Quasi-inversion of qubit channels

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(Received 27 September 2019; accepted 18 February 2020; published 16 March 2020)

In general quantum operations, or quantum channels cannot be inverted by physical operations, i.e., by completely positive trace-preserving maps. An arbitrary state passing through a quantum channel loses its fidelity with the input. Given a quantum channel $\mathcal{E}$, we discuss the concept of its quasi-inverse as a completely positive trace-preserving map $\mathcal{E}^{qi}$ which when composed with $\mathcal{E}$ increases its average input-output fidelity in an optimal way. The channel $\mathcal{E}^{qi}$ comes as close as possible to the inverse of a quantum channel. We give a complete classification of such maps for qubit channels and provide quite a few illustrative examples.

DOI: 10.1103/PhysRevA.101.032109

I. INTRODUCTION

Unitary dynamics of quantum systems is an idealization which almost never occur in reality. There are always inevitable and unknown couplings with the environment which destroy the coherence and purity of a quantum state and hence the information encoded into a quantum system. One of the central results in quantum theory is that a general nonunitary dynamics of an open quantum system can be characterized by operators acting entirely within the quantum system [1,2]. This general dynamics is aptly called a quantum channel to signify the passage of quantum states (i.e., photons) through noisy environment (optical fibers or free air).

The most important goal of quantum communication is to combat this quantum noise which has led to whole subfields in quantum information science, like quantum error correction [3], decoherence free subspaces [4–6], and pre- and postprocessing [7–10] by weak measurements [11,12]. A quantum channel being completely specified by operators inside a system, raises the natural and highly important question if it can be inverted by some other set of operators, that is, if we can invert a quantum channel and retrieve the input state in the same way that we do for unitary dynamics. We stress that by the inverse map in this paper, we mean a physically implementable map, that is, a completely positive trace-preserving map. If this inversion is possible, it can simply replace or at least complement other techniques for quantum state protection. It is, however, well known that quantum channels cannot be inverted unless they are simple unitary channels of the form $\rho \longrightarrow U\rho U^\dagger$.

In this paper we ask to what extent we can come close to a complete inversion and introduce the concept of quasi-inversion of a quantum channel. We formulate this question in a precise form, based on the notion of average fidelity of a channel. Given a completely positive trace-preserving (CPT) map or quantum channel $\mathcal{E}$, its overall performance can be measured through the average input-output fidelity,

$$\overline{F}(\mathcal{E}) := \int d\phi \, \langle\phi|\mathcal{E}(|\phi\rangle\langle\phi|)|\phi\rangle, \qquad (1)$$

where the integral is taken over all input states. The measure of the integral is taken to be unitary invariant, i.e., $d\psi = d\phi$ for $|\psi\rangle = U|\phi\rangle$, and normalized to $\int d\phi = 1$. We now ask if it is possible to perform a quantum operation at the output, which increases this average fidelity independently of the input state and in an optimal way.

Definition 1. Let $\mathcal{E} : \rho \longrightarrow \mathcal{E}(\rho)$ be a channel. Its quasi-inverse, denoted by $\mathcal{E}^{qi}$, is any channel fulfilling the following condition:

$$\overline{F}(\mathcal{E}^{qi} \circ \mathcal{E}) \geqslant \overline{F}(\mathcal{E}' \circ \mathcal{E}) \quad \forall \ \mathcal{E}'. \qquad (2)$$

for any channel $\mathcal{E}'$. Obviously for the special case of $\mathcal{E}'$ being the identity channel, this condition implies that $\overline{F}(\mathcal{E}^{qi} \circ \mathcal{E}) \geqslant \overline{F}(\mathcal{E})$, hence the quasi-inverse increases the fidelity of the channel. Furthermore Eq. (2) implies that the quasi-inverse is the optimal channel which does this.

In this paper we will restrict our study to qubit channels which will be shown to have already a rather rich structure. We will prove that the quasi-inverse of a qubit channel can always be taken to be a unitary map $\mathcal{E}^{qi}(\rho) = V\rho V^\dagger$, and that it is both a left and a right quasi-inverse. We then show how it can be determined explicitly and illustrate the method by several classes of examples. Finally we note that our approach to channel quasi-inversion differs from the ones based on “Petz recovery map” [13–16] and allows for concrete applications and characterization, as we shall see below.

The case of qubit channels is of interest for at least two reasons. The first one is of practical importance since qubits are the main carriers of quantum information (along with continuous variable quantum states) and much is known about qubit channels which model various kinds of noises and physical errors affecting the transmission of quantum states. The second reason concerns the almost complete classification of qubit channels through many works in the literature [17–19]. While a comparable knowledge is absent for higher dimensional channels, the results we present will certainly be useful to extend them to multiqubit channels.

The structure of the paper is as follows: In Sec. II we gather the definitions and all the technical tools and in Sec. III we prove that the quasi-inverse of an arbitrary qubit is a unitary channel, basing upon a unique property of qubit channels and prove some general properties of the quasi-inverse. In Sec. IV we show how the explicit form of the quasi-inverse of a given channel can be found, and in Sec. V we study several important examples, including the Pauli channel and the amplitude damping channel. Interestingly, we show that the important characteristic of unitality of a channel plays no role in fixing its quasi-inverse and that it is instead the symmetry or nonsymmetry of the affine matrix corresponding to the channel which plays an important role. Finally in Sec. VI we give a geometric picture of the quasi-inverse and discuss the uniqueness of the quasi-inverse which turns out to be unique except for a set of measure zero in the space of all qubit channels. We conclude the paper with a discussion in which we elaborate one related work present in the literature.

II. PRELIMINARIES

In this section we review the basic knowledge on the structure of qubit channels [17,18]. A quantum channel $\mathcal{E}$ [a completely positive trace-preserving (CPT) map] can be characterized by its Kraus representation,

$$\rho \longrightarrow \mathcal{E}(\rho) = \sum_i K_i \rho K_i^\dagger, \tag{3}$$

subject to the condition $\sum_i K_i^\dagger K_i = I$. Such a map is unital, $(\mathcal{E}(\mathbb{1}) = \mathbb{1})$ if $\sum_i K_i K_i^\dagger = I$. For qubits, where $\rho = \frac{1}{2}(\mathbb{1} + \boldsymbol{r}\cdot\boldsymbol{\sigma})$, this map induces an affine map on the Bloch sphere in the form,

$$\boldsymbol{r} \longrightarrow \boldsymbol{r}' = M\boldsymbol{r} + \boldsymbol{t}. \tag{4}$$

Here $M = [M_{\alpha\beta}]$ is a real 3×3 matrix and $\boldsymbol{t}$ a vector in $\mathbb{R}^3$ with components $(\alpha,\ \beta = 1, 2, 3)$:

$$M_{\alpha\beta} = \tfrac{1}{2}\mathrm{Tr}(\sigma_\alpha\, \mathcal{E}(\sigma_\beta)), \quad t_\alpha = \tfrac{1}{2}\mathrm{Tr}(\sigma_\alpha\, \mathcal{E}(\mathbb{1})), \tag{5}$$

where $\sigma_\alpha,\ \alpha \in \{1, 2, 3\}$ are the standard Pauli matrices.

Remark 1. Throughout the paper we use the letter $i \in \{1, 2, \cdots\})$ for indexing the Kraus operators and $\alpha, \beta, \gamma \in \{1, 2, 3\}$ for indexing the components of three-dimensional vectors and Pauli matrices.

