On stability of small solitons of the 1–D NLS with a trapping delta potential

Scipio Cuccagna, Masaya Maeda

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Abstract

We consider a Nonlinear Schrödinger Equation (NLS) with a very general nonlinear term and with a trapping δ potential on the line. We then discuss the asymptotic behavior of all its small solutions, generalizing a recent result by Masaki *et al.* [33] by means of virial–like inequalities. We give also a result of dispersion in the case of defocusing equations with a non–trapping delta potential.

1 Introduction

In this paper we consider the Nonlinear Schrödinger Equation (NLS):

$$\mathrm{i}\dot{u} = H_1 u + g(|u|^2)u, \quad (t,x) \in \mathbb{R}\times\mathbb{R}, \text{with } u(0) = u_0 \in H^1(\mathbb{R},\mathbb{C}), \tag{1.1}$$

with the Schrödinger operator (here $\delta(x)$ is the Dirac δ centered in 0)

$$H_q = -\partial_x^2 - q\delta(x) \text{ for } q \in \mathbb{R}\backslash\{0\} \tag{1.2}$$

defined by $H_q := -\partial_x^2$ with domain

$$D(H_q) = \{u \in H^1(\mathbb{R},\mathbb{C}) \cap H^2(\mathbb{R}\setminus\{0\},\mathbb{C}) \mid \partial_x u(0^+) - \partial_x u(0^-) = -qu(0)\}. \tag{1.3}$$

For the nonlinearity, we assume $g \in C([0,\infty),\mathbb{R}) \cap C^3((0,\infty),\mathbb{R})$ and that there exist $p > 0$ and $C > 0$ such that for $k = 0, 1, 2, 3$ we have

$$|g^{(k)}(s)| \le C|s|^{p-k} \text{ for all } s \in (0,1]. \tag{1.4}$$

In particular, we have $g(0) = 0$ and the primitive G of g defined by

$$G'(s) = g(s) \text{ and } G(0) = 0. \tag{1.5}$$

satisfies $|G(s)| \lesssim |s|^{p+1}$ for all $s \in (0,1)$.

Remark 1.1. A typical example we have in mind is $g(s) = \lambda s^p$ with $p > 0$ and $\lambda \in \{\pm 1\}$. In this case, our NLS can be written taking $q = 1$ in the form

$$\mathrm{i}\dot{u} = H_q u + \lambda |u|^{2p} u, \tag{1.6}$$

which was considered by Masaki *et al.* [33] for the case $p \ge 2$. They also considered the cubic NLS for the cubic NLS, $p = 1$, with repulsive potential $q < 0$ in [34], where they proved dispersion, that is $\|u(t)\|_{L^\infty(\mathbb{R})} \lesssim t^{-\frac{1}{2}}$ as $t \to +\infty$, for appropriate very small solutions.

We recall, see [33], that the operator in (1.2) for $q>0$ satisfies

$$\sigma_d(H_q)=\{-q^2/4\} \text{ with } \ker\left(H_q+q^2/4\right)=\mathrm{Sp}(\varphi_q) \text{ where } \varphi_q:=\sqrt{q/2}e^{-\frac{q}{2}|x|}\ , \tag{1.7}$$

with $\mathrm{Sp}(\varphi_q):=\mathbb{C}\varphi_q$. Furthermore the point 0 is neither an eigenvalue nor a resonance for H_q, that is to say, the only $u_0\in L^2(\mathbb{R})\cup L^\infty(\mathbb{R})$ such that $H_qu_0=0$ is $u_0=0$.
We also have a spectral (orthogonal) decomposition

$$L^2(\mathbb{R})=\mathrm{Sp}(\varphi_q)\oplus L^2_c(H_q) \tag{1.8}$$

with $L^2_c(H_q)$ the continuous spectrum component associated to H_q. We will consider the case $q=1$ and denote

$$L^2_c:=L^2_c(H_1) \text{ and } \varphi:=\varphi_1.$$

We will denote by P_c the projection onto L^2_c. In particular,

$$P_cu:=u-\left(\int_{\mathbb{R}}u\varphi\,dx\right)\varphi=u-\langle u,\varphi\rangle\varphi-\langle u,\mathrm{i}\varphi\rangle\mathrm{i}\varphi,$$

where

$$\langle f,g\rangle=\mathrm{Re}\int_{\mathbb{R}}f(x)\overline{g}(x)dx \text{ for } f,g:\mathbb{R}\to\mathbb{C}\ . \tag{1.9}$$

We will also use the following notation.

- Given a Banach space X, $v\in X$ and $\varepsilon>0$ we set $D_X(v,\varepsilon):=\{x\in X\mid\|v-x\|_X<\varepsilon\}$.
- For $\gamma\in\mathbb{R}$ we set

$$L^2_\gamma:=\{u\in\mathcal{S}'(\mathbb{R},\mathbb{C})\mid\|u\|_{L^2_\gamma}:=\|e^{\gamma|x|}u\|_{L^2}<\infty\}, \tag{1.10}$$
$$H^1_\gamma:=\{u\in\mathcal{S}'(\mathbb{R},\mathbb{C})\mid\|u\|_{H^1_\gamma}:=\|e^{\gamma|x|}u\|_{H^1}<\infty\}. \tag{1.11}$$

- For $f:\mathbb{C}\to X$ for some Banach space X, we set $D_1f=\partial_{\mathrm{Re}\,z}f$ and $D_2f=\partial_{\mathrm{Im}\,z}f$.

The eigenvalue of H_1 yields by bifurcation a family of standing waves solutions.
As in [6, 15, 33], we have the following, which we prove in the appendix.

Proposition 1.2 (Bound states)**.** *Let $p>0$. Then there exist $\gamma_0>0$, $a_0>0$ and $C>0$ such that there exists a unique $Q\in C^1(D_{\mathbb{C}}(0,a_0),H^1_{\gamma_0})$ satisfying the gauge property*

$$Q[e^{\mathrm{i}\theta}z]=e^{\mathrm{i}\theta}Q[z], \tag{1.12}$$

such that there exists $E\in C([0,a_0^2),\mathbb{R})$ such that

$$H_1Q[z]+g(|Q[z]|^2)Q[z]=E(|z|^2)Q[z], \tag{1.13}$$

and for $j=1,2$,

$$\|Q[z]-z\varphi\|_{H^1_{\gamma_0}}\le C|z|^{2p+1},\ \|D_jQ[z]-\mathrm{i}^{j-1}\varphi\|_{H^1_{\gamma_0}}\le C|z|^{2p},\ \left|E(|z|^2)+\frac{1}{4}\right|\le C|z|^{2p}. \tag{1.14}$$

Moreover, if $p>1/2$ we have $Q[z]\in C^2\left(D_{\mathbb{C}}(0,a_0),H^1_{\gamma_0}\right)$ and

$$\|D_jD_kQ[z]\|_{H^1_{\gamma_0}}\le C|z|^{2p-1},\quad j,k=1,2. \tag{1.15}$$

Remark 1.3. In the case of power type nonlinearities $g(s) = s^p$, there is an explicit formula for $Q[z]$. See [12, 33].

Our first result is the following, related to [15, 33, 36], see [6] for more references.

Theorem 1.4. *Assume $p > 0$ in (1.4). Then there exist $\epsilon_0 > 0$, $\gamma > 0$ and $C > 0$ such that for $\epsilon := \|u(0)\|_{H^1} < \epsilon_0$ the solution $u(t)$ of (1.1) can be written uniquely for all times as*

$$u(t) = Q[z(t)] + \xi(t) \text{ with } \xi(t) \in P_c H^1, \tag{1.16}$$

such that we have

$$|z(t)| + \|\xi(t)\|_{H^1} \le C\epsilon \text{ for all } t \in [0, \infty), \tag{1.17}$$

$$\int_0^\infty \|\xi\|^2_{H^1_{-\gamma}} dt \le C\epsilon^2. \tag{1.18}$$

In [38] it is shown that if $p < 1/2$ and $\xi(t) = e^{\mathrm{i}t\partial_x^2}\xi_+ + o(1)$ in $L^2(\mathbb{R})$ with $o(1) \xrightarrow{t\to\infty} 0$ in $L^2(\mathbb{R})$, then $\xi_+ = 0$. In this paper we do not discuss scattering. Notice that $\int_0^\infty \|e^{\mathrm{i}t\partial_x^2}\xi_+\|^2_{H^1_{-\gamma}} dt = \infty$ for $(1 + |x|^2)\xi_+ \in L^2(\mathbb{R})$ with $\widehat{\xi}_+(0) \neq 0$.

Theorem 1.4 claims that solutions with sufficiently small H^1 norm converge asymptotically to the set formed by the $Q[z]$. Indeed formula (1.18) is stating that, in an averaged sense, $\xi \xrightarrow{t\to\infty} 0$ locally in space. In Theorem 1.4 there is no proof of *selection of ground state*: we do not prove that up to a phase, $z(t)$ has a limit as $t \to +\infty$. However, if we strengthen the hypotheses of the nonlinearity $g(s)$, we obtain also the selection of ground states. This will be our second result. It requires a more subtle representation of $u(t)$ than the one in (1.16), due to Gustafson *et al.* [15].

Definition 1.5. Consider the $a_0 > 0$ in Proposition 1.2.

$$\mathcal{H}_c[z] := \left\{\eta \in L^2(\mathbb{R}) : \langle \mathrm{i}\,\eta, D_1 Q\rangle = \langle \mathrm{i}\,\eta, D_2 Q\rangle = 0\right\}. \tag{1.19}$$

It is immediate that $\mathcal{H}_c[0] = L^2_c$. Our second result is the following.

