

't Hooft anomalies and the holomorphy of supersymmetric partition functions

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ABSTRACT: We study the dependence of supersymmetric partition functions on continuous parameters for the flavor symmetry group, G_F , for 2d $\mathcal{N} = (0, 2)$ and 4d $\mathcal{N} = 1$ supersymmetric quantum field theories. In any diffeomorphism-invariant scheme and in the presence of G_F 't Hooft anomalies, the supersymmetric Ward identities imply that the partition function has a non-holomorphic dependence on the flavor parameters. We show this explicitly for the 2d torus partition function, Z_{T^2} , and for a large class of 4d partition functions on half-BPS four-manifolds, $Z_{\mathcal{M}_4}$ — in particular, for $\mathcal{M}_4 = S^3 \times S^1$ and $\mathcal{M}_4 = \Sigma_g \times T^2$. We propose a new expression for $Z_{\mathcal{M}_{d-1} \times S^1}$, which differs from earlier holomorphic results by the introduction of a non-holomorphic “Casimir” pre-factor. The latter is fixed by studying the “high temperature” limit of the partition function. Our proposal agrees with the supersymmetric Ward identities, and with explicit calculations of the absolute value of the partition function using a gauge-invariant zeta-function regularization.

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1 Introduction

In this paper, we explore intriguing properties of supersymmetric partition functions in even space-time dimensions.¹ In two dimensions, we consider the T^2 partition function of 2d $\mathcal{N} = (0, 2)$ supersymmetric theories, or elliptic genus [2]. In four dimensions, we consider the $S^3 \times S^1$ supersymmetric partition function, which computes the $\mathcal{N} = 1$ supersymmetric index [3, 4].² In both cases, the partition functions depend non-trivially on some continuous parameters:

$$Z_{T^2}(\nu, \tau), \quad Z_{S^3 \times S^1}(\nu, \tau). \tag{1.1}$$

Here, there is one “flavor parameter,” ν , for each $U(1)$ current, with $\prod_{\alpha} U(1)_{\alpha}$ the maximal torus of the flavor (non- R) symmetry group, G_F , of the field theory. The parameter τ , on the other hand, denotes a “geometric parameter,” corresponding to the complex structure modulus of the torus in $2d$, or to a complex structure modulus on $S^3 \times S^1$ seen as a Hopf surface [6]. This setup preserves two supercharges of opposite R -charges, \mathcal{Q} and $\tilde{\mathcal{Q}}$.

Through a number of explicit computations — see in particular [7–11] —, it was found that these supersymmetric partition functions are locally *holomorphic* in the complex parameters ν and τ . Moreover, there is a simple *a-priori* argument for the holomorphy of (1.1), which is that the anti-holomorphic parameters, $\bar{\nu}$ and $\bar{\tau}$, couple to \mathcal{Q} -exact operators in conserved-current supermultiplets [6].

The holomorphy of (1.1) in the flavor parameters ν , however, clashes with gauge invariance whenever the flavor symmetry, G_F , has non-vanishing ‘t Hooft anomalies. To see why, consider the effective action for the quantum field theory coupled to a background gauge field for the flavor symmetry:

$$W[a] \equiv -\log Z[a], \tag{1.2}$$

$$Z[a] = \int [D\varphi] e^{-\int d^d x (\mathcal{L}_0[\varphi] + a_{\mu} j^{\mu} + \dots)},$$

where the ellipsis denotes additional couplings designed to make the full classical action gauge-invariant. A ‘t Hooft anomaly is a violation of the gauge invariance of the effective action (1.2) which cannot be removed by a local counterterm.³ Under a gauge transformation $\delta_{\alpha} a_{\mu} = D_{\mu} \alpha$, with $\alpha = \alpha(x)$ a gauge parameter, it takes the form:

$$\delta_{\alpha} W[a] = i \mathcal{A}_F \int_{\mathcal{M}_d} \alpha \omega_d(a), \tag{1.3}$$

on any d -manifold \mathcal{M}_d (with d even), where the real d -form $\omega_d(a)$ is local in the background gauge field. The form ω_d is universal and well-known, in any dimension and for any compact Lie group G_F . The coefficient $\mathcal{A}_F \in \mathbb{Z}$ in (1.3) depends on the particular theory; in UV-free theories, it is determined by the flavor charges of the chiral fermions.

¹For a review of exact results for supersymmetric partition functions, see e.g. [1].

²We will focus on the $\mathbf{q} = \mathbf{p}$ limit of the three-sphere index. We will also consider a more general class of generalized indices, replacing the S^3 with $\mathcal{M}_{g,p}$, a degree- p circle bundle over a Riemann surface of genus g [5].

³See, for instance, section 2 of [12] for a nice and detailed discussion of ‘t Hooft anomalies.

The crucial aspect of (1.3), for our purpose, is that it is *purely imaginary*, given a real background gauge field a_μ . Therefore, the absolute value of the partition function, $|Z|$, is gauge invariant, as in the well-known case of free fermions. In fact, we expect $|Z|$ to be invariant under both small and large gauge transformations, provided we work in a scheme that respects the consistent anomaly equation (1.3). (This expectation will also be borne out by explicit computations.)

