On the periodic Ambrosetti-Prodi problem for a class of ODEs with nonlinearities indefinite in sign *

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Abstract

We prove a result of Ambrosetti-Prodi type for the scalar periodic ODE $x' = f(t, x) - s$, where, seemigly for the first time in the literature, $f(\cdot, x)$ is allowed to have indefinite sign as $|x| \to +\infty$. Our result requires that f satisfies a one-sided growth control; in case such a control fails, non-existence occurs for large $s > 0$, although multiplicity of solutions can still be detected provided $f(\cdot, 0) = 0$ and $s > 0$ is small enough.

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1 Statements

The Ambrosetti-Prodi problem for an equation of the form

$$F(x) = s \tag{1.1}$$

consists of determining how varying the parameter s affects the number of solutions x. Usually, an Ambrosetti-Prodi type result yields the existence of a number s_0 such that (1.1) has zero, at least one or at least two solutions according to $s < s_0$, $s = s_0$ or $s > s_0$. This terminology has become current after the founding work by A. Ambrosetti and G. Prodi [1] in 1972. Since then Ambrosetti-Prodi type results have been proved for several classes of boundary value problems: a thorough bibliography would include nearly two hundred titles.

Here, we revisit the simplest case of the scalar periodic ODE

$$x' = f(t, x) \tag{1.2}$$

and the associated periodic Ambrosetti-Prodi problem

$$x' = f(t, x) - s. \tag{1.3}$$

Throughout we assume that $s \in \mathbb{R}$ is a parameter and

(h_1) *$f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is T-periodic with respect to the first variable and satisfies the L^1-Carathéodory conditions.*

Hereafter, by a T-periodic solution of (1.2) or (1.3) it is meant a T-periodic function $x : \mathbb{R} \to \mathbb{R}$ which is locally absolutely continuous and satisfies the equation for a.e. $t \in \mathbb{R}$.

Under the coercivity condition

$$f(t, x) \to +\infty, \text{ as } |x| \to +\infty \text{ uniformly a.e. in } t, \tag{1.4}$$

the periodic Ambrosetti-Prodi problem for (1.3) has been investigated by several authors, since the early eighties until very recent years: we refer to the bibliografies in [5, 6, 7] for a rather complete list of references. Thanks to its simplicity, (1.3) is in fact a quite good sample problem: manifold techniques can be effectively tested on it and the obtained results can suggest possible extensions to more general and complicated contexts.

In the case where f is a Bernoulli-type nonlinearity, i.e.,

(h_2) *there exist $a, b \in L^1(0, T)$ and $p > 0$ such that $f(t, x) = a(t)|x|^p + b(t)$ for a.e. $t \in [0, T]$ and all $x \in \mathbb{R}$,*

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the coercivity assumption (1.4) amounts to requiring that $\operatorname{ess\,inf}_{[0,T]} a > 0$. However, when modeling, for instance, population dynamics, it is interesting to include cases where the function a vanishes on sets of positive measure or changes sign, in order to describe the occurrence of seasonal periods which inhibit or adversely affect the growth rate of the population under consideration. A real outbreak of papers devoted to the study of nonlinear problems which are indefinite in sign dates back to the eighties of the last century both in the PDEs and the ODEs settings, together with a parallel renewed interest towards ecological models (see, e.g., the monograph [2]).

First relevant progresses in relaxing the uniform coercivity assumption (1.4) were achieved in the recent papers [7, 8, 3]; precisely, the following result for equation (1.3) was obtained in [7].