Theorem 1.6. *Let $p > 1/2$ in (1.4). Then there exist $\epsilon_0 > 0$, $\gamma > 0$ and $C > 0$ such that for $\epsilon := \|u(0)\|_{H^1} < \epsilon_0$ the solution $u(t)$ of (1.1) can be written uniquely for all times as*

$$u(t) = Q[z(t)] + \eta(t) \text{ with } \eta(t) \in \mathcal{H}_c[z(t)], \tag{1.20}$$

such that we have

$$|z(t)| + \|\eta(t)\|_{H^1} \le C\epsilon \text{ for all } t \in [0, \infty), \tag{1.21}$$

$$\int_0^\infty \|\eta\|^2_{H^1_{-\gamma}} dt \le C\epsilon^2. \tag{1.22}$$

and there exists a $z_+ \in \mathbb{C}$ such that

$$\lim_{t\to+\infty} z(t) e^{\mathrm{i}\int_0^t E[z(s)]ds} = z_+. \tag{1.23}$$

We don't know if the last statement, with the limit (1.23), is true for $p \le 1/2$. We will prove Theorem 1.4 in Sect. 2 and Theorem 1.6 in Sect. 3. In Sect. 4 we will also give a very simple proof of the following result for defocusing equations (1.1) with non–trapping δ potential, which to our knowledge is not in the literature.

Theorem 1.7. *Consider equation* (1.1) *with* $q < 0$, $g \geq 0$ *everywhere and* $sg(s) - G(s) \geq 0$ *for any* $s \geq 0$ *for* G *defined in* (1.5). *Then for any* $\gamma > 0$ *there exists a* $C_\gamma > 0$ *such that for any* $u_0 \in H^1(\mathbb{R}, \mathbb{C})$ *the corresponding strong solution* $u(t)$ *satisfies*

$$\int_0^\infty \|u\|^2_{H^1_{-\gamma}} dt < C_\gamma \left((E(u_0)Q(u_0))^{\frac{1}{2}} + Q(u_0) \right). \tag{1.24}$$

Equations like (1.1) and its particular case (1.6) represent an interesting special type of the NLS in 1–D. Related models, obtained eliminating the linear δ potential and replacing $g(|u|^2)u$ with $\sum_{j=1}^n \delta(x - x_j)g(|u|^2)u$, in some cases have been shown to satisfy very satisfactory characterizations of the global time behavior for all their finite energy solutions; see [23]–[26], which solved the *Soliton Risolution Conjecture* in these cases.

Returning to equation (1.1), Goodman *et al.* [14] and Holmer *et al.* [16]–[19] have shown in the cubic case interesting patterns involving solitons, usually for finite intervals of time or numerically. Some of these results have been proved for global times by Deift and Park [7], using the Inverse Scattering Transform. Masaki *et al.* [33] is a transposition to (1.6) of a result similar to Theorem 1.6, but for more regular potentials, by Mizumachi [36]. Similarly, the result in Masaki *et al.* [34] we described under (1.6) transposes to the case of δ potentials work on dispersion of very small solutions for NLS's with a non–trapping and quite regular potential in [9, 13, 39]. Even though [9, 13, 34, 39] are usually motivated by the problem of stability of solitons, currently it is not so clear how to get from them results of the type $\|\xi(t)\|_{L^\infty(\mathbb{R})} \lesssim t^{-\frac{1}{2}}$ as $t \to +\infty$ for the error term $\xi(t)$ in (1.16) or for the $\eta(t)$ in (1.20). Such kind of transformation of results around 0 into results around a soliton exists in the context of the theory of Integrable Systems, where there are appropriate coordinate changes named Bäcklund and Darboux transforms, see for instance Deift and Park [7]. However for non–integrable perturbations of the cubic NLS such coordinate changes represent an open question, c.f.r. the discussion in Mizumachi and Pelinovsky [37].

The main motivation for this paper is then to show the promise of an alternative method, involving positive commutators, which is classical in Quantum Mechanics, see for example Reed and Simon [42, pp. 157–163] and [2, 10]. In the nonlinear setting the method is also classical and has been extensively used to prove dispersion, like for example Morawetz estimates, see [5], or in the analysis of blow up, see for example [35, 40]. The method, which is also referred as *virial inequality*, has been extensively used in the study of KdV–like equations, see [31] and therein, and represents the tool of choice to complete the last step, often referred as *Liouville property*, of the proof in the theory of stabilization developed by Kenig and Merle [20] (a possible alternative, the *energy channel method* of Duyckaerts *et al.* [11], has not been adapted yet to NLS's). In this paper we are inspired by the study in Kowalczyk *et al.* [27, 29] on the stability of various patterns for wave like equations. The main point here, is that this method can be applied rather simply in the proof of Theorems 1.4, 1.6 and 1.7.

The main feature of the method, and the reason of its robustness, is that a positive commutator, which is linear, allows to ignore small nonlinear terms, which are much smaller. Even large nonlinear defocusing terms are not an obstruction to the method, as we will se for Theorem 1.7.

The most insidious problem of the method comes from the fact that the commutators often have some negative eigenvalues. An important part of the proof in papers such as [27] consists in showing that analogues of the ξ in (1.16) or of the η in (1.20), live where the commutator is positive. If in (1.1) we replace the δ potential with a more regular one, this step appears to be mostly open. See [47] for related problems. In the case of a δ potential in 1–D, we show that this is easy to solve (see also Banica and Visciglia [3], Ikeda and Inui [30] and Richard [43]; for a different but similarly favorable set up we refer to [28, 32]). This allows us to cover with a rather simple proof cases outside the reach of the theory in Masaki *et al.* [33], where the use of Strichartz estimates restricts consideration to $p \geq 2$. In this sense we go beyond the results for more regular potentials considered by Mizumachi [36], in turn related to [15, 41, 44, 45]. In some of these older papers there is a clear interest at obtaining the largest possible set of values for the exponent p. However they are severely restricted by their dependence on dispersive and/or Strichartz estimates, not always sufficiently robust in nonlinear settings, see [46]. The commutator method can be more robust, as results such as [27, 29] show. In the literature some partial stability results have been obtained for mass subcritical nonlinearities in 2–D and 3–D by Kirr *et al.* [21, 22] using dispersive estimates. But exactly like for the dispersion results on small solutions of cubic NLS's with potentials in [9, 13, 34, 39] or for the theory initiated by Deift and Zhou [8] on the Scattering Transform in non–Integrable Systems, the ultimate test will be how pliable, widely utilizable and not too technically complicated will they be. Our point here is that the theory in [27, 29] seems the most promising.

2 Proof of Theorem 1.4

2.1 Notation and coordinates

We have the following ansatz, which is an elementary consequence of the Implicit Function Theorem.

Lemma 2.1. *There exist $c_0 > 0$ and $C > 0$ such that for all $u \in H^1$ with $\|u\|_{H^1} < c_0$, there exists a unique pair $(z, \xi) \in \mathbb{C} \times P_c H^1$ such that*

$$u = Q[z] + \xi \text{ with } |z| + \|\xi\|_{H^1} \leq C\|u\|_{H^1}. \tag{2.1}$$

The map $u \to (z, \xi)$ is in $C^1(D_{H^1}(0, c_0), \mathbb{C} \times H^1)$.

Proof. Set

$$F(z, u) := \begin{pmatrix} \langle u - Q[z], \varphi \rangle \\ \langle u - Q[z], \mathrm{i}\varphi \rangle \end{pmatrix}.$$

Then, by Proposition 1.2, we see that $F \in C^1(D_{\mathbb{C}}(0, a_0) \times H^1, \mathbb{R}^2)$ and moreover

$$\left.\frac{\partial F}{\partial(z_R, z_I)}\right|_{(z,u)=(0,0)} = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.2}$$

Therefore, by implicit function theorem, we have the conclusion. □

For $z, w \in \mathbb{C}$, we will use the notation

$$DQ[z]w := \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} Q[z + \varepsilon w]. \tag{2.3}$$

Notice that by $Q[e^{\mathrm{i}\theta} z] = e^{\mathrm{i}\theta} Q[z]$, we have

$$\mathrm{i}Q[z] = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} e^{\mathrm{i}\varepsilon} Q[z] = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} Q[e^{\mathrm{i}\varepsilon} z] = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} Q[z + \varepsilon \mathrm{i} z] = DQ[z]\mathrm{i}z.$$

Further, for $w \in \mathbb{C}$, we set

$$f(w) := g(|w|^2)w. \tag{2.4}$$

The well posedness of problem (1.1) in $H^1(\mathbb{R})$ is considered by Goodman *et al.* [14], Fukuizumi *et al.* [12] and [1]. The energy and mass conservation imply the global well posedness of our small solutions, with representation (1.16) valid for all times along with the bound (1.17). So we can write the equation (1.1) in terms of the ansatz (1.16) and obtain the system:

$$\left\langle \mathrm{i}DQ[z](\dot{z}+\mathrm{i}Ez), \mathrm{i}^{j-1}\varphi \right\rangle = \left\langle f(\xi)+\tilde{f}(z,\xi), \mathrm{i}^{j-1}\varphi \right\rangle \text{ for } j=1,2, \tag{2.5}$$

$$\mathrm{i}\dot{\xi} = H_1\xi + f(\xi) + \tilde{f}(z,\xi) - \mathrm{i}DQ[z](\dot{z}+\mathrm{i}Ez), \tag{2.6}$$

where

$$\tilde{f}(z,\xi) := f(Q[z]+\xi) - f(\xi) - f(Q[z]) \tag{2.7}$$

In order to prove the estimate (1.18) of Theorem 1.4, we will use the method considered by Kowalczyk *et al.* in [27] and in their very recent paper [29].