For unital channels, one has obviously $\boldsymbol{t} = 0$. Complete positivity of the map, puts stringent requirements on the parameters of M and $\boldsymbol{t}$. Therefore while a quantum channel can be characterized by the pair $(M, \boldsymbol{t})$, not every affine map corresponds to a quantum channel. Moreover from (4) one easily finds how composition of quantum channels is reflected in the composition of their corresponding affine maps:

$$\mathcal{E}' \equiv (M', \boldsymbol{t}'), \quad \mathcal{E} \equiv (M, \boldsymbol{t}) \longrightarrow \mathcal{E}' \circ \mathcal{E} \equiv (M'M, M'\boldsymbol{t} + \boldsymbol{t}'). \tag{6}$$

The conditions on the affine map imposed by completely positivity of the qubit channel $\mathcal{E}$ are obtained by first proving that $\mathcal{E}$ can be decomposed in the canonical form [17,18]:

$$\mathcal{E} = \mathcal{U} \circ \mathcal{E}_c \circ \mathcal{V}, \tag{7}$$

or

$$\mathcal{E}(\rho) = U\mathcal{E}_c(V\rho V^{-1})U^{-1}, \tag{8}$$

where U and V are unitary matrices, and $\mathcal{E}_c$ is a channel with a diagonal M matrix denoted as $\Lambda_c = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$. Correspondingly, the M matrix of $\mathcal{E}$ can be rewritten as $M = R_U \Lambda_c R_V$, where R_U and R_V are SO(3) representations of U and V. The parameters λ_1, λ_2, and λ_3 are real and satisfy $|\lambda_i| \leqslant 1\ \forall\ i$ and $(1 \pm \lambda_3)^2 \geqslant (\lambda_1 \pm \lambda_2)^2$ which constrain the vector $(\lambda_1, \lambda_2, \lambda_3)$ to lie inside a tetrahedron [17,18]. For unital channels (when $\boldsymbol{t} = 0$), these are necessary and sufficient conditions, but for general channels (when $\boldsymbol{t} \neq 0$), these are only necessary conditions which should be supplemented by other inequalities involving $\boldsymbol{t}$ [17].

Finally let us express the average fidelity of the channel in terms of its affine map. A pure state $|\phi\rangle\langle\phi| = \frac{1}{2}(\mathbb{1} + \boldsymbol{n}\cdot\boldsymbol{\sigma})$ where $\boldsymbol{n}$ is a unit vector on the surface of the Bloch sphere is transformed to $\rho := \mathcal{E}(|\phi\rangle\langle\phi|) = \frac{1}{2}(\mathbb{1} + (M\boldsymbol{n} + \boldsymbol{t})\cdot\boldsymbol{\sigma})$. Using the equality $\mathrm{Tr}(\sigma_\alpha\sigma_\beta) = 2\delta_{\alpha\beta}$, the fidelity of input and output state is found to be

$$\langle\phi|\rho|\phi\rangle = \mathrm{Tr}(|\phi\rangle\langle\phi|\rho) = \tfrac{1}{2}[1 + \boldsymbol{n}\cdot(M\boldsymbol{n} + \boldsymbol{t})]. \tag{9}$$

Averaging this fidelity over the surface of Bloch sphere and using

$$\int d\boldsymbol{n}\, \boldsymbol{n} = 0, \quad \int d\boldsymbol{n}\, n_\alpha\, n_\beta = \frac{1}{3}\delta_{\alpha\beta}, \tag{10}$$

where $d\boldsymbol{n}$ is the integration measure on the surface of the Bloch sphere, we find the average fidelity of the qubit channel,

$$\overline{F}(\mathcal{E}) = \tfrac{1}{2}\big(1 + \tfrac{1}{3}\,\mathrm{Tr}(M)\big). \tag{11}$$

It is desirable to express the average fidelity directly in terms of the Kraus operators of the channel. Let the Kraus operators be given by $K_i = a_i + \mathbf{b}_i\cdot\boldsymbol{\sigma}$, where a_i and $\mathbf{b}_i$ are in general complex numbers and vectors, respectively. The trace-preserving condition $\sum_i K_i^\dagger K_i = \mathbb{1}$ imposes the constraints,

$$\langle a^* a\rangle + \langle\boldsymbol{b}^*\cdot\boldsymbol{b}\rangle = 1, \quad \langle a\boldsymbol{b}^*\rangle + \langle a^*\boldsymbol{b}\rangle + i\langle\boldsymbol{b}^*\times\boldsymbol{b}\rangle = 0, \tag{12}$$

where we have introduced the shorthand notations $\langle c\rangle = \sum_i c_i$, $\langle\boldsymbol{d}\rangle = \sum_i \boldsymbol{d}_i$.

To express the average fidelity in terms of Kraus operators, we note from (1), that

$$\overline{F}(\mathcal{E}) = \sum_i \int d\phi |\langle\phi|K_i|\phi\rangle|^2 \tag{13}$$

and use

$$\langle\phi|K_i|\phi\rangle = \tfrac{1}{2}\mathrm{Tr}[K_i\,(\mathbb{1} + \boldsymbol{n}\cdot\boldsymbol{\sigma})] = a_i + \boldsymbol{b}_i\cdot\boldsymbol{n}, \tag{14}$$

where again we have used $|\phi\rangle\langle\phi| = \frac{1}{2}(\mathbb{1} + \boldsymbol{n}\cdot\boldsymbol{\sigma})$. Using (10) this leads to

$$\overline{F}(\mathcal{E}) = \langle a^* a\rangle + \tfrac{1}{3}\langle\boldsymbol{b}\cdot\boldsymbol{b}^*\rangle, \tag{15}$$

which of the trace-preserving property (12) can also be written as

$$\overline{F}(\mathcal{E}) = 1 - \tfrac{2}{3}\langle\boldsymbol{b}\cdot\boldsymbol{b}^*\rangle = \tfrac{1}{3}(1 + 2\langle a^* a\rangle). \tag{16}$$

We will see in the following that the matrix $B = [B_{\alpha\beta}]$, with

$$B_{\alpha\beta} = \tfrac{1}{2}\langle b_\alpha b_\beta^* + b_\alpha^* b_\beta\rangle, \tag{17}$$

plays a central role in determining the quasi-inverse of a channel. In terms of this matrix the average fidelity reads

$$\overline{F}(\mathcal{E}) = 1 - \tfrac{2}{3}\,\mathrm{Tr}(B). \tag{18}$$

Finally the connection between the Kraus representation and the affine map is obtained through Eq. (5) which gives

$$\boldsymbol{t} = \langle a^*\boldsymbol{b} + a\boldsymbol{b}^* + i\boldsymbol{b}\times\boldsymbol{b}^*\rangle, \tag{19}$$

and $M = S + A$, where the real symmetric matrix $S = [S_{\alpha\beta}]$ is given by

$$S_{\alpha\beta} = (1 - 2\langle\boldsymbol{b}\cdot\boldsymbol{b}^*\rangle)\delta_{\alpha\beta} + \langle\boldsymbol{b}_\alpha\boldsymbol{b}^*_\beta + \boldsymbol{b}^*_\alpha\boldsymbol{b}_\beta\rangle, \tag{20}$$

and the real antisymmetric matrix $A = [A_{\alpha\beta}]$ by

$$A_{\alpha\beta} = -\sum_{\gamma=1}^{3}\epsilon_{\alpha\beta\gamma}\boldsymbol{v}_\gamma, \quad \boldsymbol{v} = i\langle a^*\boldsymbol{b} - a\boldsymbol{b}^*\rangle, \tag{21}$$

where $\epsilon_{\alpha,\beta,\gamma}$ is the Levi-Civita symbol. In the sequel, we sometimes denote a quantum channel $\mathcal{E}$ with affine map pair $(M,\boldsymbol{t})$ as $\mathcal{E}_{M,t}$ or simply by the pair $(M,\boldsymbol{t})$ itself. It should also be noted that while it is straightforward to obtain the affine map from its Kraus representation, the converse is not easy at all.

Note in passing that from (20) and (17) $\mathrm{Tr}(M)=3-4\,\mathrm{Tr}(B)$ which implies the equality of the two expressions (11) and (18) for the average fidelity. Also note that when M is symmetric, we can write

$$B = \tfrac{1}{4}[2M + \mathbb{1} - \mathrm{Tr}(M)]. \tag{22}$$

This relation will be important when we discuss the quasi-inverse of qubit channels with symmetric affine maps.

III. THE QUASI-INVERSE AND ITS GENERAL PROPERTIES

We now state one of the main results of this paper.

Theorem 1. The quasi-inverse of any qubit channel can always be taken to be a unitary map.

