Qualitative analysis of a curvature equation modeling MEMS with vertical loads *

Chiara Corsato
Università degli Studi di Trieste
Dipartimento di Scienze Economiche, Aziendali, Matematiche e Statistiche
Piazzale Europa 1, 34127 Trieste, Italy
E-mail: ccorsato@units.it

Colette De Coster
Université Polytechnique Hauts-de-France
EA 4015 - LAMAV - FR CNRS 2956
F-59313 Valenciennes, France
E-mail: colette.decoster@uphf.fr

Franco Obersnel
Università degli Studi di Trieste
Dipartimento di Matematica e Geoscienze
Via A. Valerio 12/1, 34127 Trieste, Italy
E-mail: obersnel@units.it

Pierpaolo Omari
Università degli Studi di Trieste
Dipartimento di Matematica e Geoscienze
Via A. Valerio 12/1, 34127 Trieste, Italy
E-mail: omari@units.it

Abstract

We investigate existence, multiplicity and qualitative properties of the solutions of the Dirichlet problem for the singularly perturbed prescribed mean curvature equation

$$\begin{cases} -(1-bu)\operatorname{div}\Big(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) = \dfrac{a}{(u-R)^2} + \dfrac{b}{\sqrt{1+|\nabla u|^2}}, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where a, b, R are given constants and Ω is a bounded regular domain in $\mathbb{R}^N$. This model appears in the theory of micro-electro-mechanical systems (MEMS) when the effects of capillarity and vertical forces are taken into account.

2010 Mathematics Subject Classification: 35J93, 35J25, 35J62, 35J67, 35B09, 35B51, 35D99, 35D10.

Keywords and Phrases: prescribed mean curvature equation; Dirichlet boundary condition; classical solution; generalized solution; existence; multiplicity; regularity; boundary behaviour; topological degree; bifurcation.

*Research performed under the auspices of Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of Istituto Nazionale di Alta Matematica (INdAM). F. Obersnel and P. Omari have been supported by Università degli Studi di Trieste - Finanziamento di Ateneo per progetti di ricerca scientifica - FRA 2018 within the project "Equazioni Differenziali: Teoria Qualitativa e Metodologie Numeriche".

Contents

1 Introduction

With the advent and the wide spreading of miniaturized technology, systems combining the effects of electrostatic and capillary forces have been receiving more and more attention in the last decades, in order to fully understand how they operate at small scales; we refer to [39] and to the recent surveys [28, 21], as general references on this topic. A classical device which has been realized to study the interplay of such forces consists of two parallel plates, the upper plate having a hole Ω which is filled with a thin membrane, such as, e.g., a soap film. If a voltage difference is applied, the film is deflected by a Coulomb force generated between the two components. In [11] a model which describes the shape of such electrostatically actuated film has been derived from minimizing the total energy of the system, represented as the sum of the elastic energy, the electric potential and the potential due to the presence of a vertical force, such as, e.g., a vertical load. Such model consists of a coupled system of partial differential equations for the shape of the deflected membrane and the electrostatic potential. After assuming that the electric fringing field has negligible effects on the film, such system can be reduced to the following single equation, supplemented with Dirichlet boundary conditions,

$$\begin{cases} (1+\varepsilon^2\beta v)\operatorname{div}\Big(\dfrac{\nabla v}{\sqrt{1+\varepsilon^2|\nabla v|^2}}\Big) = \dfrac{\lambda}{(1+v)^2} + \dfrac{\beta}{\sqrt{1+\varepsilon^2|\nabla v|^2}}, & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

The variables and parameters appearing in the equation have the following meaning: v is an adimensional variable describing the shape of the membrane, $\varepsilon > 0$ represents the ratio between the distance of the two plates and the diameter of the membrane, and is generally small, $\lambda > 0$ measures the relative importance of the electric field versus the surface tension and β measures the relative importance of the vertical force acting over the membrane, versus the surface tension; of course, β can be of either sign, according to the orientation of this force with respect to the direction of the electric field, or zero, if absent; all parameters also incorporate information on the geometry of the system. Making the change of variable $u(x) = -\varepsilon v(x)$, $a = \varepsilon^3\lambda$, $b = \varepsilon\beta$, $R = \varepsilon$, we can rewrite (1.1) in the equivalent form

$$\begin{cases} -(1-bu)\operatorname{div}\Big(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) = \dfrac{a}{(u-R)^2} + \dfrac{b}{\sqrt{1+|\nabla u|^2}}, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Hereafter, we assume that

(H_1) *a, b, R, with $R > 0$, are given constants and Ω is a bounded domain in $\mathbb{R}^N$, with a boundary $\partial\Omega$ of class C^2, in case $N \geq 2$, and $\Omega =]-r, r[$, with $r > 0$, in case $N = 1$.*

The parameter a is generally supposed to be positive, although in the course of our mathematical discussion a will sometimes be allowed to take non-positive values, while the parameter b can be either positive, or negative, or zero. Since $R = \varepsilon$ is assumed to be positive and small, it is natural to suppose, when $b > 0$, that the condition $Rb \leq 1$ holds. However, we will also discuss the case where $Rb > 1$, so that a singularity on the left of R appears in the equation.

Problem (1.2) appears quite complex, because it simultaneously incorporates the mean curvature operator

$$\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) \tag{1.3}$$

and the two singular terms

$$\frac{a}{(u-R)^2} \quad \text{and} \quad \frac{1}{1-bu}, \tag{1.4}$$

the latter one being determined, as already noticed, by the presence of the factor $1 - bu$ in front of the left-hand side of the equation. Therefore, in the literature the model has often been simplified in various different ways. For instance, in the works [11, 8], in order to discuss the effects in the problem of the gravity, or possibly of an external pressure, the equation in (1.2) has been replaced by

$$-\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) = \frac{a}{(u-R)^2} + b. \tag{1.5}$$

Of course, if these effects are neglected, then the term b is dropped: this situation has been investigated in [9, 8]. On the other hand, approximating the mean curvature operator $\operatorname{div}\big(\nabla u/\sqrt{1+|\nabla u|^2}\big)$ by its linearization Δu at 0 leads to the most popular model in the literature, sometimes referred to as the "canonical model",

$$\begin{cases} -\Delta u = \dfrac{a}{(u-R)^2}, & \text{in } \Omega, \\ \quad u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

Although there is a large amount of literature devoted to the existence of positive solutions for the singular semilinear elliptic problem (1.6), a thorough discussion being given, e.g., in [22], no result seems to be available for the complete problem (1.2). The absence in the existing literature of any contribution in this direction might be also attributable to the presence into the equation of the mean curvature operator (1.3) and of the two singular terms (1.4). The analysis of such non-uniformly elliptic equations is indeed rather delicate and sophisticated, being fraught with a number of technical difficulties which do not arise when dealing with non-degenerate problems. In addition, the mean curvature operator has a relevant impact on the morphology of the solutions and, in general, one cannot expect that either the solutions be regular, up to the formation of discontinuities, or the boundary conditions be attained. This is the reason for which it is natural, when dealing with problem (1.2), to introduce a suitable notion of generalized solution, as the one given below which is inspired from [42, 29, 26] and has been recently considered in [13, 14, 12].

Definition 1.1 (**Notions of solution**).
(I) *A function* $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ *is a* (generalized) solution of (1.2) *if the following conditions hold:*

(*a*) $\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}} \in C^0(\overline{\Omega})$;

(*b*) $u(x) < R$ *and* $bu(x) < 1$ *for all* $x \in \Omega$;

(*c*) u *satisfies the equation in* (1.2) *for all* $x \in \Omega$;

(*d*) *for every* $x \in \partial\Omega$,

- *either* $u(x) = 0$,
- *or* $u(x) > 0$ *and* $\dfrac{\nabla u(x) \cdot \nu(x)}{\sqrt{1 + |\nabla u(x)|^2}} = -1$,
- *or* $u(x) < 0$ *and* $\dfrac{\nabla u(x) \cdot \nu(x)}{\sqrt{1 + |\nabla u(x)|^2}} = 1$,

where $\nu(x)$ *is the unit outer normal to* Ω *at* $x \in \partial\Omega$.

(II) *A solution* u *of* (1.2) *is* classical *if* $u(x) = 0$ *for all* $x \in \partial\Omega$.

Remark 1.1 The structure of the equation in (1.2) and an iterated application of [25, Theorem 9.19] implies that $u \in C^\infty(\Omega)$ and actually (real) analytic by [34, Theorem 5.8.6].

Remark 1.2 It is worthy to point out that the actual occurrence of generalized non-classical solutions can be detected even in the simpler case where $N = 1$ and $b = 0$, so that problem (1.2) reduces to

$$\begin{cases} -\Big(\dfrac{u'}{\sqrt{1+|u'|^2}}\Big)' = \dfrac{a}{(u-R)^2}, & \text{in }]-r, r[, \\ u(-r) = 0, \ \ u(r) = 0, \end{cases}$$

whenever $\frac{r}{R} < \rho^* \approx 0.34996$ and $a \in]a_*, a_{**}[$, with $0 < a_* < a_{**}$ (cf. [8, 10, 37]).

As problem (1.2) has not been attacked yet in its full generality, we aim in this work to begin its analysis, providing several results concerning the qualitative properties of the solutions, as well as their existence and multiplicity. Our approach relies on the use of various tools of nonlinear analysis, such as the implicit function theorem, topological degree, bifurcation, lower and upper solutions, combined with PDE and ODE techniques. Instead, we do not explicitly use here the variational structure of problem (1.2), which can indeed be formally interpreted as the Euler equation of the following singularly perturbed anisotropic area functional

$$\int_\Omega (1 - bu)\sqrt{1 + |\nabla u|^2}\, dx - \int_\Omega \frac{a}{R - u}\, dx.$$

One of the reasons is that, in order to deal properly with this functional and the corresponding critical points, it seems appropriate, as discussed in [4], to settle the problem in the non-standard space $BV_2(\Omega)$ and to introduce there a suitable notion of weak solution. Thus, to keep this paper within a reasonable length we prefer to leave this study to a forthcoming work.

Now we turn to describe the structure of this paper. In Section 2 we perform a qualitative analysis of the generalized solutions of problem (1.2), providing various information about the sign and the symmetry of the possible solutions, as well as, in the one-dimensional case, about their concavity properties.

Section 3 is devoted to derive some existence and non-existence results for problem (1.2) in the N-dimensional case. We start proving a quantitative necessary condition for the existence of generalized solutions, which provides an estimate for the pull-in voltage [7, p.11-12] in the presence of a vertical force. Next, we pass to study the solvability of problem (1.2). We start proving the existence of regular classical solutions which stem from any point of the line $\{(0, a, -\frac{a}{R^2}) \mid a \in \mathbb{R}\}$, provided $\frac{2a}{R^3} \notin \Sigma$, where $\Sigma = \{\lambda_n \mid n \in \mathbb{N}^+\}$ is the spectrum of $-\Delta$ in $H_0^1(\Omega)$. This conclusion is achieved by a simple application of the implicit function theorem. Next, we look at problem (1.2) as a perturbation of

$$\begin{cases} -(1 - bu)\,\mathrm{div}\Big(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) = \dfrac{b}{\sqrt{1+|\nabla u|^2}}, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

We first show, also exploiting some ideas from [20], that this problem admits non-degenerate one-sign solutions, i.e., with non-zero fixed point index, for all small $|b|$, and no solutions at all, for all large $|b|$. Then, by using this information and taking a as a homotopy parameter, a branch of regular classical solutions of (1.2) is found for each b in a suitable interval. In the one-dimensional case, additional conclusions are achieved. Namely, we can show that, for all small a, problem (1.2) actually admits two regular classical solutions for b in a suitable, explicitly computable, interval. It is worthy to observe that this multiplicity result is determined by the presence in the equation of the additional singular term $\frac{1}{1-bu}$.

In Section 4 we investigate the complementary situation where $\frac{2a_n}{R^3} \in \Sigma$, for some $n \in \mathbb{N}^+$. Namely, if $\frac{2a_n}{R^3}$ is a simple eigenvalue, we prove the existence of a branch of regular classical solutions (u, a) of problem (1.2), bifurcating from the line of the trivial solutions $\{(0, a, -\frac{a}{R^2}) \mid a \in \mathbb{R}\}$ at the point $(0, a_n, -\frac{a_n}{R^2})$, and lying in the hyperplane $\{(u, a, -\frac{a}{R^2}) \mid u \in C^1(\overline{\Omega}), a \in \mathbb{R}\} \subset C^1(\overline{\Omega}) \times \mathbb{R} \times \mathbb{R}$. In particular, bifurcation of positive solutions is detected if $n = 1$, i.e., $\frac{2a_1}{R^3}$ is the principal eigenvalue λ_1. Exhaustive information about the nodal properties of the solutions within the bifurcating branches are also obtained when dealing with radially symmetric solutions over balls, or in the one-dimensional case.

Section 5 focuses on the study of the existence and the multiplicity of classical and generalized one-sign solutions for the one-dimensional counterpart of problem (1.2)

$$\begin{cases} -(1-bu)\Big(\dfrac{u'}{\sqrt{1+|u'|^2}}\Big)' = \dfrac{a}{(u-R)^2} + \dfrac{b}{\sqrt{1+|u'|^2}}, & \text{in }]-r, r[, \\ u(-r) = 0, \; u(r) = 0. \end{cases} \tag{1.7}$$

This section is divided into two subsections, where existence and multiplicity are separately discussed. In the former section, we prove a general existence and non-existence result for both classical and generalized positive solutions of (1.7). In particular, we show that the minimum positive generalized solution is always classical and we carefully discuss its regularity. All these conclusions rely on an existence result for generalized solutions, basically obtained by an approximation argument combined with the upper and lower solutions method and some subtle comparison principles, which may have an independent interest. In the latter subsection, we discuss the existence of a second positive generalized solution and we single out the situations where multiplicity occurs, in terms of the values of the parameters a, b, R. Our analysis also displays the different nature of the multiplicity phenomena, depending on whether they originate from the one or the other singular term, $\frac{a}{(u-R)^2}$ or $\frac{1}{1-bu}$, present into the equation. Existence and multiplicity of negative solutions are treated as well following similar lines.

We finally point out that, taking $b = 0$, problem (1.5) becomes a special case of (1.2); hence the results in this paper provide a proof of some conclusions, drawn in [8] by using [38, Theorem 1.1], whose proof does not seem complete.

Notation. We denote the open interval $]0, +\infty[$ by $\mathbb{R}^+$ and we set $\mathbb{N}^+ = \mathbb{N} \cap \mathbb{R}^+$. The symbol δ_{ij} indicates the Kronecker delta. $|\Omega|$ stands for the Lebesgue measure in $\mathbb{R}^N$ of Ω, $|\partial\Omega|$ denotes the $\mathcal{H}^{N-1}$-measure in $\mathbb{R}^N$ of $\partial\Omega$ and ω_N indicates the measure of the unit ball in $\mathbb{R}^N$. $\Sigma = \{\lambda_n \mid n \in \mathbb{N}^+\}$ is the spectrum of $-\Delta$ in $H^1_0(\Omega)$. For any function $u : \overline{\Omega} \to \mathbb{R}$, we write

- $u \geq 0$ if, for all $x \in \overline{\Omega}$, $u(x) \geq 0$,
- $u > 0$ if $u \geq 0$ and $u \neq 0$,
- $u \gg 0$ if, for all $x \in \Omega$, $u(x) > 0$ and, for all $x \in \partial\Omega$, either $u(x) > 0$, or both $u(x) = 0$ and $\limsup_{t \to 0^-} \frac{u(x+t\nu(x))}{t} < 0$.

2 Qualitative properties of solutions

In this section we begin the qualitative analysis of problem (1.2). Our first result provides some general information about the sign and the symmetry of the possible solutions of (1.2). The following terminology is adopted.

Definition 2.1 (**Positive solution**).

(I) *A solution u of* (1.2) *is said* positive *(respectively,* negative*) if $u > 0$ (respectively, $u < 0$).*

(II) *A solution u of* (1.2) *is said* strictly positive *(respectively,* strictly negative*) if $u \gg 0$ (respectively, $u \ll 0$).*

We recall that λ_1 denotes the principal eigenvalue of $-\Delta$ in $H_0^1(\Omega)$.

Theorem 2.1. *Assume (H_1). Then, the following conclusions hold.*

(*a*) *If $b \geq 0$ and $\frac{a}{R^2} + b \geq 0$, then every non-trivial solution of* (1.2) *is strictly positive.*

(*b*) *If $a > 0$, $b < 0$ and $|b|a \geq (1 + R|b|)^2$, then every non-trivial solution of* (1.2) *is strictly positive.*

(*c*) *If $0 < a < \frac{R^3}{2}\lambda_1$ and $-\frac{a}{R^2} \leq b < 0$, then there exists $\varepsilon = \varepsilon(a, b) > 0$ such that every non-trivial solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ of* (1.2)*, with $\|u\|_{C^1} < \varepsilon$, is strictly positive.*

(*d*) *If $a \leq 0$ and $\frac{a}{R^2} + b \leq 0$, then every non-trivial solution of* (1.2) *is strictly negative.*

(*e*) *If $0 < a < \frac{R^3}{2}\lambda_1$ and $b \leq -\frac{a}{R^2}$, then there exists $\varepsilon = \varepsilon(a, b) > 0$ such that every non-trivial solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ of* (1.2)*, with $\|u\|_{C^1} < \varepsilon$, is strictly negative.*

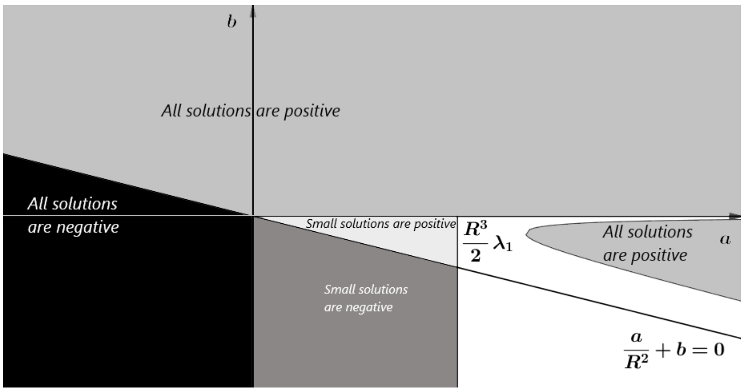

Figure 1: Positivity or negativity of possible solutions

Proof. Throughout this proof u denotes a non-trivial solution of (1.2). Let us set, for $i, j = 1, \ldots, N$,

$$a_{ij} = \frac{(1 + |\nabla u|^2)\delta_{ij} - \partial_i u\, \partial_j u}{\left(\sqrt{1 + |\nabla u|^2}\right)^3},$$

with $a_{ij} \in C^1(\Omega)$. Since, for all $\xi \in \mathbb{R}^N$,

$$\frac{|\xi|^2}{\sqrt{1 + |\nabla u|^2}} \geq \sum_{i,j=1}^{N} a_{ij}\xi_i\xi_j = \frac{(1 + |\nabla u|^2)|\xi|^2 - (\nabla u \cdot \xi)^2}{\left(\sqrt{1 + |\nabla u|^2}\right)^3} \geq \frac{|\xi|^2}{\left(\sqrt{1 + |\nabla u|^2}\right)^3},$$

the linear differential operator $\sum_{i,j=1}^{N} a_{ij}\partial_{ij}v$ is elliptic in Ω and uniformly elliptic in any subdomain Ω', with $\overline{\Omega'} \subset \Omega$. Then, we rewrite problem (1.2) in the form

$$\begin{cases} -\sum_{i,j=1}^{N} a_{ij}\partial_{ij}u = \dfrac{1}{1-bu}\Big(\dfrac{a}{(u-R)^2} + \dfrac{b}{\sqrt{1+|\nabla u|^2}}\Big), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Next, we proceed with the proof of items $(a)-(e)$.

Case (a). $b \geq 0$ *and* $\dfrac{a}{R^2} + b \geq 0$. Suppose that $a \geq 0$. As we have

$$\frac{a}{(u(x)-R)^2} + \frac{b}{\sqrt{1+|\nabla u(x)|^2}} \geq 0,$$

for all $x \in \Omega$, the weak maximum principle [25, Theorem 3.1] implies that

$$\min_{\overline{\Omega}} u = \min_{\partial\Omega} u.$$

Assume, by contradiction, that there is $x_0 \in \partial\Omega$ such that $u(x_0) = \min_{\partial\Omega} u < 0$. From the definition of (generalized) solution we have

$$\frac{\nabla u(x_0)\cdot \nu(x_0)}{\sqrt{1+|\nabla u(x_0)|^2}} = 1.$$

Hence, there exists $x_1 \in \Omega$ such that $u(x_1) < u(x_0) = \min_{\overline{\Omega}} u$, which is impossible. We have thus proved that $u > 0$. Let us show that $u \gg 0$. If there exists $x_0 \in \Omega$ such that $u(x_0) = 0 = \min_{\overline{\Omega}} u$, then the strong maximum principle implies that u vanishes in any subdomain Ω', with $\overline{\Omega'} \subset \Omega$ and $x_0 \in \Omega'$, and hence $u = 0$ in Ω, which is a contradiction, as $u \neq 0$. If there exists $x_0 \in \partial\Omega$ such that $u(x_0) = 0 = \min_{\overline{\Omega}} u$ and $\partial_\nu u(x_0) = 0$, then

$$\nabla u(x_0) = 0,$$

because $u(x_0) = \min_{\partial\Omega} u$. In this case, as by assumption $\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \in C^0(\overline{\Omega})$, we can find a ball $B \subset \Omega$, with $\partial\Omega \cap \partial B = \{x_0\}$, such that $\nabla u \in C^0(\overline{B})$. Since the differential operator on the left-hand side of the equation in (2.1) is uniformly elliptic in B, the Hopf boundary point lemma yields $\partial_\nu u(x_0) < 0$, which is a contradiction.

Suppose that $a < 0$. If there exists $x_0 \in \Omega$ such that $\min_{\overline{\Omega}} u = u(x_0) < 0$, then from (1.2) we get

$$0 \geq \frac{a}{(u(x_0)-R)^2} + b > \frac{a}{R^2} + b \geq 0,$$

which is impossible. Similarly as above, we verify that there cannot exist $x_0 \in \partial\Omega$ such that $u(x_0) = \min_{\overline{\Omega}} u < 0$. Thus, we conclude that $u \geq 0$ and, as $u \neq 0$, $u > 0$. Let us show that $u \gg 0$. First, we notice that, due to the definition of (generalized) solution, $\max_{\overline{\Omega}} u > 0$ must be attained at an interior point $x_1 \in \Omega$, which yields

$$0 \leq \frac{a}{(u(x_1)-R)^2} + b < \frac{a}{R^2} + b. \tag{2.2}$$

Next, if we suppose by contradiction the existence of $x_0 \in \Omega$ such that $\min_{\overline{\Omega}} u = u(x_0) = 0$, then, by (2.2),

$$\frac{a}{(u(x_0)-R)^2} + \frac{b}{\sqrt{1+|\nabla u(x_0)|^2}} = \frac{a}{R^2} + b > 0,$$

and we can find a maximal subdomain Ω', with $x_0 \in \Omega'$, such that, for all $x \in \Omega'$,

$$\frac{a}{(u(x)-R)^2} + \frac{b}{\sqrt{1+|\nabla u(x)|^2}} > 0.$$

As $a < 0$, we have $\nabla u \in L^\infty(\Omega')$ and hence the differential operator on the left-hand side of the equation in (2.1) is uniformly elliptic in Ω'. The strong maximum principle yields $u = 0$ in Ω', thus contradicting the maximality of Ω', in case $\Omega' \neq \Omega$, or $u(x) > 0$ for all $x \in \Omega$, in case $\Omega' = \Omega$. Similarly as above, we finally prove that if there exists $x_0 \in \partial\Omega$ such that $u(x_0) = 0$, then $\partial_\nu u(x_0) \in [-\infty, 0[$.

Case (b). $a > 0$, $b < 0$ *and* $|b|a \geq (1 + R|b|)^2$. We first notice that, as $\min_{\overline{\Omega}} u > -\frac{1}{|b|}$,

$$\frac{a}{(u(x)-R)^2} + \frac{b}{\sqrt{1+|\nabla u(x)|^2}} > \frac{a}{(\frac{1}{|b|}+R)^2} + b \geq 0,$$

for all $x \in \Omega$. Next, we argue like in case (a) to prove that $u \gg 0$.

Case (c). $0 < a < \frac{R^3}{2}\lambda_1$ *and* $-\frac{a}{R^2} \leq b < 0$. Let us introduce the function $\eta :]-\infty, R[\to \mathbb{R}$ defined by

$$\frac{1}{(R-s)^2} = \frac{1}{R^2} + \frac{2}{R^3}s + \eta(s)s. \tag{2.3}$$

Clearly, η satisfies

$$\eta(s)s \geq 0, \quad \text{for all } s < R, \qquad \text{and} \qquad \lim_{s\to 0} \eta(s) = 0.$$

Pick $\varepsilon > 0$ such that

$$a\varepsilon < R^2 \qquad \text{and} \qquad \frac{2a}{R}\frac{\sqrt{1+\varepsilon^2}}{R^2 - a\varepsilon} < \lambda_1. \tag{2.4}$$

Let u be a solution of (1.2) satisfying $\|u\|_{C^1} < \varepsilon$. Thus, we also have that $u \in H_0^1(\Omega)$. Multiplying the equation in (1.2) by $\frac{-u^-}{1-bu}$ and integrating by parts, by (2.3), we get, as $\dfrac{a}{R^2} + b \geq 0$,

$$\begin{aligned}\frac{1}{\sqrt{1+\varepsilon^2}}\int_\Omega |\nabla u^-|^2\,dx &\leq \int_\Omega \frac{|\nabla u^-|^2}{\sqrt{1+|\nabla u|^2}}\,dx = -\int_\Omega \Big(\frac{a}{(R-u)^2} + \frac{b}{\sqrt{1+|\nabla u|^2}}\Big)\frac{u^-}{1-bu}\,dx \\ &\leq \int_\Omega \Big(-\frac{a}{R^2} + \frac{2a}{R^3}u^- - b\Big)\frac{u^-}{1+bu^-}\,dx \\ &\leq \int_\Omega \frac{2a}{R^3}\frac{(u^-)^2}{1+bu^-}\,dx \leq \frac{2a}{R}\frac{1}{R^2-a\varepsilon}\int_\Omega |u^-|^2\,dx.\end{aligned}$$

Condition (2.4) and the Poincaré inequality yield $u^- = 0$ and hence $u \geq 0$. As $u \neq 0$ and

$$\frac{a}{(u(x)-R)^2} + \frac{b}{\sqrt{1+|\nabla u(x)|^2}} \geq \frac{a}{R^2} + b \geq 0,$$

for all $x \in \Omega$, the strong maximum principle and the Hopf boundary point lemma finally imply that $u \gg 0$.