Theorem 1.1. [7, Theorem 3.3] *Assume* (h_1),

(h_3) *there exist* $a, b \in L^1(0,T)$ *such that* $f(t,x) \geq a(t)|x| + b(t)$ *for a.e.* $t \in [0,T]$ *and all* $x \in \mathbb{R}$,

(h_4) *there exists* $\overline{x} \in \mathbb{R}$ *such that* $\operatorname{ess\,sup}_{t\in[0,T]} f(t,\overline{x}) < +\infty$,

(h_5) *for every* $K_1, K_2, \sigma \in]0,+\infty[$, *there exists* $d > 0$ *such that, for every* $x \in C^0([0,T])$ *with* $x(0) = x(T)$, *if*

$$\max_{[0,T]} |x| \leq K_1 \min_{[0,T]} |x| + K_2 \tag{1.5}$$

and either $\min_{[0,T]} x \geq d$ *or* $\max_{[0,T]} x \leq -d$, *then* $\int_0^T f(t,x)\,dt > \sigma$.

Then, there exists $s_0 \in \mathbb{R}$ *such that equation* (1.3) *has zero, at least one or at least two* T*-periodic solutions according to* $s < s_0$, $s = s_0$ *or* $s > s_0$.

It is easy to check (see, e.g., [7, Corollary 4.1]) that (h_5) holds whenever the function a which appears in (h_3) satisfies both

(h_6) $\quad a(t) \geq 0$ *for a.e.* $t \in [0,T]$

and

(h_7) $\quad \int_0^T a(t)\,dt > 0$.

Accordingly, condition (h_5) permits to consider nonlinearities which are just locally coercive, although bounded from below by a L^1-function.

In this short note we want to push further into the direction of relaxing the coercivity assumption on f, by showing that the non-negativity condition (h_6) can be dropped at all, while still achieving all the conclusions of Theorem 1.1. Namely, we can prove the following result.

Theorem 1.2. *Assume* (h_1), (h_4),

(h_8) *there exist* $a, b \in L^1(0,T)$ *and* $p \in]0,1]$ *with* $f(t,x) \geq a(t)|x|^p + b(t)$ *for a.e.* $t \in [0,T]$ *and all* $x \in \mathbb{R}$,

and (h_7). *Then, there exists* $s_0 \in \mathbb{R}$ *such that equation* (1.3) *has zero, at least one or at least two* T*-periodic solutions according to* $s < s_0$, $s = s_0$ *or* $s > s_0$.

Assumptions (h_8) and (h_7) basically require f being coercive on the average and allow that

both $\lim_{|x|\to+\infty} f(t,x) = +\infty$ and $\lim_{|x|\to+\infty} f(t,x) = -\infty$ on sets of positive measure.

It is worth stressing on the other hand that condition (h_5) prevents f from exhibiting this behavior, at least if f has the Bernoulli-type structure (h_2), as expressed by the following statement.

Proposition 1.3. *Assume* (h_2). *Then, condition* (h_5) *is equivalent to conditions* (h_6) *and* (h_7).

Remark 1.1 The proof of Theorem 1.2 is based on the direct construction of lower and upper solutions. Thus, from the results in [5], it is possible to infer various information on the qualitative properties of the obtained solutions. Indeed, for each $s > s_0$, equation (1.3) has at least one T-periodic solution which is weakly asymptotically stable from below, at least one T-periodic solution which is weakly asymptotically stable from above and at least one weakly stable T-periodic solution (all these solutions may possibly coincide), as well as, in addition, at least one unstable T-periodic solution, while for $s = s_0$ it has at least one unstable solution.

A question that may arise looking at Theorem (1.2) is whether or not one can assume $p > 1$ in condition (h_8). The answer is in general negative as shown by the following statement.

Proposition 1.4. *Assume* (h_1) *and*

(h_{10}) *there exist* $p > 1$, $I = [t_1, t_2] \subseteq [0, T]$ *and* $\delta > 0$ *such that* $f(t, x) \leq -\delta|x|^p$ *for a.e.* $t \in I$ *and all* $x \in \mathbb{R}$.

Then, there exists $\sigma \in \mathbb{R}$ *such that, for all* $s \geq \sigma$, *equation* (1.3) *has no* T*-periodic solutions.*

In spite of the negative result of Proposition 1.4, we may still prove a positive result provided that $f(\cdot, 0) = 0$ and s is sufficiently small.