2.2 The commutator method

Following [29], we introduce an even smooth function $\chi : \mathbb{R} \to [-1,1]$ such that

$$\chi = 1 \text{ in } [-1,1],\ \chi = 0 \text{ in } \mathbb{R}\setminus[-2,2], \chi' \leq 0 \text{ in } \mathbb{R}_+. \tag{2.8}$$

For $A \gg 1$ large enough which will be fixed later, we set

$$\zeta_A(x) := \exp\left(-\frac{|x|}{A}(1-\chi(x))\right) \text{ and } \psi_A(x) = \int_0^x \zeta_A^2(t)dt. \tag{2.9}$$

One can easily verify

$$e^{-\frac{|x|}{A}} \leq \zeta_A(x) \leq 2e^{-\frac{|x|}{A}} \text{ and } |\psi_A(x)| \leq 2A \text{ for } A \geq 4. \tag{2.10}$$

To each function $\xi \in H^1(\mathbb{R})$ we can associate

$$w := \zeta_A \xi. \tag{2.11}$$

Notice that there exist fixed constants C and A_0 such that

$$\left\langle (-\partial_x^2+\delta)w, w \right\rangle \leq C\|\xi\|_{H^1}^2 \text{ for all } A \geq A_0. \tag{2.12}$$

Also, we have

$$\|w'\|_{L^2}^2 + \|\langle x\rangle^{-2} w\|_{L^2}^2 \leq C\left\langle (-\partial_x^2+\delta)w, w \right\rangle, \tag{2.13}$$

where $\langle x\rangle := (1+|x|^2)^{1/2}$ and $C = 12$. To prove (2.13), it suffices to bound the second term. Since

$$w(x) = w(0) + \int_0^x w'(s)\,ds,$$

from Hölder inequality we have

$$|w(x)| \leq |w(0)| + |x|^{1/2}\|w'\|_{L^2} \leq 2\langle x\rangle^{1/2}\left\langle (-\partial_x^2+\delta)w, w \right\rangle^{\frac{1}{2}}. \tag{2.14}$$

Thus, we have

$$\| \langle x\rangle^{-2} w(x)\|_{L^2}^2 \le 4 \int_{\mathbb{R}} \langle x\rangle^{-3} \, dx \left\langle(-\partial_x^2+\delta)w, w\right\rangle \le 12 \left\langle(-\partial_x^2+\delta)w, w\right\rangle .$$

We consider now the quadratic form

$$\mathcal{J}(\xi) := 2^{-1}\left\langle \mathrm{i}\xi, \left(\frac{\psi_A'}{2}+\psi_A\partial_x\right)\xi\right\rangle . \tag{2.15}$$

From (2.10), one obtains the upper bound

$$|\mathcal{J}(\xi)| \le 2A\|\xi\|_{H^1}^2 \text{ for } A\ge 4. \tag{2.16}$$

By the well posedness of (1.1) we can consider

$$\xi \in C^0(\mathbb{R}, D(H_1))\cap C^1(\mathbb{R}, L^2(\mathbb{R},\mathbb{C})) \subset C^0(\mathbb{R}, H^1(\mathbb{R},\mathbb{C}))\cap C^1(\mathbb{R}, L^2(\mathbb{R},\mathbb{C})) =: Y.$$

We claim that $\mathcal{J}(\xi)\in C^1(\mathbb{R},\mathbb{R})$ with

$$\frac{d}{dt}\mathcal{J}(\xi) = \left\langle \mathrm{i}\dot{\xi}, \left(\frac{\psi_A'}{2}+\psi_A\partial_x\right)\xi\right\rangle . \tag{2.17}$$

Indeed we can consider a sequence $\{\xi_n\}$ in $C^0(\mathbb{R}, H^2(\mathbb{R},\mathbb{C}))\cap C^1(\mathbb{R}, H^2(\mathbb{R},\mathbb{C}))$ converging to ξ in Y uniformly for t on compact sets. The functions $\mathcal{J}(\xi_n)$ belong to $C^1(\mathbb{R},\mathbb{R})$ and their derivatives satisfy (2.17) with ξ replaced by ξ_n. From this formula we derive that the sequence $\{\frac{d}{dt}\mathcal{J}(\xi_n)\}$ converges uniformly on compact sets to the r.h.s. of (2.17). Since $\mathcal{J}(\xi_n) \xrightarrow{n\to\infty} \mathcal{J}(\xi)$ uniformly on compact sets, we conclude that $\mathcal{J}(\xi)\in C^1(\mathbb{R},\mathbb{R})$ and that formula (2.17) is correct.
From (2.17) we obtain

$$\begin{aligned}&\frac{d}{dt}\mathcal{J}(\xi) + \left\langle \mathrm{i}DQ(\dot{z}+\mathrm{i}Ez), \left(\frac{\psi_A'}{2}+\psi_A\partial_x\right)\xi\right\rangle \\ &\quad= \left\langle H_1\xi, \left(\frac{\psi_A'}{2}+\psi_A\partial_x\right)\xi\right\rangle + \left\langle f(\xi), \left(\frac{\psi_A'}{2}+\psi_A\partial_x\right)\xi\right\rangle + \left\langle \tilde{f}(z,\xi), \left(\frac{\psi_A'}{2}+\psi_A\partial_x\right)\xi\right\rangle .\end{aligned} \tag{2.18}$$

The main result of this subsection is the following.

Proposition 2.2. *There exist values $1\gg a_0>0$ and $A\gg 1$ such that for $\xi\in P_cH^1$ and $|z|+\|\xi\|_{H^1}<a_0$ we have*

$$\text{r.h.s. of } (2.18) \ge \frac{1}{12}\left(\|w'\|_{L^2}^2 + \|\langle x\rangle^{-2} w\|_{L^2}^2\right) \text{ for } w=\zeta_A\xi. \tag{2.19}$$

The rest of this subsection is devoted to the proof of Proposition 2.2.
The key and single most important term is the quadratic form singled out in the following lemma, see [27, 29].

Lemma 2.3. *For $w=\zeta_A\xi$ we have the equality*

$$\left\langle \left(\frac{\psi_A'}{2}+\psi_A\partial_x\right)\xi, H_1\xi\right\rangle = \left\langle H_{\frac{1}{2}}w, w\right\rangle + \frac{1}{2A}\langle Vw, w\rangle \text{ for } V(x) := \chi''(x)\ |x| + 2\chi'(x)\ \frac{x}{|x|}. \tag{2.20}$$

Proof. Integrating by parts, see Corollary 8.10 [4], we obtain

$$\left\langle \left(\frac{\psi_A'}{2} + \psi_A \partial_x \right) \xi, H_1 \xi \right\rangle = -\frac{1}{2} \operatorname{Re} \int_{\mathbb{R}} \psi_A' \xi \overline{\xi}'' dx - \frac{1}{2} \int_{\mathbb{R}} \psi_A (|\xi'|^2)' dx$$

$$= \int_{\mathbb{R}} \psi_A' |\xi'|^2 dx + \frac{1}{4} \int_{\mathbb{R}} \psi_A'' (|\xi|^2)' dx + \frac{1}{2} \operatorname{Re} \left(\psi_A'(0) \xi(0) \left(\overline{\xi}'(0^+) - \overline{\xi}'(0^-) \right) \right)$$

$$= \langle \psi_A' \xi', \xi' \rangle - \frac{1}{4} \langle \psi_A''' \xi, \xi \rangle - \frac{1}{2} \langle \delta \xi, \xi \rangle , \tag{2.21}$$

where we used $\xi'(0^+) - \xi'(0^-) = -\xi(0)$, $\psi_A(0) = \psi_A''(0) = 0$ and $\psi_A'(0) = 1$.
For the first term in the r.h.s. of (2.21), we have

$$\langle \psi_A' \xi', \xi' \rangle = \left\langle \zeta_A^2 \left(\frac{w}{\zeta_A} \right)', \left(\frac{w}{\zeta_A} \right)' \right\rangle = \left\langle w' - \frac{\zeta_A'}{\zeta_A} w, w' - \frac{\zeta_A'}{\zeta_A} w \right\rangle$$

$$= \langle w', w' \rangle + \left\langle \left(\frac{\zeta_A'}{\zeta_A} \right)^2 w, w \right\rangle - 2 \left\langle w', \frac{\zeta_A'}{\zeta_A} w \right\rangle$$

$$= \langle w', w' \rangle + \left\langle \left(\left(\frac{\zeta_A'}{\zeta_A} \right)' + \left(\frac{\zeta_A'}{\zeta_A} \right)^2 \right) w, w \right\rangle = \langle w', w' \rangle + \left\langle \frac{\zeta_A''}{\zeta_A} w, w \right\rangle ,$$

and for the second term we have

$$-\frac{1}{4} \langle \psi_A''' \xi, \xi \rangle = -\frac{1}{4} \left\langle \frac{(\zeta_A^2)''}{\zeta_A^2} w, w \right\rangle = -\frac{1}{2} \left\langle \left(\frac{\zeta_A''}{\zeta_A} + \left(\frac{\zeta_A'}{\zeta_A} \right)^2 \right) w, w \right\rangle .$$

Summing up we obtain

$$\left\langle \left(\frac{\psi_A'}{2} + \psi_A \partial_x \right) \xi, H_1 \xi \right\rangle = \left\langle H_{\frac{1}{2}} w, w \right\rangle + \frac{1}{2} \left\langle \left(\frac{\zeta_A''}{\zeta_A} - \left(\frac{\zeta_A'}{\zeta_A} \right)^2 \right) w, w \right\rangle . \tag{2.22}$$

Finally, from

$$\zeta_A' = \frac{1}{A} \left(\chi'(x)|x| + (\chi(x) - 1) \frac{x}{|x|} \right) \zeta_A \quad \text{and}$$

$$\zeta_A'' = \frac{1}{A^2} \left(\chi'(x)|x| + (\chi(x) - 1) \frac{x}{|x|} \right)^2 \zeta_A + \frac{1}{A} \left(\chi''(x)|x| + 2\chi'(x) \frac{x}{|x|} \right) \zeta_A,$$

we conclude

$$A \left(\frac{\zeta_A''}{\zeta_A} - \left(\frac{\zeta_A'}{\zeta_A} \right)^2 \right) = \chi''(x)|x| + 2\chi'(x) \frac{x}{|x|} = V(x). \tag{2.23}$$

Substituting (2.23) into (2.22) we obtain (2.20). □

The main step in the proof of Proposition 2.2 is the following lemma.

