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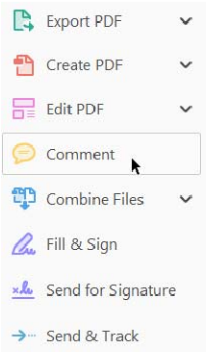

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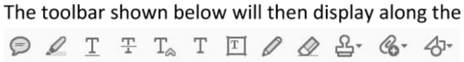

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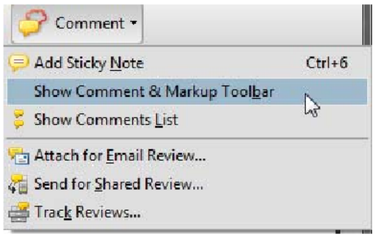

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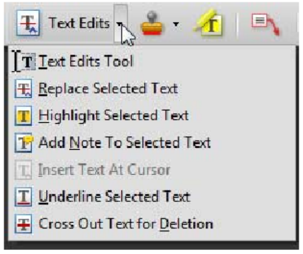

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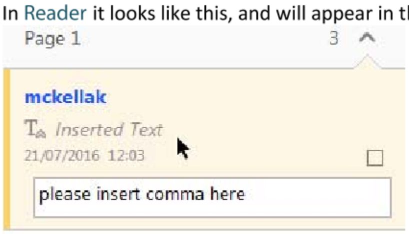

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Trigonal Deformations of Rank One and Jacobians

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We study the infinitesimal deformations of a trigonal curve that preserve the trigonal series and such that the associate infinitesimal variation of Hodge structure is of rank 1. We show that if $g \geq 8$ or $g = 6, 7$ and the curve is Maroni general, this locus is zero dimensional. Moreover, we complete the result [10, Theorem 1.6]. We show in fact that if $g \geq 6$, the hyperelliptic locus $\mathcal{M}^1_{g,2}$ is the only $2g-1$-dimensional sub-locus $\mathcal{Y}$ of the moduli space $\mathcal{M}_g$ of curves of genus g, such that for the general element $[C] \in \mathcal{Y}$, its Jacobian $J(C)$ is dominated by a hyperelliptic Jacobian of genus $g' \geq g$.

Introduction

In this paper we study infinitesimal deformations that come from families of trigonal curves and in particular the ones having the associated infinitesimal variation of Hodge structure (IVHS) of rank 1.

Our main motivation was to complete the characterization of $2g-1$-dimensional families $\mathcal{Y}$ of curves with Jacobian family dominated by a hyperelliptic Jacobian family. Indeed, in [10, Theorem 1.6] it is shown that if $\mathcal{Y}$ is a $2g-1$-dimensional closed irreducible subvariety of $\mathcal{M}_g$ with $g \geq 5$, such that the Jacobian of its generic element

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is dominated by a hyperelliptic Jacobian, then $\mathcal{Y}$ is contained either in the hyperelliptic or in the trigonal locus. In this paper we rule out the trigonal case.

Following the frequently used notation, we shall denote by $\mathcal{M}^1_{g,3}$ the locus in the moduli space $\mathcal{M}_g$ of smooth curves of genus g given by points $[C] \in \mathcal{M}_g$ such that C admits a linear series g^1_3, and by $\mathcal{M}^1_{g,2}$ the hyperelliptic locus, consisting of curves admitting a g^1_2; the locus $\mathcal{M}^1_{g,3}$ can be seen as the union of the hyperelliptic and of the trigonal loci.

Our main result is the following:

Theorem 0.1. If $g \geq 6$ then $\mathcal{M}^1_{g,2}$ is the unique closed irreducible subvariety $\mathcal{Y} \subset \mathcal{M}_g$ of dimension $2g-1$ such that for its generic element $[C] \in \mathcal{Y}$ there exists $[D] \in \mathcal{M}^1_{g',2}$ such that $J(D) \twoheadrightarrow J(C)$.

We point out that, in principle, we have to consider all the codimension-2 subvarieties of $\mathcal{M}^1_{g,3}$. The argument goes as follows. If C is smooth and trigonal, the Babbage–Enriques–Petri theorem gives that the intersection of all the quadric that contain the canonical image of C is a ruled surface $S \subset \mathbb{P}^{g-1}$. This surface S can be also embedded, by an extension $\kappa_2 : S \to \mathbb{P}H^1(T_C)$ of the bicanonical morphism of C, in the projective space $\mathbb{P}H^1(T_C)$, where T_C is the tangent sheaf of C. We identify $H^1(T_C)$ with the tangent space of $\mathcal{M}_g$ at $[C]$, by possibly adding a level structure or by considering the moduli stack $\mathscr{M}_g$, if $[C]$ is a singular point of the moduli space. By the work of Griffiths, the points of the surface $\kappa_2(S)$ correspond to the locus of infinitesimal deformations with IVHS of rank one (see [8] and Lemma 3.2).

