or sublinear at infinity (on the right).

The main aim of this work is establishing the existence of two connected components, $\mathcal{C}_0^>$ and $\mathcal{C}_{\lambda_0}^+$, of the closure of the set of positive bounded variation solutions of problem (1.1),

$$\mathcal{S}^> = \{(\lambda, u) \in [0, +\infty) \times BV(0,1) : u > 0 \text{ is a solution of (1.1)}\} \cup \{(0,0), (\lambda_0, 0)\}, \tag{1.3}$$

emanating from the line $\{(\lambda, 0) : \lambda \in \mathbb{R}\}$ of the trivial solutions, at the two principal eigenvalues $\lambda = 0$ and $\lambda = \lambda_0$ of the linearization of (1.1) at $u = 0$,

$$\begin{cases} -u'' = \lambda a(x) u & \text{in } (0,1), \\ u'(0) = u'(1) = 0. \end{cases} \tag{1.4}$$

Precisely, our main global bifurcation theorem can be stated as follows.

Theorem 1.1. *Assume that $f \in C^1(\mathbb{R})$ satisfies $f(s)s > 0$ for all $s \neq 0$, $f'(0) = 1$, and, for some constants $\kappa > 0$ and $p > 2$, $|f'(s)| \leq \kappa\,(|s|^{p-2} + 1)$ for all $s \in \mathbb{R}$. Moreover, suppose that a satisfies $\int_0^1 a(x)\,dx < 0$ and there is $z \in (0,1)$ such that $a(x) > 0$ a.e. in $(0,z)$ and $a(x) < 0$ a.e. in $(z,1)$. Then, there exist two subsets of $\mathcal{S}^>$, $\mathcal{C}_0^>$ and $\mathcal{C}_{\lambda_0}^>$, such that*

- *$\mathcal{C}_0^>$ and $\mathcal{C}_{\lambda_0}^>$ are maximal in $\mathcal{S}^>$ with respect to the inclusion, are connected with respect to the topology of the strict convergence in $BV(0,1)$[1], and are unbounded in $\mathbb{R} \times L^p(0,1)$;*

[1] See [4, Definition 3.14]

- $(0,0) \in \mathcal{C}_0^>$ *and* $(\lambda_0, 0) \in \mathcal{C}_{\lambda_0}^>$*;*
- $\{(0,r) : r \in [0,+\infty)\} \subseteq \mathcal{C}_0^>$*;*
- *if* $(\lambda, u) \in \mathcal{C}_0^> \cup \mathcal{C}_{\lambda_0}^>$ *and* $u \neq 0$*, then* $\operatorname{ess\,inf} u > 0$*;*
- *if* $(\lambda, 0) \in \mathcal{C}_0^> \cup \mathcal{C}_{\lambda_0}^>$ *for some* $\lambda > 0$*, then* $\lambda = \lambda_0$*;*
- *either* $\mathcal{C}_0^> \cap \mathcal{C}_{\lambda_0}^> = \emptyset$*, or* $(\lambda_0, 0) \in \mathcal{C}_0^+$ *and* $(0,0) \in \mathcal{C}_{\lambda_0}^>$ *and, in such case,* $\mathcal{C}_0^> = \mathcal{C}_{\lambda_0}^>$*;*
- *there exists a neighborhood* U *of* $(0,0)$ *in* $\mathbb{R} \times L^p(0,1)$ *such that* $\mathcal{C}_0^> \cap U$ *consists of regular solutions of* (1.1)*;*
- *there exists a neighborhood* V *of* $(\lambda_0, 0)$ *in* $\mathbb{R} \times L^p(0,1)$ *such that* $\mathcal{C}_{\lambda_0}^> \cap V$ *consists of regular solutions of* (1.1)*.*

Theorem 1.1 appears to be the first global bifurcation result for a quasilinear elliptic problem driven by the mean curvature operator in the setting of bounded variation functions. The absence in the existing literature of any previous result in this direction might be attributable to the fact that mean curvature problems are fraught with a number of serious technical difficulties which do not arise when dealing with other non-degenerate quasilinear problems. As a consequence, our proof of Theorem 1.1 is extremely delicate, even though the problem (1.1) is one-dimensional. The main technical difficulties coming from the eventual lack of regularity of the solutions of (1.1) as they grow, which does not allow us to work neither in spaces of differentiable functions, nor in Sobolev spaces. Instead, this lack of regularity forces us to work in the frame of the Lebesgue spaces L^p, where the cone of positive functions has empty interior and most of the global path-following techniques in bifurcation theory fail. Thus, to get most of the conclusions of Theorem 1.1, a number of highly non-trivial technical issues must be previously overcome. Among them count the reformulation of (1.1) as a suitable fixed point equation, the proof of the differentiability of the associated underlying operator, the search for the most appropriate global bifurcation setting, as well as solving the tricky problem of the preservation of the positivity of the solutions along both components, for as in the L^p context a positive solution, a priori, could be approximated by changing sign solutions. Naturally, none of these rather pathological situations cannot arise when dealing with classical regular problems, like those considered in [41].

The structure of this paper is organized as follows. Section 2 introduces the three notions of solutions, with increasing generality, that we are going to use in this work: strong, weak, and of bounded variation. Then, it discusses their reciprocal relations, providing some useful variational characterizations. The contents of Section 2 are slightly inspired by [6]. Naturally, once reformulated (1.1) as a variational inequality in the space of bounded variation functions, one might be tempted to invoke to the available bifurcation results for variational inequalities as described, e.g., in [39]. However, since in our opinion no apparent advantage seems to come from this alternative approach, in this paper we have preferred to adopt a different, more classical, treatment of this problem based on the fact that it can be equivalently written as a fixed point equation for a completely continuous operator, where one can apply the abstract unilateral theorems of [41, Chapter 6].

Section 3 is devoted to the study of the regularity of the bounded variation solutions of (1.1). It begins by characterizing the existence of the strong solutions of the problem

$$\begin{cases} -\left(\dfrac{u'}{\sqrt{1+u'^2}}\right)' = h(x) \quad \text{in } (0,1), \\ u'(0) = 0, \;\; u'(1) = 0, \end{cases} \tag{1.5}$$

where $h \in L^1(0,1)$ is given. As a by-product, any bounded variation solution of (1.5) must be strong if $\|h\|_{L^1} < 1$. Then, Section 3 analyzes the fine regularity properties of the bounded variation solutions of (1.5), by establishing that the only singularities that they can exhibit are jumps, which, necessarily, must be located at the interior points where $h(x)$ changes sign. Thus, when the set of nodal points of h is discrete, the presence of a Cantor part in the distributional derivative of the bounded variation solutions of (1.5) is ruled out. In other words, the solutions are special functions of bounded variation, as defined in [4, Ch. 4].

In Section 4 we introduce the auxiliary problem

$$\begin{cases} -\left(\dfrac{u'}{\sqrt{1+u'^2}}\right)' + k(u) = h(x) \quad \text{in} \quad (0,1), \\ u'(0) = 0, \;\; u'(1) = 0, \end{cases} \tag{1.6}$$

where $k : \mathbb{R} \to \mathbb{R}$ is a function of class C^1, strictly increasing and odd, which satisfies

$$k'(0) = 1, \qquad \lim_{|s| \to +\infty} \frac{k'(s)}{|s|^{p-2}} = 1,$$

for some $p \geq 2$ and $h \in L^q(0,1)$, with $q = \frac{p}{p-1}$. Under these circumstances, we can establish that the associated solution operator $\mathcal{P} : L^q(0,1) \to L^p(0,1)$, which maps h onto the unique bounded variation solution $u = \mathcal{P}h$ of (1.6), is completely continuous and Fréchet differentiable at $h = 0$. In addition, we show that the derivative at 0 of $\mathcal{P}$ is given by the linear operator $\mathcal{P}_1 : L^q(0,1) \to L^p(0,1)$ which sends any function h onto the unique solution $u = \mathcal{P}_1 h \in W^{2,q}(0,1)$ of the linear problem

$$\begin{cases} -u'' + u = h(x) \quad \text{in } (0,1), \\ u'(0) = 0, \;\; u'(1) = 0. \end{cases}$$

The proof of the differentiability of $\mathcal{P}$ at 0 is far from being obvious and strongly relies on the previous regularity results delivered in Section 2.

Having all these conclusions in hand, in the subsequent Section 5 one can reformulate the problem (1.1) as an abstract operator equation

$$\mathcal{N}(\lambda, u) = 0,$$

in the space $L^p(0,1)$, provided that there are constants $\kappa > 0$ and $p > 2$ such that

$$|f'(s)| \leq \kappa(|s|^{p-2} + 1) \qquad \text{for all } \; s \in \mathbb{R}.$$

Precisely, the operator $\mathcal{N} : \mathbb{R} \times L^p(0,1) \to L^p(0,1)$ is defined by

$$\mathcal{N}(\lambda, u) = \mathcal{P}\left(k(u) + \lambda a f(u)\right) - u,$$

with k as above. Thus, it is a compact perturbation of the identity. Moreover, it can be expressed in the form

$$\mathcal{N}(\lambda, u) = \mathcal{L}(\lambda)u + \mathcal{R}(\lambda, u),$$

where

$$\mathcal{L}(\lambda) = \mathcal{P}_1\big((1 + \lambda a)\mathcal{I}\big) - \mathcal{I},$$

with $\mathcal{I}$ the identity map, is the Fréchet derivative $D_u\mathcal{N}(\lambda, 0)$ of $\mathcal{N}(\lambda, u)$, with respect to u, at $u = 0$, and

$$\lim_{\|u\|_p \to 0} \frac{\|\mathcal{R}(\lambda, u)\|_p}{\|u\|_p} = 0 \qquad \text{uniformly in } \; \lambda \in J,$$

for any compact subinterval J of $\mathbb{R}$. Hence, it is not difficult to verify that we are within the functional setting suited for applying the abstract unilateral bifurcation theorem [41, Theorem 6.4.3], at both principal eigenvalues, 0 and λ_0, of the weighted eigenvalue problem (1.4). By [41, Theorem 6.4.3] there exist two connected components of the set of the solutions of (1.1) emanating from 0 and λ_0, respectively. The remainder of the proof is then basically devoted to prove that each of these components contains an unbounded subcomponent, consisting of positive solutions, which are regular near the bifurcation points. This is achieved through an elegant topological argument combined with some sophisticated, very delicate, convergence results for sequences of bounded variation solutions of (1.1), where the special nodal structure of the function a plays a crucial role.

We conclude Section 5 by providing, under an additional regularity condition on f, some further information about the fine structure of the components of positive solutions near their respective bifurcation points from $(\lambda, 0)$.

Finally, Section 6 ends the paper with a short list of open questions and conjectures.

2 Notions of solution

Throughout this section we consider the boundary value problem

$$\begin{cases} -\left(\dfrac{u'}{\sqrt{1+u'^2}}\right)' = h(x,u) \quad \text{in } (0,1), \\ u'(0)=0, \ u'(1)=0, \end{cases} \tag{2.1}$$

where $h : (0,1) \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory conditions:

- $h(\cdot, s)$ is measurable for all $s \in \mathbb{R}$,
- $h(x, \cdot) \in C^0(\mathbb{R};\mathbb{R})$ for a.e. $x \in (0,1)$,
- for each $r > 0$ there exists $h_r \in L^1(0,1)$ such that $|h(x,s)| \leq h_r(x)$ for a.e. $x \in (0,1)$ and all $s \in (-r, r)$.

We also set

$$\psi(s) = \frac{s}{\sqrt{1+s^2}} \qquad \text{for all } s \in \mathbb{R}. \tag{2.2}$$

Definition 2.1 (Strong solution)**.** *A strong solution of problem* (2.1) *is a function* $u \in W^{2,1}(0,1)$ *which satisfies the differential equation in* (2.1) *a.e. in* $(0,1)$ *and the Neumann boundary conditions.*

Remark 2.1 Any strong solution u clearly satisfies the differential equation

$$-u'' = h(x,u)\big(1+u'^2\big)^{3/2} \quad \text{a.e. in } (0,1). \tag{2.3}$$

Moreover, integrating in $(0,1)$ the differential equation in (2.1), we find

$$\int_0^1 h(x,u)\, dx = 0 \tag{2.4}$$

for any strong solution of (2.1).

Definition 2.2 (Weak solution)**.** *A weak solution of problem* (2.1) *is a function* $u \in W^{1,1}(0,1)$ *such that*

$$\int_0^1 \frac{u'\phi'}{\sqrt{1+u'^2}}\,dx = \int_0^1 h(x,u)\phi\,dx \tag{2.5}$$

for all $\phi \in W^{1,1}(0,1)$.

Remark 2.2 By making the choice $\phi = 1$ as test function, it follows that (2.4) also holds for every weak solution u of (2.1). For these solutions, we infer from (2.2) and (2.5) that

$$\int_0^1 \psi(u')\phi'\,dx = \int_0^1 h(x,u)\phi\,dx$$

for all $\phi \in W^{1,1}(0,1)$. Thus it follows that $\psi(u') \in W^{1,1}(0,1)$ and

$$-\left(\psi(u')\right)' = h(\cdot,u) \quad \text{a.e. in } (0,1). \tag{2.6}$$

Hence, we have

$$\psi(u'(x)) = -\int_0^x h(t,u)\,dt \quad \text{ in } (0,1)$$

and therefore, taking into account (2.4),

$$\psi(u'(0)) = \psi(u'(1)) = 0,$$

which, in turn, implies

$$u'(0) = u'(1) = 0. \tag{2.7}$$

In particular, since $\psi(u') \in C^0[0,1]$, we see that

$$u' : [0,1] \to [-\infty,+\infty]$$

is continuous. Actually, the condition $\psi(u') \in W^{1,1}(0,1)$ implies that $u' \in W^{1,1}(0,1)$ if and only if $\|\psi(u')\|_\infty < 1$. Therefore, as the derivative u' of a weak solution u might develop singularities, we conclude that, in general, a weak solution is not necessarily a strong solution. Nevertheless, it is clear that if a weak solution u of (1.1) lies in $C^1[0,1]$, then it is strong. Of course, the converse is always true: any strong solution is a weak one.

The next variational characterization of the weak solutions of (2.1) can be easily derived by using the convexity of the length integral.

Lemma 2.1. *Assume that* $h : (0,1) \times \mathbb{R} \to \mathbb{R}$ *satisfies the Carathéodory conditions. A function* $u \in W^{1,1}(0,1)$ *is a weak solution of* (2.1) *if and only if it satisfies the variational inequality*

$$\int_0^1 \sqrt{1+v'^2}\,dx \geq \int_0^1 \sqrt{1+u'^2}\,dx + \int_0^1 h(x,u)(v-u)\,dx,$$

for all $v \in W^{1,1}(0,1)$*, or, equivalently, it is a global minimizer in* $W^{1,1}(0,1)$ *of the associated convex functional*

$$\mathcal{I}_u(v) = \int_0^1 \sqrt{1+v'^2}\,dx - \int_0^1 h(x,u)v\,dx.$$

The next notion of solution is more sophisticated. It basically goes back to [6, 7] and it has been extensively used and discussed later (see, e.g., [46, 48, 49, 50, 51, 43]).

Definition 2.3 (Bounded variation solution). *A bounded variation solution of problem* (2.1) *is a function* $u \in BV(0,1)$ *such that*

$$\int_0^1 \frac{Du^a D\phi^a}{\sqrt{1+(Du^a)^2}}\,dx + \int_0^1 \frac{Du^s}{|Du^s|}\,D^s\phi = \int_0^1 h(x,u)\phi\,dx \tag{2.8}$$

for all $\phi \in BV(0,1)$ *such that* $|D\phi^s|$ *is absolutely continuous with respect to* $|Du^s|$.

Remark 2.3 By taking $\phi = 1$ as test function, it follows that (2.4) also holds for every bounded variation solution u of (2.1).

In Definition 2.3, as well as throughout the rest of this paper, the following notations are used for every $v \in BV(0,1)$ (we refer to, e.g., [4, 17] for any required additional details):

- $Dv = Dv^a dx + Dv^s$ is the Lebesgue–Nikodym decomposition of the Radon measure Dv in its absolutely continuous part $Dv^a dx$, with density function Dv^a, and its singular part Dv^s, with respect to the Lebesgue measure dx in $\mathbb{R}$.
- $|Dv|$, $|Dv^a|$ and $|Dv^s|$ stand for the absolute variations of the measures Dv, Dv^a and Dv^s, respectively; thus, the Lebesgue–Nikodym decomposition of $|Dv|$ is given by
$$|Dv| = |Dv|^a dx + |Dv|^s = |Dv^a|dx + |Dv^s|.$$
- $\frac{Dv}{|Dv|}$ and $\frac{Dv^s}{|Dv^s|}$ denote the density functions of Dv and Dv^s, respectively, with respect to their absolute variations $|Dv|$ and $|Dv^s|$.
- $Dv^s = Dv^j + Dv^c$ stands for the decomposition of the singular part Dv^s of Dv in its *jump part* Dv^j and its *Cantor part* Dv^c.