Proposition 1.5. *Assume* (h_1),

(h_{11}) $f(\cdot, 0) = 0$ *and there exist* $a \in L^1(0, T)$ *and* $p > 1$ *such that* $f(t, x) \geq a(t)|x|^p$ *for a.e.* $t \in [0, T]$ *and all* $x \in \mathbb{R}$,

and (h_7). *Then, there exists* $\sigma > 0$ *such that, for all* $s \in]0, \sigma[$, *problem* (1.3) *has at least one positive* T*-periodic solution and at least one negative* T*-periodic solution.*

Remark 1.2 It remains open the question if conclusions similar to the above can be proven for boundary value problems associated with second order ODEs or PDEs: a preliminary step in this direction is given by the perturbative result established in [3, Proposition 5.1].

2 Proofs

Proof of Theorem 1.2. As already announced the proof of Theorem 1.2 is based on the construction of lower and upper solutions and the application of the existence results in [5], together with an extensive use of Lemma 2.1 in [7].

Step 1. We verify that for every $s \in \mathbb{R}$ there is $\xi_0 \in \mathbb{R}$ such that, for all $\xi \geq \xi_0$, any solution x of the Cauchy problem

$$x' = a(t)|x|^p + b(t) - s, \quad x(0) = \xi, \tag{2.1}$$

is a proper lower solution of the T-periodic problem for the equation

$$x' = a(t)|x|^p + b(t) - s, \tag{2.2}$$

and hence, by (h_8), a proper lower solution of the T-periodic problem for (1.3).

We begin with the case $p \in]0, 1[$ and start by proving the following claim.
Claim. For every $m \in \mathbb{R}$ *there exists* $\xi_m \geq m$ *such that, for every* $\xi \geq \xi_m$, *any solution* x *of* (2.1) *satisfies* $\min_{[0,T]} x > m$. Assume, by contradiction, that there exists $m_0 \in \mathbb{R}$ such that, for every $n \in \mathbb{N}$, with $n > m_0$, there is a global solution x_n of (2.1) satisfying $x_n(0) \geq n$ and $\min_{[0,T]} x_n \leq m_0$. Let $s_n, t_n \in [0, T]$ be such that $s_n < t_n$, $m_0 \leq x_n(t) \leq n$ on $[s_n, t_n]$, $x_n(s_n) = n$ and $x_n(t_n) = m_0$. We have

$$n - m_0 = x_n(s_n) - x_n(t_n) \leq \int_{s_n}^{t_n} |a(t)|\, |x_n(t)|^p \, dt + \int_{s_n}^{t_n} |b(t) - s| \, dt \leq \|a\|_{L^1} n^p + \|b - s\|_{L^1}.$$

Letting n go to $+\infty$ we get a contradiction, thus proving our claim.
To conclude the proof of Step 1 in case $p \in]0, 1[$ suppose, by contradiction, that there exist sequences $(\xi_n)_n$ in $\mathbb{R}$, with $\lim_{n \to +\infty} \xi_n = +\infty$, and $(x_n)_n$ of global solutions of (2.1), with $\xi = \xi_n$, satisfying $x_n(T) \leq x_n(0)$, for all n. Possibly relabelling the sequences $(\xi_n)_n$ and $(x_n)_n$, by the claim above we can suppose that $\min_{[0,T]} x_n \geq n$. Hence we find

$$0 \geq \int_0^T \frac{x_n'(t)}{\big(x_n(t)\big)^p} \, dt = \int_0^T a(t) \, dt + \int_0^T \frac{b(t) - s}{\big(x_n(t)\big)^p} \, dt \geq \int_0^T a(t) \, dt - n^{-p} \int_0^T \big|b(t) - s\big| \, dt.$$