Case (d). $a \leq 0$ *and* $\frac{a}{R^2} + b \leq 0$. The proof is similar to the one of Case (a) and therefore is omitted.

Case (e). $0 < a < \frac{R^3}{2}\lambda_1$ *and* $b \le -\frac{a}{R^2}$. Let η be the function defined in (2.3). Pick $\varepsilon \in]0, R[$ such that

$$\frac{\varepsilon|b|}{2(1+\varepsilon|b|)} < \frac{1}{\sqrt{1+\varepsilon^2}} \tag{2.5}$$

and

$$\frac{a\left(\frac{2}{R^3} + \eta_\varepsilon\right)}{\frac{1}{\sqrt{1+\varepsilon^2}} - \frac{\varepsilon|b|}{2(1+\varepsilon|b|)}} < \lambda_1, \tag{2.6}$$

where

$$\eta_\varepsilon = \sup_{|s|\le\varepsilon} |\eta(s)|.$$

Let u be a solution of (1.2) satisfying $\|u\|_{C^1} < \varepsilon$. Hence, we have $u \in H^1_0(\Omega)$. Multiplying the equation in (1.2) by $\frac{u^+}{1-bu}$ and integrating by parts, we get

$$\begin{aligned}
\frac{1}{\sqrt{1+\varepsilon^2}} \int_\Omega |\nabla u^+|^2\,dx &\le \int_\Omega \frac{|\nabla u^+|^2}{\sqrt{1+|\nabla u|^2}}\,dx = \int_\Omega \Big(\frac{a}{(R-u)^2} + \frac{b}{\sqrt{1+|\nabla u|^2}}\Big)\frac{u^+}{1-bu}\,dx \\
&= \int_\Omega \Big(\frac{a}{R^2} + b + \frac{2a}{R^3}u^+ + a\,\eta(u^+)u^+ - \frac{b|\nabla u^+|^2}{\sqrt{1+|\nabla u|^2}(1+\sqrt{1+|\nabla u|^2})}\Big)\frac{u^+}{1+|b|u^+}\,dx \\
&\le a\Big(\frac{2}{R^3} + \eta_\varepsilon\Big)\int_\Omega |u^+|^2 dx + \int_\Omega \frac{|\nabla u^+|^2}{2}\,\frac{|b|u^+}{1+|b|u^+}\,dx.
\end{aligned}$$

As the function $f(s) = \frac{|b|s}{1+|b|s}$ is increasing, we conclude that

$$\frac{1}{\sqrt{1+\varepsilon^2}} \int_\Omega |\nabla u^+|^2\,dx \le a\Big(\frac{2}{R^3} + \eta_\varepsilon\Big)\int_\Omega |u^+|^2 dx + \frac{|b|\varepsilon}{2(1+|b|\varepsilon)}\int_\Omega |\nabla u^+|^2\,dx.$$

Conditions (2.5) and (2.6), together with the Poincaré inequality, yield $u^+ = 0$. As $u \ne 0$, arguing as in Case (a), with $a < 0$, we infer that $u \ll 0$. $\square$

Remark 2.1 It is clear that, if $\frac{a}{R^2} + b = 0$ and $a \le 0$, then problem (1.2) has only the trivial solution. Moreover, if $\frac{a}{R^2} + b = 0$ and $a < \frac{R^3}{2}\lambda_1$, then the trivial solution is isolated in $C^1(\bar\Omega)$. On the other hand, if $\frac{a}{R^2} + b \ne 0$, then any solution of problem (1.2) is non-trivial.

Remark 2.2 Note that, in Case (c), if $\varepsilon > 0$ fulfills the conditions in (2.4) for some $0 < a < \frac{R^3}{2}\lambda_1$, then it satisfies them for all $0 < \bar a < a$. In the same way, in Case (e), if $\varepsilon \in]0, 1[$, then (2.5) holds for all b; further, if $\varepsilon \in]0, 1[$ fulfills (2.6) for some $0 < a < \frac{R^3}{2}\lambda_1$ and $b < -\frac{a}{R^2}$, then it satisfies it for all $0 < \bar a < a$ and $\bar b \le -\frac{\bar a}{R^2}$, with $\bar b \ge b$. Hence, we conclude that, in Cases (c) and (e) respectively,

- if $a, b, \bar a, \bar b$ satisfy $0 < \bar a \le a < \frac{R^3}{2}\lambda_1$, $-\frac{a}{R^2} \le b < 0$, $-\frac{\bar a}{R^2} \le \bar b < 0$, then $\varepsilon(a, b) \le \varepsilon(\bar a, \bar b)$

  and

- if $a, b, \bar a, \bar b$ satisfy $0 < \bar a \le a < \frac{R^3}{2}\lambda_1$, $b \le -\frac{a}{R^2}$, $\bar b \le -\frac{\bar a}{R^2}$, $b \le \bar b$, then $\varepsilon(a, b) \le \varepsilon(\bar a, \bar b)$.

Next, we give some information about the symmetries of the possible solutions of (1.2).

Lemma 2.2. *Assume* (H_1). *Suppose that* $\mathbb{M}$ *is an orthogonal matrix such that* $\mathbb{M}(\Omega) = \Omega$. *Let* u *be a solution of* (1.2) *and set* $v(x) = u(\mathbb{M}x)$ *for all* $x \in \overline{\Omega}$. *Then,* v *is a solution of* (1.2).

Proof. For all $x \in \Omega$, we have
$$\nabla v(x) = \mathbb{M}^T \nabla u(\mathbb{M}x)$$
and
$$J\Big(\frac{\nabla v(x)}{\sqrt{1+|\nabla v(x)|^2}}\Big) = J\Big(\frac{\mathbb{M}^T \nabla u(\mathbb{M}x)}{\sqrt{1+|\mathbb{M}^T \nabla u(\mathbb{M}x)|^2}}\Big) = \mathbb{M}^T J\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big)(\mathbb{M}x)\mathbb{M},$$
where $Jf(x)$ denotes the Jacobian matrix of the function f evaluated at x. Hence, as $\mathbb{M}^T = \mathbb{M}^{-1}$, we get
$$\begin{aligned}\operatorname{div}\Big(\frac{\nabla v(x)}{\sqrt{1+|\nabla v(x)|^2}}\Big) &= \operatorname{tr}\Big(J\Big(\frac{\nabla v(x)}{\sqrt{1+|\nabla v(x)|^2}}\Big)\Big)\\ &= \operatorname{tr}\Big(\mathbb{M}^T J\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big)(\mathbb{M}x)\mathbb{M}\Big)\\ &= \operatorname{tr}\Big(J\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big)(\mathbb{M}x)\Big)\\ &= \operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big)(\mathbb{M}x).\end{aligned}$$
Thus, we conclude that v satisfies the differential equation in (1.2). Further, as $\nu(x) = \mathbb{M}^T \nu(\mathbb{M}x)$, we have
$$\nabla v(x)\cdot \nu(x) = \mathbb{M}^T \nabla u(\mathbb{M}x)\cdot \mathbb{M}^T \nu(\mathbb{M}x) = \nabla u(\mathbb{M}x)\cdot \nu(\mathbb{M}x).$$
Therefore, v satisfies the boundary conditions in (1.2) as well. □

Additional information can be obtained in the one-dimensional case.

Lemma 2.3. *Assume (H_1). Let $u :]\alpha,\omega[\to \mathbb{R}$ be a maximal solution of*
$$\begin{cases} -(1-bu)\Big(\dfrac{u'}{\sqrt{1+|u'|^2}}\Big)' = \dfrac{a}{(u-R)^2} + \dfrac{b}{\sqrt{1+|u'|^2}},\\ u(x_0) = u_0,\ u'(x_0) = 0,\end{cases} \tag{2.7}$$
for some $x_0 \in]\alpha,\omega[$ and $u_0 \in \mathbb{R}$. Then, u is symmetric with respect to x_0, i.e., $x \in]\alpha,\omega[$ if and only if $2x_0 - x \in]\alpha,\omega[$ and $u(x) = u(2x_0 - x)$.

Proof. For all $x \in]2x_0-\omega, 2x_0-\alpha[$, define $v(x) = u(2x_0 - x)$. Due to the structure of the equation, v is a solution of (2.7) too. Hence, by uniqueness, we conclude that either $]\alpha,\omega[= \mathbb{R}$, or $]\alpha,\omega[$ is bounded and $x_0 = \frac{1}{2}(\alpha+\omega)$, with $u(x) = u(2x_0-x)$ for all $x \in]\alpha,\omega[$. □

Remark 2.3 Due to the autonomous character of problem (1.2), we can always suppose, in the one-dimensional case, that Ω is a symmetric interval, e.g., $\Omega =]-r, r[$.

Proposition 2.4. *Assume (H_1). Let u be a solution of*
$$\begin{cases} -(1-bu)\Big(\dfrac{u'}{\sqrt{1+|u'|^2}}\Big)' = \dfrac{a}{(u-R)^2} + \dfrac{b}{\sqrt{1+|u'|^2}}, & \text{in }]-r,r[,\\ u(-r) = 0,\ u(r) = 0.\end{cases} \tag{2.8}$$
Then, then following conclusions hold:

(a) u has a finite number of zeros in $[-r, r]$;

(*b*) *for each maximal interval* $]x_1, x_2[$ *such that* $u(x) \neq 0$ *for all* $x \in]x_1, x_2[$, u *is symmetric with respect to the midpoint* $\frac{x_1+x_2}{2}$ *of* $]x_1, x_2[$*;*

(*c*) *if* u *has a zero, then* u *is a classical solution of* (2.8)*, with further* $u \in C^1([-r, r])$*, in case this is an interior zero;*

(*d*) *if the number of zeros of* u *in* $[-r, r]$ *is even, then* u *is even.*

Proof. Part (*a*). We rewrite the equation in (2.8) in the form

$$-u'' = g(u, u')$$

with

$$g(s, \xi) = \Big(\frac{a}{(s-R)^2} + \frac{b}{\sqrt{1+\xi^2}}\Big)\frac{(1+\xi^2)^{3/2}}{1-bs}.$$

As g is real analytic, the Cauchy theorem implies that u is real analytic (cf., e.g., [5, p. 196]); hence u has a finite number of zeros in $[-r, r]$.

Part (*b*). Consider a maximal interval $]x_1, x_2[\subseteq [-r, r]$ such that $u(x) \neq 0$ for all $x \in]x_1, x_2[$. Suppose that, for all $x \in]x_1, x_2[$, $u(x) > 0$, the other case being treated similarly. By definition of solution, there exists $x_0 \in]x_1, x_2[$ such that $u'(x_0) = 0$. Lemma 2.3 implies that u is symmetric with respect to x_0. To fix ideas, let us assume that $x_0 - x_1 \leq x_2 - x_0$. Then, in case $u(x_1) = 0$, we have also $u(2x_0 - x_1) = 0$. As $u(x) > 0$ in $]x_1, x_2[$, we conclude that $2x_0 - x_1 = x_2$, i.e., $x_0 - x_1 = x_2 - x_0$ and u is symmetric with respect to the midpoint of $]x_1, x_2[$. On the other hand, in case $x_1 = -r$, $u(-r) > 0$ and $u'(-r^+) = +\infty$, we have $u(2x_0 + r) > 0$ and $u'(2x_0 + r^-) = -\infty$, which implies $2x_0 + r = r$, i.e., $x_0 = 0$ and again the point of symmetry of u is the midpoint of the interval.

Part (*c*). Assume first that u vanishes at the boundary of the domain only, that is, $u(-r) \cdot u(r) = 0$, and $u(x) \neq 0$ for all $x \in]-r, r[$. By Part (*b*), we deduce that u is symmetric with respect to 0, therefore $u(-r) = 0 = u(r)$ and u is a classical solution of (2.8). Second, assume that u has an interior zero x_1. We can suppose that x_1 is the smallest interior zero of u. Then, applying Part (*b*), with reference to the interval $]-r, x_1[$, we see that $u(-r) = 0$ and $|u'(-r)| < +\infty$. The same argument shows that $u(r) = 0$.

Part (*d*). Assume that u has four zeros. By Part (*c*), u is a classical solution and hence the four zeros can be labelled as $-r < x_1 < x_2 < r$. Set $x_0 = \frac{x_1+x_2}{2}$. By Part (*b*), we know that $x_0 - x_1 = x_2 - x_0$ and $u(x) = u(2x_0 - x)$. Suppose that $x_0 \leq 0$, the other case being treated similarly. Then, we have $0 = u(-r) = u(2x_0 + r)$. As $u(x) \neq 0$ in $]x_2, r[$ and $2x_0 + r \in]x_2, r]$, we conclude that $2x_0 + r = r$ and hence $x_0 = 0$. In the general case, where the number of zeros of u is even, the conclusion is achieved by a recursive argument. □

We end this section with the following result.

Lemma 2.5. *Assume* (H_1)*. Let* (a, b) *be such that* $\dfrac{a}{R^2} + b \geq 0$*. Then, every positive solution of* (2.8) *is concave, even and strictly positive.*

Proof. Let u be a positive solution of (2.8).

In case $a \geq 0$ and $\dfrac{a}{R^2} + b \geq 0$, we have, for all $x \in]-r, r[$,

$$\frac{a}{(u(x)-R)^2} + \frac{b}{\sqrt{1+|u'(x)|^2}} \geq 0, \qquad \text{if } b \geq 0,$$

$$\frac{a}{(u(x)-R)^2} + \frac{b}{\sqrt{1+|u'(x)|^2}} \geq \frac{a}{R^2} + b \geq 0, \qquad \text{if } b < 0.$$

This implies that u is concave and hence strictly positive. The symmetry of u then follows from Proposition 2.4.

In case $a < 0$ and $\dfrac{a}{R^2} + b \geq 0$, Theorem 2.1 - Case (a) guarantees that u is strictly positive and Proposition 2.4 implies that it is even. Let us prove that u is concave. Let $x_0 \in \,]-r, r[$ be such that $u(x_0) = \max u$. We have $u''(x_0) \leq 0$ and hence

$$\frac{a}{(\max u - R)^2} + b \geq 0.$$

Note that, for any $x_1 \in [-r, r]$, we have

$$\frac{a}{(u(x_1) - R)^2} + b \geq \frac{a}{(\max u - R)^2} + b \geq 0.$$

In particular, if x_1 is any critical point of u, we have $u''(x_1) \leq 0$, with $u''(x_1) < 0$ whenever $u(x_1) < \max u$; thus, in any case, x_1 is a local maximum point of u. This implies that $u'(x) \geq 0$ in $]-r, 0]$ and $u'(x) \leq 0$ in $[0, r[$. Furthermore, there is $r_1 \in [0, r[$ such that $u'(x) > 0$ in $]-r, -r_1[$, $u'(x) = 0$ if $-r_1 \leq x \leq r_1$, $u'(x) < 0$ in $]r_1, r[$. As u is analytic we have indeed $r_1 = 0$. Let us show that u' in decreasing in $[-r, 0]$. Assume, by contradiction, that this is false, i.e., there exist $x_1, x_2 \in [-r, 0[$, with $x_1 < x_2$, such that $u'(x_1) < u'(x_2)$. Hence, u' must have a local maximum point $x_0 \in]-r, 0[$. Computing the third derivative of u at x_0 and using $u''(x_0) = 0$ and hence $\frac{a}{(u(x_0)-R)^2} + \frac{b}{\sqrt{1+|u'(x_0)|^2}} = 0$, we obtain, from the equation in (2.8),

$$u'''(x_0) = -\frac{2au'(x_0)}{(R - u(x_0))^3} \frac{(1 + u'(x_0)^2)^{3/2}}{1 - bu(x_0)} > 0,$$

which is a contradiction. Thus, we conclude, by symmetry, that u is concave in $[-r, r]$. □

3 Existence and non-existence of solutions

A necessary condition for the solvability

We state in this section a quantitative necessary condition for the existence of solutions of problem (1.2). This condition provides an estimate for the pull-in voltage [8, p.12], as expressed by Remark 3.1.

Theorem 3.1. *Assume (H_1). Then, for every $b \in \mathbb{R}$ there exists $\hat{a}(b) \in \mathbb{R}^+$ such that problem* (1.2) *has no solutions for $a > \hat{a}(b)$. In addition, no solution may exist for $a \geq 0$ and $b \geq \frac{|\partial\Omega|}{|\Omega|}$.*

Proof. Let us observe that, if u is a solution of (1.2), then the structure of the equation implies that $\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) \in L^\infty(\Omega)$. Hence, according to [3], the vector field $\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}$ belongs to the space $X(\Omega)_N$ and the weak trace $\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \cdot \nu$ on $\partial\Omega$ of the component of $\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}$ with respect to the unit outer normal ν to Ω is defined. Thus, the integration by parts formula holds by [3, Theorem 1.1].

Assume first $b \geq 0$. Let u be a non-trivial solution of (1.2) for some $a \geq 0$. We know, from Theorem 2.1 - Case (a), that $u \gg 0$. Integrating in Ω the equation in (1.2), we get

$$\begin{aligned} &-\int_\Omega (1 - bu) \operatorname{div}\Big(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\Big)\, dx \\ &\quad = -\int_{\partial\Omega} (1 - bu) \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \cdot \nu \, d\mathcal{H}^{N-1} - b \int_\Omega \frac{|\nabla u|^2}{\sqrt{1 + |\nabla u|^2}}\, dx \end{aligned}$$

$$= -\int_{\partial\Omega}(1-bu)\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\cdot\nu\, d\mathcal{H}^{N-1} - b\int_\Omega \sqrt{1+|\nabla u|^2}\,dx + \int_\Omega \frac{b}{\sqrt{1+|\nabla u|^2}}\,dx$$
$$= \int_\Omega \frac{a}{(R-u)^2}\,dx + \int_\Omega \frac{b}{\sqrt{1+|\nabla u|^2}}\,dx$$

and hence

$$\begin{aligned} |\partial\Omega| &\geq -\int_{\partial\Omega}(1-bu)\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\cdot\nu\, d\mathcal{H}^{N-1} \\ &= \int_\Omega \frac{a}{(R-u)^2}\,dx + b\int_\Omega \sqrt{1+|\nabla u|^2}\,dx > \big(\frac{a}{R^2}+b\big)|\Omega|. \end{aligned}$$

This implies that, if $a \geq 0, b \geq 0$, $\frac{a}{R^2}+b \geq \frac{|\partial\Omega|}{|\Omega|}$ problem (1.2) has no solutions.

Assume next $b<0$. Let u be a non-trivial solution of (1.2) for some $a>0$. Suppose $a \geq \frac{(1+|b|R)^2}{|b|}$. We know, from Theorem 2.1 - Case (b), that $u \gg 0$. Integrating in Ω the equation in (1.2), we get

$$\begin{aligned} |\partial\Omega| &\geq -\int_{\partial\Omega}\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\cdot\nu\, d\mathcal{H}^{N-1} \\ &= \int_\Omega \frac{1}{1+|b|u}\Big(\frac{a}{(R-u)^2}+\frac{b}{\sqrt{1+|\nabla u|^2}}\Big)\,dx \\ &> \frac{1}{1+|b|R}\Big(\frac{a}{R^2}-|b|\Big)|\Omega|. \end{aligned}$$

This implies that, if $a \geq \max\big\{\big(\frac{|\partial\Omega|}{|\Omega|}(1+|b|R)+|b|\big)R^2, \frac{(1+|b|R)^2}{|b|}\big\}$, problem (1.2) has no solutions. $\square$

Remark 3.1 From the proof of Theorem 3.1, we get the estimate

$$\hat{a}(b) \leq \begin{cases} \max\big\{\big(\frac{|\partial\Omega|}{|\Omega|}(1+|b|R)+|b|\big)R^2, \frac{(1+|b|R)^2}{|b|}\big\}, & \text{if } b<0, \\ \big(\frac{|\partial\Omega|}{|\Omega|}-b\big)R^2, & \text{if } b \geq 0. \end{cases}$$

It is apparent from the structure of (1.2) that $\hat{a}(b) \to +\infty$, as $b \to -\infty$.

Existence of solutions perturbing from $(0, a, -\frac{a}{R^2})$

We discuss in this section the existence of classical solutions stemming from the trivial solution at the points $(a, -\frac{a}{R^2})$. We recall again that $\Sigma = \{\lambda_n \mid n \in \mathbb{N}^+\}$ denotes the spectrum of $-\Delta$ in $H_0^1(\Omega)$.

Lemma 3.2. *Assume (H_1). Then, for each $(a_0, b_0) \in \mathbb{R}\times\mathbb{R}$, with $\frac{a_0}{R^2}+b_0 = 0$ and $\frac{2a_0}{R^3} \notin \Sigma$, there exists $\delta_0 > 0$ such that, for any $(a,b) \in \mathbb{R}\times\mathbb{R}$, with $|a-a_0|+|b-b_0| < \delta_0$, problem (1.2) has a classical solution $u = u(a,b) \in C^1(\overline{\Omega})$, which continuously depends on the parameters (a,b) in the topology of $C^1(\overline{\Omega})$ and satisfies*

$$\lim_{(a,b)\to(a_0,b_0)} \|u(a,b)\|_{C^1} = 0.$$

In addition, if $a_0 < \frac{R^3}{2}\lambda_1$, then $\delta_0>0$ can be chosen so that

$$u(a,b_0) \gg 0, \quad \text{for all } a_0 < a < a_0+\delta_0, \tag{3.1}$$

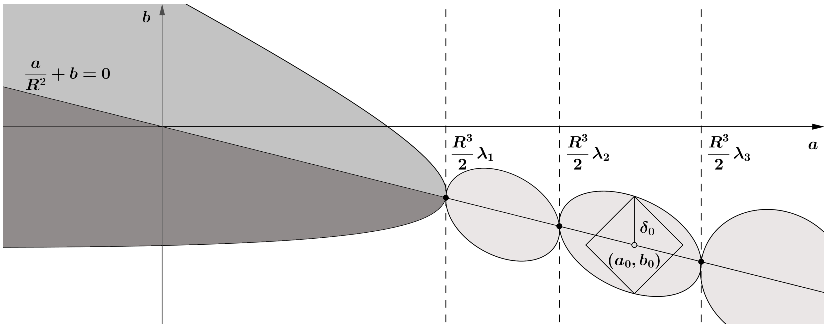

Figure 2: Regions of existence of classical solutions, coloured with different shades of grey. If $a < \frac{R^3}{2}\lambda_1$, the solutions are strictly positive above the line $\frac{a}{R^2} + b = 0$ (light grey region), strictly negative below (dark grey region).

and

$$u(a, b_0) \ll 0, \quad \textit{for all } a_0 - \delta_0 < a < a_0. \tag{3.2}$$

Finally, if $a_0 < \frac{R^3}{2}\lambda_1$*, then* u *is (locally exponentially asymptotically) whereas, if* $a_0 > \frac{R^3}{2}\lambda_1$*, then* u *is unstable.*

Proof. Fix $p > N$ and define, in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \times \mathbb{R} \times \mathbb{R}$, the open set

$$\mathcal{V} = \{(u, a, b) \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \times \mathbb{R} \times \mathbb{R} \mid u(x) < R \text{ and } b\, u(x) < 1 \text{ in } \overline{\Omega}\} \tag{3.3}$$

and the operator $\mathcal{F} : \mathcal{V} \to L^p(\Omega)$ by

$$\mathcal{F}(u, a, b) = \operatorname{div}\Big(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\Big) + \Big(\frac{a}{(u - R)^2} + \frac{b}{\sqrt{1 + |\nabla u|^2}}\Big)\frac{1}{1 - bu}. \tag{3.4}$$

From the Addendum A, we know that $\mathcal{F}$ is of class C^∞, with partial derivative

$$\begin{aligned}\partial_u \mathcal{F}(u, a, b)[v] = \operatorname{div}\Big(\frac{\nabla v}{\sqrt{1 + |\nabla u|^2}} - \frac{\nabla u \cdot \nabla v}{\big(\sqrt{1 + |\nabla u|^2}\big)^3}\nabla u\Big) - \Big(\frac{2av}{(u - R)^3} + \frac{b\,\nabla u \cdot \nabla v}{\big(\sqrt{1 + |\nabla u|^2}\big)^3}\Big)\frac{1}{1 - bu} \\ + \Big(\frac{a}{(u - R)^2} + \frac{b}{\sqrt{1 + |\nabla u|^2}}\Big)\frac{bv}{(1 - bu)^2},\end{aligned}$$

for all $v \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. If $\frac{a_0}{R^2} + b_0 = 0$, we have

$$\mathcal{F}(0, a_0, b_0) = 0 \quad \text{and} \quad \partial_u \mathcal{F}(0, a_0, b_0)[v] = \Delta v + \frac{2a_0}{R^3}v.$$

As $\frac{2a_0}{R^3} \notin \Sigma$, the Fredholm alternative [23, p. 303] and the open mapping theorem imply that

$$\partial_u \mathcal{F}(0, a_0, b_0) : W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \to L^p(\Omega)$$

is a linear homeomorphism. Hence the implicit function theorem yields the existence of a constant $\delta_0 > 0$ and a map $U : \{(a, b) \in \mathbb{R}^2 \mid |a - a_0| + |b - b_0| < \delta_0\} \to W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ of class C^∞ such that, for all $(u, a, b) \in \mathcal{V}$, with $\|u\|_{W^{2,p}} < \delta_0$, $|a - a_0| + |b - b_0| < \delta_0$,

$$\mathcal{F}(u, a, b) = 0 \quad \text{if and only if} \quad u = U(a, b).$$

For simplicity we write $u(a,b)$ in place of $U(a,b)$ to indicate the dependence of the solution u on the parameters a, b. As $W^{2,p}(\Omega)$ is embedded into $C^{1,\alpha}(\overline{\Omega})$ for any $\alpha \in \,]0, 1-\frac{N}{p}[$, it follows that $u \in C^{1,\alpha}(\overline{\Omega})$. Then, we infer, from [25, Theorem 9.19], that $u \in C^{2,\alpha}(\Omega)$ and is a classical solution.

