Inference with Nearly-Linear Uncertainty Models

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Abstract

Several simplified uncertainty models are derived from a given probability P_0 of which they are a perturbation. Among these, we introduced in previous work Nearly-Linear (NL) models. They perform a linear affine transformation of P_0 with barriers, obtaining a couple of conjugate lower/upper probabilities, and generalise several well known neighbourhood models. We classified NL models, partitioning them into three subfamilies, and established their basic consistency properties in [5]. In this paper we investigate how to extend NL models that avoid sure loss by means of their natural extension, a basic, although operationally not always simple, inferential procedure in Imprecise Probability Theory. We obtain formulae for computing directly the natural extension in a number of cases, supplying a risk measurement interpretation for one of them. The results in the paper also broaden our knowledge of NL models: we characterise when they avoid sure loss, express some of them as linear (or even convex) combinations of simpler models, and explore relationships with interval probabilities.

Keywords. Nearly-Linear Models, Pari-Mutuel Model, Total Variation Model, natural extension, coherent lower probabilities, risk measures.

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1 Introduction

Uncertainty models that are not (precise) probabilities may sometimes be derived from a reference probability P_0. A classical example is the *Pari-Mutuel Model* (PMM), that obtains an upper probability

$$\overline{P}_{PMM} = \min\{(1+\delta)P_0, 1\} \tag{1}$$

given P_0 and a parameter $\delta > 0$ [14, 22, 25]. A common motivation for introducing these models, called *neighbourhood models* [2, Section 4.7], is that P_0 might be not accurate enough, due to a number of causes (poor or conflicting information, low-quality data, etc.). Yet, the models may be useful even when P_0 is fully reliable. Suppose for instance that a bookie, or the organiser of a game (in a loose sense, might be, say, an insurer) must fix the unit selling price $\overline{P}(A)$ for an event A, meaning that s/he returns 1 monetary unit if A is true, 0 otherwise, to a counterpart who paid $\overline{P}(A)$. The bookie will not choose $\overline{P}(A) = P_0(A)$, not even when P_0 is perfectly known, like in symmetric games (Lotto, roulette, etc.), because this would not reasonably guarantee a positive gain perspective in the long run. Rather, $\overline{P} = \overline{P}_{PMM}$ in (1) could be a more convincing choice, since $\overline{P}_{PMM} \geq P_0$, and has been in fact employed, for instance, in betting systems on horse races.

Next to these considerations, neighbourhood models can be more easily explained to non-experts than other, more complex imprecise probability models, and allow for nimbler computations, inferences, etc.

In previous work [4, 5], we investigated a family of neighbourhood models that we termed *Nearly-Linear* (NL), since they are obtained by a linear affine transformation of P_0, with barriers to ensure that the result is within the $[0, 1]$ interval. Thus, a NL model returns an uncertainty measure $\mu(A)$, that can be 0, 1 or $bP_0(A) + a$, with a, b fixed real parameters. This is what is done also by neo-additive capacities [3, 10], although with remarkable differences (discussed in [5, Section 7.2]). For instance, the sets of μ-measure 0 or 1 are settled a priori and with some constraints with neo-additive capacities, unlike NL models, where they are determined by the model itself. NL models generalise the PMM, as well as the ε-contamination or linear-vacuous model [25], the Total Variation model [12], and other neighbourhood models.

In [5] we studied basic properties of NL models. In particular:

- NL models are partitioned into 3 families, depending on the values of parameters a, b: the Vertical Barrier Model (VBM), the Horizontal Barrier Model (HBM), the Restricted Range Model (RRM). Each model consists of a couple of conjugate upper and lower probabilities.

- The VBM is coherent, while the remaining models satisfy the weaker condition of 2-coherence. The HBM may be coherent (in special cases), the RRM only in a trivial instance.

- NL models can express a certain variety of an assessor's attitudes towards the given P_0, ranging from more prudential evaluations than the PMM to conflicting beliefs towards tail P_0-probability events (see also [5, Sections 5.1 and 6.1]). The latter do not necessarily imply lack of coherence.

In this paper, we explore how NL models perform in inferential matters, with the primary aim of introducing manageable formulae for inference. In Imprecise Probability theory, the *natural extension* is the fundamental inferential procedure [2, 24, 25]. It both corrects incoherent evaluations and extends them or coherent ones to larger domains in a least-committal way (see also Section 2.2). Unfortunately, finding the natural extension may be operationally hard, if the model is generic or not simple, and may require using algorithmic procedures [2, Section 16.2] [16, 26]. In the present paper we introduce formulae that simplify this task in most subcases of NL models. In general, let $\mathbb{P}$ be a partition of the sure event, $\mathcal{A}(\mathbb{P})$ the set of events logically dependent on $\mathbb{P}$ (the powerset of $\mathbb{P}$) and $\mathcal{L}(\mathbb{P})$ the set of all gambles (bounded random variables) defined on $\mathbb{P}$. Then the generic NL model is defined on $\mathcal{A}(\mathbb{P})$. We determine its natural extension:

i) on $\mathcal{A}(\mathbb{P})$, when the model is not already coherent there. Here the natural extension is a (least-committal) coherent correction to the model;

ii) on $\mathcal{L}(\mathbb{P})$, performing thus a real extension.

Next to the main task, we tackle other problems, whose solution contributes to a better knowledge of NL models:

- The natural extension procedure requires starting from an evaluation that avoids sure loss. The VBM always avoids sure loss being coherent, but it was not established in [4, 5] when a HBM or a RRM does so. In this paper, we characterise the condition of avoiding sure loss for these models (Propositions 5.1 (b), 5.5, Lemma 6.1 and Proposition 6.2).

- We prove that the VBM and the RRM can be thought of as linear (sometimes also convex) combinations of simpler models (Section 3 and Equation (60)). The VBM is also itself the natural extension of a more naive model (Section 4.1).

- If $\mathbb{P}$ is a finite partition, NL models can be interval probabilities in some special cases (characterised in [5]). Generally, they are not, yet we show that HBMs and RRMs that avoid sure loss are inferentially equivalent to special probability intervals (Proposition 5.3, Remark 6.1).

The paper is organised as follows: Section 2 recalls the basic notions useful in the sequel, regarding especially the consistency concepts employed (Section 2.1), the natural extension and known procedures to determine it (Section 2.2), NL models (Section 2.3). In Section 3, the VBM is shown to include several known neighbourhood models as special cases while being also either a linear or a convex linear combination of some of them, the vacuous model and either the Total Variation or the Pari-Mutuel Model. This suggests also a new interpretation of the VBM, akin to that of the ε-contamination model. Section 4 discusses both the VBM as a natural extension (Section 4.1) and the natural extension of a VBM onto $\mathcal{L}(\mathbb{P})$ (Section 4.2), which may be given also an interpretation in terms of risk measures (Section 4.3). The interpretation is based on that of risk measures as imprecise previsions [2, Section 12.3.1], and introduces a coherent risk measure more general than Tail Value at Risk. We recall that Tail Value at Risk is a known coherent risk measure [9] that can be viewed as the natural extension of a PMM [22]. Section 5 determines when a HBM avoids sure loss and establishes explicit formulae for its natural extension on $\mathcal{A}(\mathbb{P})$ and on $\mathcal{L}(\mathbb{P})$ when $\mathbb{P}$ is finite, or on $\mathcal{L}(\mathbb{P})$, whatever $\mathbb{P}$ is, when the HBM is coherent on $\mathcal{A}(\mathbb{P})$. Section 6 illustrates a similar work with the RRM, which is shown to avoid sure loss only when $\mathbb{P}$ is finite, and under an additional condition. It also investigates further features of the RRM: among these, it is a convex linear combination of P_0 and a degenerate NL model, and the condition of avoiding sure loss is equivalent to C-convexity for it, but not for a HBM. Our conclusions are summarised in Section 7. This paper extends, with proofs and additional material, a preliminary paper [20] presented at ISIPTA 2019 - International Symposium on Imprecise Probabilities: Theory and Applications.

2 Preliminaries

Let $\mathcal{S}$ be a set of gambles (bounded random variables). A map $\underline{P} : \mathcal{S} \to \mathbb{R}$ is a *lower prevision* on $\mathcal{S}$ if (behaviourally) $\forall X \in \mathcal{S}$, $\underline{P}(X)$ is a supremum buying price for X, while an *upper prevision* $\overline{P} : \mathcal{S} \to \mathbb{R}$ represents an infimum selling price for any $X \in \mathcal{S}$ [25]. When $\mathcal{S}$ is made of (indicators of) events only, we preferably speak of lower and upper probabilities of events, instead of previsions of their indicators. Thus, for instance, if A is an event, $\underline{P}(A)$ is its lower probability and is equal to the lower prevision $\underline{P}(I_A)$ of its indicator I_A.

It is possible to refer to lower or alternatively upper previsions only, by the conjugacy relation $\overline{P}(X) = -\underline{P}(-X)$, if $X \in \mathcal{S} \Rightarrow -X \in \mathcal{S}$. In the case of probabilities, conjugacy is written as

$$\overline{P}(A) = 1 - \underline{P}(\neg A),$$

assuming that $A \in \mathcal{S} \Rightarrow \neg A \in \mathcal{S}$.

2.1 Consistent Lower/Upper Previsions

Lower (and upper) previsions may satisfy different consistency requirements. In the next definition we recall the ones relevant to this paper, stated for lower previsions.

Definition 2.1. *[17, 18, 25] Let $\underline{P} : \mathcal{S} \to \mathbb{R}$ be a given lower prevision and denote with $\mathbb{N}$ the set of natural numbers (including 0), $\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$.*

(*a*) $\underline{P}$ *is a* coherent lower prevision *on $\mathcal{S}$ iff, $\forall n \in \mathbb{N}$, $\forall s_i \geq 0$, $\forall X_i \in \mathcal{S}$, $i = 0, 1, \ldots, n$, defining*

$$\underline{G} = \sum_{i=1}^{n} s_i\big(X_i - \underline{P}(X_i)\big) - s_0\big(X_0 - \underline{P}(X_0)\big),$$

it holds that $\sup \underline{G} \geq 0$.

(*b*) $\underline{P}$ *is a* convex lower prevision *on $\mathcal{S}$ iff* (*a*) *holds with the additional convexity constraint $\sum_{i=1}^{n} s_i = s_0 = 1$.*

$\underline{P}$ *is centered convex or* C-convex *iff it is convex, $\emptyset \in \mathcal{S}$ and $\underline{P}(\emptyset) = 0$.*

(*c*) $\underline{P}$ avoids sure loss *on $\mathcal{S}$ iff $\forall n \in \mathbb{N}^+$, $\forall s_i \geq 0$, $\forall X_i \in \mathcal{S}$, $i = 1, \ldots, n$, defining $\underline{G}_{ASL} = \sum_{i=1}^{n} s_i \left(X_i - \underline{P}(X_i)\right)$, it holds that* $\sup \underline{G}_{ASL} \geq 0$.

(*d*) $\underline{P}$ *is* 2-coherent *on $\mathcal{S}$ iff, $\forall s_1 \geq 0$, $\forall s_0 \in \mathbb{R}$, $\forall X_0, X_1 \in \mathcal{S}$, defining $\underline{G}_2 = s_1\big(X_1 - \underline{P}(X_1)\big) - s_0\big(X_0 - \underline{P}(X_0)\big)$, it holds that* $\sup \underline{G}_2 \geq 0$.

Historically, these notions are derived from de Finetti's coherence for previsions [8]:

Definition 2.2. *$P : \mathcal{S} \to \mathbb{R}$ is a* coherent prevision *on S iff, $\forall n \in \mathbb{N}^+$, $\forall s_i \in \mathbb{R}$, $\forall X_i \in \mathcal{S}$, $i = 1, \ldots, n$, defining $G = \sum_{i=1}^{n} s_i(X_i - P(X_i))$, it holds that* $\sup G \geq 0$.

When $\mathcal{S}$ is a set of (indicators of) events, the coherent prevision P is a *dF-coherent probability* as we shall often say to better distinguish coherence for precise and imprecise measures. DF-coherent probabilities are both lower and upper coherent probabilities. If $\mathcal{S}$ is an algebra, P is a dF-coherent probability iff it is a finitely additive probability.

The definitions above are axiomatic. Yet, it is customary to give them a behavioural interpretation: they all require that certain gambles termed *gains* ($\underline{G}$, $\underline{G}_{\mathrm{ASL}}$, $\underline{G}_2$ and G) are uniformly not negative. In each subcase in Definition 2.1, the admissible gains are different as for the restrictions on the sign and number of the coefficients s_i, but each gain is a linear combination of *elementary gains* $X_i - \underline{P}(X_i)$. $X_i - \underline{P}(X_i)$ is what a subject gains buying a gamble X_i at the price $\underline{P}(X_i)$. Thus the various definitions ensure that a subject exchanging gambles in $\mathcal{S}$ according to the rules of the consistency concept s/he adopts avoids being a sure loser (i.e., avoids suffering losses bounded away from 0, whatever happens).

The same consistency definitions with upper previsions can be obtained by conjugacy and a generic elementary gain is now of the type $\overline{P}(X_i) - X_i$. It is the random outcome from selling the gamble X_i for $\overline{P}(X_i)$. We recall here just the definition of avoiding sure loss:

Definition 2.3. *$\overline{P} : \mathcal{S} \to \mathbb{R}$ is an upper prevision that* avoids sure loss *on $\mathcal{S}$ iff, $\forall n \in \mathbb{N}^+$, $\forall s_i \geq 0$, $\forall X_i \in \mathcal{S}$, $i = 1, \ldots, n$, defining $\overline{G}_{\mathrm{ASL}} = \sum_{i=1}^{n} s_i(\overline{P}(X_i) - X_i)$, it holds that* $\sup \overline{G}_{\mathrm{ASL}} \geq 0$.

In Definition 2.1, coherence for lower/upper previsions (and probabilities) is the strongest requirement, and implies the other ones. Coherence also has a statistical robustness interpretation, which we state for probabilities: if we are unsure about which is the 'true' probability in a set $\mathcal{M}$, $\underline{P}$ ($\overline{P}$) is the lower (upper) envelope of this set:

Theorem 2.1. *(Envelope theorem) [25] $\underline{P} : \mathcal{S} \to \mathbb{R}$ ($\overline{P} : \mathcal{S} \to \mathbb{R}$) is a coherent lower (upper) probability on $\mathcal{S}$ iff*

$$\underline{P}(A) = \min_{P \in \mathcal{M}} P(A), \quad \forall A \in \mathcal{S} \tag{2}$$

$$\big(\overline{P}(A) = \max_{P \in \mathcal{M}} P(A), \quad \forall A \in \mathcal{S}\big),$$

where $\mathcal{M}$ is a (non-empty) set of dF-coherent probabilities on $\mathcal{S}$.

The following result, to be compared with Theorem 2.1, highlights different properties of the conditions of convexity and avoiding sure loss.

Theorem 2.2. *Given $\underline{P} : \mathcal{S} \to \mathbb{R}$ ($\overline{P} : \mathcal{S} \to \mathbb{R}$),*

(*a*) *[17] $\underline{P}$ ($\overline{P}$) is a convex lower probability (convex upper probability) on $\mathcal{S}$ iff there exist a non-empty set $\mathcal{M}$ of dF-coherent probabilities on $\mathcal{S}$ and a function $\alpha : \mathcal{M} \to \mathbb{R}$ such that*

$$\underline{P}(A) = \inf_{P \in \mathcal{M}} \big\{P(A) + \alpha(P)\big\}, \quad \forall A \in \mathcal{S}. \tag{3}$$

$$(\overline{P}(A) = \sup_{P \in \mathcal{M}} \big\{P(A) + \alpha(P)\big\}, \quad \forall A \in \mathcal{S}.) \tag{4}$$

Moreover, for any $A \in \mathcal{S}$, the infimum in (3) (the supremum in (4)) is attained.