Proof. First consider the definition of average fidelity. An important property of this quantity is its linearity, that follows from its definition in Eq. (1), whence

$$\overline{F}\left(\sum_i \lambda_i\mathcal{E}_i\right) = \sum_i \lambda_i\overline{F}(\mathcal{E}_i), \tag{23}$$

where $\sum_i \lambda_i = 1$ and $\lambda_i \geqslant 0\,,\ \forall\lambda_i$.

We now use a theorem of [19] according to which a necessary and sufficient condition for a qubit channel to be a random unitary channel, namely a convex combination of unitaries, is that it should be unital. Note that this theorem is not true for higher dimensions and holds only for qubit channels. Suppose now that the quasi-inverse $\mathcal{E}^{qi}$ is unital. This means that $\mathcal{E}^{qi} = \sum_i p_i\,\mathcal{U}_i$, where $\mathcal{U}_i(\rho) = U_i\rho\,U_i^\dagger$ is a unitary map, and

$$\overline{F}(\mathcal{E}^{qi}\circ\mathcal{E}) \geqslant \overline{F}(\mathcal{E}). \tag{24}$$

Therefore we have

$$\overline{F}\left[\sum_i p_i\,\mathcal{U}_i\circ\mathcal{E}\right] \geqslant \overline{F}(\mathcal{E}). \tag{25}$$

Let $\mathcal{U}_{\max}$ be the unitary map which has the highest contribution on the left-hand side. Then it is obvious that if we replace all the random unitaries on the left-hand side by $\mathcal{U}_{\max}$, we get an even higher average fidelity:

$$\overline{F}[\mathcal{U}_{\max}\circ\mathcal{E}] \geqslant \overline{F}\left[\sum_i p_i\,\mathcal{U}_i\circ\mathcal{E}\right] = \overline{F}[\mathcal{E}^{qi}\circ\mathcal{E}] \geqslant \overline{F}(\mathcal{E}). \tag{26}$$

Therefore for any qubit channel whose quasi-inverse is unital, we can always take the quasi-inverse to be a simple unitary.

It now remains to see under what circumstances the quasi-inverse is unital. To solve this problem, it is useful to work with the channel representation in terms of affine maps. Let Δ be the admissible domain of the parameters of the affine map defined by the pair $(M,\boldsymbol{t})$ and let $\mathcal{E}_{M,t}$ be the corresponding channel. Assume that $\mathcal{E}^{qi}_{N_0,\boldsymbol{t}_0}$ is its quasi-inverse; according to Definition 1, this implies that

$$\overline{F}\big(\mathcal{E}^{qi}_{N_0,\boldsymbol{t}_0}\circ\mathcal{E}_{M,t}\big) \geqslant \overline{F}(\mathcal{E}_{M,t}), \tag{27}$$

and that for any other channel $\mathcal{E}_{N',t'}$, $(N',\boldsymbol{t}')\in\Delta$, one has

$$\overline{F}\big(\mathcal{E}^{qi}_{N_0,\boldsymbol{t}_0}\circ\mathcal{E}_{M,t}\big) \geqslant \overline{F}(\mathcal{E}_{N',t'}\circ\mathcal{E}_{M,t}). \tag{28}$$

In view of Eq. (11) these two conditions are equivalent to

$$\mathrm{Tr}(N_0\,M) \geqslant \mathrm{Tr}(M), \tag{29}$$

and

$$\mathrm{Tr}(N_0M) \geqslant \mathrm{Tr}(N'\,M). \tag{30}$$

Note that although $\boldsymbol{t}_0$ does not appear on the right-hand side of this inequality, it affects the allowable range of N_0. However, if $\mathcal{E}^{qi}_{N_0,\boldsymbol{t}_0}$ is a CPT map, then $\mathcal{E}^{qi}_{N_0,0}$ is also a CPT map (the converse is not true, since the inclusion of the parameters $\boldsymbol{t}$ restricts the allowable range of parameters of M [17]). Therefore the average fidelity of the map $\mathcal{E}_{N_0,0}\circ\mathcal{E}_{M,t}$ is the same as the average fidelity of the map $\mathcal{E}_{N_0,\boldsymbol{t}_0}\circ\mathcal{E}_{M,t}$ and both are determined by $\mathrm{Tr}(N_0\,M)$. Thus, the two conditions (27) and (28) can be rewritten as

$$\overline{F}\big(\mathcal{E}^{qi}_{N_0,0}\circ\mathcal{E}_{M,t}\big) \geqslant \overline{F}(\mathcal{E}_{M,t}), \tag{31}$$

and

$$\overline{F}\big(\mathcal{E}^{qi}_{N_0,0}\circ\mathcal{E}_{M,t}\big) \geqslant \overline{F}(\mathcal{E}_{N',t'}\circ\mathcal{E}_{M,t}), \quad \forall(N',\boldsymbol{t}')\in\Delta. \tag{32}$$

Therefore if the channel $(N_0,\boldsymbol{t}_0)$ is the quasi-inverse for the channel $(M,\boldsymbol{t})$, then the unital channel $(N_0,0)$ is also the quasi-inverse for that channel with the same improvement of fidelity. This means that we can always take the quasi-inverse of a qubit channel to be unital and hence unitary according to the first part of the proof. ■

Given the canonical decomposition (8), one may be tempted to relate the quasi-inverses of $\mathcal{E}$ and $\mathcal{E}_c$. Theorem 2 below and the subsequent remark elaborate this point. We first need a lemma.

Lemma 1. Let $\mathcal{E}_2$ and $\mathcal{E}_1$ be related as $\mathcal{E}_2 = \mathcal{U}\circ\mathcal{E}_1\circ\mathcal{U}^{-1}$, i.e.,

$$\mathcal{E}_2(\rho) = U\mathcal{E}_1(U^{-1}\rho\,U)U^{-1}. \tag{33}$$

Then

$$\overline{F}(\mathcal{E}_2) = \overline{F}(\mathcal{E}_1). \tag{34}$$

Proof. The proof is straightforward once we use the definition of the average fidelity, make a change of variable $U|\phi\rangle \longrightarrow |\psi\rangle$, and use the invariance of the integration measure $d\phi = d\psi$. ■

Theorem 2. The quasi-inverse of the map $\mathcal{E} = \mathcal{U}\circ\mathcal{E}_c\circ\mathcal{U}^{-1}$ is given by $\mathcal{E}^{qi} = \mathcal{U}\circ\mathcal{E}_c^{qi}\circ\mathcal{U}^{-1}$.

Proof. From the above lemma, it immediately follows that if

$$\mathcal{E} = \mathcal{U}\circ\mathcal{E}_c\circ\mathcal{U}^{-1},$$

then $\overline{F}(\mathcal{E}) = \overline{F}(\mathcal{E}_c)$. We now note that the definition of quasi-inverse for the channel $\mathcal{E}_c$ implies

$$\overline{F}\big(\mathcal{E}_c^{qi}\circ\mathcal{E}_c\big) \geqslant \overline{F}_{\mathcal{E}_c}, \tag{35}$$

and for all other channels $\mathcal{E}'$,

$$\overline{F}\big(\mathcal{E}_c^{qi}\circ\mathcal{E}_c\big) \geqslant \overline{F}(\mathcal{E}'\circ\mathcal{E}_c). \tag{36}$$

Define

$$\mathcal{E}^{qi} := \mathcal{U}\circ\mathcal{E}_c^{qi}\circ\mathcal{U}^{-1}. \tag{37}$$

Then one finds

$$\begin{aligned}\mathcal{E}^{qi}\circ\mathcal{E} &= \big(\mathcal{U}\circ\mathcal{E}_c^{qi}\circ\mathcal{U}^{-1}\big)\circ(\mathcal{U}\circ\mathcal{E}_c\circ\mathcal{U}^{-1})\\ &= \mathcal{U}\circ\big(\mathcal{E}_c^{qi}\circ\mathcal{E}_c\big)\circ\mathcal{U}^{-1},\end{aligned} \tag{38}$$

and using Lemma 1 once more, one obtains

$$\overline{F}(\mathcal{E}^{qi}\circ\mathcal{E}) = \overline{F}\big(\mathcal{E}_c^{qi}\circ\mathcal{E}_c\big) \geqslant \overline{F}(\mathcal{E}_c) = \overline{F}(\mathcal{E}). \tag{39}$$