The identities

$$Dv = Dv^a dx + Dv^s, \qquad Dv^s = Dv^j + Dv^c,$$

induce the decompositions

$$v = v^a + v^s = v^a + v^j + v^c,$$

with

$$v^a(x) = v(0) + \int_0^x Dv^a, \qquad v^j(x) = \int_0^x Dv^j, \qquad v^c(x) = \int_0^x Dv^c,$$
$$v^s(x) = \int_0^x Dv^s = v^j(x) + v^c(x),$$

for a.e. $x \in (0,1)$. Throughout this paper, for any given $v \in BV(0,1)$, we set

$$\int_0^1 \sqrt{1+|Dv|^2} = \int_0^1 \sqrt{1+|Dv^a|^2}\,dx + \int_0^1 |Dv^s|, \tag{2.9}$$

or, equivalently,

$$\int_0^1 \sqrt{1+|Dv|^2} = \sup\left\{ \int_0^1 (vw_1{}' + w_2) : \; w_1, w_2 \in C_0^1(0,1), \; \|w_1^2 + w_2^2\|_\infty \le 1 \right\}.$$

Remark 2.4 It is natural to interpret $\int_0^1 \sqrt{1+|Dv|^2}$ as the length of the graph of the bounded variation function v. From its definition we immediately conclude the lower semicontinuity of the length functional with respect to the L^1-convergence in the space $BV(0,1)$ (see, e.g., [26]).

The next result, complementing Lemma 2.1, is a direct consequence of [6].

Lemma 2.2. *Assume that $h : (0,1) \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory conditions. A function $u \in BV(0,1)$ is a bounded variation solution of (2.1) if and only if it satisfies the variational inequality*

$$\int_0^1 \sqrt{1+|Dv|^2} \geq \int_0^1 \sqrt{1+|Du|^2} + \int_0^1 h(x,u)(v-u)\,dx \tag{2.10}$$

for all $v \in BV(0,1)$, or, equivalently, it is a global minimizer in $BV(0,1)$ of the associated convex functional

$$\mathcal{I}_u(v) = \int_0^1 \sqrt{1+|Dv|^2} - \int_0^1 h(x,u)v\,dx.$$

The next result is a simple, but useful, consequence of Definitions 2.2 and 2.3.

Lemma 2.3. *Assume that $h : (0,1) \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory conditions. Suppose u is a bounded variation solution of (2.1). Then, the function $v = u^a \in W^{1,1}(0,1)$ is a weak solution of*

$$\begin{cases} -\left(\dfrac{v'}{\sqrt{1+v'^2}}\right)' = h(x,u) \quad \text{in } (0,1), \\ v'(0) = 0, \ v'(1) = 0. \end{cases} \tag{2.11}$$

In particular, $\psi(v') \in W^{1,1}(0,1)$ and it satisfies

$$-\left(\psi(v')\right)' = h(\cdot,u) \quad \text{a.e. in } (0,1), \quad v'(0) = v'(1) = 0. \tag{2.12}$$

Moreover, u is a weak solution of (2.1) if and only if it is a bounded variation solution of (2.1) satisfying $Du^s = 0$.

Proof. Recall that a function $w \in W^{1,1}(0,1)$ if and only if $w \in BV(0,1)$ and satisfies $D^s w = 0$. Therefore, let u be a bounded variation solution of (2.1) and set $v = u^a \in W^{1,1}(0,1)$. Particularizing (2.8) at any $\phi \in W^{1,1}(0,1)$ yields

$$\int_0^1 \frac{v'\phi'}{\sqrt{1+v'^2}}\,dx = \int_0^1 h(x,u)\phi\,dx.$$

Hence, v is a weak solution of (2.11). The fact that $\psi(v') \in W^{1,1}(0,1)$, as well as (2.12) holds, follows from the arguments given in Remark 2.2. This shows in particular that, if $Du^s = 0$, then $u = u^a$ is a weak solution of (2.1). The converse implication follows by noting again that, if u is a weak solution, then $Du^s = 0$. Hence, all test functions $\phi \in BV(0,1)$ must satisfy $D^s\phi = 0$, i.e., they belongs to $W^{1,1}(0,1)$, and thus (2.5) holds. □

Definition 2.4 (Positive solution)**.** *A strong, or weak, or bounded variation, solution of problem (2.1) is respectively said to be non-negative if* ess inf $u \geq 0$*, positive if* ess inf $u \geq 0$ *and* ess sup $u > 0$*, and strictly positive if* ess inf $u > 0$.

Throughout the rest of this paper, for any function $u \in L^1(0,1)$, we write $u \geq 0$ if ess inf $u \geq 0$, $u > 0$ if ess inf $u \geq 0$ and ess sup $u > 0$, and $u \gg 0$ if ess inf $u > 0$.

3 Regularity of the bounded variation solutions

This section analyzes the regularity of the bounded variation solutions of the problem

$$\begin{cases} -\left(\dfrac{u'}{\sqrt{1+u'^2}}\right)' = h(x) \quad \text{in} \quad (0,1), \\ u'(0)=0,\ u'(1)=0, \end{cases} \tag{3.1}$$

where $h \in L^1(0,1)$. The next result establishes some necessary conditions for the existence of a bounded variation solution of (3.1). Hereafter, by a Caccioppoli subset B of $(0,1)$ it is meant a Borel set B such that $\chi_B \in BV(0,1)$, where χ_B stands for the characteristic function of B.

Lemma 3.1. *Assume $h \in L^1(0,1)$. Suppose that problem* (3.1) *has a bounded variation solution u. Then, for every Caccioppoli set $B \subseteq (0,1)$,*

$$\left|\int_0^1 h\chi_B\,dx\right| \leq \int_0^1 |D\chi_B| \tag{3.2}$$

holds; in particular, $\displaystyle\int_0^1 h\,dx = 0.$

Proof. Let u be a bounded variation solution of (2.1). Then, for every $\phi \in BV(0,1)$ such that $|D\phi^s|$ is absolutely continuous with respect to $|Du^s|$,

$$\int_0^1 \frac{Du^a D\phi^a}{\sqrt{1+(Du^a)^2}}\,dx + \int_0^1 \frac{Du^s}{|Du^s|}\,D^s\phi = \int_0^1 h\phi\,dx. \tag{3.3}$$

Choosing $\phi = 1$ yields $\int_0^1 h = 0$. To establish (3.2), let $B \subseteq (0,1)$ be a Caccioppoli set. Then, set $v = u \pm \chi_B \in BV(0,1)$ and substitute it in (2.10). We find that

$$\pm\int_0^1 h\chi_B\,dx \leq \int_0^1 \sqrt{1+|D(u\pm\chi_B)|^2} - \int_0^1 \sqrt{1+|Du|^2} \leq \int_0^1 |D\chi_B|,$$

where the last inequality easily follows from (2.9). Indeed, we have

$$\begin{aligned} \int_0^1 \sqrt{1+|D(u\pm\chi_B)|^2} &= \int_0^1 \sqrt{1+|Du^a \pm D\chi_B^a|^2} + \int_0^1 |Du^s \pm D\chi_B^s| \\ &\leq \int_0^1 \sqrt{1+|Du^a|^2} + \int_0^1 |Du^s| + \int_0^1 |D\chi_B^s| \\ &= \int_0^1 \sqrt{1+|Du|^2} + \int_0^1 |D\chi_B|, \end{aligned}$$

which ends the proof. □

The next result complements Lemma 3.1 in a special case of interest.

Lemma 3.2. *Assume $h \in L^1(0,1)$. Let $u \in W^{1,1}(0,1)$ be a weak solution of* (3.1)*, which is not a strong solution of* (3.1)*. Then, there exists an interval $B = (0,z)$ such that*

$$\left|\int_0^1 h\chi_B\,dx\right| = 1 = \int_0^1 |D\chi_B|.$$

Proof. As $u' \notin W^{1,1}(0,1)$, Remark 2.2 implies that $\|\psi(u')\|_\infty = 1$ and, as $\psi(u') \in C^0[0,1]$, there exists $z \in (0,1)$ such that $|\psi(u'(z))| = 1$. Therefore, integrating the differential equation $-(\psi(u'))' = h$ in $B = (0,z)$ yields

$$\int_0^z h\,dx = \int_0^1 h\chi_B\,dx = -\psi(u'(z)) + \psi(u'(0)) = -\psi(u'(z)) = \pm 1 = \pm \int_0^1 |D\chi_B|,$$

which ends the proof. □

Thanks to Lemmas 3.1 and 3.2, the next result is very natural: it characterizes the existence of strong solutions for (3.1).

Proposition 3.3. *Assume $h \in L^1(0,1)$. Then, problem* (3.1) *has a strong solution if and only if*

(h_1) *there exists a constant $\kappa \in (0,1)$ such that*

$$\Big|\int_0^1 h\chi_B\,dx\Big| \leq \kappa \int_0^1 |D\chi_B|$$

for every Caccioppoli set $B \subseteq (0,1)$.

Proof. The proof is divided into three steps:

Step 1. *If problem* (3.1) *has a strong solution, then* (h_1) *holds.* Let u be a strong solution of (3.1). Take a Caccioppoli set $B \subseteq (0,1)$ and multiply the equation in (3.1) by χ_B. Using [5, Theorem 1.9 and Corollary 1.6], we get

$$\Big|\int_0^1 h\chi_B\,dx\Big| = \Big|\int_0^1 \psi(u')D\chi_B\Big| \leq \|\psi(u')\|_\infty \int_0^1 |D\chi_B|.$$

The conclusion follows by setting $\kappa = \|\psi(u')\|_\infty < 1$.

Step 2. *If* (h_1) *holds, then* (3.1) *has a bounded variation solution.* Set

$$\mathcal{W} = \Big\{w \in BV(0,1) : \int_0^1 w\,dx = 0\Big\}.$$

By the Poincaré inequality (see, e.g., [4, Remark 3.50]), $\mathcal{W}$ is a Banach space if we endow it with the norm

$$\|w\|_{\mathcal{W}} = \int_0^1 |Dw|.$$

According to Lemma 2.2, the bounded variation solutions of (3.1) are the global minimizers in $BV(0,1)$ of the convex functional $\mathcal{I} : BV(0,1) \to \mathbb{R}$ defined by

$$\mathcal{I}(v) = \int_0^1 \sqrt{1 + |Dv|^2} - \int_0^1 hv\,dx.$$

It is a classical fact (see, e.g., [26]) that $\mathcal{I}$ is lower semicontinuous with respect to the L^1-convergence in $BV(0,1)$. Let denote by $\mathcal{I}_{\mathcal{W}}$ its restriction to $\mathcal{W}$. We claim that, for every $w \in \mathcal{W}$,

$$\mathcal{I}_{\mathcal{W}}(w) \geq (1-\kappa)\int_0^1 |Dw|. \tag{3.4}$$

To prove (3.4) we proceed as follows. Fix $w \in \mathcal{W}$ and, for each $t \in \mathbb{R}$, consider the super-level set

$$E_t = \{x \in (0,1) : w(x) > t\}\,;$$

E_t is a Caccioppoli set for a.e. $t \in (0,1)$ (see, e.g., [4, Theorem 3.40]). Then, the representation formula

$$w(x) = \int_{-\infty}^{+\infty} \varphi_{E_t}(x)\; dt \tag{3.5}$$

holds for a.e. $x \in (0,1)$, where $\varphi_{E_t} \in BV(0,1)$ is the function defined by

$$\varphi_{E_t}(x) = \begin{cases} \chi_{E_t}(x) & \text{if } t > 0, \\ \chi_{E_t}(x) - 1 = -\chi_{(0,1)\setminus E_t}(x) & \text{if } t \leq 0. \end{cases}$$

The proof of (3.5) is elementary. Obviously, for every $x \in (0,1)$, we have

$$\begin{aligned} \int_{-\infty}^{+\infty} \varphi_{E_t}(x)\; dt &= \int_{0}^{+\infty} \varphi_{E_t}(x)\; dt + \int_{-\infty}^{0} \varphi_{E_t}(x)\; dt \\ &= \int_{0}^{+\infty} \chi_{E_t}(x)\, dt - \int_{-\infty}^{0} \chi_{(0,1)\setminus E_t}(x)\, dt. \end{aligned} \tag{3.6}$$

Suppose $w(x) \geq 0$. Then, we get

$$\int_{0}^{+\infty} \chi_{E_t}(x)\, dt = \int_{0}^{w(x)} dt = w(x), \qquad \int_{-\infty}^{0} \chi_{(0,1)\setminus E_t}(x)\, dt = 0.$$

Similarly, when $w(x) \leq 0$, we find

$$\int_{0}^{+\infty} \chi_{E_t}(x)\, dt = 0, \qquad \int_{-\infty}^{0} \chi_{(0,1)\setminus E_t}(x)\, dt = \int_{w(x)}^{0} dt = -w(x).$$

Thus, in any circumstances, substituting these identities into (3.6), the identity (3.5) holds. Similarly, the next co-area formula holds

$$|Dw(x)| = \int_{-\infty}^{+\infty} |D\varphi_{E_t}|\, dt \tag{3.7}$$

(see, e.g., [4, Theorem 3.40]). Hence, by Fubini theorem, it follows from (3.5) that

$$\begin{aligned} \int_0^1 hw\; dx &= \int_0^1 h \int_{-\infty}^{+\infty} \varphi_{E_t}(x)\; dt\; dx \\ &= -\int_{-\infty}^{0} \Big(\int_{(0,1)\setminus E_t} h\; dx\; \Big) dt + \int_0^{+\infty} \Big(\int_{E_t} h\; dx\; \Big) dt. \end{aligned}$$

So, by (h_1) and (3.7), we obtain

$$\begin{aligned} \int_0^1 hw\; dx &\leq \kappa \Big(\int_{-\infty}^{0} \int_0^1 |D\chi_{(0,1)\setminus E_t}|\;\; dt + \int_0^{+\infty} \int_0^1 |D\chi_{E_t}|\;\; dt \Big) \\ &= \kappa \Big(\int_{-\infty}^{+\infty} \int_0^1 |D\varphi_{E_t}|\, dx\, dt \Big) = \kappa \int_0^1 |Dw|. \end{aligned}$$

Therefore, we infer

$$\mathcal{I}(w) = \int_0^1 \sqrt{1+|Dw|^2} - \int_0^1 hw\,dx \ge \int_0^1 |Dw| - \kappa \int_0^1 |Dw| = (1-\kappa)\int_0^1 |Dw|,$$

which provides us with (3.4). This condition entails that $\mathcal{I}_{\mathcal{W}}$ is bounded from below and coercive. Since $\mathcal{I}_{\mathcal{W}}$ is lower semicontinuous with respect to the L^1-convergence in $\mathcal{W}$, $\mathcal{I}_{\mathcal{W}}$ has a global minimizer $u \in \mathcal{W}$. As, for every $v \in BV(\Omega)$, we have $\mathcal{I}(v) = \mathcal{I}_{\mathcal{W}}(w)$, where $w = v - \int_0^1 v\,dx \in \mathcal{W}$, we can conclude that u is a minimizer of $\mathcal{I}$ in $BV(0,1)$. Therefore, it is a bounded variation solution of (3.1).

Step 3. *If condition (h_1) holds, then any bounded variation solution of* (3.1) *is a strong solution.* Let u be a bounded variation solution of (3.1), consider the decomposition $u = u^a + u^s$ and take $\phi = u^s$ as a test function in (2.8). Then, proceeding exactly as in Step 1, we find

$$\int_0^1 |D^s u| = \int_0^1 \frac{D^s u}{|D^s u|} D^s u = \int_0^1 h u^s\,dx \le \kappa \int_0^1 |D^s u|,$$

which implies that $D^s u = 0$. Thus, we have $u \in W^{1,1}(0,1)$ and, by Lemma 2.3, it is a weak solution of (3.1). For each $z \in (0,1)$, integrating (2.6) in $B = (0,z)$, and using (2.7) and (h_1), we obtain

$$|\psi(u'(z))| = \Big|\int_0^z (\psi(u'))'\,dx\Big| = \Big|\int_0^1 h\chi_B\,dx\Big| \le \kappa \int_0^1 |D\chi_B| = \kappa < 1.$$

This entails $\|\psi(u')\|_\infty < 1$ and hence, by Remark 2.2, it is clear that u is a strong solution of (3.1). This ends the proof. □

The next result provides us with a very simple sufficient condition for (h_1).

Lemma 3.4. *Assume $h \in L^1(0,1)$. Suppose that h satisfies $\int_0^1 h\,dx = 0$ and $\|h\|_1 < 1$. Then, (h_1) holds.*

Proof. Let us set $\kappa = \|h\|_1 < 1$. Take any Caccioppoli set $B \subseteq (0,1)$. In case $B = (0,1)$, up to a set of measure zero, we have $\int_0^1 |D\chi_B| = 0$ and hence

$$\Big|\int_0^1 h\,dx\Big| = 0 = \kappa \int_0^1 |D\chi_B|.$$

Otherwise, from [4, Proposition 3.52], we infer that either $\int_0^1 |D\chi_B| \ge 2$, or, up to a set of measure zero, $B = [a,b] \subseteq [0,1]$, with $a = 0$ or $b = 1$. In case $\int_0^1 |D\chi_B| \ge 2$, we get

$$\Big|\int_0^1 h\chi_B\,dx\Big| \le \kappa \le \kappa \int_0^1 |D\chi_B|.$$

In case either $a = 0$ and $b < 1$, or $a > 0$ and $b = 1$, we find $\int_0^1 |D\chi_B| = 1$ and hence

$$\Big|\int_B h\,dx\Big| = \Big|\int_0^1 h\chi_B\,dx\Big| \le \int_0^1 |h|\,dx = \kappa = \kappa \int_0^1 |D\chi_B|.$$

Therefore, the inequality in (h_1) is anyhow satisfied. □

The following simple regularity result holds.

