How to Teach Mathematics: Some Suggestions from Herbartian Tradition

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ABSTRACT
The German Johann Friedrich Herbart (1776-1841), recognized in mathematical psychology as a forerunner in the measurement of mental phenomena – in a study of psychic events which was not only qualitative but also quantitative, based upon mathematics (specifically, calculus) –, stood out also for his activity in the psycho-pedagogical field and for his theory of education, in which mathematics played a fundamental role. His contribution was, in particular, considerable for mathematics education between 1800 and 1900 in the world of German culture, where important reformist tendencies in the study of teaching problems matured. After a brief sketch of Herbart's life and work, the paper will focus on these aspects, giving methodological suggestions which can be relevant and useful for mathematics education of today.

1. JOHANN FRIEDRICH HERBART (1776-1841), A BRIEF SKETCH OF HIS LIFE AND WORK

Born in Oldenburg, capital of the Grand Duchy of the same name (now in Lower Saxony), Herbart studied philosophy at the University of Jena, where he was a pupil of Fichte (see Raapke, 1976). Employed as a private tutor in Bern, Switzerland, Herbart visited the Burgdorf Institute founded by Johann Heinrich Pestalozzi (1746-1827) and established a friendship with him. Having been "Privatdozent" first of pedagogy and then of philosophy at the University of Göttingen, in 1809 he accepted the call to take up Kant's chair in Königsberg (now Kaliningrad; at that time Eastern Prussia). There, Herbart found a favourable environment for his work and collaborated, with government support, in commissions and school reform projects. In 1833, he went back to Göttingen, where the chair of philosophy was vacant.

1.1 HERBART'S THEORY OF EDUCATION

Besides his philosophy, which he developed in opposition to idealism (at that time dominating the German world of culture) as an elaboration of concepts obtained by experience, Herbart stood out also for his activity in the psycho-pedagogical field and for his theory of education.

In this context, he conceived "pedagogy" as an organic system of concepts concerning the aims and the methods of education (see Benner, 1976; Blaß, 1976; Geißler, 1976; Raapke, 1976). In order to be a science, pedagogy needs a basic philosophical doctrine, specifically "psychology", able to provide necessary knowledge about nature and development laws of the human soul, and also "ethics", capable of showing the ideal aims of life to which the process of training the student's personality should be directed.

Education can be accomplished, according to Herbart, only through "instruction", and "educational instruction", to be such, must be able to arouse what he calls "interest". Interest is conceived in this regard not only as an educational tool to ensure that what is taught is more readily understood and learned by the student, but also and above all as an end to which education should be directed. It must be considered as an aptitude to grasp the value of the aspects of life and reality and, consequently, to assume a corresponding practical attitude. Educational instruction must make interest grow in all directions, in a simultaneous, coordinated, and harmonious way. The student must be educated in his entirety, at all times, so as to realize the organic unity and development continuity of his inner life, with the purpose of "multilateral culture".

2. HERBART AND MATHEMATICS

"Mathematics, a foreground discipline": So Herbart's position on mathematics can be outlined.

Herbart is well known in mathematical psychology as a forerunner in the measurement of mental phenomena, in a study of psychic events which was not only qualitative but also quantitative, based upon mathematics (specifically, calculus) (see, for more details, Leary, 1980; Romano, 1976; Zudini, 2009a, 2009b). In fact, inspired by Kant's dictum that science was necessarily mathematical and in answer to his famous interdiction – which affirmed the impossibility of a mathematization of psychical facts, since they took place along one dimension, time, and were devoid of spatial extension, being intensive magnitudes – Herbart affirmed not only the possibility but the necessity of applying