On the other hand, the absolute value of a supersymmetric partition function that would be truly *holomorphic* in the flavor parameters cannot be fully gauge invariant and consistent with (1.3). Consider, for instance, the supersymmetric partition function on $S^3 \times S^1$. It depends on the background U(1) gauge field through two real parameters, which we denote by a_t and a_ψ . They correspond to holonomies of the gauge field through the S^1 factor and through an S^1 Hopf fiber γ inside S^3 , respectively:

$$a_t = \frac{1}{2\pi} \int_{S^1} a, \quad a_\psi = \frac{1}{2\pi} \int_\gamma a. \tag{1.4}$$

The complex flavor parameter ν is defined in terms of these real parameters as:

$$\nu = \tau a_\psi - a_t. \tag{1.5}$$

The parameter a_t is only defined modulo the identification $a_t \sim a_t - 1$ (equivalently, $\nu \sim \nu + 1$), which corresponds to a *large gauge transformation* along the S^1 . Under it, the $S^3 \times S^1$ partition function computed in [10] transforms as:⁴

$$\mathcal{I}_{S^3 \times S^1}(\nu + 1, \tau) = e^{\pi i \left(\frac{\mathcal{A}_{qqq}}{\tau^2} (\nu^2 + \nu + \frac{1}{3}) - \frac{\mathcal{A}_q}{6} \right)} \mathcal{I}_{S^3 \times S^1}(\nu, \tau), \tag{1.6}$$

where \mathcal{A}_{qqq} and \mathcal{A}_q are U(1) 't Hooft anomalies (cubic and mixed U(1)-gravitational, respectively). Since the whole object $\mathcal{I}_{S^3 \times S^1}(\nu, \tau)$ is holomorphic in ν , its absolute value is clearly not — and could not be — gauge invariant under the large gauge transformation $\nu \sim \nu + 1$.

A completely analogous puzzle arises in the simpler case of the torus partition function, $Z_{T^2}(\nu, \tau)$, of a 2d $\mathcal{N} = (0, 2)$ gauge theory. The explicit localization result of [7, 8] transforms non-trivially under the large gauge transformations $\nu \sim \nu + 1$ and $\nu \sim \nu + \tau$, and its absolute value is not gauge invariant. It is also interesting to study modular transformations of the torus; we expect the absolute value of the partition function to be modular invariant, which is not the case of the explicitly holomorphic answer for the elliptic genus.

The holomorphy of supersymmetric partition functions in the flavor parameters follow from a supersymmetric Ward identity [6, 13], as already mentioned. We have:

$$\frac{\partial}{\partial \bar{\nu}} W(\nu, \tau) = \langle \mathcal{J} \rangle = \langle \{ \mathcal{Q}, \tilde{j} \} \rangle = 0, \tag{1.7}$$

schematically. Here, the operator \mathcal{J} , which happens to be the supersymmetry variation of a fermionic operator \tilde{j} , is the component of the conserved current j^μ that couples to the anti-holomorphic parameter $\bar{\nu}$. In that last equality, we then use the fact that the expectation value of any \mathcal{Q} -exact operator vanishes. Thus, we seem to have a tension between:

⁴There are additional contributions to these anomalous transformations from mixed anomalies between the flavor group and the R -symmetry, which we will discuss further in the main text.

- (i) Gauge invariance of the absolute value of the supersymmetric partition function $Z_{\mathcal{M}_d}$, whenever the flavor symmetry has 't Hooft anomalies; then, $Z_{\mathcal{M}_d}$ should be compatible with the anomalous G_F gauge variation (1.3).
- (ii) Holomorphy of the supersymmetric partition function in the flavor parameters, which follows from supersymmetry according to (1.7).

In this paper, we resolve this apparent contradiction. The resolution is based on the fact that, whenever we insist on working in a diffeomorphism-invariant scheme,⁵ the G_F 't Hooft anomalies induce specific quantum corrections to the supersymmetric Ward identities, which are determined by the Wess-Zumino consistency conditions. As a consequence, the relation (1.7) is modified, and the partition function acquires a non-holomorphic dependence on the flavor parameters, that is precisely as required to make its absolute value gauge-invariant.

These quantum corrections to supersymmetric Ward identities, also referred to as “supersymmetry anomalies,” were studied long ago in [14, 15], and more recently in [16–22]. They follow from an anomalous transformation of the effective action:

$$\delta_\zeta W \neq 0, \tag{1.8}$$

where δ_ζ denotes a supersymmetry variation in the Wess-Zumino (WZ) gauge [23] for the background vector multiplet. This requires some explaining, however, to avoid possible confusions. First, the anomalous supersymmetry variation (1.8) does not appear with any new anomaly coefficient; instead, the non-vanishing value of $\delta_\zeta W$ is entirely determined by the bosonic 't Hooft anomalies for the flavor currents. Secondly, one can understand (1.8) as a consequence of working in the Wess-Zumino gauge, as explained e.g. in [24]. Recall that, in the WZ gauge, the supersymmetry transformations themselves depend on the (background) gauge field, because they are defined as the “bare” supersymmetry transformation, $\delta_\zeta^{(0)}$, plus a compensating gauge transformation, $\delta_{\Omega(\zeta)}$, namely:

$$\delta_\zeta = \delta_\zeta^{(0)} + \delta_{\Omega(\zeta)}. \tag{1.9}$$

Here, $\delta_{\Omega(\zeta)}$ is a specific supersymmetric gauge transformation whose chiral-multiplet-valued gauge parameter, $\Omega(\zeta)$, depends explicitly on a_μ . In this language, the supersymmetry variation (1.8) follows from:

$$\delta_\zeta^{(0)} W = 0, \quad \delta_{\Omega(\zeta)} W \propto \mathcal{A}_F \neq 0, \tag{1.10}$$

as we will explain in detail — in this language, (1.8) is a straightforward consequence of the ordinary 't Hooft anomaly in the presence of supersymmetry. Of course, our physical conclusions below must be the same in any gauge; it is merely convenient to fix the WZ gauge from the start, as usually done in the discussion of supersymmetric partition functions.