(*b*) *[25] $\underline{P}$ ($\overline{P}$) avoids sure loss iff there exists a dF-coherent probability P such that $P \geq \underline{P}$ ($P \leq \overline{P}$) on $\mathcal{S}$.*

Thus, we see that convexity allows modifying each probability in $\mathcal{M}$ before taking the minimum. This could be the case of a correction of some P given by an expert, believed to be somewhat biased. Convex probabilities (actually previsions) are also in one-to-one correspondence with convex risk measures [17]. In general, if $\underline{P}$ is a convex lower probability, it does not necessarily avoid sure loss: when $\emptyset \in \mathcal{S}$, it does iff $\underline{P}(\emptyset) \leq 0$ [17, Proposition 3.5 (e)]. Hence, $\underline{P}$ avoids sure loss when it is C-convex.

As for avoiding sure loss, this condition guarantees by Theorem 2.2 (b) that the set $\mathcal{M}$ in Theorem 2.1 is non-empty, but not that $\underline{P}(A)$ can be obtained for each $A \in \mathcal{S}$ by (2).

2-coherence is a rather minimal consistency notion, which does not imply (nor is implied by) avoiding sure loss. 2-coherence of $\underline{P}$ ensures that

$$\underline{P} \text{ is monotone:} \quad A \Rightarrow B \;\rightarrow\; \underline{P}(A) \leq \underline{P}(B),$$
$$\underline{P} \text{ is normalised: } \underline{P}(\emptyset) = 0,\ \underline{P}(\Omega) = 1,$$

and similarly with $\overline{P}$. Thus, the properties of 2-coherence are comparable to those of capacities. If $A \in \mathcal{S} \Rightarrow \neg A \in \mathcal{S}$ and $\underline{P}, \overline{P}$ are conjugate, which is a common assumption in the sequel, 2-coherence implies further that $\underline{P} \leq \overline{P}$ [21].

2.2 Inference with natural extensions

It is well-known [25] that coherent lower/upper probabilities or previsions defined on a set $\mathcal{S}$ allow coherent, and generally not unique, extensions on any superset $\mathcal{S}' \supset \mathcal{S}$. There is anyway a special such extension, termed *natural extension*, which is *least-committal*, in the sense that it infers the value of $\underline{P}$ (or $\overline{P}$) on additional gambles exploiting only the information given by $\underline{P}$ on $\mathcal{S}$. Formally, and following [24, Section 3], the natural extension $\underline{E}$ of a lower prevision $\underline{P}$ is defined as follows:

Given $\underline{P} : \mathcal{S} \rightarrow \mathbb{R}$, $\forall \mathcal{S}' \supset \mathcal{S}$, $\forall X \in \mathcal{S}'$, the natural extension $\underline{E}(X)$ of $\underline{P}$ on X is

$$\underline{E}(X) = \sup \Big\{ \alpha \in \mathbb{R} : X - \alpha \geq \sum_{j=1}^{n} s_j \big(X_j - \underline{P}(X_j) \big),$$
$$\text{for some } n \geq 0,\ X_j \in \mathcal{S}, s_j \geq 0,\ j = 1, \ldots, n \Big\}.$$

The natural extension is well-defined (i.e., it is always finite) iff $\underline{P}$ avoids sure loss on $\mathcal{S}$, in which case it also 'corrects' $\underline{P}$ to a coherent lower prevision on $\mathcal{S}$ (whenever $\underline{P}$ is not already coherent there). It has the following properties, relevant for the sequel [24, 25]:

Theorem 2.3. *Let $\underline{P} : \mathcal{S} \rightarrow \mathbb{R}$ avoid sure loss on $\mathcal{S}$. Then, $\forall \mathcal{S}' \supset \mathcal{S}$, there exists its natural extension $\underline{E} : \mathcal{S}' \rightarrow \mathbb{R}$, which is coherent and such that:*

(a) *$\underline{E} \geq \underline{P}$ and $\underline{E} = \underline{P}$ iff $\underline{P}$ is coherent on $\mathcal{S}$;*

(b) *$\forall \underline{Q}$, coherent lower prevision on $\mathcal{S}'$ such that $\underline{Q} \geq \underline{P}$ on $\mathcal{S}$, we have that $\underline{Q} \geq \underline{E}$ on $\mathcal{S}'$.*

Similarly, let $\overline{P} : \mathcal{S} \rightarrow \mathbb{R}$ avoid sure loss on $\mathcal{S}$. Then, $\forall \mathcal{S}' \supset \mathcal{S}$, there exists its natural extension $\overline{E} : \mathcal{S}' \rightarrow \mathbb{R}$, which is coherent and such that:

(a') *$\overline{E} \leq \overline{P}$ and $\overline{E} = \overline{P}$ iff $\overline{P}$ is coherent on $\mathcal{S}$;*

(b') $\forall \overline{Q}$, *coherent upper prevision on* $\mathcal{S}'$ *such that* $\overline{Q} \leq \overline{P}$ *on* $\mathcal{S}$, *we have that* $\overline{Q} \leq \overline{E}$ *on* $\mathcal{S}'$.

Finally, if $\underline{P}, \overline{P}$ *are conjugate, so are their natural extensions* $\underline{E}, \overline{E}$.

Remark 2.1. *It can be easily proven that the natural extension is also characterised as follows. If* $\underline{E} : \mathcal{S}' \to \mathbb{R}$ *is a coherent lower prevision such that* $\underline{E} \geq \underline{P}$ *on* $\mathcal{S}$ *and Theorem 2.3* (b) *holds, then* $\underline{E}$ *is the natural extension of* $\underline{P}$ *on* $\mathcal{S}'$.

We can not apply the natural extension procedure when $\underline{P}$ does not avoid sure loss, which may be the case if $\underline{P}$ is 2-coherent. Interestingly, even in such a situation there exists an analogue of the natural extension, termed 2-coherent natural extension. It guarantees that the assessment $\underline{P}$ can be extended on any $\mathcal{S}'$ preserving 2-coherence and is least-committal within the set of 2-coherent extensions [19]. Although some of the models we consider in this paper may be only 2-coherent, we shall not focus on 2-coherent natural extensions but rather on the (usual) natural extensions, that ensure the more desirable property of coherence. This requires establishing when the 2-coherent models also avoid sure loss, a problem that will be solved in the next sections.

In a general situation, when $\underline{P}$ (or $\overline{P}$) is defined on an arbitrary set of gambles $\mathcal{S}$, finding its natural extension may be operationally not easy at all. However, in the sequel we shall be concerned with some special situations that make this task simpler by supplying explicit formulae.

Precisely, let $\mathbb{P}$ be a partition of the sure event, $\mathcal{A}(\mathbb{P})$ the set of events logically dependent on $\mathbb{P}$ (the powerset of $\mathbb{P}$), $\mathcal{L}(\mathbb{P})$ the set of all gambles defined on $\mathbb{P}$. We shall consider lower probabilities $\underline{P}$ defined on $\mathcal{A}(\mathbb{P})$ that are coherent and 2-*monotone*, i.e. such that

$$\underline{P}(A \vee B) + \underline{P}(A \wedge B) \geq \underline{P}(A) + \underline{P}(B), \quad \forall A, B \in \mathcal{A}(\mathbb{P}) \tag{5}$$

(correspondingly, $\overline{P}$ is 2-*alternating* if $\overline{P}(A \vee B) + \overline{P}(A \wedge B) \leq \overline{P}(A) + \overline{P}(B)$, $\forall A, B \in \mathcal{A}(\mathbb{P})$).

The natural extension of a lower probability $\underline{P} : \mathcal{A}(\mathbb{P}) \to \mathbb{R}$ to the gambles in $\mathcal{L}(\mathbb{P})$ is related with the *Choquet integral* as follows:

Proposition 2.1. *[24, Theorem 8.14] Given a lower probability* $\underline{P} : \mathcal{A}(\mathbb{P}) \to \mathbb{R}$ *which avoids sure loss, is monotone and such that* $\underline{P}(\emptyset) = 0$, $\underline{P}(\Omega) = 1$, *it holds that*

$$\underline{E}(X) \geq (C) \int X d\underline{P}, \quad \forall X \in \mathcal{L}(\mathbb{P}). \tag{6}$$

where $(C) \int X d\underline{P}$ *is the Choquet integral of* X *with respect to* $\underline{P}$.

There is equality in (6) *for all* $X \in \mathcal{L}(\mathbb{P})$ *iff* $\underline{P}$ *is* 2*-monotone. A symmetric statement applies to upper probabilities.*

The Choquet integral $(C) \int X d\mu$ of a gamble $X \in \mathcal{L}(\mathbb{P})$ with respect to a monotone measure μ on $\mathcal{A}(\mathbb{P})$, with $\mu(\emptyset) = 0$, always exists and is defined in terms of improper Riemann integrals (see e.g. [24, Appendix C.2]). In the Choquet integrals in this paper, μ is a lower or upper probability, and $\mu(\Omega) = 1$.

Then, known formulae are available for computing $(C)\int X d\underline{P}$, and therefore also, by Proposition 2.1, the natural extension of a 2-monotone $\underline{P}$ (of a 2-alternating $\overline{P}$) [11, 24]:

$$(C)\int X d\mu = \inf X + \int_{\inf X}^{\sup X} \mu(X > x)dx \tag{7}$$

and, if X is a simple gamble with distinct values $x_1 < x_2 < \ldots < x_m$,

$$(C)\int X d\mu = \sum_{h=1}^{m} x_h(\mu(X \geq x_h) - \mu(X \geq x_{h+1})), \tag{8}$$

where by definition $(X \geq x_{m+1}) = \emptyset$.

A further instance where 2-monotonicity may simplify the computation of the natural extension is the following:

Proposition 2.2. *Let $\underline{P}_1, \underline{P}_2, \ldots, \underline{P}_n$ be 2-monotone coherent lower probabilities on $\mathcal{A}(\mathbb{P})$ with natural extensions (on $\mathcal{L}(\mathbb{P})$) $\underline{E}_1, \underline{E}_2, \ldots, \underline{E}_n$, respectively. Let $\underline{P}$ be a monotone and 2-monotone linear combination of $\underline{P}_1, \underline{P}_2, \ldots, \underline{P}_n$, $\underline{P} = \sum_{i=1}^{n} \beta_i \underline{P}_i$, such that $\sum_{i=1}^{n} \beta_i = 1$. The natural extension $\underline{E}$ of $\underline{P}$ on $\mathcal{L}(\mathbb{P})$ is given by*

$$\underline{E}(X) = \sum_{i=1}^{n} \beta_i \underline{E}_i(X), \ \forall X \in \mathcal{L}(\mathbb{P}). \tag{9}$$

Analogously, let $\overline{P}_1, \overline{P}_2, \ldots, \overline{P}_n$ be 2-alternating and coherent upper probabilities on $\mathcal{A}(\mathbb{P})$ with natural extensions (on $\mathcal{L}(\mathbb{P})$) $\overline{E}_1, \overline{E}_2, \ldots, \overline{E}_n$, respectively. Let $\overline{P}$ be a monotone and 2-alternating linear combination of $\overline{P}_1, \overline{P}_2, \ldots, \overline{P}_n$, $\overline{P} = \sum_{i=1}^{n} \beta_i \overline{P}_i$, with $\sum_{i=1}^{n} \beta_i = 1$. The natural extension $\overline{E}$ of $\overline{P}$ on $\mathcal{L}(\mathbb{P})$ is

$$\overline{E}(X) = \sum_{i=1}^{n} \beta_i \overline{E}_i(X), \ \forall X \in \mathcal{L}(\mathbb{P}).$$

Proof. It suffices to prove the lower probability part of the thesis, the remaining one being analogous.

If $\underline{P}$ is monotone and 2-monotone, $\underline{P}$ is also coherent since $\underline{P}(\emptyset) = 0$, $\underline{P}(\Omega) = 1$, by [24, Corollary 6.16]. Then, the natural extension of $\underline{P}$ on $X \in \mathcal{L}(\mathbb{P})$ is given by (7). Applying to (7) the linearity property of the Riemann integral and then (7) itself we obtain Equation (9):

$$\begin{aligned}
\underline{E}(X) &= \inf X + \int_{\inf X}^{\sup X} \underline{P}(X > x)\,dx \\
&= \sum_{i=1}^{n} \beta_i \inf X + \int_{\inf X}^{\sup X} \sum_{i=1}^{n} \beta_i \underline{P}_i(X > x)\,dx \\
&= \sum_{i=1}^{n} \beta_i \left(\inf X + \int_{\inf X}^{\sup X} \underline{P}_i(X > x)\,dx \right) = \sum_{i=1}^{n} \beta_i \underline{E}_i(X).
\end{aligned}$$

□

Remark 2.2. *In a less general version, Proposition 2.2 holds replacing the hypothesis*

(a) $\underline{P}$ *is a monotone and* 2*-monotone linear combination of* $\underline{P}_1, \underline{P}_2, \dots, \underline{P}_n$*,* $\underline{P} = \sum_{i=1}^{n} \beta_i \underline{P}_i$*, such that* $\sum_{i=1}^{n} \beta_i = 1$

with

(b) $\underline{P}$ *is a convex linear combination of* $\underline{P}_1, \underline{P}_2, \dots, \underline{P}_n$*,* $\underline{P} = \sum_{i=1}^{n} \beta_i \underline{P}_i$*, such that* $\beta_i \geq 0$*,* $i = 1, \dots, n$*, and* $\sum_{i=1}^{n} \beta_i = 1$*.*

while keeping the other assumptions (and similarly with $\overline{P}$*).*

In fact, (*b*) *implies* (*a*)*: if* (*b*) *is true, then* $\underline{P}$ *is coherent, hence monotone, being a convex linear combination of coherent lower probabilities [24, Proposition 4.19 (ii)]. Also, it follows trivially from* (5) *that a convex linear combination of* 2*-monotone lower probabilities (coherent or not) is* 2*-monotone.*

We shall apply Proposition 2.2 in the proofs of Propositions 4.4, 4.5, 6.4 (*b*)*, 6.5. In most cases, these propositions satisfy condition* (*b*)*. However, Proposition 4.5 does not: its proof involves Equation* (22)*, where* $\overline{P}$ *is a linear combination, with coefficients* b *and* $1 - b$*, of coherent and* 2*-alternating upper probabilities. Yet, coefficient* $1 - b$ *there may be negative, since* b *can be greater than* 1*.*

In our commitment to determine simple ways of computing the natural extension of Nearly-Linear models, we shall also be concerned with *probability intervals* [7], which are lower and upper probability assignments on a *finite* partition $\mathbb{P}$. Denoting a probability interval with $I = [\underline{P}(\omega), \overline{P}(\omega)]_{\omega \in \mathbb{P}}$, $0 \leq \underline{P}(\omega) \leq \overline{P}(\omega) \leq 1$, $\forall \omega \in \mathbb{P}$, it is well-known, see e.g. [24, Section 7.1], that

$$I \text{ avoids sure loss (on } \mathbb{P}) \text{ iff } \sum_{\omega \in \mathbb{P}} \underline{P}(\omega) \leq 1 \leq \sum_{\omega \in \mathbb{P}} \overline{P}(\omega). \tag{10}$$

Further, and importantly for us, if I avoids sure loss its natural extensions on $\mathcal{A}(\mathbb{P})$, $\underline{E}$ and its conjugate $\overline{E}$, are well-known. They are, respectively, 2-monotone and 2-alternating, and given by

$$\underline{E}(A) = \max\Big\{ \sum_{\omega \Rightarrow A} \underline{P}(\omega), 1 - \sum_{\omega \Rightarrow \neg A} \overline{P}(\omega) \Big\} \tag{11}$$

$$\overline{E}(A) = \min\Big\{ \sum_{\omega \Rightarrow A} \overline{P}(\omega), 1 - \sum_{\omega \Rightarrow \neg A} \underline{P}(\omega) \Big\}. \tag{12}$$

2.3 Nearly-Linear Models

Denoting with $\mu : \mathcal{A}(\mathbb{P}) \to \mathbb{R}$ either a lower or an upper probability, we have

Definition 2.4. $\mu : \mathcal{A}(\mathbb{P}) \to \mathbb{R}$ *is a* Nearly-Linear *imprecise probability iff* $\mu(\emptyset) = 0$*,* $\mu(\Omega) = 1$*, and given a probability* P_0 *on* $\mathcal{A}(\mathbb{P})$*,* $a \in \mathbb{R}$*,* $b > 0$*, it holds that* $\forall A \in \mathcal{A}(\mathbb{P}) \setminus \{\emptyset, \Omega\}$*,*

$$\mu(A) = \min\{\max\{bP_0(A) + a, 0\}, 1\}.$$

In short, we write that μ is $NL(a,b)$. NL models are closed with respect to conjugacy [5, Proposition 3.1]: if μ is $NL(a,b)$, then its conjugate $\mu^c(A) = 1 - \mu(\neg A)$ is $NL(c,b)$, with

$$c = 1 - (a+b). \tag{13}$$

Thus, every NL submodel is made up of a couple of conjugate imprecise probabilities. By convention we identify the lower probability $\underline{P}$ with the parameters (a,b), the upper probability $\overline{P}$ with (c,b). It has been shown in [5, Section 3.1] that NL models whose lower/upper probabilities are at least 2-coherent can be partitioned into 3 submodels, described in the next subsections.