This proves that $\mathcal{E}^{qi}$ as in (37) increases the average fidelity of $\mathcal{E}$. Now let $\mathcal{E}'$ be any other channel. We have

$$\begin{aligned}\overline{F}(\mathcal{E}'\circ\mathcal{E}) &= \overline{F}(\mathcal{E}'\circ\mathcal{U}\circ\mathcal{E}_c\circ\mathcal{U}^{-1})\\ &= \overline{F}(\mathcal{U}\circ[\mathcal{U}^{-1}\circ\mathcal{E}'\circ\mathcal{U}\circ\mathcal{E}_c]\circ\mathcal{U}^{-1}),\end{aligned} \tag{40}$$

and using again Lemma 1, we find

$$\overline{F}(\mathcal{E}'\circ\mathcal{E}) = \overline{F}([\mathcal{U}^{-1}\circ\mathcal{E}'\circ\mathcal{U}\circ\mathcal{E}_c]) = \overline{F}(\mathcal{E}''\circ\mathcal{E}_c), \tag{41}$$

where $\mathcal{E}'' := \mathcal{U}^{-1}\circ\mathcal{E}'\circ\mathcal{U}$. Using Eq. (36), we have

$$\overline{F}(\mathcal{E}'\circ\mathcal{E}) \leqslant \overline{F}\big(\mathcal{E}_c^{qi}\circ\mathcal{E}_c\big) = \overline{F}(\mathcal{E}^{qi}\circ\mathcal{E}), \tag{42}$$

where (38) has also been used. ■

Remark 2. We should stress that for general channels of the form $\mathcal{E} = \mathcal{U}\circ\mathcal{E}_c\circ\mathcal{V}$, one cannot simply write the quasi-inverse as $\mathcal{E}^{qi} = \mathcal{V}^{-1}\circ\mathcal{E}_c^{qi}\circ\mathcal{U}^{-1}$. It is true that

$$\mathcal{E}^{qi}\circ\mathcal{E} = \mathcal{V}^{-1}\circ\mathcal{E}_c^{qi}\circ\mathcal{E}_c\circ\mathcal{V}, \tag{43}$$

and hence according to Lemma 1 and Eq. (35),

$$\overline{F}(\mathcal{E}^{qi}\circ\mathcal{E}) = \overline{F}\big(\mathcal{E}_c^{qi}\circ\mathcal{E}_c\big) \geqslant \overline{F}(\mathcal{E}_c), \tag{44}$$

where in the inequality we have used the definition of quasi-inverse of the channel $\mathcal{E}_c$. However, we can no longer use the equality of $\overline{F}(\mathcal{E}_c)$ and $\overline{F}(\mathcal{E})$, since this equality is not valid when $\mathcal{U}$ and $\mathcal{V}^{-1}$ in the canonical decomposition of the channel are different.

IV. EXPLICIT FORM OF THE QUASI-INVERSE

To find the explicit form of this quasi-inverse, let the quasi-inverse be $\mathcal{E}^{qi}(\rho) = V\rho V^\dagger$. The average fidelity of the combined channel,

$$\mathcal{E}^{qi}\circ\mathcal{E} = \sum_i (VK_i)\rho\,(VK_i)^\dagger,$$

can be obtained from (16). We simply need to determine the scalar coefficients of the new Kraus operators $VK_i = a_i'\mathbb{1} + \boldsymbol{b}_i'\cdot\sigma$.

The unitary matrix V can be taken of the form $V = x_0\mathbb{1} + i\boldsymbol{x}\cdot\boldsymbol{\sigma}$,with x_0 and $\mathbf{x}$ real and $x_0^2 + \mathbf{x}\cdot\mathbf{x} = 1$, the possible presence of an overall phase in V is here not playing any role. Therefore we find

$$VK_i = (x_0\mathbb{1} + i\boldsymbol{x}\cdot\boldsymbol{\sigma})(a_i\mathbb{1} + \boldsymbol{b}_i\cdot\boldsymbol{\sigma}) = a_i'\mathbb{1} + \boldsymbol{b}'_i\cdot\boldsymbol{\sigma}, \tag{45}$$

where $a_i' = x_0 a_i + i\boldsymbol{x}\cdot\boldsymbol{b}_i$. Using (16), the fidelity of the combined channel is $\overline{F} = \frac{1}{3}(1 + 2\langle a'^* a'\rangle)$ which can be rewritten as

$$\begin{aligned}\overline{F}(\mathcal{E}^{qi}\circ\mathcal{E}) = 1 - \frac{2}{3}\,\mathrm{Tr}(B) + \frac{2}{3}\boldsymbol{x}^T\cdot\widehat{B}\cdot\boldsymbol{x}\\ + \frac{2i}{3}\,x_0\,\langle a^*\boldsymbol{b} - a\boldsymbol{b}^*\rangle\cdot\boldsymbol{x},\end{aligned} \tag{46}$$

where

$$\widehat{B} \equiv B - \mathbb{1} + \mathrm{Tr}(B). \tag{47}$$

By combining (47) and (22) we find $\widehat{B} = \frac{1}{2}(M - \mathrm{Tr}(M))$. Note that setting $x_0 = 1$ and $\boldsymbol{x} = 0$ ($V = \mathbb{1}$), one gets back the fidelity of the original channel. Recalling the definition of the vector $\boldsymbol{v}$ in (21) and also the average fidelity of the original channel in (18), the increase of average fidelity $\Delta\overline{F}(\mathcal{E}) \equiv \overline{F}(\mathcal{E}^{qi}\circ\mathcal{E}) - \overline{F}(\mathcal{E})$ can then be written as

$$\Delta\overline{F}(\mathcal{E}) = \tfrac{2}{3}(\boldsymbol{x}^T\cdot\widehat{B}\cdot\boldsymbol{x} + x_0\,\boldsymbol{v}\cdot\boldsymbol{x}). \tag{48}$$

Maximizing its value over all unitary maps, i.e., maximizing over the real parameters $(x_0, \boldsymbol{x})$, subject to the constraint $x_0^2 + \boldsymbol{x}\cdot\boldsymbol{x} = 1$, determines the quasi-inverse of the channel. It is convenient to rewrite the right-hand side of (48) in quadratic form:

$$\Delta\overline{F}(\mathcal{E}) = \frac{2}{3}(x_0\ \boldsymbol{x}^T)Q\begin{pmatrix}x_0\\ \boldsymbol{x}\end{pmatrix}, \tag{49}$$

where

$$Q = \frac{1}{2}\begin{pmatrix}0 & \boldsymbol{v}^T\\ \boldsymbol{v} & 2\widehat{B}\end{pmatrix}; \tag{50}$$

its maximum value is given by

$$\Delta\overline{F}(\mathcal{E}) = \tfrac{2}{3}\mathrm{Max}(\lambda_{\max},\ 0), \tag{51}$$

where $\lambda_{\max}$ is the largest eigenvalue of the 4×4 matrix Q. The normalized eigenstate $(x_0, \boldsymbol{x})^T$ corresponding to this largest eigenvalue will determine the quasi-inverse of $\mathcal{E}$, i.e., the unitary rotation $V = x_0\mathbb{1} + i\boldsymbol{x}\cdot\boldsymbol{\sigma}$, or equivalently $V = e^{i\phi\hat{\boldsymbol{x}}\cdot\boldsymbol{\sigma}}$, with $\cos\phi = x_0$ and $\boldsymbol{x} = \sin\phi\,\hat{\boldsymbol{x}}$, with $\hat{\boldsymbol{x}}$ the unit vector along $\boldsymbol{x}$.