Lemma 2.4. *There exist a fixed constant $A_0 > 1$ such that for $A \geq A_0$ we have*

$$\left\langle \left(\frac{\psi_A'}{2} + \psi_A \partial_x \right) \xi, H_1 \xi \right\rangle \geq \frac{1}{5} \left\langle (-\partial_x^2 + \delta) w, w \right\rangle \ \text{for } w = \zeta_A \xi \text{ and all } \xi \in \mathcal{H}_c[0]. \tag{2.24}$$

Proof. We use formula (2.20) singling out the the 1st term in the r.h.s. and writing it as

$$\left\langle H_{\frac{1}{2}} w, w \right\rangle = \frac{1}{4} \left\langle \left(-\partial_x^2 + \delta\right) w, w \right\rangle + \frac{3}{4} \left\langle H_1 w, w \right\rangle .$$

We now make the following two claims.

Claim 2.5. There exist $A_0 > 0$ and $C_0 > 0$ such that for $A \geq A_0$ we have

$$\frac{1}{2A} \| \langle Vw, w \rangle \| \leq \frac{C_0}{A} \left\langle (-\partial_x^2 + \delta) w, w \right\rangle \text{ for all } w \in H^1. \tag{2.25}$$

Claim 2.6. There exist $A_0 > 1$ and $C_0 > 0$ such that for $A \geq A_0$ we have

$$\langle H_1 w, w \rangle \geq -\frac{C_0}{A^2} \left\langle (-\partial_x^2 + \delta) w, w \right\rangle \text{ for all } w = \zeta_A \xi \text{ with } \xi \in L_c^2. \tag{2.26}$$

Let us assume Claims 2.5 and 2.6. Then we conclude

$$\begin{aligned} \left\langle \left(\frac{\psi_A'}{2} + \psi_A \partial_x \right) \xi, H_1 \xi \right\rangle &= \left\langle H_{\frac{1}{2}} w, w \right\rangle + \frac{1}{2A} \langle Vw, w \rangle \\ &= \frac{1}{4} \left\langle \left(-\partial_x^2 + \delta\right) w, w \right\rangle + \frac{3}{4} \langle H_1 w, w \rangle + \frac{1}{2A} \langle Vw, w \rangle \\ &\geq \left(\frac{1}{4} - C_0 \left(\frac{3}{4} A^{-2} + A^{-1} \right) \right) \left\langle \left(-\partial_x^2 + \delta\right) w, w \right\rangle \end{aligned}$$

which yields immediately (2.24). This, up to the proof of Claims 2.5 and 2.6, completes the proof of Lemma 2.4. □

Proof of Claim 2.5. By multiplying $V(x)$ to the square of (2.14) and integrating, we obtain

$$\frac{1}{2A} \| \langle Vw, w \rangle \| \leq \frac{2}{A} \int_{\mathbb{R}} |V(x)| \langle x \rangle \, dx \left\langle (-\partial_x^2 + \delta) w, w \right\rangle \text{ for all } w \in H^1.$$

□

Proof of Claim 2.6. Since $w = \left(\int_{\mathbb{R}} w \varphi_1 \, dx \right) \varphi_1 + P_c w$ and $\langle H_1 P_c w, P_c w \rangle \geq 0$, we have

$$\langle H_1 w, w \rangle = -\frac{1}{4} \left| \int_{\mathbb{R}} w \varphi_1 dx \right|^2 + \langle H_1 P_c w, P_c w \rangle \geq -\frac{1}{4} \left| \int_{\mathbb{R}} w \varphi_1 dx \right|^2 \text{ for all } w \in H^1.$$

On the other hand, for w as in (2.26) we have

$$\int_{\mathbb{R}} w \varphi_1 dx = \int_{\mathbb{R}} \xi \zeta_A \varphi_1 \, dx = \int_{\mathbb{R}} \xi \left(\zeta_A \varphi_1 - \varphi_1 \right) \, dx = -\int_{\mathbb{R}} w \varphi_1 \left(\frac{1}{\zeta_A} - 1 \right) \, dx.$$

Since $e^{-\frac{|x|}{A}} \leq \zeta_A(x) \leq 1$ and $e^{|x|} - 1 \leq |x| e^{|x|}$ for all $x \in \mathbb{R}$, for $A \geq 4$

$$\begin{aligned} \left| \int_{\mathbb{R}} w \varphi_1 dx \right| &\leq \int_{\mathbb{R}} |w| \varphi_1 \left(\frac{1}{\zeta_A} - 1 \right) dx \leq \frac{1}{\sqrt{2}} \int_{\mathbb{R}} |w| e^{-\frac{|x|}{2}} \left(e^{\frac{|x|}{A}} - 1 \right) dx \\ &\leq \frac{1}{\sqrt{2} A} \int_{\mathbb{R}} |w| e^{-\frac{|x|}{2} + \frac{|x|}{A}} |x| dx \leq \frac{1}{\sqrt{2} A} \int_{\mathbb{R}} |w| e^{-\frac{|x|}{4}} |x| dx. \end{aligned}$$

Furthermore, by (2.14) we obtain the following, which immediately leads to the lower bound (2.26):

$$\frac{1}{\sqrt{2} A} \int_{\mathbb{R}} |w| e^{-\frac{|x|}{4}} |x| dx \leq \frac{\sqrt{2}}{A} \int_{\mathbb{R}} \langle x \rangle^{3/2} e^{-\frac{|x|}{4}} \, dx \left\langle (-\partial_x^2 + \delta) w, w \right\rangle^{1/2} .$$

□

By Lemma 2.4 we have found a lower bound on the the 1st term in the r.h.s. of (2.18).

We now examine the contribution to (2.19) of the term with $f(\xi) = g(|\xi|^2)\xi$.

Lemma 2.7. *For any $\varepsilon_0 > 0$ and $A \gg 1$ there exists $a_0 > 0$ such that for $\|\xi\|_{H^1} \le a_0$ we have*

$$\left| \left\langle \left(\frac{\psi_A'}{2} + \psi_A \partial_x \right) \xi, f(\xi) \right\rangle \right| \le \varepsilon_0 \|w'\|_{L^2}^2 \text{ for } w = \zeta_A \xi \ . \tag{2.27}$$

Proof. We follow [27, 29]. Recall that $f(\xi) = g(|\xi|^2)\xi$. Consider the G in (1.5). Then, we have

$$\langle \psi_A \partial_x \xi, g(|\xi|^2)\xi \rangle = \mathrm{Re} \int_{\mathbb{R}} \psi_A g(|\xi|^2) \bar{\xi} \partial_x \xi \, dx = \frac{1}{2} \int_{\mathbb{R}} \psi_A \partial_x G(|\xi|^2) \, dx = -\frac{1}{2} \int_{\mathbb{R}} \psi_A' G(|\xi|^2) \, dx.$$

Thus, by $\psi_A' = \zeta_A^2$, we have

$$\left| \left\langle \left(\frac{\psi_A'}{2} + \psi_A \partial_x \right) \xi, f(\xi) \right\rangle \right| \le \int_{\mathbb{R}} \zeta_A^2 \left(|g(|\xi|^2)|\xi|^2| + 2^{-1} |G(|\xi|^2)| \right) \, dx \le C \int_{\mathbb{R}} \zeta_A^2 |\xi|^{2(p+1)} \, dx.$$

Let $q = \frac{2p}{3} > 0$. Then, by the embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$, we have

$$\int_{\mathbb{R}} \zeta_A^2 |\xi|^{2(p+1)} \, dx \lesssim \|\xi\|_{H^1}^q \int_{\mathbb{R}} \zeta_A^2 |\xi|^{2(p+1)-q} dx.$$

Therefore, it suffices to prove

$$\int_{\mathbb{R}} \zeta_A^2 |\xi|^{2(p+1)-q} \, dx \lesssim \|w'\|_{L^2}^2. \tag{2.28}$$

Following p. 793 [27], since $2(2(p-q)+1) = 2(p+1) - q$,

$$\begin{aligned}
\int_{\mathbb{R}} \zeta_A^2 |\xi|^{2(p+1)-q} dx &= \int_{\mathbb{R}} \zeta_A^{-(2p-q)} |w|^{2(p+1)-q} dx \\
&\lesssim \int_0^\infty e^{\frac{2p-q}{A}x} |w|^{2(p+1)-q} dx + \int_{-\infty}^0 e^{-\frac{2p-q}{A}x} |w|^{2(p+1)-q} dx \\
&\le -\frac{2A}{2p-q} |w(0)|^{2(p+1)-q} + \frac{A}{2p-q} \int_{\mathbb{R}} e^{\frac{2p-q}{A}|x|} \left| (|w|^{2(p+1)-q})' \right| \, dx \\
&\lesssim \frac{A}{p} \int_{\mathbb{R}} \zeta_A^{-2p+q} |w|^{2p-q+1} |w'| \, dx = \frac{A}{p} \int_{\mathbb{R}} \zeta_A |\xi|^{2p-q+1} |w'| \, dx \\
&\le \frac{A}{p} \|\xi\|_{H^1}^q \int_{\mathbb{R}} \zeta_A |\xi|^{2(p-q)+1} |w'| \, dx \\
&\le \frac{A}{p} \|\xi\|_{H^1}^q \left(\int_{\mathbb{R}} \zeta_A^2 |\xi|^{2(p+1)-q} \, dx \right)^{\frac{1}{2}} \|w'\|_{L^2} \\
&\le \|w'\|_{L^2}^2 + \frac{1}{4p^2} A^2 \|\xi\|_{H^1}^{2q} \int_{\mathbb{R}} \zeta_A^2 |\xi|^{2(p+1)-q} \, dx.
\end{aligned}$$

Thus, taking $\frac{1}{4p^2} A^2 a_0^{2q} \ll 1$, we have (2.28). □

We now examine the contribution to (2.19) of the term with $\tilde{f}(z, \xi)$.