2.3.1 The Vertical Barrier Model

Definition 2.5. *A* Vertical Barrier Model (VBM) *is a NL model where a, b satisfy*

$$0 \leq a+b \leq 1, \quad a \leq 0 \tag{14}$$

and c is given by (13), hence $\underline{P}$ and its conjugate $\overline{P}$ are given by:

$$\underline{P}(A) = \begin{cases} \max\{bP_0(A)+a, 0\} & \text{if } A \in \mathcal{A}(\mathbb{P}) \setminus \{\Omega\} \\ 1 & \text{if } A = \Omega \end{cases} \tag{15}$$

$$\overline{P}(A) = \begin{cases} \min\{bP_0(A)+c, 1\}, & \text{if } A \in \mathcal{A}(\mathbb{P}) \setminus \{\emptyset\} \\ 0 & \text{if } A = \emptyset \end{cases}.$$

In a VBM, $\underline{P}$ is coherent and 2-monotone, $\overline{P}$ is coherent and 2-alternating. If $b = 1+\delta > 1$, $a = -\delta < 0$, we obtain

$$\underline{P}_{\mathrm{PMM}}(A) = \max\{(1+\delta)P_0(A) - \delta, 0\},$$
$$\overline{P}_{\mathrm{PMM}}(A) = \min\{(1+\delta)P_0(A), 1\}$$

which is the Pari-Mutuel Model, a special VBM. It is easy to see that $\overline{P}(A) \downarrow c \geq 0$ as $P_0(A) \downarrow 0$, and that $c = 0$ in the PMM. Thus, $\overline{P}(A)$ does generally not tend to 0 with P_0, meaning that a VBM with $c > 0$ ensures a minimum positive selling price c for any event $A \neq \emptyset$, even those very unlikely (according to P_0). This is reasonable in realistic situations, since the seller may stand some fixed costs for any event, irrespective of its uncertainty evaluation. In the $(P_0, \overline{P})$-plane, this difference between VBM and PMM is visualised by the vertical barrier given by the segment with endpoints $(0,0)$, $(0,c)$ on the $\overline{P}$-axis, after which is named the model, see Fig. 1, 1).

2.3.2 The Horizontal Barrier Model

Definition 2.6. *A* Horizontal Barrier Model (HBM) *is a NL model where a, b satisfy the constraints $a+b > 1$, $b+2a \leq 1$ and c is given by (13):*

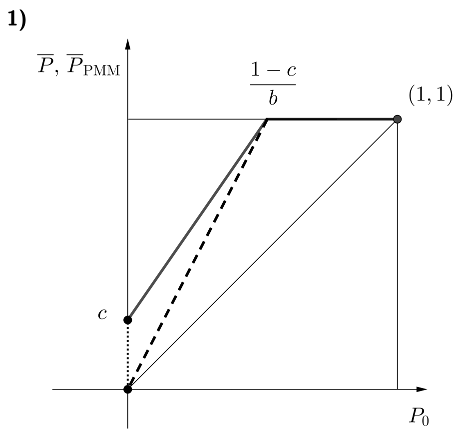

Figure 1: Plots of $\overline{P}$ $NL(c,b)$ (continuous bold line) and $\overline{P}_{\mathrm{PMM}}$ $NL(0,b')$ (dashed bold line), with $b' = \frac{b}{1-c}$, against P_0 ($\overline{P}, \overline{P}_{\mathrm{PMM}}$ overlap on the line $\overline{P} = \overline{P}_{\mathrm{PMM}} = 1$): 1) in the VBM 2) in the HBM.

$\forall A \in \mathcal{A}(\mathbb{P}) \setminus \{\emptyset, \Omega\}$, *it holds that*

$$\underline{P}(A) = \min\{\max\{bP_0(A) + a, 0\}, 1\}, \tag{16}$$
$$\overline{P}(A) = \max\{\min\{bP_0(A) + c, 1\}, 0\}. \tag{17}$$

It is easy to see that $a < 0$, $c < 0$, $b > 1$ in a HBM. Further, in this model an agent acting as a seller obtains her/his selling prices from P_0 in conflicting ways. This is especially patent with tail P_0-probability events: the agent's $\overline{P}(A)$ is 1 iff $P_0(A) \geq \frac{1-c}{b}$, is 0 iff $P_0(A) \leq -\frac{c}{b}$. As a selling price evaluation, P_0 is reputed too low, with high P_0-probability events (regarded as practically sure by the agent), too high, with low P_0-probability ones (considered practically impossible). An analogous, conflicting behaviour is implied by $\underline{P}$ as a buying price evaluation. See also Fig. 1, 2), where a comparison between the HBM and the PMM upper probabilities shows that the HBM introduces a horizontal barrier, the segment connecting $(0,0)$, $(-\frac{c}{b}, 0)$ on the P_0-axis, that names the model.

Despite its conveying an agent's somewhat contradictory uncertainty evaluation, the HBM is not always incoherent. Although $\underline{P}$ and $\overline{P}$ in a HBM are generally only 2-coherent, it can be shown [5] that

Proposition 2.3. *$\overline{P}$ in a HBM is a coherent upper probability iff it is subadditive (i.e. iff $\overline{P}(A) + \overline{P}(B) \geq \overline{P}(A \vee B), \forall A, B \in \mathcal{A}(\mathbb{P})$). When $\overline{P}$ is coherent, it is* 2-*alternating too.*

There are also instances of HBMs where $\underline{P} = \overline{P} = P$, and P may or not be a dF-coherent probability [5, Example 5.1 and Section 5.3].

2.3.3 The Restricted Range Model

Definition 2.7. *A* Restricted Range Model (RRM) *is a NL model where* a, b *satisfy*

$$a > 0,\ b + 2a \leq 1$$

and c *is given by* (13)*, hence* $\forall A \in \mathcal{A}(\mathbb{P}) \setminus \{\emptyset, \Omega\}$

$$\underline{P}(A) = bP_0(A) + a, \tag{18}$$

$$\overline{P}(A) = bP_0(A) + c. \tag{19}$$

The name of this model arises from its property $\underline{P}(A) \in [a, a+b] \subset [0, 1]$, $\forall A \in \mathcal{A}(\mathbb{P}) \setminus \{\emptyset, \Omega\}$. It can be seen that an agent using it has conflicting attitudes towards high and low P_0-probability events. This feature is shared with the HBM, but a RRM is less prudential, in the sense that, unlike the HBM, a subject adopting a RRM is committed to buy or to sell any non-trivial event. $\underline{P}$ and $\overline{P}$ are almost never coherent in the RRM: they are iff $\mathbb{P}$ is a partition of cardinality two, i.e. never in significant problems.

3 Vertical Barrier and Other Models

The VBM includes various well-known neighbourhood models as special cases. We point out the following:

- If $a = 0$, $0 < b < 1$ (hence $c = 1 - b > 0$), the *ε-contamination model* (termed linear-vacuous mixture model in [25] - here $\varepsilon = b$):

$$\begin{aligned} &\underline{P}_\varepsilon(A) = bP_0(A), && \forall A \neq \Omega, && \underline{P}_\varepsilon(\Omega) = 1, \\ &\overline{P}_\varepsilon(A) = bP_0(A) + 1 - b, && \forall A \neq \emptyset, && \overline{P}_\varepsilon(\emptyset) = 0. \end{aligned}$$

- If $a + b = 0$ (hence $c = 1$), the *vacuous lower/upper probability model* [25]:

$$\begin{aligned} &\underline{P}_V(A) = 0, && \forall A \neq \Omega, && \underline{P}_V(\Omega) = 1, \\ &\overline{P}_V(A) = 1, && \forall A \neq \emptyset, && \overline{P}_V(\emptyset) = 0. \end{aligned}$$

  Note that we would obtain the same model also for $a + b < 0$. Because of this, we may require, without losing generality, VBMs to satisfy the condition $a + b \geq 0$.

- If $b = 1 + \delta > 1$, $a = -\delta < 0$, the *Pari-Mutuel Model* [13, 14, 22, 25], as already seen in Section 2.3.1.

- If $b = 1$, $-1 < a < 0$ (hence $c = -a$), the *Total Variation Model* [12, 22]:

$$\begin{aligned} &\underline{P}_{\text{TVM}}(A) = \max\{P_0(A) - c, 0\}, && \forall A \neq \Omega, \\ &\overline{P}_{\text{TVM}}(A) = \min\{P_0(A) + c, 1\}, && \forall A \neq \emptyset, \\ &\underline{P}_{\text{TVM}}(\Omega) = 1, \quad \overline{P}_{\text{TVM}}(\emptyset) = 0. \end{aligned}$$

In this model, $\underline{P}_{\mathrm{TVM}}$ ($\overline{P}_{\mathrm{TVM}}$) is the lower (upper) envelope of the probabilities whose total variation distance from P_0 does not exceed c ($\in]0,1[$).[1]

- If $a \geq -1$, a model introduced by Rieder [23] in 1977, in the realm of statistical robustness.

It is also interesting to note that the VBM *can be written as a linear combination of some of its special cases*. Specifically, it can be expressed as a linear combination of a suitable TVM and the vacuous model or as a convex combination of a suitable PMM and the vacuous model. In the rest of this section we prove these relationships.

For this, starting from Equation (15), we get, $\forall A \in \mathcal{A}(\mathbb{P}) \setminus \{\Omega\}$,

$$\begin{aligned}\underline{P}(A) &= b \max\left\{P_0(A) + \frac{a}{b}, 0\right\} \\ &= b \max\left\{P_0(A) + \frac{a}{b}, 0\right\} + (1-b)\underline{P}_V(A).\end{aligned} \tag{20}$$

Since, by (14), $a' = \frac{a}{b} \in]-1,0[$,[2] (20) gives $\underline{P}$ in the VBM as a linear combination of a $\underline{P}_{TVM}$ which is $NL(a',1)$ and of $\underline{P}_V$. We may also write

$$\underline{P}(A) = b\underline{P}_{TVM}(A) + (1-b)\underline{P}_V(A), \quad \forall A \in \mathcal{A}(\mathbb{P}),$$

which applies to $A = \Omega$, too. Analogously, with the VBM upper probability $\overline{P}$ we get, $\forall A \in \mathcal{A}(\mathbb{P}) \setminus \{\emptyset\}$ and using (13) at the last equality,

$$\begin{aligned}\overline{P}(A) = \min\{bP_0(A) + c, 1\} &= b \min\left\{P_0(A) + \frac{c}{b}, \frac{1}{b}\right\} \\ &= b\left(\min\left\{P_0(A) + \frac{c}{b} + 1 - \frac{1}{b}, \frac{1}{b} + 1 - \frac{1}{b}\right\} - \left(1 - \frac{1}{b}\right)\right) \\ &= b \min\left\{P_0(A) - \frac{a}{b}, 1\right\} + (1-b).\end{aligned} \tag{21}$$

Hence, $\overline{P}$ in the VBM is a linear combination of a $\overline{P}_{TVM}$ which is $NL(c',1)$, with $c' = -a' = -\frac{a}{b}$, and of $\overline{P}_V$. We may write

$$\overline{P}(A) = b\overline{P}_{TVM}(A) + (1-b)\overline{P}_V(A), \quad \forall A \in \mathcal{A}(\mathbb{P}). \tag{22}$$

As for the relationship between a VBM and a suitable PMM, we start again from (15), assuming $a + b > 0$:[3] $\forall A \in \mathcal{A}(\mathbb{P}) \setminus \{\Omega\}$,

$$\begin{aligned}\underline{P}(A) &= (a+b) \max\left\{\frac{b}{a+b}P_0(A) + \frac{a}{a+b}, 0\right\} \\ &= (a+b) \max\left\{\frac{b}{a+b}P_0(A) + \frac{a}{a+b}, 0\right\} + \big(1 - (a+b)\big)\underline{P}_V(A).\end{aligned} \tag{23}$$

Since, again by (14),[4] we have that $a'' = \frac{a}{a+b} < 0$, $b'' = \frac{b}{a+b} > 1$, Equation (23)

[1] We can neglect the cases $c = 0$, $c = 1$ which return, respectively, P_0 and the vacuous model.

[2] We skip the cases $a' = 0$, $a' = -1$ which return, respectively, the ε-contamination and the vacuous model.

[3] We neglect here the case $a + b = 0$, since it implies $\underline{P} = \underline{P}_V$.

[4] We assume here $a < 0$. If $a = 0$, the VBM reduces to an ε-contamination model.

gives $\underline{P}$ in the VBM as a convex combination of a $\underline{P}_{PMM}$ which is $NL(a'', b'')$ and of $\underline{P}_V$. We may also write

$$\underline{P}(A) = (a+b)\underline{P}_{PMM}(A) + \big(1 - (a+b)\big)\underline{P}_V(A), \ \forall A \in \mathcal{A}(\mathbb{P}). \tag{24}$$

Analogously, for $\overline{P}$ in a VBM, we get, $\forall A \in \mathcal{A}(\mathbb{P}) \setminus \{\emptyset\}$.

$$\begin{aligned}
\overline{P}(A) &= \min\{bP_0(A) + c, 1\} \\
&= (a+b)\min\left\{\frac{b}{a+b}P_0(A) + \frac{c}{a+b}, \frac{1}{a+b}\right\} \\
&= (a+b)\left(\min\left\{\frac{b}{a+b}P_0(A), \frac{1-c}{a+b}\right\} + \frac{c}{a+b}\right) \\
&= (a+b)\min\left\{\frac{b}{a+b}P_0(A), 1\right\} + \big(1-(a+b)\big). \qquad (25)
\end{aligned}$$

We conclude that

$$\overline{P}(A) = (a+b)\overline{P}_{PMM}(A) + \big(1-(a+b)\big)\overline{P}_V(A), \quad \forall A \in \mathcal{A}(\mathbb{P}), \tag{26}$$

where $\overline{P}_{PMM}$ is $NL(0, b'')$. By equating (20) and (23) we easily obtain a relationship between $\underline{P}_{TVM}$ and $\underline{P}_{PMM}$:

$$\underline{P}_{TVM}(A) = \frac{a+b}{b}\underline{P}_{PMM}(A) = \frac{1}{b''}\underline{P}_{PMM}(A), \quad \forall A \in \mathcal{A}(\mathbb{P}) \setminus \{\Omega\}.$$

Analogously, by equating (21) and (25), we get

$$\overline{P}_{TVM}(A) = \frac{a+b}{b}\overline{P}_{PMM}(A) - \frac{a}{b} = \frac{1}{b''}\overline{P}_{PMM}(A) - \frac{a''}{1-a''}, \quad \forall A \in \mathcal{A}(\mathbb{P}) \setminus \{\emptyset\}$$

(cf. also [22, Section 3.2]).