In general, the supersymmetric Ward identity (1.8) receives contributions from both the “flavor sector” and the “geometric sector” (in particular, from the $U(1)_R$ symmetry

⁵Such a scheme always exists in four-dimensions. In 2d, we might have a gravitational anomaly. Our discussion can be generalized to that case as well.

't Hooft anomalies). The background gauge fields for the former sector sit in vector multiplets, while the background fields for the latter (including the $U(1)_R$ gauge field) are part of a supergravity multiplet.⁶ In this work, for simplicity, we focus on the flavor sector and, thus, on the consequences of the flavor 't Hooft anomalies in the presence of supersymmetry.

't Hooft anomalies and supersymmetry. Let us then study supersymmetric partition functions with background vector multiplets in the Wess-Zumino (WZ) gauge [23]. Any background vector multiplet, \mathcal{V}_F , has components:

$$\mathcal{V}_F = (a_\mu, \lambda, \tilde{\lambda}, D), \tag{1.11}$$

in WZ gauge (in 2d or 4d), with $\lambda, \tilde{\lambda}$ the gauginos. Then, the supersymmetry variation (1.8) of the effective action takes the schematic form:

$$\delta_\zeta W = \mathcal{A}_F \int d^d x \sqrt{g} \zeta \Psi(\lambda, \tilde{\lambda}, a), \tag{1.12}$$

where ζ is a supersymmetry parameter of R -charge ± 1 and dimension $-\frac{1}{2}$, and the coefficient, \mathcal{A}_F , is the same 't Hooft anomaly coefficient that appears in (1.3). The local function $\Psi(\lambda, \tilde{\lambda}, a)$ in (1.12) must be fermionic and of engineering dimension $d + \frac{1}{2}$; in particular, it is at least linear in the gauginos (it is quadratic in the vector multiplet in 2d, and cubic in 4d). This term was first computed long ago [14, 15], while a systematic analysis in 4d $\mathcal{N} = 1$ theories was recently carried out in [22].

In the WZ gauge for \mathcal{V}_F , the existence of the anomalous supersymmetry variation (1.12) is implied by the Wess-Zumino consistency conditions for the anomalies [14]:

$$[\delta_\alpha, \delta_\zeta]W = 0. \tag{1.13}$$

Since the supersymmetry variation of the gauge anomaly (1.3) is non-trivial:

$$\delta_\zeta(\delta_\alpha W) \neq 0, \tag{1.14}$$

it follows that $\delta_\alpha(\delta_\zeta W) \neq 0$, therefore $\delta_\zeta W$ itself must be non-zero. Alternatively, one can simply compute (1.12) by fixing the WZ gauge, as in (1.10).

We then study $\delta_\zeta W$ on a *fixed* geometric background — more precisely, a fixed supergravity background *à la* Festuccia-Seiberg [25]. We will consider the torus in 2d, and half-BPS four-manifolds \mathcal{M}_4 in 4d [26, 27]. For instance, for 4d $\mathcal{N} = 1$ theories, the cubic 't Hooft anomaly:

$$\delta_\alpha W = \frac{i\mathcal{A}_{qqq}}{24\pi^2} \int_{\mathcal{M}_4} \alpha f \wedge f, \tag{1.15}$$

for a $U(1)$ flavor symmetry, with $f = da$ the field strength of the background gauge field a_μ and $\mathcal{A}_{qqq} \in \mathbb{Z}$ the cubic anomaly coefficient, implies:

$$\delta_\zeta W = -\frac{\mathcal{A}_{qqq}}{24\pi^2} \int_{\mathcal{M}_4} d^A x \sqrt{g} \left(\epsilon^{\mu\nu\rho\sigma} \zeta \sigma_\mu \tilde{\lambda} a_\nu f_{\rho\sigma} - 3i\zeta \lambda \tilde{\lambda} \tilde{\lambda} \right), \tag{1.16}$$

which happens to take the same form as in flat space [14].

⁶The geometric-sector contributions to $\delta_\zeta W$ can be understood as arising from fixing the WZ gauge in supergravity, similarly to the flavor sector contributions.

The “gauge-invariant,” non-holomorphic partition function. In this work, we distinguish between two distinct expressions for the supersymmetric partition function on \mathcal{M}_d :

- (i) The “gauge-invariant partition function,” which we will generally simply call “the partition function,” and denote by $Z(\nu, \tau)$. Here, by “gauge-invariant,” we mean that $|Z(\nu, \tau)|^2$ is completely gauge invariant. This partition function is computed in a diffeomorphism-invariant (diff-invariant) scheme compatible with the ’t Hooft anomaly (1.3).
- (ii) The “holomorphic partition function,” which is the one studied so far in the literature. We denote it by $\mathcal{I}(\nu, \tau)$, to distinguish it from $Z(\nu, \tau)$. It is implicitly computed in a scheme that violates gauge and diffeomorphism invariance.