Taking the limit we get the contradiction $0 \geq \int_0^T a(t) \, dt > 0$.
The validity of Step 1 when $p = 1$ can be verified by a direct inspection. Indeed, set $A(t) = \int_0^t a(\tau) \, d\tau$ and define the function

$$x(t) = e^{A(t)} \Big(\xi + \int_0^t e^{-A(\tau)} \big(b(\tau) - s\big) \, d\tau \Big).$$

Choose

$$\xi > \max \Big\{ \max_{t \in [0,T]} \Big(- \int_0^t e^{-A(\tau)} \big(b(\tau) - s\big) \, d\tau \Big), \quad (e^{-A(T)} - 1)^{-1} \int_0^T e^{-A(\tau)} \big(b(\tau) - s\big) \, d\tau \Big\}.$$

Then, we have $x(t) > 0$ on $[0,T]$, $x(T) > x(0)$, and x is a solution of (2.1).

Step 2. A symmetric counterpart of Step 1 holds. For every $s \in \mathbb{R}$ there is $\eta_0 \in \mathbb{R}$ such that, for all $\xi \leq \eta_0$, any solution x of the Cauchy problem (2.1) is a proper lower solution of the T-periodic problem for the equation (2.2) and hence, by (h_8), a proper lower solution of the T-periodic problem for (1.3).

Step 3. We show that there exists $s^* \in \mathbb{R}$ such that, for all $s > s^*$, equation (1.3) has at least two T-periodic solutions. Indeed, it is easily verified that there exists $s^* \in \mathbb{R}$ such that, for all $s > s^*$, the constant $\beta = \overline{x}$ defined in (h_4) is a proper upper solution of the T-periodic problem for equation (1.3). Furthermore, by the results proved in Step 1 and Step 2, the T-periodic problem for equation (1.3) admits also two proper lower solutions α_1, α_2 satisfying $\alpha_1(t) < \beta(t) < \alpha_2(t)$ for all $t \in [0,T]$. Therefore, equation (1.3) has at least two T-periodic solultions x_1, x_2, sastisfying $\alpha_1(t) \leq x_1(t) \leq \beta(t) \leq x_2(t) \leq \alpha_2(t)$ for all $t \in [0,T]$ and $x_1 \neq \alpha_1, \beta$, $x_2 \neq \alpha_2, \beta$.

Step 4. We prove that the set of the parameters s for which equation (1.3) has at least one T-periodic solution is bounded from below. Let us introduce the set

$$\mathscr{S} = \{s \in \mathbb{R} : \text{equation (1.3) has at least one } T\text{-periodic solution}\}$$

and define $s_0 = \inf \mathscr{S}$. We claim that $s_0 \in \mathbb{R}$. Assume, by contradiction, that $\inf \mathscr{S} = -\infty$. Then, there exist a sequence $(s_n)_n$ in $\mathbb{R}$, with $\lim_{n\to+\infty} s_n = -\infty$, and a sequence $(x_n)_n$ of T-periodic solutions of equation (1.3) with $s = s_n$. We claim that $\lim_{n\to+\infty} \|x_n\|_\infty = +\infty$. Indeed, otherwise, we would get $0 = \int_0^T x_n'(t)\,dt = \int_0^T f(t,x_n(t))\,dt - s_n T$, and, by (h_1), there would exist a function $\varphi \in L^1(0,T)$ such that $|s_n T| = \bigl|\int_0^T f(t,x_n(t))\,dt\bigr| \leq \int_0^T \varphi(t)\,dt < +\infty$, which is a contradiction. Moreover, by (h_8) we have $x_n'(t) = f(t,x_n(t)) - s_n \geq f(t,x_n(t)) \geq a(t)|x_n(t)|^p + b(t)$ a.e. in $[0,T]$. After reversing time into the equation, we can apply [7, Lemma 2.1] (with $\lambda = 1$, $\psi = f$, $a_0(t) = |a(t)|$, $b_0(t) = |a(t)| + |b(t)|$). It follows that the sequence $(x_n)_n$ diverges uniformly either to $+\infty$ or to $-\infty$ in $[0,T]$. Assume, e.g., that the former condition holds and that $p \in \,]0,1[$, the proof being similar in the other cases. We find

