Teaching Mathematics With The Use Of The History Of Mathematics: Some Opportunities Offered By The World Of Psychophysics

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Abstract
The history of mathematics may be useful in education and represent a support in mathematics teaching mainly in two directions: on the one hand, arousing the students' interest in mathematics and, on the other hand, stimulating the reflection about mathematical concepts and methods. This work shows how the figure of Gustav Theodor Fechner (1801-1887) and the discipline of psychophysics which he founded should rightly constitute a chapter of the history of nineteenth century mathematics to be adequately re-evaluated also in the context of secondary school and university mathematics teaching (including teacher training). We will focus on the opportunities which a deep analysis of Fechner's contribution can offer in mathematics education of today.

1. Fechner And Psychophysics

Historians of science credit Gustav Theodor Fechner (1801-1887) with having founded the discipline of psychophysics; his work, which was entitled *Elemente der Psychophysik* and published in 1860, is the first official text of this "new" discipline, presented as a complete and mature science.

Owing to his theory, Fechner is recognized as the scholar who developed the first quantitative methodology of psychology. Fechner, in his work, aims constantly to unite psychology and mathematics, conceived as an essential component in achieving a measurement of mental variables. Mathematics plays in Fechner's theory a key role as a tool that can enable the realization of his scientific project and is a fundamental element for the sheer existence of the discipline of psychophysics, applied at a high level.

The figure of Fechner was central to the issue of mental measurement and his contribution (in terms of the measurement model he proposed) had a strong impact on the reality of nineteenth century measurement in its entirety, having influence up to today. So, his figure and work deserve a prominent place in the landscape of (applied) mathematics of that period.

Focusing on Fechner and psychophysics is of interest in so far as they can be adequately re-evaluated also in the context of secondary school and university mathematics teaching (including teacher training). The fascinating example of Fechner, the person, may significantly stimulate students' interest; an analysis of the concepts and methods (e.g., functions and calculus) used by him and by the subsequent scholars who have dealt with psychophysics provides a good aid to see mathematics from a new point of view and in a new work environment (particularly engaging, concerning our human nature, and different from the traditional approaches: Regarding the interaction of the domains of mathematics education and the history of mathematics in the process of mathematics teaching, see, e.g., Furinghetti & Radford, 2008, and Fenaroli, Furinghetti & Somaglia, 2014).

1.1 Fechner, A Fascinating Figure

Fechner's conception of psychophysics stems from his education, as a physicist and a philosopher, and his life and personality: The study of his figure and the origin of his thought is fundamental for a correct interpretation of his psychophysical theory, in which science and philosophy fuse, and is a very good example for showing (secondary school and university) students how we work when doing (applied) mathematics.

A singularly eclectic figure, when he left school, Fechner attended the Academy of Medicine in Dresden and enrolled at the Faculty of Medicine at the University of Leipzig, where he followed lectures also in logic, botany, zoology, physics, chemistry, pharmacy, anatomy, physiology, obstetrics, and algebra.

Fechner's decision to study medicine soon proved to be an unhappy choice. Although he completed his studies, Fechner did not feel totally fulfilled in this discipline and decided against practising it. Such a negative reaction towards medicine found expression in the satirical contributions which he published under the pseudonym of "Dr. Mises" (see Antonelli & Zudini, 2011). However, none of this undermined Fechner's enthusiasm for science, which remained the fundamental interest of his life.

Fechner was physicist, physician, and philosopher, professor at the University of Leipzig. Initially attracted by the "philosophy of nature" ("Naturphilosophie") - which at that time went against Newton's doctrine, supporting instead the idea of the universe as an animated organism -, he extended his interests to physiology, physics, chemistry, and meteorology. In 1834 he became professor of physics and in 1835 founded the first physics institute in Germany, in the newly-built "Augusteum" in Leipzig. Fechner carried out research mainly on the theory of electricity, electromagnetism and electrical chemistry. Among the first scholars to recognize the importance of Ohm's law (1827), he distinguished himself by using the quantification method and by collecting a great amount

of experimental data, which he then analysed and discussed in his research. In 1832 he published the "Repertorium der Experimental-Physik", forerunner of the journal "Fortschritte der Physik", which was founded in 1845 (for more details regarding Fechner's figure and work see, e.g., Brožek & Gundlach, 1988, Heidelberger, 1993, and Zudini, 2009).

1.2 The Definition Of Psychophysics

The central role which mathematics would have in psychophysics is clear from the very definition of it at the beginning of *Elemente*: Psychophysics is conceived by Fechner as the "exact doctrine" of the "functional or dependence relationships" between body and soul, more generally, between the bodily and spiritual, physical and psychical worlds (Fechner, 1860, vol. I, p. 8).