To interpret the relations we derived, recall that of the ε-contamination model, adopted by a subject who believes that P_0 is the 'true' model with probability $(\varepsilon =)b$, while having no information about alternative models and therefore opting for the vacuous one with probability $1-b$. In the VBM interpretation, suggested by the equalities (24), (26), the role of P_0 is played by a PMM (replacing b with $a+b$). The same interpretation applies also to the representation of a VBM in terms of a TVM (this time with b instead of $a+b$), but only when $b < 1$. Note also that, for given parameters a, b ($b < 1$), the assessor is less sure that the appropriate model may be a PMM rather than a TVM, since $a + b < b$.

Later on (see Equation (60) and Comment, (iv) in Section 6) we shall find a different decomposition as a convex linear combination also for the RRM.

4 Vertical Barrier Models and Natural Extensions

Since Vertical Barrier Models are coherent, we can discuss natural extensions under two different perspectives: next to considering the natural extension of a VBM to $\mathcal{L}(\mathbb{P})$, we may wonder whether the VBM itself is, or plays a role in, the natural extension of something else.

4.1 Vertical Barrier Models as Natural Extensions

We begin with this latter aspect, and consider the lower probability $\underline{Q}(A) = bP_0(A) + a$ which is the first term of the maximum defining $\underline{P}$ in Definition 2.5. It is proven in [20, Proposition 3.1] that

Proposition 4.1. *The lower probability*

$$\underline{Q}(A) = bP_0(A) + a, \quad \forall A \in \mathcal{A}(\mathbb{P}) \tag{27}$$

in Equation (15) *(where a, b satisfy* (14)*) avoids sure loss; $\underline{Q}$ is convex iff $b = 1$. Its natural extension on $\mathcal{A}(\mathbb{P})$ is precisely the lower probability $\underline{P}$ of the VBM it originates from.*

Thus, a VBM corrects the naive evaluation $\underline{Q}$ via natural extension, by introducing barriers to its values.

More generally, it can be shown that a generalisation of the VBM is the natural extension of a class of non-centered convex probabilities. The next result is useful for this.

Proposition 4.2. *Let I be a set of indexes and, for any $\alpha \in I$, let $P_\alpha : \mathcal{A}(\mathbb{P}) \to \mathbb{R}$ be a dF-coherent probability, $a_\alpha \leq 0$, such that $\inf_{\alpha \in I} a_\alpha \in \mathbb{R}$. Define*

$$\underline{P}_\alpha = P_\alpha + a_\alpha, \quad \underline{P} = \inf_{\alpha \in I} \underline{P}_\alpha.$$

Then, $\underline{P}$ is convex and avoids sure loss. Letting $\underline{E}_\alpha$ be the natural extension of $\underline{P}_\alpha$, $\underline{E}_{\underline{P}}$ the natural extension of $\underline{P}$, it holds that

$$\underline{E}_{\underline{P}} = \inf_{\alpha \in I} \underline{E}_\alpha. \tag{28}$$

Proof. By Theorem 2.2 (a), the lower probabilities $\underline{P}_\alpha$ and $\underline{P}$ are convex, $\forall \alpha \in I$; since $\underline{P}_\alpha(\emptyset) = a_\alpha \leq 0$, $\underline{P}(\emptyset) = \inf_{\alpha \in I} a_\alpha \leq 0$, they also avoid sure loss [17]. Thus the natural extensions $\underline{E}_\alpha$, $\underline{E}_{\underline{P}}$ exist and are finite. Their properties (Theorem 2.3 (a)) imply that

$$\underline{P} \leq \underline{E}_{\underline{P}}, \quad \underline{P}_\alpha \leq \underline{E}_\alpha, \qquad \forall \alpha \in I. \tag{29}$$

Since any $\underline{E}_\alpha$ is coherent, also $\inf_{\alpha \in I} \underline{E}_\alpha$ is coherent [24, Proposition 4.20 (ii)]. From this, the inequality $\underline{P} = \inf_{\alpha \in I} \underline{P}_\alpha \leq \inf_{\alpha \in I} \underline{E}_\alpha$ (implied by (29)), and property (b), Theorem 2.3, we obtain

$$\underline{E}_{\underline{P}}(A) \leq \inf_{\alpha \in I} \underline{E}_\alpha(A), \quad \forall A \in \mathcal{A}(\mathbb{P}). \tag{30}$$

Clearly, the equality holds in (30) for $A = \Omega$: $\underline{E}_{\underline{P}}(\Omega) = \inf_{\alpha \in I} \underline{E}_\alpha(\Omega) = 1$.

Now take any $A \in \mathcal{A}(\mathbb{P})$, $A \neq \Omega$. If there exists $\bar{\alpha} \in I$ such that $\underline{E}_{\bar{\alpha}}(A) = 0$, then $0 \leq \underline{E}_{\underline{P}}(A) \leq \inf_{\alpha \in I} \underline{E}_\alpha(A) = 0$, hence $\underline{E}_{\underline{P}}(A) = \inf_{\alpha \in I} \underline{E}_\alpha(A) = 0$.

Otherwise, if $\underline{E}_\alpha(A) > 0$, $\forall \alpha \in I$, then $\underline{E}_\alpha(A) = P_\alpha(A) + a_\alpha$, by Proposition 4.1 (with $b = 1$). Using this, and (30) at the last inequality, we have

$$\inf_{\alpha \in I} \underline{E}_\alpha(A) = \inf_{\alpha \in I} \big(P_\alpha(A) + a_\alpha\big) = \underline{P}(A) \leq \underline{E}_{\underline{P}}(A) \leq \inf_{\alpha \in I} \underline{E}_\alpha(A),$$

implying again the equality $\underline{E}_{\underline{P}}(A) = \inf_{\alpha \in I} \underline{E}_\alpha(A)$. $\square$

	ω_1	ω_2	ω_3	$\omega_1 \vee \omega_2$
P_1	0.3	0.25	0.45	0.55
$P_1 + a_1$	0.25	0.2	0.4	0.5
P_2	0.15	0.05	0.8	0.2
$P_2 + a_2$	0.25	0.15	0.9	0.3
$\underline{P}$	0.25	0.15	0.4	0.3

Table 1: Values of P_1, $P_2, \underline{P}$ in Remark 4.1.

By Proposition 4.2, $\underline{P}$ is a convex lower probability; since $\underline{P}(\emptyset) = \inf_{\alpha \in I} a_\alpha \leq 0$, $\underline{P}$ is also non-centered, outside the limiting situation $a_\alpha = 0$, $\forall \alpha \in I$. We may apply Proposition 4.1 to write explicitly $\underline{E}_\alpha$ in (28), since any $\underline{P}_\alpha$ is a lower probability of the type (27), with $b = 1$. We obtain that $\underline{E}_{\underline{P}}(\Omega) = 1$, and, $\forall A \in \mathcal{A}(\mathbb{P}) \setminus \{\Omega\}$,

$$\underline{E}_{\underline{P}}(A) = \inf_{\alpha \in I} \max\{P_\alpha(A) + a_\alpha, 0\}. \tag{31}$$

From (31) (and the coherence condition $\underline{E}_{\underline{P}}(A) \geq 0$), $\underline{E}_{\underline{P}}(A) = 0$ if $\exists \bar{\alpha} \in I : P_{\bar{\alpha}}(A) + a_{\bar{\alpha}} < 0$, while $\underline{E}_{\underline{P}}(A) = \inf_{\alpha \in I}(P_\alpha(A) + a_\alpha)$ otherwise. Thus, we may rewrite (31) as follows:

$$\underline{E}_{\underline{P}}(A) = \begin{cases} 1 & \text{if } A = \Omega \\ \max\{\inf_{\alpha \in I}(P_\alpha(A) + a_\alpha), 0\} & \text{otherwise.} \end{cases} \tag{32}$$

From (32), the natural extension of the lower envelope $\underline{P}$ of a given set of lower probabilities $\underline{P}_\alpha = P_\alpha + a_\alpha$ that ensure the condition $a_\alpha \leq 0$, $\forall \alpha \in I$ (with $\inf_{\alpha \in I} a_\alpha > -\infty$), is formally analogue to a VBM. It differs from it because it replaces the lower probability that avoids sure loss $bP_0(A) + a$ with $\inf_{\alpha \in I}(P_\alpha(A) + a_\alpha)$, a convex lower probability still avoiding sure loss.

Remark 4.1. *The conclusion above does not apply to those convex lower probabilities $\underline{P} = \inf_{\alpha \in I} \underline{P}_\alpha$ such that, for some $\bar{\alpha} \in I$, $a_{\bar{\alpha}}$ is positive, while still being $0 \geq \inf_{\alpha \in I} a_\alpha > -\infty$. One might wonder whether in such cases $\underline{P}$ could be also obtained as a lower envelope of a different set $\{\underline{P}'_\alpha\}_{\alpha \in I'}$ such that $a_\alpha \leq 0$, $\forall \alpha \in I'$. An affirmative answer would imply that Equation (32) characterises the natural extensions of* all *convex lower probabilities that avoid sure loss on $\mathcal{A}(\mathbb{P})$, but unfortunately this is not the case. To illustrate, let $\mathbb{P} = \{\omega_1, \omega_2, \omega_3\}$ and define $\underline{P} = \min\{P_1 - 0.05, P_2 + 0.1\}$, see Table 1. Whatever is the set $\mathcal{M}' = \{\underline{P}'_\alpha\}_{\alpha \in I'}$ of which $\underline{P}$ is the lower envelope, by Theorem 2.2 there is at least one $\underline{P}'_{\bar{\alpha}} \in \mathcal{M}'$ such that*

$$\underline{P}'_{\bar{\alpha}}(\omega_1 \vee \omega_2) = P'_{\bar{\alpha}}(\omega_1 \vee \omega_2) + a_{\bar{\alpha}} = P'_{\bar{\alpha}}(\omega_1) + P'_{\bar{\alpha}}(\omega_2) + a_{\bar{\alpha}} = \underline{P}(\omega_1 \vee \omega_2) = 0.3. \tag{33}$$

However, $\underline{P}'_{\bar{\alpha}}$ must also satisfy the inequalities

$$\underline{P}'_{\bar{\alpha}}(\omega_1) = P'_{\bar{\alpha}}(\omega_1) + a_{\bar{\alpha}} \geq 0.25 = \underline{P}(\omega_1),$$
$$\underline{P}'_{\bar{\alpha}}(\omega_2) = P'_{\bar{\alpha}}(\omega_2) + a_{\bar{\alpha}} \geq 0.15 = \underline{P}(\omega_2).$$

Thus, $P'_{\bar{\alpha}}(\omega_1) + P'_{\bar{\alpha}}(\omega_2) + 2a_{\bar{\alpha}} \geq 0.4$*. Using* (33)*, this implies* $a_{\bar{\alpha}} \geq 0.1 > 0$.

4.2 The Natural Extension of a Vertical Barrier Model

Let us consider now the problem of extending a VBM (that is, extending its lower probability $\underline{P}$ or the conjugate upper probability $\overline{P}$) from $\mathcal{A}(\mathbb{P})$ to $\mathcal{L}(\mathbb{P})$. Take for instance $\underline{P}$: since it is coherent and 2-monotone, its natural extension $\underline{E}(X)$ on a gamble $X \in \mathcal{L}(\mathbb{P})$ is a Choquet integral by Proposition 2.1 and may be written in general in the form (7), with $\mu = \underline{P}$.

In the case of a VBM, Equation (7) may be further specialised, by the following proposition, proven in [20, Proposition 3.3].

Proposition 4.3. *Let* $\underline{P} : \mathcal{A}(\mathbb{P}) \to \mathbb{R}$ *be the lower probability of a VBM. If* $a < 0$*, for any* $X \in \mathcal{L}(\mathbb{P})$ *define*

$$\tilde{x} = \sup\left\{x \in \mathbb{R} : P_0(X \leq x) \leq 1 + \frac{a}{b}\right\},$$

$(\tilde{x} - X)^+ = \max\{\tilde{x} - X, 0\}$*. Then*

$$\underline{E}(X) = (a+b)\tilde{x} + \big(1 - (a+b)\big)\inf X - bE^{P_0}\big((\tilde{x} - X)^+\big), \tag{34}$$

where $E^{P_0}\big((\tilde{x} - X)^+\big)$ *is the (precise) natural extension of* P_0 *to* $(\tilde{x} - X)^+$*.*

If $a = 0$*, we get instead*

$$\underline{E}(X) = (1-b)\inf X + bE^{P_0}(X), \tag{35}$$

with $E^{P_0}(X)$ *(precise) natural extension of* P_0 *to* X*.*

If $a = -\delta < 0$, $b = 1 + \delta$, the VBM is a Pari-Mutuel Model, and $\underline{E}(X)$ in (34) boils down to

$$\underline{E}(X) = \tilde{x} - (1+\delta)E^{P_0}\big((\tilde{x} - X)^+\big)$$

which is in fact an expression for the natural extension of a PMM $\underline{P}$ that may be found in [25, p. 131].

In the special case $a = 0$ the VBM is instead an ε-contamination model (defined on $\mathcal{A}(\mathbb{P})$). Here (35) states that its natural extension is again an ε-contamination model (defined on $\mathcal{L}(\mathbb{P})$). Putting $1 - b = \delta$, $\underline{E}(X)$ in (35) is rewritten in fact in the form $\underline{E}(X) = \delta \inf X + (1-\delta)E^{P_0}(X)$, appeared in [25].

Lastly, let us see how the natural extension of a VBM $\underline{P}$, which is $NL(a, b)$, varies when modifying its parameters into $a' = ka$, $b' = kb$, $k \in]0, 1[$, getting a VBM $\underline{P}'$, which is $NL(a', b')$. This analysis is interesting because such a choice of a', b' does not affect $\tilde{x}$, which remains the same in Proposition 4.3 for both $\underline{P}$ and $\underline{P}'$.

Proposition 4.4. *Let* $\underline{P}$*,* $\underline{P}'$ *be VBM lower probabilities, with* $\underline{P}$ $NL(a, b)$ *given by* (15)*,* $\underline{P}'$ $NL(a', b')$ *identified by* $a' = ka$*,* $b' = kb$*,* $k \in]0, 1[$*. Terming* $\underline{E}$*,* $\underline{E}'$ *the natural extensions of* $\underline{P}$*,* $\underline{P}'$ *on* $\mathcal{L}(\mathbb{P})$*, we have*

$$\underline{E}'(X) = k\underline{E}(X) + (1-k)\inf X. \tag{36}$$

Proof. Since, $\forall A \in \mathcal{A}(\mathbb{P}) \setminus \{\Omega\}$, we have that $\underline{P}'(A) = \min\{kbP_0(A) + ka, 0\} = k\underline{P}(A)$, it holds that

$$\underline{P}'(A) = k\underline{P}(A) + (1-k)\underline{P}_V(A) \ \ \forall A \in \mathcal{A}(\mathbb{P}). \tag{37}$$

By Proposition 2.2 (Equation (9)) and since the natural extension of $\underline{P}_V$ is, for all $X \in \mathcal{L}(\mathbb{P})$, $\underline{E}(X) = \inf X$ [25], we obtain (36). □

Note that the more k is closer to 0, the more $\underline{E}'$ tends to be vacuous, as appears from Equation (36).