Finally, the sign properties of the solutions can be proved as follows. Fix (a_0, b_0), with $\frac{a_0}{R^2} + b_0 = 0$. According to Theorem 2.1, only the case $a_0 > 0$ requires a proof. Therefore, assume $a_0 \in]0, \frac{R^3}{2}\lambda_1[$, and pick $\bar{a} \in \,]a_0, \frac{R^3}{2}\lambda_1[$. From Theorem 2.1 and Remark 2.2, we can find $\varepsilon = \varepsilon(\bar{a}, b_0) > 0$ such that, for all $a \in \,]a_0, \bar{a}[$, all solutions of (1.2), with

$$\|u\|_{C^1} < \varepsilon,$$

satisfy $u \gg 0$. Choosing $\delta_0 > 0$ sufficiently small and using the Sobolev embedding theorem yield (3.1). A similar argument allows to prove (3.2).

The stability conclusions follows from the linearized stability-instability principle in, e.g., [33, Section 9.1]. □

From Lemma 3.2 we derive the following existence result for problem (1.2). Here we just consider the situation $a \geq 0$ we are mainly concerned with.

Theorem 3.3. *Assume (H_1). Then, there exists $b^* \in \mathbb{R}^+$ such that for all $b < b^*$, with $b \notin \{-\frac{R}{2}\lambda_n \mid n \in \mathbb{N}^+\}$, there are constants $a_*(b)$, $a^*(b)$, with $0 \leq a_*(b) < a^*(b) \leq \hat{a}(b)$ ($\hat{a}(b)$ being defined in Theorem 3.1), such that, for all $a \in \,]a_*(b), a^*(b)[$, problem (1.2) has a classical solution $u = u(a,b) \in C^1(\overline{\Omega})$. In addition, we have that*

- *(a) if $b \geq 0$, then $u \gg 0$;*
- *(b) if $-\frac{R}{2}\lambda_1 < b < 0$ and $a_*(b) < a < -R^2 b$, then $u \ll 0$;*
- *(c) if $-\frac{R}{2}\lambda_1 < b < 0$ and $-R^2 b < a < a^*(b)$, then $u \gg 0$.*

Finally, u is stable if $b > -\frac{R}{2}\lambda_1$, unstable otherwise.

Existence of solutions perturbing from $(u, 0, b)$

We discuss in this section the existence of classical solutions obtained by perturbing problem (1.2) with $a = 0$. We consider at first the case $a = 0$ and then the case $a > 0$. Special attention is devoted to the one-dimensional case where multiple solutions are detected.

The case $a = 0$, N arbitrary. Consider problem (1.2) with $a = 0$, that is,

$$\begin{cases} -\mathrm{div}\Big(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) = \dfrac{b}{1-bu}\dfrac{1}{\sqrt{1+|\nabla u|^2}}, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.5}$$

Assume (H_1) and fix any $\alpha \in \,]0,1[$. Set, for convenience,

$$A = c(N)|\Omega|^{1/N}(2^{N/2}+1) \tag{3.6}$$

and

$$\hat{b} = \frac{1}{c(N)|\Omega|^{1/N}(2^{N/2}+1)^2}, \tag{3.7}$$

where $c(N) = N^{-1}\omega_N^{-1/N}$ and ω_N is the measure of the unit sphere in $\mathbb{R}^N$. For any given $b \in]0, \hat{b}[$, define the open bounded subset of $C^{1,\alpha}(\overline{\Omega})$

$$\mathcal{U}_M = \{u \in C^{1,\alpha}(\overline{\Omega}) \mid \|u\|_\infty < \tfrac{1}{b} - A, \ \|\nabla u\|_{C^{0,\alpha}} < M\}, \tag{3.8}$$

where $M > 0$ is a constant that will be determined in Proposition 3.4 below. Pick $p > \frac{N}{1-\alpha}$ and, for each $v \in \overline{\mathcal{U}}_M$ and $\sigma \in [0, b]$, denote by $u = \mathcal{T}(\sigma, v) \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ the unique solution of the problem

$$\begin{cases} -\displaystyle\sum_{i,j=1}^{N} \frac{(1+|\nabla v|^2)\delta_{ij} - \partial_i v\, \partial_j v}{\left(\sqrt{1+|\nabla v|^2}\right)^3} \partial_{ij} u = \frac{\sigma}{1-\sigma v} \frac{1}{\sqrt{1+|\nabla v|^2}}, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Such a solution $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ does exist by, e.g., [25, Theorem 9.15], since the linear differential operator $\mathcal{L} : W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \to L^p(\Omega)$, defined by

$$\mathcal{L}u = \sum_{i,j=1}^{N} a_{ij} \partial_{ij} u,$$

with coefficients

$$a_{ij} = \frac{(1+|\nabla v|^2)\delta_{ij} - \partial_i v\, \partial_j v}{\left(\sqrt{1+|\nabla v|^2}\right)^3} \in C^{0,\alpha}(\overline{\Omega}) \quad \text{for } i, j = 1, \ldots, N,$$

is uniformly elliptic in Ω, as

$$\frac{|\xi|^2}{\sqrt{1+|\nabla v|^2}} \geq \sum_{i,j=1}^{N} a_{ij}\xi_i\xi_j = \frac{(1+|\nabla v|^2)|\xi|^2 - (\nabla v \cdot \xi)^2}{\left(\sqrt{1+|\nabla v|^2}\right)^3} \geq \frac{|\xi|^2}{\left(\sqrt{1+|\nabla v|^2}\right)^3} \quad \text{for all } \xi \in \mathbb{R}^N,$$

and

$$\frac{\sigma}{1-\sigma v} \frac{1}{\sqrt{1+|\nabla v|^2}} \in C^{0,\alpha}(\overline{\Omega}).$$

By using the L^p elliptic regularity theory (see, e.g., [25, Lemma 9.17]) and the compact embedding of $W^{2,p}(\Omega)$ into $C^{1,\alpha}(\overline{\Omega})$, it is a standard matter to prove that $\mathcal{T} : [0, b] \times \overline{\mathcal{U}}_M \to C^{1,\alpha}(\overline{\Omega})$ is completely continuous. It is also clear that the fixed points $u \in \overline{\mathcal{U}}_M$ of $\mathcal{T}(b, \cdot)$ are precisely the classical solutions $u \in \overline{\mathcal{U}}_M$ of (3.5).

Proposition 3.4. *Assume (H_1) and, in case $N \geq 2$, further suppose that*

(H_2) for all $x \in \partial\Omega$, the mean curvature $H(x)$ of $\partial\Omega$ at x satisfies $H(x) \geq 0$.

Let $\hat{b}$ be given by (3.7). Then, for each $b \in]0, \hat{b}[$, there exists a constant $M > 0$ such that

$$\deg(\mathcal{I} - \mathcal{T}(b, \cdot), \mathcal{U}_M, 0) = 1, \tag{3.9}$$

where $\mathcal{U}_M$ is defined in (3.8) and $\mathcal{I}$ is the identity operator. In particular, problem (3.5) has at least one classical solution $u \in \mathcal{U}_M$ with $u \gg 0$.

Proof. We fix $b \in]0, \hat{b}[$ and look for a constant $M > 0$ such that, defining $\mathcal{U}_M$ through (3.8), condition (3.9) holds. To this end, for any $\sigma \in [0, b]$, we consider the problem

$$\begin{cases} -\operatorname{div}\left(\dfrac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \dfrac{\sigma}{1-\sigma u} \dfrac{1}{\sqrt{1+|\nabla u|^2}}, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.10}$$

Let u be a classical solution of (3.10), for some $\sigma \in [0, b]$, satisfying

$$\|u\|_\infty \leq \frac{1}{b} - A.$$

By Theorem 2.1 we have $u \geq 0$, with $u = 0$ if and only if $\sigma = 0$.

Step 1. $\|u\|_\infty \leq 2^{\frac{N}{2}} A < \frac{1}{b} - A$. We follow some ideas from [20, Lemma]. For $\delta \in [0, \frac{1}{b} - A[$, let us consider the function $v_\delta = \max\{u - \delta, 0\}$ and the set $A_\delta = \{x \in \Omega \mid u(x) \geq \delta\}$. Using v_δ as test function in (3.10), we obtain

$$\int_{A_\delta} \frac{|\nabla v_\delta|^2}{\sqrt{1 + |\nabla v_\delta|^2}}\, dx = \int_{A_\delta} \frac{\sigma}{1 - \sigma u} \frac{v_\delta}{\sqrt{1 + |\nabla v_\delta|^2}}\, dx$$

and hence

$$\int_{A_\delta} \sqrt{1 + |\nabla v_\delta|^2}\, dx - \int_{A_\delta} \frac{1}{\sqrt{1 + |\nabla v_\delta|^2}}\, dx = \int_{A_\delta} \frac{\sigma}{1 - \sigma u} \frac{v_\delta}{\sqrt{1 + |\nabla v_\delta|^2}}\, dx.$$

Observing that

$$\frac{\sigma}{1 - \sigma u} \leq \frac{b}{1 - bu} \leq \frac{b}{1 - b(\frac{1}{b} - A)} = \frac{1}{A},$$

we then deduce that

$$\begin{aligned}
\int_{A_\delta} |\nabla v_\delta|\, dx &\leq \int_{A_\delta} \sqrt{1 + |\nabla v_\delta|^2}\, dx \\
&= \int_{A_\delta} \frac{1}{\sqrt{1 + |\nabla v_\delta|^2}}\, dx + \int_{A_\delta} \frac{\sigma}{1 - \sigma u} \frac{v_\delta}{\sqrt{1 + |\nabla v_\delta|^2}}\, dx \\
&\leq |A_\delta| + \frac{1}{A} \int_{A_\delta} v_\delta\, dx.
\end{aligned}$$

Using Sobolev inequality on the left-hand side and Hölder inequality on the right-hand side, we obtain

$$\|v_\delta\|_{L^{\frac{N}{N-1}}} \leq \frac{A\, c(N)}{A - c(N)\, |\Omega|^{1/N}} |A_\delta| \tag{3.11}$$

where $c(N)$ is the Sobolev constant such that, for all $u \in W_0^{1,1}(\Omega)$,

$$\|u\|_{L^{N/N-1}} \leq c(N) \|\nabla u\|_{L^1}.$$

Let $\delta_1 \geq \delta_2$. Observe that

$$\|v_{\delta_2}\|_{L^{\frac{N}{N-1}}} = \Big(\int_{A_{\delta_2}} v_{\delta_2}^{\frac{N}{N-1}}\, dx \Big)^{\frac{N-1}{N}} \geq \Big(\int_{A_{\delta_1}} v_{\delta_2}^{\frac{N}{N-1}}\, dx \Big)^{\frac{N-1}{N}} \geq (\delta_1 - \delta_2) |A_{\delta_1}|^{\frac{N-1}{N}}. \tag{3.12}$$

From (3.11) and (3.12), we obtain

$$(\delta_1 - \delta_2)^{\frac{N}{N-1}} |A_{\delta_1}| \leq \Big(\frac{A\, c(N)}{A - c(N)\, |\Omega|^{1/N}} \Big)^{\frac{N}{N-1}} |A_{\delta_2}|^{\frac{N}{N-1}}.$$

Applying [27, Lemma B.1] yields

$$\|u\|_\infty \leq \frac{Ac(N)|\Omega|^{1/N} 2^N}{A - c(N)|\Omega|^{1/N}} = 2^{N/2} A. \tag{3.13}$$

The conclusion follows because, for $b \in]0, \hat{b}[$, we have

$$\frac{1}{b} - A > \frac{1}{\hat{b}} - A = 2^{N/2} A.$$

Step 2. There is a constant $K > 0$, only depending on N, Ω, A, b, such that

$$|\nabla u(x)| \leq K \quad \text{for all } x \in \partial\Omega. \tag{3.14}$$

To this end, we rewrite the equation in (3.10) in the form

$$(1+|\nabla u|^2)\Delta u - \sum_{i,j=1}^{N} \partial_i u\, \partial_j u\, \partial_{ij} u + \frac{\sigma}{1-\sigma u}(1+|\nabla u|^2) = 0.$$

Note that the structure conditions (14.43) and (14.54) in [25] are satisfied, as well as, by (H_2), the inequalities (14.51). Indeed, adopting the notation introduced in [25, Section 14.3], we can take, in (14.43), (14.54) and (14.51),

$$a^{ij} = \Lambda a_\infty^{ij} + a_0^{ij}, \qquad a_\infty^{ij} = \delta_{ij} - \frac{p_i p_j}{|p|^2}, \qquad a_0^{ij} = \frac{p_i p_j}{|p|^2}, \qquad \text{for } i,j=1,\dots,N,$$

$$b = |p|\Lambda b_\infty + b_0, \qquad b_\infty = 0, \qquad b_0 = \frac{\sigma}{1-\sigma z}(1+|p|^2),$$

$$\Lambda = 1+|p|^2, \qquad \lambda = 1, \qquad \mathcal{K}^{\pm} = (N-1)H.$$

Hence, we can apply [25, Corollary 14.7] and conclude, as $\sigma \in [0,b]$ and $\|u\|_\infty \le 2^{\frac{N}{2}} A$, that (3.14) holds.

Step 3. There exists a constant $L>0$, only depending on N,Ω,A,b,K, such that

$$\|\nabla u\|_\infty < L.$$

To this end, we show that all assumptions of [25, Theorem 15.2] are satisfied. We write now the equation in (3.10) in the form

$$\Delta u - \sum_{i,j=1}^{N} \frac{\partial_i u\, \partial_j u}{1+|\nabla u|^2}\, \partial_{ij} u + \frac{\sigma}{1-\sigma u} = 0.$$

Following the notation used in [25, Sections 15.1, 15.2], we take

$$a^{ij} = \delta_{ij} - \frac{p_i\, p_j}{1+|p|^2}, \qquad a_*^{ij} = \delta_{ij}, \qquad \lambda_* = 1, \qquad c_i = -\frac{p_i}{1+|p|^2}, \qquad \text{for } i,j=1,\dots,N,$$

$$b = \frac{\sigma}{1-\sigma z}, \qquad r = -1, \qquad s = 0, \qquad \mathcal{E} = \frac{|p|^2}{1+|p|^2},$$

and, setting

$$\delta = \partial_z + \sum_{i=1}^{N} \frac{p_i \partial_{x_i}}{|p|^2} \qquad\qquad \overline{\delta} = \sum_{i=1}^{N} p_i \partial_{p_i},$$

by calculations, we get

$$\alpha = \frac{2}{1+|p|^2} - 1, \qquad \beta = -\frac{1+|p|^2}{|p|^2}\frac{\sigma}{1-\sigma z}, \qquad \gamma = \frac{1+|p|^2}{|p|^2}\Big(\frac{\sigma}{1-\sigma z}\Big)^2,$$

and hence

$$a = -1, \qquad b = \sup_{[0,\frac{1}{b}-A]} -\frac{\sigma}{1-\sigma z}, \qquad c = \sup_{[0,\frac{1}{b}-A]} \Big(\frac{\sigma}{1-\sigma z}\Big)^2.$$

In addition, we have that

$$(\overline{\delta} + r + 1)a_*^{ij} = (\delta + s)a_*^{ij} = 0$$

and all quantities

$$\overline{\delta}\mathcal{E} = 2\Big(\frac{|p|}{1+|p|^2}\Big)^2, \quad \delta\mathcal{E} = 0, \quad (\overline{\delta}+r)b = -\frac{\sigma}{1-\sigma z}, \quad (\delta+s)b = \Big(\frac{\sigma}{1-\sigma z}\Big)^2$$

are uniformly bounded for $|z| \le \frac{1}{b} - A$. Hence, the structure conditions (15.32) and the condition $a \le 0$ are satisfied and Step 3 follows from [25, Theorem 15.2].

Step 4. Conclusion. Finally, from Theorem 13.2 and the successive note at page 323 in [25], we infer the existence of constants $\beta \in]0,1[$ and $M > 0$, only depending on N, Ω, A, b, L, such that

$$\|\nabla u\|_{C^{0,\beta}} < M.$$

By replacing α with β in the definition of $\mathcal{U}_M$ in case $\beta < \alpha$, we conclude that, for every $\sigma \in [0, b]$, $\mathcal{T}(\sigma, \cdot)$ has no fixed point on $\partial\mathcal{U}_M$. The homotopy invariance of the degree implies that $\deg(\mathcal{I} - \mathcal{T}(\sigma, \cdot), \mathcal{U}_M, 0)$ is constant in $[0, b]$ and hence

$$\deg(\mathcal{I} - \mathcal{T}(b, \cdot), \mathcal{U}_M, 0) = \deg(\mathcal{I} - \mathcal{T}(0, \cdot), \mathcal{U}_M, 0) = \deg(\mathcal{I}, \mathcal{U}_M, 0) = 1.$$

In particular, problem (3.5) has, for all $b \in]0, \hat{b}[$, a classical solution $u \in \mathcal{U}_M$, which satisfies $u \gg 0$, by Theorem 2.1. □

By the symmetry properties of problem (3.5), we can draw similar conclusions for $b \in]-\hat{b}, 0[$ as well.

Proposition 3.5. *Assume* (H_1) *and, in case* $N \geq 2$, (H_2). *Then, for any given* $b \in]-\hat{b}, 0[$, *there exists a constant* $M > 0$ *such that*

$$\deg(\mathcal{I} - \mathcal{T}(b, \cdot), \mathcal{U}_M, 0) = 1,$$

where $\mathcal{U}_M = \{u \in C^{1,\alpha}(\overline{\Omega}) \mid \|u\|_\infty < -\frac{1}{b} - A,\ \|\nabla u\|_{C^{0,\alpha}} < M\}$. *In particular, problem* (3.5) *has at least one classical solution* $u \in \mathcal{U}_M$ *with* $u \ll 0$.

Remark 3.2 Our existence results slightly improve [20, Theorem]; better conclusions are achieved in [19] by other methods that, however, do not provide the degree information given in (3.9). It is worth noticing that, according to Theorem 3.1, problem (3.5) has no solution for $|b| \geq \frac{|\partial\Omega|}{|\Omega|}$.

The case $a = 0$, $N = 1$. In the one-dimensional case much more precise conclusions, concerning the solvability of problem (3.5), can be obtained by direct calculations. Let us denote by $t^\# \approx 1.19997$ the unique positive solution of the equation $t \tanh(t) = 1$ and set

$$b^\# = \frac{1}{r \sinh(t^\#)}. \tag{3.15}$$

Remark 3.3 Since, if $N = 1$, $|\Omega| = 2r$ and $|\partial\Omega| = 2$, we have $b^\# < \frac{1}{r} = \frac{|\partial\Omega|}{|\Omega|}$.

Proposition 3.6. *Assume* (H_1). *Then, the following conclusions hold:*

(*a*) *if* $0 < |b| < b^\#$, *then the problem*

$$\begin{cases} -(1 - bu)\Big(\dfrac{u'}{\sqrt{1 + |u'|^2}}\Big)' = \dfrac{b}{\sqrt{1 + |u'|^2}}, & \text{in }]-r, r[, \\ u(-r) = 0,\ u(r) = 0 \end{cases} \tag{3.16}$$

has exactly two classical solutions $u_1, u_2 \in C^2([-r, r])$, *with* $u_1, u_2 \gg 0$ *for* $b > 0$ *and* $u_1, u_2 \ll 0$ *for* $b < 0$;

(*b*) *if* $|b| = b^\#$, *then problem* (3.16) *has exactly one classical solution* $u_1 \in C^2([-r, r])$, *with* $u_1 \gg 0$ *for* $b = b^\#$ *and* $u_1 \ll 0$ *for* $b = -b^\#$;

(c) *if* $|b| > b^{\#}$, *then problem* (3.16) *has no solutions.*

Proof. For all (b, v) satisfying $1 - bv > 0$, the solution of the initial value problem

$$\begin{cases} -(1-bu)\Big(\dfrac{u'}{\sqrt{1+|u'|^2}}\Big)' = \dfrac{b}{\sqrt{1+|u'|^2}}, \\ u(0) = v,\ u'(0) = 0 \end{cases} \tag{3.17}$$

is given by

$$u(x) = u(x; b, v) = \begin{cases} \dfrac{1}{b} - \dfrac{1-bv}{b}\cosh\Big(\dfrac{bx}{1-bv}\Big) & \text{if } b \neq 0, \\ v & \text{if } b = 0. \end{cases}$$

In order that $u(\cdot; b, v)$ be a solution of (3.16), the couple (b, v) must satisfy $\varphi(b, v) = 0$, where

$$\varphi(b, v) = u(r; b, v) = \begin{cases} \dfrac{1}{b} - \dfrac{1-bv}{b}\cosh\Big(\dfrac{br}{1-bv}\Big) & \text{if } b \neq 0, \\ v & \text{if } b = 0. \end{cases} \tag{3.18}$$

The function φ is odd symmetric and analytic, with first order partial derivatives

$$\partial_b\varphi(b, v) = \begin{cases} \dfrac{1}{b^2}\Big(-1 + \cosh\Big(\dfrac{br}{1-bv}\Big) - \dfrac{br}{1-bv}\sinh\Big(\dfrac{br}{1-bv}\Big)\Big) & \text{if } b \neq 0, \\ -\dfrac{1}{2}r^2 & \text{if } b = 0 \end{cases}$$

and

$$\partial_v\varphi(b, v) = \cosh\Big(\frac{br}{1-bv}\Big) - \frac{br}{1-bv}\sinh\Big(\frac{br}{1-bv}\Big).$$

It is apparent that $\nabla\varphi$ never vanishes. Elementary calculations also show that

- $\varphi(0, 0) = 0$ and, for $b > 0$, $\varphi(b, 0) < 0$ and $\lim_{v\to 1/b} \varphi(b, v) = -\infty$;
- in case $b \geq \frac{t^{\#}}{r}$, $\partial_v\varphi(b, v) \leq 0$, if $v \in [0, \frac{1}{b}[$;
- in case $0 < b < \frac{t^{\#}}{r}$, $\partial_v\varphi(b, v) > 0$, if $v \in [0, \frac{1}{b} - \frac{r}{t^{\#}}[$, $\partial_v\varphi(b, \frac{1}{b} - \frac{r}{t^{\#}}) = 0$ and $\partial_v\varphi(b, v) < 0$, if $v \in]\frac{1}{b} - \frac{r}{t^{\#}}, \frac{1}{b}[$;
- $\partial_b\varphi(b, v) = \frac{1}{b^2}\big(-1 + \partial_v\varphi(b, v)\big) < 0$, if $b > 0$ and $0 < v < \frac{1}{b}$;
- $\varphi(b, \frac{1}{b} - \frac{r}{t^{\#}}) = \frac{1}{b} - \frac{r}{t^{\#}}\cosh t^{\#} > 0$, if $b \in]0, b^{\#}[$, and $\varphi(b, \frac{1}{b} - \frac{r}{t^{\#}}) = 0$, if $b = b^{\#}$.

Hence, we conclude, by the implicit function theorem, that the set $\{(b, v) \mid 1 - bv > 0, \varphi(b, v) = 0\}$ is the graph of an odd analytic function $\psi :]-\infty, +\infty[\to [-b^{\#}, b^{\#}]$, which is strictly increasing in $[0, \frac{1}{b} - \frac{r}{t^{\#}}[$, strictly decreasing in $]\frac{1}{b} - \frac{r}{t^{\#}}, +\infty[$, and satisfies $\psi(0) = 0$ and $\lim_{v\to+\infty}\psi(v) = 0$.

Therefore, all the stated conclusions hold; in particular, u solves (3.16) for some b if and only if $-u$ solves (3.16) with b replaced by $-b$. □

Remark 3.4 From the proof of Proposition 3.6, we get the following additional information about the solutions of (3.17) and, hence, of (3.16):

(a) for any $b \in]0, b^{\#}[$, we have

$$0 < u_1(0;b) < \frac{1}{b} - \frac{r}{t^{\#}} < u_2(0;b) < \frac{1}{b} \quad \text{and} \quad u_1(0;b^{\#}) = \frac{1}{b^{\#}} - \frac{r}{t^{\#}},$$

$$\lim_{b \to 0^+} u_1(0;b) = 0, \qquad \lim_{b \to 0^+} u_2(0;b) = +\infty,$$

$u_1(0;b)$ is strictly increasing in $]0, b^{\#}[$ and $u_2(0;b)$ is strictly decreasing in $]0, b^{\#}[$;

(b) for any $b \in] - b^{\#}, 0[$, we have

$$0 > u_1(0;b) > \frac{1}{b} + \frac{r}{t^{\#}} > u_2(0;b) > \frac{1}{b} \quad \text{and} \quad u_1(0;-b^{\#}) = -\frac{1}{b^{\#}} + \frac{r}{t^{\#}},$$

$$\lim_{b \to 0^-} u_1(0;b) = 0, \qquad \lim_{b \to 0^-} u_2(0;b) = -\infty,$$

$u_1(0;b)$ is strictly increasing in $] - b^{\#}, 0[$ and $u_2(0;b)$ is strictly decreasing in $] - b^{\#}, 0[$;

(c) for any $b \in]0, b^{\#}[$, we have

- if $v \in [0, u_1(0;b)[\cup]u_2(0;b), \frac{1}{b}[$, then $u(r;b,v) < 0$,
- if $v \in]u_1(0;b), u_2(0;b)[$, then $u(r;b,v) > 0$;

(d) for any $b \in] - b^{\#}, 0[$, we have

- if $v \in]\frac{1}{b}, u_2(0;b)[\cup]u_1(0;b), 0]$, then $u(r;b,v) > 0$,
- if $v \in]u_2(0;b), u_1(0;b)[$, then $u(r;b,v) < 0$.

The case $a > 0$, N arbitrary. From Proposition 3.4 we deduce the following result, which complements Theorem 3.3.

Theorem 3.7. *Assume (H_1) and, in case $N \geq 2$, (H_2). Let A and $\hat{b}$ be given by (3.6) and (3.7), respectively. Suppose that*

$$R > 2^{\frac{N}{2}} A = c(N)|\Omega|^{1/N}(2^{N/2}+1)2^{N/2}.$$

Then, for each $b \in \mathbb{R}$, with $|b| \in]0, \hat{b}[$, there exists $\hat{a}(b) > 0$ such that, for all $a \in]0, \hat{a}(b)[$, problem (1.2) *has at least one classical solution $u \in C^1(\overline{\Omega})$, with $u \gg 0$ if $b > 0$ and $u \ll 0$ if $b < 0$.*

Remark 3.5 The constant $2^{\frac{N}{2}} A = c(N)|\Omega|^{1/N}(2^{N/2}+1)2^{N/2}$ is the bound obtained in (3.13) and, by (3.6) and (3.7), we have $2^{N/2}A = \frac{1}{\hat{b}} - A$.