mathematics to psychology (Herbart, 1822), conceived as "science, newly founded on experience, metaphysics and mathematics" (Herbart, 1824-1825).
According to Herbart, if psychical magnitudes, considered singly, cannot be measured, nevertheless, we can measure the variation of these magnitudes in a direct way, through calculus (the "mathesis intensorum" of that period). In case of shortage and imprecision of empirical observations, we can follow an indirect way, using the predictive power of the mathematical formulation.
The magnitudes which Herbart put at the basis of his calculation were the force of representations and the grade of contrast between them: The representations, which constitute our mental life, clash and, as a consequence of these interactions with stronger or weaker representations, fuse with or inhibit one another, reach or leave consciousness when they exceed or go below the threshold value ("Schwelle"). Once this value has been reached, representations remain in a sort of state of latency, waiting to return to consciousness. All that takes place in the context of a sort of mental "statics" and "mechanics" (Heidelberger, 1993). So "consciousness threshold" (or "limen") is a fundamental point of Herbart's theory: He introduced this concept in psychology and defined it, with calculation methods (concerning the history of the concept and its application in psychology, see Corso, 1963).
In Herbart's vision, mathematics is therefore fundamental but not only in this respect. It has a basic role also in education, as "gymnastics" of thought, for the formation of the mind, in all its degrees (Herbart, 1804).
In fact, the mind, as well as the body, needs to find a "gym", to test the "muscles" and renew the "elasticity". In this context, the importance of mathematics is affirmed for all scientific disciplines; the teaching of mathematics should not be reserved for the last years of formation, but imparted from the earliest years in order to avoid a late, superficial, and hence fruitless learning of it.
Thus, mathematics plays an irreplaceable role in a complete education of young people.

3. FROM MATHEMATICS TO MATHEMATICS EDUCATION

Since mathematics was a foreground discipline, mathematics teaching became a major matter of study for students' education as a whole.
The trace of Herbart's influence is clear and recognized in the German cultural world between 1800 and 1900, and, specifically, his contribution was significant for mathematics education of that time, where important reformist tendencies in the study of teaching problems matured. It was a very fruitful period of research and deepening the understanding of these problems, in particular in the Middle-European cultural world, both for the innovation of the teaching methods and for the introduction of new subjects in the secondary school, first of all calculus (see Zuccheri & Zudini, 2014). In Göttingen, where Herbart had been professor, a prominent scholar of didactical problems was the mathematician Felix Klein (1849-1925) (see Corry, 2004).
An opportunity to examine and discuss the situation of mathematics education in various countries, with reference also to psycho-pedagogical considerations, was the Fourth International Congress of Mathematicians held in Rome (April 6-11, 1908). On that occasion, the "Commission Internationale de l'Enseignement Mathématique" (CIEM) or "Internationale Mathematische Unterrichtskommission" (IMUK) – thereafter, "International Commission on Mathematical Instruction" (ICMI) – was established, presided over by Klein, with the aim of promoting and spreading the interest of the mathematicians in school education. Together with this commission, the journal "L'Enseignement Mathématique" played an important role allowing communication and cooperation among scholars at international level (see Schubring, 2003, 2008; Furinghetti, 2003; Furinghetti et al., 2008).
Focusing on these aspects is of interest in so far as suggestions, which were drawn at that time, were destined to have rich developments, still resounding in current research in education and being relevant and useful for mathematics education of today.

3.1 METHODOLOGICAL SUGGESTIONS FOR MATHEMATICS TEACHING

Herbart had a significant influence on elementary school education. Traces of this influence were evident in most geometry textbooks for elementary schools of Klein's time. In preschools ("Kindergarten"), according to Pestalozzi's theory of intuition ("Anschauung"), picked up by Herbart, young children learned about the simplest spatial forms by playing with suitable objects.
These pedagogical ideas had been applied also in higher level schools. The school curricula proposed by Franz Serafin Exner (1802-1853) and by Hermann Bonitz (1814-1888) in Austria already around the middle of the nineteenth century adopted the "new" intuitive methods.
The same trends came to the fore, at the beginning of the seventies of the nineteenth century, in Prussia and, in general, in northern Germany. Instead of a rigid theoretical complex, like that of Euclid's, a natural theoretical path, stimulating

the students' experience, should be preferred. In the specific case of geometry, the starting point was represented by drawing and construction, giving particular value to the formation of space intuition (see, in general, concerning geometry teaching and its history, Barbin & Menghini, 2014).

In Germany, as well as in the other countries, around 1890, new orientations were present, beside these ideas.

Firstly, a strong movement had set up, which wished for a deeper understanding of the applications of mathematics in all branches of natural sciences, particularly in technology, and of its importance for all aspects of human life. With this movement reformative trends were connected, which saw in the teaching of the concept of function, of graphic methods and fundamentals of calculus new stimuli for geometric teaching (see Zuccheri & Zudini, 2007, 2008). Klein himself, on the one hand, stressed the need to adjust the teaching methods and content to contemporary cultural trends and to demonstrate the way mathematics was applied in natural sciences and technology. On the other hand, he was convinced that there should not be too-clear distinctions, in teaching, between the various sections of mathematics, actually citing arithmetic and geometry as examples. He also highlighted the possibility of introducing students fairly early to the concept of function, using analytical geometry (see Klein, 1925, pp. 226ff.).