A simple calculation from Eq. (45) shows that the value of a_i' for both VK_i and K_iV are equal. This means that if we had sought a right quasi-inverse, we would have reached the same equations as in (49) and (51). This can also be seen from the affine map picture. Let $\mathcal{E}^{qi}$ and $\mathcal{E}$ induce, respectively, the affine maps $(N, \boldsymbol{t}')$ and $(M, \boldsymbol{t})$. Then

$$\overline{F}(\mathcal{E}^{qi}\circ\mathcal{E}) \equiv \tfrac{1}{2}\big(1 + \tfrac{1}{3}\mathrm{Tr}(NM)\big), \tag{52}$$

which is symmetric with respect to the interchange of the two channels. Therefore the quasi-inverse of a qubit channel is both a right and a left quasi-inverse. We now study further properties of quasi-inverses.

Theorem 3. For all qubit channels $\mathcal{E}$ whose affine matrix is symmetric and positive, the quasi-inverse is the identity map, i.e., their average fidelity cannot be increased.

Proof. A symmetric matrix is diagonalizable. Therefore in a suitable basis it is in the form,

$$M \equiv \Lambda_c = \text{diag}(\lambda_1,\ \lambda_2,\ \lambda_3). \tag{53}$$

In the same basis the matrix $\widehat{B}$ is of the form,

$$\widehat{B} = -\tfrac{1}{2}\text{diag}(\lambda_2+\lambda_3,\ \lambda_1+\lambda_3,\ \lambda_1+\lambda_2), \tag{54}$$

which in view of (51) implies that if all λ_i's are nonnegative, then $\Delta\overline{F}(\mathcal{E}) = 0$. Therefore such a channel has a nontrivial quasi-inverse only if at least one of the eigenvalues of M, i.e., one of λ_i's is negative. ■

V. EXAMPLES

In this section, we present several examples. First it is crucial to note from the relation $M = R_U \Lambda_c R_V$ that the affine matrix of a channel is symmetric if and only if it is of the form $\mathcal{E} = \mathcal{U}\circ\mathcal{E}_c\circ\mathcal{U}^{-1}$. This connection drastically differentiates between the quasi-inverse of qubit channels with the symmetric affine matrix (for which $U = V^{-1}$ in their canonical form) and qubit channels with the nonsymmetric affine matrix (for which $U \neq V^{-1}$). Interestingly, the unitality of the channel does not play any role in this distinction, except for the implicit role that the transition vector $\mathbf{t}$ plays in determining the range of the parameters λ_i [17]. In fact there is a basic difference between channels with symmetric and nonsymmetric affine matrices. In the symmetric case, $\boldsymbol{v} = 0$ and the eigenvectors of the matrix Q in (50) are of the form $(0, \hat{\boldsymbol{x}})^T$ with $\hat{\boldsymbol{x}}$ a unit vector. Therefore the quasi-inverse of such a channel, if different from identity, is an inversion (a π rotation) around some axis, i.e., $\mathcal{E}^{qi}(\rho) = V\rho V^\dagger$, with $V = \hat{\boldsymbol{x}}\cdot\boldsymbol{\sigma}$, and $\hat{\boldsymbol{x}}$ a unit vector. In the nonsymmetric case (21), $\boldsymbol{v} \neq 0$ and the corresponding eigenvector will not necessarily have $x_0 = 0$ and hence the quasi-inverse will be a rotation with a specific angle depending on the channel parameters. Below we will present one example of each kind.

A. The Pauli channel

This is a channel with the symmetric affine matrix.

$$\mathcal{E}(\rho) = p_0\rho + p_1\sigma_x\rho\sigma_x + p_2\sigma_y\rho\sigma_y + p_3\sigma_z\rho\sigma_z, \tag{55}$$

with $p_i \geqslant 0$ and $\sum_{i=0}^3 p_i = 1$. This leads to

$$B = \text{diag}(p_1,\ p_2,\ p_3), \tag{56}$$

and

$$Q = \text{diag}(0, p_1 - p_0, p_2 - p_0, p_3 - p_0). \tag{57}$$

The average fidelity of the channel is given by

$$\overline{F}(\mathcal{E}) \equiv 1 - \tfrac{2}{3}\text{Tr}(B) = \tfrac{1}{3}(1 + 2p_0). \tag{58}$$

After combining with the quasi-inverse, the increase of average fidelity (51) is given by

$$\Delta\overline{F}(\mathcal{E}) = \tfrac{2}{3}\text{Max}(\lambda_{\max}, 0) = \tfrac{2}{3}\text{Max}(p_{\max} - p_0, 0), \tag{59}$$

where $p_{\max}$ is the largest of the probabilities p_1, p_2, and p_3. Therefore if $p_{\max} > p_0$, the fidelity of the channel can be increased from $\frac{1}{3}(1+2p_0)$ to $\frac{1}{3}(1+2p_{\max})$. The quasi-inverse V is now a reflection with respect to the axis corresponding to $p_{\max}$ (i.e., the x axis if p_1 is the largest probability). Moreover, we find that if $p_0 \leqslant \frac{1}{2}$ and $p_{\max} \geqslant \frac{1}{2}$, then $\overline{F}(\mathcal{E}) \leqslant \frac{2}{3}$ and $\overline{F}(\mathcal{E}^{qi}\circ\mathcal{E}) \geqslant \frac{2}{3}$. This means that the quasi-inverse can indeed increase the average fidelity of a noisy channel which is below the value of 2/3 corresponding to that of a "classical" random channel, to above this value. *Note that it is not always the case that the inversion is determined by one of the Kraus operators. An example where this is not the case is given in Sec. V D.*

B. The amplitude damping channel

The amplitude damping channel $\mathcal{E}_{AD}$ is a nonunital characterized by the following Kraus operators,

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & \sqrt{1-\gamma^2} \\ 0 & 0 \end{pmatrix}. \tag{60}$$

The Q matrix is given by

$$Q = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\gamma(\gamma+1) & 0 & 0 \\ 0 & 0 & -\gamma(\gamma+1) & 0 \\ 0 & 0 & 0 & -2\gamma \end{pmatrix}. \tag{61}$$

As seen from above, for this channel, $\boldsymbol{v} = 0$ and hence this is a channel with a symmetric affine matrix. Later on we will consider a slightly twisted version of it which has a nonsymmetric affine matrix. It is readily seen that if $\gamma > 0$, then $\lambda_{\max}$ is negative and hence no increase in average fidelity is possible, i.e., no nontrivial quasi-inverse exists. However, for $\gamma < 0$, the largest eigenvalue is $\lambda_{\max} = -\gamma$ and $\Delta\overline{F} = -\frac{2}{3}\gamma$ implying that the quasi-inverse is $V = \sigma_z$. The fidelity of the channel itself is given from Eq. (18) as $\overline{F}(\mathcal{E}_{\text{AD}}) = \frac{1}{2} + \frac{1}{6}\gamma^2 + \frac{1}{3}\gamma$ and the fidelity of the combined channel is $\overline{F}(\mathcal{E}^{qi}\circ\mathcal{E}_{\text{AD}}) = \frac{1}{2} + \frac{1}{6}\gamma^2 - \frac{1}{3}\gamma$.

Remark 3. If we denote the amplitude damping channel with negative γ by $\mathcal{E}_{\text{AD}}^-$ and that with positive γ, which is the standard amplitude damping channel, by $\mathcal{E}_{\text{AD}}^+$, then from the form of their Kraus operators, it is evident that $\mathcal{E}_{\text{AD}}^- = \mathcal{Z}\circ\mathcal{E}_{\text{AD}}^+$, where $\mathcal{Z}\rho = \sigma_z\rho\sigma_z$. However, from this relation, one cannot infer any conclusion between their quasi-inverses, since the concept of quasi-inverse as defined in this paper doesn't lead to a relation like $(\Phi\circ\mathcal{E})^{qi} = \mathcal{E}^{qi}\circ\Phi^{qi}$.

One may ask why a simple change of sign $\gamma \longrightarrow -\gamma$ makes so much difference in the quasi-inverse of a channel. The answer is best seen when we look at the affine transformation associated with the amplitude damping channel: $M = \frac{1}{2}\text{diag}(\gamma, \gamma, \gamma^2)$ and $\boldsymbol{t} = (0, 0, 1-\gamma^2)^T$. When $\gamma > 0$, the Bloch sphere is only shrunk and translated, but when $\gamma < 0$, it is also reflected with respect to the z axis (see Fig. 1). The quasi-inverse compensates for this reflection and increases the average fidelity of the $\mathcal{E}_{\text{AD}}^-$, an action which if applied to $\mathcal{E}_{\text{AD}}^+$ decreases the average fidelity instead of increasing it.