It is also possible to derive a formula for the natural extension of the upper probability $\overline{P}$ of a VBM:

Proposition 4.5. *Let $\overline{P} : \mathcal{A}(\mathbb{P}) \to \mathbb{R}$ be the upper probability of a VBM. If $c < 1$, for any $X \in \mathcal{L}(\mathbb{P})$ define*

$$\tilde{x} = \sup\left\{x \in \mathbb{R} : P_0(X > x) \geq 1 + \frac{a}{b}\right\}, \tag{38}$$

$(X - \tilde{x})^+ = \max\{X - \tilde{x}, 0\}$.

Then, if $c < 1$,

$$\overline{E}(X) = (1-c)\tilde{x} + c\sup X + bP_0\big((X-\tilde{x})^+\big), \tag{39}$$

where $E^{P_0}((X-\tilde{x})^+)$ is the (precise) natural extension of P_0 to $(X-\tilde{x})^+$.

If $c = 1$, we get instead

$$\overline{E}(X) = \sup X.$$

Proof. In the case that $c = 1$, we have that $\overline{P} = \overline{P}_V$, whose natural extension on $\mathcal{L}(\mathbb{P})$ is well-known: $\overline{E}_V(X) = \sup X$, $\forall X \in \mathcal{L}(\mathbb{P})$ [25].

If $c < 1$, we recall (22) and apply Proposition 2.2, since also the natural extension $\overline{E}_{TVM}$ of $\overline{P}_{TVM}$ (with $c = -\frac{a}{b}$) is known [22, Equation (14)]:

$$\overline{E}_{TVM} = \tilde{x} + P_0\left((X-\tilde{x})^+\right) - \frac{a}{b}(\sup X - \tilde{x}).$$

Thus, we obtain using also (13):

$$\begin{aligned}\overline{E}(X) &= b\tilde{x} + bP_0\left((X-\tilde{x})^+\right) - a(\sup X - \tilde{x}) + (1-b)\sup X \\ &= (1-(a+b))\sup X + (a+b)\tilde{x} + bP_0\left((X-\tilde{x})^+\right) \\ &= c\sup X + (1-c)\tilde{x} + bP_0\left((X-\tilde{x})^+\right),\end{aligned}$$

which is Equation (39). □

4.3 An Interpretation in Terms of Risk Measures

Any upper prevision $\overline{P}$ defined on a set of gambles $\mathcal{D}$ induces a risk measure [2, Section 12.3.1]. Indeed, given a gamble Y, $\overline{P}(-Y)$ measures the riskiness of Y, that is, it represents the amount to be provided in order to manage possible losses from Y. Here, we shall refer the risk measures to $X = -Y$; this corresponds, when $Y \leq 0$, to thinking in terms of losses, a common practice for instance in insurance.

Thus, since a risk measure is an upper prevision, we may apply to risk measures the consistency notions developed in the theory of Imprecise Probabilities. In particular, we say that $\overline{P}$ is a coherent risk measure on $-\mathcal{D} = \{Y : X = -Y \in \mathcal{D}\}$ iff $\overline{P}$ is a coherent upper prevision on $\mathcal{D}$. If $\mathcal{D}$ is a linear space of gambles, this is equivalent to the definition given in [1] through a set of axioms.

Within NL models, VBMs are the most interesting ones from the perspective of coherent risk measures. This is because they are always coherent, and because they generalise the PMM, whose natural extension $\overline{E}_{\mathrm{PMM}}$ to $\mathcal{L}(\mathbb{P})$ was shown in [22] to originate a well-known coherent risk measure termed *Tail Value at Risk*, see below.

Let us now discuss the natural extension $\overline{E}$ of $\overline{P}$ in the VBM, given by (39), from a risk measurement viewpoint.

For this, we remark that $\tilde{x}$, given by (38), coincides with $\sup\{x \in \mathbb{R} : P_0(X \leq x) \leq -\frac{a}{b}\}$, which is a well-known risk measure [9], the *Value at Risk* of X at level $-\frac{a}{b}$, $VaR_{-\frac{a}{b}}(X)$.

As for $P_0((X - \tilde{x})^+)$, it is the *Expected Shortfall* of X at level $-\frac{a}{b}$, denoted by $ES_{-\frac{a}{b}}(X)$. As a consequence, Equation (39) can be equivalently written as follows:

$$\overline{E}(X) = (1 - c)VaR_{-\frac{a}{b}}(X) + c \sup X + bES_{-\frac{a}{b}}(X). \tag{40}$$

In the case of the PMM, where $a = -\delta < 0$, $b = 1 + \delta$, hence $c = 1 - (a + b) = 0$ and $-\frac{a}{b} = \frac{\delta}{1+\delta}$, (40) boils down to

$$\overline{E}_{\mathrm{PMM}}(X) = VaR_{\frac{\delta}{1+\delta}}(X) + (1 + \delta)ES_{\frac{\delta}{1+\delta}}(X). \tag{41}$$

As already hinted, $\overline{E}_{\mathrm{PMM}}(X)$ is a known risk measure, termed Tail Value at Risk, $TailVaR$ or $TVaR_{\frac{\delta}{1+\delta}}$. It corrects VaR adding to it a term proportional to the Expected Shortfall, i.e. proportional to how insufficient we expect VaR to be in covering losses from X (the losses not covered by VaR are given by $(X - \tilde{x})^+$).

Passing from the PMM to a generic VBM, we see from (40) and (41) that the role of VaR gets weaker. In fact, VaR is replaced by a convex combination of VaR itself and of $\sup X$, $(1 - c)VaR_{-\frac{a}{b}}(X) + c \sup X > VaR_{-\frac{a}{b}}(X)$, while the shortfall correction term is unchanged. Hence, $\overline{E}(X)$ corresponds to a more prudential risk measure than $\overline{E}_{\mathrm{PMM}}(X)$, since it requires a higher amount than $\overline{E}_{\mathrm{PMM}}(X)$ to cover risks arising from X. Recall also that $\sup X$ is the most prudential choice for a risk measure of X, that covering all losses that may arise from X. It is also remarkable that, replacing VaR with $(1 - c)VaR$ and adding $c \sup X$ when passing from (41) to (40), we still obtain a coherent upper prevision, or equivalently a coherent risk measure, that may be viewed as a generalisation of $TailVaR$.

5 Natural Extensions of Horizontal Barrier Models

Unlike the VBM, a HBM may be coherent (in rather special cases, as recalled in Section 2.3.2) or not. In general, it is only guaranteed to be 2-coherent. Thus, in order to apply the usual natural extension procedure to a HBM, we need to know when it (i.e., its $\underline{P}$ or $\overline{P}$) avoids sure loss, since this condition is necessary and sufficient for the natural extension $\underline{E}(X)$ to be finite, $\forall X \in \mathcal{L}(\mathbb{P})$ [25]. In the case that partition $\mathbb{P}$ is finite, the following proposition answers this question and determines the natural extension $\overline{E}$ of $\overline{P}$ on $\mathcal{A}(\mathbb{P})$. Results are stated for upper probabilities, since most formulae with HBMs are more manageable for them. Since $\overline{P}$ is already defined on $\mathcal{A}(\mathbb{P})$, we stress that $\overline{E}$ is actually a least-committal correction of $\overline{P}$ rather than a real extension, in the case that $\overline{P}$ avoids sure loss but is not coherent on $\mathcal{A}(\mathbb{P})$.

Proposition 5.1. *Let $\overline{P} : \mathcal{A}(\mathbb{P}) \to \mathbb{R}$ be the upper probability of a HBM, with $\mathbb{P}$ finite.*

(*a*) *Define, $\forall A \in \mathcal{A}(\mathbb{P})$*

$$\overline{E}(A) = \min\left\{\sum_{\omega \Rightarrow A} \overline{P}(\omega), 1\right\}. \tag{42}$$

If $\sum_{\omega\in\mathbb{P}} \overline{P}(\omega) \geq 1$, then $\overline{E}$ is a coherent and 2-alternating upper probability.

(*b*) *$\overline{P}$ avoids sure loss on $\mathcal{A}(\mathbb{P})$ iff*

$$\sum_{\omega\in\mathbb{P}} \overline{P}(\omega) \geq 1. \tag{43}$$

(*c*) *If $\overline{P}$ avoids sure loss, $\overline{E}$ in (42) is its natural extension on $\mathcal{A}(\mathbb{P})$.*

Proof. (*a*) If $\sum_{\omega\in\mathbb{P}} \overline{P}(\omega) \geq 1$, then the probability interval $[0, \overline{P}(\omega)]_{\omega\in\mathbb{P}}$ avoids sure loss by (10) (while being not necessarily coherent). Then, by Equation (12) its natural extension coincides with $\overline{E}$ in (42), and as such, $\overline{E}$ is coherent and 2-alternating.

(*b*) If $\overline{P}$ avoids sure loss, by Theorem 2.2 (*b*) there is a probability P such that $P(A) \leq \overline{P}(A)$, $\forall A \in \mathcal{A}(\mathbb{P})$. Hence, $1 = \sum_{\omega\in\mathbb{P}} P(\omega) \leq \sum_{\omega\in\mathbb{P}} \overline{P}(\omega)$.

Conversely, assume $\sum_{\omega\in\mathbb{P}} \overline{P}(\omega) \geq 1$ holds. Then we may deduce that $\overline{P}$ avoids sure loss provided that we can show that

(*i*) $\overline{E}$ is coherent

(*ii*) $\overline{E}(A) \leq \overline{P}(A), \forall A \in \mathcal{A}(\mathbb{P})$, because then there is a probability P such that $P \leq \overline{E} \leq \overline{P}$, so $\overline{P}$ avoids sure loss, again by Theorem 2.2 (b). Statement (i) is implied by (a). As for (ii), it is obvious if $\overline{P}(A) = 1$. If $\overline{P}(A) = 0$, by monotonicity of $\overline{P}$ also $\overline{P}(\omega) = 0$, $\forall \omega : \omega \Rightarrow A$. It follows applying (42) that

$$\overline{E}(A) = \min\{0, 1\} = 0 = \overline{P}(A).$$

If $0 < \overline{P}(A) < 1$, let $S_+(A) = \{\omega \in \mathbb{P} : \omega \Rightarrow A, \overline{P}(\omega) > 0\}$, and let m_A be the cardinality of $S_+(A)$. Then

$$\begin{aligned}\sum_{\omega \Rightarrow A} \overline{P}(\omega) &= \sum_{\omega \in S_+(A)} \overline{P}(\omega) = \sum_{\omega \in S_+(A)} \big(bP_0(\omega) + c\big) \\ &= bP_0\Big(\bigvee_{\omega \in S_+(A)} \omega\Big) + m_A\, c \leq bP_0(A) + m_A c.\end{aligned}$$

At this point, if $m_A = 0$ then $\sum_{\omega \Rightarrow A} \overline{P}(\omega) = 0$ and by (42) $\overline{E}(A) = 0 \leq \overline{P}(A)$; if $m_A > 0$, recalling that $c < 0$ in the HBM, we get $\sum_{\omega \Rightarrow A} \overline{P}(\omega) \leq bP_0(A) + m_A\, c \leq bP_0(A) + c = \overline{P}(A) < 1$. Therefore by (42) $\overline{E}(A) = \sum_{\omega \Rightarrow A} \overline{P}(\omega) \leq \overline{P}(A)$.

(c) Let $\overline{E}^*$ be the natural extension of $\overline{P}$ on $\mathcal{A}(\mathbb{P})$. By the upper probability version of Theorem 2.3, we have that $\overline{E}^*(\omega) \leq \overline{P}(\omega)$, $\forall \omega \in \mathbb{P}$. Consequently, since $\overline{E}^*$ is coherent, applying its subadditivity property [25, Section 2.7.4 (e)] it holds that, $\forall A \in \mathcal{A}(\mathbb{P})$,

$$\overline{E}^*(A) = \overline{E}^*\Big(\bigvee_{\omega \Rightarrow A} \omega\Big) \leq \sum_{\omega \Rightarrow A} \overline{E}^*(\omega) \leq \sum_{\omega \Rightarrow A} \overline{P}(\omega).$$

Since also $\overline{E}^*(A) \leq 1$, we have that $\overline{E}^* \leq \overline{E}$ by (42). But from (b) $\overline{E}$ is coherent, and $\overline{E} \leq \overline{P}$. Thus by Theorem 2.3 (b') $\overline{E} \leq \overline{E}^*$. It follows that $\overline{E} = \overline{E}^*$.

□

Thus, condition (43) characterises a HBM that avoids sure loss, with $\mathbb{P}$ finite. To see what (43) means in terms of the HBM parameters b, c, note preliminarily that if there exists $\omega \in \mathbb{P}$ such that $\overline{P}(\omega) = 1$, then trivially $\overline{P}$ avoids sure loss. Otherwise, we have that

Proposition 5.2. *Under the assumptions of Proposition 5.1, suppose further that $\overline{P}(\omega) < 1$, $\forall \omega \in \mathbb{P}$. Let also $M = \{\omega \in \mathbb{P} : 0 < bP_0(\omega) + c\ (< 1)\}$ and denote its cardinality by m. Then, it is necessary for $\overline{P}$ to avoid sure loss on $\mathcal{A}(\mathbb{P})$ that $m > 1$ and*

$$b + mc \geq 1. \tag{44}$$

Proof. When $m = 0$ then $\sum_{\omega \in \mathbb{P}} \overline{P}(\omega) = 0 < 1$, so $\overline{P}$ incurs sure loss by (43). Analogously, when $m = 1$, $M = \{\varpi\}$ and $\sum_{\omega \in \mathbb{P}} \overline{P}(\omega) = \overline{P}(\varpi) < 1$, so $\overline{P}$ incurs sure loss by (43) again.

Let then $m > 1$. Applying (17),

$$\sum_{\omega \in \mathbb{P}} \overline{P}(\omega) = \sum_{\omega \in M} (bP_0(\omega) + c) \leq b + mc.$$

From (43), the thesis follows. $\square$

For *given* b, c, we may deduce from Proposition 5.2 that:

- $\overline{P}$ incurs sure loss if $b + mc < 1$, and $b + mc \to -\infty$ as $m \to +\infty$ ($c < 0$ in a HBM). Thus, a HBM incurs sure loss if there are enough $\omega \in \mathbb{P}$ such that $0 < \overline{P}(\omega) < 1$ (and no one with $\overline{P}(\omega) = 1$).
- Even when m is low, condition (44) is not sufficient. In fact, (44) always obtains when $m = 2$ since, by (13), it is then equivalent to the true condition $b + 2a \leq 1$. Yet, taking $\mathbb{P} = \{\omega_1, \omega_2, \omega_3\}$, $P_0(\omega_1) = P_0(\omega_2) = 0.4$, $P_0(\omega_3) = 0.2$, $a = -0.6$, $b = 2$ (hence $c = -0.4$), we obtain the HBM upper probability $\overline{P}(\omega_1) = \overline{P}(\omega_2) = 0.4$, $\overline{P}_(\omega_3) = 0$. Here $m = 2$, but $\sum_{i=1}^{3} \overline{P}(\omega_i) = 0.8 < 1$, and $\overline{P}$ does not avoid sure loss.

If, following [24, Section 4.5.2], we say that two upper (alternatively lower) probabilities that avoid sure loss are *equivalent* if they have the same natural extension, an interesting follow-up of Proposition 5.1 is

Proposition 5.3. *Under the assumptions of Proposition 5.1, $\overline{P}$ and the probability interval $[0, \overline{P}(\omega)]_{\omega \in \mathbb{P}}$ are equivalent.*

Proof. Follows from the proof of Proposition 5.1. $\square$

Thus, we may say that from an inferential viewpoint a HBM that avoids sure loss is equivalent to a probability interval. In general, however, NL models and (natural extensions of) probability intervals do not overlap. As proven in [5, Section 7.3], a NL model is the natural extension on $\mathcal{A}(\mathbb{P})$ of a coherent probability interval in special instances only (including the PMM - which was already shown in [14] - and the ε-contamination model).