In all the examples we will consider, the supersymmetric background is of the form $\mathcal{M}_d \cong \mathcal{M}_{d-1} \times S^1$, and the supersymmetric partition function can then be computed as an index, by canonical quantization on \mathcal{M}_{d-1} . In 4d, the holomorphic partition function $\mathcal{I}_{\mathcal{M}_3 \times S^1}$ can be understood as a generalized 4d $\mathcal{N} = 1$ index, as discussed e.g. in [5, 11, 28, 29] for a large class of geometries. For the $S^3 \times S^1$ partition function, we have the holomorphic result:

$$\mathcal{I}_{S^3 \times S^1}(\nu, \tau) = e^{2\pi i \tau \mathcal{E}(\nu, \tau)} \mathbf{I}_{S^3}(\nu, \tau). \tag{1.17}$$

Here, $\mathbf{I}_{S^3}(\nu, \tau)$ is the ordinary $\mathcal{N} = 1$ supersymmetric index [3, 4], and the function $\mathcal{E}(\nu, \tau)$ in (1.17) is the supersymmetric Casimir energy [10, 30, 31]:

$$\mathcal{E}(\nu, \tau) = \frac{\mathcal{A}_{qqq} \nu^3}{6\tau^3} - \frac{\mathcal{A}_q \nu}{12\tau}. \tag{1.18}$$

The non-trivial large-gauge transformation in (1.6) comes entirely from the latter.⁷

We should compare this result to the expected properties of the “gauge invariant” partition function. Working in a diff-invariant scheme, we have an anomalous supersymmetry variation (1.12), which leads to an anomaly in the decoupling of some \mathcal{Q} -exact operators [16]. Indeed, consider the naive Ward identity (1.7). The fermionic operator \tilde{j} is precisely the operator that couples to the gaugino $\tilde{\lambda}$ in the background vector multiplet. Then, in the presence of any non-trivial contribution to $\delta_\zeta W$, we have:

$$\frac{\partial}{\partial \bar{\nu}} W(\nu, \tau) = \langle \mathcal{J} \rangle = \langle \{ \mathcal{Q}, \tilde{j} \} \rangle = \frac{\partial}{\partial \tilde{\lambda}} (\delta_\zeta W) \neq 0, \tag{1.19}$$

schematically. This “holomorphy anomaly” equation determines the non-holomorphic dependence of the partition function on the flavor parameters $\nu, \bar{\nu}$.

For instance, for the $S^3 \times S^1$ partition function, the contribution (1.16) from the cubic U(1) anomaly to $\delta_\zeta W$ implies:

$$\frac{\partial W}{\partial \bar{\nu}} = -\frac{\pi i \mathcal{A}_{qqq}}{6\tau_2} \nu (\nu - \bar{\nu}). \tag{1.20}$$

⁷This is specific to this simple case. On spaces of slightly more general topology, such as $\mathcal{M}_{g,p} \times S^1$, there are other large gauge transformations under which the “pure index,” $\mathbf{I}_{\mathcal{M}_3}$, also transforms non-trivially.

We should then revisit the supersymmetric localization computation of the $S^3 \times S^1$ partition function for $\mathcal{N} = 1$ gauge theories [10] in light of the supersymmetric Ward identity (1.20).

Importantly, the above discussion does not invalidate the localization argument in any way — it is still true that one can localize supersymmetric gauge theories and compute at the weakly-coupled UV fixed point.⁸ The only subtle point is that various one-loop determinants have to be computed carefully, with appropriate regulators, depending on which scheme we decide to work in. To compute the partition function $Z(\nu, \tau)$, we must work in a scheme consistent with the anomalous variations (1.15) and (1.16), and with diff-invariance. We then obtain results for the one-loop determinants that differ from the fully holomorphic results obtained in [9, 10]. The non-holomorphic dependence is determined by (1.20).

We show that the absolute value squared of the supersymmetric partition function $Z_{S^3 \times S^1}$ takes the explicit form:

$$|Z_{S^3 \times S^1}(\nu, \tau)|^2 = e^{-\frac{2\pi\tau_2}{3} \left(\mathcal{A}_{qqq} a_\psi^3 - \frac{\mathcal{A}_q}{2} a_\psi \right)} |\mathbf{I}_{S^3}(\nu, \tau)|^2, \tag{1.21}$$

with \mathbf{I}_{S^3} the ordinary letter-counting index, and a_ψ related to ν as in (1.5). This is to be compared with the expression (1.17) for the holomorphic partition function $\mathcal{I}_{S^3 \times S^1}(\nu, \tau)$. Note that the result (1.21) does not admit an “holomorphic square root,” therefore the supersymmetric partition function $Z_{S^3 \times S^1}(\nu, \tau)$ cannot be holomorphic. The expression (1.21) is completely gauge-invariant, by construction. Similar formulae hold for other supersymmetric partition functions.

While the result (1.21) determines the absolute value of the “gauge-invariant” partition function, we will further argue that the phase is also fully determined, thanks to supersymmetry, essentially because the anomalous Ward identity (1.20) relates the real and imaginary parts of the supersymmetric effective action.

The small- β limit of supersymmetric partition functions. We also obtain new general constraints on supersymmetric partition functions in the case of any half-BPS manifold $\mathcal{M}_d = \mathcal{M}_{d-1} \times S^1$ (for $d = 2$, this is the elliptic genus; for $d = 4$ we have a generalized index on \mathcal{M}_3).⁹ In that context, we consider the “high-temperature limit,” sending the circle radius β to zero, up to order β in the small- β expansion:

$$W \underset{\beta \rightarrow 0}{\sim} \frac{1}{\beta} W_{d-1}^{(-1)} + W_{d-1}^{(0)} + \beta W_{d-1}^{(1)} + \dots \tag{1.22}$$

Crucially, the effective action is expected to be *local* in the background gauge fields, at each order in β , essentially because the d -dimensional theory compactified on the S^1 (and with appropriate chemical potentials) is generically gapped [32, 33]. The term of order $1/\beta$ gives the Cardy-like contribution discussed in [33],¹⁰ while the finite term essentially gives

⁸This is because the supersymmetric localization argument relies on the decoupling of \mathcal{Q} -exact terms in the gauge sector — in terms of dynamical gauge fields —, and the relevant \mathcal{Q} -exact operators do decouple because the gauge anomalies must vanish.