Now, in [10, Proof of Theorem 1.6, p. 908–909], it is shown that, under the conditions of theorem (0.1), if the general curve $[C] \in \mathcal{Y}$ is trigonal, there must exist a non-degenerate rational curve $Z \subset S \subset \mathbb{P}^{g-1}$, which corresponds to a 1-dimensional component of the intersection $\Gamma = \kappa_2(S) \cap \mathbb{P}(T_{\mathcal{M}^1_{g,3},[C]}) \subset \mathbb{P}H^1(T_C)$, where $T_{\mathcal{M}^1_{g,3},[C]}$ denotes the tangent space to $\mathcal{M}^1_{g,3}$ at $[C]$. We show that Γ cannot contain such a component, by analyzing the geometry on the Hirzebruch surface $\mathbb{F}_n$ isomorphic to S. We recall that the integer $n \geq 0$ is classically called the Maroni invariant of C; we shall perform our study in terms of the Maroni degree $k = \frac{(g-2-n)}{2}$ for trigonal curves, that is in terms of the degree of a non-positive section in the extended canonical embedding in $\mathbb{P}^{g-1}$, see subsection 2. We show:

Theorem 0.2. If $g \geq 8$, or $6 \leq g \leq 7$ and Maroni degree $k = 2$, or $g = 6$ and $k \neq 1$, then Γ has dimension 0.

The proof will follow from Proposition 3.11 and Proposition 3.12.

In the special cases $g = 7, 6$ and $k = 1$ we need to deepen our analysis. We consider the locus $\mathcal{D}_1 \subset \mathcal{M}^1_{g,3} \setminus \mathcal{M}^1_{g,2}$ of trigonal curves with Maroni degree $k = 1$ and we define $\Gamma^1 \subset \Gamma$ the intersection $\Gamma \cap \mathbb{P}(T_{\mathcal{D}_1,[C]})$. We show:

Theorem 0.3. If $g = 6$ or $g = 7$ and Maroni degree $k = 1$ and C is generic, then Γ is irreducible and Γ^1 has dimension 0.

Moreover, if Γ is reducible, then Γ^1 does not contain any rational curve Z_1, such that $\kappa_2^{-1}(Z_1) \subset \mathbb{P}^{g-1}$ is non-degenerate.

The result will be a consequence of Proposition 4.2 for $g = 7$ and Proposition 4.3 if $g = 6$.

From the two theorems we deduce that Γ contains no non-degenerate rational curves, which implies our main result.

Our point of view is also related to problems concerning the relative irregularity of families of curves, see for instance [2], [5], [6], and [7], since our result gives an infinitesimal version of a result of Xiao [12, Corollary 4, page 462]. To shorten the paper we do not include this here as well as some partial computation on genus 5 case, but we only finally remark that hopefully, this can shed some new light on the slope problem as treated in [3] and [4].

1 A Gaussian Lemma

Let C be a smooth curve and let L be line bundle of degree $d \geq 1$ on C. Consider a subspace $V \subset H^0(C, L)$ corresponding to a base point free linear system. We can associate two objects with V, the Gaussian section and the Euler class, which we recall.

Let $f\colon C \to \mathbb{P}(V^\vee) \equiv \mathbb{P}^r$ be the projective morphism given by V and

$$0 \to T_C \xrightarrow{df} f^*(T_{\mathbb{P}^r}) \to N_f \to 0 \tag{1.1}$$

the exact sequence associated with its differential. We tensor it by the canonical sheaf ω_C:

$$0 \to \mathcal{O}_C \to \omega_C \otimes f^*(T_{\mathbb{P}^r}) \to \omega_C \otimes N_f \to 0. \tag{1.2}$$

Then we can consider the section

$$\Omega_V \in H^0(C, \omega_C \otimes f^*(T_{\mathbb{P}^r}))$$

obtained as the image of $1 \in H^0(C, \mathcal{O}_C)$, and represents the differential df.

Definition 1.1. We call Ω_V the Gaussian section of the morphism $f\colon C \to \mathbb{P}^r$.

Next we will relate Ω_V to the Euler class, whose definition we briefly recall. It is well-known that $H^1(\mathbb{P}^r, \Omega^1_{\mathbb{P}^r}) = \mathbb{C}$ and that the Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}^r} \to (\mathcal{O}_{\mathbb{P}^r}(1))^{\oplus r+1} \to T_{\mathbb{P}^r} \to 0$$

is given by a non-trivial class $\eta_{\mathrm{eul}} \in H^1(\mathbb{P}^r, \Omega^1_{\mathbb{P}^r})$. The pull-back to C of the Euler sequence gives

$$0 \to \mathcal{O}_C \to V \otimes L \to f^*(T_{\mathbb{P}^r}) \to 0. \tag{1.3}$$

The extension class of the sequence (1.3) determines an element $\eta_V \in H^1(C, f^*(\Omega^1_{\mathbb{P}^r}))$: we call it the *Euler class of* $f\colon C \to |V|^\vee$.

Following [1, Page 804], we recall that with a line bundle L on a smooth curve C we can associate a sheaf Σ_L, by considering its Chern class $c_1(L) \in H^1(C, \omega_C) \cong Ext^1(T_C, \mathcal{O}_C)$ and the extension determined by $c_1(L)$:

$$0 \to \mathcal{O}_C \to \Sigma_L \xrightarrow{\tau} T_C \to 0. \tag{1.4}$$

By taking into account the sequence (1.1), we get a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 \to & \mathcal{O}_C & \to & & \Sigma_L & \xrightarrow{\tau} & T_C & & \to 0 \\
 & \| & & & \downarrow & & \downarrow\ \Omega_V & & \\
0 \to & \mathcal{O}_C & \to & & V \otimes L & \to & f^*(T_{\mathbb{P}^r}) & & \to 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{N}_f & = & \mathcal{N}_f & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array} \tag{1.5}$$

We observe that the differential df determines a map of complexes, and the image of the bottom extension is the upper extension.