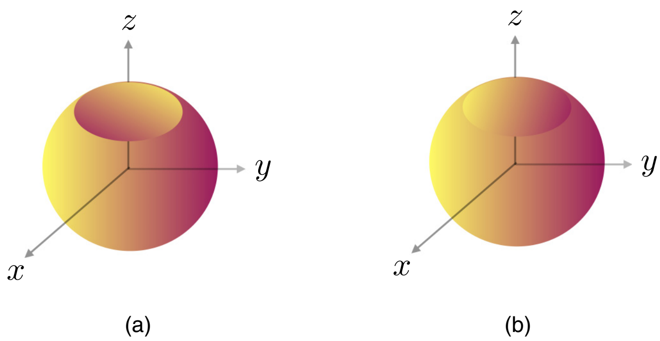

FIG. 1. The transformation of the Bloch sphere under the amplitude damping channels $\mathcal{E}_{AD}^{-}$ (the small elipsoid) (a) before the quasi-inverse and (b) after the quasi-inverse. The contrast in colors is meant to show the lower average fidelity compared to (b) where the quasi-inverse reflects the Bloch sphere with respect to the z axis and raises the average fidelity.

In fact this is a general feature of channels which can be quasi-inverted in this sense. Figure 2 shows how two generic points in the Bloch sphere move under the channel $\mathcal{E}$ to the opposite side of the Bloch sphere and how the quasi-inverse bring them back to the original side of the Bloch sphere where they are closer to their initial place. Note that in both operations, the distance of the two states decreases under the CPT maps $\mathcal{E}$ and $\mathcal{E}^{qi}$, as they should; however, the average fidelity increases under the quasi-inverse channel.

Consider now a slight modification of this channel when A_0 is changed to

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & i\gamma \end{pmatrix}. \tag{62}$$

The corresponding channel is still trace preserving and nonunital but has a nonsymmetric associated affine matrix.

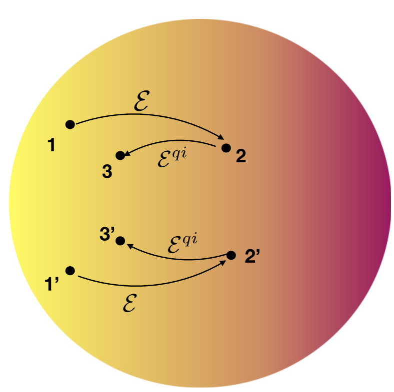

FIG. 2. The increase of average fidelity is consistent with the contraction property of both the channel and its quasi-inverse. $\mathcal{E}$ decreases the distance between the states 1 and 1′, $\mathcal{E}^{qi}$ decreases their distance still further, but the final distance between the input and output states (1 and 3) or (1′ and 3′) is lowered compared with the distance between (1 and 2) or (1′ and 2′) when only $\mathcal{E}$ acts.

The matrix Q is now given by

$$Q = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & \gamma \\ 0 & -\gamma^2 & 0 & 0 \\ 0 & 0 & -\gamma^2 & 0 \\ \gamma & 0 & 0 & 0 \end{pmatrix}, \tag{63}$$

so that $\lambda_{\max} = \frac{|\gamma|}{2}$ and hence $\Delta\overline{F} = \frac{|\gamma|}{3}$. Then, the quasi-inverse is the unitary $V = e^{i\frac{\pi}{4}\sigma_z}$. The average fidelity of this amplitude damping channel before applying the quasi-inverse is

$$\overline{F}(\mathcal{E}_{AD}) = 1 - \tfrac{2}{3}\mathrm{Tr}(B) = \tfrac{1}{2} + \tfrac{1}{6}\gamma^2, \tag{64}$$

and

$$\overline{F}(\mathcal{E}^{qi} \circ \mathcal{E}_{AD}) = \frac{1}{2} + \frac{1}{6}\gamma^2 + \frac{|\gamma|}{3}, \tag{65}$$

which is larger than the average fidelity of the original channel for all values of γ.

C. A mixed unitary channel

This is a channel with the nonsymmetric affine matrix.

$$\mathcal{E}(\rho) = p_0\rho + p\sum_{i=1}^{3} U_i\rho U_i^{\dagger}, \tag{66}$$

where $U_i = e^{-i\frac{\theta}{2}\sigma_i}$ is a rotation around the $\boldsymbol{x}_i$ axis with angle θ and $p_0 + 3p = 1$. The matrix Q is now given by

$$Q = \begin{pmatrix} 0 & v/2 & v/2 & v/2 \\ v/2 & q & 0 & 0 \\ v/2 & 0 & q & 0 \\ v/2 & 0 & 0 & q \end{pmatrix}, \tag{67}$$

where $v = p\sin\theta$ and $q = 4p\sin^2\frac{\theta}{2} - 1$. For $q \geqslant 0$, the largest eigenvalue of this matrix is $\lambda_{\max} = \frac{1}{2}(q + \sqrt{q^2 + 3v^2})$, with the corresponding eigenvector given by $|\lambda_{\max}\rangle \propto (\frac{3v}{2\lambda_{\max}} \quad 1 \quad 1 \quad 1)^T$. This means that the quasi-inverse of the channel is given by the unitary $V = e^{i\phi\,\boldsymbol{n}\cdot\boldsymbol{\sigma}}$, where

$$\cos\phi = \frac{\sqrt{3}v}{\sqrt{3v^2 + 4\lambda_{\max}^2}}, \qquad \boldsymbol{n} = \frac{1}{\sqrt{3}}(\boldsymbol{x} + \boldsymbol{y} + \boldsymbol{z}). \tag{68}$$

The increase in average fidelity is given by $\Delta\overline{F}(\mathcal{E}) = \frac{2}{3}\lambda_{\max}$, which is plotted in Fig. 3 as a function of parameters p and θ.

D. A channel whose quasi-inverse is different from one of its own Kraus operators

The Pauli channel is a special channel for which the quasi-inverse turns out to be one of the Kraus operators of the channel, i.e., the Pauli matrices. There are many other channels for which this is not the case. In order to remain within the domain of analytical solutions and avoid numerical methods, we define a new channel and call it the tetrahedron channel. The channel is defined by

$$\mathcal{E}(\rho) = q\rho + \sum_{i=0}^{3} p_i(\boldsymbol{u}_i\cdot\boldsymbol{\sigma})\,\rho\,(\boldsymbol{u}_i\cdot\boldsymbol{\sigma}), \tag{69}$$

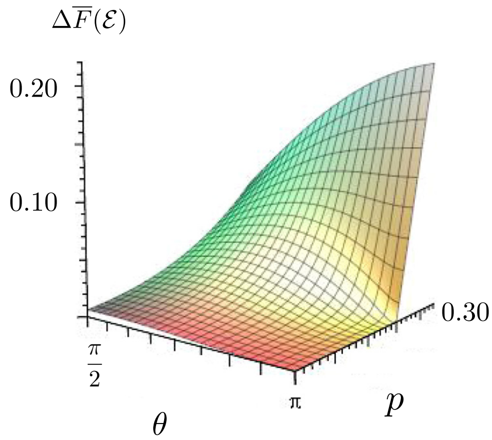

FIG. 3. The increase of average fidelity for the random unitary channel given in (66).