$$0 = \int_0^T \frac{x_n'(t)}{\bigl(x_n(t)\bigr)^p}\,dt \geq \int_0^T a(t)\,dt + \int_0^T \frac{b(t)}{\bigl(x_n(t)\bigr)^p}\,dt,$$

which, letting $n \to +\infty$ and using (h_7), yields the contradiction $0 \geq \int_0^T a(t)\,dt > 0$.

Step 5. We show the existence of at least one T-periodic solution of equation (1.3) for $s = s_0$. Let $(s_n)_n$ be a sequence in $\mathscr{S}$ converging to s_0 and let $(x_n)_n$ be the corresponding sequence of T-periodic solutions of equation (1.3) with $s = s_n$. Let us verify that there is $R > 0$ such that $\|x_n\|_\infty \leq R$ for all n. Indeed, otherwise, we can find a subsequence of $(x_n)_n$, we still denote by $(x_n)_n$, such that $\lim_{n\to+\infty} \|x_n\|_\infty = +\infty$. Arguing similarly as in the proof of Step 4, by using again [7, Lemma 1], we see that the sequence $(x_n)_n$ diverges uniformly either to $+\infty$ or to $-\infty$ in $[0,T]$, thus easily leading to a contradiction as above. Therefore $(x_n)_n$ is bounded in $L^\infty(0,T)$. Hence, by (h_1), there exists a function $\varphi \in L^1(0,T)$ such that, for all large n and every $t_1, t_2 \in [0,T]$,

$$\begin{aligned}\bigl|x_n(t_1) - x_n(t_2)\bigr| &= \Bigl|\int_{t_1}^{t_2} x_n'(t)\,dt\Bigr| = \Bigl|\int_{t_1}^{t_2} \bigl(f(t,x_n(t)) - s_n\bigr)\,dt\Bigr| \leq \Bigl|\int_{t_1}^{t_2} \bigl(|f(t,x_n(t))| + |s_n|\bigr)\,dt\Bigr| \\ &\leq \Bigl|\int_{t_1}^{t_2} \varphi(t)\,dt\Bigr| + \bigl(|s_0|+1\bigr)|t_1 - t_2| = \bigl|\Phi(t_1) - \Phi(t_2)\bigr| + \bigl(|s_0|+1\bigr)|t_1 - t_2|,\end{aligned}$$

where $\Phi(t) = \int_0^t \varphi(\tau)\,d\tau$. This implies that $(x_n)_n$ is also uniformly equicontinuous and therefore there exists $x_0 \in C^0([0,T])$, with $x_0(0) = x_0(T)$, such that $\lim_{n\to+\infty} x_n(t) = x_0(t)$ uniformly in $[0,T]$. In addition, we have that, for a.e. $t \in [0,T]$, $\lim_{n\to+\infty} f\bigl(t,x_n(t)\bigr) = f\bigl(t,x_0(t)\bigr)$ and, for all large n, $\bigl|f(t,x_n(t)))\bigr| \leq \varphi(t)$. The dominated convergence theorem guarantees that the sequence $\bigl(f(\cdot,x_n) - s_n\bigr)_n$, and thus $(x_n')_n$, converges in $L^1(0,T)$ to the function $f(\cdot,x_0) - s_0$. Therefore, we see that x_0 is absolutely continuous in $[0,T]$ and satisfies $x_0' = f(t,x_0) - s_0$, $x_0(T) = x_0(0)$, that is, x_0 defines a T-periodic solution of equation (1.3), with $s = s_0$.