Lemma 1.2 (Gaussian Lemma). Consider the natural map given by the cup product and duality

$$H^0(C, \omega_C \otimes f^*(T_{\mathbb{P}^r})) \times H^1(C, f^*(\Omega^1_{\mathbb{P}^r})) \to H^1(C, \omega_C).$$

Then we have

$$\Omega_V \cdot \eta_V = c_1(L) \neq 0.$$

As a consequence, the Gaussian section Ω_V does not belong to the image of the map

$$H^0(C, V \otimes \omega_C \otimes L) \to H^0(C, f^*(T_{\mathbb{P}^r}) \otimes \omega_C).$$

Remark 1.3 We will use Lemma 1.2 in the case where L induces a g^1_3 on C. We stress that if $r = 1$ then the ramification scheme R_V of $f \colon C \to \mathbb{P}^1$ is the zero locus of the Gaussian section, which is a section

$$\Omega_V \in H^0(C, \omega_C \otimes L^{\otimes 2}).$$

2 Trigonal Curves and Gaussian Sections

Let $C \subset \mathbb{P}^{g-1}$ be a canonical trigonal curve of genus $g \geq 5$, which has no g^2_5, and let I_C be its graded ideal. By the Babbage–Enriques–Petri theorem the hyperquadrics containing C intersect in a smooth rational ruled surface S of minimal degree $g-2$ in $\mathbb{P}^{g-1}$ and the trigonal series is cut on C by the ruling of S.

Let us fix some notations and recall some known results concerning the Hirzebruch surfaces $\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$. The Picard group of $\mathbb{F}_n$ satisfies $\mathrm{Pic}(\mathbb{F}_n) = [B]\mathbb{Z} \oplus [R]\mathbb{Z}$, where B is a section of minimal self-intersection and R a fiber in the ruling of the projection $\pi : \mathbb{F}_n \to \mathbb{P}^1$. The basic intersection formulae are

$$B^2 = -n, \; BR = 1, \; R^2 = 0. \tag{2.1}$$

Definition 2.1. Let C be a trigonal curve of genus $g \geq 6$ and let L be the line bundle of degree 3 computing the unique trigonal series. We set $V := H^0(C, L)$. The Maroni degree $k \in \mathbb{N}$ of C can be characterized as the unique number such that

$$h^0(C, L^{\otimes k+1}) = k+2, \qquad h^0(C, L^{\otimes k+2}) > k+3.$$

The following bounds on k have been established by Maroni [9] and are well-known

$$\frac{g-4}{3} \leq k \leq \frac{g-2}{2}. \tag{2.2}$$

It turns out that S is an embedding of the Hirzebruch surface $\mathbb{F}_{g-2-2k}$ via the linear system $|H| = |B + (g-2-k)R|$. Moreover, we have

$$C \in |3B + (2g-2-3k)R|, \qquad K_C = H_{|C}. \tag{2.3}$$

We recall that $H^0(\mathbb{F}_n, \mathcal{O}_{\mathbb{F}_n}(R)) \cong H^0(\mathbb{P}^1, \pi_\star \mathcal{O}_{\mathbb{F}_n}(R)) \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$. A class $M = kB + sR$ is ample if and only if $MR = k > 0$ and $MB = -nk + s > 0$, that is $s > nk$, and respectively nef if $s \geq nk \geq 0$. Finally, if $s \geq nk > 0$, then M is big and nef.

The following results will be useful in the sequel:

Lemma 2.2. If $m \geq 0$ and $s \geq n(m+1)+1$, then the multiplication map

$$\mu\colon H^0(\mathbb{F}_n, \mathcal{O}_{\mathbb{F}_n}(R)) \otimes H^0(\mathbb{F}_n, \mathcal{O}_{\mathbb{F}_n}(mB + sR)) \to H^0(\mathbb{F}_n, \mathcal{O}_{\mathbb{F}_n}(mB + (s+1)R)) \tag{2.4}$$

is surjective.

Proof. Set $V := H^0(\mathbb{F}_n, \mathcal{O}_{\mathbb{F}_n}(R))$. We tensor the sequence

$$0 \to \mathcal{O}_{\mathbb{F}_n}(-R) \to V \otimes \mathcal{O}_{\mathbb{F}_n} \to \mathcal{O}_{\mathbb{F}_n}(R) \to 0 \tag{2.5}$$

by $\mathcal{O}_{\mathbb{F}_n}(mB+sR)$ where $m \geq 0$ and $s \geq n(m+1)+1$. By the free pencil trick the cokernel of μ injects into $H^1(\mathbb{F}_n, \mathcal{O}_{\mathbb{F}_n}(mB+(s-1)R))$, which is the dual of $H^1(\mathbb{F}_n, \mathcal{O}_{\mathbb{F}_n}(-(m+2)B-(s+n-1)R))$ by Serre duality and the fact that the canonical divisor is $K_{\mathbb{F}_n} \sim -2B-(2+n)R$. By the hypotheses on m and s, the divisor $M := (m+2)B + (s+n-1)R$ is big and nef, therefore $H^1(\mathbb{F}_n, \mathcal{O}_{\mathbb{F}_n}(-M)) = 0$. ■

Proposition 2.3. The map $\mu : H^0(S, H+R) \otimes H^0(S, R) \to H^0(S, H+2R)$ is surjective.