⁹In fact, our results will also be valid if the S^1 is fibered non-trivially over \mathcal{M}_{d-1} .

¹⁰Recently, there was a renewed interest in similar Cardy-like limits in the context of black-hole microstate counting [34–38] (see also [39]). We remark that our small- β limit is distinct from the one considered in those recent references (while it is the same as in [33]).

the partition function of the dimensionally-reduced $(d - 1)$ -dimensional theory on \mathcal{M}_{d-1} — see e.g. [40, 41]. We will study in detail the order- β term, $W_{d-1}^{(1)}$. That term must reproduce all the consistent flavor anomalies (1.3) of the d -dimensional theory, as well as their consequences (1.12), because the anomalies themselves are local d -dimensional functionals, and therefore only contribute at order β [32].

There indeed exists $(d - 1)$ -dimensional local terms that reproduce the dimensional reduction of the flavor 't Hooft anomalies, and are consistent with supersymmetry. For instance, in 4d $\mathcal{N} = 1$ theories on $\mathcal{M}_3 \times S^1$, the cubic 't Hooft anomaly (1.15), together with the supersymmetry variation (1.16), uniquely determine the 3d local term:

$$W_{3d}^{(1)} = \frac{\mathcal{A}_{qqq}}{12\pi} \int_{\mathcal{M}_3} d^3x \sqrt{g} \left(-i\sigma \epsilon^{\mu\nu\rho} A_\mu F_{\nu\rho} + 3\sigma^2 D - 6i\sigma \tilde{\lambda} \lambda + \sigma^2 A_\mu \mathbf{V}^\mu + \sigma^3 \mathbf{H} \right). \quad (1.23)$$

Here, the fields $(\sigma, A_\mu, \lambda, \tilde{\lambda}, D)$ are part of a 3d $\mathcal{N} = 2$ vector multiplet (which descends from the 4d $\mathcal{N} = 1$ vector multiplet), and \mathbf{V}_μ and \mathbf{H} are background supergravity fields necessary to preserve supersymmetry [42]. Analogous 3d local terms are necessary to match any other 't Hooft anomalies.

Plugging the supersymmetric values for the background supergravity and vector multiplet fields into (1.23) — and its analogues for other 't Hooft anomalies —, we determine *a priori* the form of the supersymmetric partition function $Z_{\mathcal{M}_3 \times S^1}$ at order β . The result agrees with our explicit computation of the absolute value of the supersymmetric partition function, and also agrees — as it should, by construction — with the anomalous Ward identity (1.20).

These considerations lead us to propose a general formula for the supersymmetric partition function $Z_{\mathcal{M}_{d-1} \times S^1}(\nu, \tau)$, which is fully consistent with gauge invariance and supersymmetry. This is based on the observation, true in a number of examples, that the holomorphic partition function $\mathcal{I}_{\mathcal{M}_{d-1} \times S^1}(\nu, \tau)$, *does not have any order- β term in its small- β expansion* — for the S^3 index, this was proven in [43]. Ref. [43] proves this result under the assumption that the S^3 partition function of the dimensionally reduced theory is finite, and moreover it shows that counterexamples exist when this assumption is not satisfied. In our paper we will always assume the finiteness of the \mathcal{M}_3 partition function.

Then, we propose that the “gauge-invariant” supersymmetric partition function, compatible with all anomalous Ward identities, takes the form:

$$Z_{\mathcal{M}_{d-1} \times S^1}(\nu, \bar{\nu}, \tau) = e^{-\beta W_{d-1}^{(1)}(\nu, \bar{\nu}, \tau)} \mathcal{I}_{\mathcal{M}_{d-1}}(\nu, \tau). \quad (1.24)$$

Here, $W_{d-1}^{(1)}(\nu, \bar{\nu}, \tau)$ denotes a function computed from the $(d-1)$ -dimensional functional $W_{d-1}^{(1)}$ — in particular, we have the contribution (1.23) for the 4d cubic 't Hooft anomaly contribution — upon evaluation onto the supersymmetric locus for the background vector multiplet.

Note also that, if we interpret the 3d local term $\beta W_{3d}^{(1)}(\nu, \bar{\nu}, \tau)$ as a 4d local term, by a trivial uplift which breaks 4d diff invariance explicitly, one can view the relation (1.24) as *a change of scheme* from the holomorphic to the “gauge invariant” form of the supersymmetric partition function. As expected, the holomorphic partition function corresponds to

a scheme that breaks diff-invariance explicitly. We shall leave a more systematic understanding of the allowed counterterms for future work.

Outlook and summary. In this paper, we focussed our attention on general constraints on supersymmetric partition functions in the presence of 't Hooft anomalies for the flavor symmetry, for 2d $\mathcal{N} = (0, 2)$ and 4d $\mathcal{N} = 1$ supersymmetric theories with an R -symmetry, $U(1)_R$. This clarifies important aspects of the dependence of these partition functions on the *flavor parameters*.

Supersymmetric partition functions can also depend non-trivially on *geometric parameters* — in particular, on the modular parameter τ discussed in this paper, but also on more subtle geometric data. It was first pointed out in [16] that the 't Hooft anomalies for the $U(1)_R$ symmetry — and, more generally, 't Hooft anomalies in the “gravity sector” — can introduce an anomalous dependence of the partition function on the geometry, thus resolving puzzles that arose from detailed holographic computations [44]. This interesting problem could also be addressed by combining the approach of this paper with the general solution to the WZ consistency conditions for the \mathcal{R} -multiplet given recently in [22]. It would also be interesting to generalize this work to include 6d $\mathcal{N} = (1, 0)$ theories, and to discuss theories in 2d and 4d with higher supersymmetry.