Step 6. To conclude the proof we verify that, for each $s \in \,]s_0,+\infty[$, equation (1.3) has at least two T-periodic solutions. Fix $s \in \,]s_0,+\infty[$. As $s_0 = \inf \mathscr{S}$, there is $\tilde{s} \in \mathscr{S}$ such that $s_0 < \tilde{s} < s$. Let $\tilde{x}$ be a T-periodic solution of equation (1.3) with $s = \tilde{s}$. As $\tilde{s} < s$, $\tilde{x}$ is a proper upper solution of the T-periodic problem for equation (1.3). Thus, from Step 1 and Step 2 we infer the existence of two T-periodic solutions x_1, x_2 of equation (1.3) satisfying $x_1(t) \leq \tilde{x}(t) \leq x_2(t)$ on $[0,T]$ and $x_1 \neq x_2$, as $x_1 \neq \tilde{x}$ and $x_2 \neq \tilde{x}$. $\square$

Proof of Proposition 1.3. In view of [7, Corollary 4.1], it is enough to show that (h_5) implies (h_6) and (h_7). We first prove that (h_7) holds. Taking $K_1 = 1$, any $K_2 > 0$ and $\sigma = \int_0^T |b(t)|\,dt + 1$, condition (h_5) implies that there exists $d > 0$ such that, for all $x \in \mathbb{R}$, with $x \geq d$,

$$\int_0^T f(t,x)\,dt = \Big(\int_0^T a(t)\,dt\Big)|x|^p + \int_0^T b(t)\,dt > \sigma = \int_0^T |b(t)|\,dt + 1.$$

Hence, we conclude that $\int_0^T a(t)\,dt > 0$. Next, we show that (h_6) holds. Assume, by contradiction, that $a(t) < 0$ on a set of positive measure or, equivalently, that $\int_0^T a^-(t)\,dt > 0$. As we already know that $\int_0^T a(t)\,dt > 0$ we have $\int_0^T a^+(t)\,dt > \int_0^T a^-(t)\,dt > 0$. Take any $K_2 > 0$ and set

$$K_1 = \Big(\frac{\int_0^T a^+(t)\,dt}{\int_0^T a^-(t)\,dt}\Big)^{\frac{1}{p}} > 1, \quad \sigma = \int_0^T |b(t)|\,dt + 1.$$

By (h_5) there exists $d > 0$ such that, if $x \in C^0([0,T])$ satisfies $x(0) = x(T)$, (1.5) and $\min_{[0,T]} x \geq d$, then

$$\int_0^T f(t,x(t))\,dt = \int_0^T a(t)|x(t)|^p\,dt + \int_0^T b(t)\,dt > \sigma. \tag{2.3}$$

Define a function u by setting, for all $t \in [0,T]$, $u(t) = K_1 d$ if $a(t) \leq 0$, $u(t) = d$ if $a(t) > 0$. The absolute continuity of the Lebesgue integral implies that for every $\varepsilon > 0$ there is $\delta > 0$ such that, for every measurable set $M \subseteq [0,T]$, if $|M| < \delta$, then $\int_M |a(t)|\,dt < \varepsilon$. Set $\varepsilon = \frac{1}{2(K_1 d)^p}$. The Lusin theorem [4, Theorem 7.10] implies that there exists $v \in C^0([0,T])$, having compact support in $]0,T[$, such that

$$\max_{[0,T]} |v| \leq \max_{[0,T]} |u| = K_1 d \quad \text{and} \quad |M| < \delta,$$

where $M = \{t \in [0,T] : v(t) \neq u(t)\}$. Observe that these conditions still hold replacing v with $x \in C^0([0,T])$ defined by $x(t) = \max\{v(t), d\}$ for all $t \in [0,T]$, and M with $N = \{t \in [0,T] : x(t) \neq u(t)\}$. Indeed, we have $N \subseteq M$, $x(0) = x(T) = d$ and $\max_{[0,T]} |x| \leq \max_{[0,T]} |v| \leq K_1 d = K_1 \min_{[0,T]} |x|$. Thus, (1.5) and hence (2.3) hold. Let us compute