Remark 5.1. *The upper probability $\overline{P}$ of a HBM avoiding sure loss has further properties on $\mathcal{A}(\mathbb{P})$, with $\mathbb{P}$ finite. Recall firstly that, as is easily proven, a generic upper probability $\overline{P}$ is coherent on a finite partition $\mathbb{P}$ iff (43) holds and $\overline{P}(\omega) \in [0, 1]$, $\forall \omega \in \mathbb{P}$ (this latter condition is satisfied by any HBM by Definition 2.6).*

Thus, within upper probabilities $\overline{P}$ such that $\overline{P}(\omega) \in [0, 1]$, $\forall \omega \in \mathbb{P}$, condition (43) is also sufficient for $\overline{P}$ to avoid sure loss on $\mathbb{P}$. In the case of a HBM, (43) has a much wider extension: it is sufficient for $\overline{P}$ to avoid sure loss on the whole $\mathcal{A}(\mathbb{P})$.

Next to justifying the simple formula (42) for computing $\overline{E}$ on $\mathcal{A}(\mathbb{P})$, Proposition 5.1 is useful also in a second natural extension problem, that of determining $\overline{E}(X)$ for any $X \in \mathcal{L}(\mathbb{P})$.

In fact, by the transitivity property of the natural extension [24, Section 4.5.4], $\overline{E}(X)$ may be thought of as the natural extension of $\overline{E}$ from $\mathcal{A}(\mathbb{P})$ to $\mathcal{A}(\mathbb{P}) \cup \{X\}$, which is given by a Choquet integral, since $\overline{E}$ is 2-alternating on $\mathcal{A}(\mathbb{P})$ by Proposition 5.1. The next proposition states the final result.

Proposition 5.4. *Let $(\underline{P}, \overline{P})$ be a HBM that avoids sure loss on $\mathcal{A}(\mathbb{P})$, $\mathbb{P} = \{\omega_1, \omega_2, \ldots, \omega_n\}$. Consider $X \in \mathcal{L}(\mathbb{P})$, taking $m \leq n$ distinct values $x_1 < x_2 < \ldots < x_m$.*

Define, for $j = 1, \ldots, m$,

$$e_j = \bigvee_{X(\omega)=x_j} \omega$$

and let $k \in \{1, \ldots, m\}$ be such that

$$\sum_{j=k}^{m} \overline{E}(e_j) \leq 1, \sum_{j=k-1}^{m} \overline{E}(e_j) \geq 1, \tag{45}$$

with the convention $x_0 = 0$, $e_0 = \emptyset$ and where $\overline{E}$ is the natural extension of $\overline{P}$. Then, we have

$$\overline{E}(X) = x_{k-1}\Big(1 - \sum_{j=k}^{m} \overline{E}(e_j)\Big) + \sum_{j=k}^{m} x_j \overline{E}(e_j). \tag{46}$$

Proof. Since by Proposition 5.1 (a) the natural extension $\overline{E}$ of $\overline{P}$ *on* $\mathcal{A}(\mathbb{P})$ is 2-alternating and coherent, the further natural extension (of $\overline{E}$) on X is the Choquet integral $(C) \int X \, d\overline{E}$ (Proposition 2.1).

To compute it applying (8), we detail, by means of (45) and (42), the terms of its summation $x_h\big(\mu(X \geq x_h) - \mu(X \geq x_{h+1})\big) = x_h\big(\overline{E}(X \geq x_h) - \overline{E}(X \geq x_{h+1})\big)$ as h varies.

For $h \geq k$, we get

$$x_h\Big(\overline{E}\big(X \geq x_h\big) - \overline{E}\big(X \geq x_{h+1}\big)\Big) = x_h\Big(\sum_{j=h}^{m} \overline{E}(e_j) - \sum_{j=h+1}^{m} \overline{E}(e_j)\Big) = x_h \overline{E}(e_h).$$

If $h = k-1$, then $x_{k-1}\big(\overline{E}\big(X \geq x_{k-1}\big) - \overline{E}\big(X \geq x_k\big)\big) = x_{k-1}\big(1 - \sum_{j=k}^{m} \overline{E}(e_j)\big)$; if $h < k-1$, we have that $x_h\big(\overline{E}\big(X \geq x_h\big) - \overline{E}\big(X \geq x_{h+1}\big)\big) = x_h(1-1) = 0$.

Substituting the non-null terms above in the summation (8) we obtain (46). □

By Proposition 5.4, in order to compute $\overline{E}$ we first have to group the atoms of $\mathbb{P}$ where $X(\omega)$ takes the same value (x_j on e_j) and to determine k (cf. [15, Theorem 7] for a similar technique). For this, we need to know every $\overline{E}(e_j)$, which is achieved easily by (42) of Proposition 5.1. If $m = n$, then obviously $e_j = \omega_j \in \mathbb{P}$, $\overline{E}(e_j) = \overline{P}(\omega_j)$, $j = 1, \ldots, n$.

For some peculiar HBMs, there may be more than one integer number playing the role of k in Equation (45). This does not affect the computation of $\overline{E}(X)$, as shown in detail in [20, Remark 4.2]: Equation (46) returns a unique $\overline{E}$ for all choices.

The final formula (46) shows that the natural extension $\overline{E}$ is very similar to a classical expectation with the probability of e_j, i.e. that X takes the value x_j, replaced by its upper probability, and this for the highest $m-k+1$ values of X. The remaining values of X do not appear in the computation (46), but for x_{k-1}. This structure of formula (46) is not quite surprising: it is derived from (8), and it is easily seen that when μ is linear (8) returns an expectation, which is what partly happens with $\overline{E}$.

A special situation arises when it is possible that $k=1$: by (45) it holds then that $\sum_{j=1}^{m} \overline{E}(e_j) = 1$. Thus, $\overline{E}$ is a precise probability on the partition $\mathbb{P}' = \{e_1, e_2, \ldots, e_m\}$ and $\overline{E}(X) = \sum_{j=1}^{m} x_j \overline{E}(e_j)$ is an expectation. However, this does not imply that the starting $\overline{P}$ in the HBM is a precise probability also outside $\mathbb{P}'$. For this, see [20, Example 4.1].

A natural question that arises at this point is whether results similar to the previous ones can be obtained when the partition $\mathbb{P}$ is infinite. In general, we lean to think that the answer is negative. In fact, while some results can be generalised to an infinite framework, other ones are tailored for the finite case. We discuss a case for both.

On the one hand, Proposition 5.1 (b) can be generalised as follows.

Proposition 5.5. *In a HBM, $\overline{P}$ (hence $\underline{P}$) avoids sure loss on $\mathcal{A}(\mathbb{P})$ if and only if for every* finite *partition $\mathbb{P}'$ coarser than $\mathbb{P}$ it holds that*

$$\sum_{e \in \mathbb{P}'} \overline{P}(e) \geq 1.$$

Proof. ($\Rightarrow$) If $\overline{P}$ avoids sure loss on $\mathcal{A}(\mathbb{P})$, by Theorem 2.2 (b) there is a dF-coherent probability P less than or equal to $\overline{P}$ for any event in $\mathcal{A}(\mathbb{P})$, hence also on the atoms of any finite partition $\mathbb{P}' = \{e_1, \ldots, e_m\} \subset \mathcal{A}(\mathbb{P})$. This implies $1 = \sum_{i=1}^{m} P(e_i) \leq \sum_{i=1}^{m} \overline{P}(e_i)$.

($\Leftarrow$) Take a generic gain $\overline{G}_{\mathrm{ASL}}$ in Definition 2.3 (where $\overline{P}$ now is an upper probability on $\mathcal{A}(\mathbb{P})$), obtained selecting a finite number of events $E_1, \ldots, E_n \in \mathcal{A}(\mathbb{P})$, and $s_1, \ldots, s_n \geq 0$:

$$\overline{G}_{\mathrm{ASL}} = \sum_{i=1}^{n} s_i \big(\overline{P}(E_i) - I_{E_i}\big).$$

Let $\mathbb{P}' = \{\bigwedge_{i=1}^{n} E_i' : E_i' = E_i \text{ or } E_i' = \neg E_i, i = 1, \ldots, n\}$ be the partition generated by $E_1, \ldots, E_n$. Note that $\overline{G}_{\mathrm{ASL}}$ is a gamble defined on $\mathbb{P}'$, that $\{E_1, \ldots, E_n\} \subset \mathcal{A}(\mathbb{P}') \subset \mathcal{A}(\mathbb{P})$, and that putting $\underline{P}' = \underline{P}|_{\mathcal{A}(\mathbb{P}')}$, $\overline{P}' = \overline{P}|_{\mathcal{A}(\mathbb{P}')}$, $(\underline{P}', \overline{P}')$ is a HBM on $\mathcal{A}(\mathbb{P}')$. Further, $\mathbb{P}'$ is a *finite* partition coarser than $\mathbb{P}$, thus by assumption $\sum_{e \in \mathbb{P}'} \overline{P}'(e) \geq 1$. By Proposition 5.1 ($b$), $\overline{P}'$ avoids sure loss on $\mathcal{A}(\mathbb{P}')$, hence $\max \overline{G}_{\mathrm{ASL}} \geq 0$. Since $\overline{G}_{\mathrm{ASL}}$ is arbitrary among the admissible gains in Definition 2.3, we conclude that $\overline{P}$ avoids sure loss on $\mathcal{A}(\mathbb{P})$. $\square$

On the other hand, consider Remark 5.1: when $\mathbb{P}$ is infinite, no analogue of (43) is necessary for $\overline{P}$ to be coherent on $\mathbb{P}$. In fact

Lemma 5.1. *Any upper probability $\overline{P} : \mathbb{P} \to [0, 1]$, $\mathbb{P}$ infinite, is coherent on $\mathbb{P}$.*

Proof. Apply Theorem 2.1: $\overline{P}$ is the upper envelope of the probabilities P_i (one for each $\omega_i \in \mathbb{P}$) given by $P_i(\omega_i) = \overline{P}(\omega_i)$, $P_i(\omega_j) = 0, \forall j \neq i$. In general, P_i is not countably additive, however each P_i is a dF-coherent probability on $\mathbb{P}$, as is easy to check applying Definition 2.2 (the result is also a special case of [6, Section 10.4.1]). □

The statement in Lemma 5.1 applies to any $\overline{P}$, not necessarily those in HBMs, and is independent of whether any summation of the upper probabilities of events of the infinite partition $\mathbb{P}$ is greater than 1 or not, see also the next example.

Example 5.1. *Given the dF-coherent probability P_0 on the infinite partition $\mathbb{P}$ defined by $P_0(\omega_1) = 0.5$, $P_0(\omega_j) = 0$, $\forall j \neq 1$, define a HBM using P_0 and the parameters $b = 1.1$, $c = -0.05$. Then, its upper probability is $\overline{P} = P_0$ and is coherent (even dF-coherent) on $\mathbb{P}$, but $\sum_{\omega \in \mathbb{P}} \overline{P}(\omega) = 0.5$.*

Let us turn now to the case that $\overline{P}$ (and $\underline{P}$) in the HBM is coherent. We can still apply (46) to compute $\overline{E}(X)$ if $\mathbb{P}$ is finite, or more generally if, even when $\mathbb{P}$ is infinite, X is a simple gamble, so that X may be defined on a finite partition $\mathbb{P}' = \{e_1, \ldots, e_m\}$, $\mathbb{P}'$ coarser than $\mathbb{P}$. In this latter situation, we consider in fact the restrictions of $\overline{P}, \underline{P}$ on $\mathcal{A}(\mathbb{P}')$. They still constitute a HBM that is coherent, hence also avoids sure loss, on $\mathcal{A}(\mathbb{P}')$, and to which Proposition 5.1 applies. Thus Proposition 5.4 may be used, as well as (46) with $\overline{E}(e_j) = \overline{P}(e_j)$. In the same assumptions ($\mathbb{P}$ finite or X simple gamble) $\underline{E}(X)$ can be computed as $\underline{E}(X) = -\overline{E}(-X)$, obtaining $\overline{E}(-X)$ as just described.

More generally, with $\mathbb{P}$ and X arbitrary, since a coherent HBM is formed by $\overline{P}$, $\underline{P}$ that are, respectively, 2-alternating and 2-monotone [5, Proposition 5.9], we can also derive $\overline{E}(X)$ or $\underline{E}(X)$ by means of (7). Unlike (46), the result applies no matter whether $\mathbb{P}$ is finite or not, and is stated in the next proposition for $\underline{P}$.

Proposition 5.6. *Let $(\underline{P}, \overline{P})$ be a HBM that is* coherent*. Let $X \in \mathcal{L}(\mathbb{P})$. Define*

$$I_u = \{x \in \mathbb{R} : bP_0(X > x) + a \geq 0\}, \quad \tilde{x}_u = \sup I_u \tag{47}$$

$$I_l = \{x \in \mathbb{R} : bP_0(X > x) + a \leq 1\}, \quad \tilde{x}_l = \inf I_l \tag{48}$$

$$Z(\omega) = \max\{\min\{X(\omega) - \tilde{x}_l, \tilde{x}_u - \tilde{x}_l\}, 0\} \quad (\omega \in \mathbb{P}). \tag{49}$$

Then,

$$\underline{E}(X) = a\tilde{x}_u + (1 - a)\tilde{x}_l + bE^{P_0}(Z), \tag{50}$$

where $E^{P_0}(Z)$ is the (precise) natural extension of P_0 to the gamble Z.

Proof. The proof consists of four major steps.

First step. Since $\underline{P}$ is coherent and 2-monotone, write $\underline{E}(X)$ by means of (7), (16) as

$$\underline{E}(X) = \inf X + \int_{\inf X}^{\sup X} \min\left\{\max\left\{bP_0(X > x) + a, 0\right\}, 1\right\} dx. \tag{51}$$

Second step. Split the Riemann integral in (51) into terms, separating those where $\underline{P}$ is equal to 0, to 1, or to $bP_0(X > x) + a$.

For this, the following facts are relevant:

(a) The set I_u in (47) is a lower unbounded interval in the real line: if $x < \tilde{x}_u$, there is $x^* \in I_u$ such that $x < x^* \leq \tilde{x}_u$, and $bP_0(X > x) + a \geq bP_0(X > x^*) + a \geq 0$, hence $x \in I_u$.

Further, if $x > \tilde{x}_u$, then $bP_0(X > x) + a < 0$. Recalling (16), we conclude that

$$\text{if } x > \tilde{x}_u, \text{ then } \underline{P}(X > x) = 0.$$

(b) Similarly, the set I_l in (48) is an upper unbounded interval (just adapt the argument in (a)), and if $x < \tilde{x}_l$, then $bP_0(X > x) + a > 1$. It follows by (16) that

$$\text{if } x < \tilde{x}_l, \text{ then } \underline{P}(X > x) = 1.$$

(c) It holds that

$$\inf X \leq \tilde{x}_l \leq \tilde{x}_u \leq \sup X. \tag{52}$$

In fact, take $x = \sup X$. Then, $bP_0(X > \sup X) + a = bP_0(\emptyset) + a = a < 0$, which implies, recalling (47) and (a), that $\tilde{x}_u \leq \sup X$.

Now take, for any $\varepsilon > 0$, $x = \inf X - \varepsilon$. Since $bP_0(X > \inf X - \varepsilon) + a = b + a > 1$ (using Definition 2.6 at the inequality), we deduce, taking account of (48) and (b), that $\tilde{x}_l \geq \inf X - \varepsilon\ \forall \varepsilon > 0$, hence $\tilde{x}_l \geq \inf X$.

Finally, we have $\tilde{x}_l \leq \tilde{x}_u$: otherwise, there would exist $x \in \mathbb{R}$, $\tilde{x}_u < x < \tilde{x}_l$. For such x, by (a) and (b), it should be both $\underline{P}(X > x) = 0$ and $\underline{P}(X > x) = 1$, a contradiction.