Corollary 3.5. *Assume that $h \in L^1(0,1)$. Suppose that $\|h\|_1 < 1$. Then, any bounded variation solution of* (3.1) *is a strong solution.*

Proof. Let u be a bounded variation solution of (3.1). From (3.3), taking $\phi = 1$, we infer $\int_0^1 h\,dx = 0$. Hence, by Lemma 3.4, h satisfies (h_1). Step 3 in the proof of Proposition 3.3 yields the conclusion. ☐

We can go further in the study of the regularity properties of the bounded variation solutions of (3.1), by establishing that the only singularities that they can exhibit are jumps at the interior points where h changes sign.

Proposition 3.6. *Assume $h \in L^1(0,1)$. Let u be a bounded variation solution of* (3.1)*.*

(a) *Let $(\alpha,\beta) \subset (0,1)$ be an interval such that $h(x) \geq 0$ a.e. in (α,β) (respectively, $h(x) \leq 0$ a.e. in (α,β)). Then, u is concave (respectively, convex) in (α,β), and its restriction to (α,β) satisfies*

$$u_{|(\alpha,\beta)} \in W^{2,1}_{\mathrm{loc}}(\alpha,\beta) \cap W^{1,1}(\alpha,\beta)$$

and

$$-\left(\frac{u'}{\sqrt{1+u'^2}}\right)' = h(x) \quad \text{a.e. in } (\alpha,\beta).$$

Moreover, $u \in W^{2,1}_{\mathrm{loc}}[0,\beta)$ and $u'(0) = 0$ if $\alpha = 0$, while $u \in W^{2,1}_{\mathrm{loc}}(\alpha,1]$ and $u'(1) = 0$ if $\beta = 1$.

(b) *Let (α,β), (β,γ) be any pair of adjacent subintervals of $(0,1)$ such that $h(x) \geq 0$ a.e. in (α,β) and $h(x) \leq 0$ a.e. in (β,γ) (respectively, $h(x) \leq 0$ a.e. in (α,β) and $h(x) \geq 0$ a.e. in (β,γ)). Then, either $u \in W^{2,1}_{\mathrm{loc}}(\alpha,\gamma)$, or*

$$u(\beta^-) \geq u(\beta^+) \quad \text{and} \quad u'(\beta^-) = -\infty = u'(\beta^+)$$

(respectively, $u(\beta^-) \leq u(\beta^+)$ and $u'(\beta^-) = +\infty = u'(\beta^+)$), where $u'(\beta^-)$ and $u'(\beta^+)$ are the left and the right Dini derivatives of u at β, respectively.

Proof. Let u be a bounded variation solution of (3.1) and consider the decomposition

$$u = u^a + u^j + u^c.$$

First, we prove Part (a). Let (α,β) be an interval such that $h(x) \geq 0$ a.e. in (α,β). The proof is divided into three steps.

Step 1. ${u^a}_{|(\alpha,\beta)} \in W^{2,1}_{\mathrm{loc}}(\alpha,\beta)$ and it is concave in (α,β).
Set $v = u^a \in W^{1,1}(0,1)$. By Lemma 2.3, we already know that $\psi(v') \in W^{1,1}(0,1)$ and

$$-\left(\psi(v')\right)' = h \quad \text{a.e. in } (0,1). \tag{3.8}$$

As $h(x) \geq 0$ a.e. in (α,β), $\psi(v')$ is decreasing in (α,β). Since, in addition, $\psi(v')$ is continuous and $v' \in L^1(0,1)$, we must have

$$|\psi(v'(x))| < 1 \quad \text{for all } x \in (\alpha,\beta). \tag{3.9}$$

This implies that

$$v'_{|(\alpha,\beta)} = \psi^{-1}(\psi(v')_{|(\alpha,\beta)}) \in W^{1,1}_{\mathrm{loc}}(\alpha,\beta)$$

and it is decreasing in (α,β), i.e., $v_{|(\alpha,\beta)} \in W^{2,1}_{\mathrm{loc}}(\alpha,\beta)$ and it is concave on (α,β).

Step 2. ${u^j}_{|(\alpha,\beta)} = 0$.
Assume that there exists a jump point $z \in (\alpha, \beta)$ of u. Set

$$\phi(x) = H(z - x) \quad \text{in} \ (0, 1),$$

where H stands for the Heaviside function. Clearly, we have

$$D\phi = D\phi^s = -\delta_z,$$

where δ_z is the Dirac measure concentrated at z. Since $|D\phi^s| = \delta_z$ is absolutely continuous with respect to $|Du^s|$ and its unique atom is z, it follows from (3.3) that

$$\begin{aligned} \int_0^z h\,dx = \int_0^1 h\phi\,dx &= \int_0^1 \frac{Du^s}{|Du^s|} D\phi^s \\ &= -\int_0^1 \frac{Du^s}{|Du^s|}\,\delta_z = -\int_0^1 \frac{Du^s}{|Du^s|}(z)\,\delta_z = -\frac{Du^s}{|Du^s|}(z). \end{aligned}$$

On the other hand, by the polar decomposition of measures (see, e.g., [4, Corollary 1.29]), we have

$$\frac{Du^s}{|Du^s|}(x) \in \{-1, 1\} \quad \text{for all} \ x \in (0, 1).$$

Thus, we see that $\int_0^z h\,dx \in \{-1, 1\}$. Hence, integrating (3.8) in $(0, z)$ yields

$$-\psi(v'(z)) = \int_0^z h\,dx \in \{-1, 1\},$$

which contradicts (3.9). Therefore, we conclude that $u^j = 0$ on (α, β).

Step 3. ${u^c}_{|(\alpha,\beta)} = 0$.
From the two previous steps, we already know that $u = u^a + u^c$ in (α, β). In particular, u can be extended by continuity onto $[\alpha, \beta]$. Let us prove that u is concave in $[\alpha, \beta]$. On the contrary, assume that there exists an interval $[\gamma, \delta] \subseteq [\alpha, \beta]$ such that

$$u(x) < u(\gamma) + \tfrac{u(\delta)-u(\gamma)}{\delta-\gamma}(x - \gamma) \quad \text{in} \ (\gamma, \delta).$$

Let us define $v \in BV(0, 1)$ by setting

$$v(x) = \begin{cases} u(\gamma) + \frac{u(\delta)-u(\gamma)}{\delta-\gamma}(x - \gamma) & \text{in} \ [\gamma, \delta], \\ u(x) \quad \text{elsewhere.} \end{cases}$$

It is clear that

$$\int_0^1 \sqrt{1 + |Dv|^2} < \int_0^1 \sqrt{1 + |Du|^2}$$

and, since $v(x) > u(x)$ in (γ, δ),

$$\int_0^1 hv\,dx \geq \int_0^1 hu\,dx.$$

Thus, we get

$$\int_0^1 \sqrt{1 + |Dv|^2} - \int_0^1 hv\,dx < \int_0^1 \sqrt{1 + |Du|^2} - \int_0^1 hu\,dx,$$

which contradicts the fact that u is a global minimizer of the functional

$$\mathcal{I}(v) = \int_0^1 \sqrt{1+|Dv|^2} - \int_0^1 hv\,dx.$$

Therefore, u being concave in (α,β), it is locally Lipschitz in (α,β) and hence, ${u^c}_{|(\alpha,\beta)} = 0$. As we have just proved that $u = u^a$ in (α,β), the conclusions follow from Step 1 and Lemma 2.3.

Next we prove Part (b). Let (α,β), (β,γ) be a pair of adjacent subintervals of $(0,1)$ such that $h(x) \geq 0$ a.e. in (α,β) and $h(x) \leq 0$ a.e. in (β,γ).

Set $v = u^a \in W^{1,1}(0,1)$. As v is concave in (α,β) and convex in (β,γ), two possibilities may occur: either $\psi(v'(\beta)) \in (-1,1)$, or $\psi(v'(\beta)) = -1$. In the former case, by the proof of Part (a), we have that

$$|\psi(v')(x)| < 1 \quad \text{for all } \ x \in (\alpha,\gamma)$$

and hence $v_{|(\alpha,\gamma)} \in W^{2,1}_{\text{loc}}(\alpha,\gamma)$. In the latter case, either u is continuous at β, or β is a jump point. Let us show that $u(\beta^-) \geq u(\beta^+)$. Indeed, like in Step 2, we set $\phi(x) = H(\beta - x)$ in $(0,1)$, where H is the Heaviside function. We have that $D\phi = -\delta_\beta$, where δ_β is the Dirac measure concentrated at β. Thus, it follows from (3.3) that

$$-\frac{Du^s}{|Du^s|}(\beta) = -\int_0^1 \frac{Du^s}{|Du^s|}(\beta)\,\delta_\beta = \int_0^\beta h\,dx.$$

On the other hand, integrating (3.8) in $(0,\beta)$, we find

$$1 = -\psi(v'(\beta)) = \int_0^\beta h\,dx.$$

Therefore, we conclude that $\frac{Du^s}{|Du^s|}(\beta) = -1$ and thus both

$$u(\beta^-) \geq u(\beta^+) \quad \text{and} \quad u'(\beta^-) = -\infty = u'(\beta^+),$$

which ends the proof. □

Hence we get the following result; hereafter by $SBV(0,1)$ we mean the space of all special functions of bounded variation, that is, of all bounded variation functions with vanishing Cantor part, as discussed in [4, Chapter 4].

Corollary 3.7. *Assume $h \in L^1(0,1)$ and*

(h_2) *there exists a decomposition*

$$[0,1] = \bigcup_{i=1}^k [\alpha_i,\beta_i], \qquad \textit{with } \alpha_i < \beta_i = \alpha_{i+1} < \beta_{i+1}, \quad \textit{for } i = 1,\dots,k-1,$$

such that either

$$(-1)^i h(x) \geq 0 \quad \textit{a.e. in } (\alpha_i,\beta_i), \quad \textit{for } i = 1,\dots,k,$$

or

$$(-1)^i h(x) \leq 0 \quad \textit{a.e. in } (\alpha_i,\beta_i), \quad \textit{for } i = 1,\dots,k.$$

Let u be a bounded variation solution of (3.1). Then, $u \in SBV(0,1)$, i.e., u is a special function of bounded variation, whose jumps may occur at the points α_i, with $i \in \{2,\dots,k\}$, at most. In addition, all conclusions of Proposition 3.6 hold on each interval, as well as on each pair of adjacent intervals of the decomposition.

The following uniqueness/non-uniqueness result can be of interest.

Lemma 3.8. *The problem* (3.1) *has at most one weak solution* u *such that*

$$\int_0^1 u\,dx = 0. \tag{3.10}$$

Moreover, if u *is a bounded variation solution with* $u^s \neq 0$*, then* $u^a + tu^s$ *is a bounded variation solution of* (3.1) *for any* $t \in [0,1]$.

Proof. Suppose $\int_0^1 h\,dx = 0$ and u_1, u_2 are weak solutions of (3.1) such that

$$\int_0^1 u_1\,dx = \int_0^1 u_2\,dx = 0. \tag{3.11}$$

As, for every $\phi \in W^{1,1}(0,1)$,

$$\int_0^1 \psi(u_1'(x))\phi'(x)\,dx = \int_0^1 h(x)\phi(x)\,dx = \int_0^1 \psi(u_2'(x))\phi'(x)\,dx,$$

we have

$$\psi(u_1') = \psi(u_2') \quad \text{a.e. in } (0,1)$$

and hence $u_1' = u_2'$ a.e. in $(0,1)$. So, $u_1 = u_2 + C$ for some constant C and, due to (3.11), $C = 0$, which implies $u_1 = u_2$ and shows the uniqueness of the weak solution.

By Lemma 2.2, the bounded variation solutions of (3.1) are the global minimizers in $BV(0,1)$ of the convex functional

$$\mathcal{I}(v) = \int_0^1 \sqrt{1+|Dv|^2} - \int_0^1 h(x)v\,dx.$$

If $u = u^a + u^s$, $u^s \neq 0$, is a global minimizer, by Lemma 2.3, u^a must be another global minimizer. Thus, by convexity, we find that for every $t \in [0,1]$ and $v \in BV(0,1)$,

$$\mathcal{I}(tu + (1-t)u^a) \leq t\mathcal{I}(u) + (1-t)\mathcal{I}(u^a) \leq t\mathcal{I}(v) + (1-t)\mathcal{I}(v) = \mathcal{I}(v).$$

Therefore, $tu + (1-t)u^a = u^a + tu^s$ provides us with a bounded variation solution of (3.1) for all $t \in [0,1]$. $\square$

Remark 3.1 It is easy to exhibit functions h with $\int_0^1 h\,dx = 0$, like, e.g., $h(x) = \frac{1}{2}\mathrm{sign}(x - \frac{1}{2})$, for which problem (3.1) admits two, and therefore infinitely many, bounded variation solutions, which can all be taken to satisfy (3.10) as well.

4 Fixed point reformulation

We start introducing the following assumption: the functions satisfying such conditions will be used in the sequel to define a class of suitable auxiliary problems.

(k_1) $k : \mathbb{R} \to \mathbb{R}$ *is a function of class* C^1*, strictly increasing and odd, which satisfies* $k'(0) = 1$ *and, for some* $p \geq 2$,

$$\lim_{|s|\to+\infty} \frac{k'(s)}{|s|^{p-2}} = 1. \tag{4.1}$$

The following conclusions are elementary.

Lemma 4.1. *Assume* (k_1). *Then, there exist constants* $\mu, \nu > 0$ *such that, for all* $s \in \mathbb{R}$,

$$|k'(s)| \le \mu(|s|^{p-2} + 1), \tag{4.2}$$
$$|k(s)| \le \mu(|s|^{p-1} + 1), \tag{4.3}$$
$$\nu|s| \le k(s)\,\mathrm{sign}(s), \tag{4.4}$$
$$\nu s^2 \le K(s) \le \mu(|s|^p + 1), \tag{4.5}$$

where $K(s) = \int_0^s k(t)\,dt$ *is the potential of* k.

Proof. By (4.1), for every $\mu > 1$ there exists $s_0 > 0$ such that

$$|k'(s)| \le \mu|s|^{p-2} \quad \text{if } |s| \ge s_0$$

and hence, for all $s \in \mathbb{R}$,

$$|k'(s)| \le \mu|s|^{p-2} + \max_{|s|\le s_0} |k'(s)|.$$

Thus, possibly taking a larger μ, we conclude that estimate (4.2) holds true for all $s \in \mathbb{R}$.

Next, pick $s > 0$. Integrating (4.2) and using $k(0) = 0$ yield

$$-\frac{\mu}{p-1}s^{p-1} - \mu s \le k(s) \le \frac{\mu}{p-1}s^{p-1} + \mu s.$$

Hence, as the function k is odd, we get, for all $s \in \mathbb{R}$,

$$|k(s)| \le \frac{\mu}{p-1}|s|^{p-1} + \mu|s|.$$

Since $p \ge 2$, possibly taking a taking a larger μ, we conclude that also estimate (4.3) holds true for all $s \in \mathbb{R}$.

As $k'(0) = 1$, for every $\nu \in (0,1)$ there exists $s_0 > 0$ such that

$$k'(s) \ge \nu \quad \text{if } |s| \le s_0.$$

Integrating this inequality and using $k(0) = 0$, we obtain

$$k(s) \ge \nu s \quad \text{if } 0 < s \le s_0$$

and hence, as k is odd,

$$k(s)\,\mathrm{sign}(s) \ge \nu|s| \quad \text{if } |s| \le s_0.$$

On the other hand, by (4.1), there exists $s_1 \ge 1$ such that

$$k'(s) \ge \nu|s|^{p-2} \ge \nu \quad \text{if } |s| \ge s_1,$$

because $p \ge 2$. Integrating this inequality yields

$$k(s) \ge \nu s + k(s_1) - \nu s_1 \quad \text{if } s \ge s_1.$$

As $k(s_1) > 0$, we can reduce $\nu > 0$ in such a way that

$$k(s) \ge \nu s \quad \text{if } s \ge s_1$$

and hence, as k is odd,

$$k(s)\,\mathrm{sign}(s) \geq \nu|s| \quad \text{if } |s| \geq s_1.$$

Since k is increasing, possibly further reducing $\nu > 0$, we conclude that estimate (4.4) holds true for all $s \in \mathbb{R}$.

The lower estimate in (4.5) follows from (4.4) by integration. Whereas, the upper estimate can be obtained arguing as done for deriving (4.3) from (4.2). □

Next we introduce the following auxiliary problem.