Finally, let us mention that the non-holomorphy of partition functions in the flavor parameters, as discussed here, is distinct from (and much simpler than) several other examples of holomorphic anomalies in the literature, such as the holomorphic anomaly of the topological string partition function [45], or of the elliptic genus of theories with continuous spectra [46, 47], or else in Donaldson-Witten theory [48]. In our case, the non-holomorphic term appears entirely as a “supersymmetric Casimir-energy term,” and not as part of the “spectrum;” the interpretation being that it essentially corresponds to a choice of scheme. In all cases, however, it is expected that the non-decoupling of the relevant \mathcal{Q} -exact operators can be understood as a boundary term in the path integral; it would be interesting to understand our results in that language.

This paper is organized as follows. In section 2, we discuss the flavor 't Hooft anomaly in 2d $\mathcal{N} = (0, 2)$ supersymmetric theories, and its consequences for the T^2 partition function. In section 3, after reviewing some necessary formalism, we discuss the analogous problem for 4d $\mathcal{N} = 1$ supersymmetric theories on a fixed half-BPS new-minimal supergravity background, and we consider the small- β expansion for theories on $\mathcal{M}_3 \times S^1$. In section 4, we use those results to revisit the computation of supersymmetric partition functions on a large class of four-dimensional backgrounds called $\mathcal{M}_{g,p} \times S^1$, which include $S^3 \times S^1$ as a special case. Additional computations and hopefully useful discussions are contained in several appendices.

2 't Hooft anomalies and the 2d $\mathcal{N} = (0, 2)$ elliptic genus

As an interesting warm-up to the four-dimensional case, let us study the partition function of a 2d $\mathcal{N} = (0, 2)$ supersymmetric theory on the torus, in the presence of flat connections for background gauge fields coupling to the flavor symmetry — this is also known as the flavored $\mathcal{N} = (0, 2)$ elliptic genus.

2.1 The free fermion and the Quillen anomaly

The simplest example of a two-dimensional $\mathcal{N} = (0, 2)$ supersymmetric theory is a free massless Fermi multiplet, which contains a single chiral fermion, λ_- , of positive chirality. Consider the partition function of this free theory on a Riemann surface Σ , with a background gauge field a_μ coupled to the $U(1)$ global symmetry:

$$\mathcal{L} = -2i\bar{\lambda}_- D_{\bar{w}} \lambda_- . \tag{2.1}$$

The Dirac operator can be written as:

$$D_{\bar{w}} = \nabla_{\bar{w}} - ia_{\bar{w}} , \tag{2.2}$$

with w some local complex coordinate. The naive formula for the partition function:

$$Z_\Sigma = \det(D_{\bar{w}}) , \tag{2.3}$$

seems to define a locally holomorphic function of the connection, i.e. it only depends on $a_{\bar{w}}$ and not on a_w . On the other hand, since the operator $D_{\bar{w}}$ flips the chirality, its determinant is ill-defined. The standard procedure is to consider $\det(D_w D_{\bar{w}})$ instead. If the latter determinant factorized holomorphically — that is, if it were a product of a holomorphic times an anti-holomorphic factor —, one would naturally obtain a holomorphic definition of the original partition function, simply by taking the “holomorphic square-root” of $\det(D_w D_{\bar{w}})$. However, an explicit calculation by means of a gauge-invariant regularization, such as Pauli-Villars or zeta-function regularization, reveals that [49]:

$$\det(D_w D_{\bar{w}}) = e^{-q} |F(a_{\bar{w}})|^2 , \tag{2.4}$$

where F is a holomorphic function, and q is the “Quillen counterterm”:

$$q = \frac{i}{2\pi} \int_\Sigma dw \wedge d\bar{w} a_w a_{\bar{w}} , \tag{2.5}$$

schematically. Due to the prefactor e^{-q} , the holomorphic factorization fails, and in particular the original partition function cannot be defined as a holomorphic function of the connection. This is known as Quillen’s holomorphic anomaly. As we will see, it is also the simplest example of the non-trivial interplay between ’t Hooft anomalies and supersymmetry.¹¹

A nice geometric interpretation of (2.4) was provided in [50]. The freedom in choosing the connection a_μ can be parametrized by the space of flat connection on the Riemann surface, i.e. the Jacobian $J(\Sigma)$. The latter is a $2g$ -dimensional torus for Σ a Riemann surface of genus g , and it inherits a complex structure from that of Σ . In the presence of a $U(1)$ ’t Hooft anomaly, the partition function of the chiral fermion is not a holomorphic function on $J(\Sigma)$ but rather a section of a holomorphic line bundle over it:

$$\mathcal{L} \longrightarrow J(\Sigma) , \tag{2.6}$$

¹¹Quillen’s anomaly can be stated much more generally in terms of Dirac operators that depend holomorphically on some moduli; presumably, the 4d $\mathcal{N} = 1$ case to be discussed in the next section could also be understood in that language.

usually called the *determinant line bundle* [49]. The expression (2.4) is then interpreted as defining a Hermitian norm on \mathcal{L} (known as the Quillen metric), and its failure to factorize is due to its curvature. The first Chern class $c_1(\mathcal{L})$ is fixed precisely by the 't Hooft anomaly coefficient.