Proof. As in the proof of Lemma (2.2) we only need to show that $H^1(S, \mathcal{O}_S(H)) = 0$. By the remarks above we have $H \sim K_S + C$. It is easy to show that C is 1-connected, so $|nC|$ gives a morphism to a surface. We conclude by applying Ramanujam's vanishing theorem [11]. ■

Now consider the exact sequence:

$$0 \to \mathcal{O}_S(H + 2R - C) \to \mathcal{O}_S(H + 2R) \to \omega_C(2L) \to 0$$

and the restriction map

$$r\colon H^0(S,\mathcal{O}_S(H+2R)) \to H^0(C,\omega_C(2L)).$$

We observe that in the trigonal case, the Gaussian section $\Omega_V \in H^0(C,\omega_C\otimes L^{\otimes 2})$.

Corollary 2.4. The Gaussian section does not belong to the image of r.

Proof. By contradiction assume that $\Omega_V = r(\alpha)$ for some $\alpha \in H^0(S,\mathcal{O}_S(H+2R))$. By Proposition 2.3 we can write $\alpha = s_1\beta_1 + s_2\beta_2$ where the s_i are a basis of $H^0(S,R)$ and $\beta_i \in H^0(S,H+R)$. By restricting the s_i and the β_i to C, this would imply that Ω_V belongs to the image of the multiplication map $\mu : H^0(C,L)\otimes H^0(C,\omega_C\otimes L) \to H^0(C,\omega_C\otimes 2L)$. This is in contradiction with the Gaussian Lemma 1.2. ■

3 The Locus of Rank 1 Infinitesimal Deformations

Let $I_2 := I_C(2)$ be the degree two part of the homogeneous ideal of a trigonal canonical curve $C \subset \mathbb{P}^{g-1}$, and let S be the ruled surface containing C. By a simple computation we see that

$$|2H| \cong |2H_{|C}| \cong |2K_C|. \tag{3.1}$$

If we consider the bicanonical map

$$C \to |2K_C|^\vee \cong \mathbb{P}(H^1(C,T_C)) = \mathbb{P}^{3g-4},$$

we observe that it extends to an embedding $\kappa_2 : S \to \mathbb{P}^{3g-4}$. Since C is not hyperelliptic, the multiplication map

$$\mu\colon \mathrm{Sym}^2 H^0(C,\omega_C) \to H^0(C,\omega_C^{\otimes 2})$$

is surjective. By definition the kernel of μ is I_2. By Serre duality we obtain:

$$0 \to H^1(C,T_C) \xrightarrow{\iota} \mathrm{Sym}^2(H^1(C,\mathcal{O}_C)) \to I_2^\vee \to 0. \tag{3.2}$$

The inclusion $\iota\colon H^1(C,T_C) \to \mathrm{Sym}^2(H^1(C,\mathcal{O}_C))$ is given by $\xi \overset{\iota}{\mapsto} q_\xi$ where q_ξ is the quadric associated with the co-boundary homomorphism

$$\partial_\xi\colon H^0(C,\omega_C) \to H^0(C,\omega_C)^\vee = H^1(C,\mathcal{O}_C)$$

of the extension class $\xi \in H^1(C,T_C) = \mathrm{Ext}^1_{\mathcal{O}_C}(\omega_C,\mathcal{O}_C)$; see [8].

Definition 3.1. We define *the rank of* ξ as the rank of its associated quadric q_ξ.

By the standard properties of the Veronese embedding

$$\nu_2 : \mathbb{P}(H^1(C, \mathcal{O}_C)) \to \mathbb{P}(\mathrm{Sym}^2(H^1(C, \mathcal{O}_C)))$$

it follows that

$$\kappa_2(S) = \nu_2(\mathbb{P}(H^1(C, \mathcal{O}_C)) \cap \iota(\mathbb{P}(H^1(C, T_C)) \subset \mathbb{P}(\mathrm{Sym}^2(H^1(C, \mathcal{O}_C))).$$

We have:

Lemma 3.2. The image of the embedding of $\kappa_2 : S \to \mathbb{P}^{3g-4}$ satisfies

$$\kappa_2(S) = \{[\xi] \in \mathbb{P}H^1(C, T_C) \mid \partial_\xi : H^0(C, \omega_C) \to H^1(C, \mathcal{O}_C) \text{ has rank } 1\}.$$

Proof. It follows from the sequence (3.2) and its dual. See also [8, p. 271]. ■

3.1 Trigonal deformations of rank 1

In the present section we shall study the locus of S corresponding to deformations that preserve the property of having a trigonal series.

As in the Introduction we denote by $\mathcal{M}^1_{g,3}$ the locus of curves admitting a g^1_3 and let $\mathcal{D}_k \subset \mathcal{M}^1_{g,3} \setminus \mathcal{M}^1_{g,2}$ be the locus of trigonal curves with Maroni degree k. We define

$$T^k_C := T_{\mathcal{D}_k,[C]} \subseteq H^1(C, T_C), \qquad T := T_{\mathcal{M}^1_{g,3},[C]} \subseteq H^1(C, T_C)$$

the tangent spaces to respectively $\mathcal{D}_k$ and $\mathcal{M}^1_{g,3}$ at $[C] \in \mathcal{M}^1_{g,3} \setminus \mathcal{M}^1_{g,2}$.