Proposition 4.2. *Fix $p \geq 2$, set $q = \frac{p}{p-1}$, and assume (k_1). Then, for each $h \in L^q(0,1)$, the problem*

$$\begin{cases} -\left(\dfrac{u'}{\sqrt{1+u'^2}}\right)' + k(u) = h(x) \quad \text{in} \quad (0,1), \\ u'(0) = 0, \; u'(1) = 0, \end{cases} \tag{4.6}$$

has a unique bounded variation solution.

Proof. Let us endow $BV(0,1)$ with the norm

$$\|v\|_{BV} = \|v\|_p + \int_0^1 |Dv|,$$

and consider the functional $\mathcal{J} : BV(0,1) \to \mathbb{R}$ defined by

$$\mathcal{J}(v) = \int_0^1 \sqrt{1+|Dv|^2} + \int_0^1 K(v) - \int_0^1 hv\,dx. \tag{4.7}$$

The proof will be divided into three steps.

Step 1. *$\mathcal{J}$ is lower semicontinuous with respect to the L^p-convergence in $BV(0,1)$.* Indeed, take a sequence $(v_n)_n$ in $BV(0,1)$ and $v \in BV(0,1)$ such that

$$\lim_{n\to+\infty} v_n = v \quad \text{in } L^p(0,1).$$

Owing to the upper estimate in (4.5), we infer from [25, Theorem 2.8] that

$$\lim_{n\to+\infty} \int_0^1 (K(v_n) - hv_n)\,dx = \int_0^1 (K(v) - hv)\,dx.$$

Moreover, by Remark 2.4, we have

$$\liminf_{n\to+\infty} \int_0^1 \sqrt{1+|Dv_n|^2} \geq \int_0^1 \sqrt{1+|Dv|^2}.$$

Thus, we get

$$\liminf_{n\to+\infty} \mathcal{J}(v_n) \geq \mathcal{J}(v),$$

which ends the proof of Step 1.

Step 2. *$\mathcal{J}$ is coercive and bounded from below in $BV(0,1)$.* By the upper estimate in (4.5), there are constants $c_1, c_2 > 0$ such that, for every $v \in BV(0,1)$,

$$\mathcal{J}(v) \geq \int_0^1 |Dv| + c_1\|v\|_p^p - \|h\|_q\|v\|_p - c_2. \tag{4.8}$$

On the other hand, there exists a constant $c_3 > 0$ such that

$$c_1|s|^p - \|h\|_q|s| - c_2 \geq c_1|s| - c_3 \quad \text{for all } \; s \in \mathbb{R}$$

and thus

$$\mathcal{J}(v) \geq \int_0^1 |Dv| + c_1\|v\|_p - c_3 \geq \min\{1, c_1\}\|v\|_{BV} - c_3$$

Therefore, $\mathcal{J}$ is coercive and bounded from below in $BV(0,1)$.

Step 3. *Problem* (4.6) *has a unique bounded variation solution.* From Steps 1 and 2 we conclude that $\mathcal{J}$ has a global minimizer $u \in BV(0,1)$, which is a bounded variation solution of (4.6). In order to prove it is unique, suppose that u_1, u_2 are bounded variation solutions of (4.6). From (2.10) we get

$$\int_0^1 \sqrt{1+|Du_1|^2} - \int_0^1 \sqrt{1+|Du_2|^2} \geq \int_0^1 (h - k(u_2))(u_1 - u_2)\, dx$$

and

$$\int_0^1 \sqrt{1+|Du_2|^2} - \int_0^1 \sqrt{1+|Du_1|^2} \geq \int_0^1 (h - k(u_1))(u_2 - u_1)\, dx.$$

Summing up we obtain

$$0 \geq \int_0^1 (k(u_1) - k(u_2))(u_1 - u_2)\, dx.$$

The strict monotonicity of the function k yields $u_1 = u_2$. □

Subsequently, we denote by $\mathcal{P} : L^q(0,1) \to L^p(0,1)$, with $p \geq 2$ and $q = \frac{p}{p-1}$, the operator sending any function $h \in L^q(0,1)$ onto the unique bounded variation solution $u = \mathcal{P}h$ of (4.6). Note that $\mathcal{P}(0) = 0$.

Proposition 4.3. *Fix $p \geq 2$, set $q = \frac{p}{p-1}$, and assume (k_1). Then, the operator $\mathcal{P} : L^q(0,1) \to L^p(0,1)$ is completely continuous.*

Proof. This proof is divided into two steps.

Step 1. *$\mathcal{P}$ is compact.* Let $(h_n)_n$ be a bounded sequence in $L^q(0,1)$ and, for every $n \geq 1$, set $u_n = \mathcal{P}h_n$. Since u_n is the global minimizer of the functional $\mathcal{J}_n : BV(0,1) \to \mathbb{R}$ defined by

$$\mathcal{J}_n(v) = \int_0^1 \sqrt{1+|Dv|^2} + \int_0^1 K(v) - \int_0^1 h_n v\, dx,$$

we have that $\mathcal{J}_n(u_n) \leq \mathcal{J}_n(0) = 1$. Thus, it follows from (4.8) that

$$\int_0^1 |Du_n| + c_1\|u_n\|_p^p - \|h_n\|_q\|u_n\|_p - c_2 \leq \mathcal{J}(u_n) \leq 1.$$

Therefore, the boundedness in $L^q(0,1)$ of $(h_n)_n$ implies the boundedness in $BV(0,1)$ of $(u_n)_n$. The compact embedding of $BV(0,1)$ into $L^p(0,1)$ yields the conclusion.

Step 2. *$\mathcal{P}$ is continuous.* Let $(h_n)_n$ be a sequence converging in $L^q(0,1)$ to some $h \in L^q(0,1)$ and set $u_n = \mathcal{P}h_n$. Pick any subsequence $(h_{n_k})_k$ of $(h_n)_n$. The boundedness of $(h_n)_n$ in $L^q(0,1)$ and the compactness of $\mathcal{P}$ yields the existence of a further subsequence $(h_{n_{k_j}})_j$ of $(h_{n_k})_k$ such that $(u_{n_{k_j}})_j$ converges in $L^p(0,1)$ to some $u \in L^p(0,1)$. As in the previous step, the next estimate holds

$$\int_0^1 |Du_{n_{k_j}}| + c_1\|u_{n_{k_j}}\|_p^p - \|h_{n_{k_j}}\|_q\|u_{n_{k_j}}\|_p - c_2 \leq \mathcal{J}(u_{n_{k_j}}) \leq 1$$

and it implies that $(u_{n_{k_j}})_j$ is bounded in $BV(0,1)$. Thus, by [4, Theorem 3.23], $u \in BV(0,1)$. Moreover, as $\mathcal{J}$ is lower semicontinuous with respect to the L^p-convergence in $BV(0,1)$, we find

$$\begin{aligned}\mathcal{J}(u) &= \int_0^1 \sqrt{1+|Du|^2} + \int_0^1 K(u) - \int_0^1 hu\,dx \\ &\leq \liminf_{j\to+\infty} \int_0^1 \sqrt{1+|Du_{n_{k_j}}|^2} + \lim_{j\to+\infty} \int_0^1 K(u_{n_{k_j}}) - \lim_{j\to+\infty} \int_0^1 h_{n_{k_j}} u_{n_{k_j}}\,dx \\ &= \liminf_{j\to+\infty} \left(\int_0^1 \sqrt{1+|Du_{n_{k_j}}|^2} + \int_0^1 K(u_{n_{k_j}}) - \int_0^1 h_{n_{k_j}} u_{n_{k_j}}\,dx \right).\end{aligned}$$

Therefore, since, by construction, $u_{n_{k_j}}$ provides us with the global minimizer in $BV(0,1)$ of the functional $\mathcal{J}_{n_{k_j}}$ it becomes apparent that, for every $v \in BV(0,1)$,

$$\begin{aligned}\mathcal{J}(u) &\leq \liminf_{j\to+\infty} \left(\int_0^1 \sqrt{1+|Dv|^2} + \int_0^1 K(v) - \int_0^1 h_{n_{k_j}} v\,dx \right) \\ &= \int_0^1 \sqrt{1+|Dv|^2} + \int_0^1 K(v) - \int_0^1 hv\,dx = \mathcal{J}(v).\end{aligned}$$

Consequently, u is the unique bounded variation solution of (4.6), that is, $u = \mathcal{P}(h)$. Since u does not depend on the sequence $(u_{n_{k_j}})_j$, we conclude that the whole sequence $(u_n)_n$ converges to u in $L^p(0,1)$. This ends the proof. □

Fix $p, q \geq 1$ and denote by $\mathcal{P}_1 : L^q(0,1) \to L^p(0,1)$ the linear operator which sends any function h onto the unique solution $u = \mathcal{P}_1 h \in W^{2,q}(0,1)$ of the linear problem

$$\begin{cases} -u'' + u = h(x) & \text{in } (0,1), \\ u'(0) = 0, \ u'(1) = 0. \end{cases} \tag{4.9}$$

The compact imbedding of $W^{2,q}(0,1)$ into $L^p(0,1)$ implies that $\mathcal{P}_1$ is a compact linear operator.

Proposition 4.4. *Fix $p \geq 2$, set $q = \frac{p}{p-1}$, and assume (k_1). Then, the operator $\mathcal{P} : L^q(0,1) \to L^p(0,1)$ is Fréchet differentiable at 0, with derivative $\mathcal{P}'(0) = \mathcal{P}_1$.*

Proof. We aim to show that, for any sequence $(h_n)_n$, with $h_n \to 0$ in $L^q(0,1)$,

$$\|h_n\|_q^{-1}(\mathcal{P}(h_n) - \mathcal{P}(0) - \mathcal{P}_1(h_n)) \to 0 \quad \text{in } L^p(0,1) \text{ as } n \to +\infty.$$

Since $\mathcal{P}(0) = 0$, this amounts to prove that, for any sequence $(v_n)_n$ in $L^q(0,1)$, with $\|v_n\|_q = 1$, and for any sequence $(s_n)_n$ in $(0,+\infty)$, with $s_n \to 0$, there holds

$$s_n^{-1}\mathcal{P}(s_n v_n) - \mathcal{P}_1(v_n) \to 0 \quad \text{in } L^p(0,1) \text{ as } n \to +\infty. \tag{4.10}$$

It suffices to establish that, for all subsequences $(v_{n_k})_k$ of $(v_n)_n$ and $(s_{n_k})_k$ of $(s_n)_n$, we can find further subsequences $(v_{n_{k_j}})_j$ of $(v_{n_k})_k$ and $(s_{n_{k_j}})_j$ of $(s_{n_k})_k$ such that

$$s_{n_{k_j}}^{-1}\mathcal{P}(s_{n_{k_j}} v_{n_{k_j}}) - \mathcal{P}_1(v_{n_{k_j}}) \to 0 \quad \text{in } L^p(0,1) \text{ as } j \to +\infty. \tag{4.11}$$

Let $(v_{n_k})_k$ be a subsequence of $(v_n)_n$ and $(s_{n_k})_k$ be a subsequence of $(s_n)_n$. Since $(v_{n_k})_k$ is bounded in $L^q(0,1)$, there exist a subsequence $(v_{n_{k_j}})_j$ of $(v_{n_k})_k$ and $v \in L^q(0,1)$ such that $v_{n_{k_j}} \to v$ weakly in

$L^q(0,1)$. Let $(s_{n_{k_j}})_j$ be the corresponding subsequence of $(s_{n_k})_k$. In there sequel, for convenience, we simply write v_j for $v_{n_{k_j}}$ and s_j for $s_{n_{k_j}}$.

By the continuity of $\mathcal{P}_1$, we have that $\mathcal{P}_1(v_j) \to \mathcal{P}_1(v)$ weakly in $L^p(0,1)$. Moreover, since $\mathcal{P}_1$ is compact and $(v_j)_j$ is bounded in $L^q(0,1)$, $(\mathcal{P}_1(v_j))_j$ is relatively compact in $L^p(0,1)$. Thus, along some subsequence, relabeled by j, we have that $\mathcal{P}_1(v_j) \to w$ in $L^p(0,1)$ for some $w \in L^p(0,1)$. Necessarily, by the uniqueness of the limit, $w = \mathcal{P}_1(v)$ and hence

$$\mathcal{P}_1(v_j) \to \mathcal{P}_1(v) \quad \text{in } L^p(0,1) \text{ as } j \to +\infty.$$

Consequently, (4.11) reduces to establishing

$$s_j^{-1}\mathcal{P}(s_j v_j) \to \mathcal{P}_1(v) \quad \text{in } L^p(0,1) \text{ as } j \to +\infty. \tag{4.12}$$

Setting, for all $j \geq 1$,

$$u_j = s_j^{-1}\mathcal{P}(s_j v_j),$$

it is clear that $s_j u_j = \mathcal{P}(s_j v_j) \in BV(0,1)$ is the unique bounded variation solution of

$$\begin{cases} -\left(\dfrac{u'}{\sqrt{1+u'^2}}\right)' + k(u) = s_j v_j & \text{in } (0,1), \\ u'(0) = 0, \ u'(1) = 0, \end{cases} \tag{4.13}$$

Since $s_j v_j \to 0$ in $L^q(0,1)$, the continuity of $\mathcal{P}$ implies that

$$s_j u_j = \mathcal{P}(s_j v_j) \to 0 \quad \text{in } L^p(0,1) \text{ as } j \to +\infty.$$

According to estimate (4.3), it follows from [25, Theorem 2.3] that

$$k(s_j u_j) \to 0 \quad \text{in } L^q(0,1) \text{ as } j \to +\infty$$

and hence

$$s_j v_j - k(s_j u_j) \to 0 \quad \text{in } L^q(0,1) \text{ as } j \to +\infty. \tag{4.14}$$

Therefore, as $q > 1$, Corollary 3.5 implies that $s_j u_j$ is a strong solution of (4.13) for all large j.

Next, we show that $(u_j)_j$ is bounded in $W^{1,1}(0,1)$. Fix any $x \in (0,1]$. Integrating over $(0,x)$ the equation in (4.13) yields

$$\psi(s_j u_j'(x)) = \frac{-s_j u_j'(x)}{\sqrt{1 + s_j^2 u_j'(x)^2}} = \int_0^x (s_j v_j - k(s_j u_j))\, dx, \tag{4.15}$$

the function ψ being defined in (2.2). Thus, as ψ is odd and increasing, we get from (4.14)

$$\psi(\|s_j u_j'\|_\infty) = \|\psi(s_j u_j')\|_\infty \leq \|s_j v_j - k(s_j u_j)\|_1 \to 0 \text{ as } j \to +\infty$$

and hence

$$\|s_j u_j'\|_\infty \to 0 \text{ as } j \to +\infty. \tag{4.16}$$

Multiplying the differential equation in (4.13) by $s_j u_j$ and integrating in $(0,1)$, we find

$$\int_0^1 \frac{s_j^2 {u_j'}^2}{\sqrt{1 + s_j^2 {u_j'}^2}}\, dx + \int_0^1 k(s_j u_j) s_j u_j \, dx = \int_0^1 s_j^2 v_j u_j \, dx. \tag{4.17}$$

We want to estimate the three terms in (4.17). As the function $q(\xi) = \xi^2(1+\xi^2)^{-\frac{1}{2}}$ is convex if $|\xi| < \sqrt{2}$, thanks to (4.16), Jensen inequality applies, for all large j, and yields

$$\frac{s_j^2\|u_j'\|_1^2}{\sqrt{1+s_j^2\|u_j'\|_1^2}} \leq \int_0^1 \frac{s_j^2 {u_j'}^2}{\sqrt{1+s_j^2 {u_j'}^2}}\,dx. \tag{4.18}$$

Condition (4.4) implies in particular that, for all $j \geq 1$,

$$\int_0^1 k(s_j u_j) s_j u_j \, dx \geq 0. \tag{4.19}$$

By Hölder inequality, we have

$$\int_0^1 s_j^2 v_j u_j \, dx \leq s_j^2 \|v_j\|_q \|u_j\|_p = s_j^2 \|u_j\|_p, \tag{4.20}$$

because, by construction, $\|v_j\|_q = 1$. Thus, substituting (4.18), (4.19) and (4.20) in (4.17) and dividing by s_j^2, we conclude that, for all large j,

$$\frac{\|u_j'\|_1^2}{\sqrt{1+s_j^2\|u_j'\|_1^2}} \leq \|u_j\|_p. \tag{4.21}$$

Since by (4.16)

$$\|s_j u_j'\|_1 \to 0 \text{ as } j \to +\infty,$$

from (4.21) we infer, for all large j,

$$\|u_j'\|_1^2 \leq \sqrt{2}\|u_j\|_p. \tag{4.22}$$

Let us set, for every $j \geq 1$,

$$r_j = \int_0^1 u_j \, dx \quad \text{and} \quad w_j = u_j - r_j.$$

From (4.22), using the Poincaré-Wirtinger inequality (see, e.g., [13, page 233]),

$$\|w_j\|_p \leq \|w_j'\|_1 = \|u_j'\|_1 \tag{4.23}$$

we obtain, for all large j,

$$\|u_j'\|_1^2 \leq \sqrt{2}(\|w_j\|_1 + |r_j|) \leq \sqrt{2}(\|w_j'\|_1 + |r_j|) = \sqrt{2}(\|u_j'\|_1 + |r_j|).$$

Hence, for any given $\varepsilon \in (0,1)$, there is $c_\varepsilon > 0$ such that

$$\|u_j'\|_1 \leq \varepsilon|r_j| + c_\varepsilon \quad \text{for all } j \geq 1. \tag{4.24}$$

Therefore, for proving that $(u_j)_j$ is bounded in $W^{1,1}(0,1)$, thanks to the Poincaré inequality (4.23), we only need to show that the sequence $(r_j)_j$ is bounded. The proof of this fact proceeds by contradiction. Thus, suppose that some subsequence of $(r_j)_j$, still labeled by j, satisfies

$$\lim_{j\to+\infty} r_j = +\infty; \tag{4.25}$$

the argument is similar in case

$$\lim_{j\to+\infty} r_j = -\infty.$$

Then, by (4.24), we have, for all large j,

$$u_j(x) = r_j + w_j(x) \geq r_j - \|w_j\|_\infty \geq r_j - \|w_j'\|_1 = r_j - \|u_j'\|_1 \geq (1-\varepsilon)r_j$$

and hence, it follows from (4.25) that

$$\lim_{j\to+\infty} u_j(x) = +\infty \quad \text{uniformly in } [0,1].$$

Integrating in $[0,1]$ the differential equation in (4.13) yields

$$1 = \|v_j\|_q \geq \int_0^1 v_j \, dx = \int_0^1 (s_j u_j)^{-1} k(s_j u_j) u_j \, dx.$$

Thus, owing to the estimate (4.4), we find that

$$1 \geq \nu \int_0^1 u_j \, dx \to +\infty \quad \text{as } j \to +\infty,$$

which is a contradiction. Therefore, we conclude that $(u_j)_j$ is bounded in $W^{1,1}(0,1)$, as claimed above.