A first element emerges immediately in this definition: the conception of psychophysics as "exact doctrine", which is characterized, in particular, as a study of the quantitative links that bind physical stimuli to mental variables (sensations) related to them. The basic idea of the conception is to consider the human perceptual system as a measuring instrument (more precisely as a physical instrument) that allows the quantification of sensory experience.

The philosophical basis for an assumption of this kind is the second major element in the definition of psychophysics: the "functional" conception of the relationships between body and soul, matter and spirit, and, more generally, between the physical and the psychical worlds, which in Fechner takes the form of an "identity view" ("Identitätsansicht"), already presented in the appendix to the second part of the work *Zend-Avesta* (Fechner, 1851; see also Heidelberger, 1993). In *Zend-Avesta, oder über die Dinge des Himmels und des Jenseits. Vom Standpunkt der Naturbetrachtung*, Fechner's panpsychist philosophy finds expression in the theory of universal animation, which goes back to Friedrich Schelling and has a strong oriental character, and the program of the future discipline of psychophysics is already presented in an embryonic way.

According to the "identity view" by Fechner, matter and spirit are expressions of a single reality which manifests itself at times in one way and at times in another, just as the two sides of the same coin or medal, depending on the (internal or external) observation point from which one looks at it. On the basis of the principle of identity view, Fechner postulates the existence of a general and constant relationship of functional dependence between physical and psychical phenomena, that can be expressed in exact terms, through a law which has, as he intends it, elementary and universal character, similar to that of the law of gravitation. This allows, in Fechner's theory, quantitative analysis of psychical dynamics and paves the way to realizing his project of a scientific foundation for the disciplines of the mind.

The appendix to the second part of the work *Zend-Avesta* is also important because, as above mentioned, it contains, in an embryonic way, a "new principle of mathematical psychology", which would be the basis of the future discipline of psychophysics (Fechner, 1851, vol. II, pp. 373-386). Fechner himself remembers the morning of October 22, 1850 (still celebrated today as the birth date of psychophysics) when, while he was in bed, he had the intuition that the relative increase in the physical vital force could be the measure of the increase in the corresponding mental intensity (Fechner, 1860, vol. II, p. 554).

The aim to carry out the experimental measurement of the sensations and the foundation of a quantitative science, which is evident from the very definition of psychophysics as "exact science", clashes with the (seemingly insurmountable) problem of the inaccessibility of sensations from direct, external observation.

As we will see, Fechner solves this problem by developing a "correlative theory" of the measurement. Instead of a direct measurement, he implements a psychical quantification by indirect means, through physical processes corresponding to the sensation, on the basis of the functional relationship of reciprocity that allows a scholar to choose the observation point which seems the better one (or, ultimately, in this specific case, the only feasible one): The scholar can measure the stimuli correlated to a sensation and determine the thresholds of sensations, using them as a unit of measurement.

1.3 Fechner's Model Of Measurement

According to Fechner, any exact doctrine has to begin with the measurement of its objects; therefore, it is necessary to show how psychophysics allows the measurement of psychical magnitudes.

Already in 1858, anticipating the treatment of *Elemente*, Fechner had considered the status of measurement in psychology and had noted the problems and the relevant polemics connected with this operation (Fechner, 1858). Fechner was convinced that a quantitative aspect was undeniable in psychical phenomena; the question was to prove the actual correlation between quantity and measure in them.

"Measuring" means describing data through numbers, and therefore using mathematical rules in investigations. Fechner says clearly how this should be done in psychophysics when he affirms, proposing an "Euclidean model of measurement" (Zudini, 2011), that:

> In general, the measurement of a magnitude consists in verifying how many times a magnitude of the same kind taken as a unit is contained in it. (Fechner, 1860, vol. I, p. 45)

Sensations are to be measured through the measurement of the stimuli that induce equally perceptible sensations, finding a method for establishing the equality between two given sensations. The specific method adopted by Fechner is the following: Two sensations of the same kind are perceived as different only when the difference between the corresponding physical variables is greater than a minimum value ("differential threshold").

As explained in Zudini (2011), Fechner faces the problem of measurement in two phases: In the first phase, he intends to indicate the conditions, which are necessary and at the same time sufficient, for saying, in general, that a magnitude, both of physical and psychical nature, is measurable - from 1858, he will call these conditions the "general principle of measurement" ("allgemeines Maßprincip"). With the second phase, Fechner seeks to put into practice the principle of measurement for psychical magnitudes, that is, to find an empirical application of this principle in the psychical field.

1.3.1 The General Principle Of Measurement And Its Application In The Psychical Field

So as to guarantee the possibility of measuring a magnitude, the general principle of measurement sets out a series of conditions, which can be expressed schematically in the following way (Heidelberger, 1993, p. 220):

1. Magnitudes have to be conceived as phenomena that can increase or decrease in a continuous way;
2. There must exist a grade of difference, which can be reproduced or verified in various situations, such that one can decide if it is equal or not to the grade of difference between any other two magnitudes;
3. One can distinguish the conditions under which a magnitude has the value zero.