Lemma 2.8. *There exist $C_0 > 0$ and $r > 0$ and a neighborhood $\mathcal{U}$ of the origin in $\mathbb{C}$ such that for any pair $z, \xi \in \mathcal{U}$ and for $\tilde{f}(z,\xi)$ defined in (2.7) we have*

$$|\tilde{f}(z,\xi)| \le C_0 |Q[z]|^r |\xi|. \tag{2.29}$$

Proof. It is enough to set $\zeta = Q[z]$ and then to prove

$$|f(\zeta+\xi) - f(\xi) - f(\zeta)| \le C_0 |\zeta|^r |\xi|. \tag{2.30}$$

We will prove (2.30) with $r = 1$ if $p \ge 1/2$ and with $r = 2p$ if $p \le 1/2$. With these values of r, then $|\zeta| \ge |\xi|$ implies $|\zeta|^r |\xi| \le |\xi|^r |\zeta|$. This means that it is enough to consider the case $|\zeta| \ge |\xi|$.
If $|\zeta| \le 2|\xi|$ we have $|\zeta| \sim |\xi|$ and it is elementary to conclude that each of the 3 terms in the l.h.s. of (2.30) is $\lesssim |\zeta|^r |\xi|$. Hence we are left with case $|\zeta| \ge 2|\xi|$. Notice that $|f(\xi)| \lesssim |\xi|^{2p+1} \le |\zeta|^r |\xi|$. So it is enough to prove that for a fixed $C_r > 0$ we have

$$|f(\zeta+\xi) - f(\zeta)| \le C_r |\zeta|^r |w|. \tag{2.31}$$

By (1.4) we obtain the following, that implies (2.31) and completes the proof of the lemma:

$$\begin{aligned}
&|f(\zeta+\xi) - f(\zeta)| \le \int_0^1 \left| \frac{d}{dt} f(\zeta + t\xi) \right| dt \\
&= \int_0^1 \left| g(|\zeta + t\xi|^2)\xi + 2g'(|\zeta + t\xi|^2)(\zeta + tw) \left(\mathrm{Re}\,(\zeta \overline{w}) + t|w|^2 \right) \right| dt \\
&\le C|\zeta|^{2p}|\xi| + C\left(|\zeta|^{2p}|\xi| + |\zeta|^{2p-1}|\zeta|^2 \right) \le 3C|\zeta|^{2p}|\xi|.
\end{aligned}$$

□

Lemma 2.9. *For any $\varepsilon_0 > 0$ and $A \gg 1$ there exists $a_0 > 0$ such that for $\|(z,\xi)\|_{\mathbb{C}\times H^1} \le a_0$ we have*

$$\left| \left\langle \left(\frac{\psi_A'}{2} + \psi_A \partial_x \right) \xi, \tilde{f}(z,\xi)) \right\rangle \right| \le \varepsilon_0 \left(\|w'\|_{L^2}^2 + \| \langle x \rangle^{-2} w \|_{L^2}^2 \right) \text{ for } w = \zeta_A \xi \ .$$

Proof. First, we have

$$\left| \left\langle \frac{\psi_A'}{2} \xi, \tilde{f}(z,\xi) \right\rangle \right| \lesssim \int_{\mathbb{R}} |Q[z]| |w|^2 \, dx \lesssim a_0 \| \langle x \rangle^{-2} w \|_{L^2}^2.$$

Next, since $|\partial_x \xi| \lesssim e^{2\frac{|x|}{A}} (|w'| + |w|)$, we have

$$\left| \left\langle \psi_A \partial_x \xi, \tilde{f}(z,\xi) \right\rangle \right| \lesssim A \int_{\mathbb{R}} |Q[z]|^r \, e^{\frac{3}{A}|x|} (|w'| + |w|) \, |w| \, dx \lesssim a_0 A \left(\|w'\|_{L^2}^2 + \| \langle x \rangle^{-2} w \|_{L^2}^2 \right).$$

Therefore, taking $a_0 A \ll 1$, we have the conclusion. □

2.3 Closure of the estimates and completion of the proof of Theorem 1.4

We fix $A \gg 1$ and $0 < a_0 \ll 1$ so that the conclusion of Proposition 2.2 holds and we have $A^{-1} < \frac{\gamma_0}{4}$, where $\gamma_0 > 0$ is the constant given in Proposition 1.2. We next take $\epsilon_1 > 0$ sufficiently small so that $C\epsilon_1 < a_0$, where C is the constant given in (1.17). Recall that (1.17) is an easy consequence of the energy and mass conservation.

We set

$$\|w\|_X^2 := \|w'\|_{L^2}^2 + \| \langle x\rangle^{-2} w\|_{L^2}^2. \tag{2.32}$$

By (1.14) and (2.3), we have

$$\left|\left\langle \mathrm{i}DQ[z](\dot z+\mathrm{i}Ez), \left(\frac{\psi_A'}{2}+\psi_A\partial_x\right)\xi\right\rangle\right| \le C^{(1)}|\dot z+\mathrm{i}Ez| \; \|e^{-\frac{\gamma_0}{2}|x|}\left(\frac{\psi_A'}{2}+\psi_A\partial_x\right)\xi\|_{L^2},$$

with $C^{(1)}$ a fixed constant s.t. $C^{(1)}_{\gamma_0} \ge 2\|\varphi\|_{H^1_{\gamma_0}} + Ca_0^{2p}$.
By $\psi_A' = \zeta_A^2$, $w=\zeta_A\xi$ and $0<\zeta_A\le 1$ and for $C^{(2)} \ge \|e^{-\frac{\gamma_0}{2}|x|}\langle x\rangle^2\|_{L^\infty(\mathbb{R})}$, we have for all $x\in\mathbb{R}$

$$\left|e^{-\frac{\gamma_0}{2}|x|}\psi_A'\xi\right| = \left|e^{-\frac{\gamma_0}{2}|x|}\zeta_A w\right| \le \left|e^{-\frac{\gamma_0}{2}|x|}w\right| \le C^{(2)}\langle x\rangle^{-2}|w|.$$

Using $|\psi_A(x)|\le|x|$ and $A\ge\frac{4}{\gamma_0}$, for a similar constant $C^{(3)}$ we have for all $x\in\mathbb{R}$

$$\begin{aligned}
&|e^{-\frac{\gamma_0}{2}|x|}\psi_A\partial_x\xi| \le |x|e^{-\frac{\gamma_0}{2}|x|}\left|\partial_x\left(\frac{w}{\zeta_A}\right)\right| \le |x|e^{-\left(\frac{\gamma_0}{2}-A^{-1}\right)|x|}|\partial_x w| + |x|e^{-\frac{\gamma_0}{2}|x|}\zeta_A^{-2}|\zeta_A'||w| \\
&\le |x|e^{-\left(\frac{\gamma_0}{2}-A^{-1}\right)|x|}\left(|\partial_x w| + \frac{1}{A}\left(\chi'(x)|x| + (\chi(x)-1)\frac{x}{|x|}\right)|w|\right) \le C^{(3)}\left(|w'| + \langle x\rangle^{-2}|w|\right).
\end{aligned}$$

In view of (2.18)–(2.19), for any $T>0$ and for $C^{(4)} = C^{(1)}\left(C^{(2)}+C^{(3)}\right)$ we have

$$\begin{aligned}
\frac{1}{12}\int_0^T \|w(t)\|_X^2\,dt &\le \mathcal{J}(\xi(T)) - \mathcal{J}(\xi(0)) + \int_0^T \left|\left\langle \mathrm{i}DQ[z](\dot z+\mathrm{i}Ez), \left(\frac{\psi_A'}{2}+\psi_A\partial_x\right)\xi\right\rangle\right| dt \\
&\le 4A\epsilon^2 + C^{(4)}\left(\int_0^T \|w\|_X^2\,dt\right)^{1/2} \|\dot z+\mathrm{i}Ez\|_{L^2(0,T)}.
\end{aligned} \tag{2.33}$$

From (1.14), (2.5), $f(\xi)=g(|\xi|^2)\xi$ and (1.4) and Lemma 2.8 and for constants appearing in these formulas, we have

$$\left(1-Ca_0^{2p}\right)|\dot z+\mathrm{i}Ez| \le C\|\xi\|_{L^\infty}^{2p}\|w\|_X + C_0|z|^{\min(2p,1)}\|w\|_X \text{ for all } t.$$

Thus, using (1.17), which we already know to be true, and Sobolev Embedding, we obtain

$$\|\dot z+\mathrm{i}Ez\|_{L^2(0,T)} \le C_1\epsilon^{\min(2p,1)}\left(\int_0^T \|w(t)\|_X^2\,dt\right)^{1/2} \text{ for all } T,$$

for a fixed constant $C_1 = C(C,p,a_0)$. Entering this in (2.33), for some $C_5 = C(C_1,A,\gamma_0)$ we obtain

$$\|w\|_{L^2([0,T],X)}^2 \le \frac{1}{2}C_5^2\left(\epsilon^2 + \epsilon^{\min(2p,1)}\|w\|_{L^2([0,T],X)}^2\right) \text{ for all } T.$$

Now, take $\epsilon_0<\epsilon_1$ so that $C_5^2\epsilon_0^{\min(2p,1)}<1$. Then, if $\epsilon<\epsilon_0$ we have $\|w\|_{L^2([0,T],X)}\le C_5\epsilon$ for any T, and so we get the following, which implies the estimate (1.18) and ends the proof of Theorem 1.4,

$$\|w\|_{L^2(\mathbb{R}_+,X)} \le C_5\epsilon. \tag{2.34}$$

3 Proof of Theorem 1.6

We have the following ansatz.

Lemma 3.1. *There exists $c_0 > 0$ such that there exists a $C > 0$ such that for all $u \in H^1$ with $\|u\|_{H^1} < c_0$, there exists a unique pair $(z, \eta) \in \mathbb{C} \times (H^1 \cap \mathcal{H}_c[z])$ such that*

$$u = Q[z] + \eta \text{ with } |z| + \|\eta\|_{H^1} \leq C\|u\|_{H^1}. \tag{3.1}$$

The map $u \to (z, \eta)$ is in $C^1(D_{H^1}(0, c_0), \mathbb{C} \times H^1)$.