From [4, Proposition 3.13, Theorem 3.23], we infer the existence of $u \in BV(0,1)$ such that, possibly passing to a subsequence, $u_j \to u$ in $L^1(0,1)$ and $u_j' \to Du$ weakly* in the sense of measures, i.e.,

$$\lim_{j\to+\infty} \int_0^1 \phi \, u_j' \, dx = \int_0^1 \phi \, Du \quad \text{for all } \phi \in C^0[0,1] \text{ with } \phi(0) = \phi(1) = 0. \tag{4.26}$$

Dividing the identity (4.15) by s_j yields

$$\frac{u_j'(x)}{\sqrt{1 + s_j^2 u_j'(x)^2}} = \int_0^x \left(s_j^{-1} k(s_j u_j) - v_j \right) dx.$$

for all $x \in [0,1]$ and $j \geq 1$. Since $s_j \|u_j\|_\infty \to 0$ as $j \to +\infty$, the conditions $k(0) = 0$ and $k'(0) = 1$ imply that

$$(s_j u_j(x))^{-1} k(s_j u_j(x)) \to 1 \quad \text{uniformly in } [0,1] \text{ as } j \to +\infty \tag{4.27}$$

and hence

$$\int_0^1 \left| (s_j \|u_j\|_\infty)^{-1} k(s_j u_j) \right| dx \leq \int_0^1 \left| (s_j u_j)^{-1} k(s_j u_j) \right| dx \leq 2,$$

for all large j. This estimate, together with the fact that $\|v_j\|_q = 1$, finally yields the existence of a constant $C > 0$ such that

$$|u_j'(x)| \leq C \sqrt{1 + s_j^2 u_j'(x)^2},$$

for all $x \in [0,1]$ and all large j. As $s_j \to 0$, we can conclude that $(u_j')_j$ is bounded in $L^\infty(0,1)$. Therefore, possibly passing to a further subsequence, still denoted by $(u_j)_j$, there exists $z \in L^\infty(0,1)$ such that $u_j' \to z$ weakly* in $L^\infty(0,1)$, i.e.

$$\lim_{j\to+\infty} \int_0^1 \phi \, u_j' \, dx = \int_0^1 \phi \, z \, dx \quad \text{for all } \phi \in L^1(0,1).$$

According to (4.26), this implies that $Du = z \, dx$ and thus $u \in W^{1,\infty}(0,1)$.

Pick any $\phi \in W^{1,1}(0,1)$ and observe that

$$\frac{\phi'}{\sqrt{1+s_j^2 {u'_j}^2}} \to \phi' \quad \text{in } L^1(0,1) \quad \text{as } \; j \to +\infty. \tag{4.28}$$

Note that, according to the weak formulation of (4.12), we have that

$$\int_0^1 u'_j \frac{\phi'}{\sqrt{1+s_j^2 {u'_j}^2}}\, dx = \int_0^1 \big(- s_j^{-1} k(s_j u_j) + v_j \big)\phi \, dx.$$

Thus, letting $j \to +\infty$ in this identity and using the boundedness of $(u'_j)_j$ in $L^\infty(0,1)$, we infer from (4.27) and (4.28) that

$$\int_0^1 u'\phi'\, dx = \int_0^1 (-u+v)\phi\, dx.$$

In other words, u is the unique solution of

$$\begin{cases} -u'' + u = v(x) & \text{in } (0,1), \\ u'(0) = 0, \; u'(1) = 0, \end{cases}$$

or, equivalently, $u = \mathcal{P}_1(v)$. Finally, the compact embedding of $W^{1,1}(0,1)$ into $L^p(0,1)$ allows us to conclude that, possibly along some subsequence,

$$u_j = s_j^{-1}\mathcal{P}(s_j v_j) \to \mathcal{P}_1(v) \quad \text{in } L^p(0,1) \quad \text{as } \; j \to +\infty.$$

Therefore, (4.12), and hence (4.10), is proven and the proof completed. □

Hereafter, we suppose that

(h_2) *$h : (0,1) \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, having a Carathéodory partial derivative $\frac{\partial h}{\partial s}$: $(0,1) \times \mathbb{R} \to \mathbb{R}$, such that there exist constants $r > 1$ and $a > 0$ and a function $b \in L^{\frac{r+1}{r-1}}(0,1)$, for which $h(\cdot, 0) \in L^{\frac{r+1}{r}}(0,1)$ and*

$$\Big|\frac{\partial h}{\partial s}(x,s)\Big| \le a|s|^{r-1} + b(x) \quad \textit{for a.e. } \; x \in (0,1) \; \textit{ and every } \; s \in \mathbb{R}. \tag{4.29}$$

Remark 4.1 Integrating (4.29) and using assumption (h_2), we see that h satisfies, for a.e. $x \in (0,1)$ and every $s \in \mathbb{R}$,

$$|h(x,s)| \le \tfrac{a}{r}|s|^r + |b(x)||s| + |h(x,0)|.$$

As the Young inequality implies that

$$|b(x)||s| \le \tfrac{1}{r}|s|^r + \tfrac{r-1}{r}|b(x)|^{\frac{r}{r-1}},$$

we conclude that

$$|h(x,s)| \le \tfrac{a+1}{r}|s|^r + \tfrac{r-1}{r}|b(x)|^{\frac{r}{r-1}} + |h(x,0)| \quad \text{for a.e. } \; x \in (0,1) \; \text{ and every } \; s \in \mathbb{R}, \tag{4.30}$$

where $\frac{r-1}{r}|b|^{\frac{r}{r-1}} + |h(\cdot,0)| \in L^{\frac{r+1}{r}}(0,1)$.

Set $p = r + 1$ and let k be a function satisfying (k_1). Let $\mathcal{S}$ denote the operator defined by

$$\mathcal{S}(u) = k(u) + h(\cdot, u),$$

for $u \in L^p(0,1)$. Then, the following result holds (see, e.g., [25, Chapter 2]).

Proposition 4.5. *Assume (k_1) and (h_2). Then, the operator $\mathcal{S}$ maps $L^p(0,1)$ into $L^q(0,1)$, with $q = \frac{p}{p-1}$, is continuous, and maps bounded sets into bounded sets. Moreover, it is continuously Fréchet differentiable, with derivative*

$$\mathcal{S}' : L^p(0,1) \to \mathcal{L}(L^p(0,1), L^q(0,1))$$

defined by

$$\mathcal{S}'(u)[v] = k'(u)v + \frac{\partial h}{\partial s}(\cdot, u)v \quad \textit{for all } u, v \in L^p(0,1).$$

By Propositions 4.3 and 4.5, the operator

$$\mathcal{M} = \mathcal{P}\mathcal{S} : L^p(0,1) \to L^p(0,1)$$

is well defined. Moreover, by construction, the fixed points of $\mathcal{M}$ are precisely the bounded variation solutions of (2.1). Combining Propositions 4.3 and 4.5 yields the following result.

Proposition 4.6. *Assume (k_1) and (h_2). Then, the operator $\mathcal{M} : L^p \to L^p$ is completely continuous and Fréchet differentiable at 0, with derivative $\mathcal{M}'(0) = \mathcal{P}_1\mathcal{S}'(0)$, that is,*

$$\mathcal{M}'(0)[v] = \mathcal{P}_1 \left(v + \frac{\partial h}{\partial s}(\cdot, 0)v \right) \quad \textit{for all } v \in L^p(0,1).$$

For our purposes in the next section, it should be noted that, assuming

(a_1) *$a \in L^\infty(0,1)$ satisfies $\int_0^1 a\,dx < 0$ and $a(x) > 0$ a.e. on a set of positive measure,*

then the eigenvalue problem

$$\begin{cases} -u'' = \lambda a(x)u & \text{in } (0,1), \\ u'(0) = u'(1) = 0, \end{cases} \tag{4.31}$$

has a discrete spectrum Σ, with exactly two principal eigenvalues: $\lambda = 0$, with principal eigenfunction 1, and $\lambda = \lambda_0 > 0$, with principal eigenfunction $\varphi_0 \gg 0$, normalized so that $\|\varphi_0\|_p = 1$, for some $p \geq 1$. A proof of these statements is given in [14]) (see also [44, Section 2]).

5 Global bifurcation

In this section we analyze the topological structure of the set of the positive solutions of (1.1). A pair (λ, u) is said to be a positive (resp. strictly positive) solution of (1.1) if u is a positive (resp. strictly positive) solution of (1.1) for some $\lambda > 0$. Of course, in each of these cases, u can be either a strong, or a weak, or a bounded variation solution of (1.1); accordingly, (λ, u) is also referred to as a strong, or a weak, or a bounded variation solution of (1.1).

Throughout this section, we assume that

(f_1) *$f \in C^1(\mathbb{R})$ satisfies $f(0) = 0$, $f'(0) = 1$, $f(s)s > 0$ for $s \neq 0$, and, for some constants $p > 2$ and $\kappa > 0$,*

$$|f'(s)| \leq \kappa(|s|^{p-2} + 1) \quad \textit{for all } s \in \mathbb{R}.$$

The functional setting

Assume (f_1) and set $r = p-1 > 1$. Let k be any function satisfying (k_1) for such p, and consider the operator $\mathcal{N} : \mathbb{R} \times L^p(0,1) \to L^p(0,1)$ given by

$$\mathcal{N}(\lambda, u) = \mathcal{P}\left(k(u) + \lambda a f(u)\right) - u, \tag{5.1}$$

where $\mathcal{P}$ is defined in Section 4. Thus, (λ, u) is a bounded variation solution of (1.1) if and only if

$$\mathcal{N}(\lambda, u) = 0. \tag{5.2}$$

Setting $h = \lambda a f$ and using the notations introduced in the last part of Section 4, we have

$$\mathcal{N} = \mathcal{M} - \mathcal{I} = \mathcal{P}\mathcal{S} - \mathcal{I}.$$

Here and in the sequel $\mathcal{I}$ stands for the identity operator in the space under consideration. From Propositions 4.3, 4.4 and 4.6 it becomes apparent that $\mathcal{I} + \mathcal{N} = \mathcal{M}$ is completely continuous and that it can be expressed in the form

$$\mathcal{N}(\lambda, u) = \mathcal{L}(\lambda)u + \mathcal{R}(\lambda, u), \tag{5.3}$$

where

$$\mathcal{L}(\lambda) = D_u\mathcal{N}(\lambda, 0) = \mathcal{P}_1\big((1+\lambda a)\mathcal{I}\big) - \mathcal{I}, \tag{5.4}$$

because $k'(0) = f'(0) = 1$. Here, $D_u\mathcal{N}(\lambda, 0)$ stands for the Fréchet derivative of $\mathcal{N}(\lambda, u)$, with respect to u, at $u = 0$. Of course,

$$\mathcal{R}(\lambda, \cdot) = \mathcal{N}(\lambda, \cdot) - \mathcal{L}(\lambda) \tag{5.5}$$

is a family of compact operators, continuously depending on λ, such that

$$\lim_{\|u\|_p \to 0} \frac{\|\mathcal{R}(\lambda, u)\|_p}{\|u\|_p} = 0 \qquad \text{uniformly in } \ \lambda \in J, \tag{5.6}$$

for any compact subinterval J of $\mathbb{R}$. Since $\mathcal{L}(\lambda)$ is a compact perturbation of the identity, it is a Fredholm operator of index zero.

Hereafter, for any given linear operator T, we denote by $N[T]$ the null space of T, and by $R[T]$ the range of T. The partial differentiation $\frac{\partial}{\partial \lambda}$, with respect to λ, will be simply indicated by $'$. The next result provides us with some fundamental properties of $\mathcal{L}(\lambda)$ at λ_0, the positive principal eigenvalue of the weighted eigenvalue problem (4.31).

Proposition 5.1. *Under assumption (a_1), the following properties hold:*

(a) $N[\mathcal{L}(\lambda_0)] = \operatorname{span}[\varphi_0]$,

(b) $N[\mathcal{L}(\lambda_0)] \oplus R[\mathcal{L}(\lambda_0)] = L^p(0,1)$,

(c) $\mathcal{L}'(\lambda_0)\left(N[\mathcal{L}(\lambda_0)]\right) \oplus R[\mathcal{L}(\lambda_0)] = L^p(0,1)$.

Proof. Part (a) follows from the fact that $\mathcal{L}(\lambda_0)\varphi = 0$ if and only if

$$\mathcal{P}_1\big((1+\lambda_0 a)\varphi\big) = \varphi,$$

that is, φ satisfies (4.31) for $\lambda = \lambda_0$. Since $\mathcal{L}(\lambda_0)$ is a Fredholm operator of index zero, it follows from Part (a) that

$$\operatorname{codim} R[\mathcal{L}(\lambda_0)] = 1 \tag{5.7}$$

Hence, in order to prove Part (b) it suffices to show that $\varphi_0 \notin R[\mathcal{L}(\lambda_0)]$. On the contrary, assume that $\varphi_0 \in R[\mathcal{L}(\lambda_0)]$. Then, there is $u \in L^p(0,1)$ such that

$$u - \mathcal{P}_1\left((1+\lambda_0 a)u\right) = \varphi_0,$$

i.e.,

$$\mathcal{P}_1\left((1+\lambda_0 a)u\right) = u - \varphi_0.$$

This equation is equivalent to the problem

$$\begin{cases} -(u-\varphi_0)'' + (u-\varphi_0) = u + \lambda_0 a u \quad \text{in} \quad (0,1), \\ u'(0) = u'(1) = 0. \end{cases}$$

that is, by rearranging terms,

$$\begin{cases} -u'' - \lambda_0 a u = -\varphi_0'' + \varphi_0 \quad \text{in} \quad (0,1), \\ u'(0) = u'(1) = 0. \end{cases}$$

Multiplying the differential equation by φ_0, integrating by parts and taking into account (4.31) with $\lambda = \lambda_0$, we find

$$0 = \int_0^1 (-u'' - \lambda_0 a u)\varphi_0 \, dx = \int_0^1 (-\varphi_0'' + \varphi_0)\varphi_0 \, dx = \int_0^1 \left((\varphi_0')^2 + {\varphi_0}^2\right) \, dx > 0.$$

This contradiction ends the proof of Part (b).

Similarly, by (5.7), in order to prove Part (c), it suffices to show that

$$\mathcal{L}'(\lambda_0)\varphi_0 \notin R[\mathcal{L}(\lambda_0)].$$

Suppose, on the contrary, that $\mathcal{L}'(\lambda_0)(\varphi_0) \in R[\mathcal{L}(\lambda_0)]$. Then, differentiating (5.4) with respect to λ yields

$$\mathcal{L}'(\lambda_0) = \mathcal{P}_1(a\mathcal{I})$$

and hence, there exists $u \in L^p(0,1)$ such that

$$\mathcal{L}(\lambda_0)u = \mathcal{P}_1\big((1+\lambda_0 a)u\big) - u = \lambda_0 \mathcal{L}'(\lambda_0)\varphi_0 = \mathcal{P}_1(\lambda_0 a \varphi_0)$$

and thus

$$\mathcal{P}_1(u + \lambda_0 a u - \lambda_0 a \varphi_0) = u.$$

Theefore, u satisfies

$$\begin{cases} -u'' - \lambda_0 a u = -\lambda_0 a \varphi_0 \quad \text{in} \quad (0,1), \\ u'(0) = u'(1) = 0. \end{cases}$$

Multiplying the differential equation by φ_0 and integrating by parts, it follows from (4.31) that

$$0 = \int_0^1 (-u'' - \lambda_0 a u)\varphi_0 \, dx = -\int_0^1 \lambda_0 a \varphi_0^2 \, dx = \int_0^1 \varphi_0''\varphi_0 = -\int_0^1 (\varphi_0')^2 \, dx < 0,$$

which is impossible. This completes the proof of Part (c). □

Similarly, the next result holds.