From the conclusions of (a) and (b) and (52), we write $\underline{E}(X)$ in (51) as

$$\begin{aligned}
\underline{E}(X) &= \inf X + \int_{\inf X}^{\tilde{x}_l} 1 dx + \int_{\tilde{x}_l}^{\tilde{x}_u} (bP_0(X > x) + a) dx + \int_{\tilde{x}_u}^{\sup X} 0\ dx \\
&= \tilde{x}_l + \int_{\tilde{x}_l}^{\tilde{x}_u} (bP_0(X > x) + a) dx \\
&= a\tilde{x}_u + (1 - a)\tilde{x}_l + b \int_{\tilde{x}_l}^{\tilde{x}_u} P_0(X > x) dx. \qquad (53)
\end{aligned}$$

Third step. Manipulate the integral in (53) in order to apply (7) to get the natural extension of the dF-coherent probability P_0, i.e. its expectation.

To begin with, take x such that

$$\tilde{x}_l < x < \tilde{x}_u. \tag{54}$$

Then, using (54) twice (so that $x - \tilde{x}_l > 0$, $\tilde{x}_u - x > 0$) and recalling (49),

$$\begin{aligned}
Z = &\max\left\{\min\left\{X - \tilde{x}_l, \tilde{x}_u - \tilde{x}_l\right\}, 0\right\} > x - \tilde{x}_l && \text{iff}\\
&\min\left\{X - \tilde{x}_l, \tilde{x}_u - \tilde{x}_l\right\} > x - \tilde{x}_l && \text{iff}\\
&\min\left\{X - \tilde{x}_l - x + \tilde{x}_l, \tilde{x}_u - \tilde{x}_l - x + \tilde{x}_l\right\} > 0 && \text{iff}\\
&\min\left\{X - x, \tilde{x}_u - x\right\} > 0 && \text{iff}\\
&X - x > 0.
\end{aligned}$$

Thus, the events $(Z > x - \tilde{x}_l)$ and $(X > x)$ are equivalent. Using this fact, and performing the substitution $z = x - \tilde{x}_l$ in the integral in (53), we obtain:

$$\mathcal{J} = \int_{\tilde{x}_l}^{\tilde{x}_u} P_0(X > x)dx = \int_{\tilde{x}_l}^{\tilde{x}_u} P_0(Z > x - \tilde{x}_l)dx = \int_0^{\tilde{x}_u - \tilde{x}_l} P_0(Z > z)dz. \tag{55}$$

Fourth step. Apply (7) to $\mathcal{J}$ in (55) to get (50).

Before applying (7) to $\mathcal{J}$, we have to make sure that this is feasible. In detail, we must prove that:

(a) $\inf Z = 0$

(b) $\sup Z = \tilde{x}_u - \tilde{x}_l$.

For this, recall that the values $Z \in \mathcal{L}(\mathbb{P})$ may take at the varying of ω in $\mathbb{P}$ are, at most:

$$Z(\omega) = \begin{cases} 0 & \text{if } X(\omega) \le \tilde{x}_l \\ X(\omega) - \tilde{x}_l & \text{if } X(\omega) \in \,]\tilde{x}_l, \tilde{x}_u[\\ \tilde{x}_u - \tilde{x}_l & \text{if } X(\omega) \ge \tilde{x}_u \end{cases} \tag{56}$$

We may suppose $\tilde{x}_u > \tilde{x}_l$ (or else the integral in (55) is 0). Define the events

$$B = (X \in \,]\tilde{x}_l, \tilde{x}_u[) \,, B_l = (X \le \tilde{x}_l) \,, B_u = (X \ge \tilde{x}_u) \tag{57}$$

Since $\tilde{x}_u > \tilde{x}_l$, by (52) we get $\tilde{x}_u > \inf X$, hence $B_l \vee B \neq \emptyset$. Then, if $B_l \neq \emptyset$, $\inf Z = 0$. Otherwise, $0 \le \inf Z = \inf X - \tilde{x}_l \le 0$, hence $\inf Z = 0$, by applying (56) and (52) at the first and the second inequality, respectively. Analogously for item (b): by (52) we get $B \vee B_u \neq \emptyset$. Then, if $B_u \neq \emptyset$, $\sup Z = \tilde{x}_u - \tilde{x}_l$. Otherwise, $\tilde{x}_u - \tilde{x}_l \ge \sup Z = \sup X - \tilde{x}_l \ge \tilde{x}_u - \tilde{x}_l$, hence $\sup Z = \tilde{x}_u - \tilde{x}_l$, by applying (56) and (52) again.

Finally, substituting the integral (55) into (53) and applying (7) to it, we obtain

$$\underline{E}(X) = a\tilde{x}_u + (1-a)\tilde{x}_l + b(E^{P_0}(Z) - \inf Z),$$

from which (50) follows. □

The natural extension $\underline{E}(X)$ can be written in alternative ways. We present now two such formulae.

Corollary 5.1. *In the assumptions of Proposition 5.6, it holds that:*

$$\underline{E}(X) = (a+b)\tilde{x}_u + (1-(a+b))\tilde{x}_l + bE^{P_0}(Z'), \tag{58}$$

with $Z' = Z + \tilde{x}_l - \tilde{x}_u$*;*

$$\underline{E}(X) = (a + bP_0(B_u))\tilde{x}_u + (1 - a - b(1 - P_0(B_l)))\tilde{x}_l + bP_0(B)E^{P_0}(X|B), \tag{59}$$

where B, B_l, B_u *are defined by (57) and whenever these events are non-impossible.*[5]

Proof. Equation (58) is obtained from (50) substituting $E^{P_0}(Z) = E^{P_0}(Z') - \tilde{x}_l + \tilde{x}_u$.

To get Equation (59), disintegrate $E^{P_0}(Z)$, apply (57) and then (56):

$$\begin{aligned} E^{P_0}(Z) &= E^{P_0}(Z|B_l)P_0(B_l) + E^{P_0}(Z|B)P_0(B) + E^{P_0}(Z|B_u)P_0(B_u) \\ &= 0 + E^{P_0}(X - \tilde{x}_l|B)P_0(B) + E^{P_0}(\tilde{x}_u - \tilde{x}_l|B_u)P_0(B_u) \\ &= E^{P_0}(X|B)P_0(B) - \tilde{x}_l P_0(B) + \tilde{x}_u P_0(B_u) - \tilde{x}_l P_0(B_u) \\ &= E^{P_0}(X|B)P_0(B) + \tilde{x}_u P_0(B_u) - x_l(1 - P_0(B_l)). \end{aligned}$$

Substituting $E^{P_0}(Z)$ with the expression above in (50) gives (59). □

Equation (58) rather then (50) may be appropriate for a comparison with the natural extension (34) of $\underline{P}$ in a VBM; we see that the latter is anyway simpler. Equation (59), unlike (50) and (58), does not involve the auxiliary gambles Z or Z'.

6 Natural Extensions of Restricted Range Models

Analogously to HBMs, with RRMs we have the preliminary problem, before looking for their natural extensions, of establishing when they avoid sure loss. It is easy to see that

Lemma 6.1. *A RRM on* $\mathcal{A}(\mathbb{P})$ *does not avoid sure loss, nor is C-convex, if* $\mathbb{P}$ *is infinite.*

Proof. Take for instance $\underline{P}$. We apply Theorem 2.2 (b) to prove that $\underline{P}$ does not avoid sure loss. In fact, since $\underline{P}(\omega) = bP_0(\omega) + a \geq a > 0$, $\forall \omega \in \mathbb{P}$, the set of dF-coherent probabilities $P \geq \underline{P}$ is empty: any such P should by additivity take values greater than 1 on events of $\mathcal{A}(\mathbb{P})$ formed by sufficiently many $\omega \in \mathbb{P}$.

Further, $\underline{P}$ cannot be C-convex, since any C-convex probability avoids sure loss [17, Proposition 3.5 (e)]. □

[5] If some of these events are impossible, $\underline{E}(X)$ is still given by (59) with the corresponding terms set to 0.

While Lemma 6.1 highlights a difference with HBMs, which do not require finiteness of $\mathbb{P}$ to avoid sure loss, it does not clarify what happens when $\mathbb{P}$ is finite. The question is solved by the subsequent Proposition 6.2, which exploits and extends an analogous result, recalled here as Proposition 6.1 and proven in [5, Proposition 6.4 (c)] for *degenerate* NL models.

Definition 6.1. *[5] A* degenerate NL model *is a couple* $(\underline{P}_d, \overline{P}_d)$*, with* $\underline{P}_d$ $NL(a, 0)$*,* $\overline{P}_d$ $NL(1-a, 0)$*,* $a \in]0, \frac{1}{2}]$.[6]

Proposition 6.1. *Let* $(\underline{P}_d, \overline{P}_d)$ *be a degenerate NL model on* $\mathcal{A}(\mathbb{P})$*. If* $\mathbb{P}$ *is finite and* $|\mathbb{P}| = n$*, the following are equivalent:*

(a) $\underline{P}_d$ *is C-convex on* $\mathcal{A}(\mathbb{P})$

(b) $\underline{P}_d$ *avoids sure loss on* $\mathcal{A}(\mathbb{P})$

(c) $a \leq \dfrac{1}{n}$.

Proposition 6.2. *Let* $(\underline{P}, \overline{P})$ *be a RRM on* $\mathcal{A}(\mathbb{P})$*,* $\mathbb{P}$ *finite partition with* $|\mathbb{P}| = n$*. The following are equivalent:*

(a) $\underline{P}$ *is C-convex on* $\mathcal{A}(\mathbb{P})$

(b) $\underline{P}$ *avoids sure loss on* $\mathcal{A}(\mathbb{P})$

(c) $b + na \leq 1$ *(equivalent to* $\sum_{i=1}^{n} \underline{P}(\omega_i) \leq 1$*).*

Proof. We prove that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a).

(a) $\Rightarrow$ (b) This is a general property of C-convex previsions [17].

(b) $\Rightarrow$ (c) Suppose that $\underline{P}$ avoids sure loss. Then, by Definition 2.1 (c) $\max \underline{G}_{ASL} \geq 0$, for any admissible $\underline{G}_{ASL}$, including the following, formed by n elementary gains, one for every atom of $\mathbb{P}$: $\underline{G}^*_{ASL} = \sum_{i=1}^{n}(I_{\omega_i} - \underline{P}(\omega_i))$. For any $\omega_j \in \mathbb{P}$, $\underline{G}^*_{ASL}(\omega_j)$ is constant:

$$\underline{G}^*_{ASL}(\omega_j) = (1 - bP_0(\omega_j) - a) - \sum_{i \neq j}^{n}(bP_0(\omega_i) + a) = 1 - b - na.$$

Therefore, $\max \underline{G}^*_{ASL} \geq 0$ iff $b + na \leq 1$. Then clearly $b + na \leq 1$ iff $\sum_{i=1}^{n} \underline{P}(\omega_i) \leq 1$, from (18).

(c) $\Rightarrow$ (a) Let $b + na \leq 1$. To prove that $\underline{P}$ is C-convex, define $\underline{P}_1$ as:

$$\underline{P}_1(A) = \frac{a}{1-b}, \ \forall A \in \mathcal{A}(\mathbb{P}) \setminus \{\emptyset, \Omega\}, \ \underline{P}_1(\emptyset) = 0, \ \underline{P}_1(\Omega) = 1.$$

Then, we recognise that $\underline{P}$ is a convex linear combination of P_0 and $\underline{P}_1$:

$$\underline{P}(A) = bP_0(A) + (1-b)\underline{P}_1(A), \ \forall A \in \mathcal{A}(\mathbb{P}). \quad (60)$$

[6] We could admit the value $a = 0$ as in [5], but we prefer not to do so to rule out the vacuous model, already included in the VBM.

In (60), P_0 is obviously (also) C-convex. As for $\underline{P}_1$, it is the lower probability of a degenerate NL model (with $\frac{a}{1-b}$ playing the role of a in Definition 6.1), because $0 \leq \frac{a}{1-b}$ since $a > 0$ and $b < 1$ in a RRM, and $\frac{a}{1-b} \leq \frac{1}{2}$ iff $2a + b \leq 1$, which is assumed true in RRMs. Moreover, $\underline{P}_1$ is also C-convex: this ensues from Proposition 6.1, since $\frac{a}{1-b} \leq \frac{1}{n}$ iff $b+na \leq 1$, true by assumption. Therefore, $\underline{P}$ is convex, being a linear combination of convex lower probabilities [17, Proposition 3.2 (b)], and is obviously centered. □

Comment The above results highlight a number of interesting features of RRMs:

(i) While in general C-convexity implies the condition of avoiding sure loss but the converse is not true, these two concepts are equivalent for RRMs, by Lemma 6.1 and Proposition 6.2.

(ii) NL degenerate models have the same property. Actually, although we kept them distinguished from non-degenerate NL models (those with $b > 0$), they could be included into RRM models allowing $b = 0$ there. In this way they would appear as a limiting situation for RRMs.

(iii) The condition for a RRM to avoid sure loss is, in the form $\sum_{i=1}^{n} \underline{P}(\omega_i) \leq 1$, seemingly symmetric to that for HBMs (Equation (43)). However, it is more restrictive: in fact, it is equivalent to $a \leq \frac{1-b}{n}$. This means that for n getting larger and b constant, a must tend to 0, which makes the RRM approach the ε-contamination model.

(iv) Equation (60) shows that *any* RRM can be viewed as the convex linear combination of the given P_0 and a degenerate NL model.

Let us determine the natural extension on $\mathcal{A}(\mathbb{P})$ of $\underline{P}_d$ in a degenerate NL model.

Proposition 6.3. *Let $(\underline{P}_d, \overline{P}_d)$ be a degenerate NL model that avoids sure loss. The natural extension $\underline{E}_d$ of $\underline{P}_d$ on $\mathcal{A}(\mathbb{P})$ is*

$$\underline{E}_d(A) = m_A \cdot a, \ \forall A \in \mathcal{A}(\mathbb{P}) \setminus \{\Omega\}, \ \underline{E}_d(\Omega) = 1, \tag{61}$$

where m_A is the number of atoms of $\mathbb{P}$ implying A. $\underline{E}_d$ is 2-monotone.

Proof. Because of the superadditivity of coherent lower probabilities [25, Section 2.7.4 (e)], any coherent $\underline{Q} \geq \underline{P}_d$ must satisfy

$$\underline{Q}(A) \geq m_a \cdot a = \underline{E}_d(A), \ \forall A \in \mathcal{A}(\mathbb{P}) \setminus \{\Omega\}$$

(and $\underline{Q}(\Omega) = \underline{E}_d(\Omega) = 1$). Thus, by Theorem 2.3 it is sufficient to prove that $\underline{E}_d$ is a coherent lower probability to conclude that $\underline{E}_d$ is the natural extension of $\underline{P}_d$. But $\underline{E}_d$ is coherent, and also 2-monotone, because by (11) it is the natural extension of the probability interval $[a, 1]_{\omega \in \mathbb{P}}$, which avoids sure loss by (10) because $na \leq 1$. □

We can now determine the natural extension of $\underline{P}$ in a RRM, both on $\mathcal{A}(\mathbb{P})$ and on $\mathcal{L}(\mathbb{P})$.

Proposition 6.4. *Let $\underline{P} : \mathcal{A}(\mathbb{P}) \to \mathbb{R}$ be the lower probability of a RRM that avoids sure loss, and $\underline{E}$ its natural extension.*

(*a*) *On $\mathcal{A}(\mathbb{P})$, $\underline{E}(\Omega) = 1$ and, for any $A = \bigvee_{i=1}^{m_A} \omega_{j_i} \in \mathcal{A}(\mathbb{P}) \setminus \{\Omega\}$, it is*

$$\underline{E}(A) = bP_0(A) + m_A a. \tag{62}$$

Further, $\underline{E}$ is 2-monotone.