Proof. Fix $b \in]0, \hat{b}[$ and consider the set $\mathcal{U}_M$ defined in (3.8). By (3.13), any solution $u \in \overline{\mathcal{U}}_M$ of (3.5), satisfies $\|u\|_\infty \leq 2^{N/2}A$ and $u \in \mathcal{U}_M$. By assumption, we then have $\|u\|_\infty < R$. Pick $\tilde{R} \in]2^{N/2}A, R[$ and define the set

$$\widetilde{\mathcal{U}}_M = \{u \in C^{1,\alpha}(\overline{\Omega}) \mid \|u\|_\infty < \min\{\tfrac{1}{b} - A, \tilde{R}\}, \ \|\nabla u\|_{C^{0,\alpha}} < M\}.$$

Hence, any solution $u \in \overline{\widetilde{\mathcal{U}}}_M$ of (3.5) belongs to $\widetilde{\mathcal{U}}_M$.

For all $v \in \overline{\widetilde{\mathcal{U}}}_M$ and $a \in \mathbb{R}$, denote by $u = \mathcal{S}(a,v) \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ the unique solution of the problem

$$\begin{cases} -\displaystyle\sum_{i,j=1}^{N} \frac{(1+|\nabla v|^2)\delta_{ij} - \partial_i v\, \partial_j v}{(1+|\nabla v|^2)^{3/2}}\, \partial_{ij} u = \frac{1}{1-bv}\left(\frac{a}{(v-R)^2} + \frac{b}{\sqrt{1+|\nabla v|^2}}\right), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

The operator $\mathcal{S} : \mathbb{R} \times \overline{\widetilde{\mathcal{U}}}_M \to C^{1,\alpha}(\overline{\Omega})$ is completely continuous and the fixed points $u \in \overline{\widetilde{\mathcal{U}}}_M$ of $\mathcal{S}(a,\cdot)$ are precisely the solutions $u \in C^2(\Omega) \cap \overline{\widetilde{\mathcal{U}}}_M$ of (1.2). As $\mathcal{S}(0,\cdot) = \mathcal{T}(b,\cdot)$ and $\mathcal{T}(b,\cdot)$ has no fixed point on the boundary of $\widetilde{\mathcal{U}}_M$, the continuity of $\mathcal{S}(a,\cdot)$ with respect to the parameter a yields the existence of $\hat{a}(b) > 0$ such that, for all $a \in \,]-\hat{a}(b), \hat{a}(b)[$, $\mathcal{S}(a,\cdot)$ has no fixed point on the boundary of $\widetilde{\mathcal{U}}_M$. Hence, the degree information provided by Proposition 3.4 and the homotopy invariance property of the degree imply the existence of a continuum $\mathcal{C} \subset \,]-\hat{a}(b), \hat{a}(b)[\, \times \widetilde{\mathcal{U}}_M$ of solutions pairs (a,u) of (1.2), with $\mathrm{proj}_{\mathbb{R}}\, \mathcal{C} = \,]-\hat{a}(b), \hat{a}(b)[$. As the solutions u of (3.5), with $b > 0$, satisfy $u \gg 0$, we infer, possibly reducing $\hat{a}(b)$, that $u \gg 0$ for all $(a,u) \in \mathcal{C}$.

The proof of the corresponding conclusions in the case where $b \in \,]-\hat{b}, 0[$ is based on Proposition 3.5 and follows similar patterns. $\square$

The case $a > 0$, $N = 1$. In the one-dimensional case, a multiplicity result for the problem

$$\begin{cases} -(1-bu)\left(\dfrac{u'}{\sqrt{1+|u'|^2}}\right)' = \dfrac{a}{(u-R)^2} + \dfrac{b}{\sqrt{1+|u'|^2}}, & \text{in }]-r,r[, \\ u(-r) = 0, \ u(r) = 0, \end{cases} \tag{3.19}$$

can be obtained using Proposition 3.6.

Remark 3.6 Note that, according to Remark 3.4, there exists a unique $\tilde{b} \in [0, b^{\#}]$ such that

- $u_1(0;\tilde{b}) = R$, if $R \le \frac{1}{b^{\#}} - \frac{r}{t^{\#}}$;
- $u_2(0;\tilde{b}) = R$, if $R > \frac{1}{b^{\#}} - \frac{r}{t^{\#}}$;

where $u_1(\cdot;b)$ and $u_2(\cdot;b)$ are respectively the smaller and the larger solution of (3.16).

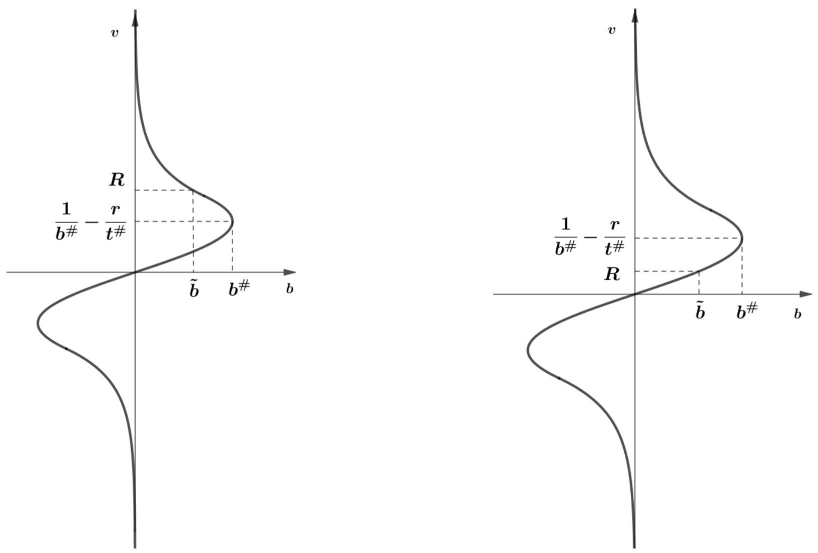

Figure 3: The null set of the function φ given by (3.18) and the definition of the constant $\tilde{b}$, according to $R > \frac{1}{b^{\#}} - \frac{r}{t^{\#}}$, on the left, or $R < \frac{1}{b^{\#}} - \frac{r}{t^{\#}}$, on the right.

Proposition 3.8. *Assume* (H_1). *Let* $t^\#$ *be the unique positive solution of the equation* $t\tanh(t) = 1$ *and let* $b^\#$ *be defined by* (3.15). *Then, the following conclusions hold:*

(a) *in case* $R < \frac{1}{b^\#} - \frac{r}{t^\#}$, *for every* $b \in]0, \tilde{b}[$, *there exists* $a^\#(b) > 0$ *such that, for all* $a \in]0, a^\#(b)[$, *problem* (3.19) *has at least one classical solution* $u \in C^2([-r, r])$, *with* $u \gg 0$;

(b) *in case* $R > \frac{1}{b^\#} - \frac{r}{t^\#}$, *for every* $b \in]0, b^\#]$, *there exists* $a^\#(b) > 0$ *such that, for all* $a \in]0, a^\#(b)[$, *problem* (3.19) *has at least one classical solution* $u \in C^2([-r, r])$, *with* $u \gg 0$, *if* $b \in]0, \tilde{b}]$, *and has at least two classical solutions* $u_1, u_2 \in C^2([-r, r])$, *with* $u_1, u_2 \gg 0$, *if* $b \in]\tilde{b}, b^\#[$;

(c) *for every* $b \in] - b^\#, 0[$, *there exists* $a^\#(b) > 0$ *such that, for all* $a \in]0, a^\#(b)[$, *problem* (3.19) *has at least two classical solutions* $u_1, u_2 \in C^2([-r, r])$, *with* $u_1, u_2 \ll 0$.

Proof. In the course of this proof we ever keep the notation used in Proposition 3.6 and in Remark 3.4.

Case (a). Fix $b \in]0, b^*[$ and pick $v_0 = v_0(b) \in]u_1(0; b), R[$, where $u_1(\cdot; b)$ is the smaller solution of (3.16). The continuous dependence on initial values and parameters imply that, for every $\varepsilon > 0$, there exists $\overline{a} = \overline{a}(b, \varepsilon) > 0$ such that, for all $a \in [0, \overline{a}]$ and $v \in [0, v_0]$, the solution $z(\cdot; a, b, v)$ of

$$\begin{cases} -(1-bu)\Big(\dfrac{u'}{\sqrt{1+|u'|^2}}\Big)' = \dfrac{a}{(u-R)^2} + \dfrac{b}{\sqrt{1+|u'|^2}}, \\ u(0) = v, \ u'(0) = 0, \end{cases} \tag{3.20}$$

is defined on $[0, r]$ and satisfies

$$\|z(\cdot; a, b, v) - z(\cdot; 0, b, v)\|_{C^1} < \varepsilon.$$

Indeed, otherwise, we could find $\varepsilon > 0$ and sequences $(a_n)_n$, with $a_n > 0$ and $a_n \to 0$, $(v_n)_n$, with $v_n \in [0, v_0]$ and $v_n \to \overline{v} \in [0, v_0]$, and $(z_n)_n$, with $z_n = z(\cdot; a_n, b, v_n)$ solution of (3.20), such that, for all n, either z_n is not defined in $[0, r]$, or

$$\|z(\cdot; a_n, b, v_n) - z(\cdot; 0, b, \overline{v})\|_{C^1} \ge \varepsilon.$$

Observe that $z(\cdot; 0, b, \overline{v})$ is defined on $[0, r]$. Hence, from [24, Theorem I-6], we infer that $z(\cdot; a_n, b, v_n)$ is defined in $[0, r]$, for all large n, and converges, in the topology of $C^1([0, r])$, to $z(\cdot; 0, b, \overline{v})$, thus yielding a contradiction.

By Remark 3.4, we know that there are v_1, $v_2 \in]0, v_0[$, with $v_1 < v_2$, such that the corresponding solutions $z_i(\cdot; 0, b, v_i)$, for $i \in \{1, 2\}$, of (3.16) satisfy

$$z_1(r; 0, b, v_1) < 0 \quad \text{and} \quad z_2(r; 0, b, v_2) > 0.$$

Hence, there exists $\overline{\varepsilon} > 0$ sufficiently small such that, setting $a^\#(b) = \overline{a}(b; \overline{\varepsilon})$, we conclude that, for any $a \in]0, a^\#(b)[$, also the corresponding solutions $z_i(\cdot; a, b, v_i)$, for $i \in \{1, 2\}$, of (3.20) satisfy

$$z_1(r; a, b, v_1) < 0 \quad \text{and} \quad z_2(r; a, b, v_2) > 0.$$

The continuous dependence property and the intermediate value theorem yield the existence of a solution $u \in C^2([0, r])$, with $u(0) \in]0, v_0[$, of the problem

$$\begin{cases} -(1-bu)\Big(\dfrac{u'}{\sqrt{1+|u'|^2}}\Big)' = \dfrac{a}{(u-R)^2} + \dfrac{b}{\sqrt{1+|u'|^2}}, & \text{in }]0, r[, \\ u'(0) = 0, \ u(r) = 0. \end{cases} \tag{3.21}$$

Since u is concave, it is decreasing and strictly positive. Finally, the even extension of u onto $[-r, r]$ is the solution of (3.19) we are looking for.

Case (b). As the first part of the conclusion can be proved like in Case (a), we only give the proof of the multiplicity result. Therefore, fix $b \in]b^*, b^\#[$ and pick $v_0 = v_0(b) \in]u_2(0;b), \min\{R, \frac{1}{b}\}[$, where $u_2(\cdot;b)$ is the larger solution of (3.16). Using Remark 3.4 again, we find $v_1,\ v_2,\ v_3 \in]0, v_0[$ with $v_1 < v_2 < v_3$ such that the corresponding solutions $z_i(\cdot;0,b,v_i)$, for $i \in \{1,2,3\}$, of (3.16) satisfy

$$z_1(r;0,b,v_1) < 0, \quad z_3(r;0,b,v_3) < 0 \quad \text{and} \quad z_2(r;0,b,v_2) > 0.$$

Hence, there exists $\overline{\varepsilon} > 0$ sufficiently small such that, setting $a^\#(b) = \overline{a}(b;\overline{\varepsilon})$, we conclude that, for any $a \in]0, a^\#(b)[$, also the corresponding solutions $z_i(\cdot;a,b,v_i)$, for $i \in \{1,2,3\}$, of (3.20) satisfy

$$z_1(r;a,b,v_1) < 0, \quad z_3(r;a,b,v_3) < 0 \quad \text{and} \quad z_2(r;a,b,v_2) > 0.$$

The continuous dependence property and the intermediate value theorem now yield the existence of two solutions $u_1,\ u_2 \in C^2([0,r])$ of (3.21), with $u_1(0),\ u_2(0) \in]0, v_0[$, which provide the desired solutions of (3.19).

Case (c). Fix $b \in]-b^\#, 0[$. Pick $v_0 = v_0(b) \in]\frac{1}{b}, u_2(0;b)[$, where $u_i(\cdot;b)$, for $i \in \{1,2\}$, are the solutions of (3.16), with $u_2(0;b) < u_1(0;b)$. Let us consider the modified problem

$$\begin{cases} -(1-bu)\Big(\dfrac{u'}{\sqrt{1+|u'|^2}}\Big)' = \dfrac{a}{(R+u^-)^2} + \dfrac{b}{\sqrt{1+|u'|^2}}, \\ u(0) = v,\ u'(0) = 0, \end{cases} \tag{3.22}$$

for some $v \in [v_0, 0]$. As in the previous steps, for every $\varepsilon > 0$, there exists $\overline{a} = \overline{a}(b;\varepsilon) > 0$ such that, for all $a \in]0, \overline{a}[$ and $v \in [v_0, 0]$, the solution $z(\cdot;a,b,v)$ of (3.22) is defined in $[0,r]$ and satisfies

$$\|z(\cdot;a,b,v) - z(\cdot;0,b,v)\|_{C^1} < \varepsilon.$$

The same shooting-type argument used above then yields the existence of $\overline{\varepsilon} > 0$ and $a^\#(b) = \overline{a}(b;\overline{\varepsilon}) > 0$, such that, for all $a \in]0, a^\#(b)[$, there exist two solutions $u_1, u_2 \in C^2([-r,r])$ of the modified problem

$$\begin{cases} -(1-bu)\Big(\dfrac{u'}{\sqrt{1+|u'|^2}}\Big)' = \dfrac{a}{(R+u^-)^2} + \dfrac{b}{\sqrt{1+|u'|^2}}, & \text{in }]-r, r[, \\ u(-r) = 0,\ u(r) = 0, \end{cases}$$

which are even and satisfy $u_1(0),\ u_2(0) \in]v_0, 0[$.

It remains to show that, for $i \in \{1,2\}$, $u_i \ll 0$ and hence is a solution of (5.3). Otherwise there should exist $\overline{x} \in]0, r]$ such that $u_i(\overline{x}) = \max u_i \geq 0$ and $u_i'(\overline{x}) = 0$. From the equation, possibly restricting $a^\#(b)$ in such a way that $a^\#(b) < R^2|b|$, we get the contradiction $u_i''(\overline{x}) = (-\frac{a}{R^2} - b)\frac{1}{1-bu_i(\overline{x})} > 0$. This proves the result. □

4 Bifurcation from simple eigenvalues

In the previous section we proved, in particular, the solvability of problem (1.2) when the parameters (a,b) vary in a neighbourhood of any point (a_0, b_0) lying on the line $a_0 + R^2 b_0 = 0$, provided that $\frac{2a_0}{R^3} \notin \Sigma$, where $\Sigma = \{\lambda_n \mid n \in \mathbb{N}^+\}$ is the spectrum of $-\Delta$ on $H_0^1(\Omega)$. Now we discuss the case where $\frac{2a_0}{R^3} \in \Sigma$. Namely, we fix a point $(0, a_n, -\frac{a_n}{R^2})$, where $\frac{2a_n}{R^3} = \lambda_n \in \Sigma$, for some $n \in \mathbb{N}^+$, and we study the bifurcation of the solution set of problem (1.2) from the line of the trivial solutions $\{(0, a, -\frac{a}{R^2}) \mid a \in \mathbb{R}\}$ at the point $(0, a_n, -\frac{a_n}{R^2})$, within the hyperplane $\{(u, a, -\frac{a}{R^2}) \mid u \in C^1(\overline{\Omega}), a \in \mathbb{R}\} \subset C^1(\overline{\Omega}) \times \mathbb{R} \times \mathbb{R}$.

Let us assume condition (H_1), fix $p > N$, with $p \geq 2$ if $N = 1$, and consider, in the space $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \times \mathbb{R}$, the open set

$$\mathcal{O} = \{(u,a) \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \times \mathbb{R} \mid u(x) < R \text{ and } R^2 + a\, u(x) > 0 \text{ in } \overline{\Omega}\}.$$

Next, define the operator $\mathcal{H} : \mathcal{O} \to L^p(\Omega)$ by setting

$$\mathcal{H}(u,a) = \mathcal{F}(u,a,-\tfrac{a}{R^2}) = \operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) + \Big(\frac{1}{(u-R)^2} - \frac{1}{R^2}\frac{1}{\sqrt{1+|\nabla u|^2}}\Big)\frac{aR^2}{R^2+au},$$

where $\mathcal{F}$ is defined in (3.4). Let us observe that $(u,a) \in \mathcal{O}$ satisfies $\mathcal{H}(u,a) = 0$ if and only if

$$\begin{cases} -\displaystyle\sum_{i,j=1}^{N} \frac{(1+|\nabla u|^2)\delta_{ij} - \partial_i u\, \partial_j u}{(1+|\nabla u|^2)^{3/2}}\, \partial_{ij} u = \Big(\frac{1}{(u-R)^2} - \frac{1}{R^2}\frac{1}{\sqrt{1+|\nabla u|^2}}\Big)\frac{aR^2}{R^2+au}, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

Similarly as in the proof of Lemma 3.2 we see that any $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ satisfying (4.1), for some a, belongs to $C^{1,\alpha}(\overline{\Omega})$, for all $\alpha \in \,]0, 1 - \frac{N}{p}[$, and is a classical solution.

Since for all $a \in \mathbb{R}$, $(0,a)$ solves (4.1), we look for non-trivial solutions of this problem by using bifurcation theory. By Addendum A, the operator $\mathcal{H}$ is of class C^∞ and its partial derivative $\partial_u \mathcal{H}$ is given by

$$\begin{aligned} \partial_u \mathcal{H}(u,a)[v] = \operatorname{div}\Big(&\frac{\nabla v}{\sqrt{1+|\nabla u|^2}} - \frac{\nabla u \cdot \nabla v}{\big(\sqrt{1+|\nabla u|^2}\big)^3}\nabla u\Big) \\ &- \Big(\frac{2v}{(u-R)^3} - \frac{1}{R^2}\frac{\nabla u \cdot \nabla v}{\big(\sqrt{1+|\nabla u|^2}\big)^3}\Big)\frac{aR^2}{R^2+au} \\ &- \Big(\frac{1}{(u-R)^2} - \frac{1}{R^2}\frac{1}{\sqrt{1+|\nabla u|^2}}\Big)\frac{a^2R^2 v}{(R^2+au)^2}, \end{aligned}$$

for all $(u,a) \in \mathcal{O}$ and $v \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$; in particular, we have

$$\partial_u \mathcal{H}(0,a)[v] = \Delta v + \frac{2a}{R^3} v.$$

Hence, bifurcation may occur only at the points $(0,a_n)$, with $a_n = \frac{R^3\lambda_n}{2}$ for some $n \in \mathbb{N}^+$. In order to state our first bifurcation result, we introduce the set of the non-trivial solutions of (4.1),

$$\mathcal{S} = \{(u,a) \in \mathcal{O} \mid u \neq 0 \text{ and } \mathcal{H}(u,a) = 0\}.$$

Theorem 4.1. *Assume (H_1). Suppose that, for some $n \in \mathbb{N}^+$, $\lambda_n \in \Sigma$ is a geometrically simple eigenvalue, with corresponding eigenfunction φ_n and set $a_n = \frac{R^3\lambda_n}{2}$. Then, the following conclusions hold:*

(a) *$(0,a_n)$ is a bifurcation point of (4.1); more precisely there exist a neighbourhood U of $(0,a_n)$ in $\mathcal{O}$, a constant $\delta > 0$, and functions*

$$\psi :]-\delta,\delta[\to \Big\{u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \mid \int_\Omega u\varphi_n \, dx = 0\Big\}, \qquad \chi :]-\delta,\delta[\to \mathbb{R}$$

of class C^∞ such that

$$\psi(0) = 0, \qquad \chi(0) = a_n,$$

and

$$\overline{\mathcal{S}} \cap U = \{(u,a) \mid u = t\varphi_n + t\psi(t),\ a = \chi(t),\ |t| < \delta\};$$

(b) *the curve* $\Gamma = \{(t\varphi_n + t\psi(t), \chi(t)) \mid |t| < \delta\}$ *is contained in a connected component* $\mathcal{C}$ *of* $\overline{\mathcal{S}}$ *such that either* $\mathcal{C}$ *is not compact in* $\mathcal{O}$, *or* $\mathcal{C}$ *contains a point* $(0, a_m)$ *with* $m \neq n$;

(c) *near* $(0, a_n)$ *the component* $\mathcal{C}$ *behaves as follows:*

(i) *if* $R^2\lambda_n \neq 4$ *and* $\int_\Omega \varphi_n^3\,dx \neq 0$, *then the bifurcation is transcritical, i.e.,* $\chi'(0) \neq 0$, *with* $\chi'(0) > 0$, *in case* $(R^2\lambda_n - 4)\int_\Omega \varphi_n^3\,dx > 0$, *and* $\chi'(0) < 0$, *in case* $(R^2\lambda_n - 4)\int_\Omega \varphi_n^3\,dx < 0$;

(ii) *if* $R^2\lambda_n = 4$, *or* $\int_\Omega \varphi_n^3\,dx = 0$, *then the bifurcation is subcritical, i.e.,* $\chi'(0) = 0$ *and* $\chi''(0) < 0$.

Remark 4.1 If $\mathcal{C}$ is not compact in $\mathcal{O}$, then either $\mathcal{C}$ meets $\partial\mathcal{O}$, or $\mathcal{C}$ is unbounded.

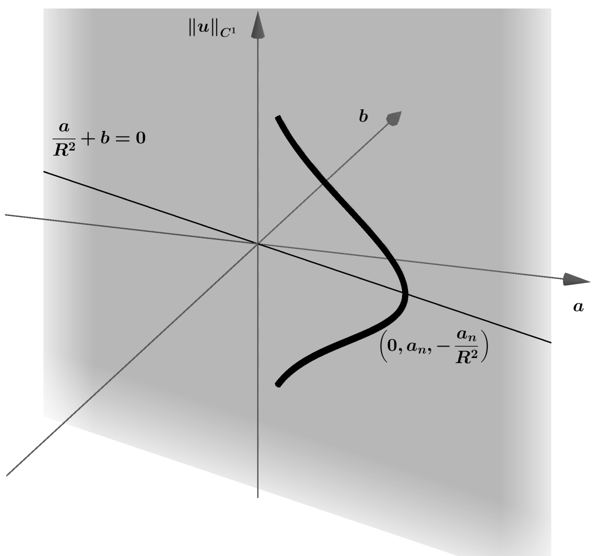

Figure 4: Bifurcation from the line of the trivial solutions $\{(0, a, -\frac{a}{R^2}) \mid a \in \mathbb{R}\}$ at the point $(0, a_n, -\frac{a_n}{R^2})$, within the hyperplane $\{(u, a, -\frac{a}{R^2}) \mid u \in C^1(\overline{\Omega}), a \in \mathbb{R})\}$: the subcritical case on the left, the transcritical case on the right.

Proof. The proof is divided into three steps.