Proposition 5.2. *Under assumption* (a_1)*, the following properties hold:*

(a) $N[\mathcal{L}(0)] = \text{span}\,[1]$,

(b) $N[\mathcal{L}(0)] \oplus R[\mathcal{L}(0)] = L^p(0,1)$,

(c) $\mathcal{L}'(0)\big(N[\mathcal{L}(0)]\big) \oplus R[\mathcal{L}(0)] = L^p(0,1)$.

Proof. By (5.4), we see that $\mathcal{L}(0)\varphi = 0$ if and only if $\mathcal{P}_1(\varphi) = \varphi$, that is, φ satisfies (4.31) for $\lambda = 0$. Hence, φ is a constant, thus proving Part (a). Moreover, since $\mathcal{L}(0)$ is a Fredholm operator of index zero, we have that

$$\text{codim}\, R[\mathcal{L}(0)] = 1. \tag{5.8}$$

In order to prove Part (b), it suffices to show that $1 \notin R[\mathcal{L}(0)]$. On the contrary, assume that there exists $u \in L^p(0,1)$ such that $\mathcal{L}(0)u = 1$. Then, by (5.4), we have $\mathcal{P}_1(u) = u + 1$, i.e.,

$$\begin{cases} -(u+1)'' + (u+1) = u & \text{in } (0,1) \\ u'(0) = u'(1) = 0. \end{cases}$$

Rearranging terms we get

$$\begin{cases} -u'' + 1 = 0 & \text{in } (0,1) \\ u'(0) = u'(1) = 0, \end{cases}$$

which is impossible, as

$$0 = \int_0^1 u'' = 1.$$

This ends the proof of Part (b).

Due to (5.8), to prove Part (c) we just show that

$$\mathcal{L}'(0)1 = \mathcal{P}_1(a) \notin R[\mathcal{L}(0)].$$

On the contrary, assume that $\mathcal{L}(0)u = \mathcal{P}_1(a)$ for some $u \in L^p(0,1)$. By (5.4), we get

$$\mathcal{P}_1(u) - u = \mathcal{P}_1(a),$$

that is, $\mathcal{P}_1(u-a) = u$. Hence we obtain

$$\begin{cases} -u'' = -a & \text{in } (0,1) \\ u'(0) = u'(1) = 0, \end{cases}$$

which is impossible, as

$$0 = \int_0^1 u'' = \int_0^1 a < 0.$$

This concludes the proof. □

Preliminary properties of the solution set

Conditions (a_1) and (f_1) are always assumed in this subsection. We start introducing a few definitions.

Definition 5.1 (Nontrivial solution)**.** *We say that* $(\lambda, u) \in \mathbb{R} \times L^p(0,1)$ *is a nontrivial solution of* (5.2) *if either* $u \neq 0$*, or* $u = 0$ *and* $\lambda \in \Sigma$*, where* Σ *denotes the spectrum of* (4.31).

Then, we set

$$\begin{aligned}\mathcal{S} &= \{(\lambda,u)\in\mathbb{R}\times L^p(0,1) : (\lambda,u) \text{ is a nontrivial solution of (5.2)}\}\\ &= \{(\lambda,u)\in\mathbb{R}\times \big(L^p(0,1)\setminus\{0\}\big) : \mathcal{N}(\lambda,u)=0\}\cup\ \{(\lambda,0):\lambda\in\Sigma\}\end{aligned}$$

and

$$\mathcal{S}^{>} = \{(\lambda,u)\in\mathcal{S} : \lambda\geq 0,\ u>0\}\cup\{(0,0),(\lambda_0,0)\}, \tag{5.9}$$

where 0 and λ_0 are the two principal eigenvalues of (4.31). We endow $\mathcal{S}$ and $\mathcal{S}^{>}$ with the topology of $\mathbb{R}\times L^p(0,1)$. Since, $(\lambda,u)\in\mathbb{R}\times L^p(0,1)$ is a solution of equation (5.2) if and only if $(\lambda,u)\in\mathbb{R}\times BV(0,1)$ is a bounded variation solution of problem (1.1), $\mathcal{S}$ and $\mathcal{S}^{>}$ are also subsets of $BV(0,1)$ and, in particular, of $L^\infty(0,1)$.

Definition 5.2 (Connected component)**.** *By a connected component of $\mathcal{S}$ (respectively, of $\mathcal{S}^{>}$), we mean a closed and connected subset of $\mathcal{S}$ (respectively, of $\mathcal{S}^{>}$) that is maximal for the inclusion.*

We want to show that the solutions of (5.2) can bifurcate in $\mathbb{R}\times L^p(0,1)$ from the line of trivial solutions $\mathbb{R}\times\{0\}$ only at $(\hat\lambda,0)$, with $\hat\lambda\in\Sigma$. Hence, the bifurcation points of the bounded variation solutions of (1.1) are precisely the bifurcation points of the strong solutions of (1.1). This basically follows from Corollary 3.5, which shows that the bounded variation solutions that are small in $L^1(0,1)$, are actually strong solutions. Since Σ is a closed subset of $\mathbb{R}$, this eventually implies that both $\mathcal{S}$ and $\mathcal{S}^{>}$ are closed in $\mathbb{R}\times L^p(0,1)$.

Lemma 5.3. *Assume (a_1) and (f_1). Then, any sequence $\big((\lambda_n,u_n)\big)_n$ in $\mathcal{S}$, with $u_n\neq 0$ for all $n\geq 1$, for which there exists $\hat\lambda\in\mathbb{R}$ such that*

$$\lim_{n\to+\infty}(\lambda_n,u_n)=(\hat\lambda,0)\quad \textit{in }\ \mathbb{R}\times L^p(0,1),$$

satisfies

$$\hat\lambda\in\Sigma\quad\textit{and}\quad \lim_{n\to+\infty}\frac{u_n}{\|u_n\|_p}=\hat\varphi\quad\textit{in }\ C^1[0,1],$$

where $\hat\varphi$ is an eigenfunction of (4.31) associated to $\hat\lambda$.

Proof. Let us set, for every $n\geq 1$,

$$v_n=\frac{u_n}{\|u_n\|_p}.$$

Since $(v_n)_n$ is bounded, for any subsequence of $(v_n)_n$ we can find a further subsequence, still labeled by n, which converges weakly in $L^p(0,1)$ to some $v\in L^p(0,1)$. From (5.2)–(5.6) and the compactness of $\mathcal{P}_1$, dividing by $\|u_n\|_p$, we find

$$v_n=\mathcal{P}_1(v_n+\lambda_n a v_n)+\frac{\mathcal{R}(\lambda_n,u_n)}{\|u_n\|_p}\to\mathcal{P}_1(v+\hat\lambda a v)\quad\text{in }\ L^p(0,1)\quad\text{as } n\to+\infty$$

and hence

$$v=\mathcal{P}_1(v+\hat\lambda a v),$$

with $\|v\|_p=1$. Thus, $\hat\lambda\in\Sigma$ and $v=\hat\varphi$ is an eigenfunction of (4.31) associated with $\hat\lambda$.

On the other hand, as u_n is, for every $n\geq 1$, a bounded variation solution of (1.1) such that

$$\lambda_n a f(u_n)\to 0\quad\text{in }\ L^p(0,1)\quad\text{as } n\to+\infty,$$

Corollary 3.5 implies that u_n is a strong solution of (1.1), for sufficiently large n. Thus, integrating the differential equation of (1.1) in $(0,z)$ yields

$$-\psi(u_n'(z)) = \lambda_n \int_0^z af(u_n)\,dx \to 0 \quad \text{as } n\to+\infty, \tag{5.10}$$

where ψ is the function defined in (2.2). Since

$$-\psi(u_n'(z)) = \|\psi(u_n')\|_\infty,$$

we find, from (5.10) and (2.2), that $\|u_n'\|_\infty \to 0$ and hence $\|u_n\|_{C^1}\to 0$, as $n\to+\infty$. Let us set

$$g(s) = \begin{cases} \frac{f(s)}{s} & \text{if } s\neq 0,\\ f'(0) & \text{if } s=0.\end{cases}$$

Since $f'(0)=1$, from (2.3) we obtain that

$$-v_n'' = \lambda_n a\frac{f(u_n)}{\|u_n\|_p}\left(1+(u_n')^2\right)^{\frac{3}{2}} = \lambda_n a g(u_n)v_n\left(1+(u_n')^2\right)^{\frac{3}{2}} \to \hat\lambda a v \ \text{ in } L^p(0,1),$$

as $n\to+\infty$. In particular, $(v_n)_n$ is bounded in $W^{2,p}(0,1)$. As any subsequence of $(v_n)_n$ contains a further subsequence converging in $C^1[0,1]$ to v, the proof is completed. □

The following conclusions are immediate consequences of Lemma 5.3.

Corollary 5.4. *Assume* (a_1) *and* (f_1). *Then, any sequence* $\big((\lambda_n,u_n)\big)_n$ *in* $\mathcal{S}$, *with* $u_n\neq 0$ *for all* $n\geq 1$, *such that*

$$\lim_{n\to+\infty}(\lambda_n,u_n) = (\lambda_0,0) \quad \text{in } \mathbb{R}\times L^p(0,1)$$

satisfies

$$\lim_{n\to+\infty}\frac{u_n}{\|u_n\|_p} = \varphi_0, \qquad \text{or} \qquad \lim_{n\to+\infty}\frac{u_n}{\|u_n\|_{L^p}} = -\varphi_0, \quad \text{in } C^1[0,1],$$

where λ_0 *and* φ_0 *are, respectively, the principal positive eigenvalue and the associated positive normalized eigenfunction of* (4.31).

Corollary 5.5. *Assume* (a_1) *and* (f_1). *Then, any sequence* $\big((\lambda_n,u_n)\big)_n$ *in* $\mathcal{S}$, *with* $u_n\neq 0$ *for all* $n\geq 1$, *such that*

$$\lim_{n\to+\infty}(\lambda_n,u_n) = (0,0) \quad \text{in } \mathbb{R}\times L^p(0,1)$$

satisfies

$$\lim_{n\to+\infty}\frac{u_n}{\|u_n\|_p} = 1, \qquad \text{or} \qquad \lim_{n\to+\infty}\frac{u_n}{\|u_n\|_{L^p}} = -1, \quad \text{in } C^1[0,1].$$

Corollary 5.6. *Assume* (a_1) *and* (f_1). *Then, any sequence* $\big((\lambda_n,u_n)\big)_n$ *in* $\mathcal{S}^>$, *such that* $u_n\neq 0$, *for all* $n\geq 1$, *and*

$$\lim_{n\to+\infty}(\lambda_n,u_n) = (\hat\lambda,0) \quad \text{in } \mathbb{R}\times L^p(0,1),$$

satisfies that either

$$\hat\lambda = \lambda_0 \quad \text{and} \quad \lim_{n\to+\infty}\frac{u_n}{\|u_n\|_p} = \varphi_0 \quad \text{in } C^1[0,1],$$

or

$$\hat\lambda = 0 \quad \text{and} \quad \lim_{n\to+\infty}\frac{u_n}{\|u_n\|_p} = 1 \quad \text{in } C^1[0,1].$$

Taking into account that Σ is a closed subset of $\mathbb{R}$, from these results we can prove that $\mathcal{S}$ and $\mathcal{S}^>$ are closed subsets of $\mathbb{R} \times L^p(0,1)$.

Proposition 5.7. *Assume* (a_1) *and* (f_1). *Then, both* $\mathcal{S}$ *and* $\mathcal{S}^>$ *are closed and locally compact subsets of* $\mathbb{R} \times L^p(0,1)$.

Proof. By Lemma 5.4 the solutions of (5.2) can bifurcate in $\mathbb{R} \times L^p(0,1)$ from the trivial line $\mathbb{R} \times \{0\}$ only at $(\hat{\lambda}, 0)$, with $\hat{\lambda} \in \Sigma$. Since, by the continuity of $\mathcal{N}$, the set of solutions of (5.2) is closed $\mathbb{R} \times L^p(0,1)$, we conclude that $\mathcal{S}$ is closed in $\mathbb{R} \times L^p(0,1)$.

Similarly, by Corollary 5.6, the solutions of (5.2), with $u > 0$, can bifurcate in $\mathbb{R} \times L^p(0,1)$ from the trivial line $\mathbb{R} \times \{0\}$ only at $(0,0)$, or at $(\lambda_0, 0)$. Since the set of the solutions of (5.2), with $u \geq 0$, is closed $\mathbb{R} \times L^p(0,1)$, we conclude that $\mathcal{S}^>$ is closed in $\mathbb{R} \times L^p(0,1)$.

The local compactness of $\mathcal{S}$ and $\mathcal{S}^>$ follows from the complete continuity of $\mathcal{I} + \mathcal{N}$. □

We conclude this subsection with some technical results, which might have their own interest. First, we establish the following convergence-in-length result.

Lemma 5.8. *Assume* (a_1) *and* (f_1). *Then, any sequence* $((\lambda_n, u_n))_n$ *in* $\mathcal{S}$, *converging to* $(\lambda, u) \in \mathcal{S}$ *in* $\mathbb{R} \times L^p(0,1)$, *satisfies*

$$\lim_{n \to +\infty} \int_0^1 \sqrt{1 + |Du_n|^2} = \int_0^1 \sqrt{1 + |Du|^2}. \tag{5.11}$$

Proof. From (2.10) we have that, for every $n \geq 1$,

$$\int_0^1 \sqrt{1 + |Du_n|^2} \leq \int_0^1 \sqrt{1 + |Dv|^2} - \int_0^1 \lambda_n a f(u_n)(v - u_n)\, dx$$

for all $v \in BV(0,1)$. Thus, taking $v = u$ and letting $n \to +\infty$, as the sequence $(f(u_n))_n$ is bounded in $L^q(0,1)$, we infer that

$$\begin{aligned} \limsup_{n \to +\infty} \int_0^1 \sqrt{1 + |Du_n|^2} &\leq \int_0^1 \sqrt{1 + |Du|^2} - \lim_{n \to +\infty} \int_0^1 \lambda_n a f(u_n)(u - u_n)\, dx \\ &= \int_0^1 \sqrt{1 + |Du|^2}. \end{aligned}$$

On the other hand, the lower semicontinuity of the length functional with respect to the L^1-convergence in $BV(0,1)$ yields

$$\liminf_{n \to +\infty} \int_0^1 \sqrt{1 + |Du_n|^2} \geq \int_0^1 \sqrt{1 + |Du|^2}.$$

Therefore, (5.11) holds. □

From Lemma 5.8 and [6, Fact 3.1] we infer the next strict convergence result, which is a pivotal technical tool for proving our main bifurcation theorem. For a discussion of the notion of strict convergence in $BV(0,1)$, the reader is sent to [4, p. 125]. Here, we just recall that the topology of the strict convergence is induced by the metric

$$d(u,v) = \|u - v\|_{L^1} + \left| \int_0^1 |Du| - \int_0^1 |Dv| \right|, \quad \text{for all } u, v \in BV(0,1),$$

and that $BV(0,1)$, endowed with this metric, is continuously embedded into $L^p(0,1)$ for all $p \in [1, \infty]$.

Corollary 5.9. *Assume* (a_1) *and* (f_1)*. Then, any sequence* $((\lambda_n, u_n))_n$ *in* $\mathcal{S}$*, converging to* $(\lambda, u) \in \mathcal{S}$ *in* $\mathbb{R} \times L^p(0,1)$*, satisfies*

$$\lim_{n\to+\infty} u_n = u \;\; \text{in} \;\; L^1(0,1) \qquad \text{and} \qquad \lim_{n\to+\infty} \int_0^1 |Du_n| = \int_0^1 |Du|, \tag{5.12}$$

i.e., $(u_n)_n$ *converges strictly to* u *in* $BV(0,1)$*.*

Remark 5.1 Proposition (5.7) and Corollary 5.9 imply, in particular, that both $\mathcal{S}$ and $\mathcal{S}^>$ are closed and locally compact subsets of $\mathbb{R} \times BV(0,1)$, when $BV(0,1)$ is endowed with the topology of the strict convergence.

Finally, the following simple fact holds true.