Remark 3.2. We note that the assumption $p > 1/2$ is needed for this lemma which requires that the map $z \mapsto Q[z]$ is C^2. See Proposition 1.2.

Proof. Set

$$F(z, u) := \begin{pmatrix} \langle u - Q[z], \mathrm{i}D_1Q[z]\rangle \\ \langle u - Q[z], \mathrm{i}D_2Q[z]\rangle \end{pmatrix}.$$

Then, since here $p > 1/2$, we have $F(z, u) \in C^1$ with formula (2.2) true for this function. We conclude by Implicit Function Theorem. □

In terms of decomposition (1.20), equation (1.1) can be expressed as follows:

$$\langle \mathrm{i}DQ[z](\dot{z} + \mathrm{i}Ez), D_jQ[z]\rangle - \langle \mathrm{i}\eta, D_jDQ[z](\dot{z} + \mathrm{i}Ez)\rangle = \langle h(z, \eta), D_jQ[z]\rangle \text{ for } j = 1, 2 \tag{3.2}$$

$$\mathrm{i}\dot{\eta} = H_1\eta + f(\eta) + \tilde{f}(z, \eta) - \mathrm{i}DQ[z](\dot{z} + \mathrm{i}Ez), \tag{3.3}$$

where

$$h(z, \eta) := f(Q[z] + \eta) - f(Q[z]) - \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} f(Q[z] + \varepsilon\eta). \tag{3.4}$$

Here, (3.3) is same as (2.6) since it can be obtained by substituting $u = Q[z]+\eta$ into (1.1). However, (3.2) differs from (2.5) because we have changed the orthogonality condition. For the derivation of (3.2), see (3.17) of [15].

Like in Sect. 2.2, we consider

$$\mathcal{J}(\eta) = 2^{-1}\left\langle \mathrm{i}\left(\frac{\psi_A'}{2} + \psi_A\partial_x\right)\eta, \eta\right\rangle. \tag{3.5}$$

Proceeding like in Sect. 2.2 we obtain

$$\begin{aligned} &\frac{d}{dt}\mathcal{J}(\eta) - \left\langle \left(\frac{\psi_A'}{2} + \psi_A\partial_x\right)\eta, \mathrm{i}DQ(\dot{z} + \mathrm{i}Ez)\right\rangle \\ &\quad = \left\langle \left(\frac{\psi_A'}{2} + \psi_A\partial_x\right)\eta, H_1\eta\right\rangle + \left\langle \left(\frac{\psi_A'}{2} + \psi_A\partial_x\right)\eta, \widetilde{f}(z, \eta)\right\rangle + \left\langle \left(\frac{\psi_A'}{2} + \psi_A\partial_x\right)\eta, f(\eta)\right\rangle. \end{aligned} \tag{3.6}$$

Then, like in Sect. 2.2, we have the following result.

Proposition 3.3. *There exist values $1 \gg a_0 > 0$ and $A \gg 1$ such that for $\xi \in P_cH^1$ and $|z| + \|\eta\|_{H^1} < a_0$ we have*

$$\text{r.h.s. of (3.6)} \geq \frac{1}{12}\left(\|w'\|_{L^2} + \|\langle x\rangle^{-2} w\|_{L^2}^2\right) \text{ for } w = \zeta_A\eta. \tag{3.7}$$

Proof. The proof of Proposition 3.3 is exactly like the proof of Proposition 2.2, except that the analogue Lemma 2.24 continues to be true with an $\eta \in \mathcal{H}[z]$ instead of $\xi \in \mathcal{H}[0]$. We skip the elementary proof of the last point. $\square$

End of the proof of Theorem 1.6. We only sketch the proof, which is similar to that in Sect. 2.3. From (3.7) we have like in (2.33),

$$\|w\|^2_{L^2((0,t),X)} \lesssim 2\epsilon^2 + \|w\|_{L^2((0,t),X)} \ \|\dot z + \mathrm{i}Ez\|_{L^2((0,t))}. \tag{3.8}$$

Exploiting the fact that the function $f(w)$ in (2.4) is in $C^2(\mathbb{C},\mathbb{C})$ when $2p>1$, we have

$$|h(z,\eta)| \lesssim \left(|Q[z]|^{2p-1} + |\eta|^{2p-1}\right)|\eta|^2 \tag{3.9}$$

Indeed setting $F(s)=f(Q[z]+s\eta)$, we have by (3.4)

$$h(z,\eta) = F(1)-F(0)-F'(0) = \int_0^1 (1-s)F''(s)\,ds.$$

Further, since

$$\begin{aligned} F''(s) =& 2g'(|Q[z]+s\eta|^2)\left(\mathrm{Re}\left(\overline{(Q[z]+s\eta)}\eta\right)\eta + |\eta|^2 Q[z]\right) \\ &+ 4g''(|Q[z]+\eta|^2)\left(\mathrm{Re}\,\overline{(Q[z]+s\eta)}\eta\right)^2 Q[z], \end{aligned}$$

by (1.4) we have

$$|F''(s)| \lesssim (|Q[z]|+|\eta|)^{2p-1}\,|\eta|^2, \ \ s\in[0,1].$$

From (3.2) and (3.9) when $p>1/2$, we have

$$\begin{aligned} \|\dot z + \mathrm{i}Ez\|_{L^2((0,t))} \lesssim & \|\dot z + \mathrm{i}Ez\|_{L^2((0,t))} \| \langle x\rangle^{-2} w\|_{L^\infty((0,t),L^2)} \\ &+ \left(\|z\|^{2p-1}_{L^\infty((0,t))} + \| \langle x\rangle^{-2} w\|^{2p-1}_{L^\infty((0,t),L^2)}\right)\|w\|^2_{L^2((0,t),X)} \\ &\lesssim \epsilon \|\dot z + \mathrm{i}Ez\|_{L^2((0,t))} + \epsilon^{2p}\|w\|_{L^2((0,t),X)}, \end{aligned}$$

and hence

$$\|\dot z + \mathrm{i}Ez\|_{L^2((0,t))} \lesssim \epsilon^{2p}\|w\|_{L^2((0,t),X)}.$$

Entering this in (3.8) we conclude, for a fixed $C_5>0$,

$$\|w\|_{L^2((0,t),X)} \le C_5\epsilon.$$

We set now $\rho(t) := z(t)e^{\mathrm{i}\int_0^t E[z(s)]ds}$. Then, since $|\dot\rho| = |\dot z + \mathrm{i}Ez|$, we have

$$\begin{aligned} &\|\dot\rho\|_{L^1((0,t))} = \|\dot z + \mathrm{i}Ez\|_{L^1((0,t))} \\ &\lesssim \|\dot\rho\|_{L^1((0,t))}\| \langle x\rangle^{-2} w\|_{L^\infty((0,t),L^2)} + \left(\|z\|^{2p-1}_{L^\infty((0,t))} + \| \langle x\rangle^{-2} w\|^{2p-1}_{L^\infty((0,t),L^2)}\right)\|w\|^2_{L^2((0,t),X)}. \end{aligned}$$

From this we derive

$$\|\dot\rho\|_{L^1(\mathbb{R}_+)} \lesssim \epsilon^{2p-1}\|w\|^2_{L^2(\mathbb{R}_+,X)} \lesssim \epsilon^{2p+1}.$$

The existence of ρ_+ and of the limit (1.23) follow. This ends the the proof of (1.22)–(1.23). $\square$

4 Proof of Theorem 1.7

We know that there exists a unique global strong solution $u \in C^0(\mathbb{R}, H^1(\mathbb{R}, \mathbb{C}))$, and furthermore that energy and mass are constant

$$\begin{aligned} E(u(t)) &= \frac{1}{2}\|\partial_x u(t)\|^2_{L^2(\mathbb{R})} + \frac{|q|}{2}|u(t,0)|^2 + \frac{1}{2}\int_{\mathbb{R}} G(|u(t)|^2)dx = E(u_0), \\ Q(u(t)) &= \frac{1}{2}\|u(t)\|^2_{L^2(\mathbb{R})} = Q(u_0). \end{aligned}$$

By well posedness and a density argument, it is enough to focus on the case $u_0 \in D(H_q)$, so that

$$u \in C^0(\mathbb{R}, D(H_1)) \cap C^1(\mathbb{R}, L^2(\mathbb{R}, \mathbb{C})).$$

Then we consider $\mathcal{J}(u)$, defined like in (2.15), and by the same argument of Sect. 2.2 we have

$$\begin{aligned} \frac{d}{dt}\mathcal{J}(u) &= \left\langle \mathrm{i}\dot{u}, \left(\frac{\psi_A'}{2} + \psi_A \partial_x\right) u \right\rangle = \left\langle \mathrm{i}\dot{u}, \left(\frac{\psi_A'}{2} + \psi_A \partial_x\right) u \right\rangle \\ &= \left\langle (-\partial_x^2 + |q|\delta(x))u + g(|u|^2)u, \left(\frac{\psi_A'}{2} + \psi_A \partial_x\right) u \right\rangle. \end{aligned}$$

By computations similar to Lemma 2.3, for $w = \zeta_A \xi$ and for the $V(x)$ in (2.20), we have

$$\begin{aligned} \left\langle (-\partial_x^2 + |q|\delta(x))u, \left(\frac{\psi_A'}{2} + \psi_A \partial_x\right) u \right\rangle &= \left\langle \left(-\partial_x^2 + \frac{|q|}{2}\delta(x)\right) w, w \right\rangle + \frac{1}{2A}\langle Vw, w\rangle \\ &\geq \frac{1}{2}\left\langle \left(-\partial_x^2 + \frac{|q|}{2}\delta(x)\right) w, w \right\rangle, \end{aligned}$$

for $A \geq A_0$ with A_0 a fixed sufficiently large constant.
On the other hand, by $\psi_A' > 0$ and the argument in the first few lines of Lemma 2.7,

$$\left\langle g(|u|^2)u, \frac{\psi_A'}{2}u \right\rangle + \langle g(|u|^2)u, \psi_A \partial_x u\rangle = \frac{1}{2}\langle g(|u|^2)|u|^2 - G(|u|^2), \psi_A'\rangle \geq 0.$$

Hence, for fixed constants

$$\int_0^T \|w(t)\|_X^2\, dt \lesssim \mathcal{J}(u(T)) - \mathcal{J}(u_0) \lesssim \sqrt{E(u_0)Q(u_0)} + Q(u_0),$$

which yields Theorem 1.7. □

A Appendix.