where $q = 1 - p_0 - p_1 - p_2 - p_3$. The vectors $\boldsymbol{u}_i$ are chosen to be the corners of a tetrahedron as

$$\boldsymbol{u}_0 = \frac{1}{\sqrt{3}}(1, 1, 1),$$
$$\boldsymbol{u}_1 = \frac{1}{\sqrt{3}}(1, -1, -1),$$
$$\boldsymbol{u}_2 = \frac{1}{\sqrt{3}}(-1, 1, -1),$$
$$\boldsymbol{u}_3 = \frac{1}{\sqrt{3}}(-1, -1, 1), \tag{70}$$

so that the corresponding $\boldsymbol{b}$ vectors are given by $\boldsymbol{b}_i = \sqrt{p_i}\,\boldsymbol{u}_i$, $i = 0, 1, 2, 3$, from which the corresponding B matrix can be computed. For simplicity we consider the special case where

$$p_1 = p_2 = p, \quad p_0 = p_3 = p', \tag{71}$$

with $p + p' \leqslant 1/2$ due to the normalization condition $q + 2p + 2p' = 1$. With this choice, one finds

$$B = \frac{1}{3}\begin{pmatrix} 2p + 2p' & 2p - 2p' & 0 \\ 2p - 2p' & 2p + 2p' & 0 \\ 0 & 0 & 2p + 2p' \end{pmatrix}, \tag{72}$$

with eigenvalues

$$\lambda_1 = \frac{4p}{3}, \quad \lambda_2 = \frac{4p'}{3}, \quad \lambda_3 = \frac{2p + 2p'}{3}, \tag{73}$$

and corresponding eigenvectors

$$\boldsymbol{e}_+ = \frac{1}{\sqrt{2}}(\boldsymbol{x} + \boldsymbol{y}), \quad \boldsymbol{e}_- = \frac{1}{\sqrt{2}}(\boldsymbol{x} - \boldsymbol{y}), \quad \boldsymbol{e}_3 = \boldsymbol{z}, \tag{74}$$

where $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ here denote the Cartesian three-dimensional unit vectors. The original average fidelity of this channel is given by

$$\overline{F}(\mathcal{E}) = 1 - \tfrac{2}{3}\,\mathrm{Tr}(B) = 1 - \tfrac{4}{3}(p + p'), \tag{75}$$

and the increase in average fidelity is given by

$$\Delta\overline{F}(\mathcal{E}) = \tfrac{2}{3}\,\mathrm{Max}[\lambda_{\max},\ 0], \tag{76}$$

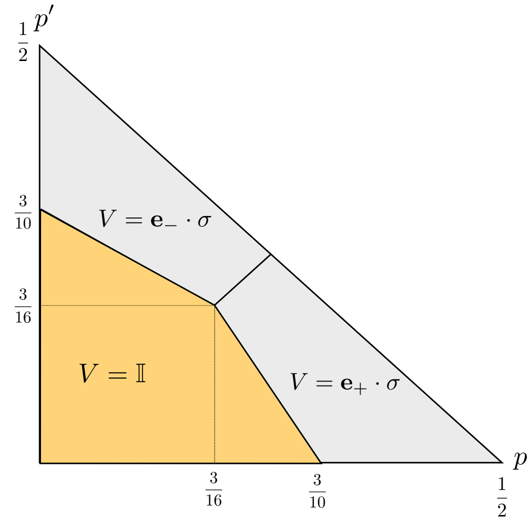

FIG. 4. The region of parameters where the tetrahedron channel has a quasi-inverse. In the colored (yellow) region where $V = \mathbb{1}$, no increase is obtained in average fidelity. In the other (gray) regions the correcting quasi-inverse (i.e., the unitary operator) is shown. Here $\boldsymbol{e}_\pm = \frac{1}{\sqrt{2}}(\boldsymbol{x} \pm \boldsymbol{y})$.

or, explicitly,

$$\Delta\overline{F}(\mathcal{E}) = \begin{cases} \frac{2}{3}\,\mathrm{Max}\{2p' - 1 + \frac{10p}{3}, 0\} & \text{if } p \geqslant p', \\ \frac{2}{3}\,\mathrm{Max}\{2p - 1 + \frac{10p'}{3}, 0\} & \text{if } p \leqslant p'. \end{cases} \tag{77}$$

The regions where an increase of fidelity is possible, and the unitary operator V that achieves it, are shown in Fig. 4.

Of course due to the symmetry of the tetrahedron channel, we can obtain, without further calculations, similar results if other pairs of probabilities are equal. When $p_0 = p_2 = p$ and $p_1 = p_3 = p'$, one finds the same results as before but with $\boldsymbol{e}_\pm = \frac{1}{\sqrt{2}}(\boldsymbol{x} \pm \boldsymbol{z})$; similarly, the same holds when $p_0 = p_1 = p$ and $p_2 = p_3 = p'$ provided $\boldsymbol{e}_\pm = \frac{1}{\sqrt{2}}(\boldsymbol{y} \pm \boldsymbol{z})$.

VI. THE GEOMETRIC PICTURE

One may ask why we have not followed entirely the approach of affine maps for finding the quasi-inverse of a qubit channel by using equation,

$$\overline{F}(\mathcal{E}^{qi} \circ \mathcal{E}) \equiv \tfrac{1}{2}\big(1 + \tfrac{1}{3}\mathrm{Tr}(N\,M)\big), \tag{78}$$

and finding the matrix N which maximizes the trace on the right-hand side. The problem is that even if one finds such a matrix by say numerical methods, it is not guaranteed that it defines a qubit channel. In fact while any qubit channel defines an affine map, not all affine maps define qubit channels. Nevertheless one can solve this problem for the special case of symmetric affine maps in a geometrical way. We note that such affine maps pertain to channels of the form $\mathcal{E} = \mathcal{U} \circ \mathcal{E}_c \circ \mathcal{U}^{-1}$, with $\mathcal{E}_c$ having a diagonal affine matrix, $\Lambda_c = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$. For complete positivity of the map (the qubit channel), these parameters are confined to be inside a tetrahedron as shown in Fig. 5. The corners of these tetrahedron correspond to $\mathcal{E}_i : \rho \longrightarrow \sigma_i \rho\, \sigma_i$ where $i \in \{0, 1, 2, 3\}$ also includes the identity matrix $\sigma_0 = \mathbb{1}$, that is, to simple conjugation by Pauli matrices. The edges, faces,

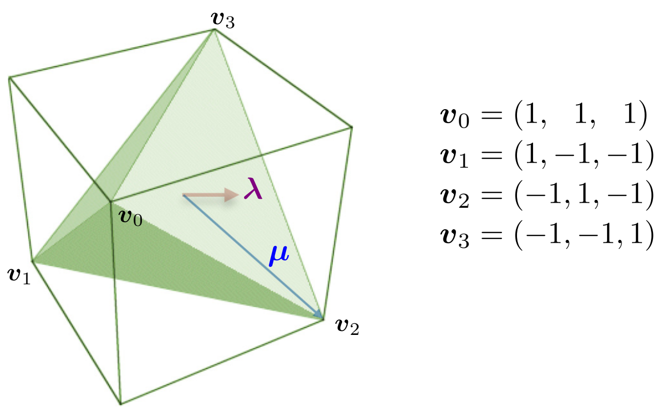

FIG. 5. The canonical qubit channel $\mathcal{E}_c$ is characterized by the vector $\boldsymbol{\lambda}$. Its quasi-inverse $\mathcal{E}_c^{qi}$ is characterized by a vector $\boldsymbol{\mu}$ which maximizes the product $\boldsymbol{\lambda}\cdot\boldsymbol{\mu}$ in the expression $\frac{1}{2}(1+\frac{1}{3}\boldsymbol{\lambda}\cdot\boldsymbol{\mu})$.

and the inside of the tetrahedron correspond, respectively, to convex combination of two, three, and four of these simple maps. According to Theorem 2, we only need to find the quasi-inverse $\mathcal{E}_c^{qi}$ whose affine matrix is again diagonal $N_c = \mathrm{diag}(\mu_1, \mu_2, \mu_3)$ with parameters in the same tetrahedron. The parameters μ_i should be chosen to maximize the fidelity,

$$\overline{F}(\mathcal{E}_c^{qi}\circ\mathcal{E}_c) = \tfrac{1}{2}\big(1+\tfrac{1}{3}\mathrm{Tr}(N_c\Lambda_c)\big) = \tfrac{1}{2}\big(1+\tfrac{1}{3}(\mu_1\lambda_1+\mu_2\lambda_2+\mu_3\lambda_3)\big). \tag{79}$$