These conditions determine the possibility of measuring in the sense of "counting equals". In fact, Condition 2 gives a value "difference" that can be compared with other differences and therefore used as a unit for measurement. Any other value can in fact be expressed as consisting of a number n of units, starting from zero (Condition 3).

The general principle of measurement has a clearly theoretical character since it does not explain, for example, how to reproduce or discover the magnitude unit, nor how to proceed to the necessary comparisons between magnitudes. This principle is therefore to be considered in the light of the various types of magnitudes and verified in its validity in every field.

In the case of mental phenomena, according to Fechner, one can see at a glance that Condition 1 is satisfied, insofar as mental phenomena such as consciousness or attention are susceptible to a variation in grade; sensation, feeling, instinct, or volition can be weak or strong. Sensations can therefore be conceived as continuous magnitudes generated according to a gradual process starting from the zero condition.

Condition 2 requires us to be able to estimate the equality of the differences of sensations, which is, in general, difficult to carry out. In the case of sensations that are present at the same time and very different in intensity, a subject is, at most, able to estimate that one is stronger than the other; but to assess the size of the difference between the two is a very different thing. The latter implies that we have successfully derived a scale of sensation strength. To find this kind of scale, we need a unit. The difficulties that recur in this operation come from the very nature of sensations, which cannot be treated as material objects. For example, they cannot be put side by side as one does when estimating the length of two objects; neither is it possible to find a standard that allows for such a comparison.

The question is to find a proper procedure for psychical magnitudes; for this purpose, it is sufficient to exploit two other conditions which are valid for physical standards of measurement and which we can formulate in the following way (Heidelberger, 1993, p. 224):

4. A definition of a standard unit of measurement of physical magnitudes can be obtained by relating this unit to other dimensions, with which it is functionally connected;
5. In order to achieve standards of measurement of physical magnitudes we have to rely on our mental impressions, which are produced in us by the material quantities.

Condition 4 means that, when we measure physical objects, we have to deal not with pure dimensions but with dimensions which are expressed through concrete standards, as happens, for example, in the case of time: Fractions of time are measured not directly through time itself but using an external, concrete standard, such as the movement of the hand of a watch.

It is clear that, when we measure units of time, we have to deal with the units of space with which they are connected, considering as a point of reference both the movements of the heavenly bodies and the hands of the watch. That happens in (classical) physics also for forces: The forces themselves are not measured directly (which is impossible); we measure, instead, magnitudes which are relevant or dependent on them, such as the velocity variation of equal masses, or the differences of masses having the same velocity.

According to this interpretation of measurement, which could be defined as "correlative" (Heidelberger, 1993; Murray, 1993), for the measurement of any dimension Q, we need a directly observable dimension R and a device or apparatus of measurement A, properly calibrated and able to represent the values of R in correlation with those of Q in a unique and monotonic way (i.e., capable of preserving the order of the values of the two dimensions).

In the above quoted case of time, which is measured through the movements of the hands of the watch, we can construct a watch in order to supply such a representation: If we know the length of a distance covered by the hand d, we have at a glance the quantity of correspondent time t, according to the formula ("measurement formula") $d = c \cdot t$, with c constant.

In the case of sensation, we have to search in an analogous way for a dimension which is correlated to it as space is to time, and that dimension is, according to Fechner and his "identity view," the external physical stimulus. The apparatus that plays here the role of the watch is the human body, which is able to correlate, as the model of measurement itself demands, the values of intensity of sensation with those of stimulus. We must only find the "measurement formula" that expresses that correlation.

Condition 5 emphasizes the importance of sensorial impressions in the process of constructing standards of measurement of physical magnitudes. Still considering the example of time, if we use a watch, calibrated in minutes, it is evident that we conclude that a minute has passed from the mental impression that the distance covered by the hand is equal to the space between two notches. In the process of derivation of standards of measurement for physical magnitudes, the consciousness of equality of a standard to the measured object is fundamental, and this consciousness depends on the subjective mental impressions which we have of concrete physical quantities. The subjective component is therefore necessary in the process of measurement, but can be made objective, restricting its role to the function of comparison of the correspondences existing between the device of measurement and the measured object.

In the process of measuring physical magnitudes we use mental impressions in order to determine the equality of standards and measured objects; therefore, we determine the magnitude of the physical by connecting the physical with the psychical. In the case of measuring psychical magnitudes, on the other hand, we have to reverse this relation. This is possible only after arranging and calibrating a physical scale in such a way that equal psychical magnitudes result from intervals of equal dimension on the scale, according to a law which allows us to connect increments of stimulus and increments of sensation and to deduce therefore, from the intensity of the stimulus, the intensity of the associated sensation.