Step 1. Local bifurcation. By Addendum A, we have, for all $(u, a) \in \mathcal{O}$ and $v \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$,

$$\partial_a \mathcal{H}(u, a) = \Big(\frac{1}{(u-R)^2} - \frac{1}{R^2}\frac{1}{\sqrt{1+|\nabla u|^2}}\Big)\frac{R^4}{(R^2+au)^2},$$

$$\begin{aligned}\partial_u \mathcal{H}(u, a)[v] = \operatorname{div}\Big(&\frac{\nabla v}{\sqrt{1+|\nabla u|^2}} - \frac{\nabla u \cdot \nabla v}{\big(\sqrt{1+|\nabla u|^2}\big)^3}\nabla u\Big) \\ &- \Big(\frac{2v}{(u-R)^3} - \frac{1}{R^2}\frac{\nabla u \cdot \nabla v}{\big(\sqrt{1+|\nabla u|^2}\big)^3}\Big)\frac{aR^2}{R^2+au} \\ &- \Big(\frac{1}{(u-R)^2} - \frac{1}{R^2}\frac{1}{\sqrt{1+|\nabla u|^2}}\Big)\frac{a^2R^2v}{(R^2+au)^2},\end{aligned}$$

$$\partial_{ua}\mathcal{H}(u,a)[v] = -\Big(\frac{2v}{(u-R)^3} - \frac{1}{R^2}\frac{\nabla u\cdot\nabla v}{\big(\sqrt{1+|\nabla u|^2}\big)^3}\Big)\frac{R^4}{(R^2+au)^2}$$
$$- \Big(\frac{1}{(u-R)^2} - \frac{1}{R^2}\frac{1}{\sqrt{1+|\nabla u|^2}}\Big)\frac{2aR^4v}{(R^2+au)^3}.$$

Let us set

$$\mathcal{L} = \partial_u\mathcal{H}(0,a_n) = \Delta + \lambda_n\mathcal{I} \qquad \text{and} \qquad \mathcal{M} = \partial_{ua}\mathcal{H}(0,a_n) = \frac{2}{R^3}\mathcal{I},$$

where $\mathcal{I}$ is the identity operator. It is clear that $\mathcal{L}$ is a Fredholm operator with index 0, with kernel

$$N(\mathcal{L}) = \operatorname{span}\{\varphi_n\}$$

and range

$$R(\mathcal{L}) = \Big\{u\in L^p(\Omega)\mid \int_\Omega u\varphi_n\,dx = 0\Big\}.$$

Further, $\mathcal{M}$ satisfies the transversality condition

$$\mathcal{M}[\varphi_n] = \frac{2}{R^3}\varphi_n \notin R(\mathcal{L}).$$

Hence, the Crandall-Rabinowitz theorem in [15, Theorem 1.7] applies and yields conclusion (a), that is, there exist a neighbourhood U of $(0,a_n)$ in $\mathcal{O}$, a constant $\delta>0$, and functions

$$\psi :]-\delta,\delta[\to \Big\{u\in W^{2,p}(\Omega)\cap W^{1,p}_0(\Omega)\mid \int_\Omega u\varphi_n\,dx = 0\Big\}, \qquad \chi :]-\delta,\delta[\to\mathbb{R}$$

of class C^∞ such that

$$\Big\{\frac{1}{2}\Big\}\,\psi(0)=0, \qquad \chi(0)=a_n,$$

and

$$\overline{\mathcal{S}}\cap U = \{(u,a)\mid u = t\varphi_n + t\psi(t),\ a=\chi(t),\ |t|<\delta\}.$$

Step 2. Global bifurcation. We are going to apply the Rabinowitz global bifurcation theorem [40], in the form of [41, Theorem 4.3] (see also [31, Chapter 6], [32]). We only need to show that $\partial_u\mathcal{H}(u,a)$ is a Fredholm operator of index 0, for all $(u,a)\in\mathcal{O}$. Let us fix $(u,a)\in\mathcal{O}$ and denote the operator $\partial_u\mathcal{H}(u,a)$ simply by $\partial_u\mathcal{H}$. Set, for $i,\,j=1,\ldots,N$,

$$a_{ij} = \frac{(1+|\nabla u|^2)\delta_{ij} - \partial_i u\partial_j u}{\big(\sqrt{1+|\nabla u|^2}\big)^3},$$
$$b_i = -\frac{a\partial_i u}{\big(\sqrt{1+|\nabla u|^2}\big)^3(R^2+au)},$$
$$c = \frac{2aR^2}{(u-R)^3(R^2+au)} + \Big(\frac{1}{(u-R)^2} - \frac{1}{R^2}\frac{1}{\sqrt{1+|\nabla u|^2}}\Big)\frac{a^2R^2}{(R^2+au)^2},$$

with $a_{ij}, b_i, c\in W^{1,p}(\Omega)$. We have, for all $v\in W^{2,p}(\Omega)\cap W^{1,p}_0(\Omega)$,

$$\partial_u\mathcal{H}[v] = \sum_{i,j=1}^N a_{ij}\partial^2_{ij}v + \sum_{i,j=1}^N \partial_i a_{ij}\partial_j v - \sum_{i=1}^N b_i\partial_i v - cv.$$

As, for all $\xi \in \mathbb{R}^N$,

$$\frac{|\xi|^2}{\sqrt{1+|\nabla u|^2}} \geq \sum_{i,j=1}^{N} a_{ij}\xi_i\xi_j = \frac{(1+|\nabla u|^2)|\xi|^2 - (\nabla u\cdot\xi)^2}{\left(\sqrt{1+|\nabla u|^2}\right)^3} \geq \frac{|\xi|^2}{\left(\sqrt{1+|\nabla u|^2}\right)^3},$$

$\partial_u\mathcal{H}$ is uniformly elliptic. Pick a constant $\sigma > 0$ so large that $c + \sigma \geq 0$ and define the operators $\mathcal{A} : C^1(\overline{\Omega}) \to L^p(\Omega)$, by

$$\mathcal{A}[v] = -\sum_{i,j=1}^{N} \partial_i a_{ij}\partial_j v,$$

and $\mathcal{B} : W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) \to L^p(\Omega)$, by

$$\mathcal{B}[v] = \partial_u\mathcal{H}[v] + \mathcal{A}[v] - \sigma v.$$

The operator $\mathcal{A}$ is continuous and, by [25, Theorem 9.15, Lemma 9.17], $\mathcal{B}$ is a linear homeomorphism. Moreover, as $\mathcal{B}^{-1} : L^p(\Omega) \to C^1(\overline{\Omega})$ is compact, $\mathcal{A}\mathcal{B}^{-1} : L^p(\Omega) \to L^p(\Omega)$ is compact too and then $\mathcal{I} - \mathcal{A}\mathcal{B}^{-1} : L^p(\Omega) \to L^p(\Omega)$ is a Fredholm operator of index 0. Writing

$$\partial_u\mathcal{H} - \sigma\mathcal{I} = \mathcal{B} - \mathcal{A} = (\mathcal{I} - \mathcal{A}\mathcal{B}^{-1})\mathcal{B},$$

we conclude that $\partial_u\mathcal{H} - \sigma\mathcal{I} : W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) \to L^p(\Omega)$ is a Fredholm operator with

$$\text{index } (\partial_u\mathcal{H} - \sigma\mathcal{I}) = \text{index } (\mathcal{I} - \mathcal{A}\mathcal{B}^{-1}) + \text{index } (\mathcal{B}) = 0.$$

The maximum principle for strong solutions [25, Theorem 9.5] implies that the equation $\partial_u\mathcal{H}[v]-\sigma v = 0$ has only the trivial solution. Hence $\partial_u\mathcal{H} - \sigma\mathcal{I} : W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) \to L^p(\Omega)$ is one-to-one. Being Fredholm of index 0, it is onto and, being continuous, it is a linear homeomorphism. In addition, $(\partial_u\mathcal{H} - \sigma\mathcal{I})^{-1} : L^p(\Omega) \to L^p(\Omega)$ is compact and therefore $\mathcal{I} + \sigma(\partial_u\mathcal{H} - \sigma\mathcal{I})^{-1} : L^p(\Omega) \to L^p(\Omega)$ is a Fredholm operator of index 0. Finally, writing

$$\partial_u\mathcal{H} = (\mathcal{I} + \sigma(\partial_u\mathcal{H} - \sigma\mathcal{I})^{-1})(\partial_u\mathcal{H} - \sigma\mathcal{I}),$$

we conclude that $\partial_u\mathcal{H} : W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) \to L^p(\Omega)$ is a Fredholm operator with

$$\text{index } (\partial_u\mathcal{H}) = \text{index } \left((\mathcal{I} + \sigma(\partial_u\mathcal{H} - \sigma\mathcal{I})^{-1}\right) + \text{index } (\partial_u\mathcal{H} - \sigma\mathcal{I}) = 0.$$

Therefore, by [41, Theorem 4.3] there exists a connected component $\mathcal{C}$ of $\overline{\mathcal{S}}$, bifurcating from $(0, a_n)$ and containing the curve

$$\Gamma = \{(t\varphi_n + t\psi(t), \chi(t)) \mid |t| < \delta\}.$$

In addition, as by Lemma 3.2 bifurcation may occur only at the points $(0, a_m)$ with $m \in \mathbb{N}^+$, it follows that either $\mathcal{C}$ is not compact in $\mathcal{O}$, or $\mathcal{C}$ contains a point $(0, a_m)$ with $m \neq n$, thus yielding conclusion (b).

Step 3. Behaviour of the bifurcation branch near $(0, a_n)$*.* Computing the derivatives of $\mathcal{H}$ (see Addendum A) and recalling that $a_n = \frac{R^3\lambda_n}{2}$, we find

$$\partial_{uu}\mathcal{H}(0, a_n)[\varphi_n][\varphi_n] = \left(\frac{6a_n}{R^4} - \frac{4a_n^2}{R^5}\right)\varphi_n^2 + \frac{a_n}{R^2}|\nabla\varphi_n|^2 = \left(\frac{3\lambda_n}{R} - R\lambda_n^2\right)\varphi_n^2 + \frac{R\lambda_n}{2}|\nabla\varphi_n|^2$$

and

$$\begin{aligned}\partial_{uuu}\mathcal{H}(0, a_n)[\varphi_n][\varphi_n][\varphi_n] = -3\,\mathrm{div}(|\nabla\varphi_n|^2\nabla\varphi_n) &+ \left(\frac{12}{R^2}\lambda_n - \frac{9}{2}\lambda_n^2 + \frac{3}{2}R^2\lambda_n^3\right)\varphi_n^3 \\ &- \frac{3}{4}R^2\lambda_n^2|\nabla\varphi_n|^2\varphi_n.\end{aligned}$$

Following [2, Chapter 5], we set

$$
\begin{aligned}
A &= \int_\Omega \varphi_n \, \partial_{ua}\mathcal{H}(0,a_n)[\varphi_n] \, dx, \\
B &= \frac{1}{2}\int_\Omega \varphi_n \, \partial_{uu}\mathcal{H}(0,a_n)[\varphi_n][\varphi_n] \, dx, \\
C &= -\frac{1}{6A}\int_\Omega \varphi_n \, \partial_{uuu}\mathcal{H}(0,a_n)[\varphi_n][\varphi_n][\varphi_n] \, dx.
\end{aligned}
$$

Next, observe that

$$
\begin{aligned}
A &= \frac{2}{R^3}\int_\Omega \varphi_n^2 \, dx > 0, \\
B &= \frac{3\lambda_n}{8R}(4 - R^2\lambda_n)\int_\Omega \varphi_n^3 \, dx,
\end{aligned}
$$

because

$$
\int_\Omega \varphi_n |\nabla\varphi_n|^2 \, dx = \frac{1}{2}\lambda_n \int_\Omega \varphi_n^3 \, dx,
$$

and

$$
C = -\frac{1}{2A}\int_\Omega \Big(|\nabla\varphi_n|^4 + \Big(\frac{5}{3}\big(\frac{R\lambda_n}{2} - \frac{9}{10R}\big)^2 + \frac{53}{20R^2}\Big)\lambda_n\varphi_n^4 \Big) \, dx < 0,
$$

because

$$
-\int_\Omega \operatorname{div}(|\nabla\varphi_n|^2\nabla\varphi_n)\varphi_n \, dx = \int_\Omega |\nabla\varphi_n|^4 \, dx \quad \text{and} \quad \int_\Omega \varphi_n^2|\nabla\varphi_n|^2 \, dx = \frac{1}{3}\lambda_n\int_\Omega \varphi_n^4 \, dx.
$$

Then, conclusion (c) follows from the representation

$$
\chi(t) = a_n - \frac{B}{A}t + Ct^2 + o(t^2).
$$

Thus, the proof is complete. $\square$

The next result is concerned with the bifurcation of positive solutions. Let us set

$$
\mathcal{S}^+ = \{(u,a) \in \mathcal{S} \mid u \gg 0\}.
$$

Corollary 4.2. (Bifurcation of positive solutions) *Assume* (H_1). *Then, there exists a connected component* $\mathcal{C}^+$ *of the set* $\overline{\mathcal{S}^+}$, *bifurcating from* $(0,a_1)$, *with* $a_1 = \frac{R^3\lambda_1}{2}$, *which is not compact in* $\mathcal{O}$. *Moreover, near* $(0,a_1)$ *the component* $\mathcal{C}^+$ *behaves as follows:*

- (i) *if* $R^2\lambda_1 > 4$, *then the bifurcation of* $\mathcal{C}^+$ *from* $(0,a_1)$ *is supercritical;*
- (ii) *if* $R^2\lambda_1 \le 4$, *then the bifurcation of* $\mathcal{C}^+$ *from* $(0,a_1)$ *is subcritical.*

Proof. As λ_1 is a simple eigenvalue of $-\Delta$ on $H_0^1(\Omega)$ and $\varphi_1 \gg 0$ in Ω, Theorem 4.1 applies and yields the existence of a neighbourhood U of $(0,a_1)$ in $\mathcal{O}\times\mathbb{R}$, a constant $\delta > 0$ and functions

$$
\psi :]-\delta,\delta[\to \Big\{ u \in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega) \mid \int_\Omega u\varphi_1 \, dx = 0 \Big\}, \qquad \chi :]-\delta,\delta[\to \mathbb{R}
$$

of class C^∞ such that

$$
\psi(0) = 0, \qquad \chi(0) = a_1,
$$

and

$$\overline{\mathcal{S}} \cap U = \Gamma = \{(t\varphi_1 + t\psi(t), \chi(t)) \mid |t| < \delta\}.$$

Let us set

$$\Gamma^+ = \{(t\varphi_1 + t\psi(t), \chi(t)) \mid 0 < t < \delta\} \quad \text{and} \quad \Gamma^- = \{(t\varphi_1 + t\psi(t), \chi(t)) \mid -\delta < t < 0\}.$$

We are going to apply the unilateral global bifurcation theorem in [41, Theorem 4.4] (see also [31, Chapter 6], [32]). As the norm in the Lebesgue space $L^p(\Omega)$, with $p \geq 2$, is of class C^1 (cf., e.g., [30, pp. 44-45]) and, hence, the norm in the Sobolev space $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ shares the same property, we only need to show that

$$(1-\tau)\partial_u \mathcal{H}(u,a) + \tau \partial_u \mathcal{H}(0,a)$$

is a Fredholm operator of index 0, for all $(u,a) \in \mathcal{O}$ and $\tau \in]0,1[$. Indeed, this operator can be written as

$$\sum_{i,j=1}^{N} \partial_i((1-\tau)a_{ij} + \tau\delta_{ij})\partial_j v) - \sum_{i=1}^{N}(1-\tau)b_i\partial_i v - ((1-\tau)c - \tau\tfrac{2}{R^3}a)v,$$

with, for all $\xi \in \mathbb{R}^N$,

$$|\xi|^2 \geq \sum_{i,j=1}^{N} \left((1-\tau)\, a_{ij} + \tau\delta_{ij}\right)\xi_i\xi_j = (1-\tau)\frac{(1+|\nabla u|^2)|\xi|^2 - (\nabla u \cdot \xi)^2}{\left(\sqrt{1+|\nabla u|^2}\right)^3} + \tau|\xi|^2 \geq \frac{|\xi|^2}{\left(\sqrt{1+|\nabla u|^2}\right)^3}.$$

Hence, the operator $(1-\tau)\partial_u \mathcal{H}(u,a) + \tau\partial_u \mathcal{H}(0,a)$ is uniformly elliptic, with coefficients in $W^{1,p}(\Omega)$. Thus, the same argument used in the second step of the proof of Theorem 4.1 yields the conclusion.

Let us denote by $\mathcal{C}^+$ the connected component of $\overline{\mathcal{S}} \setminus \Gamma^-$, which contains Γ^+, whose existence is guaranteed by [41, Theorem 4.4].

Let us prove that every $(u,a) \in \mathcal{C}^+ \setminus \{(0,a_1)\}$ satisfies $u \gg 0$. Assume, by contradiction, that this is false. Since, thanks to the conclusion (b) of Theorem 4.1, $\mathcal{C}^+ \setminus \{(0,a_1)\}$ is connected, there must exist $(u,a) \in \mathcal{C}^+ \setminus \{(0,a_1)\}$ and a sequence $((u_k,a_k))_k$ in $\mathcal{C}^+ \setminus \{(0,a_1)\}$, with $u_k \gg 0$ for all k, which converges to (u,a) in $W^{2,p}(\Omega) \times \mathbb{R}$, such that:

- either $u = 0$ and $a = a_m = \frac{R^3}{2}\lambda_m$, for some $m > 1$,
- or $u > 0$, but not $u \gg 0$; in this case, by Remark 2.1, $a > 0$.

In the former case, we set, for all k, $v_k = \frac{u_k}{\|u_k\|_{C^1}}$. Each v_k belongs to $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and satisfies $v_k \gg 0$. Moreover, observing that

$$\frac{1}{(u-R)^2} - \frac{1}{R^2}\frac{1}{\sqrt{1+|\nabla u|^2}} = \frac{2R-u}{R^2(u-R)^2}u + \frac{|\nabla u|^2}{R^2\sqrt{1+|\nabla u|^2}(1+\sqrt{1+|\nabla u|^2})},$$

by (4.1), v_k satisfies

$$\begin{aligned} -\Delta v_k = \sum_{i,j=1}^{N} \frac{-\partial_i u_k}{1+|\nabla u_k|^2}\,\partial_j v_k\,\partial_{ij}u_k + \frac{2R-u_k}{R^2(u_k-R)^2}v_k\frac{a_kR^2\sqrt{1+|\nabla u_k|^2}}{R^2+a_ku_k} \\ + \frac{|\nabla u_k|}{R^2\sqrt{1+|\nabla u_k|^2}(1+\sqrt{1+|\nabla u_k|^2})}\,|\nabla v_k|\frac{a_kR^2\sqrt{1+|\nabla u_k|^2}}{R^2+a_ku_k}, \quad \text{in } \Omega. \end{aligned} \tag{4.2}$$

It is easily checked that the right-hand side of this equation is uniformly bounded with respect to k in $L^p(\Omega)$. Hence $(v_k)_k$ is bounded in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and thus there exist a subsequence of $(v_k)_k$, still denoted by $(v_k)_k$, and $v \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ such that $(v_k)_k$ converges weakly in $W^{2,p}(\Omega)$ and strongly in $C^1(\overline{\Omega})$ to v, with $\|v\|_{C^1} = 1$. Since $(u_k)_k$ converges to 0 in $C^1(\overline{\Omega})$, letting $k \to +\infty$ in (4.2), we conclude that v satisfies

$$\begin{cases} -\Delta v = \lambda_m v, & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega. \end{cases}$$

This yields a contradiction, because $v \gg 0$, whereas all eigenfunctions corresponding to λ_m, with $m > 1$, change sign in Ω.

In the latter case, from (4.1) we infer

$$-\mathrm{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big) \geq \Big(\frac{1}{(u-R)^2} - \frac{1}{R^2}\Big)\frac{aR^2}{R^2+au} \geq 0, \qquad \text{in } \Omega.$$

The strong maximum principle and the Hopf boundary point lemma yield $u \gg 0$, which is a contradiction, thus proving that $\mathcal{C}^+ \setminus \{(0, a_1)\} \subseteq \mathcal{S}^+$.

Therefore, [41, Theorem 4.4] implies that $\mathcal{C}^+$ is not compact in $\mathcal{O}$.

Finally, the behaviour of $\mathcal{C}^+$, as described by (i) and (ii), follows from Theorem 4.1. □

Next, we discuss the bifurcation of nodal solutions. We first consider the one-dimensional case.

Corollary 4.3. (Bifurcation of nodal solutions in case $N = 1$**)** *Assume* (H_1)*. Then, for all* $n \in \mathbb{N}^+$*, conclusion* (a) *of Theorem 4.1 holds for the problem*

$$\begin{cases} -\Big(\dfrac{u'}{\sqrt{1+|u'|^2}}\Big)' = \Big(\dfrac{1}{(u-R)^2} - \dfrac{1}{R^2}\dfrac{1}{\sqrt{1+|u'|^2}}\Big)\dfrac{aR^2}{R^2+au}, & \text{in }]-r, r[, \\ u(-r) = 0, \ u(r) = 0, \end{cases} \tag{4.3}$$

with $a_n = \frac{R^3 n^2 \pi^2}{8r^2}$ *and* $\varphi_n(x) = \sin(\frac{n\pi(x+r)}{2r})$*.*

In addition, the curve $\Gamma = \{(t\varphi_n + t\psi(t), \chi(t)) \mid |t| < \delta\}$ *is contained in a connected component* $\mathcal{C}$ *of* $\overline{\mathcal{S}}$*, which is not compact in* $\mathcal{O}$*. Moreover, for every* $(u, a) \in \mathcal{C} \setminus \{(0, a_n)\}$*,* u *has exactly* $n+1$ *zeros in* $[-r, r]$*.*

Finally, near $(0, a_n)$ *the component* $\mathcal{C}$ *behaves as follows:*

(i) *if* $R\,n\,\pi \neq 4\,r$ *and* n *is odd, then the bifurcation is transcritical, i.e.,* $\chi'(0) \neq 0$*, with* $\chi'(0) > 0$*, in case* $R\,n\,\pi > 4\,r$*, and* $\chi'(0) < 0$*, in case* $R\,n\,\pi < 4\,r$*;*

(ii) *if* $R\,n\,\pi = 4\,r$ *or* n *is even, then the bifurcation is subcritical, i.e.,* $\chi'(0) = 0$ *and* $\chi''(0) < 0$*.*

Proof. In case $\Omega =]-r, r[$, for each $n \in \mathbb{N}^+$, $\lambda_n = \frac{n^2\pi^2}{4r^2}$ is a simple eigenvalue, with eigenfunction $\varphi_n(x) = \sin(\frac{n\pi(x+r)}{2r})$. Then, the existence of $\mathcal{C}$ follows from Theorem 4.1.

Let us prove the nodal properties of the solutions in $\mathcal{C}$. By the uniqueness property of the Cauchy problem, every non-trivial solution of (4.3) has only simple zeros. Since the map that counts the number of simple zeros of a function is continuous with respect to the topology of $C^1([-r, r])$, we conclude that the number of zeros of the non-trivial solutions is constant on $\mathcal{C}$ and equals $n+1$. Hence, in particular, $\mathcal{C}$ does not contain another point $(0, a_m)$ with $a_m \neq a_n$.

Next, conclusions (i) and (ii) follow from Theorem 4.1 observing that

$$\int_{-r}^{r} \sin^3\Big(\frac{n\pi(x+r)}{2r}\Big)dx = \begin{cases} 0, & \text{if } n \text{ is even}, \\ \frac{8r}{3n\pi}, & \text{if } n \text{ is odd}. \end{cases}$$

This concludes the proof. □

Remark 4.2 In the case $n = 1$, combining the arguments of the proof of Corollary 4.2 with the nodal properties of the solutions yields the existence of a connected component $\mathcal{C}^-$ of the set $\mathcal{S}^- = \{(u, a) \in \mathcal{S} \mid u \ll 0\}$, bifurcating from $(0, \frac{R^3\pi^2}{8r^2})$, which is not compact in $\mathcal{O}$.

Remark 4.3 Recall that by Proposition 2.4, if n is odd, we know that u is even, i.e., u is radially symmetric.

Now we deal with radially symmetric solutions of (4.1) on balls. In the following statement, J_α denotes the Bessel function of the first kind of order α and $\{\mu_{n,\alpha} \mid n \in \mathbb{N}^+\}$ is the set of the positive zeros of J_α.

Corollary 4.4. (Bifurcation of nodal radial solutions in case $N \geq 2$**)** *Assume* (H_1)*, where* $\Omega = B_r$ *is the ball of center* 0 *and radius* r *in* $\mathbb{R}^N$*, with* $N \geq 2$*. Then, for all* $n \in \mathbb{N}^+$*, conclusion* (a) *of Theorem 4.1 holds, where*

$$a_n = \frac{R^3}{2}\left(\frac{\mu_{n,\alpha}}{r}\right)^2, \quad \textit{with } \alpha = \tfrac{N-2}{2},$$

and $\varphi_n(x) = |x|^{-\alpha} J_\alpha(\sqrt{\lambda_n}|x|)$*.*

In addition, the curve $\Gamma = \{(t\varphi_n + t\psi(t), \chi(t)) \mid |t| < \delta\}$ *is contained in a connected component* $\mathcal{C}$ *of* $\overline{\mathcal{S}}$*, which is not compact in* $\mathcal{O}$*. Moreover, for every* $(u, a) \in \mathcal{C}$*, there exists* v*, with exactly* n *zeros in* $[0, r]$ *such that* $u(x) = v(|x|)$ *in* Ω*.*

Finally, near $(0, a_n)$ *the component* $\mathcal{C}$ *behaves as follows:*

- (i) *if* $R^2\lambda_n = 4$*, then the bifurcation is subcritical, i.e.,* $\chi'(0) = 0$ *and* $\chi''(0) < 0$*;*
- (ii) *if* $N \geq 3$ *and* $R^2\lambda_n \neq 4$*, then the bifurcation is transcritical, i.e.,* $\chi'(0) \neq 0$*, with* $\chi'(0) > 0$*, in case* $R^2\lambda_n > 4$*, and* $\chi'(0) < 0$*, in case* $R^2\lambda_n < 4$*.*

Proof. Recall that the eigenvalues and the corresponding eigenfunctions of the problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } B_r, \\ u = 0 & \text{on } B_r, \\ u \text{ radially symmetric} & \end{cases}$$

are given (see, e.g., [18]) by

$$\lambda_n = \left(\frac{\mu_{n,\alpha}}{r}\right)^2 \quad \text{and} \quad \varphi_n(x) = |x|^{-\alpha} J_\alpha(\sqrt{\lambda_n}|x|), \quad \text{with } \alpha = \tfrac{N}{2} - 1 \text{ and } n \in \mathbb{N}^+.$$

In particular, all the eigenvalues are simple. Hence, we can apply Theorem 4.1. The nodal properties of the solutions in $\mathcal{C}$ can be proved as in Corollary 4.3, passing to the ordinary differential equation equivalent to (4.1). In order to prove the claimed information on the local behaviour of $\mathcal{C}$ near the bifurcation point, we need to evaluate

$$\int_{B_r} \varphi_n^3(x)\, dx = \omega_N \int_0^r t^{N-1} \varphi_n^3(t)\, dt = \omega_N \int_0^r t^{1-\alpha} J_\alpha^3\left(\frac{\mu_{n,\alpha}}{r} t\right) dt$$

where the fact that $N - 1 - 3\alpha = N - 1 - 3(\frac{N}{2} - 1) = 1 - \alpha$ is used. The change of variable

$$s = \frac{\mu_{n,\alpha}}{r} t$$

yields

$$\int_0^r t^{1-\alpha} J_\alpha^3\left(\frac{\mu_{n,\alpha}}{r} t\right) dt = \left(\frac{r}{\mu_{n,\alpha}}\right)^{2-\alpha} \int_0^{\mu_{n,\alpha}} s^{1-\alpha} J_\alpha^3(s)\, ds.$$

By [6, Lemma 3.2] we conclude

$$\int_0^t s^{1-\alpha} J_\alpha^3(s)\,ds > 0 \quad \text{for all } t > 0,$$

provided that $\alpha \geq \frac{1}{2}$, i.e., $N \geq 3$. □

Remark 4.4 In case $N = 2$, we can prove that the conclusion (ii) holds only for n large enough. This is due to the fact that, in order to get (ii), we need to show that

$$\int_0^{\mu_{n,0}} sJ_0^3(s)\,ds \neq 0.$$

By [35, Formula 10.22.74], we have

$$\int_0^{+\infty} sJ_0^3(s)\,ds > 0.$$

Since $\mu_{n,0} \to +\infty$, as $n \to +\infty$, we obtain

$$\int_0^{\mu_{n,0}} sJ_0^3(s)\,ds > 0,$$

that is,

$$\int_{B_r} \varphi_n^3(x)\,dx > 0,$$

for all large n large. Actually, we have the numerical evidence that

$$\int_0^t sJ_0^3(s)\,ds > 0 \quad \text{for all } t > 0.$$

Hence, should this inequality be true, we could conclude that transcritical bifurcation occurs, in dimension $N = 2$, whenever $\lambda_n R^2 \neq 4$.

5 Existence and multiplicity of one-sign solutions in case $N = 1$

This section is divided into two subsections, where existence and multiplicity of one-sign solutions for the one-dimensional problem are separately discussed.