Consider the natural homomorphism

$$H^1(C, T_C) \otimes H^0(C, \omega_C \otimes L^{\otimes 2}) \to H^1(C, L^{\otimes 2})$$

given by the cup-product $\xi \otimes \sigma \mapsto \xi \cdot \sigma$. It holds:

Lemma 3.3. If $[C] \in \mathcal{M}^1_{g,3} \setminus \mathcal{M}^1_{g,2}$ then

$$T = \{\zeta \in H^1(C, T_C) : \zeta \cdot \Omega_V = 0 \in H^1(C, 2L)\}. \tag{3.3}$$

Proof. Let $\mathcal{C}_\xi \to \mathrm{Spec}(\mathbb{C}[\epsilon]/(\epsilon^2))$ be the infinitesimal family associated with $\xi \in T$. We have that the trigonal morphism $f\colon C \to \mathbb{P}^1$ lifts to $\mathcal{C}_\xi$. By standard arguments it follows that the Gaussian section lifts to $\mathcal{C}_\xi$. This implies that the cup product $\xi \cdot \Omega_V = 0$. Since $h^1(C, L^{\otimes 2}) = g - 4$ and since the cup product $\cdot\Omega_V\colon H^1(C, T_C) \to H^1(C, L^{\otimes 2})$ is easily seen to be surjective by dualizing the map, we have that

$$\dim_{\mathbb{C}}\{\zeta \in H^1(C, T_C) : \zeta \cdot \Omega = 0 \in H^1(C, 2L)\} = 2g + 1.$$

On the other hand $\dim \mathcal{M}^1_{g,3} = 2g + 1$. Then the claim follows. ■

Remark 3.4. Dually, the annihilator of T in $H^1(C, T_C)^\vee \cong H^0(C, \omega_C^{\otimes 2})$ is $H^0(C, \omega_C^{\otimes 2}(-R_V))$, where R_V is the ramification divisor.

By taking into account the isomorphisms in (3.1), such a space corresponds to the subspace

$$H^0(S, \mathcal{O}_S(2H) \otimes \mathcal{I}_{R_V,S}),$$

where $\mathcal{I}_{R_V,S}$ is the ideal sheaf of the subscheme $R_V \subset S$.

We shall denote by

$$\Lambda := \mathbb{P}(H^0(S, \mathcal{O}_S(2H) \otimes \mathcal{I}_{R_V,S})).$$

Next we would like to determine the intersection of $\mathbb{P}(T)$ with the surface $\kappa_2(S)$.

Definition 3.5. The locus $\Gamma = \mathbb{P}(T) \cap \kappa_2(S)$ is called the locus of trigonal deformation of rank 1.

3.2 Proof of Theorem 0.2

In what follows with k we shall always denote the Maroni degree. For the reader's convenience we fix the relations and classes in $\mathrm{Pic}(S) = \mathrm{Pic}(\mathbb{F}_{g-2-2k})$, which we shall use:

- R is the ruling inducing the trigonal linear system on C;
- B is a section of minimal self-intersection of $\mathbb{F}_{g-2-2k}$, and $B^2 = 2k + 2 - g$;
- $A = B + (g - 2 - 2k)R$ is the tautological divisor of $\mathbb{F}_{g-2-2k}$;
- $K_S = -2B - (g - 2k)R$;
- $H = B + (g - 2 - k)R$ is the hyperplane divisor of the canonical embedding, $H_{|C} \sim K_C$, and $H^2 = g - 2$;

- $C \in |3B + (2g - 2 - 3k)R|$.

Recalling the bounds on k given in (2.2), we shall distinguish two cases: $g = 2k+2$ or $2k + 2 < g \leq 3k + 4$.

3.2.1 *The case $g = 2k + 2$ and $k \geq 2$: curves of even genus with Maroni invariant zero*

We shall show that in this case the subscheme $R_V \subset S$ is a complete intersection of two divisors $G_1, G_2 \sim 2B + (k+2)R$, and since $h^0(S, \mathcal{O}_S(2H - G_i)) \neq 0$, they determine a pencil in Λ with base locus exactly R_V; as a consequence we will obtain that the base locus $\mathrm{Bs}(\Lambda)$ of Λ satisfies $\mathrm{Bs}(\Lambda) = R_V$.

Let C be a trigonal curve of genus $g = 2k + 2$. In this case we have that

$$S \cong \mathbb{P}^1 \times \mathbb{P}^1, \quad C \in |3B + (k+2)R|, \quad \Lambda \subseteq |2H| = |2B + 2kR|.$$

Proposition 3.6. If $g = 2k+2$ with $k \geq 2$, then the base locus of Λ satisfies $\dim \mathrm{Bs}(\Lambda) = 0$ and $\mathrm{Bs}(\Lambda) = R_V$.

Proof. We observe that

$$h^0(S, \mathcal{O}_S(2B + (k+2)R) \otimes \mathcal{I}_{R_V,S}) \neq 0.$$

Indeed, we have

$$(2B + (k+2)R)_{|C} \sim H_{|C} + 2R_{|C} + B_{|C} \sim R_V + B_{|C},$$

and since $-C + 2B + (k+2)R \sim -B$, we have $h^0(S, \mathcal{O}_S(-C + 2B + (k+2)R)) = h^1(S, \mathcal{O}_S(-C + 2B + (k+2)R)) = 0$, so there is an isomorphism

$$H^0(\mathcal{O}_S(2B + (k+2)R)) \cong H^0(\mathcal{O}_C(K_C + 2L + B_{|C})). \tag{3.4}$$