In the case of measuring time through a watch, the determination of such a law is simple, since equal intervals of time correspond to equal distances covered by the hand. That does not happen for the measurement of sensation through the stimulus connected with it; in such a case, equal differences in sensation do not correspond to equal differences in the stimulus.

The question is to find a function that expresses the reality of the process. The solution to the problem proposed by Fechner follows, by analogy, the procedure used in (classical) physics for measuring forces: In this case too, we can try to measure the magnitude of the sensation produced by equal stimuli or the magnitude of the stimuli which induce an equal sensation. The first possibility is, however, not practicable in psychophysics, since sensation is not susceptible to direct measurement; therefore, the second possibility is left, which consists of assuming the variable sensation as constant and in measuring the physical stimuli for which it remains constant. Here, the distinction made by Fechner between absolute sensitivity and differential sensitivity comes into play. With the first, he means the organic response to the absolute value of a given stimulus; the second refers, on the other hand, to the response to the relative difference between two stimuli and varies according to the variation of the absolute magnitude of the stimulus.

1.3.2 The Measurement Of Sensations

With such conditions posed for measurement, we have to be able, according to a model which follows Euclid (Zudini, 2011), to find a homogeneous unit of measurement and to determine the number of times in which it is contained in the magnitude to be measured. As time is measured through the hand of the watch and space through the ruler, so we can deal with sensation, using the stimulus for its measurement: We proceed to subdivide sensation into equal sections (i.e., in the equal increments which its growth from zero consists of) and to consider the number of these equal sections as determined by the number of the correspondent increments of the stimulus, which can provoke equal increments of sensation ("Maßprincip").

These increments in sensation, determined through physical units of variable magnitude which are able to provoke equal increments in sensation, are assumed as units of measurement.

The sensations, as we know them, satisfy, therefore, in Fechner's system, the conditions posed for measuring a magnitude. In particular, the fulfilment of those conditions is strictly connected with Fechnerian thought in its totality and achieved on the basis of the parallelism between the physical and psychical and of the functional relation of reciprocity which justifies the scholar in choosing the point of view which he considers more proper (or the only practicable). If we cannot measure sensations directly, nevertheless we can measure the stimuli that provoke them and determine the thresholds of sensations, especially the differential ones, using them as the unit

of measurement; we have to measure sensations through the measure of the stimuli that induce equally noticeable sensations, finding a method for determining the equality of two given sensations.

For this purpose, Fechner uses the results on differential sensitivity obtained years earlier by the physiologist Ernst Heinrich Weber (1795-1878), according to which equal relative increments in stimulus correspond to equal increments in sensation (see Weber, 1834, 1846); he generalizes them and, also with the aim of statistical methods, sets out what he calls "Weber's law" ("Weber'sches Gesetz").

1.3.3 Fechner's Measurement Formula

Starting from Weber's law, Fechner takes what he considers to be an experimental result - the fact that the just noticeable difference in sensation is constant - and applies the differential and integral calculus to sensation conceived as a phenomenon which increases in time and is susceptible to infinitesimal variations.

Moving from the differential equation:

$$d\gamma = k\,\frac{d\beta}{\beta}$$

by integrating both members of the equation and imposing the condition of the threshold for the stimulus magnitude - i.e, for the threshold value of the stimulus magnitude, the sensation magnitude is zero -, Fechner obtains his famous logarithmic "measurement formula" ("Maßformel"):

$$\gamma = k\,\log\frac{\beta}{b}$$

where γ is the sensation magnitude, β the stimulus magnitude, b the threshold value of the stimulus magnitude β (called the "absolute threshold" of the stimulus magnitude), and k is a constant (called "Weber's constant", depending on the sensory modality) (see Fechner, 1860, vol. II, pp. 10-13; for in-depth analysis of this formula, see Masin, Zudini & Antonelli, 2009, and its references).

The formula of measurement proposed by Fechner is called, in current psychophysics and, generally, in science, "Fechner's law" (see Figure 1) and is considered the first explicit, quantitative formulation connecting sensations with stimuli (Algom, 2003).
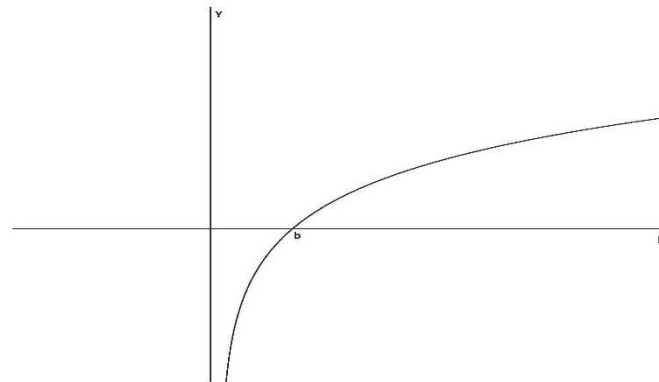

Figure 1. "Fechner's law"

Fechner arrives at his result giving credit to several scholars who preceded him (see Fechner, 1860, vol. II, pp. 548ff.): first of all, the aforementioned Ernst Heinrich Weber, who, according to Fechner, gave unity to various events observed in the psychophysical sphere and brought new evidence to the discipline, making it an exact and "connected" science. It was he who set out the first clear and, in some way, general enunciation of the psychophysical law, determined by using a method of measurement of sensation in almost all the fields of perception and called by Fechner "Weber's law".