Existence of one-sign solutions in case $N = 1$

We start studying the existence of positive solutions for the problem

$$\begin{cases} -u'' = g(x, u, u'), & \text{in }]-r, r[, \\ u(-r) = 0, \ u(r) = 0. \end{cases} \tag{5.1}$$

The notion of solution of problem (5.1) is similar to the one introduced for problem (1.2). Precisely, a function $u \in C^2(]-r, r[) \cap C^0([-r, r])$ is a *(generalized) solution of* (5.1) if the following conditions hold:

- $-u''(x) = g(x, u(x), u'(x))$ for all $x \in]-r, r[$;
- either $u(-r) = 0$, or $u(-r) > 0$ and $u'(-r^+) = +\infty$, or $u(-r) < 0$ and $u'(-r^+) = -\infty$;

- either $u(r) = 0$, or $u(r) > 0$ and $u'(r^-) = -\infty$, or $u(r) < 0$ and $u'(r^-) = +\infty$.

A solution u of (5.1) is *classical* if $u(-r) = 0 = u(r)$.

Theorem 5.1. *Let β_1, $\beta_2 \in C^2(]-r,r[) \cap C^0([-r,r])$ and $\beta = \min\{\beta_1, \beta_2\} \geq 0$. Let $D = \{(x,s,\xi) \in [-r,r] \times \mathbb{R} \times \mathbb{R} \mid 0 \leq s \leq \beta(x)\}$. Suppose that*

(a) $g : D \to \mathbb{R}$ is continuous;

(b) $g(x,s,\xi) \geq 0$ for all $(x,s,\xi) \in D$;

(c) uniqueness holds for the Cauchy problems associated with the equation in (5.1);

(d) for $i = 1,2$, $-\beta_i''(x) \geq g(x, \beta_i(x), \beta_i'(x))$ for every $x \in]-r,r[$.

Then there exists a solution u of (5.1) satisfying $0 \leq u \leq \beta$ and u is the minimum among all solutions v of (5.1) satisfying $0 \leq v \leq \beta$.

In case $u > 0$ we also have $u \gg 0$.

Proof. For all n, there exists $N_n \in \mathbb{R}$ such that, for all $(x,s,\xi) \in D$ with $|\xi| \leq n$, we have $g(x,s,\xi) \leq N_n$. For every n, we consider the function

$$g_n(x,s,\xi) = \min\{g(x,s,\xi), N_n\}$$

and the problem

$$\begin{cases} -u'' = g_n(x,u,u'), & \text{in }]-r,r[, \\ u(-r) = 0, \ u(r) = 0. \end{cases} \tag{5.2}$$

Observe that, for $i = 1,2$, β_i is an upper solution of (5.2). Moreover, 0 is a lower solution of (5.2). Hence, by [16, Theorem II-1.6], there exists a solution $v_n \in C^2([-r,r])$ of (5.2) satisfying $0 \leq v_n \leq \beta$ which is the minimum among all solutions v of (5.2) satisfying $0 \leq v \leq \beta$.

Step 1. Convergence. By the sign assumption on g, v_n is concave in $]-r,r[$. Hence, there exists $M > 0$ such that, for all n, $\|v_n\|_\infty + \|v_n'\|_{L^1} \leq M$. Therefore there exists a subsequence of $(v_n)_n$, we still denote by $(v_n)_n$, converging in $L^1(-r,r)$ and a.e. in $]-r,r[$ to a function $u \in BV(-r,r)$ (see e.g. [1, Theorem 3.23]). Let us verify the regularity in $]-r,r[$ of u. Fix ε with $0 < \varepsilon < r$. The concavity and the boundedness of v_n imply that

$$v_n'(-r+\varepsilon) \leq \frac{M}{\varepsilon} \text{ and } v_n'(r-\varepsilon) \geq -\frac{M}{\varepsilon}.$$

By the monotonicity of v_n' we conclude that

$$|v_n'(x)| \leq \frac{M}{\varepsilon}$$

for every n and all $x \in [-r+\varepsilon, r-\varepsilon]$. As, by continuity of g, there exists $L > 0$ such that,

$$0 \leq g(x,s,\xi) \leq L, \qquad \text{for all } (x,s,\xi) \in D \text{ with } |\xi| \leq \tfrac{M}{\varepsilon},$$

we have also, for every $n \geq \frac{M}{\varepsilon}$ and all $x \in [-r+\varepsilon, r-\varepsilon]$,

$$0 \leq g_n(x, v_n(x), v_n'(x)) = g(x, v_n(x), v_n'(x)) \leq L.$$

Since v_n satisfies the equation in (5.2), by the Arzelà-Ascoli theorem, the sequence $(v_n)_n$ converges in $C^1([-r+\varepsilon, r-\varepsilon])$ to u. Taking the limit we conclude that u satisfies the equation

$$-u'' = g(x,u,u'),$$

in $[-r+\varepsilon, r-\varepsilon]$. Note that then $u \in C^2(]-r,r[)$ and satisfies

$$-u'' = g(x,u,u'), \qquad \text{in }]-r,r[.$$

As u is the pointwise limit of a sequence of non-negative concave functions, we know that u is non-negative concave and either $u \equiv 0$ or $u \gg 0$.

Step 2. Behaviour at the boundary. To prove that u is a solution of (5.1), it remains to consider the behaviour of u at the points $-r$ and r. Since u is non-negative and concave, there exist the limits

$$\lim_{x\to -r^+} u(x) = u(-r^+) \in [0,+\infty[, \qquad \lim_{x\to r^-} u(x) = u(r^-) \in [0,+\infty[,$$

and

$$\lim_{x\to -r^+} u'(x) = u'(-r^+) \in [0,+\infty[\cup\{+\infty\}, \qquad \lim_{x\to r^-} u'(x) = u'(r^-) \in]-\infty,0] \cup \{-\infty\}.$$

Suppose that $u(-r^+) > 0$. We aim to show that $u'(-r^+) = +\infty$. By contradiction, suppose that $u'(-r^+) \in \mathbb{R}$. Consider the Cauchy problems

$$\begin{cases} -w'' = g_n(x,w,w'), \\ w(0) = v_n(0), \quad w'(0) = v_n'(0), \end{cases}$$

and

$$\begin{cases} -w'' = g(x,w,w'), \\ w(0) = u(0), \quad w'(0) = u'(0). \end{cases}$$

The continuous dependence on parameters and initial conditions, which follows from (c), yields

$$0 = \lim_{n\to+\infty} v_n(-r) = u(-r^+) > 0,$$

which is a contradiction.

In a parallel way, we obtain that $u'(r^-) = -\infty$ in case $u(r^-) > 0$.

Step 3. Minimality of the solution u. Let us prove that u is is the minimum among all solutions v of (5.1) satisfying $0 \le v \le \beta$. Let v be such a function. Then, for each n, v is an upper solution of (5.2) with $v \ge 0$, hence, by the minimality of v_n, we conclude that $v_n \le v$ and therefore $u \le v$, in $]-r,r[$. □

Remark 5.1 If, in addition to the hypotheses of Theorem 5.1, we assume that $\beta(-r) = \beta(r) = 0$, then the solution u we have found is classical.

Corollary 5.2. *Fix $d \in \mathbb{R}^+ \cup \{+\infty\}$ and let $g : [-r,r] \times [0,d[\times\mathbb{R} \to \mathbb{R}$ be a non-negative continuous function. Assume that uniqueness holds for the Cauchy problems associated with the equation in* (5.1)*. If there exists a solution u of* (5.1)*, then there exists the minimum solution u_m among all solutions of* (5.1)*.*

Proof. First observe that, as g is non-negative, every solution of (5.1) is non-negative.

By Theorem 5.1 applied with $\beta = u$, there exists the minimum solution u_m of (5.1) among all solutions v satisfying $0 \le v \le u$. If u_m is not the minimum solution of (5.1), then there exists a solution v of (5.1) with $\min(v - u_m) < 0$. By Theorem 5.1, there exists the minimum solution u_0 of (5.1) among all solutions v satisfying $0 \le u_0 \le \min\{u_m, v\}$. This implies that $0 \le u_0 < u_m \le u$ which contradicts the fact that u_m is the minimum solution lying between 0 and u. □

Let us consider problem (1.2) in case $N = 1$ and $\Omega =]-r, r[$, i.e.,

$$\begin{cases} -(1-bu)\Big(\dfrac{u'}{\sqrt{1+|u'|^2}}\Big)' = \dfrac{a}{(u-R)^2} + \dfrac{b}{\sqrt{1+|u'|^2}}, & \text{in }]-r,r[, \\ u(-r) = 0, \ u(r) = 0, \end{cases} \tag{5.3}$$

where $a, b \in \mathbb{R}$ are parameters and $R > 0$ is a fixed constant.

We associate with the equation in (5.3) the function

$$g(x,s,\xi) = \Big(\frac{a}{(s-R)^2} + \frac{b}{\sqrt{1+\xi^2}}\Big)\frac{(1+\xi^2)^{3/2}}{1-bs}. \tag{5.4}$$

We prove now a comparison principle for the solutions of (5.3).

Lemma 5.3. *Assume* (H_1)*. Suppose that* $\dfrac{a_1}{R^2} + b_1 \geq 0$ *holds, with* $a_1 \geq 0$*, and take any* $a_2 \geq a_1$ *and* $b_2 \geq b_1$*. Suppose further that problem* (5.3) *has positive solutions* u_i *for* $(a,b) = (a_i, b_i)$*, with* $i = 1,2$*, and that* $u_1(-r) = u_2(-r)$ *and* $u_1'(-r) = u_2'(-r)$ *(or, equivalently,* $u_1(r) = u_2(r)$ *and* $u_1'(r) = u_2'(r)$*), possibly infinite. Then, we have* $u_2 \leq u_1$*. In particular, if* u_1 *is the minimum positive solution of* (5.3) *for* $a = a_1$ *and* $b = b_1$*, then* $a_1 = a_2$*,* $b_1 = b_2$ *and* $u_1 = u_2$*.*

Proof. Set $c = u_1(-r) = u_2(-r)$ and $d = 1/u_1'(-r) = 1/u_2'(-r)$ if $u_1'(-r) \neq +\infty$, $d = 0$ otherwise. Observe that, by Lemma 2.5 and by uniqueness for the Cauchy problems associated with the equation in (5.3), we have that u_1 and u_2 are both strictly positive, concave, even and increasing on $[-r, 0]$. Denote by φ_1 and φ_2 the inverse functions of the restrictions to $[-r, 0]$ of u_1 and u_2, respectively, and set $y_1 = \varphi_1'$, $y_2 = \varphi_2'$. Observe that the function $y_1 = y_1(z)$ is a solution of the initial value problem

$$\begin{cases} y' = \Big(\dfrac{a_1}{(R-z)^2} + \dfrac{b_1 y}{\sqrt{1+y^2}}\Big)\dfrac{(1+y^2)^{3/2}}{1-b_1 z}, \\ y(c) = d, \end{cases}$$

and the function $y_2 = y_2(z)$ is a solution of the initial value problem

$$\begin{cases} y' = \Big(\dfrac{a_2}{(R-z)^2} + \dfrac{b_2 y}{\sqrt{1+y^2}}\Big)\dfrac{(1+y^2)^{3/2}}{1-b_2 z}, \\ y(c) = d. \end{cases} \tag{5.5}$$

We claim that $y_1(z) \leq y_2(z)$ for all $z \in [c, \omega[$, where $\omega = \min\{u_1(0), u_2(0)\}$. By contradiction, suppose that there exists $z_0 \in]c, \omega[$ where $y_1(z_0) > y_2(z_0)$. As y_2 is a solution of (5.5) and y_1 satisfies

$$y_1' \leq \Big(\frac{a_2}{(R-z)^2} + \frac{b_2 y_1}{\sqrt{1+y_1^2}}\Big)\frac{(1+y_1^2)^{3/2}}{1-b_2 z},$$

using, e.g., [36, Lemma 2.1], for all $p \in]y_2(z_0), y_1(z_0)[$, there exists a solution $y : [c, z_0] \to \mathbb{R}$ of the equation

$$y' = \Big(\frac{a_2}{(R-z)^2} + \frac{b_2 y}{\sqrt{1+y^2}}\Big)\frac{(1+y^2)^{3/2}}{1-b_2 z}$$

satisfying $\min\{y_1, y_2\} \leq y \leq \max\{y_1, y_2\}$ and $y(z_0) = p$. As $p > y_2(z_0)$ we obtain a solution of the initial value problem (5.5) which is different from y_2, thus yielding a contradiction with the uniqueness of the solution of the Cauchy problem. Since

$$\varphi_2(z) - \varphi_1(z) = \int_c^z \big(y_2(s) - y_1(s)\big)\,ds \geq 0,$$

for all $z \in [c, \omega[$, we conclude that $\varphi_1 \leq \varphi_2$ on $[c, \omega[$. Therefore $u_2 \leq u_1$ which concludes the first part of the proof.

To prove the second claim of the statement we apply Theorem 5.1 for g defined in (5.4), with $\beta = u_2$, $a = a_1$ and $b = b_1$. Hence, the minimum positive solution u_1 of (5.3), for $a = a_1$ and $b = b_1$, satisfies also $u_1 \leq u_2$ and, consequently, we have $u_1 = u_2$ as well as $a_1 = a_2$ and $b_1 = b_2$. □

Theorem 5.4. *Assume (H_1). Suppose that $\dfrac{a_1}{R^2} + b_1 \geq 0$ holds, with $a_1 \geq 0$, and take any $a_2 \geq a_1$ and $b_2 \geq b_1$. Let u_1 be the minimum positive solution of (5.3) for $a = a_1$ and $b = b_1$, in case $\dfrac{a_1}{R^2} + b_1 > 0$, and $u_1 = 0$ in case $\dfrac{a_1}{R^2} + b_1 = 0$. Suppose further that u_2 is a positive solution of (5.3) for $a = a_2$ and $b = b_2$, with $u_1 \neq u_2$. Then, we have $u_1 \ll u_2$.*

Proof. By applying Theorem 5.1 for g defined in (5.4), with $\beta = u_2$, $a = a_1$ and $b = b_1$, we have that $u_1 \leq u_2$.

Let us prove that, for all $x \in]-r, r[$, $u_1(x) < u_2(x)$. In case $a_1 = a_2$ and $b_1 = b_2$, the claim follows by uniqueness for the Cauchy problems associated with the equation in (5.3). In case $(a_1, b_1) \neq (a_2, b_2)$ assume, by contradiction, that $\min(u_2 - u_1) = u_2(x_0) - u_1(x_0) = 0$. Then we have $u_2'(x_0) - u_1'(x_0) = 0$ and $u_2''(x_0) - u_1''(x_0) \geq 0$. From the equation satisfied by u_2 and u_1 we obtain the contradiction $u_2''(x_0) - u_1''(x_0) < 0$.

By Lemma 5.3, we infer that either $u_1(-r) < u_2(-r)$ or $u_1'(-r) < u_2'(-r)$. Altogether this gives $u_1 \ll u_2$. □

Proposition 5.5. *Assume (H_1). Suppose that u is the minimum positive solution of (5.3) for some $a \geq 0$ and $b \geq -\frac{2}{R}$, with $\dfrac{a}{R^2} + b > 0$. Then, u is classical.*

Proof. Recall that, by Lemma 2.5, u is concave and even. By contradiction, suppose that $u(-r) = u(r) = \overline{\delta} > 0$.

Observe that, for any fixed $x \in [-r, r]$, the function

$$f_1(\delta) = \frac{b}{1 - b(u(x) - \delta)}$$

is decreasing on $[0, \overline{\delta}]$ and, due to the assumption $b \geq -\frac{2}{R}$, the function

$$f_2(\delta) = \frac{a}{(u(x) - \delta - R)^2(1 - b(u(x) - \delta))}$$

is decreasing on $[0, \overline{\delta}]$. Hence $\beta = u - \overline{\delta}$ satisfies

$$-\beta'' \geq \Big(\frac{a}{(\beta - R)^2} + \frac{b}{\sqrt{1 + |\beta'|^2}}\Big)\frac{(1 + |\beta'|^2)^{3/2}}{1 - b\,\beta}$$

and we can apply Theorem 5.1 for g defined in (5.4), with $\beta = u - \overline{\delta}$. Therefore, problem (5.3) has a solution u_1 satisfying $0 \leq u_1 \leq u - \overline{\delta}$, thus contradicting the minimality of u. □

Proposition 5.6. *Assume (H_1). Suppose there exist $a_2 \geq a_1 \geq 0$ and $b_2 \geq b_1$ with $(a_2, b_2) \neq (a_1, b_1)$, $\dfrac{a_1}{R^2} + b_1 > 0$, such that (5.3), with $a = a_2$ and $b = b_2$, has a positive classical solution. Then the minimum positive solution u of (5.3), with $a = a_1$ and $b = b_1$, is classical and $u \in C^2([-r, r])$.*

Proof. By Theorem 5.4, the minimum positive solution u of (5.3), with $a = a_1$ and $b = b_1$, is classical. As $u_2 \neq u$, by Lemma 5.3 we have that $u'(-r) < u_2'(-r)$. From the equation, we conclude that $u \in C^2([-r, r])$. $\square$

Corollary 5.7. *Assume* (H_1). *Suppose that* u *is the minimum positive solution of* (5.3), *for some* $a \geq 0$ *and* $b < -\frac{2}{R}$, *with* $\dfrac{a}{R^2} + b \geq 0$. *Assume there exists* $\overline{b} \geq -\frac{2}{R}$ *such that* (5.3), *with* $b = \overline{b}$, *has a positive solution* $\overline{u}$. *Then,* u *is classical and* $u \in C^2([-r, r])$.

Proof. Without loss of generality we can assume that $\overline{u}$ is the minimum solution of (5.3) for a and $b = \overline{b}$. By Proposition 5.5, $\overline{u}$ is classical. It also satisfies

$$-\overline{u}'' \geq \Big(\frac{a}{(\overline{u} - R)^2} + \frac{b}{\sqrt{1 + |\overline{u}'|^2}}\Big)\frac{(1 + |\overline{u}'|^2)^{3/2}}{1 - b\,\overline{u}}.$$

Hence, we can apply Theorem 5.1 for g defined in (5.4), with $\beta = \overline{u}$, and therefore u is classical. The rest of the proof follows exactly as in Proposition 5.6. $\square$

Theorem 5.8. *Assume* (H_1). *Let us set*

$$b^* = \sup\{b \mid \textit{problem } (3.16) \textit{ has a solution } u \textit{ with } u(0) < R\}$$

and recall that $\lambda_1 = \frac{\pi^2}{4r^2}$. *Then, we have* $0 < b^* < +\infty$ *and there exists a function* $a^* :]-\frac{R}{2}\lambda_1, b^*[\to \mathbb{R}^+$ *such that*

(*a*) *for all* $b \in]-\frac{R}{2}\lambda_1, 0[$, $a^*(b) > -R^2 b$;

(*b*) a^* *is strictly decreasing and right-continuous;*

(*c*) *problem* (5.3) *has at least one positive solution* u_1 *in the following cases:*

- $0 \leq b < b^*$ *and* $0 < a \leq a^*(b)$,
- $-\frac{R}{2}\lambda_1 < b < 0$ *and* $-R^2 b < a \leq a^*(b)$;

(*d*) *problem* (5.3) *has no positive solution in the following cases:*

- $b \geq b^*$ *and* $a > 0$,
- $-\frac{R}{2}\lambda_1 < b < b^*$ *and* $a > a^*(b)$.

Moreover, the solution u_1 *we have found in* (*c*) *is classical and also, if* $a < a^*(b)$, $u_1 \in C^2([-r, r])$, *in the following cases:*

- $0 \leq b < b^*$ *and* $0 < a \leq a^*(b)$,
- $\pi R \leq 4r$, $-\frac{R}{2}\lambda_1 < b < 0$ *and* $-R^2 b < a \leq a^*(b)$,
- $\pi R > 4r$, $-\frac{2}{R} \leq b < 0$ *and* $-R^2 b < a \leq a^*(b)$,
- $\pi R > 4r$, $-\frac{R}{2}\lambda_1 < b < -\frac{2}{R}$ *and* $-R^2 b < a \leq a^*(-\frac{2}{R})$.

Remark 5.2 According to Remark 3.6, we have $b^* = \tilde{b}$, if $R \leq \frac{1}{b^\#} - \frac{r}{t^\#}$, and $b^* = b^\#$, if $R > \frac{1}{b^\#} - \frac{r}{t^\#}$.

Remark 5.3 By Theorem 3.1, recall that $b^* \leq \frac{1}{r}$.

Remark 5.4 Observe that, if $\pi R \leq 4r$, then all solutions u_1 whose existence is proved above are classical.

Remark 5.5 If $a > 0$ and $b \geq 0$, by Theorem 2.1 any solution of problem (5.3) is positive. Hence we deduce that, for all $b \in]0, b^*[$ and $a > a^*(b)$, problem (5.3) has no solutions at all.

Proof. The proof is divided into several steps.

Step 1. Existence of a positive solution $u \in C^2([-r,r])$ for $0 \leq b < b^*$ and small $a > 0$. Fix $b \in [0, b^*[$, choose $\bar{b} \in]b, b^*[$ and let $\overline{u} \in C^2([-r,r])$ be a positive solution of (3.16) for $b = \bar{b}$, with $\overline{u}(0) < R$. Then, for $a > 0$ small enough we have, for all $x \in]-r, r[$,

$$\frac{\bar{b}}{\sqrt{1+|\overline{u}'(x)|^2}} \frac{1}{1-\bar{b}\overline{u}(x)} > \Big(\frac{a}{(\overline{u}(x)-R)^2} + \frac{b}{\sqrt{1+|\overline{u}'(x)|^2}}\Big)\frac{1}{1-b\overline{u}(x)}.$$

Hence $\beta = \overline{u}$ satisfies the assumptions of Theorem 5.1, for g defined in (5.4), and therefore problem (5.3) has a positive solution $u \leq \overline{u}$. This proves that the set

$$A_b = \{a > 0 \mid \text{problem (5.3) has a positive solution}\}$$

is not empty. Define $a^*(b) = \sup A_b$. Recall that, by Remark 3.1, $a^*(b) \leq (\frac{1}{r} - b)R^2$. Moreover, by definition of $a^*(b)$, problem (5.3) has no positive solution for $a > a^*(b)$.

Step 2. Existence of a positive solution $u \in C^2([-r,r])$ for $-\frac{R}{2}\lambda_1 < b < 0$, $a > -R^2 b$ with $a + R^2 b$ small. This can be deduced from Lemma 3.2. Define A_b as above and set $a^*(b) = \sup A_b$. By Theorem 3.1, we have that $a^*(b) < +\infty$. Moreover, by definition of $a^*(b)$, problem (5.3) has no positive solution for $a > a^*(b)$.

Step 3. Non-existence of solution for $b > b^*$ and $a > 0$. Assume, by contradiction, that β is a positive solution of (5.3) with $b > b^*$ and $a > 0$. Then β satisfies the assumptions of Theorem 5.1 for g defined by (5.4) with $a = 0$ and, hence, there exists a positive solution u of problem (3.16) with $u(0) < R$ for $b > b^*$, thus contradicting the definition of b^*.

Step 4. Non-existence of solution for $b = b^*$ and $a > 0$. Assume, by contradiction, that $\bar{u}$ is a positive solution of (5.3) with $b = b^*$ and $\bar{a} > 0$. Then, for every $a \in]0, \bar{a}[$, $\bar{u}$ satisfies the assumptions of Theorem 5.1 for g defined by (5.4) with a and $b = b^*$ and, hence, there exists the minimum positive solution u of problem (5.3) with $u(0) < R$. By Proposition 5.5 u is classical and, by Proposition 5.6, possibly replacing a with $\tilde{a} < a$, $u \in C^2([-r,r])$. Take $b > b^*$ such that

$$1 - bu(x) > 0 \qquad \text{and} \qquad \frac{b - b^*}{1 - bu(x)} \leq \frac{a}{R^2},$$

for all $x \in [-r, r]$. Then u satisfies the assumptions of Theorem 5.1 for g defined by (5.4) with $a = 0$ and, hence, there exists a positive solution $\hat{u}$ of problem (3.16) with $\hat{u}(0) < R$ for $b > b^*$, thus contradicting the definition of b^*.

Step 5. Existence of the minimum positive solution u_1 and its regularity for either $0 \leq b < b^*$ and $0 < a < a^*(b)$ or $-\frac{R}{2}\lambda_1 < b < 0$ and $-R^2 b < a < a^*(b)$. Choose $\hat{a} \in]a, a^*(b)]$ such that problem (5.3), for $\hat{a}$ and b, has a positive solution $\hat{u}$. Then $\beta = \hat{u}$ satisfies the assumptions of Theorem 5.1 for g as in (5.4) and therefore problem (5.3) has a minimum positive solution u_1. Its regularity in case either $b \geq 0$, or $\pi R > 4r$, $-\frac{2}{R} \leq b < 0$ and $-R^2 b < a \leq a^*(b)$, or $\pi R > 4r$, $-\frac{R}{2}\lambda_1 < b < -\frac{2}{R}$ and $-R^2 b < a \leq a^*(-\frac{2}{R})$, can be deduced from Proposition 5.5, Proposition 5.6, and Corollary 5.7.

Step 6. Existence of a positive classical solution for $0 \le b < b^*$ *and* $a = a^*(b)$. We pick an increasing sequence $(a_n)_n$ with $0 < a_1 < a^*(b)$ converging to $a^*(b)$. For each n we denote by u_n the minimum positive solution of problem (5.3) with $a = a_n$ and b. We claim that the sequence $(u_n)_n$ is bounded away from R. Take $\delta \in]0, R/2[$ satisfying

$$4\delta(R-\delta) < a_1 r. \tag{5.6}$$

We aim to show that $u_n(x) < R - \delta$ for all $x \in [-r, r]$ and all n. Suppose, by contradiction, that $\max u_n = u_n(0) \ge R - \delta$ for some n. By the concavity of u_n, we have

$$u_n(x) \ge \frac{u_n(0)}{r}(x+r) \ge \frac{R-\delta}{r}(x+r) \qquad \text{for all } x \in [-r, 0].$$

Set $x_0 = -\frac{r\delta}{R-\delta}$. Then we have

$$u_n(x) \ge R - 2\delta, \qquad \text{for all } x \in [x_0, 0].$$

Integrating the equation in (5.3) on the interval $[x_0, 0]$ we obtain, by (5.6),

$$\frac{u_n'(x_0)}{\sqrt{1+|u_n'(x_0)|^2}} \ge \int_{x_0}^0 \frac{a_n}{(2\delta)^2}\,dx \ge \frac{a_1}{4\delta^2}\frac{r\delta}{R-\delta} > 1,$$

which is a contradiction.