The mathematicians of Klein's time had, more than in the past, to deal with theories that were the result of "modern" psychological research, particularly in the field of experimental psychology. Beside Herbart's research, for example, the study of memory and of fatigue, conducted at that time by Hermann Ebbinghaus (1850-1909) (see Ebbinghaus, 1895) and by Georg Elias Müller (1850-1934), the director of the laboratory in Göttingen and successor, after Rudolph Hermann Lotze, to the chair of philosophy that Herbart had occupied (see Müller, 1911-1917), should be mentioned.

An important subject, precisely concerning mathematics, was that of individual differences in "talent" and intelligence. After a first period in which people had believed that "mathematical talent" existed – meaning with that expression that only students gifted in mathematics were able to understand mathematics – following the educational plans of Exner and Bonitz, a greater value was given to pedagogy and the opposite opinion developed, affirming that every student with good will and some effort (even by the teacher) should be able to learn mathematics.

In this context, also studies of the different mathematical "species" of mathematical talent were included, that is, for example, the commonly observed fact that a mathematician is more gifted from an arithmetic, abstract point of view than the one more oriented to work with intuitive shapes, from a geometric point of view. Studies had already been done on people who had developed remarkable capabilities in a well-defined field, such as those with significant computational capacity or chess players (see Binet, 1894).

REFERENCES

Barbin, E., & Menghini, M. (2014). History of teaching geometry. In A. Karp & G. Schubring (Eds.), Handbook on history of mathematics education (pp. 473-492). New York: Springer.

Benner, D. (1976). Herbart als Schultheoretiker. Zur Bedeutung seiner Konzeption des „erziehenden Unterrichts“ für eine Entschulung der Schule. In F. W. Busch & H.-D. Raapke (Eds.), Johann Friedrich Herbart. Leben und Werk in den Widersprüchen seiner Zeit. Neun Analysen (pp. 53-66). Oldenburg: Holzberg.

Binet, A. (1894). La psychologie des grands calculateurs et joueurs d'échecs. Paris: Hachette.

Blaß, J. L. (1976). Systemtechnik und pädagogisches Denken bei Johann Friedrich Herbart. In F. W. Busch & H.-D. Raapke (Eds.), Johann Friedrich Herbart. Leben und Werk in den Widersprüchen seiner Zeit. Neun Analysen (pp. 67-78). Oldenburg: Holzberg.

Corry, L. (2004). David Hilbert and the axiomatization of physics (1898-1918). From Grundlagen der Geometrie to Grundlagen der Physik. Dordrecht: Kluwer.

Corso, J. F. (1963). A theoretico-historical review of the threshold concept. Psychological Bullettin, 60(4), 356-370.

Ebbinghaus, H. (1885). Über das Gedächtnis. Leipzig: Duncker & Humblot.

Furinghetti, F. (2003). Mathematical instruction in an international perspective: The contribution of the journal L'Enseignement Mathématique. In D. Coray, F. Furinghetti, H. Gispert, B. R. Hodgson, & G. Schubring (Eds.), One Hundred Years of L'Enseignement Mathématique. Moments of Mathematics Education in the Twentieth Century. Proceedings of the EM–ICMI Symposium (Geneva, 20-22 October 2000) (pp. 19-46). Geneva: L'Enseignement Mathématique.

Furinghetti, F., Menghini, M., Arzarello, F., & Giacardi, L. (2008). ICMI Renaissance: The emergence of new issues in mathematics education. In M. Menghini, F. Furinghetti, L. Giacardi, & F. Arzarello (Eds.), The First Century of the International Commission on Mathematical Instruction (1908-2008). Reflecting and Shaping the World of Mathematics Education (pp. 131-147). Roma: Istituto della Enciclopedia Italiana fondata da Giovanni Treccani.