According to Fechner, a fundamental role in the development of his psychical measurement theory ("Maßlehre") was also played by researchers who studied the mathematical (logarithmic) function through which psychical magnitudes are linked to physical quantities in some specific areas: Among these, Daniel Bernoulli is to be mentioned with his determination of the dependency relationship of "moral fortune" on "physical fortune" (see Bernoulli, 1738).

By elaborating all these contributions in the context of a very particular panpsychist philosophy, Fechner comes to the determination of the functional relationship between the physical and mental worlds, between body and soul,

formulating a law which expresses their relationship in precise and quantitative terms and, in this way, achieves a psychical "measurement".

Fechnerian psychophysics, whose principal result is Fechner's law, represents therefore a composite theory in which different elements or "ingredients" are harmoniously fused together (Zudini, 2009) according to the following schema (Figure 2):
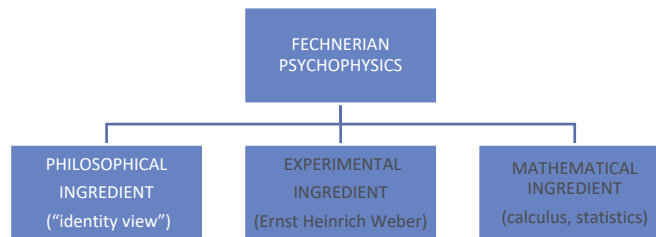

Figure 2. The structural system of Fechner's psychophysics

The mathematical ingredient, as reported in this scheme, assumes a key role inasmuch as it enables Fechner to treat mathematically mental phenomena and to pursue his aim of building a psychological metric, anchored in the physical world (see Romano, 1976).

2. Fechner's Heritage
The impact of Fechner's work was immediately great on the scientific community: For the first time, a rigorous project of empirical and experimental research, which was guaranteed by the possibility of measuring mental phenomena, based upon mathematics, was begun and carried out (see Zudini, 2011).

2.1 The "Fechner Case" And The Debate On It: From Then Up To The Present Day
Fechner's project was the object of lively discussion, in particular in the Mitteleuropean cultural world (extending to France and Belgium) among scholars from very different disciplines, who dealt with the "Fechner case": Among them, many scientists of the nineteenth and twentieth centuries, such as Joseph Antoine Ferdinand Plateau, Hermann von Helmholtz, Ernst Mach, Jules and Paul Tannery, Joseph Delboeuf, and David Hilbert, are to be mentioned (see Zudini, 2009, 2011; Antonelli & Zudini, 2012; Zudini & Zuccheri, 2016).

The model of measurement proposed by Fechner became that of reference, which no scholar could neglect: a model to criticize, correct, or confute, in the methodological aspects of its (empirical and mathematical) procedures or even in its psychophysical, physiological, or, in a strict sense, psychological value itself; in certain cases, it was a model to reject in a radical way, on the basis of the assumption that it was impossible to measure sensations and, in general, psychical magnitudes and therefore to make a scientific study on them.

So, as described in Zudini (2011), Fechner's destiny would be analogous, in a certain way, to that of Euclid, whose model was considered for centuries as the "summa" of geometrical knowledge and of logical rigour, as the correct idealization of the properties of physical space and of the figures of this space, and was then called into question in particular with the advent of non-Euclidean geometries in the nineteenth century. Fechner's attempt to apply a model of measurement to sensation magnitudes which goes back to Euclid himself (Zudini, 2011) - and the debate that followed - generated ideas, concepts and theories which were destined to have rich developments in the scientific field of the twentieth century and that still animate and are relevant in current psychophysics.

The discussion on Fechner's work was initially developed from three types of issues: The first concerned the correctness of the law proposed by Fechner on the basis of the experimental data and the mathematical techniques which he used; the second was related to the very nature of the law; the third regarded the possibility of measuring sensations and mental magnitudes in general. In particular, it is to be said that the response given by the cultural world of the French language to Fechner's psychophysics was negative both from a theoretical and a mathematical point of view, based on the conviction that the application of mathematical models to the psychical sphere required further analysis and experimentation.