We consider in particular the pencil $R_V + |B_{|C}|$ in $|K_C + 2L + B_{|C}$, and the corresponding pencil $|G|$ in $|2B + (k+2)R|$ under the isomorphism (3.4). By construction, we have that the base locus $\mathrm{Bs}|G| \supseteq R_V$. We claim that

$$\mathrm{Bs}|G| = R_V.$$

We first show that $\mathrm{Bs}|G|$ contains no divisors. If by contradiction it contains a divisor Γ in some linear system $|aB+bR|$ with

$$0 \le a \le 2, \qquad 0 \le b \le k+2, \tag{3.5}$$

then we must have

$$\Gamma|_C < \mathrm{Bs}|K_C + 2L + B_{|C} - R_V| = R_V,$$

therefore

$$\Gamma \cdot C = 3b + (k+2)a \le \deg R_V = 2g+4 = 4k+8. \tag{3.6}$$

On the other hand, the base points of the moving part of the pencil must contain the residual part of R_V, so we must have

$$\Gamma \cdot C + (G-\Gamma)^2 \ge \deg R_V = 4k+8.$$

This gives the inequality

$$3b + (k+2)a + 2(2-a)(k+2-b) \ge 4k+8. \tag{3.7}$$

We observe that if $a=0$, then (3.7) gives $b=0$.

If $a = 1$, then by (3.7) and (3.5) we get $b = k+2$; in this case we would have $\Gamma \sim H + 2R$ and $\Gamma|_C = R_V$, but this is not the case since by Corollary 2.4 the Gaussian section $\Omega \notin \mathrm{Image}(H^0(S, \mathcal{O}_S(H+2R)) \to H^0(C, \omega_C(2L))$.

Finally, observe that the case $a=2$ cannot occur, as, by construction, the moving part of the pencil $R_V + |B_{|C}|$ is not cut out by fibers R of the ruling.

Therefore $\dim \mathrm{Bs}(|G|) = 0$. Since $G^2 = 4k+8 = \deg R_V$, we get the statement of the Proposition. ■

3.2.2 *The case* $2k+2 < g \le 3k+4$, $k \ge 1$ *and* $g \ge 6$

The argument is similar to the one of the previous section. We shall show that in this case the subscheme $R_V \subset S$ is a complete intersection of two divisors $Q_V \sim 2B + (g-k)R$ and $Q_1 \sim 2B + (2g-3k-2)R$, and since $H \sim B + (g-2-k)R$, the bounds (2.2) imply $h^0(S, \mathcal{O}_S(2H - Q_V)) \ne 0$ and with the additional hypothesis $k \ge 2$ we have also

$$h^0(S, \mathcal{O}_S(2H - Q_1)) \ne 0. \tag{3.8}$$

As a consequence we will obtain that the base locus of Λ is $\mathrm{Bs}(\Lambda) = R_V$.

Finally, we shall prove that in the remaining cases $g = 6$, $g = 7$, and $k = 1$, the linear system Λ has a one dimensional fixed component.

Since we are assuming $g > 2k+2$, the ruled surface $S \cong \mathbb{F}_{g-2k-2}$ admits a negative section $B^2 < 0$.

Lemma 3.7. There exists a unique divisor $Q_V \in |2B+(g-k)R|$ containing R_V. Moreover, we have

$$Q_{V|C} = B_{|C} + R_V,$$

and B is not a component of Q_V.

Proof. By observing that $-C+2B+(g-k)R \sim -A$ one can easily see that the restriction morphism $|2B+(g-k)R| \to |B_{|C}+R_V|$ is an isomorphism.

To prove the second claim, we note that $-C+B = -(2B+(2g-2-3k)R)$. Since $2B+(2g-2-3k)R$ is big and nef, then $h^1(S, \mathcal{O}_S(-C+B)) = 0$. On the other hand we have $h^0(S, \mathcal{O}_S(B)) = 1$; therefore $h^0(C, \mathcal{O}_C(B_{|C})) = 1$, which concludes the proof. ■

Next we consider the linear system $|2B+(2g-3k-2)R|$.

Lemma 3.8. The restriction map $|2B+(2g-3k-2)R| \to |B_{|C}+R_V+(g-2k-2)L|$ is an isomorphism.

Proof. Note that $2B+(2g-3k-2)R-C = -B$. The claim easily follows since B is an irreducible curve and S is a regular surface. ■

Next we note that the sublinear system $R_V+|B_{|C}+(g-2k-2)L|$ of $|R_V+B_{|C}+(g-2k-2)L|$ has dimension

$$\dim(R_V+|B_{|C}+(g-2k-2)L|) = g-2k-1. \tag{3.9}$$

Indeed, we have $K_C = H_{|C} \sim B_{|C}+(g-2-k)L$. Hence,

$$h^1(C, \mathcal{O}_C(B_{|C}+(g-2k-2)L)) = h^0(C, \mathcal{O}_C(kL)) = k+1.$$

By Riemann–Roch for curves the claim follows.

Now we consider the sublinear system $\Lambda' < |2B+(2g-3k-2)R|$ on S which is isomorphic to the sublinear system $R_V+|B_{|C}+(g-2k-2)L|$ on C.

Corollary 3.9. There exists $Q_1 \in \Lambda'$ such that

$$\Lambda' = \langle Q_V + |(g-2k-2)R|, Q_1\rangle.$$

Proof. Note that $Q_V + |(g-2k-2)R|$ is a $(g-2k-2)$-dimensional sublinear system of Λ. Hence, by (3.9) the claim follows. ∎

Proposition 3.10. The divisors Q_V and Q_1 have no common component.