If it were not for the constraint that the vector $\boldsymbol{\mu}$ should be inside the tetrahedron, it could have simply been taken parallel to $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$. However, with this constraint and with our knowledge from Theorem 1 that the quasi-inverse can be a unitary map, it is enough to take the vector $\boldsymbol{\mu}$ to correspond to one of the vertices $\boldsymbol{v}_0, \boldsymbol{v}_1, \boldsymbol{v}_2$, or $\boldsymbol{v}_3$ depending on which one has the smallest Euclidean distance from $\boldsymbol{\lambda}$. Inserting the coordinates of these vertices from Fig. 5 into the following formulas,

$$||\boldsymbol{v}_0-\boldsymbol{\lambda}||,\quad ||\boldsymbol{v}_1-\boldsymbol{\lambda}||,\quad ||\boldsymbol{v}_2-\boldsymbol{\lambda}||,\quad ||\boldsymbol{v}_3-\boldsymbol{\lambda}||, \tag{80}$$

and simplifying, we find that the comparison of these distances amounts to comparing the following expressions and determining which one is the maximum,

$$\lambda_1+\lambda_2+\lambda_3\ ,\ \lambda_1-\lambda_2-\lambda_3\ ,\ \lambda_2-\lambda_1-\lambda_3\ ,\ \lambda_3-\lambda_1-\lambda_2. \tag{81}$$

The maximality of these terms correspond, respectively, from left to right to the quasi-inverse being the identity operator or conjugation by σ_1, σ_2, and σ_3. More concretely, when all the λ_i's are nonnegative, $\lambda_1+\lambda_2+\lambda_3$ is the largest of the above terms, which implies that $\boldsymbol{\lambda}$ is closest to v_0 and hence the quasi-inverse is the identity map. When $\lambda_1 \geqslant \lambda_2, \lambda_3$, then the second term in (81) is the largest term and $\boldsymbol{\lambda}$ is closest to $\boldsymbol{v}_1$ implying that the quasi-inverse is σ_1, etc.

Once the quasi-inverse of the canonical map $\mathcal{E}_c$ is obtained as one of the σ_i's, the quasi-inverse of the map $\mathcal{E}$ is obtained from $\mathcal{E}^{qi}$ as $U\sigma_i U^\dagger = \boldsymbol{x}\cdot\boldsymbol{\sigma}$ where $\boldsymbol{x}$ is the eigenvector corresponding to the largest eigenvalue of its matrix M or $\widehat{B}$.

VII. THE PROBLEM OF UNIQUENESS

We have stated in Theorem 1 that the quasi-inverse can be taken to be unitary. Indeed a better statement is that the quasi-inverses of all, but a set of measure zero qubit channels, are unitary. The point is that the matrix Q of a channel $\mathcal{E}$ has two equal largest eigenvalues corresponding to two different unitary operators V_1 and V_2 as quasi-inverses, where $\overline{F}(\mathcal{V}_1\circ\mathcal{E}) = \overline{F}(\mathcal{V}_2\circ\mathcal{E})$. This then leads to a one-parameter family of quasi-inverses $\mathcal{E}_p^{qi} = (1-p)\mathcal{V}_1 + p\mathcal{V}_2$, not all members of which are unitary. For generic channels the quasi-inverse is unique and unitary, since this degeneracy happens only for a set of measure zero in the space of all qubit channels. From the geometric picture in the previous section, we can now elaborate more on this point for the case of channels with symmetric affine matrices. Consider the tetrahedron in Fig. 5. We see that unless the tip of the affine vector $\boldsymbol{\lambda}$ is equidistant to the corners of the tetrahedron, there is always a unique quasi-inverse which is a unitary (corresponding to the vertex closest to the tip of $\boldsymbol{\lambda}$). Only at this set of measure zero, we have degeneracy of quasi-inverses, where the convex combination of these quasi-inverses also leads to the same average fidelity and hence we have unital quasi-inverses which are no longer pure unitary. As an example, consider the channel $\mathcal{E} = \frac{1}{2}(\sigma_x\rho\sigma_x + \sigma_y\rho\sigma_y)$, corresponding to the middle of an edge of the tetrahedron, corresponding to the affine map $M = \mathrm{diag}(0, 0, -1)$. The quasi-inverse of this channel is $\mathcal{E}_p^{qi} = (1-p)\sigma_x\rho\sigma_x + p\sigma_y\rho\sigma_y$ for any p, leading to the channel $(\mathcal{E}_p^{qi}\circ\mathcal{E})(\rho) = \frac{1}{2}(\rho + \sigma_z\rho\sigma_z)$ for which the affine matrix is $NM = \mathrm{diag}(0, 0, 1)$.

VIII. DISCUSSION

We have introduced the concept of quasi-inverse of quantum channels and have proved several of its properties for qubit channels, including their unitarity, and equality of left and right inverses. A concrete formalism for finding the quasi-inverse of general qubit channels has been introduced and several classes of examples have been studied in detail. In relation to these results, various possible extensions can be considered. The first one is a possible generalization to higher dimensional channels. While we have considered general unital and nonunital qubit channels, the proof of the unitarity of the quasi-inverse hinges on a very specific property of unital qubit channels, [19], according to which any unital qubit channel is a random unitary channel. A counter example to this theorem in higher dimensions was first found by [20,21]. A specific example is given by Landau and Streater [22] in odd dimensions,

$$\mathcal{E}(\rho) = \frac{1}{j(j+1)}[J_x\rho J_x^\dagger + J_y\rho J_y^\dagger + J_z\rho J_z^\dagger]. \tag{82}$$

Moreover, the main difficulty for generalizing these results to higher dimensions is that contrary to the qubit case, where a rather complete characterization of the space of all channels is available [17,18], our knowledge about higher dimensional channels is very limited.

Furthermore, let us remark that the results presented here using the average fidelity should hold also for other linear fidelity functions. In addition, it is plausible to believe that, as in the case of qubit channels, the quasi-inverse of any general channel should be an extreme point on the space of all channels. However, the problem of finding the extreme points of the space of higher dimensional channels is a highly nontrivial open problem.

A different, possible extension of our work regards the analysis of the quasi-inverses of classical stochastic or bi-stochastic maps. Here we have a fairly well-developed knowledge on the space itself and its extreme points [23], but then the difficulty is in the fidelity function between two classical probability distributions which is no longer linear. We plan to report on these issues in the future.

Connections of our results with the error correction theory is also worth mentioning. We have found quasi-inverses of the Pauli and the amplitude damping channels as two common error models which of practical relevance. While it is true that error-correcting codes are used to recover the encoded information transmitted through noisy channels, the effectiveness of error correction depends on whether the encoded information is affected drastically or slightly, i.e., on the error probabilities of the combined channel $\mathcal{E}^{qi} \circ \mathcal{E}$. Applying the quasi-inverse at the end of a channel certainly lowers the error rate and makes any error correction scheme more reliable and effective.

We would also like to briefly comment on other notions of inverses that have appeared in the literature. The quasi-inverse of a channel $\mathcal{E}^{qi}$ is different from its time reversal $\mathcal{E}^R$ introduced in [24] in the context of entropy production in open quantum systems. The operation R is not unitary, rather an involution $(\mathcal{E}^R)^R = \mathcal{E}$, in direct contrast to the quasi-inverse channel as defined in this paper. Furthermore, as already observed in the Introduction, our approach also differs from those based on the "Petz recovery map" [13–16]; indeed, our quasi-inverse is defined through the average fidelity, while the "Petz recovery map" depends on specific reference states.

Finally it would be interesting to investigate how the quasi-inverse of a channel can partially restore the coherence of input states of different channels [25–27], an area which has recently seen intensive research.

ACKNOWLEDGMENTS

We would like to thank S. Filippov, S. Raeisi, K. Zyczkowski, and L. Memarzadeh for their valuable comments and suggestions. The work of V.K. was partially supported by Grant No. 96011347 from the Iran National Science Foundation and Grant No. G950222 from Sharif University of Technology. F.B. and R.F. acknowledge that their research has been conducted within the framework of the Trieste Institute for Theoretical Quantum Technologies.

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