As for the first aspect, namely the correctness of the mathematical law obtained by Fechner - the aspect which is the most interesting in this context and on which we focus, also considering the development of psychophysics -, the contribution of Joseph Antoine Ferdinand Plateau (1801-1883), a Belgian physicist and professor at the University of Gand (Ghent), is to be mentioned. Starting from experiments which he conducted on the bisection of sensorial intervals (Plateau, 1872), Plateau, unlike Fechner, derived a power function as psychophysical law (which was also proposed by the German philosopher and psychologist Franz Brentano (1838-1917): See Brentano, 1874). Plateau's power function would come to the fore again about a century later, in the so-called "modern psychophysics" of the American S. S. Stevens (Stevens, 1957). The latter liked to mention Plateau among those who, in the nineteenth century, had anticipated his power law.

The general form of this law is

$$\psi(I) = kI^{a}$$

where I is the magnitude of the physical stimulus, $\psi(I)$ is the subjective magnitude of the sensation evoked by the stimulus, a is an exponent which depends on the type of stimulation, and k is a proportionality constant which depends on the units used.

**2.2 Toward A "New" Conception Of Mental Measurement**
Starting from the 18th century, the question of the possibility of measuring the intensity of mental variables was much discussed. Fechner, Plateau, and other contemporary scholars treated this question considering the two "worlds", the physical and the mental ones, as arranged according to the following schema:

PHYSICAL   MENTAL

Φ ———————————————→ Ψ
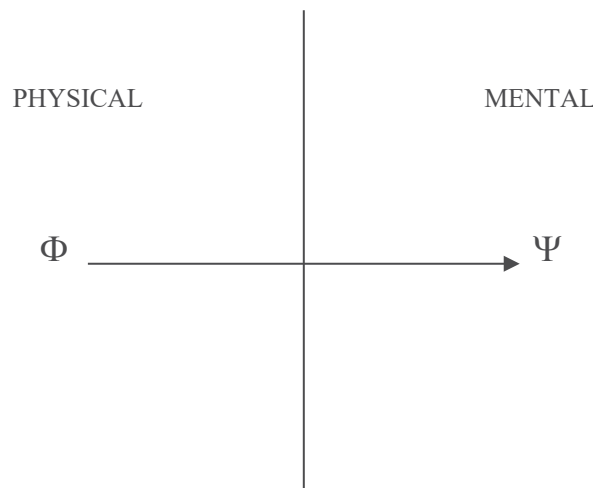
In this schema, a mental variable Ψ corresponds to every physical variable Φ. The problem is then reduced to finding the function ("psychophysical law") that connects Φ and Ψ.
It is evident that in this formulation, which dominated the history of psychology from Fechner until the 1960s, the problem is somewhat simplified because of the (undoubtedly limited) way of conceiving physical and mental variables. It is clear, in fact, that, for certain mental variables Ψ, there is no physical correspondent Φ. The very search for the function that connects Φ and Ψ creates difficulties, related to the arbitrary nature of the choice of the formula, among the various existing possibilities, and the measurement units used from time to time (see Masin, 2003, 2004).
Starting from the 1960s (Attneave, 1962; Curtis, Attneave & Harrington, 1968), a new way of dealing with the problem was stated, characterized by a decisive "leap in quality"; the schema became more complex, involving a further "world", that of the responses (of a subject):

PHYSICAL   MENTAL   RESPONSES

Φ Ψ J
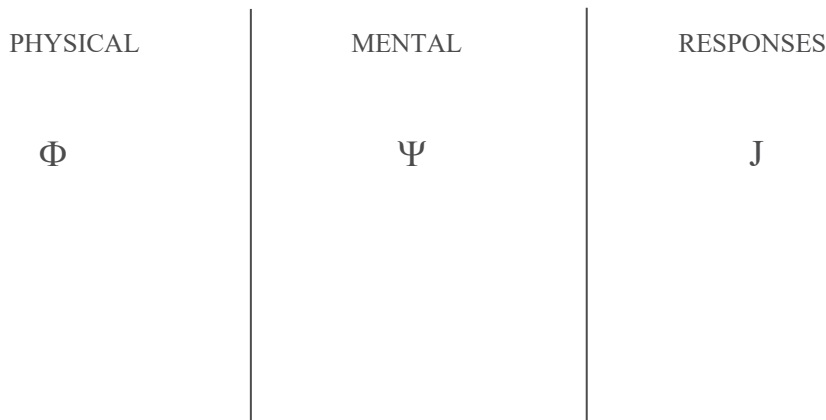
Responses played an important role and had to be taken into account, as the world of responses of a subject is accessible, unlike the mental one.
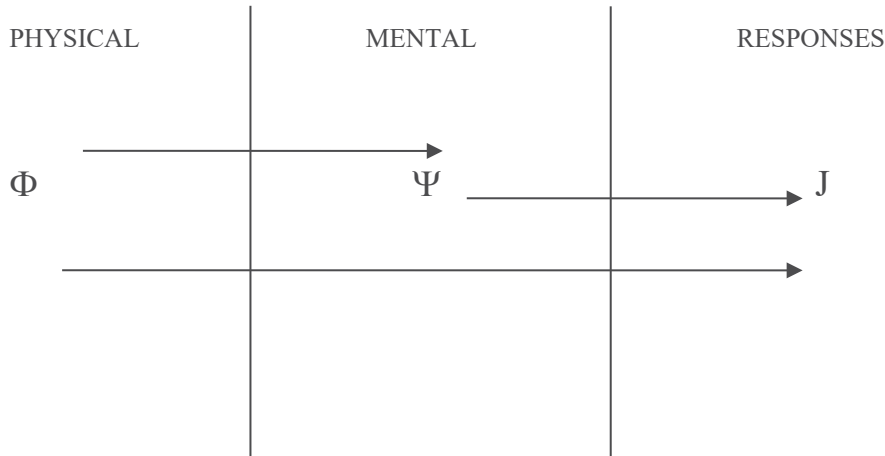
The relationship between Φ and Ψ is what Fechner sought, while the relation between Φ and J is known, since it is possible to measure the variables Φ and the variables J. For a psychologist, it is more important to determine the relation between Ψ and J.