$$\begin{aligned}\int_0^T a(t)\,|x(t)|^p\,dt &= \int_0^T a(t)\,x(t)^p\,dt = \int_0^T a(t)\,u(t)^p\,dt + \int_0^T a(t)\,\big(x(t)^p - u(t)^p\big)\,dt \\ &\leq \int_0^T a^+(t)\,d^p\,dt - \int_0^T a^-(t)\,(K_1 d)^p\,dt + \int_N |a(t)|\,\big|x(t)^p - u(t)^p\big|\,dt \\ &\leq d^p\Big(\int_0^T a^+(t)\,dt - K_1^p \int_0^T a^-(t)\,dt\Big) + 2(K_1 d)^p \int_N |a(t)|\,dt \leq 1.\end{aligned}$$

On the other hand, (2.3) implies that $\int_0^T a(t)|x(t)|^p\,dt > 1$, thus getting a contradiction. □

Proof of Proposition 1.4. Let us suppose, for simplicity, that $\delta = 1$ and $t_1 = 0$. Pick $x_0 \in \mathbb{R}$ satisfying $x_0 > \Big(\frac{8}{t_2(p-1)}\Big)^{\frac{1}{p-1}}$. Set $\tau_1 = \frac{t_2}{4}$, $\tau_2 = \frac{3}{4}t_2$. Note that the maximal solution x_1 of the Cauchy problem $x' = -|x|^p$, $x(\tau_1) = x_0$ is the function $x_1(t) = \Big(x_0^{1-p} + (p-1)(t - \frac{t_2}{4})\Big)^{\frac{1}{1-p}}$, which is defined on the interval

$$I_1 = \Big]\,\frac{t_2}{4} - \frac{x_0^{1-p}}{p-1}, +\infty\Big[\subset \Big]\,\frac{t_2}{8}, +\infty\Big[.$$

Similarly, the maximal solution x_2 of the Cauchy problem $x' = -|x|^p$, $x(\tau_2) = -x_0$ is the function $x_2(t) = -\Big(x_0^{1-p} - (p-1)(t - \frac{3}{4}t_2)\Big)^{\frac{1}{1-p}}$, which is defined on the interval

$$I_2 = \Big] -\infty, \frac{3}{4}t_2 + \frac{x_0^{1-p}}{p-1}\Big[\subset \Big] -\infty, \frac{7}{8}t_2\Big[.$$

Set $\sigma = \frac{8}{t_2}x_0$ and let $s \geq \sigma$. We claim that no solution x of the equation (1.3) exists on the whole interval $[0, t_2]$. Indeed, consider such a solution and assume it is defined up to $\tau_0 = \frac{t_2}{2}$, with $x(\tau_0) = \phi$.
If $|\phi| \leq x_0$, as $x'(t) \leq -s$ on $[\tau_1, \tau_0]$, we have $x(\tau_1) \geq x_0$. Since $x'(t) \leq -|x(t)|^p$ on $]\frac{t_2}{8}, \tau_1]$, x is not defined for $t \leq \frac{t_2}{8}$.

If $\phi > x_0$, since $x'(t) \le -|x(t)|^p$ on $]\frac{t_2}{8}, \tau_0]$, again we conclude that x is not defined for $t \le \frac{t_2}{8}$.
If $\phi < -x_0$, since $x'(t) \le -|x(t)|^p$ on $[\tau_0, \frac{7}{8}t_2[$, we conclude that x is not defined for $t \ge \frac{7}{8}t_2$. $\square$