As the sequence $(u_n)_n$ is uniformly upper bounded by $R - \delta$, arguing as in Step 1 and Step 2 of the proof of Theorem 5.1, we see that there exists a subsequence converging to some function $u \in C^2(]-r, r[) \cap C^0([-r, r])$ which satisfies $0 \le u \le R - \delta$ in $[-r, r]$ and is a solution of (5.3) for $a = a^*(b)$. By Proposition 5.5, the minimum positive solution is a classical solution of (5.3) for $a = a^*(b)$.

Step 7. Existence of a positive solution for $-\frac{R}{2}\lambda_1 < b < 0$ *and* $a = a^*(b)$. We argue as in Step 6 choosing $\delta \in]0, R/2[$ such that

$$\Big(\frac{a_1}{4\delta^2} + b\Big)\frac{1}{1-bR}\frac{r\delta}{R-\delta} > 1.$$

Taking the minimum positive solution, by Proposition 5.5, we obtain a positive solution of (5.3) for $a = a^*(b)$, which is classical if $b \ge -\frac{2}{R}$.

Step 8. The function a^* *is decreasing.* Suppose that $b_1 < b_2 < b^*$ and assume that, for some $a_1 > 0$, there exists a positive solution β of (5.3) with $a = a_1$ and $b = b_2$. Observe that β satisfies the assumptions of Theorem 5.1 for g as in (5.4) with $a = a_1$ and $b = b_1$. Therefore, problem (5.3), with $a = a_1$ and $b = b_1$, has a solution u_1, thus showing that $a^*(b_1) \ge a^*(b_2)$.

Step 9. The function a^* *is strictly decreasing.* Suppose, by contradiction, that $b_1 < b_3$ and $a^*(b_1) = a^*(b_3)$. Denote by u_3 the minimum positive solution of (5.3) for $b = b_3$ and $a = a^*(b_1)$. Take $b_2 \in]b_1, b_3[$ and let β be the minimum positive solution of (5.3) for $b = b_2$ and $a = a^*(b_2)$ $(= a^*(b_1) = a^*(b_3)$ by the previous step). By Proposition 5.6, either β is not classical, or β is classical. In the latter case, possibly replacing β with the minimum positive solution of (5.3), where b_2 is substituted by $\tilde b_2 \in]b_1, b_2[$, we also have $\beta \in C^2([-r, r])$. This implies that we have, for all $x \in [-r, r]$,

$$\Big(\frac{a^*(b_1)}{(\beta(x)-R)^2} + \frac{b_2}{\sqrt{1+|\beta'(x)|^2}}\Big)\frac{1}{1-b_2\beta(x)} > \Big(\frac{a^*(b_1)}{(\beta(x)-R)^2} + \frac{b_1}{\sqrt{1+|\beta'(x)|^2}}\Big)\frac{1}{1-b_1\beta(x)}.$$

Hence, there exists $\varepsilon > 0$ such that β satisfies the assumptions of Theorem 5.1 for g defined by

$$g(x, s, \xi) = \Big(\frac{a^*(b_1)+\varepsilon}{(s-R)^2} + \frac{b_1}{\sqrt{1+\xi^2}}\Big)\frac{(1+\xi^2)^{3/2}}{1-b_1 s}.$$

This implies that problem (5.3), for $a = a^*(b_1) + \varepsilon$ and $b = b_1$, has a solution, thus contradicting the definition of $a^*(b_1)$.

Step 10. The function a^* is right-continuous. As a^* is decreasing, we know that $\lim_{b\to b_0^+} a^*(b) \le a^*(b_0)$. Assume, by contradiction, that $\lim_{b\to b_0^+} a^*(b) < a^*(b_0)$ and choose $\varepsilon < a^*(b_0) - \lim_{b\to b_0^+} a^*(b)$. Observe that, for all $b > b_0$, we have

$$a^*(b) < a^*(b_0) - \varepsilon. \tag{5.7}$$

Let u_0 satisfy

$$\begin{cases} -u_0'' = \Big(\dfrac{a^*(b_0)}{(u_0-R)^2} + \dfrac{b_0}{\sqrt{1+|u_0'|^2}}\Big)\dfrac{(1+|u_0'|^2)^{3/2}}{1-b_0\,u_0}, & \text{in }]-r,r[, \\ u_0(-r)=0,\ \ u_0(r)=0. \end{cases}$$

We can choose $\delta > 0$ small enough such that, for all $x \in [-r,r]$,

$$-u_0''(x) \ge \Big(\frac{a^*(b_0)-\varepsilon}{(u_0(x)-R)^2} + \frac{b_0+\delta}{\sqrt{1+|u_0'(x)|^2}}\Big)\frac{(1+|u_0'(x)|^2)^{3/2}}{1-(b_0+\delta)\,u_0(x)}.$$

By Theorem 5.1 for g defined in (5.4), problem (5.3), with $a = a^*(b_0)-\varepsilon$ and $b = b_0+\delta$, has a solution and, hence, we infer that $a^*(b_0+\delta) \ge a^*(b_0) - \varepsilon$, thus contradicting (5.7). □

We provide some estimates on $a^*(b)$, which refine the ones given in Remark 3.1.

Proposition 5.9. *Assume (H_1). Let $a^* :]-\frac{R}{2}\lambda_1, b^*[\to \mathbb{R}^+$ be the function defined in Theorem 5.8. Then,*

$$a^*(b) \le \min\Big\{(\lambda_1 - b^2)\frac{R^3}{2+bR}, \big(\frac{1}{r} - b\big)R^2\Big\}, \quad \text{if } 0 \le b < b^*, \tag{5.8}$$

and

$$a^*(b) \le \frac{\lambda_1 R^3}{2}(1-bR), \quad \text{if } -\frac{R}{2}\lambda_1 < b < 0.$$

In case $b > 0$, the following lower estimate also holds:

$$a^*(0)(1-bR) - bR^2 \le a^*(b).$$

Proof. The proof is divided into two parts.

Part 1. Upper estimate on $a^*(b)$. In case $a \ge 0$ and $b \ge 0$, if u is a solution of (5.3), then u satisfies

$$\begin{aligned} -u'' &= \Big(\frac{a}{(u-R)^2} + \frac{b}{\sqrt{1+|u'|^2}}\Big)\frac{(1+|u'|^2)^{3/2}}{1-bu} \\ &\ge \Big(\frac{a}{(u-R)^2} + b\Big)\frac{1}{1-bu} \ge \Big(\frac{a}{R^2} + b\Big) + \Big(\frac{2a}{R^3} + \big(\frac{a}{R^2}+b\big)b\Big)u \ge \Big(\frac{2a}{R^3} + \big(\frac{a}{R^2}+b\big)b\Big)u. \end{aligned} \tag{5.9}$$

As u is concave and even, we have

$$\int_{-r}^{r} |u'(x)|\,dx = 2(u(0)-u(r)),$$

so that $u' \in L^1(-r,r)$, and

$$u(x) - u(-r) \ge u'(x)(x+r) \ge 0, \text{ for all } x \in [-r,0],$$

$$u(x) - u(r) \ge u'(x)(x-r) \ge 0, \text{ for all } x \in [0,r].$$

Hence, as $\varphi_1(x) = \sin(\frac{\pi}{2r}(x+r))$, we infer

$$\lim_{x\to\mp r} u'(x)\varphi_1(x) = \lim_{x\to\mp r} u'(x)(x\pm r)\frac{\varphi_1(x)}{x\pm r} = 0.$$

Integrating by parts, we obtain

$$\begin{aligned}
-\lim_{x\to r}\int_{-x}^{x} u''\varphi_1\,ds &= \lim_{x\to r}\big[-u'(x)\varphi_1(x) + u'(-x)\varphi_1(-x)\big] + \lim_{x\to r}\int_{-x}^{x} u'\varphi_1'\,ds \\
&= \int_{-r}^{r} u'\varphi_1'\,ds = u(r)\varphi_1'(r) - u(-r)\varphi_1'(-r) - \int_{-r}^{r} u\varphi_1''\,ds = \lambda_1\int_{-r}^{r} u\varphi_1\,ds.
\end{aligned}$$

From (5.9) we deduce

$$\lambda_1\int_{-r}^{r} u\varphi_1\,ds \geq \Big(\frac{2a}{R^3} + \big(\frac{a}{R^2}+b\big)b\Big)\int_{-r}^{r} u\varphi_1\,ds,$$

and the first estimate in (5.8) follows. The second estimate follows from Remark 3.1.

In case $b<0$ and $\dfrac{a}{R^2}+b>0$, if u is a positive solution of (5.3), then u satisfies

$$\begin{aligned}
-u'' &= \Big(\frac{a}{(u-R)^2} + \frac{b}{\sqrt{1+|u'|^2}}\Big)\frac{(1+|u'|^2)^{3/2}}{1-bu} \\
&\geq \Big(\frac{a}{(u-R)^2}+b\Big)\frac{1}{1-bR} \geq \Big(\frac{a}{R^2}+b\Big)\frac{1}{1-bR} + \frac{2a}{R^3}\frac{1}{1-bR}u \geq \frac{2a}{R^3}\frac{1}{1-bR}u.
\end{aligned}$$

Hence arguing as above we obtain the estimate $a \leq \lambda_1\frac{R^3}{2}(1-bR)$.

Part 2. Lower estimate on $a^*(b)$. We finally prove that the inequality

$$a^*(b) \geq a^*(0)(1-bR) - bR^2$$

holds for all $b>0$. We shall assume $0<b<\frac{1}{R}$, otherwise the right-hand side is negative and the inequality trivially holds. Assume, by contradiction, the existence of b with

$$\frac{a^*(b)+bR^2}{1-bR} < a^*(0).$$

Let $\varepsilon>0$ be such that

$$\frac{a^*(b)+\varepsilon+bR^2}{1-bR} < a^*(0)$$

and u^* be the solution of (5.3) with $b=0$ and $a=a^*(0)$. We have

$$\begin{aligned}
-\Big(\frac{u^{*\prime}}{\sqrt{1+|u^{*\prime}|^2}}\Big)' = \frac{a^*(0)}{(u^*-R)^2} &\geq \frac{a^*(b)+\varepsilon}{(1-bR)(u^*-R)^2} + \frac{bR^2}{(1-bR)(u^*-R)^2} \\
&\geq \frac{a^*(b)+\varepsilon}{(1-bu^*)(u^*-R)^2} + \frac{b}{(1-bu^*)\sqrt{1+|u^{*\prime}|^2}}.
\end{aligned}$$

Then $\beta = u^*$ satisfies the assumptions of Theorem 5.1 for g as in (5.4) with $a=a^*(b)+\varepsilon$ and hence there exists a solution of (5.3) with $a=a^*(b)+\varepsilon$, which contradicts the definition of $a^*(b)$. $\square$

We conclude this section discussing the existence of negative solutions. In the following theorem the constant $b^\#$ comes from (3.15).

Theorem 5.10. *Assume* (H_1)*. For all* b*, with* $-b^{\#} < b < 0$ *and* a*, with* $0 < a < -R^2 b$*, problem* (5.3) *has at least one solution* $u \in C^2([-r,r])$ *with* $u \ll 0$*.*

Proof. Fix $b \in [-b^{\#}, 0[$. By Proposition 3.6 there exists a solution $u_1(\cdot; 0, b) \ll 0$ of (3.16). For any $a \in]0, -R^2 b[$, $u_1(\cdot; 0, b)$ is a lower solution and 0 is an upper solution of (5.3). Observing that

$$g(s,\xi) = a\frac{(1+|\xi|^2)^{3/2}}{(s-R)^2(1-b\,s)} + b\frac{1+|\xi|^2}{1-b\,s} \geq b\frac{1+|\xi|^2}{1-b\,s}$$

satisfies a one-sided Nagumo condition, we can apply [16, Theorem II-3.1] and get the existence of a solution $u(\cdot; a, b) \in C^2([-r,r])$ of (5.3). As 0 is not a solution, we have $u < 0$. Actually, we have $u \ll 0$. Indeed, otherwise, there should exist $\overline{x} \in [-r,r]$ such that $u(\overline{x}) = \max u = 0$ and $u'(\overline{x}) = 0$. From the equation we obtain the contradiction $u''(\overline{x}) = -\frac{a}{R^2} - b > 0$. □

Multiplicity of one-sign solutions in case $N = 1$

We start this section studying the multiplicity of positive solutions for problem (5.3).

Theorem 5.11. *Assume* (H_1)*. Suppose that problem* (5.3) *has a positive solution* $\hat{u}$*, for* $a = \hat{a} \geq 0$ *and* $b = \hat{b} \leq \frac{1}{R}$*, with* $\dfrac{\hat{a}}{R^2} + \hat{b} > 0$*. Then, for any* a *and* b*, with* $0 \leq a \leq \hat{a}$*,* $b \leq \hat{b}$*,* $\dfrac{a}{R^2} + b \geq 0$*,* $(\hat{a}, \hat{b}) \neq (a, b)$ *problem* (5.3) *has at least two solutions* u_1 *and* u_2*, satisfying*

(*a*) $0 \ll u_1 \ll u_2$ *and* u_1 *is the minimum among all positive solutions of* (5.3)*, in case* $\dfrac{a}{R^2} + b > 0$*;*

(*b*) $0 = u_1 \ll u_2$*, in case* $\dfrac{a}{R^2} + b = 0$*.*

Remark 5.6 In case $b = 0$, this multiplicity result was first established in [7] (see also [9, 37]).

Proof. In case $a = 0$, the result follows from Proposition 3.6. So, let us assume that $a > 0$.

Fix $0 < a \leq \hat{a}$ and $b \leq \hat{b}$ with $\dfrac{a}{R^2} + b \geq 0$ and $(\hat{a}, \hat{b}) \neq (a, b)$. In case $\dfrac{a}{R^2} + b = 0$ we set $u_1 = 0$. In case $\dfrac{a}{R^2} + b > 0$ we observe that the function $\beta = \hat{u}$ satisfies the assumptions of Theorem 5.1 for g defined by (5.4), hence there exists a minimum positive solution u_1 of (5.3) with $0 \ll u_1 < \hat{u}$. By Theorem 5.4 we actually have $u_1 \ll \hat{u}$. We aim to prove the existence of a second solution u_2 of (5.3); this will be achieved in several steps. Since g is independent of x, for sake of simplicity we shall write $g(s,\xi)$ instead of $g(x,s,\xi)$ from now on.

Step 1. Construction of a large lower solution of (5.3). We first consider the case $b \geq 0$. Observe that, for all $s \in [0, R[$ and $\xi \in \mathbb{R}$, we have

$$g(s,\xi) \geq \frac{a}{(R-s)^2}.$$

We set

$$M = \max \hat{u} < R \leq \frac{1}{\hat{b}}$$

and choose $c > 0$ satisfying

$$\frac{R}{2cr^2} < 1, \quad 2ac\left(\frac{r^4}{R^4}\right) > 1, \quad R - \frac{R^2}{4cr^2} > M. \tag{5.10}$$

We set

$$d = R - \frac{R^2}{4cr^2}$$

and define

$$\alpha(x) = \begin{cases} \frac{R}{r}x + R, & \text{if } x \in [-r, -\frac{R}{2cr}[, \\ d - cx^2, & \text{if } x \in [-\frac{R}{2cr}, \frac{R}{2cr}], \\ -\frac{R}{r}x + R, & \text{if } x \in]\frac{R}{2cr}, r]. \end{cases}$$

Note that $\alpha \in W^{2,\infty}(-r,r)$, $\alpha(-r) = 0$ and $\alpha(r) = 0$ and, for all $x \in [-r,r]$, $(\alpha(x), \alpha'(x)) \in \operatorname{dom} g$, as $R \leq \frac{1}{b}$. For all $x \in]-\frac{R}{2cr}, \frac{R}{2cr}[$, we have $\alpha(x) \geq \alpha(-\frac{R}{2cr}) = R - \frac{R^2}{2cr^2} > 0$ and

$$g\big(\alpha(x), \alpha'(x)\big) \geq \frac{a}{\big(R - \alpha(x)\big)^2} \geq a\Big(\frac{2cr^2}{R^2}\Big)^2 > 2c = -\alpha''(x).$$

Furthermore, if $x \in [-r, -\frac{R}{2cr}[\cup]\frac{R}{2cr}, r]$, we also have

$$-\alpha''(x) = 0 < g(\alpha(x), \alpha'(x)).$$

Note that, for all $x_0 \in [-r,r]$, there exists $\varepsilon > 0$ such that, for a.e. $x \in]x_0 - \varepsilon, x_0 + \varepsilon[\cap [-r,r]$, $u \in [\alpha(x), \alpha(x) + \varepsilon]$, $v \in [\alpha'(x) - \varepsilon, \alpha'(x) + \varepsilon]$ we have

$$-\alpha''(x) \leq g(u,v).$$

Applying [16, Proposition III-2.3], we conclude that α is a strict lower solution of

$$\begin{cases} -u'' = \Big(\dfrac{a}{(u-R)^2} + \dfrac{b}{\sqrt{1+|u'|^2}}\Big)\dfrac{(1+u'^2)^{\frac{3}{2}}}{1-bu}, & \text{in }]-r,r[, \\ u(-r) = 0, \ u(r) = 0. \end{cases} \tag{5.11}$$

In case $-\frac{a}{R^2} \leq b < 0$, we observe that, for all $(s,\xi) \in [0,R[\times \mathbb{R}$, we have

$$g(s,\xi) \geq \Big(\frac{a}{(R-s)^2} + b\Big)\frac{1}{1+|b|R},$$

with strict inequality if $\xi \neq 0$ or $s \neq 0$. We set again $M = \max \hat{u} < R$. In this case we choose $c > 0$ such that

$$\frac{R}{2cr^2} < 1, \quad \Big(a\Big(\frac{2cr^2}{R^2}\Big)^2 + b\Big)\frac{1}{1+|b|R} > 2c, \quad R - \frac{R^2}{4cr^2} > M. \tag{5.12}$$

We define d and α as in case $b \geq 0$ and, arguing as in the previous case, we conclude that α is a strict lower solution of (5.11) as well.

Step 2. Construction of an upper solution β of (5.11) *with $\beta(x) > \hat{u}(x)$ for all $x \in [-r,r]$.* As $\hat{a} \geq a$ and $\hat{b} \geq b$, with $(\hat{a}, \hat{b}) \neq (a,b)$, possibly replacing $(\hat{a}, \hat{b})$ with some $(\tilde{a}, \tilde{b})$ satisfying $\hat{a} \geq \tilde{a} \geq a$, $\hat{b} \geq \tilde{b} \geq b$, $(\hat{a}, \hat{b}) \neq (\tilde{a}, \tilde{b})$ and $(\tilde{a}, \tilde{b}) \neq (a,b)$, by Proposition 5.6 we see that either $\hat{u}$ is not classical and $\hat{u}(-r) > 0$, or $\hat{u}$ is classical and $\hat{u} \in C^2([-r,r])$. In both cases we have, for all $x \in [-r,r]$,

$$\Big(\frac{\hat{a}}{\big(\hat{u}(x) - R\big)^2} + \frac{\hat{b}}{\sqrt{1+|\hat{u}'(x)|^2}}\Big)\frac{1}{1-\hat{b}\hat{u}(x)} > \Big(\frac{a}{\big(\hat{u}(x) - R\big)^2} + \frac{b}{\sqrt{1+|\hat{u}'(x)|^2}}\Big)\frac{1}{1-b\hat{u}(x)}.$$

Take $\varepsilon > 0$ so small that, for all $x \in [-r,r]$,

$$\Big(\frac{\hat{a}}{\big(\hat{u}(x) - R\big)^2} + \frac{\hat{b}}{\sqrt{1+|\hat{u}'(x)|^2}}\Big)\frac{1}{1-\hat{b}\hat{u}(x)} > \Big(\frac{a}{\big(\hat{u}(x) + \varepsilon - R\big)^2} + \frac{b}{\sqrt{1+|\hat{u}'(x)|^2}}\Big)\frac{1}{1-b\big(\hat{u}(x)+\varepsilon\big)}.$$

Therefore $\beta_\varepsilon = \hat{u} + \varepsilon$ is a strict upper solution of (5.11) (see [16, Proposition III-2.2]).

Step 3. A modified problem. Let $(R_n)_n$ be an increasing sequence converging to R and satisfying $R_1 > d$ and let $(K_n)_n$ be an increasing divergent sequence satisfying $K_1 > \frac{R}{r}$. For all n and $(s,\xi) \in \mathbb{R} \times \mathbb{R}$ we define

$$\tilde{g}_n(s,\xi) = \begin{cases} g(0,\xi) & \text{if } s < 0, \\ g(s,\xi) & \text{if } 0 \leq s \leq R_n, \\ g(R_n,\xi) & \text{if } s > R_n, \end{cases}$$

and

$$g_n(s,\xi) = \begin{cases} \min\{\tilde{g}_n(s,\xi),\, \tilde{g}_n(s,-K_n)\} & \text{if } \xi < -K_n, \\ \tilde{g}_n(s,\xi) & \text{if } -K_n \leq \xi \leq K_n, \\ \min\{\tilde{g}_n(s,\xi),\, \tilde{g}_n(s,K_n)\} & \text{if } \xi > K_n. \end{cases}$$

Next we consider the problem

$$\begin{cases} -u'' = g_n(u,u'), & \text{in }]-r,r[, \\ u(-r) = 0, \ u(r) = 0. \end{cases} \tag{5.13}$$

We first observe that 0 is a lower solution of (5.13). Next, since $K_n > \frac{R}{r}$ and $R_n > d$, we note that $g_n\big(\alpha,\alpha'\big) = g\big(\alpha,\alpha'\big)$ and α is still a strict lower solution of (5.13). Finally, if we choose $\varepsilon \in]0, d-M]$, then $\max \beta_\varepsilon = M + \varepsilon \leq d < R_1 < R_n$, and, for every $x \in [-r,r]$,

$$-\beta_\varepsilon''(x) > g\big(\beta_\varepsilon(x),\beta_\varepsilon'(x)\big) = \tilde{g}_n(\beta_\varepsilon(x),\beta_\varepsilon'(x)) \geq g_n\big(\beta_\varepsilon(x),\beta_\varepsilon'(x)\big).$$

By [16, Proposition III-2.2], we conclude that β_ε is a strict upper solution of (5.13).

Since $\max \alpha = d > \max \beta_\varepsilon$ and the function g_n is bounded, by [17, Theorem 4.1], there exists a positive solution $v_n \in C^2([-r,r])$ of (5.13), satisfying, for some $\hat{x}_n, \hat{y}_n \in [-r,r]$, $v_n(\hat{x}_n) < \alpha(\hat{x}_n)$ and $v_n(\hat{y}_n) > \beta_\varepsilon(\hat{y}_n)$. Notice that, by Lemma 2.5, v_n is even and concave. In particular, we may assume that $\hat{x}_n, \hat{y}_n \in [-r,0]$.

Step 4. Estimates on v_n in case $b > 0$. In this step we aim to show that $v_n(x) \leq R - \frac{7R^2}{32cr^2}$, for all $x \in [-r,r]$.

In case $\max v_n = v_n(0) \leq R - \frac{R^2}{4cr^2} = d$, the result is proved. Therefore, let us suppose that $\max v_n = v_n(0) > d$.

Let $-r < x_1 < x_2 < 0$ be points where

$$v_n(x_1) = R - \frac{R^2}{2cr^2}, \quad v_n(x_2) = d.$$

Observe that, for all $x \in [x_1,x_2]$, we have $v_n(x) \leq d < R_n$ and hence, by the definition of g_n,

$$g_n(v_n(x),v_n'(x)) \geq \frac{a}{\big(R - v_n(x)\big)^2}.$$

Therefore, using (5.10), we obtain, for $x \in [x_1,x_2]$,

$$-v_n''(x) = g_n\big(v_n(x),v_n'(x)\big) \geq \frac{a}{\big(R - v_n(x)\big)^2} \geq \frac{a}{\big(R - v_n(x_1)\big)^2} = a\Big(\frac{2cr^2}{R^2}\Big)^2 > 2c. \tag{5.14}$$

On the other hand, as $R_n \geq d = v_n(x_2)$, again by the definition of g_n we have also, for all $x \in [x_2,0]$,

$$-v_n''(x) = g_n\big(v_n(x),v_n'(x)\big) \geq \frac{a}{\big(R - v_n(x_2)\big)^2} = a\Big(\frac{4cr^2}{R^2}\Big)^2 > 8c.$$

Integrating $-v_n''$ between x_1 and x_2 yields

$$v_n'(x_1) - v_n'(x_2) = \int_{x_1}^{x_2} -v_n''(t)\,dt > 2c(x_2 - x_1), \tag{5.15}$$

while integrating $-v_n''$ between x_2 and 0 yields

$$v_n'(x_2) = \int_{x_2}^{0} -v_n''(t)\,dt > -8cx_2. \tag{5.16}$$

Case 1. $v_n'(x_1) \le \frac{R}{r}$. By concavity of v_n, we have

$$\frac{R^2}{4cr^2} = v_n(x_2) - v_n(x_1) = \int_{x_1}^{x_2} v_n'(s)\,ds \le v_n'(x_1)(x_2 - x_1) \le \frac{R}{r}(x_2 - x_1).$$

This implies that

$$x_2 - x_1 \ge \frac{R}{4cr}$$

and, using (5.15), we obtain

$$v_n'(x_2) \le v_n'(x_1) - 2c(x_2 - x_1) \le \frac{R}{r} - \frac{R}{2r} = \frac{R}{2r}.$$

By (5.16), we have also

$$-x_2 \le \frac{1}{8c} v_n'(x_2) \le \frac{R}{16cr}.$$

These two inequalities imply that

$$v_n(0) = \max v_n = v_n(x_2) + \int_{x_2}^{0} v_n'(s)\,ds \le R - \frac{R^2}{4cr^2} + v_n'(x_2)|x_2| \le R - \frac{R^2}{4cr^2} + \frac{R^2}{32cr^2} = R - \frac{7R^2}{32cr^2}.$$

Case 2. $v_n'(x_1) > \frac{R}{r}$. Since $v_n'(x) > \frac{R}{r}$ in $[-r, x_1]$ and $v_n(x_1) = R - \frac{R^2}{2cr^2}$, we obtain $x_1 < -\frac{R}{2cr}$. On the other hand, as $\alpha'(x) = \frac{R}{r}$ in $[-r, -\frac{R}{2cr}]$, $v_n'(x) > \frac{R}{r}$ in $[-r, x_1]$ and $\alpha(-\frac{R}{2cr}) = R - \frac{R^2}{2cr^2} = v_n(x_1)$, we deduce that $v_n(x) > \alpha(x)$ in $]-r, x_1]$.