Lemma 5.10. *Let* $(u_n)_n$ *be a sequence in* $L^\infty(0,1)$ *which converges to* $u \in L^\infty(0,1)$ *a.e. in* $[0,1]$*. Then, we have*

$$\limsup_{n\to+\infty} (\text{ess inf}\, u_n) \leq \text{ess inf}\, u \quad \text{and} \quad \liminf_{n\to+\infty} (\text{ess sup}\, u_n) \geq \text{ess sup}\, u. \tag{5.13}$$

Proof. We will prove the first inequality. Assume, by contradiction, that there exists $k \in \mathbb{R}$ such that

$$\limsup_{n\to+\infty} (\text{ess inf}\, u_n) > k > \text{ess inf}\, u. \tag{5.14}$$

Let E be a set of positive measure such that $u(x) < k$ in E and let $(u_{n_j})_j$ be a subsequence of $(u_n)_n$ such that

$$\lim_{j\to+\infty} (\text{ess inf}\, u_{n_j}) = \limsup_{n\to+\infty} (\text{ess inf}\, u_n).$$

Lastly, let F be a set of measure zero such that, for every $x \in [0,1] \setminus F$,

$$u_{n_j}(x) \geq \text{ess inf}\, u_{n_j} \qquad \text{and} \qquad \lim_{j\to+\infty} u_{n_j}(x) = u(x).$$

Pick $x \in E \setminus F$. By the definition of E, we have $u(x) < k$. Thus, by the definition of F, we get

$$\limsup_{n\to+\infty} (\text{ess inf}\, u_n) = \lim_{j\to+\infty} (\text{ess inf}\, u_{n_j}) \leq \lim_{j\to+\infty} u_{n_j}(x) = u(x) < k,$$

which contradicts (5.14) and ends the proof of the first estimate of (5.13). As the second one can be proven similarly, we omit the technical details of its proof. □

The bifurcation theorems

In order to state the main global bifurcation result of this paper we assume that, besides (a_1), the weight function a also satisfies

(a_2) *there is* $z \in (0,1)$ *such that either* $a(x) > 0$ *a.e. in* $(0,z)$ *and* $a(x) < 0$ *a.e. in* $(z,1)$*, or* $a(x) < 0$ *a.e. in* $(0,z)$ *and* $a(x) > 0$ *a.e. in* $(z,1)$*.*

Thanks to assumption (a_2) the one-signed bounded variation solutions of (1.1) enjoy the special properties listed in the next result.

Proposition 5.11. *Assume* (f_1) *and suppose that* $a \in L^\infty(0,1)$ *satisfies* $a(x) > 0$ *a.e. in* $(0,z)$ *and* $a(x) < 0$ *a.e. in* $(z,1)$*. Let* (λ, u) *be a bounded variation solution of* (1.1) *with either* $u > 0$*, or* $u < 0$*. Then, one of the following three alternatives holds:*

- $\lambda = 0$ *and then* u *is constant;*
- $\lambda u > 0$ *and then* $u \gg 0$ *if* $\lambda > 0$*, or* $u \ll 0$ *if* $\lambda < 0$*; moreover, regardless its sign,* u *is decreasing in* $[0,1]$*, concave in* $[0,z)$*, convex in* $(z,1]$*, and either* $u \in W^{2,1}(0,1)$*, or* $u \in W^{2,1}_{\mathrm{loc}}[0,z) \cap W^{1,1}(0,z)$*,* $u \in W^{2,1}_{\mathrm{loc}}(z,1] \cap W^{1,1}(z,1)$*,* $u'(z^-) = -\infty = u'(z^+)$*.*
- $\lambda u < 0$ *and then* $u \gg 0$ *if* $\lambda < 0$*, or* $u \ll 0$ *if* $\lambda > 0$*; moreover, regardless its sign,* u *is increasing in* $[0,1]$*, convex in* $[0,z)$*, concave in* $(z,1]$*, and either* $u \in W^{2,1}(0,1)$*, or* $u \in W^{2,1}_{\mathrm{loc}}[0,z) \cap W^{1,1}(0,z)$*,* $u \in W^{2,1}_{\mathrm{loc}}(z,1] \cap W^{1,1}(z,1)$*,* $u'(z^-) = +\infty = u'(z^+)$*.*

In all cases, u *satisfies*

$$-\left(\frac{u'(x)}{\sqrt{1+(u'(x))^2}}\right)' = \lambda\, a(x) f(u(x)) \quad \textit{a.e. in } (0,1) \quad \textit{and} \quad u'(0) = u'(1) = 0. \tag{5.15}$$

If, in addition, we assume (a_1) *and*

(f_2) f *is increasing in* $\mathbb{R}$*,*

then the third alternative cannot occur.

Proof. Let us suppose that $\lambda u > 0$. Condition (f_1) yields $\lambda f(u) > 0$. Hence, setting $h = \lambda\, a f(u)$, Proposition 3.6 and Corollary 3.7 imply that u is concave in $[0,z)$, convex in $(z,1]$, and, moreover, either $u \in W^{2,1}(0,1)$, or $u \in W^{2,1}_{\mathrm{loc}}[0,z) \cap W^{1,1}(0,z)$, $u \in W^{2,1}_{\mathrm{loc}}(z,1] \cap W^{1,1}(z,1)$, $u(z^-) \geq u(z^+)$, and $u'(z^-) = -\infty = u'(z^+)$. In any case u satisfies (5.15). In particular, we have that u is decreasing in $[0,1]$.

Similarly, we show that if $\lambda u < 0$, then u is increasing in $[0,1]$, convex in $[0,z)$, and concave in $(z,1]$. In addition, either $u \in W^{2,1}(0,1)$, or $u \in W^{2,1}_{\mathrm{loc}}[0,z) \cap W^{1,1}(0,z)$, $u \in W^{2,1}_{\mathrm{loc}}(z,1] \cap W^{1,1}(z,1)$, $u(z^-) \leq u(z^+)$, $u'(z^-) = +\infty = u'(z^+)$, and anyhow u satisfies (5.15).

Next, let us suppose that $\lambda > 0$ and $u > 0$. we want to show that $u \gg 0$. Assume, by contradiction, that

$$u > 0 \quad \text{and} \quad \operatorname{ess\,inf} u = 0.$$

Since u is decreasing in $[0,1]$ and continuous in $[0,z) \cup (z,1]$, we see that

$$0 = \operatorname{ess\,inf} u = \min u = u(1).$$

As, in addition, $u'(1) = 0$, the uniqueness of solution for the Cauchy problem

$$\begin{cases} -\left(\dfrac{u'}{\sqrt{1+u'^2}}\right)' = \lambda a(x) f(u) \\ u(1) = 0,\ u'(1) = 0, \end{cases}$$

guaranteed by (f_1), entails that either $u = 0$ in $[0,1]$, if u is continuous in $[0,1]$, or $u = 0$ in $(z,1]$, if u is discontinuous at z. The first case cannot occur, because we are assuming that $u > 0$. Thus, u is discontinuous at z and vanishes on $(z,1]$, which is impossible, because $u'(z^+) = -\infty$. Therefore, we conclude that $u \gg 0$.

Similarly, we can prove that if $\lambda < 0$ and $u < 0$, then $u \ll 0$, or if $\lambda > 0$ and $u < 0$, then $u \ll 0$, or if $\lambda < 0$ and $u > 0$, then $u \gg 0$.

Finally, let us further suppose that (a_1) and (f_2) hold. We want to show that if (λ, u) is a bounded variation solution of (1.1), with $u > 0$, then $\lambda \geq 0$. Suppose, by contradiction, that $\lambda < 0$. We know that

u is increasing in $[0,1]$ and $u \gg 0$. From the differential equation in (5.15), using (f_1), we get

$$\begin{aligned}\lambda a(x) &= -\left(\frac{u'(x)}{\sqrt{1+(u'(x))^2}}\right)' \frac{1}{f(u(x))}\\ &= -\left(\frac{1}{f(u(x))}\frac{u'(x)}{\sqrt{1+(u'(x))^2}}\right)' + \left(\frac{1}{f(u(x))}\right)' \frac{u'(x)}{\sqrt{1+(u'(x))^2}} \quad \text{a.e. in } (0,1).\end{aligned}$$

Integrating in $(0,z)$ and in $(z,1)$, respectively, using the condition $u'(0) = u'(1) = 0$ and $u'(z^-) = u'(z^+) = +\infty$, and summing up, we find that

$$\begin{aligned}\lambda \int_0^1 a\,dx &= -\frac{1}{f(u(z^-))} + \frac{1}{f(u(z^+))} + \int_0^1 \left(\frac{1}{f(u(x))}\right)' \frac{u'(x)}{\sqrt{1+(u'(x))^2}}\,dx\\ &= \frac{f(u(z^-)) - f(u(z^+))}{f(u(z^-))f(u(z^+))} - \int_0^1 \frac{f'(u(x))}{f^2(u(x))}\frac{(u'(x))^2}{\sqrt{1+(u'(x))^2}}\,dx \le 0,\end{aligned}$$

because $u(z^-) \le u(z^+)$ and (f_2) holds. Therefore, as $\lambda < 0$ and, by (a_1), $\int_0^1 a\,dx < 0$, we get a contradiction.

Similarly, we show that if (λ, u) is a bounded variation solution of (1.1), with $u < 0$, then $\lambda \le 0$.

This allows us to conclude that, for one-signed bounded variation solutions (λ, u) of (1.1), the alternative $\lambda = 0$ or $\lambda u > 0$ must holds. □

The following symmetric counterpart of Proposition 5.11 holds.

Proposition 5.12. *Assume (f_1) and suppose that $a \in L^\infty(0,1)$ satisfies $a(x) < 0$ a.e. in $(0,z)$ and $a(x) > 0$ a.e. in $(z,1)$. Let (λ,u) be a bounded variation solution of (1.1), with either $u > 0$, or $u < 0$. Then, the following three alternatives hold:*

- *$\lambda = 0$ and then u is constant;*
- *$\lambda u > 0$ and then $u \gg 0$ if $\lambda > 0$, or $u \ll 0$ if $\lambda < 0$; moreover, regardless its sign, u is increasing in $[0,1]$, convex in $[0,z)$, concave in $(z,1]$, and either $u \in W^{2,1}(0,1)$, or $u \in W^{2,1}_{\mathrm{loc}}[0,z) \cap W^{1,1}(0,z)$, $u \in W^{2,1}_{\mathrm{loc}}(z,1] \cap W^{1,1}(z,1)$, and $u'(z^-) = +\infty = u'(z^+)$;*
- *$\lambda u < 0$ and then $u \gg 0$ if $\lambda < 0$, or $u \ll 0$ if $\lambda > 0$; moreover, regardless its sign, u is decreasing in $[0,1]$, concave in $[0,z)$, convex in $(z,1]$, and either $u \in W^{2,1}(0,1)$, or $u \in W^{2,1}_{\mathrm{loc}}[0,z) \cap W^{1,1}(0,z)$, $u \in W^{2,1}_{\mathrm{loc}}(z,1] \cap W^{1,1}(z,1)$, and $u'(z^-) = -\infty = u'(z^+)$.*

In all cases, u satisfies (5.15). If, in addition, we assume (a_1) and (f_2), then the third alternative cannot occur.

Remark 5.2 Proposition 5.11 implies that if $(\lambda,u) \in \mathcal{S}^>$ and $u \neq 0$, then $u \gg 0$.

Our main global bifurcation result establishes the existence of two unbounded connected components $\mathcal{C}_0^>$ and $\mathcal{C}_{\lambda_0}^>$ of the set $\mathcal{S}^>$ of the positive solutions of (1.1), as defined in (5.9), bifurcating from $(\lambda,0)$ at $\lambda = 0$ and $\lambda = \lambda_0$, respectively.

Theorem 5.13. *Assume (f_1), (a_1) and (a_2). Then, there exist two connected components $\mathcal{C}_0^>$ and $\mathcal{C}_{\lambda_0}^>$ of $\mathcal{S}^>$ such that:*

- *$\mathcal{C}_0^>$ and $\mathcal{C}_{\lambda_0}^>$ are unbounded in $\mathbb{R} \times L^p(0,1)$;*

- $\mathcal{C}_0^>$ *and* $\mathcal{C}_{\lambda_0}^>$ *are closed and connected subsets of* $BV(0,1)$*, endowed with the topology of the strict convergence;*
- $(0,0) \in \mathcal{C}_0^>$ *and* $(\lambda_0, 0) \in \mathcal{C}_{\lambda_0}^>$*;*
- $\{(0,r) : r \geq 0\} \subseteq \mathcal{C}_0^>$*;*
- *if* $(\lambda, u) \in \mathcal{C}_0^> \cup \mathcal{C}_{\lambda_0}^>$ *and* $u \neq 0$*, then* $u \gg 0$*;*
- *if* $(\lambda, 0) \in \mathcal{C}_0^> \cup \mathcal{C}_{\lambda_0}^>$ *for some* $\lambda > 0$*, then* $\lambda = \lambda_0$*;*
- *either* $\mathcal{C}_0^> \cap \mathcal{C}_{\lambda_0}^> = \emptyset$*, or* $(\lambda_0, 0) \in \mathcal{C}_0^>$ *and* $(0,0) \in \mathcal{C}_{\lambda_0}^>$ *and, in such case,* $\mathcal{C}_0^> = \mathcal{C}_{\lambda_0}^>$*;*
- *there exists a neighborhood* U *of* $(0,0)$ *in* $\mathbb{R} \times L^p(0,1)$ *such that* $\mathcal{C}_0^> \cap U$ *consists of strong solutions of* (1.1)*;*
- *there exists a neighborhood* V *of* $(\lambda_0, 0)$ *in* $\mathbb{R} \times L^p(0,1)$ *such that* $\mathcal{C}_{\lambda_0}^> \cap V$ *consists of strong solutions of* (1.1)*.*

Proof. We suppose here that the first alternative holds in (a_2), that is, we assume that there is $z \in (0,1)$ such that $a(x) > 0$ a.e. in $(0,z)$ and $a(x) < 0$ a.e. in $(z,1)$. The argument in the other case follows similar patterns.

The proof is divided into two parts.

Part 1. Bifurcation from $(\lambda_0, 0)$: existence and properties of $\mathcal{C}_{\lambda_0}^>$.
We are going to apply the unilateral global bifurcation theorem [41, Theorem 6.4.3] to the equation (5.2) in $L^p(0,1)$, with $p > 2$. Following [41, Chapter 6] we introduce the closed subspace

$$Y = \Big\{ y \in L^p(0,1) : \int_0^1 y \varphi_0 \, dx = 0 \Big\},$$

and, for every $\varepsilon > 0$ and $\eta \in (0,1)$, we consider the open wedges

$$Q_{\varepsilon,\eta}^+(\lambda_0) = \Big\{ (\lambda, u) \in \mathbb{R} \times L^p(0,1) : |\lambda - \lambda_0| < \varepsilon, \quad \int_0^1 u \varphi_0 \, dx > \eta \|u\|_p \Big\},$$
$$Q_{\varepsilon,\eta}^-(\lambda_0) = \Big\{ (\lambda, u) \in \mathbb{R} \times L^p(0,1) : |\lambda - \lambda_0| < \varepsilon, \quad \int_0^1 u \varphi_0 \, dx < -\eta \|u\|_p \Big\}.$$

Thanks to Proposition 5.1, we infer from [41, Lemma 6.4.1] that, for every $\varepsilon > 0$ and $\eta \in (0,1)$, there exists a neighborhood V of $(\lambda_0, 0)$ in $\mathbb{R} \times L^p(0,1)$ such that

$$(\mathcal{S} \cap V) \setminus \{(\lambda_0, 0)\} \subset Q_{\varepsilon,\eta}^+(\lambda_0) \cup Q_{\varepsilon,\eta}^-(\lambda_0).$$

Due to (a_1) and (f_1), by possibly reducing the size of V, we can also suppose that

$$\|\lambda a f(u)\|_1 < 1 \quad \text{for all } \ (\lambda, u) \in \mathcal{S} \cap V.$$

Thus, by Corollary 3.5, $\mathcal{S} \cap V$ consists of strong solutions of (1.1).

Let us fix $\varepsilon > 0$ and $\eta \in (0,1)$. By Proposition 5.1 and [41, Theorem 5.6.2], all the assumptions of [41, Theorem 6.4.3] hold true with reference to λ_0. Thus, there is a connected component $\mathcal{C}_{\lambda_0}$ of $\mathcal{S} \setminus (Q_{\varepsilon,\eta}^-(\lambda_0) \cap V)$, with $(\lambda_0, 0) \in \mathcal{C}_{\lambda_0}$, such that one of the following non-excluding options holds:

(A1) $\mathcal{C}_{\lambda_0}$ is unbounded in $\mathbb{R} \times L^p(0,1)$, or

(A2) there exists $\hat{\lambda} \in \Sigma \setminus \{\lambda_0\}$ such that $(\hat{\lambda}, 0) \in \mathcal{C}_{\lambda_0}$,

or

(A3) there exists $(\lambda, y) \in \mathcal{C}_{\lambda_0} \cap \big(\mathbb{R} \times (Y \setminus \{0\})\big)$, i.e., $y \neq 0$ and $\int_0^1 y\varphi_0\, dx = 0$.

If $\mathcal{C}_{\lambda_0} \cap ((-\infty, 0) \times L^p(0,1)) \neq \emptyset$, then by connectedness there exists $u \in L^p(0,1)$ such that $(0, u) \in \mathcal{C}_{\lambda_0}$. Since u must be constant and $\mathcal{C}_{\lambda_0}$ is a maximal connected subset of $\mathcal{S} \setminus (Q^-_{\varepsilon,\eta}(\lambda_0) \cap V)$, $\mathcal{C}_{\lambda_0}$ contains the vertical line $\{(0, r) : r \in \mathbb{R}\}$ and hence $\mathcal{C}_{\lambda_0} \cap ([0, +\infty) \times L^p(0,1))$ is unbounded. Accordingly, if we set

$$\mathcal{C}^+_{\lambda_0} = \mathcal{C}_{\lambda_0} \cap ([0, +\infty) \times L^p(0,1)),$$

we see that, in any case, $\mathcal{C}^+_{\lambda_0}$ is a maximal connected subset of $\mathcal{S} \cap ([0, +\infty) \times L^p(0,1))$, satisfying either (A1), or (A2), or (A3).