(*b*) *For any $X \in \mathcal{L}(\mathbb{P})$ we have that*

$$\underline{E}(X) = (1 - na - b) \min X + bE^{P_0}(X) + naE^{P_u}(X), \tag{63}$$

where $E^{P_0}(X)$, $E^{P_u}(X)$ are the (usual) expectations of X referring to, respectively, P_0 and the probability P_u uniform on $\mathbb{P}$ ($P_u(\omega) = \frac{1}{n}$, $\forall \omega$).

Proof. (*a*) $\underline{E}$ is coherent and 2-monotone because by (11) it is the natural extension on $\mathcal{A}(\mathbb{P})$ of the probability interval $J = [bP_0(\omega) + a, 1]_{\omega \in \mathbb{P}}$. Note that J avoids sure loss by (10), since $\sum_{\omega \in \mathbb{P}}(bP_0(\omega) + a) = b + na \leq 1$ by Proposition 6.2.

It also holds that

(*a*1) $\underline{E} \geq \underline{P}$.
Trivial for $A = \emptyset$, $A = \Omega$. Otherwise, we get from (62) $\underline{E}(A) = bP_0(A) + m_A a \geq bP_0(A) + a \geq \underline{P}(A)$.

(*a*2) If $\underline{Q}$ is a coherent lower probability on $\mathcal{A}(\mathbb{P})$ and $\underline{Q} \geq \underline{P}$, then $\underline{Q} \geq \underline{E}$ on $\mathcal{A}(\mathbb{P})$.
To prove this statement, let $\mathcal{M}_{\underline{Q}}$ be the *credal set* of $\underline{Q}$, i.e. the set of all dF-coherent probabilities P such that $P \geq \underline{Q}$ on $\mathcal{A}(\mathbb{P})$. It holds that

$$P \in \mathcal{M}_{\underline{Q}} \Rightarrow P(A) \geq \underline{E}(A), \forall A \in \mathcal{A}(\mathbb{P}). \tag{64}$$

In fact, this holds trivially for $A = \emptyset$, $A = \Omega$. Otherwise, we get $P(A) = \sum_{\omega \Rightarrow A} P(\omega) \geq \sum_{\omega \Rightarrow A} \underline{Q}(\omega) \geq \sum_{\omega \Rightarrow A} \underline{P}(\omega) = \sum_{\omega \Rightarrow A}(bP_0(\omega) + a) = bP_0(A) + m_A a = \underline{E}(A)$.
In terms of credal sets, (64) ensures that $\mathcal{M}_{\underline{Q}} \subseteq \mathcal{M}_{\underline{E}}$. This justifies the inequality in the next development, while the equalities are due to Theorem 2.1:

$$\underline{Q}(A) = \inf_{P \in \mathcal{M}_{\underline{Q}}} P(A) \geq \inf_{P \in \mathcal{M}_{\underline{E}}} P(A) = \underline{E}(A), \ \forall A \in \mathcal{A}(\mathbb{P}).$$

Coherence of $\underline{E}$, (*a*1) and (*a*2) (note that (*a*2) is a special case of Theorem 2.3 (*b*), with $\mathcal{S} = \mathcal{S}' = \mathcal{A}(\mathbb{P})$) imply by Remark 2.1 that $\underline{E}$ is the natural extension of $\underline{P}$, and we have already seen that it is 2-monotone.

(*b*) From (62), let us rewrite $\underline{E}$ as

$$\underline{E}(A) = (1 - b - na)\underline{P}_V(A) + bP_0(A) + naP_u(A).$$

Thus, since $1 - b - na \geq 0$ by Proposition 6.2, $b > 0$ and $na \geq 0$, $\underline{E}$ is a convex linear combination of the 2-monotone coherent lower probabilities $\underline{P}_V$, P_0, P_u. Applying Proposition 2.2 gives (63).

$\square$

Remark 6.1. *It ensues from the proof of Proposition 6.4 (a) that $\underline{P}$ in a RRM avoiding sure loss and the probability interval $[bP_0(\omega) + a, 1]_{\omega\in\mathbb{P}}$ are* equivalent*, i.e. they induce the same natural extension. This property is similar to that of HBMs that avoid sure loss on finite partitions, see Proposition 5.3.*

A result analogous to Proposition 4.4 holds for RRMs;

Proposition 6.5. *Let $\underline{P}$, $\underline{P}'$ be RRM lower probabilities, with $\underline{P}$ $NL(a, b)$ given by (18) and avoiding sure loss, $\underline{P}'$ $NL(a', b')$ with $a' = ka$, $b' = kb$, $k \in]0, 1[$. Terming $\underline{E}$, $\underline{E}'$ the natural extensions of $\underline{P}$, $\underline{P}'$ on $\mathcal{L}(\mathbb{P})$, we have*

$$\underline{E}'(X) = k\underline{E}(X) + (1 - k) \min X. \tag{65}$$

Proof. Since $\underline{P}$ avoids sure loss, $b+na \leq 1$ by Proposition 6.2. Hence, $b'+na' \leq 1$, thus $\underline{P}$, that is still a RRM, avoids sure loss and $\underline{E}'$ is well-defined. From here, the proof is analogous to that of Proposition 4.4 and is omitted. $\square$

Like the VBM, also with the RRM $\underline{E}'$ tends to be vacuous as $k \to 0^+$.

The results in this section confirm that RRMs ensure weaker consistency properties than the other NL models, also as for the condition of avoiding sure loss. In particular, unlike HBMs (and of course VBMs) they cannot avoid sure loss if $\mathbb{P}$ is infinite.

Further, by Lemma 6.1 and Proposition 6.2, *the condition of avoiding sure loss and C-convexity are equivalent* with RRMs. We point out that the equivalence does not extend to HBMs, as the next example illustrates.

Example 6.1. *Let $\mathbb{P} = \{\omega_1, \omega_2, \omega_3\}$. Given the probability $P_0 : \mathcal{A}(\mathbb{P}) \to \mathbb{R}$ such that:*

$$P_0(\omega_1) = \frac{7}{24}, P_0(\omega_2) = \frac{1}{3}, P_0(\omega_3) = \frac{3}{8},$$

take $b = \frac{9}{2}$, $a = -\frac{5}{2}$.

Parameters a, b satisfy the constraints in Definition 2.6, thus applying (16) and (17) we can compute $\underline{P}$ and $\overline{P}$ for the corresponding HBM. In particular, we obtain

$$\underline{P}(\omega_1) = \underline{P}(\omega_2) = \underline{P}(\omega_3) = 0,$$
$$\underline{P}(\omega_1 \vee \omega_2) = \frac{5}{16}, \underline{P}(\omega_1 \vee \omega_3) = \frac{1}{2}, \underline{P}(\omega_2 \vee \omega_3) = \frac{11}{16}.$$

Using the values above to compute the conjugate $\overline{P}$ *of* $\underline{P}$ *on* $\mathbb{P}$*, it ensues that* $\overline{P}$ *avoids sure loss, because* (43) *holds:*

$$\sum_{i=1}^{3} \overline{P}(\omega_i) = \left(1 - \frac{11}{16}\right) + \left(1 - \frac{1}{2}\right) + \left(1 - \frac{5}{16}\right) = \frac{3}{2} > 1.$$

On the other hand, $\underline{P}$ *is not C-convex: it suffices for this to find a gain* $\underline{G}$*, complying with Definition 2.1 (b) and such that* $\sup \underline{G} < 0$*. For instance, take*

$$\begin{aligned}\underline{G} &= \frac{1}{2}\left(I_{\omega_2 \vee \omega_3} - \underline{P}(\omega_2 \vee \omega_3)\right) + \frac{1}{2}\left(I_{\omega_1 \vee \omega_3} - \underline{P}(\omega_1 \vee \omega_3)\right) - \left(I_{\omega_3} - \underline{P}(\omega_3)\right) \\ &= \frac{1}{2}\left(I_{\omega_2} + I_{\omega_3}\right) - \frac{1}{2} \cdot \frac{11}{16} + \frac{1}{2}\left(I_{\omega_1} + I_{\omega_3}\right) - \frac{1}{2} \cdot \frac{1}{2} - I_{\omega_3} \\ &= -\frac{1}{2} I_{\omega_3} - \frac{3}{32} \leq -\frac{3}{32} < 0.\end{aligned}$$

When a RRM avoids sure loss, its natural extensions on $\mathcal{A}(\mathbb{P})$ and on $\mathcal{L}(\mathbb{P})$ can be easily computed by means of, respectively, equations (62) and (63).

Note that, from Equation (61), $\underline{E}(\omega) = \underline{P}(\omega)$, so $\underline{P}$ is coherent on $\mathbb{P}$. Instead, $\overline{P}$ is coherent on $\mathbb{P}$ iff $n = 2$: this follows, recalling (62) at the second equality, (13) and (19) at the fourth, from $\overline{E}(\omega) = 1 - \underline{E}(\neg\omega) = 1 - bP_0(\neg\omega) - (n-1)a = 1 - b(1 - P_0(\omega)) - (n-1)a = \overline{P}(\omega) - (n-2)a = \overline{P}(\omega)$ iff $n = 2$. In [5, Proposition 6.1], we proved that $\overline{P}$ is coherent on $\mathcal{A}(\mathbb{P})$ iff $n = 2$; here we learn that not even the restriction of $\overline{P}$ on $\mathbb{P}$ is coherent for $n > 2$.

7 Conclusions

In this paper, we primarily introduced formulae for computing the natural extension of those NL models that (at least) avoid sure loss. Our results allow computing easily the natural extension in most cases, either $\underline{E}$ or its conjugate $\overline{E}$ (the other one may be obtained by conjugacy). When the NL model does not avoid sure loss, it is nevertheless 2-coherent. Thus, it is possible to extend it to $\mathcal{L}(\mathbb{P})$ with the 2-coherent natural extension $\underline{E}_{2c}$, using the formulae introduced in [19]. However, these formulae do not significantly simplify with NL models, and for this reason we do not recall them explicitly here.

The paper also contains results that broaden our knowledge of NL models, in particular concerning their consistency properties, summarised in Table 2.

Type of consistency	VBM	HBM	RRM
Coherent	YES [5]	Characterised (Prop. 2.3 and [5])	Iff $\|\mathbb{P}\| = 2$ [5]
Avoids sure loss (ASL)	YES	Characterised (Prop.s 5.1 (*b*), 5.5)	Characterised (Lemma 6.1, Prop. 6.2)
C-convex	YES	More restrictive than ASL (Example 6.1)	Equivalent to ASL (Lemma 6.1, Prop. 6.2)
2-coherent	YES	YES [5]	YES [5]

Table 2: Consistency of NL models.

Further results concern the relationships between NL models and interval probabilities, and the representation of VBMs and RRMs as linear combinations of simpler models.

Among the topics that remain to be investigated, we mention conditioning and dilation with NL models. In particular, it could be interesting to explore how the results on this for the PMM in [22] can be extended. Other interesting questions, strictly related to recent work on neighbourhood models in a finite framework [13, 14], regard an analysis of the vertexes of the credal set of a coherent NL model, and interpretations of NL models in terms of distances.

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References

References

[1] Artzner, P. (1999). Application of coherent risk measures to capital requirements in insurance. *North American Actuarial Journal*, 3(2):11–25.

[2] Augustin, T., Coolen, F. P., de Cooman, G., and Troffaes, M. C., editors (2014). *Introduction to imprecise probabilities*. Wiley Series in Probability and Statistics. John Wiley and Sons.

[3] Chateauneuf, A., Eichberger, J., and Grant, S. (2007). Choice under uncertainty with the best and worst in mind: Neo-additive capacities. *J. Econ. Th.*, 137:538–567.

[4] Corsato, C., Pelessoni, R., and Vicig, P. (2019a). Generalising the Pari-Mutuel Model. In Destercke, S., Denoeux, T., Gil, M., Grzegorzewski, P., and Hryniewicz, O., editors, *Uncertainty Modelling in Data Science, Soft Methods Prob. Stat. 2018, AISC 832*, pages 216–223.

[5] Corsato, C., Pelessoni, R., and Vicig, P. (2019b). Nearly-Linear Uncertainty Measures. *Int. J. Approx. Reason.*, 114:1–28.

[6] Crisma, L. (2006). *Introduzione alla teoria delle probabilità coerenti*, volume 1. EUT. Freely available at http://hdl.handle.net/10077/2223.

[7] De Campos, L. M., Huete, J. F., and Moral, S. (1994). Probability intervals: a tool for uncertain reasoning. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 2(2):167–196.

[8] de Finetti, B. (1974). *Theory of Probability.* Wiley.

[9] Denuit, M., Dhaene, J., Goovaerts, M., and Kaas, R. (2005). *Actuarial Theory for Dependent Risks: Measures, Orders and Models.* Wiley.

[10] Eichberger, J., Grant, S., and Lefort, P. (2012). Generalized neo-additive capacities and updating. *Int. J. Econ. Theory*, 8:237–257.

[11] Grabisch, M. (2016). *Set Functions, Games and Capacities in Decision Making.* Springer.

[12] Herron, T., Seidenfeld, T., and Wasserman, L. (1997). Divisive conditioning: Further results on dilation. *Philos. Sci.*, 64(3):411–444.

[13] Miranda, E., Montes, I., and Destercke, S. (2019). A Unifying Frame for Neighbourhood and Distortion Models. In De Bock, J., de Cooman, G., de Campos, C. P., Quaeghebeur, E., and Wheeler, G., editors, *PMLR*, volume 103, pages 304–313.

[14] Montes, I., Miranda, E., and Destercke, S. (2019). Pari-Mutuel probabilities as an uncertainty model. *Information Sciences*, 481:550–573.

[15] Nakharutai, N., Caiado, C. C., and Troffaes, M. C. (2019). Evaluating betting odds and free coupons using desirability. *International Journal of Approximate Reasoning*, 106:128–145.

[16] Pelessoni, R. and Vicig, P. (1998). A Consistency Problem for Imprecise Conditional Probability Assessments. In *Proceedings of IPMU'98*, volume 2, pages 1478–1485.

[17] Pelessoni, R. and Vicig, P. (2003). Convex Imprecise Previsions. *Rel. Comput.*, 9:465–485.

[18] Pelessoni, R. and Vicig, P. (2016). 2-coherent and 2-convex conditional lower previsions. *Int. J. Approx. Reason.*, 77:66–86.

[19] Pelessoni, R. and Vicig, P. (2017). Weakly consistent extensions of lower previsions. *Fuzzy Sets and Systems*, 328:83 – 106.

[20] Pelessoni, R., Vicig, P., and Corsato, C. (2019). Extending Nearly-Linear Models. In De Bock, J., de Cooman, G., de Campos, C. P., Quaeghebeur, E., and Wheeler, G., editors, *PMLR*, volume 103, pages 82–90.

[21] Pelessoni, R., Vicig, P., Miranda, E., and Montes, I. (2016). Bivariate p-boxes. *Internat. J. Uncertain. Fuzziness Knowledge-Based Systems*, 24:229–263.

[22] Pelessoni, R., Vicig, P., and Zaffalon, M. (2010). Inference and risk measurement with the pari-mutuel model. *Int. J. Approx. Reason.*, 51:1145–1158.

[23] Rieder, H. (1977). Least favourable pairs for special capacities. *Annals Stat.*, 5(5):909–921.

[24] Troffaes, M. and de Cooman, G. (2014). *Lower Previsions.* Wiley.

[25] Walley, P. (1991). *Statistical Reasoning with Imprecise Probabilities.* Chapman & Hall.

[26] Walley, P., Pelessoni, R., and Vicig, P. (2004). Direct algorithms for checking consistency and making inferences from conditional probability assessments. *Journal of Statistical Planning and Inference*, 126(1):119–151.