For example, if the relationship between Ψ and J is the identity, we would have

$$J = \Psi$$

as Stevens, essentially, affirmed.

Other authors have believed, on the contrary, that

$$J \neq \Psi$$

A possible hypothesis is that J is equal to Ψ, up to an unknown constant, namely

$$J = a \cdot \Psi$$

with a an unknown constant.

Another possibility, less "good" (from the operative point of view, for those who do research in psychology) than the former, is that

$$J = a \cdot \Psi + b$$

with a and b unknown constants.

Further, it is possible that there are more complex relationships between J and Ψ, namely:

$$J = F(\Psi),$$

with F a non-linear function.

In psychology, scholars are interested in knowing which relationship is the correct one, in order to calculate Ψ.

Today, in the context of mental measurement, the studies by Norman H. Anderson, carried out starting from the 1960s, are an example of interest. Anderson has proposed a theory called "information integration theory" (hereafter, IIT) which "illustrates" the "modern aims" of psychophysics (Masin, 2004).

Through this theory (see, e.g., Anderson, 1981, 1982, 1991), Anderson intends to "enter" into the human mind more than Fechner and others before him did: He "reaches" not only Ψ, but "arrives" at measuring the mental quantities which are generated "before" Ψ, according to a procedure appreciably different from that of classical psychophysics.

Anderson formulates two axioms, the axiom of "purposiveness" and the axiom of "integration", and starting from these develops, on an inductive basis, his theory. This is proposed as a "unified" theory (Anderson, 1996) and aims at explaining, by using the very concepts of everyday life, the multiplicity of factors that contribute, combining (or indeed integrating) with each other, to the processing of information in various mental processes, in the context of a vision of a human being as an active elaborator of information in order to reach different goals.

IIT gives, according to Anderson (2007), a solution to the age-old problem of measuring the sensations in psychophysics, going beyond the question of determining the "psychophysical law" between stimulus (physical variable) and sensation (mental variable) which was typical of classical psychophysics.

IIT aims not only at measuring the intensity of sensations, but also at discovering the laws governing the process of integrating information in the context of the genesis of the sensations themselves; the measurements of this

information are obtained on the basis of these integration laws (which are said to be simple: additions, multiplications, weighted averages, or combinations of these - hence the idea of a "cognitive algebra" as a tool not only in psychophysics, but in the entire psychology).

IIT can be schematized using a diagram ("functional measurement diagram") (Figure 3), where the (observable) stimuli, indicated by S_1, S_2, S_3, on the left side of the diagram, are factors of the (observable) response, denoted by R, on the right side of the diagram.
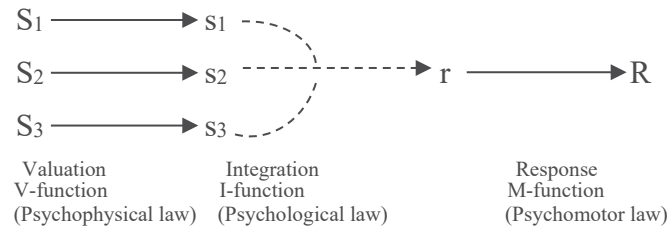

Figure 3. The functional measurement diagram, which is based on Anderson (1981, p. 5)

Between each of the stimuli S_i and R there are three functions: the valuation function (or "psychophysical law"), the integration function (or "psychological law"), and the response function (or "psychomotor law"), in symbols V, I, and M, respectively, representing the transition from S_i to R. The S_i are transformed, through the valuation function V, into the corresponding (not observable) psychological stimuli s_i; these stimuli s_i combine with each other, through the integration function I, into the unitary, implicit (unobservable) response r, which is then transformed by the response function M into the (observable) R response, according to

$$V(S_i) = s_i$$
$$I\{s_i\} = r$$
$$M(r) = R$$

Three unobservable entities are present in the diagram: the integration function I; the psychological stimuli s_i, corresponding to the physical ones S_i and internal to the body, often unconscious; the implicit answer r, which can be conscious, but it is also internal to the body.