Recall that $v_n(x) > d = \max \alpha$ in $]x_2, 0]$ and $v_n(\hat{x}_n) < \alpha(\hat{x}_n)$ at some point $\hat{x}_n \in]-r, 0]$. Actually this implies that $x_1 < \hat{x}_n < x_2$ and there must be points y_1 and y_2, with $x_1 < y_1 < \hat{x}_n < y_2 < x_2$, satisfying

$$v_n(y_1) = \alpha(y_1) \text{ and } v_n'(y_1) \le \alpha'(y_1)$$

and

$$v_n(y_2) = \alpha(y_2) \text{ and } v_n'(y_2) \ge \alpha'(y_2),$$

respectively. Observe that

$$\alpha'(y_1) - \alpha'(y_2) = \int_{y_1}^{y_2} -\alpha''(s)\,ds \le 2c(y_2 - y_1).$$

Then, recalling (5.14), we obtain the contradiction

$$2c(y_2 - y_1) = \int_{y_1}^{y_2} 2c\,dt < \int_{y_1}^{y_2} -v_n''(t)\,dt = v_n'(y_1) - v_n'(y_2) \le \alpha'(y_1) - \alpha'(y_2) \le 2c(y_2 - y_1).$$

Step 5. Estimates on v_n in case $-\frac{a}{R^2} < b < 0$. In this step we aim to show that $v_n(x) \le R - \frac{R^2}{8cr^2}$, for all $x \in [-r, r]$.

In case $\max v_n = v_n(0) \le d$, the result is proved. Therefore, let us suppose that $\max v_n = v_n(0) > d$.

Let $-r < x_1 < x_2 < 0$ be points where

$$v_n(x_1) = R - \frac{R^2}{2cr^2}, \quad v_n(x_2) = d = R - \frac{R^2}{4cr^2}.$$

Observe that, for all $x \in [x_1, x_2]$, we have $v_n(x) \le d < R_n$ and hence, by the definition of g_n,

$$g_n\big(v_n(x), v_n'(x)\big) \ge \Big(\frac{a}{\big(R - v_n(x)\big)^2} + b\Big)\frac{1}{1 + |b|R}.$$

Therefore, using (5.12), we obtain, for $x \in [x_1, x_2]$,

$$-v_n''(x) = g_n\big(v_n(x), v_n'(x)\big) \ge \Big(\frac{a}{\big(R - v_n(x_1)\big)^2} + b\Big)\frac{1}{1 + |b|R} \ge \Big(a\Big(\frac{2cr^2}{R^2}\Big)^2 + b\Big)\frac{1}{1 + |b|R} > 2c. \quad (5.17)$$

On the other hand, as $R_n \ge d = v_n(x_2)$ and by the definition of g_n, we have also, for all $x \in [x_2, 0]$,

$$\begin{aligned} -v_n''(x) \ge \Big(\frac{a}{\big(R - v_n(x_2)\big)^2} + b\Big)\frac{1}{1 + |b|R} &= \Big(a\Big(\frac{2cr^2}{R^2}\Big)^2 + b\Big)\frac{1}{1 + |b|R} + 12a\Big(\frac{cr^2}{R^2}\Big)^2\frac{1}{1 + |b|R} \\ &> 2c + \frac{12ac^2r^4}{R^4(1 + |b|R)}. \end{aligned}$$

Integrating $-v_n''$ between x_1 and x_2 yields

$$v_n'(x_1) - v_n'(x_2) = \int_{x_1}^{x_2} -v_n''(t)\,dt > 2c(x_2 - x_1), \quad (5.18)$$

while integrating $-v_n''$ between x_2 and 0 yields

$$v_n'(x_2) = \int_{x_2}^{0} -v_n''(t)\,dt > \int_{x_2}^{0}\Big(2c + \frac{12ac^2r^4}{R^4(1 + |b|R)}\Big)\,dt = -\Big(2c + \frac{12ac^2r^4}{R^4(1 + |b|R)}\Big)x_2. \quad (5.19)$$

Case 1. $v_n'(x_1) \le \frac{R}{r}$. By concavity of v_n, we have

$$\frac{R^2}{4cr^2} = v_n(x_2) - v_n(x_1) = \int_{x_1}^{x_2} v_n'(s)\,ds \le v_n'(x_1)(x_2 - x_1) \le \frac{R}{r}(x_2 - x_1).$$

This implies that

$$x_2 - x_1 \ge \frac{R}{4cr}$$

and, using (5.18), we obtain

$$v_n'(x_2) \le v_n'(x_1) - 2c(x_2 - x_1) \le \frac{R}{r} - \frac{R}{2r} = \frac{R}{2r}.$$

By (5.19), we have also

$$-x_2 \le \frac{1}{2c + \frac{12ac^2r^4}{R^4(1+|b|R)}}v_n'(x_2) \le \frac{1}{2c + \frac{12ac^2r^4}{R^4(1+|b|R)}}\frac{R}{2r}.$$

These two inequalities imply that

$$v_n(0) = \max v_n = v_n(x_2) + \int_{x_2}^{0} v_n'(s)\,ds \le R - \frac{R^2}{4cr^2} + v_n'(x_2)|x_2|$$
$$\le R - \frac{R^2}{4cr^2} + \Big(\frac{R}{2r}\Big)^2 \frac{1}{2c + \frac{12ac^2r^4}{R^4(1+|b|R)}} < R - \frac{R^2}{8cr^2}.$$

Case 2. $v_n'(x_1) > \frac{R}{r}$. This can be proved exactly as in Step 4, using (5.17) instead of (5.14).

Step 6. Convergence to a solution. Let us set, by convenience of notation, $\overline{R} = R - \frac{R^2}{8cr^2}$. We proved in Steps 4 and 5 that $v_n(x) \le \overline{R}$ for all $x \in [-r, r]$. As $\|v_n'\|_{L^1} = 2\|v_n\|_\infty \le 2\overline{R}$, the sequence $(v_n)_n$ is bounded in $W^{1,1}(-r, r)$. Therefore there exists a subsequence of $(v_n)_n$, we still denote by $(v_n)_n$, converging in $L^1(-r, r)$ and a.e. in $]-r, r[$ to a function $u_2 \in BV(-r, r)$.

Fix σ with $0 < \sigma < r$. By concavity of v_n we have $|v_n'(x)| \le \frac{\overline{R}}{\sigma}$, for all $x \in [-r+\sigma, r-\sigma]$. Suppose n is so large that $R_n > \overline{R}$ and $K_n > \frac{\overline{R}}{\sigma}$ and hence $g_n(v_n, v_n') = g(v_n, v_n')$ on $[-r+\sigma, r-\sigma]$. Arguing as in Step 1 and Step 2 of the proof of Theorem 5.1, we prove that $u_2 \in C^2(]-r, r[) \cap C^0([-r, r])$ and is a solution of (5.3).

Step 7. $u_1 \ll u_2$. By Step 3 we know that, for each n, there exists $\hat{y}_n \in [-r, 0]$ satisfying $v_n(\hat{y}_n) \ge \beta_\varepsilon(\hat{y}_n)$. As a consequence there exists $y_0 \in [-r, 0]$ satisfying $u_2(y_0) \ge \beta_\varepsilon(y_0) = \hat{u}(y_0) + \varepsilon$. Since $u_1 \ll \hat{u}$ this implies that $u_1 \ne u_2$. The result then follows from Theorem 5.4. $\square$

Remark 5.7 Recall that, by Theorem 5.8, if problem (5.3) for $a = \hat{a} > 0$ and $b = \hat{b} > 0$ has a solution, then $\hat{b} < b^* \le b^\#$, with $b^\#$ the number defined in (3.15). Hence, the assumption $\hat{b} \le \frac{1}{R}$ is satisfied in particular in case $R \le r\sinh(t^\#)$. On the other hand, in case $R > r\sinh(t^\#)$, according to Proposition 3.8, for all $b \in]\tilde{b}, b^\#[$ there exists $a^\# = a^\#(b) > 0$ such that, for all $a \in]0, a^\#[$, problem (5.3) has at least two solutions u_1, $u_2 \in C^2([-r, r])$, with $u_i \gg 0$ for $i = 1, 2$. Observe that $\tilde{b} < R^{-1}$, as otherwise, if $\tilde{b} \ge R^{-1}$, by Remarks 3.4-(a) and 3.6, we get the contradiction

$$R = u_2(0; \tilde{b}) < \frac{1}{\tilde{b}} \le R.$$

Further, according to Remark 5.2, for $R > r\sinh(t^\#)$ we have $b^\# = b^*$. Hence, for all $b \in]R^{-1}, b^*[$, we obtain, in particular, the existence of $a^\# = a^\#(b) > 0$ such that, for all $a \in]0, a^\#[$, problem (5.3) has at least two solutions u_1, $u_2 \in C^2([-r, r])$, with $u_i \gg 0$ for $i = 1, 2$.

Theorem 5.12. *Assume* (H_1)*. Define* b^* *and* $a^*(b)$ *as in Theorem 5.8. Then, problem* (5.3) *has at least two solutions* u_1 *and* u_2*, satisfying* $0 \le u_1 \ll u_2$*,* u_1 *being the minimum non-negative solution of* (5.3)*, in the following cases:*

- $0 \le b < \min\{b^*, \frac{1}{R}\}$ *and* $0 < a < a^*(b)$*,*
- $-\frac{R}{2}\lambda_1 < b < 0$ *and* $-R^2 b \le a < a^*(b)$*.*

We have $u_1 \gg 0$ *in case* $\frac{a}{R^2} + b > 0$ *and* $u_1 = 0$ *in case* $\frac{a}{R^2} + b = 0$*. Moreover, the solution* u_1 *is classical and* $u_1 \in C^2([-r, r])$ *in the following cases:*

- $0 \le b < \min\{b^*, \frac{1}{R}\}$ *and* $0 < a < a^*(b)$*,*
- $\pi R \le 4r$*,* $-\frac{R}{2}\lambda_1 < b < 0$ *and* $-R^2 b \le a < a^*(b)$*,*

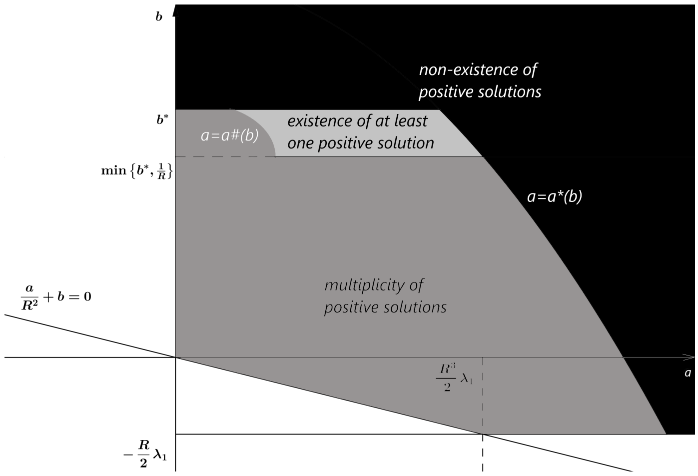

Figure 5: Existence and multiplicity of positive solutions

- $\pi R > 4r$, $-\frac{2}{R} \le b < 0$ *and* $-R^2 b \le a < a^*(b)$,
- $\pi R > 4r$, $-\frac{R}{2}\lambda_1 < b < -\frac{2}{R}$ *and* $-R^2 b \le a \le a^*(-\frac{2}{R})$.

Proof. Take $\hat{a} \in]a, a^*(b)[$ such that problem (5.3), for $\hat{a}$ and b, has a solution. By Theorem 5.11, problem (5.3) has at least two solutions u_1 and u_2, satisfying $u_1 \ll u_2$, and $u_1 = 0$ if $a = -R^2 b$, u_1 is the minimum among all positive solutions of (5.3) if $a > -R^2 b$. The regularity of u_1 is already obtained in Theorem 5.8. □

Remark 5.8 The existence of multiple negative solutions has been discussed in Proposition 3.8.

A Addendum

In this Addendum, we provide a few details about the differentiability properties and the expressions of some partial derivatives of the operator $\mathcal{F} : \mathcal{V} \to L^p(\Omega)$, where $p > N$ is fixed and $\mathcal{F}$ and $\mathcal{V}$ are respectively defined by (3.4) and (3.3). Indeed, by combining [43, Chapter II, Section 4] with the continuity, from $W^{1,p}(\Omega)$ to $L^p(\Omega)$, of the linear operators mapping u onto $\operatorname{div} u$ and u onto ∇u, we infer that $\mathcal{F}$ is of class C^∞. Moreover, for all $(u, a, b) \in \mathcal{V}$ and $v, w, z \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, elementary, but tedious, calculations yield:

$$\begin{aligned}\partial_u \mathcal{F}(u,a,b)[v] = \operatorname{div}\Big(\frac{\nabla v}{\sqrt{1+|\nabla u|^2}} - \frac{\nabla u\cdot\nabla v}{\big(\sqrt{1+|\nabla u|^2}\big)^3}\nabla u\Big) - \Big(\frac{2av}{(u-R)^3} + \frac{b\,\nabla u\cdot\nabla v}{\big(\sqrt{1+|\nabla u|^2}\big)^3}\Big)\frac{1}{1-bu} \\ + \Big(\frac{a}{(u-R)^2} + \frac{b}{\sqrt{1+|\nabla u|^2}}\Big)\frac{bv}{(1-bu)^2},\end{aligned}$$

$$\begin{aligned}\partial_{uu}\mathcal{F}(u,a,b)[v][w] = \operatorname{div}\Big(-\frac{\nabla u\cdot\nabla w}{(\sqrt{1+|\nabla u|^2})^3}\nabla v - \frac{\nabla w\cdot\nabla v}{(\sqrt{1+|\nabla u|^2})^3}\nabla u - \frac{\nabla u\cdot\nabla v}{(\sqrt{1+|\nabla u|^2})^3}\nabla w + 3\frac{(\nabla u\cdot\nabla v)\,(\nabla u\cdot\nabla w)}{(\sqrt{1+|\nabla u|^2})^5}\nabla u\Big) \\ + \Big(\frac{6\,a\,v\,w}{(u-R)^4} - \frac{b\,\nabla w\cdot\nabla v}{(\sqrt{1+|\nabla u|^2})^3} + 3b\frac{(\nabla u\cdot\nabla v)\,(\nabla u\cdot\nabla w)}{(\sqrt{1+|\nabla u|^2})^5}\Big)\frac{1}{1-bu} \\ - \Big(\frac{2a\,v}{(u-R)^3} + \frac{b\,\nabla u\cdot\nabla v}{(\sqrt{1+|\nabla u|^2})^3}\Big)\frac{bw}{(1-bu)^2} - \Big(\frac{2a\,w}{(u-R)^3} + \frac{b\,\nabla u\cdot\nabla w}{(\sqrt{1+|\nabla u|^2})^3}\Big)\frac{bv}{(1-bu)^2} \\ + \Big(\frac{a}{(u-R)^2} + \frac{b}{\sqrt{1+|\nabla u|^2}}\Big)\frac{2b^2 v\,w}{(1-bu)^3}\end{aligned}$$

and

$$
\begin{aligned}
\partial_{uuu}\mathcal{F}(u,a,b)[v][w][z] = \operatorname{div}\Big(& -\frac{\nabla z\cdot\nabla w}{(\sqrt{1+|\nabla u|^2})^3}\nabla v + 3\frac{(\nabla u\cdot\nabla z)\,(\nabla u\cdot\nabla w)}{(\sqrt{1+|\nabla u|^2})^5}\nabla v - \frac{\nabla w\cdot\nabla v}{(\sqrt{1+|\nabla u|^2})^3}\nabla z \\
& + 3\frac{(\nabla w\cdot\nabla v)\,(\nabla z\cdot\nabla u)}{(\sqrt{1+|\nabla u|^2})^5}\nabla u - \frac{\nabla z\cdot\nabla v}{(\sqrt{1+|\nabla u|^2})^3}\nabla w + 3\frac{(\nabla u\cdot\nabla v)\,(\nabla u\cdot\nabla z)}{(\sqrt{1+|\nabla u|^2})^5}\nabla w \\
& + 3\frac{(\nabla z\cdot\nabla v)\,(\nabla u\cdot\nabla w)}{(\sqrt{1+|\nabla u|^2})^5}\nabla u + 3\frac{(\nabla u\cdot\nabla v)\,(\nabla z\cdot\nabla w)}{(\sqrt{1+|\nabla u|^2})^5}\nabla u + 3\frac{(\nabla u\cdot\nabla v)\,(\nabla u\cdot\nabla w)}{(\sqrt{1+|\nabla u|^2})^5}\nabla z \\
& - 15\frac{(\nabla u\cdot\nabla v)\,(\nabla u\cdot\nabla w)\,(\nabla u\cdot\nabla z)}{(\sqrt{1+|\nabla u|^2})^7}\nabla u\Big) \\
& - \Big(\frac{24a\,v\,w\,z}{(u-R)^5} - 3b\frac{(\nabla w\cdot\nabla v)\,(\nabla u\cdot\nabla z)}{(\sqrt{1+|\nabla u|^2})^5} - 3b\frac{(\nabla z\cdot\nabla v)\,(\nabla u\cdot\nabla w)}{(\sqrt{1+|\nabla u|^2})^5} \\
& \qquad - 3b\frac{(\nabla u\cdot\nabla v)\,(\nabla z\cdot\nabla w)}{(\sqrt{1+|\nabla u|^2})^5} + 15b\frac{(\nabla u\cdot\nabla v)\,(\nabla u\cdot\nabla w)\,(\nabla u\cdot\nabla z)}{(\sqrt{1+|\nabla u|^2})^7}\Big)\frac{1}{1-bu} \\
& + \Big(\frac{6a\,v\,w}{(u-R)^4} - \frac{b\,\nabla w\cdot\nabla v}{(\sqrt{1+|\nabla u|^2})^3} + 3b\frac{(\nabla u\cdot\nabla v)\,(\nabla u\cdot\nabla w)}{(\sqrt{1+|\nabla u|^2})^5}\Big)\frac{b\,z}{(1-bu)^2} \\
& + \Big(\frac{6a\,v\,z}{(u-R)^4} - \frac{b\,\nabla z\cdot\nabla v}{(\sqrt{1+|\nabla u|^2})^3} + 3b\frac{(\nabla u\cdot\nabla v)\,(\nabla u\cdot\nabla z)}{(\sqrt{1+|\nabla u|^2})^5}\Big)\frac{b\,w}{(1-bu)^2} \\
& - \Big(\frac{2a\,v}{(u-R)^3} + \frac{b\,\nabla u\cdot\nabla v}{(\sqrt{1+|\nabla u|^2})^3}\Big)\frac{2b^2wz}{(1-bu)^3} \\
& + \Big(\frac{6a\,w\,z}{(u-R)^4} - \frac{b\,\nabla z\cdot\nabla w}{(\sqrt{1+|\nabla u|^2})^3} + 3b\frac{(\nabla u\cdot\nabla w)\,(\nabla u\cdot\nabla z)}{(\sqrt{1+|\nabla u|^2})^5}\Big)\frac{b\,v}{(1-bu)^2} \\
& - \Big(\frac{2a\,w}{(u-R)^3} + \frac{b\,\nabla u\cdot\nabla w}{(\sqrt{1+|\nabla u|^2})^3}\Big)\frac{2b^2vz}{(1-bu)^3} \\
& - \Big(\frac{2az}{(u-R)^3} + \frac{b\,\nabla u\cdot\nabla z}{(\sqrt{1+|\nabla u|^2})^3}\Big)\frac{2b^2vw}{(1-bu)^3} \\
& + \Big(\frac{a}{(u-R)^2} + \frac{b}{\sqrt{1+|\nabla u|^2}}\Big)\frac{6b^3v\,w\,z}{(1-bu)^4}.
\end{aligned}
$$

References

[1] L. Ambrosio, N. Fusco and D. Pallara, Functions of Bounded Variation and Free Discontinuity Problems, Clarendon Press, Oxford, 2000.

[2] A. Ambrosetti and G. Prodi, A Primer of Nonlinear Analysis, Cambridge University Press, Cambridge, 1993.

[3] G. Anzellotti, Pairings between measures and bounded functions and compensated compactness, Ann. Mat. Pura Appl. 135 (1983), 293–318.

[4] J. Bemelmans and U. Dierkes, On a singular variational integral with linear growth. I. Existence and regularity of minimizers. Arch. Rational Mech. Anal. 100 (1987), 83–103.

[5] G. Birkhoff and G.-C. Rota, Ordinary Differential Equations, Fourth Edition, Wiley, New York, 1989.

[6] D. Bonheure, C. Grumiau and C. Troestler, Multiple radial positive solutions of semilinear elliptic problems with Neumann boundary conditions, Nonlinear Analysis 147 (2016), 236–273.

[7] D. Bonheure, P. Habets, F. Obersnel and P. Omari, Classical and non-classical positive solutions of a prescribed curvature equation with singularities, Rend. Istit. Mat. Univ. Trieste 39 (2007), 63–85.

[8] N.D. Brubaker, Mathematical Theory of Electro-Capillary Surfaces, Ph.D. Thesis, University of Delaware, ProQuest Dissertations Publishing, 2013.

[9] N.D. Brubaker and J.A. Pelesko, Nonlinear elastic effects on canonical MEMS models, European J. Appl. Math. 22 (2011) 455–470.

[10] N.D. Brubaker and J.A. Pelesko, Analysis of a one-dimensional prescribed mean curvature equation with singular nonlinearity, Nonlinear Analysis 75 (2012), 5086–5102.

[11] N.D. Brubaker, J.I. Siddique, E. Sabo, R. Deaton and J.A. Pelesko, Refinements to the study of electrostatic deflections: theory and experiment, European J. Appl. Math. 24 (2013), 343–370.

[12] C. Corsato, C. De Coster, N. Flora and P. Omari, Radial solutions of the Dirichlet problem for a class of quasilinear elliptic equations arising in optometry, Nonlinear Anal. 181 (2019), 9–23.

[13] C. Corsato, C. De Coster and P. Omari, The Dirichlet problem for a prescribed anisotropic mean curvature equation: existence, uniqueness and regularity of solutions, J. Differential Equations 260 (2016), 4572–4618.

[14] C. Corsato, C. De Coster, F. Obersnel, P. Omari and A. Soranzo, A prescribed anisotropic mean curvature equation modeling the corneal shape: a paradigm of nonlinear analysis. Discrete Contin. Dyn. Syst. Ser. S 11 (2018), 213–256.

[15] M.C. Crandall and P. Rabinowitz, Bifurcation from simple eigenvalues, J. Funct. Analysis 8 (1971), 321–340.

[16] C. De Coster and P. Habets, Two-point Boundary Value Problems: Lower and Upper Solutions, Elsevier, Amsterdam, 2006.

[17] C. De Coster and M. Henrard, Existence and localization of solution for second order elliptic BVP in presence of lower and upper solutions without any order, J. Differential Equations 145 (1998), 420–452.

[18] C. De Coster, S. Nicaise and C. Troestler, Nodal properties of eigenfunctions of a generalized buckling problem on balls, Positivity 19 (2015), 843–875.

[19] U. Dierkes, On the regularity of solutions for a singular variational problem, Math. Z. 225 (1997), 657–670.

[20] U. Dierkes and G. Huisken, The n-dimensional analogue of the catenary: existence and non-existence, Pacific J. Math. 141 (1990), 47–54.

[21] J. Escher and C. Lienstromberg, A survey on second-order free boundary value problems modelling MEMS with general permittivity profile, Discrete Contin. Dyn. Syst. Ser. S 10 (2017), 745–771.

[22] P. Esposito, N. Ghoussoub and Y. Guo, Mathematical Analysis of Partial Differential Equations Modeling Electrostatic MEMS, American Mathematical Society, Providence, RI, 2010.

[23] L. Evans, Partial Differential Equations, American Mathematical Society, Providence, RI, 1998.

[24] A.F. Filippov, Differential Equations with Discontinuous Right-hand Sides, Kluwer Academic Publishers, Dordrecht, 1988.

[25] D. Gilbarg and N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer, New York, 2001.

[26] E. Giusti, On the equation of surfaces of prescribed mean curvature. Existence and uniqueness without boundary conditions, Invent. Math. 46 (1978), 111–137.

[27] D. Kinderlehrer and G. Stampacchia, An Introduction to Variational Inequalities and their Applications, Academic Press, New York-London, 1980.

[28] P. Laurençot and C. Walker, Some singular equations modeling MEMS, Bull. Amer. Math. Soc. 54 (2017), 437–479.

[29] A. Lichnewsky, Solutions généralisées du problème des surfaces minimales pour des données au bord non bornées, J. Math. Pures Appl. 57 (1978), 231–253.

[30] E.H. Lieb and M. Loss, Analysis, American Mathematical Society, Providence, RI, 1997.

[31] J. López Gómez, Spectral Theory and Nonlinear Functional Analysis, CRC Press, Boca Raton, FL, 2001.

[32] J. López Gómez, Global bifurcation for Fredholm operators, Rend. Istit. Mat. Univ. Trieste 48 (2016), 539–564.

[33] A. Lunardi, Analytic Semigroups and Optimal Regularity in Parabolic problems, Birkhäuser, Basel, 1995.

[34] C.B. Morrey Jr., Multiple Integrals in the Calculus of Variations, Springer, New York, 1966.

[35] National Institute of Standards and Technology, Digital Library of Mathematical Functions, http://dlmf.nist.gov/, December, 2018.

[36] F. Obersnel and P. Omari, Old and new results for first order periodic ODEs without uniqueness: a comprehensive study by lower and upper solutions, Adv. Nonlinear Stud. 4 (2004), 323–376.

[37] H. Pan and R. Xing, Exact multiplicity results for a one-dimensional prescribed mean curvature problem related to MEMS models, Nonlinear Anal. Real World Appl. 13 (2012), 2432–2445.

[38] H. Pan and R. Xing, Sub- and supersolution methods for prescribed mean curvature equations with Dirichlet boundary conditions, J. Differential Equations 254 (2013), 1464–1499.

[39] J.A. Pelesko and D.H. Bernstein, Modeling MEMS and NEMS, CRC Press, Boca Raton, FL, 2003.

[40] P.H. Rabinowitz, Some global results for nonlinear eigenvalue problems, J. Funct. Analysis 7 (1971), 487–513.

[41] J. Shi and X. Wang, On global bifurcation for quasilinear elliptic systems on bounded domains, J. Differential Equations 246 (2009), 2788–2812.

[42] R. Temam, Solutions généralisées de certaines équations du type hypersurfaces minima, Arch. Rational Mech. Anal. 44 (1971/72), 121–156.

[43] T. Valent, Boundary Value Problems of Finite Elasticity, Springer, New York, 1988.