The case $\Sigma = T^2$. In anticipation to the discussion of the elliptic genus, let us consider more explicitly the case of a chiral fermion on a torus. We choose coordinates $x \sim x + 2\pi$ and $y \sim y + 2\pi$, with the complex coordinate:

$$w = x + \tau y, \quad \tau \equiv \tau_1 + i\tau_2, \quad \tau_2 = \frac{\beta_2}{\beta_1}, \quad (2.7)$$

for some fixed complex structure modulus τ . The metric on the torus reads:

$$ds^2(T^2) = \beta_1^2 dw d\bar{w} = \beta_1^2 (dx + \tau_1 dy)^2 + \beta_2^2 dy^2. \quad (2.8)$$

In this case, the Jacobian is itself a torus, parametrized by the complex variable:

$$\nu \equiv a_x \tau - a_y = 2i\tau_2 a_{\bar{w}}, \quad (2.9)$$

subject to the identifications:

$$\nu \sim \nu + 1, \quad \nu \sim \nu + \tau, \quad (2.10)$$

under large gauge transformations. In appendix C.1, for review the calculation of the determinant $\det(D_w D_{\bar{w}})$ via zeta-function regularization. From that calculation, we obtain the following result for the partition function of a chiral fermion:

$$Z_{T^2}^\lambda \propto e^{-2\pi\tau_2 \left(\frac{a_x^2}{2} - \frac{a_x}{2} + \frac{1}{12} \right)} \theta_0(\nu; \tau) \quad (2.11)$$

where the coefficient of proportionality is a phase factor (possibly a function of the real parameters a_x, a_y, τ_1, τ_2). Here, $\theta_0(\nu; \tau)$ is the “reduced” theta-function; its definition and properties can be found in appendix B. The theta-function is an holomorphic section of the determinant line bundle on $J(T^2)$, in agreement with the discussion above; in particular, it transforms non-trivially under large gauge transformations. The non-holomorphic prefactor in Z_λ takes the form of a Casimir energy, being the dominant contribution in the limit $\tau_2 \rightarrow \infty$. By construction, the absolute value of the partition function, $|Z_\lambda|$, is invariant under the large gauge transformations $\nu \rightarrow \nu + 1$ and $\nu \rightarrow \nu + \tau$, in agreement with the requirement of gauge-invariance described in the introduction.

2.2 The elliptic genus and the torus partition function

Let us now consider a two-dimensional field theory with $\mathcal{N} = (0, 2)$ supersymmetry. We are interested in its elliptic genus, which may be defined as the supersymmetric index:

$$\mathcal{I}(\nu, \tau) = \text{Tr} \left((-1)^F q^H \bar{q}^{\bar{H}} y^{Q_F} \right), \quad q = e^{2\pi i \tau}, \quad y = e^{2\pi i \nu}. \quad (2.12)$$

Here, Q_F denotes the generator of a $U(1)$ global symmetry, and y is the associated fugacity — more generally, one introduces one fugacity y_α for each $U(1)_\alpha$ in the maximal torus of the global symmetry group G_F .

The elliptic genus can also be understood as the supersymmetric partition function on T^2 with complex structure modulus τ , as in (2.7). The chemical potential ν corresponds to a flat background gauge field for the $U(1)$ global symmetry, $a_\mu dx^\mu = a_w dw + a_{\bar{w}} d\bar{w}$, with:

$$\nu \equiv a_x \tau - a_y, \quad a_x \equiv \frac{1}{2\pi} \int_{\gamma_x} a, \quad a_y \equiv \frac{1}{2\pi} \int_{\gamma_y} a, \quad (2.13)$$

with γ_x and γ_y the one-cycles along x and y , respectively. As we will review below, a simple argument shows that the derivatives with respect to $\bar{\nu}$ and $\bar{\tau}$ are Q -exact and therefore the partition function $Z_{T^2}(\nu, \tau)$ is expected to be holomorphic in ν ; a similar argument shows that it should be holomorphic in the geometric parameter τ , as well.

Explicit expression for ‘Lagrangian’ theories. The elliptic genus can easily be computed for free theories. For a free chiral multiplet Φ or for a free Fermi multiplet Λ , of charge $q = 1$ under the $U(1)$ background gauge field, we have:

$$\mathcal{I}_\Phi(\nu, \tau) \equiv \frac{i}{\theta(\nu; \tau)}, \quad \mathcal{I}_\Lambda(\nu, \tau) \equiv i\theta(\nu; \tau), \quad (2.14)$$

respectively. Here, $\theta(\nu, \tau)$ is the ordinary Jacobi theta function — see appendix B.

Thanks to supersymmetric localization, we also have an explicit expression for supersymmetric gauged linear σ -models (GLSM) [7, 8, 51]. Consider an $\mathcal{N} = (0, 2)$ vector multiplet for the gauge group \mathbf{G} , coupled to chiral multiplets Φ_i and Fermi multiplets Λ_I in representations \mathfrak{R}_i and \mathfrak{R}_I of \mathbf{G} , respectively. The localization formula for the elliptic genus takes the form:

$$\mathcal{I}(\nu, \tau) = \oint_{\text{JK}} \prod_{a=1}^{\text{rk}(\mathbf{G})} \frac{du_a}{2\pi i} I(u, \nu, \tau) \quad (2.15)$$

where the contour integral denotes the Jeffrey-Kirwan residue [52]. The integrand reads:

$$\begin{aligned} I(u, \nu, \tau) = & \prod_i \prod_{\rho_i \in \mathfrak{R}_i} \mathcal{I}_\Phi(\rho_i(u) + \omega_i(\nu), \tau) \prod_I \prod_{\rho_I \in \mathfrak{R}_I} \mathcal{I}_\Lambda(\rho_I(u) + \omega_I(\nu), \tau) \\ & \times (-2\pi i \eta(\tau)^2)^{\text{rk}(\mathbf{G})} \prod_{\alpha \in \mathfrak{g}} \mathcal{I}_\Lambda(\alpha(u), \tau). \end{aligned} \quad (2.16)$$