Functional measurement solves the problem of "the three unobservable entities" through the joint solution of the following three related problems:

1. the measurement of the psychological stimuli s_i;
2. the measurement of the implicit response r;
3. the determination of the integration function I.

2.3 Psychophysics And "Virtual Reality": Looking Ahead To The Future

Today's psychophysics - obviously evolved from Fechner's time in terms of physiological and methodological knowledge, as we have briefly seen - is a lively area of research that analyses the perceptual processes by studying the effects of physical stimuli on a subject. These stimuli are constructed by an investigator according to his research needs and have controlled and defined physical properties. Stimuli can be of visual, auditory, tactile and olfactory type.

Psychophysical research is aimed primarily at determining the "perceptual threshold" value below which sensory stimuli are not perceived. Scholars have to systematically vary the magnitude of the stimuli on the basis of different methodologies, in order to define "psychometric" functions, which describe the link between the magnitude in question and the extent of the perception that human beings have of it. Therefore, they proceed, in the context of an "experiment", collecting "experimental data", which express, through their numerical value, the perceptions that subjects have according to the variation of the magnitude of the stimuli. An experiment places subjects in a "small virtual reality": Subjects' tasks will be to indicate (often simply responding to a question with only two forced answers) their own level of perception of the magnitude in its variation.

With the advent of computers, computational capabilities have been revolutionized: In just a few years, it has become possible to obtain performances that greatly reduce the time required to simulate, analyse and solve "systems"; the number of variables that can be used in the experiments has been significantly increased and the quality of the response of the simulated systems has radically improved. The electronic calculation methods, gradually developed due to technology, have become essential instruments for the realization of "virtual realities", which are able to simulate systems in almost all their aspects.

In the research field of psychophysics, experiments can actually be considered "small virtual realities"; subjects interact with these and respond on the basis of their own perceptions and cognitions. Using a programming language, combined with graphics and sound libraries, scholars can create virtual environments on the computer with highly realistic graphic and sound details. "Optimal" software tools for the realization of experiments of

visual and acoustic perception with the use of an electronic calculator, and then for the creation of small virtual realities, are "MATLAB" and the "Psychtoolbox" libraries (here the term "optimal" refers to a balance between the high calculation performances obtained and the low production times of the final result: This is possible thanks to "MATLAB", which makes available almost all the mathematical tools through a very intuitive and simplified language; therefore, one can initialize and process large amounts of data with a minimum design effort. See, e.g., Brainard, 1997, Pelli, 1997, and Shreiner et al., 2012).

Psychophysical experiments require the use of mathematical tools (such as analytic geometry, numerical analysis, trigonometry, Boolean algebra, complex analysis, statistics) primarily for defining and processing the data which are representative of the stimuli and reworking the output data (i.e., the magnitudes related to subjects' responses).

3. Conclusion

Following the way of "mathematization" indicated by Fechner and developed afterwards, psychophysics has elaborated over time increasingly rigorous methods for studying mental events, namely for evaluating and modelling them. These methods have been incorporated into many different disciplines (e.g., sensory systems, cognition, memory, psycholinguistics) and - often being modified to meet the specific needs of each discipline - have contributed, in a significant and successful way, to the state of knowledge in them.

Fechner's project of combining mathematics and psychology to achieve a mathematical treatment of the mind and constituting a mathematical science of humans has therefore borne fruit. We can hazard that psychophysics, as a discipline, has gone beyond the most optimistic expectations of Fechner, who was, after all, a son of the physiological and technical knowledge of his time.

As the present work has shown, in the development of psychophysics, mathematics has always played, from its origins until today, a central role in order to unite the physical world and the mental world - as the name of psychophysics itself suggests.

Those who do psychophysical research can realize it every day. And Fechner deserves to be remembered for his mathematical contribution, just as the psychophysical scholars, members of ISP ("International Society for Psychophysics"), have organized every year, since 1985, a "Fechner Day" in order to discuss psychophysics and its developments or trends. It is no coincidence that, when possible, "Fechner Day" takes place on the anniversary of the intuition, mentioned at the beginning of the present work, that Fechner had on October 22, 1850 - namely, that the relationship between the stimulus and the sensation was capable of being expressed in mathematical terms (specifically, with a logarithmic formula) - and from which psychophysics would originate.

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