Let us also observe that, by Corollary 5.4, possibly shortening V, we have that $u \gg 0$ for all $(\lambda, u) \in (\mathcal{C}^+_{\lambda_0} \cap V) \setminus \{(\lambda_0, 0)\}$; however, we cannot guarantee that $\mathcal{C}^+_{\lambda_0}$ does not contain any negative, or sign-changing, solution. The remainder of the proof of this part is devoted to showing that an unbounded component $\mathcal{C}^>_{\lambda_0}$ of $\mathcal{C}^+_{\lambda_0}$, constituted by positive solutions, actually exists.

Let us define $\mathcal{C}^>_{\lambda_0}$ as the component of $\mathcal{S}^>$ such that $(\lambda_0, 0) \in \mathcal{C}^>_{\lambda_0}$. Proposition 5.11 and the subsequent Remark 5.2 guarantee that $u \gg 0$ for all $(\lambda, u) \in \mathcal{C}^>_{\lambda_0}$ with $u \neq 0$. We know that

$$\mathcal{C}^>_{\lambda_0} = \mathcal{C}^+_{\lambda_0} \quad \text{in } V, \tag{5.16}$$

since $u \gg 0$ for all $(\lambda, u) \in \mathcal{C}^+_{\lambda_0} \cap V$ with $u \neq 0$. Moreover, by construction, we have $\mathcal{C}^>_{\lambda_0} \subseteq \mathcal{C}^+_{\lambda_0}$. Actually, the following result holds.

Claim. *$\mathcal{C}^>_{\lambda_0}$ is unbounded in $\mathbb{R} \times L^p(0,1)$.*

To prove this claim, we distinguish two cases, according to either $(0,0) \in \mathcal{C}^>_{\lambda_0}$, or $(0,0) \notin \mathcal{C}^>_{\lambda_0}$.

In case $(0,0) \in \mathcal{C}^>_{\lambda_0}$, $\mathcal{C}^>_{\lambda_0}$ is unbounded in $\mathbb{R} \times L^p(0,1)$, because, being a component, it must contain the whole vertical half-line $\{(0, r) : r \in [0, +\infty)\}$.

In case $(0,0) \notin \mathcal{C}^>_{\lambda_0}$, we will show that

$$\mathcal{C}^+_{\lambda_0} = \mathcal{C}^>_{\lambda_0}. \tag{5.17}$$

Consequently, as the component $\mathcal{C}^+_{\lambda_0} = \mathcal{C}^>_{\lambda_0}$ cannot satisfy alternatives (A2) and (A3) above, $\mathcal{C}^>_{\lambda_0}$ must satisfy (A1), i.e., it is unbounded in $\mathbb{R} \times L^p(0,1)$.

In order to prove (5.17), we suppose on the contrary that $\mathcal{C}^>_{\lambda_0}$ is a proper subset of $\mathcal{C}^+_{\lambda_0}$. Being components, $\mathcal{C}^+_{\lambda_0}$ is connected and $\mathcal{C}^>_{\lambda_0}$ is closed; hence, there exist a sequence $((\lambda_n, u_n))_n$ in $\mathcal{C}^+_{\lambda_0} \setminus \mathcal{C}^>_{\lambda_0}$ and a solution $(\lambda_\omega, u_\omega) \in \mathcal{C}^>_{\lambda_0}$ such that

$$\lim_{n \to +\infty} (\lambda_n, u_n) = (\lambda_\omega, u_\omega) \quad \text{in } \mathbb{R} \times L^p(0,1).$$

As $(\lambda_\omega, u_\omega) \in \mathcal{S}^>$, the definition of $\mathcal{S}^>$ implies that one of the following three cases occurs: either $(\lambda_\omega, u_\omega) = (0,0)$, or $(\lambda_\omega, u_\omega) = (\lambda_0, 0)$, or $u > 0$.

The first case, $(\lambda_\omega, u_\omega) = (0,0)$, is immediately ruled out because we are supposing $(0,0) \notin \mathcal{C}^>_{\lambda_0}$.

The second case, $(\lambda_\omega, u_\omega) = (\lambda_0, 0)$, cannot occur, because otherwise

$$(\lambda_n, u_n) \in (\mathcal{C}^+_{\lambda_0} \setminus \mathcal{C}^>_{\lambda_0}) \cap V, \quad \text{for all large } n,$$

which is impossible by (5.16).

Thus, $u_\omega > 0$ must hold, and actually, due to Proposition 5.11, $u_\omega \gg 0$. If in a neighborhood of $(\lambda_\omega, u_\omega)$ the component $\mathcal{C}^+_{\lambda_0}$ consisted of solutions of the form (λ, v) with $v > 0$, $\mathcal{C}^>_{\lambda_0}$ would not be maximal for the inclusion in $\mathcal{S}^>$ and hence, could not be a component. Therefore, without loss of generality, we can assume that, for every $n \geq 1$, either $u_n \leq 0$, or u_n changes sign.

On the other hand, since $u_n \to u_\omega$ in $L^p(0,1)$, there is a subsequence, relabeled by n, such that $u_n(x) \to u_\omega(x)$ a.e. in $[0,1]$. If there existed a subsequence of $((\lambda_n, u_n))_n$, still labeled by n, such that $u_n \leq 0$ for all n, it would necessarily follow that $u \leq 0$. Therefore, since $u \gg 0$, u_n must change sign for all large n. Thus, by Corollary 5.9 and Lemma 5.10, possibly along some subsequence, we find that (5.12) and (5.13) hold. As $u \gg 0$, we also have, by Proposition 5.11, that u is decreasing. Hence, we find

$$\begin{aligned}
\operatorname{ess\,sup} u - \operatorname{ess\,inf} u &= \int_0^1 |Du| = \lim_{n\to+\infty} \int_0^1 |Du_n| \\
&\geq \liminf_{n\to+\infty} (\operatorname{ess\,sup} u_n - \operatorname{ess\,inf} u_n) \\
&\geq \liminf_{n\to+\infty} (\operatorname{ess\,sup} u_n) - \limsup_{n\to+\infty} (\operatorname{ess\,inf} u_n) \\
&\geq \liminf_{n\to+\infty} (\operatorname{ess\,sup} u_n) \geq \operatorname{ess\,sup} u,
\end{aligned}$$

which is impossible because $\operatorname{ess\,inf} u > 0$. This contradiction shows that $\mathcal{C}^+_{\lambda_0} = \mathcal{C}^>_{\lambda_0}$. The proof of our claim is therefore complete.

Therefore, we have proved that, in all circumstances, $\mathcal{C}^>_{\lambda_0}$ is a connected component of $\mathcal{S}^>$, unbounded in $\mathbb{R} \times L^p(0,1)$, as claimed by Theorem 5.13. Actually, a slightly stronger conclusion holds: $\mathcal{C}^>_{\lambda_0}$ is a connected subset of $BV(0,1)$, endowed with the topology of the strict convergence. Indeed, otherwise we could partition $\mathcal{C}^>_{\lambda_0}$ into two disjoint subsets, closed in $\mathbb{R} \times BV(0,1)$ with respect to the topology of the strict convergence. Since, by Corollary 5.9, these sets should be closed in $\mathbb{R} \times L^p(0,1)$ as well, a contradiction would follow.

Part 2. Bifurcation from $(0,0)$: existence and properties of $\mathcal{C}^>_0$.
The proof of Part 1 can be adapted, with some simplifications, to construct $\mathcal{C}^>_0$. Therefore we will omit some details of such construction, not to be repetitive. Indeed, in this case we can define $\mathcal{C}^>_0$ as the component of $\mathcal{S}^>$ such that $(0,0) \in \mathcal{C}^>_0$. Since $\mathcal{S}^>$ contains the vertical half-line $\{(0,r) : r \in [0,+\infty)\}$, we see that $\mathcal{C}^>_0$ is unbounded in $\mathbb{R} \times L^p(0,1)$. Next, Proposition 5.11 and Remark 5.2 guarantee that if $(\lambda, u) \in \mathcal{C}^>_0$ and $u \neq 0$, then $u \gg 0$. Further, Lemma 5.3 implies that if $(\lambda, 0) \in \mathcal{C}^>_0$ for some $\lambda > 0$, then $\lambda = \lambda_0$, because λ_0 is the only positive eigenvalue of (4.31) with positive eigenfunctions. Finally, Corollary 3.5 shows that there exists a neighborhood U of $(0,0)$ in $\mathbb{R} \times L^p(0,1)$ such that $\mathcal{C}^>_0 \cap U$ consists of strong solutions. Exactly as in Part 1, we also see that $\mathcal{C}^>_{\lambda_0}$ is a connected subset of $BV(0,1)$, endowed with the topology of the strict convergence.

Finally, the maximality and the connectedness of both $\mathcal{C}^>_0$ and $\mathcal{C}^>_{\lambda_0}$ yield the following alternative: either $\mathcal{C}^>_0 \cap \mathcal{C}^>_{\lambda_0} = \emptyset$, or $(\lambda_0, 0) \in \mathcal{C}^>_0$ and $(0,0) \in \mathcal{C}^>_{\lambda_0}$ and, in such case, $\mathcal{C}^>_0 = \mathcal{C}^>_{\lambda_0}$. This ends the proof of Theorem 5.13. □

We conclude this section, and this paper, remarking that, under an additional regularity condition on f, some further information can be obtained about the fine structure of the connected components $\mathcal{C}^>_0$ and $\mathcal{C}^+_{\lambda_0}$ near their respective bifurcation points from the trivial line. More precisely, the next result follows easily by combining Corollary 3.5 with Theorem 5.13 and the analysis already done in [44, Section 4]. As they can be easily reproduced, the technical details of its proof are omitted here.

Theorem 5.14. *Assume* (a_1), (f_1), *and*

(f_3) *there are* $\ell \geq 2$ *and* $\eta > 0$ *such that* $f \in C^\ell(-\eta, \eta)$.

Then, there exists a neighborhood U *of* $(0,0)$ *in* $\mathbb{R}\times L^p(0,1)$ *such that if* $(\lambda,u)\in U$ *is a bounded variation solution of* (1.1)*, then either* $u=0$*, or* $\lambda=0$ *and* $u=r$ *for some* $r\in\mathbb{R}\setminus\{0\}$*. In particular, there is* $r_0>0$ *such that* $\mathcal{C}_0^{>}\cap U$ *consist of* $\{(0,r):r\in[0,r_0)\}$*.*

Furthermore, there exist a neighborhood V *of* $(\lambda_0,0)$ *in* $\mathbb{R}\times L^p(0,1)$*,* $\varepsilon>0$ *and two maps of class* $C^{\ell-1}$*,*

$$\lambda:(-\varepsilon,\varepsilon)\to\mathbb{R},\qquad z:(-\varepsilon,\varepsilon)\to Z,$$

where

$$Z=\Big\{z\in C^1[0,1]:z'(0)=z'(1)=0,\ \int_0^1 z\,\varphi_0\,dx=0\Big\}$$

is endowed with the topology of $\mathbb{R}\times C^1[0,1]$*, such that*

- $\lambda(0)=\lambda_0$ *and* $z(0)=0$*;*
- $\big(\lambda(s),s(\varphi_0+z(s))\big)$ *is a strong solution of* (1.1) *for all* $s\in(-\varepsilon,\varepsilon)$*;*
- *if* $(\lambda,u)\in V$ *is a bounded variation solution of* (1.1)*, then either* $u=0$*, or* $\lambda=\lambda(s)$ *and* $u=s(\varphi_0+z(s))$ *for some* $s\in(-\varepsilon,\varepsilon)$*; in particular,* $\mathcal{C}_{\lambda_0}^{>}\cap V$ *is precisely the curve* $\big(\lambda(s),s(\varphi_0+z(s))\big)$ *with* $s\in[0,\varepsilon)$*.*

Finally, the bifurcation at λ_0 *is transcritical if* $f''(0)\neq0$*; in particular, the bifurcation of positive solutions is supercritical if* $f''(0)<0$ *and subcritical if* $f''(0)>0$*. Suppose, further, that* $\ell\geq3$ *in* (f_3)*. Then, a subcritical pitchfork bifurcation occurs at* λ_0 *if* $f''(0)=0$*.*

6 Conclusions, conjectures and open questions

In this paper the topological structure of the set of positive solutions of the one-dimensional quasilinear indefinite Neumann problem (1.1) has been analyzed in the special case when $f(0)=0$ and $f'(0)=1$. For the first time in the literature, a unilateral bifurcation theorem in the space of bounded variation functions has been established for an elliptic problem driven by the mean curvature operator. According to it, there exist two global connected components of the set of positive solutions emanating from the line of the trivial solutions at the two principal eigenvalues of the linearized problem around 0.

As already predicted by the analysis carried out in [44, Section 8], the solutions on these components are regular as long as they are sufficiently small, while they may develop jump singularities at the nodes of the weight function, a, as they become sufficiently large. Thus, we have established, in the general setting of this paper, the existence of components consisting, simultaneously, of regular and singular solutions, which might be a breakthrough in "global bifurcation theory"as applied to study more general quasilinear equations and systems. However, a number of important questions still remain open that fall outside the general scope of this paper, but deserve some further effort to gain insight into the problem of ascertaining the fine structure of the bounded variation solutions of (1.1). A very relevant one consists in clarifying the hidden relationships between the regular and the singular solutions of (1.1), with special attention towards the problem of understanding the precise mechanisms generating the formation of jump singularities along the λ-paths of regular solutions. We have a strong heuristic evidence that the local regularity of the weight function a at its nodes should play a significant role to describe the transition from regular to singular solutions, i.e., in explaining the underlying formation of singularities on the small regular solutions.

Nevertheless, in some particular, but pivotal, examples we already know that the global bifurcation diagram of bounded variation solutions looks like shows Figure 1. Namely, when the associated potential $F(s)=\int_0^s f(t)\,dt$ of f is superlinear at infinity, then the component of positive bounded variation solutions $\mathcal{C}_{\lambda_0}^{>}$ bifurcating from $(\lambda,0)$ at $\lambda=\lambda_0$ looses the a priori bounds in $C^1[0,1]$ at some $\lambda^*>0$, where the solutions become singular and fill in a subcontinuum consisting of singular bounded variation solutions bifurcating from infinity at $\lambda = 0$. Instead, when the potential F is sublinear at infinity, then the component $\mathcal{C}^{>}_{\lambda_0}$ remains separated away from the vertical line $\mathbb{R} \times \{0\}$ and looses the a priori bounds in $C^1[0,1]$ at some $\lambda_* > 0$, where it links another unbounded subcontinuum of singular bounded variation solutions whose λ-projection contains $(\lambda^*, +\infty)$. We conjecture that, actually, these are the only admissible global bifurcation diagrams under the assumptions of Theorem 5.13, at least, topologically, in the sense that the underlying global bifurcation diagrams should be homeomorphic to those shown by Figure 1, though the number of solutions of (1.1) for a fixed value of λ on the component $\mathcal{C}^{>}_{\lambda_0}$ might be arbitrarily large according, e.g., to the number of interior nodes and the relative size of the weight a on each of the nodal subintervals.

For simplicity, here we have restricted ourselves to deal with the simplest situation when the function a possesses a single interior node z, and thus the positive solutions of (1.1) are monotone. As our proof of Theorem 5.13 relies, on a pivotal basis, on this special feature, getting a proof of this theorem in the general case when a has an intricate nodal behavior might be a real challenge plenty of technical difficulties. Nevertheless, in spite of these technical troubles, we still conjecture the validity of Theorem 5.13, at least, under the assumptions imposed to the weight a in Corollary 3.7. The validity of Theorem 5.13 in more general settings remains therefore an open problem here.

A further challenge, of a rather different vein, consists in describing the precise asymptotic profile of the bounded variation solutions of (1.1) as $\lambda \to 0$, or $\lambda \to +\infty$, according to the behavior of the associated potential F at infinity. In some particular cases of interest, we already know that the derivatives of the solutions of (1.1) approximate, asymptotically, the profile of the solution of the problem

$$\begin{cases} -\left(\dfrac{v}{\sqrt{1+v^2}}\right)' = b(x) & \text{in } (0,1), \\ v(0)=0, \ v(1)=0, \end{cases}$$

where

$$b(x) = \begin{cases} \dfrac{a(x)}{\int_0^z a(t)\,dt} & \text{in } (0,z), \\ \dfrac{-a(x)}{\int_z^1 a(t)\,dt} & \text{in } (z,1), \end{cases}$$

and z is the unique interior node of the function a. This feature should be relevant to establish in various cases the non-existence of positive regular solutions of (1.1); however this analysis, being outside the scope of this paper, is postponed here and will be carried out elsewhere.

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