We prove Proposition 1.2.

Lemma A.1. *Set* $R := \left((H_1 + \frac{1}{4})\big|_{P_c L^2} \right)^{-1}$. *Then, for sufficiently small* $\gamma > 0$, R *is a bounded operator from* L^2_γ *to* H^1_γ.

Proof. For case $\gamma = 0$ see Lemma 2.12 of [33]. For the case $\gamma > 0$, set $\chi_A(x) := \chi(x/A)$ where χ is given in (2.8). Set $\mu_{\gamma,A}(x) := e^{\gamma\sqrt{1+|x|^2}}\chi_A(x)$. Then, multiplying $H_1 Ru = u$ by μ_A, we obtain

$$H_1 \mu_{\gamma,A} Ru = [H_1, \mu_{\gamma,A}]Ru + \mu_{\gamma,A} u.$$

Notice that there exists a $C > 0$ such that

$$\|[H_1, \mu_{\gamma,A}]u\| \le C\ \gamma\ \|\mu_{\gamma,A} u\|_{H^1} \text{ for all } \gamma \in [0,1] \text{ and } A \in [1,\infty).$$

This implies that for sufficiently small $\gamma > 0$,

$$\|\mu_{\gamma,A} Ru\|_{H^1} \lesssim \|\mu_{\gamma,A} u\|_{L^2} \lesssim \|u\|_{L^2_\gamma}.$$

Thus, taking $A \to \infty$, we have $\|Ru\|_{H^1_\gamma} \lesssim \|u\|_{L^2_\gamma}$. □

We consider $\tilde{h} : [0,1] \times \mathbb{R} \to \mathbb{R}$ defined by

$$\tilde{h}(\rho,\mu) := g(\rho\mu^2)\mu. \tag{A.1}$$

For $\gamma < \frac{1}{2}$, we set $h : [0,1] \times H^1_\gamma(\mathbb{R},\mathbb{R}) \to L^2_\gamma(\mathbb{R},\mathbb{R})$ by

$$h(\rho,q)(x) := \tilde{h}(\rho, q(x)) = g(\rho q(x)^2)q(x). \tag{A.2}$$

Notice that q in (A.1) is a number but q in (A.2) is a function.

Lemma A.2. *We have $\tilde{h} \in C([0,1] \times \mathbb{R},\mathbb{R}) \cap C^1((0,1] \times \mathbb{R},\mathbb{R})$ and the estimates*

$$|\tilde{h}(\rho,\mu)| \lesssim \rho^p |\mu|^{2p+1} \tag{A.3}$$

and

$$|\partial_\rho \tilde{h}(\rho,\mu)| \lesssim \rho^{p-1}|\mu|^{2p+1}, \quad |\partial_\mu \tilde{h}(\rho,\mu)| \lesssim \rho^p |\mu|^{2p}. \tag{A.4}$$

Furthermore, for $\rho\mu \neq 0$, h is three times differentiable and we have

$$|\partial_\rho^2 \tilde{h}(\rho,\mu)| \lesssim \rho^{p-2}|\mu|^{2p+1}, \quad |\partial_\rho\partial_\mu \tilde{h}(\rho,\mu)| \lesssim \rho^{p-1}|\mu|^{2p}, \quad |\partial_\mu^2 \tilde{h}(\rho,\mu)| \lesssim \rho^p|\mu|^{2p-1} \tag{A.5}$$

and

$$|\partial_\rho^3 \tilde{h}(\rho,\mu)| \lesssim \rho^{p-3}|\mu|^{2p+1}, \quad |\partial_\mu \partial_\rho^3 h(\rho,\mu)| \lesssim \rho^{p-1}\mu^{2p}. \tag{A.6}$$

If $p > \frac{1}{2}$, we have $\tilde{h} \in C^2((0,1] \times \mathbb{R},\mathbb{R})$.

Proof. By the definition of $\tilde{h}$, we have $C([0,1] \times \mathbb{R},\mathbb{R}) \cap C^3((0,1] \times (\mathbb{R} \setminus \{0\}),\mathbb{R})$. Also, (A.3) is immediate from (1.4) and (A.1). At $(\rho,q) = (\rho,0)$ with $\rho > 0$, $\tilde{h}$ is differentiable w.r.t. ρ and μ having $\partial_\rho \tilde{h} = \partial_q \tilde{h} = 0$. One can see this easily from $\tilde{h}(\rho,0) = 0$ and

$$\tilde{h}(\rho+\epsilon,0) = 0,\ |\tilde{h}(\rho,\epsilon)| \lesssim \epsilon^{2p+1}.$$

Further, since for $\mu \neq 0$,

$$\partial_\rho \tilde{h}(\rho,q) = g'(\rho\mu^2)\mu^3, \quad \partial_\mu \tilde{h}(\rho,\mu) = g(\rho\mu^2) + 2\rho g'(\rho\mu^2)\mu^2, \tag{A.7}$$

we have (A.4) from (1.4), which imply that $\partial_\rho \tilde{h}$ and $\partial_\mu \tilde{h}$ are continuous at $(\rho,0)$.

Differentiating (A.7) for $\rho, \mu \neq 0$, we have

$$\partial_\rho^2 \tilde{h}(\rho, q) = g''(\rho\mu^2)\mu^5, \quad \partial_\rho\partial_\mu \tilde{h}(\rho, \mu) = 3g'(\rho\mu^2)\mu^2 + 2\rho g''(\rho\mu^2)\mu^4,$$
$$\partial_\mu^2 \tilde{h}(\rho, \mu) = 6\rho g'(\rho\mu^2)\mu + 4\rho^2 g''(\rho\mu^2)\mu^3.$$

and

$$\partial_\rho^3 \tilde{h}(\rho, q) = g'''(\rho\mu^2)\mu^7, \quad \partial_\mu\partial_\rho^2 \tilde{h}(\rho, q) = 2\rho g'''(\rho\mu^2)\mu^6.$$

This implies that $\rho, \mu \neq 0$, we have (A.5) and (A.6). By (A.5), for the case $p > 1/2$, we see that h is twice continuously differentiable at $(\rho, 0)$ $(\rho \neq 0)$ with

$$\partial_\rho^2 \tilde{h}(\rho, 0) = \partial_\rho\partial_\mu \tilde{h}(\rho, 0) = \partial_\mu^2 \tilde{h}(\rho, 0) = 0.$$

Therefore, we have the conclusion. □

Lemma A.3. *Let $\gamma \geq 0$. Let $\tilde{h}$, h be the functions given in* (A.1) *and* (A.2). *Then,*

$$h \in C([0,1] \times H^1_\gamma, L^2_\gamma) \cap C^1((0,1] \times H^1_\gamma, L^2_\gamma) \tag{A.8}$$

and

$$\partial_\rho h(\rho, q)(x) = \partial_\rho \tilde{h}(\rho, q(x)), \ \left(\partial_q h(\rho, q)v\right)(x) = \partial_\mu \tilde{h}(\rho, q(x))v(x). \tag{A.9}$$

Proof. First of all, for $q \in H^1_\gamma$, we have $h(\rho, q) \in L^2_\gamma$. Indeed, from (A.3),

$$\|h(\rho, q)\|_{L^2_\gamma} = \|\tilde{h}(\rho, q(\cdot))\|_{L^2_\gamma} \lesssim \rho^p \|q\|_{L^\infty}^{2p} \|q\|_{L^2_\gamma} \lesssim \rho^p \|q\|_{H^1_\gamma}^{2p+1}, \tag{A.10}$$

which implies also that h is continuous at $\{0\} \times H^1$.

Next, we show (A.9). For $(\rho, q) \in (0,1] \times H^1_\gamma$, and $|\epsilon| < \rho$,

$$\begin{aligned}
&|\epsilon|^{-1} \|h(\rho+\epsilon, q) - h(\rho, q) - \epsilon\partial_\rho \tilde{h}(\rho, q)\|_{L^2_\gamma} = |\epsilon|^{-1} \|\tilde{h}(\rho+\epsilon, q) - \tilde{h}(\rho, q) - \epsilon\partial_\rho \tilde{h}(\rho, q)\|_{L^2_\gamma} \\
&= \int_0^1 \|\partial_\rho \tilde{h}(\rho + \tau_1\epsilon, q)\, d\tau - \partial_\rho \tilde{h}(\rho, q)\|_{L^2_\gamma}\, d\tau_1 \\
&\lesssim \int_0^1 \int_0^1 \tau_1 |\epsilon| \|\partial_\rho^2 \tilde{h}(\rho + \tau_1\tau_2\epsilon, q)\|_{L^2_\gamma(\{x \in \mathbb{R} | q(x) \neq 0\})}\, d\tau_1 d\tau_2 \\
&\lesssim |\epsilon| \int_0^1 \int_0^1 \tau_1 (\rho + \tau_1\tau_2\epsilon)^{p-2}\, d\tau_1 d\tau_2 \|q\|_{H^1_\gamma}^{2p+1} \to 0 \text{ as } \epsilon \to 0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\|v\|_{H^1_\gamma}^{-1} \|h(\rho, q+v) - h(\rho, q) - \partial_\mu \tilde{h}(\rho, q)v\|_{L^2_\gamma} = \|v\|_{H^1_\gamma}^{-1} \| \int_0^1 \left(\partial_\mu \tilde{h}(\rho, q+\tau v) - \partial_\mu \tilde{h}(\rho, q)\right) v \|_{L^2_\gamma} \\
&\leq \sup_{\tau \in [0,1]} \|\partial_\mu \tilde{h}(\rho, q+\tau v) - \partial_\mu \tilde{h}(\rho, q)\|_{L^\infty} \to 0 \text{ as } \|v\|_{H^1_\gamma} \to 0.
\end{aligned} \tag{A.11}$$