Proof. Assume by contradiction that there exists a component $\Gamma \in |aB + bR|$ such that $\Gamma < Q_V$ and $\Gamma < Q_1$ where $g > 2k+2$ and $k > 1$.

We observe that Γ can not be a bisecant divisor on S. Indeed, we first note that we can't have $\Gamma = Q_V$, since $Q_1 \neq Q_V + (g-2k-2)R$ by Corollary 3.9.

Assume now that

$$\Gamma \sim 2B + bR, \quad Q_V = \Gamma + (g-k-b)R, \quad Q_1 = \Gamma + (2g-3k-2-b)R \tag{3.10}$$

for some $b \leq g-k-1$. As R_V is a subscheme of both Q_V and Q_1 and as $R^2 = 0$, it would follow that R_V is necessarily a subscheme of Γ.

On the other hand, we have $Q_{V|C} = B_{|C} + R_V$ by construction, so the relations in (3.10) would imply that the subscheme $(g-k-b)R_{|C}$ is contained in $B_{|C}$, which is a contradiction.

Next we claim that we have the following bounds:

1. if $a = 0$, then $2g-4k-4 \geq b$;
2. if $a = 1$, then $b > k+2$.

Indeed, as R_V is the base locus of $R_V + |B_{|C} + (g-2k-2)L|$, we have in particular $R_V > \Gamma_{|C}$. Consider the sublinear system

$$\Lambda'' := \langle (Q_V - \Gamma) + |(g-2k-2)R|, (Q_1 - \Gamma)\rangle.$$

The subscheme $R_V - \Gamma_{|C}$ of C is contained in the base locus of Λ''. Hence, we have

$$\Gamma \cdot C + (Q_V - \Gamma)^2 \geq 2g + 4.$$

Assume now $a = 0$ and $2g - 4k - 4 \geq b$. In this case $\Gamma \in |bR|$. This would imply that there exists a fiber R of $S \to \mathbb{P}^1$ such that the subscheme R_V of C contains the subscheme $R_{|C}$, that is a divisor in the trigonal series. This is impossible.

Finally, assume $a = 1$ and $b > k + 2$. In this case $Q_V = U + \Gamma$ where $U \in |B + (g - k - b)R|$. Since $b > k + 2$, then $U = B + U'$ where $U' \in |(g - k - b)R|$. This implies that $B < Q_V$. Then $R_V = (Q_V - B)_{|C}$ and this contradicts Lemma 2.4. ∎

Corollary 3.11. The subscheme R_V is a complete intersection of Q_V and Q_1.

As a consequence we have $\mathrm{Bs}(\Lambda) = R_V$.

Proof. We have $(2B + (g - k)R) \cdot (2B + (2g - 3k - 2)R) = 2g + 4$. Since R_V is a subscheme both of Q_V and Q_1 and it is of length $2g + 4$, the claim follows.

Finally, as Λ' can be embedded in Λ by multiplication of a base point free linear system we have the last assertion. ∎

3.3 Proof of Theorem 0.3

In this section we shall treat the remaining cases $k = 1$ and $g = 6, 7$.

Proposition 3.12. If $g = 7$ and $k = 1$ then the base locus of Λ satisfies $\mathrm{Bs}(\Lambda) = \Gamma \sim 2B + 6R$.

Proof. In this case we have $S \cong \mathbb{F}_3$, $H \sim B + 4R$ and $C \sim 3B + 9R = 3A$, where $A \sim B + 3R$ denotes the tautological divisor of $\mathbb{F}_3$.

Note that $\deg R_V = 2g + 4 = 18 = C \cdot 2A$. Since $2A_{|C} \equiv K_C + 2L$, and $H^1(S, \mathcal{O}_S(-A)) = 0$, by computing the cohomology of $0 \to \mathcal{O}_S(-A) \to \mathcal{O}_S(2A) \to \mathcal{O}_C(K_C + 2L) \to 0$ it easily follows that it exists a unique $\Gamma \in |2A|$ such that $\Gamma_{|C} = R_V$. ∎

We can describe very explicitly the linear subspace $T \subset H^1(C, T_C)$. Let $\langle t_0, t_1 \rangle = H^0(S, \mathcal{O}_S(R))$ be a basis and let $X_1 \in H^0(S, \mathcal{O}_S(A))$ be an irreducible section. If $X_\infty \in H^0(S, \mathcal{O}_S(B))$, then any trigonal curve C can be written with an equation of the type

$$X_1^3 + \alpha_3(t_0, t_1) X_1^2 X_\infty + \alpha_6(t_0, t_1) X_1 X_\infty^2 + \alpha_9(t_0, t_1) X_\infty^3 = 0, \tag{3.11}$$

where $\alpha_j(t_0, t_1) \in \mathbb{C}[t_0, t_1]_{[j]}$ are general homogeneous polynomials of degree $j = 3, 6, 9$, so that C is smooth. A simple computation shows that the equation of Γ is given by

$$3X_1^2 + 2\alpha_3(t_0, t_1) X_1 X_\infty + \alpha_6(t_0, t_1) X_\infty^2 = 0. \tag{3.12}$$

Remark 3.13. By Proposition 3.12 we can write $T = \langle I_2, \mathbb{C}[t_0, t_1]_{[2]} \cdot \Gamma \rangle^{\perp}$.

Proposition 3.14. If $g = 6$ and $k = 1$ then we have Bs $\Lambda = \Gamma \sim 2B + 5R$.