Geißler, E. E. (1976). Herbarts Lehre vom erziehenden Unterricht. In F. W. Busch & H.-D. Raapke (Eds.), Johann Friedrich Herbart. Leben und Werk in den Widersprüchen seiner Zeit. Neun Analysen (pp. 79-88). Oldenburg: Holzberg.
Heidelberger, M. (1993). Die innere Seite der Natur: Gustav Theodor Fechners wissenschaftlich-philosophische Weltauffassung. Frankfurt am Main: Klostermann. (English edition Nature from within: Gustav Theodor Fechner and his psychophysical worldview. Pittsburgh: University of Pittsburgh Press, 2004.)
Herbart, J. F. (1804). Pestalozzis Idee eines ABC der Anschauung untersucht und wissenschaftlich ausgeführt. 2. Edition. In E. von Sallwürk (Ed.) (1906), Joh. Friedr. Herbarts Pädagogische Schriften. Volume 2 (pp. 81-219). 7. Edition. Langensalza: Beyer & Söhne.
Herbart, J. F. (1822). Über Möglichkeit und Notwendigkeit, Mathematik auf Psychologie anzuwenden. In K. Kehrbach & O. Flügel (Eds.) (1887-1912), Johann Friedrich Herbart. Sämtliche Werke, in chronologischer Reihenfolge. Volume 5 (pp. 91-122). Langensalza. (Reprint Aalen: Scientia, 1964.)
Herbart, J. F. (1824-1825). Psychologie als Wissenschaft, neu gegründet auf Erfahrung, Metaphysik und Mathematik. Königsberg: Unzer. Also in K. Kehrbach & O. Flügel (Eds.) (1887-1912), Johann Friedrich Herbart. Sämtliche Werke, in chronologischer Reihenfolge. Volume 5 (pp. 177-434) and Volume 6 (pp. 1-340). Langensalza. (Reprint Aalen: Scientia, 1964.)
Klein, F. (1925). Elementarmathematik vom höheren Standpunkte aus. II. Band. Geometrie. 3. Edition. Berlin: Springer.
Leary, D. E. (1980). The historical foundation of Herbart's mathematization of psychology. Journal of the History of the Behavioral Sciences, 16, 150-163.
Müller, G. E. (1911-1917). Zur Analyse der Gedächtnistätigkeit und des Vorstellungsverlaufes. Leipzig: Barth.
Raapke, H.-D. (1976). Johann Friedrich Herbart – Pädagoge in den Widersprüchen seiner Zeit. In F. W. Busch & H.-D. Raapke (Eds.), Johann Friedrich Herbart. Leben und Werk in den Widersprüchen seiner Zeit. Neun Analysen (pp. 11-22). Oldenburg: Holzberg.
Romano, D. F. (1976). Der Beitrag Herbarts zur Entwicklung der modernen Psychologie. Eine geschichtskritische Analyse. In F. W. Busch & H.-D. Raapke (Eds.), Johann Friedrich Herbart. Leben und Werk in den Widersprüchen seiner Zeit. Neun Analysen (pp. 89-103). Oldenburg: Holzberg.
Schubring, G. (2003). L'Enseignement Mathématique and the first International Commission (IMUK): The emergence of international communication and cooperation. In D. Coray, F. Furinghetti, H. Gispert, B. R. Hodgson, & G. Schubring (Eds.), One Hundred Years of L'Enseignement Mathématique. Moments of Mathematics Education in the Twentieth Century. Proceedings of the EM–ICMI Symposium (Geneva, 20-22 October 2000) (pp. 47-65). Geneva: L'Enseignement Mathématique.
Schubring, G. (2008). The origins and early incarnations of ICMI. In M. Menghini, F. Furinghetti, L. Giacardi, & F. Arzarello (Eds.), The First Century of the International Commission on Mathematical Instruction (1908-2008). Reflecting and Shaping the World of Mathematics Education (pp. 113-130). Roma: Istituto della Enciclopedia Italiana fondata da Giovanni Treccani.
Zuccheri, L., & Zudini, V. (2007). On the influence of cognitive theories in the teaching of calculus in Austrian secondary schools at the beginning of the 20th century. Rendiconti dell'Istituto di Matematica dell'Università di Trieste, 39, 347-357.
Zuccheri, L., & Zudini, V. (2008). The "Jacob Method": An example of application of cognitive theories in the period of the introduction of calculus in Austrian secondary mathematics instruction. The International Journal for the History of Mathematics Education, 3(2), 57-64.
Zuccheri, L., & Zudini, V. (2014). History of teaching calculus. In A. Karp & G. Schubring (Eds.), Handbook on history of mathematics education (pp. 493-513). New York: Springer.
Zudini, V. (2009a). I numeri della mente. Sulla storia della misura in psicologia. Trieste: EUT.
Zudini, V. (2009b). Psicologia & matematica nel pensiero di Johann Friedrich Herbart. Teorie & Modelli, 14(2), 91-103.