Here we have used the fact that $\partial_\mu \tilde{h}$ is uniformly continuous in $[\frac{\rho}{2}, 1] \times [-\|q\|_{L^\infty} - 1, \|q\|_{L^\infty} + 1]$ and $\|v\|_{L^\infty} \to 0$ if $\|v\|_{H^1_\gamma} \to 0$. By similar estimate, we see that $\partial_\rho h$ and $\partial_q h$ are continuous in $(0,1] \times H^1_\gamma$. From this, we have (A.8). □

Lemma A.4. *Let $p > 1/2$. Let $\tilde{h}$, h be the functions given in* (A.1) *and* (A.2). *Then,*

$$h \in C([0,1] \times H^1_\gamma, L^2_\gamma) \cap C^2((0,1] \times H^1_\gamma, L^2_\gamma), \tag{A.12}$$

and

$$\begin{aligned} &\partial_\rho^2 h(\rho,q)(x) = \partial_\rho^2 \tilde{h}(\rho, q(x)),\ \left(\partial_\rho \partial_q h(\rho,q) v\right)(x) = \partial_\rho \partial_\mu \tilde{h}(\rho, q(x)) v(x), \\ &\partial_q^2 h(\rho,q)(v,w)(x) = \partial_\mu^2 \tilde{h}(\rho, q(x)) v(x) w(x). \end{aligned} \tag{A.13}$$

Proof. Since the argument is similar to the proof of Lemma A.3 we omit it. □

Lemma A.5. *Let $\gamma \in [0, \frac{1}{2})$ and set*

$$\mathfrak{e}(\rho, q) := \langle h(\rho, \varphi + q), \varphi \rangle \,.$$

Then, $\mathfrak{e} \in C^1((0,1) \times H^1_\gamma(\mathbb{R},\mathbb{R}), \mathbb{R})$. Moreover, if $p > 1/2$, we have $\mathfrak{e} \in C^2((0,1) \times H^1_\gamma(\mathbb{R},\mathbb{R}), \mathbb{R})$.

Proof. Set $F \in C^\infty(L^2_\gamma, \mathbb{R})$ by $F(h) := \langle h, \varphi \rangle$. Since $\mathfrak{e}(\rho,q) = F \circ h(\rho + \varphi, q)$, we immediately have the conclusion from Lemmas A.3 and A.4. □

Lemma A.6. *Let $\gamma \in [0, \frac{1}{2})$. Set*

$$\Phi(\rho, q) := \mathfrak{e}(\rho,q)(\varphi + q) - h(\rho, \varphi + q). \tag{A.14}$$

Then, $\Phi \in C^1((0,1] \times P_c H^1_\gamma(\mathbb{R},\mathbb{R}), P_c L^2_\gamma(\mathbb{R},\mathbb{R}))$. Moreover, if $p > 1/2$, we have $\Phi \in C^2((0,1] \times P_c H^1_\gamma(\mathbb{R},\mathbb{R}), P_c L^2_\gamma(\mathbb{R},\mathbb{R}))$

Proof. From Lemmas A.3, A.4 and A.5, it suffices to show $\Phi(\rho, q) \in P_c L^2_\gamma$. However, from the definition of $\mathfrak{e}$ we obtain the following, which yields the conclusion:

$$\langle \Phi(\rho,q), \varphi \rangle = \mathfrak{e}(\rho,q) - \langle h(\rho, \varphi+q), \varphi \rangle = 0.$$

□

Lemma A.7. *Take $\gamma_0 \in (0, 1/2)$ such that the conclusion of Lemma A.1 holds. Then there exists $\rho_0 > 0$ such that there exists a unique $q \in C^k((0,\rho_0), H^1_\gamma)$ with $k = 1$ and $k = 2$ if $p > \frac{1}{2}$ satisfying*

$$q(\rho) = R\Phi(\rho, q(\rho)), \tag{A.15}$$

and

$$\|q(\rho)\|_{H^1_\gamma} \lesssim \rho^p,\ \|\partial_\rho q(\rho)\|_{H^1_\gamma} \lesssim \rho^{p-1} \text{ and } |\mathfrak{e}(\rho, q(\rho))| \lesssim \rho^p. \tag{A.16}$$

Moreover, if $p > 1/2$ we have

$$\|\partial_\rho^2 q(\rho)\|_{H^1_\gamma} \lesssim \rho^{p-2}. \tag{A.17}$$

Proof. By Lemma A.1 and Lemma A.3, we have

$$\begin{aligned} &\|R\Phi(\rho,q_1) - R\Phi(\rho,q_2)\|_{H^1_\gamma} \lesssim \|\Phi(\rho,q_1) - \Phi(\rho,q_2)\|_{L^2_\gamma} \lesssim \|h(\rho,\varphi+q_1) - h(\rho,\varphi+q_2)\|_{L^2_\gamma} \\ &\le \int_0^1 \|\partial_q h(\rho, \varphi + q_2 + \tau(q_1 - q_2))\|_{L^\infty}\, d\tau \|q_1 - q_2\|_{L^2_\gamma} \lesssim \rho^p \|q_1 - q_2\|_{H^1_\gamma}, \end{aligned}$$

for $q_1, q_2 \in \overline{D_{H^1_\gamma}(0,1)}$. Therefore, there exists $\rho_0 > 0$ such that

$$\|R\Phi(\rho, q_1) - R\Phi(\rho, q_2)\|_{H^1_\gamma(\mathbb{R},\mathbb{R})} \le \frac{1}{2}\|q_1 - q_2\|_{H^1_\gamma},$$

for all $\rho \in (0, \rho_0)$ and $q_1, q_2 \in \overline{D_{H^1_\gamma(\mathbb{R},\mathbb{R})}(0,1)}$. Thus, by contraction mapping principle, there exists a unique $q \in \overline{D_{H^1_\gamma(\mathbb{R},\mathbb{R})}(0,1)}$ satisfying (A.15). We call $q(\rho)$ the fixed point of $R\Phi(\rho, \cdot)$ and set

$$F(\rho, q) := q - R\Phi(\rho, q).$$

Since one can show $\partial_q F|_{(\rho,q)=(\rho,q(q))}$ is invertible by using the estimate we have prepared, by the Implicit Function Theorem and by Lemma A.6 we have $q \in C^k((0,\rho_0), H^1_\gamma)$ with $k = 1$ and $k = 2$ if $p > \frac{1}{2}$.

We now prove (A.16). First, by the fact that $q(\rho)$ is the fixed point of $R\Phi(\rho, \cdot)$, Lemma (A.1) and (A.3) with the definition of h, Φ, we have

$$\|q(\rho)\|_{H^1_\gamma} = \|R\Phi(\rho)\|_{H^1_\gamma} \lesssim \|\Phi(\rho)\|_{L^2_\gamma} \lesssim \|h(\rho, \varphi + q(\rho))\|_{L^2_\gamma} \lesssim \rho^p.$$

Next, by the definition of $\mathfrak{e}$, we have

$$|\mathfrak{e}(\rho, q(\rho))| \le \|h(\rho, \varphi + q(\rho))\|_{L^2} \lesssim \rho^p.$$

Finally, since

$$\partial_\rho q = R\partial_q \Phi \partial_\rho q + R\partial_\rho \Phi,$$

and by the above argument, for sufficietly small $\rho > 0$, we have $\|(\mathrm{Id} - R\partial_q\Phi)^{-1}\|_{H^1_\gamma \to H^1_\gamma} \le 2$, we have the 2nd estimate of (A.16) by

$$\|\partial_\rho q\|_{H^1_\gamma} = \|(\mathrm{Id} - R\partial_q\Phi)^{-1} R\partial_q\Phi\|_{H^1_\gamma} \lesssim \|\partial_q\Phi\|_{L^2_\gamma} \lesssim \|\partial_q h\|_{L^2_\gamma} \lesssim \rho^{p-1}.$$

The estimate (A.17) can be proved similarly. □

Proof of Proposition 1.2. Set $a_0 = \rho_0^{\frac{1}{2}}$ where $\rho_0 > 0$ is given in Lemma A.7. Set

$$Q[z] := z(\varphi + q(|z|^2)) \text{ and } E(|z|^2) := -\frac{1}{4} + \mathfrak{e}(|z|^2, q(|z|^2)),$$

where $q \in C^1((0, a_0^2), H^1_\gamma(\mathbb{R},\mathbb{R}))$ is given in Lemma A.7. Then, (1.12) and (1.13) are immediate from the definition of Q, q and $\mathfrak{e}$. The first and third inequality of (1.14) follow from (A.16).

By Lemma A.7, we have $Q \in C(D_{\mathbb{C}}(0, a_0), H^1_\gamma) \cap C^k(D_{\mathbb{C}}(0, a_0) \setminus \{0\}, H^1_\gamma)$ for $k = 1$ and $k = 2$ for $p > \frac{1}{2}$. However, since

$$\|D_j Q[z] - \mathrm{i}^{j-1}\varphi\|_{H^1_\gamma} = \|\mathrm{i}^{j-1} q(|z|^2) + 2q'(|z|^2) z z_j\|_{H^1_\gamma} \lesssim |z|^{2p}, \ j = 1, 2,$$

we see that $Q[z]$ is also continuously differentiable at $z = 0$. Here, we have set $z = z_1 + \mathrm{i}z_2$ for $z_1, z_2 \in \mathbb{R}$. Similarly, if $p > 1/2$, we see that $Q[z]$ is twice continuously differentiable at the origin and satisfying the estimate (1.15). This finishes the proof. □

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Department of Mathematics and Geosciences, University of Trieste, via Valerio 12/1 Trieste, 34127 Italy. *E-mail Address*: `scuccagna@units.it`

Department of Mathematics and Informatics, Faculty of Science, Chiba University, Chiba 263-8522, Japan. *E-mail Address*: `maeda@math.s.chiba-u.ac.jp`