Proof. In this case we have $S \cong \mathbb{F}_2$, $H \sim B + 3R$, $C \sim 3B + 7R$. This case differs from the analogue case where $g = 7$ because all quadrics vanishing on R_V actually vanish also on a scheme of length 17 on C; note that $16 = \deg R_V$. We observe that the point $B \cap C = \{p\}$ is a subscheme of $\Gamma_{|C} = p + R_V$. So we can conclude in a similar way as in Proposition 3.12. ∎

Remark 3.15. Note that if $g = 6$ and $k = 1$, by Proposition 3.14 we can write $T = \langle I_2, \mathbb{C}[t_0, t_1]_{[1]} \cdot \Gamma \rangle^{\perp}$.

4 Hyperelliptic Families and Trigonal Deformations

Let $\mathcal{Y} \hookrightarrow \mathcal{M}_g$ be a closed irreducible subvariety where $g \geq 5$ and $\dim \mathcal{Y} = 2g - 1$. Assume that for a very general $[C] \in \mathcal{Y}$ there exists a dominant morphism $J(D) \twoheadrightarrow J(C)$ where $[D]$ belongs to the hyperelliptic locus $\mathcal{M}^1_{g',2}$ for some $g' \geq g$. By standard arguments we can assume the existence of a family of surjective maps of Jacobians:

$$\begin{array}{ccc} \mathcal{JD} & \xrightarrow{\;f\;} & \mathcal{JC} \\ & {\scriptstyle \rho}\searrow \quad \swarrow{\scriptstyle \rho'} & \\ & \mathcal{U} & \end{array} \tag{4.1}$$

such that the moduli map $\Phi \colon \mathcal{U} \to \mathcal{M}_g$ induces a generically finite dominant map $\mathcal{U} \to \mathcal{Y}$. Moreover, we can also assume that $f_u \colon J(D_u) \twoheadrightarrow J(C_u)$ and $[D_u] \in \mathcal{M}^1_{g',2}$, for every $u \in \mathcal{U}$. Let $W_u < H^0(D_u, \omega_{D_u})$ be the isomorphic image of $H^0(C_u, \omega_{C_u})$ via the codifferential of f_u. In [10, Proof of Theorem 1.6, p. 899] they show that if C_u is not hyperelliptic there exists a rational dominant map $D_u \dashrightarrow Z \subseteq |W_u|^* = \mathbb{P}(H^1(C_u, \mathcal{O}_{C_u}))$ *where Z is a curve contained in the locus of rank* 1 *trigonal deformations of C_u.*

Proposition 4.1. If $g \geq 8$ or $g = 6, 7$ and $k > 1$ then $\mathcal{M}^1_{g,2}$ is the unique closed irreducible subvariety $\mathcal{Y} \subset \mathcal{M}_g$ of dimension $2g - 1$ such that for its generic element $[C] \in \mathcal{Y}$ there exists $[D] \in \mathcal{M}^1_{g',2}$ such that $J(D) \twoheadrightarrow J(C)$.

Proof. Assume that the general $[C] \in \mathcal{Y}$ is not hyperelliptic. By [10, Theorem 1.6] it follows that C is trigonal. In particular the curve Z recalled above is a curve contained inside the fix part of Λ. This contradicts Corollary 3.11. ■

4.1 Rational curves in the locus of rank-1 trigonal deformations

If $g = 6, 7$ and $k = 1$ there can exist rational curves in the locus of rank 1 trigonal deformations, but we claim that they cannot be non-degenerate. By [10, Theorem 1.6] and by Proposition 4.1, to show our claim, we have to study the rational curves inside the schematic intersection $\Gamma^1 := \Gamma \cap \mathbb{P}(T^1_C)$. Note that Γ^1 is always a proper subscheme of Γ.

Proposition 4.2. If $g = 7$, $k = 1$, and C is generic then Γ is smooth, irreducible and Γ^1 is a finite scheme. Moreover, if Γ is reducible then Γ^1 does not contain any non-degenerate curve.

Proof. By Proposition 3.12 and the explicit equations (3.11) and (3.12) the first claim follows. Assume now that C is smooth but non-generic and that Γ is a union of at least two components. By Lemma 2.4, Γ cannot contain B as one of its components. Therefore $\Gamma = D_1 + D_2$ where $D_1, D_2 \in |A|$ since $R_{|C}$ is not a subdivisor of R_V. This implies that both components of Γ are degenerate curves for the embedding $\phi_{|H|}\colon S \to \mathbb{P}^6$. In particular $\Gamma^1 = \mathbb{P}(T^k_C)_{|\Gamma}$ does not contain non-degenerate rational curves. ■

Proposition 4.3. If $g = 6$, $k = 1$ then either Γ is irreducible or if it is reducible it does not contain any non-degenerate rational curve. In particular Γ^1 does not contain any non-degenerate rational curve.

Proof. The proof is similar to the one of Proposition 4.2 by using Proposition 3.14. ■

4.2 The proof of the main Theorem 0.1

By Proposition 4.1 we have to consider only the cases where $k = 1$, $g = 6, 7$. We consider the diagram (4.1). Note that Z is a non-degenerate curve since it is obtained by projection of the canonical image of D, which is a rational normal curve since D is hyperelliptic. On the other hand we also have $Z \hookrightarrow \Gamma^1$. By Proposition 4.2 and by Proposition 4.3 we see that such a non-degenerate curve Z cannot exist.

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