Proof of Proposition 1.5. Let us extend the function a by T-periodicity onto $\mathbb{R}$ and set $A(t) = \int_0^t a(\tau)\,d\tau$. Without loss of generality, we may assume $A(t) > 0$ for all $t \in]0,T]$. If this is not the case, we set

$$-m = \min\{A(t) : t \in]0,T]\} \le 0 \quad \text{and} \quad t_0 = \max\{t \in]0,T] : A(t) = -m\} < T.$$

We claim that $\int_{t_0}^t a(\tau)\,d\tau > 0$ for all $t \in]t_0, t_0+T]$. Indeed, if $t_0 < t \le T$, we have

$$-m < \int_0^t a(\tau)\,d\tau = \int_0^{t_0} a(\tau)\,d\tau + \int_{t_0}^t a(\tau)\,d\tau = -m + \int_{t_0}^t a(\tau)\,d\tau$$

and hence $\int_{t_0}^t a(\tau)\,d\tau > 0$. If $T < t \le t_0 + T$, we have

$$\int_{t_0}^t a(s)\,ds = \int_{t_0}^T a(s)\,ds + \int_T^t a(s)\,ds = A(T) + m + \int_0^{t-T} a(s)\,ds \ge A(T) > 0.$$

Replacing the interval $[0,T]$ with $[t_0, t_0+T]$ yields the conclusion.
Pick $K > 0$ such that $A(t) < K$ for all $t \in [0,T]$. Take any z_0 satisfying $0 < z_0 < \big((p-1)K\big)^{\frac{1}{1-p}}$ and let z be the solution of the Cauchy problem $z' = a(t)|z|^p,\ z(0) = z_0$, whose solution is $z(t) = z_0\big(1 - (p-1)z_0^{p-1}A(t)\big)^{-\frac{1}{p-1}}$. By our choice of z_0 we easily verify that z is defined on $[0,T]$ and $z(t) > z_0 > 0$ for all $t \in]0,T]$. By the continuous dependence of the solutions of the Cauchy problem with respect to initial data there exists $\varepsilon_1 > 0$ such that the solution u of the problem $z' = a(t)|z|^p - \varepsilon_1,\ z(0) = z_0$, is still defined on $[0,T]$ and satisfies $u(t) > 0$ for all $t \in]0,T]$ and, in particular, $u(T) > u(0)$. Accordingly, u is a proper lower solution of the T-periodic problem for equation (1.3), with $s = \varepsilon_1$. As $\beta = 0$ is a proper upper solution of the T-periodic problem for equation (1.3), and $\beta(t) < u(t)$ on $[0,T]$, there exists a T-periodic solution x of (1.3), with $s = \varepsilon_1$, satisfying $0 \le x(t) \le u(t)$ on $[0,T]$, $x \ne 0, x \ne u$.
Finally, let σ_1 be the supremum of the set of numbers s such that the equation (1.3) has a positive T-periodic solution. Then, for all $\varepsilon_1 < \sigma$ there is η_1 satisfying $\varepsilon_1 < \eta_1 \le \sigma$ and a positive T-periodic solution x_{η_1} of (1.3) with $s = \eta_1$. The function x_{η_1} is a lower solution of the T-periodic problem for equation (1.3), with $s = \varepsilon_1$. As 0 is an upper solution of the same problem, there exists a positive T-periodic solution x_{ε_1} of (1.3) with $s = \varepsilon_1$.
In order to show the existence of negative solutions we can argue in a similar way, by choosing z_0 satisfying $-\big((p-1)K\big)^{\frac{1}{1-p}} < z_0 < 0$. We can check that the solution of the Cauchy problem is given by $z(t) = z_0\big(1 + (p-1)(-z_0)^{p-1}A(t)\big)^{-\frac{1}{p-1}}$, which satisfies $z_0 \le z(t) < 0$ on $[0,T]$. As in the previous case, we use lower and upper solutions to obtain a number $\sigma_2 > 0$ such that equation (1.3) has a negative T-periodic solution for all $s \in]0, \sigma_2[$. $\square$

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