The products over the indices i and I , on the first line, run over the chiral and Fermi multiplets, respectively. The fugacities associated to the gauge group are denoted by u_a , and are integrated over; the flavor fugacities are denoted by ν . Here, $\rho = (\rho^\alpha)$ denote the weights of the representations \mathfrak{R} under the gauge symmetry. We will also use indices α, β, \dots for the Cartan subalgebra \mathfrak{h}_F of the flavor symmetry, so that e.g. $\omega(\nu) = \omega^\alpha \nu_\alpha$, with $\omega = (\omega^\alpha)$ a flavor weight and $\nu = (\nu_\alpha) \in (\mathfrak{h}_F)_\mathbb{C}$. The second line in (2.16) is the contribution from the vector multiplet, with the product over the non-zero roots α of the gauge algebra \mathfrak{g} .

Behavior under large gauge transformations. It is instructive to consider the behavior of the elliptic genus under large gauge transformations for the flavor fugacities:

$$(a_x, a_y) \sim (a_x - m, a_y + n) \quad \Leftrightarrow \quad \nu \sim \nu + n + m\tau, \quad (2.17)$$

for any $n, m \in \mathbb{Z}$. For instance, for a free Fermi multiplet of charge $q \in \mathbb{Z}$ under the $U(1)$ flavor symmetry, we have:

$$\begin{aligned} \mathcal{I}_{\Lambda, q}(\nu, \tau) &= \mathcal{I}_{\Lambda}(q\nu, \tau), \\ \mathcal{I}_{\Lambda, q}(\nu + n + m\tau, \tau) &= (-1)^{q(n+m)} e^{-\pi i q^2 (2m\nu + m^2\tau)} \mathcal{I}_{\Lambda, q}(\nu, \tau). \end{aligned} \quad (2.18)$$

The behavior under large gauge transformations is governed by $\mathcal{A}_{qq} = q^2$, the 't Hooft anomaly coefficient for a free Fermi multiplet. More generally, for any GLSM, we have the quadratic $U(1)_{\alpha}$ 't Hooft anomaly coefficients:

$$\mathcal{A}_{qq}^{\alpha\beta} = - \sum_i \omega_i^{\alpha} \omega_i^{\beta} + \sum_I \omega_I^{\alpha} \omega_I^{\beta}, \quad (2.19)$$

where the integers ω_i^{α} and ω_I^{α} denotes the $U(1)_{\alpha}$ flavor charges of the chiral and Fermi multiplets, respectively. In a well-defined GLSM, all the gauge and gauge-flavor anomalies should vanish:

$$\mathcal{A}_{qq}^{ab} = - \sum_i \rho_i^a \rho_i^b + \sum_I \rho_I^a \rho_I^b = 0, \quad \mathcal{A}_{qq}^{a\beta} = - \sum_i \rho_i^a \omega_i^{\beta} + \sum_I \rho_I^a \omega_I^{\beta} = 0. \quad (2.20)$$

This implies that the integrand (2.16) is single-valued under $u \sim u + 1 \sim u + \tau$. Under an arbitrary large gauge transformation:

$$\nu_{\alpha} \rightarrow \nu_{\alpha} + n_{\alpha} + m_{\alpha}\tau, \quad n_{\alpha}, m_{\alpha} \in \mathbb{Z}, \quad (2.21)$$

for the flavor parameters, the elliptic genus transforms as:¹²

$$\mathcal{I}(\nu + n + m\tau, \tau) = (-1)^{\mathcal{A}_q^{\alpha}(n_{\alpha} + m_{\alpha})} e^{-\pi i \mathcal{A}^{\alpha\beta} (m_{\alpha} \nu_{\beta} + \nu_{\alpha} m_{\beta} + m_{\alpha} m_{\beta} \tau)} \mathcal{I}(\nu, \tau), \quad (2.22)$$

where we sum over repeated indices. The fact that the behavior under large gauge transformations is governed by the 't Hooft anomalies can be understood on general ground; we discuss some aspects of this in appendix D.1.

Holomorphy versus gauge invariance. As emphasized in the introduction, the transformation property (2.22) is somewhat puzzling when $\mathcal{I}(\nu, \tau)$ is viewed as a partition function on T^2 , because the real part of $\log \mathcal{I}$ transforms non-trivially under large gauge transformations. On the other hand, we expect that there exists a scheme in which the absolute value of the partition function, $Z_{T^2}(\nu, \tau)$, is fully gauge invariant:

$$|Z_{T^2}(\nu + n + m\tau, \tau)| = |Z_{T^2}(\nu, \tau)|. \quad (2.23)$$

¹²Here we also introduced the “linear anomaly” coefficient:

$$\mathcal{A}_q^{\alpha} = - \sum_i \omega_i^{\alpha} + \sum_I \omega_I^{\alpha},$$

which governs the sign on the right-hand-side of (2.22). This sign is a more subtle effect, related to the parity anomaly in 1d, which we will not discuss in this paper.

Consider again a free Fermi multiplet of unit charge. According to (2.11), we have:

$$|Z_{T^2}^\Lambda(\nu, \tau)| = e^{-\pi\tau_2 a_x^2} |\mathcal{I}_\Lambda(\nu, \tau)|. \quad (2.24)$$

More generally, we will argue that:

$$Z_{T^2}(\nu, \tau) = e^{-\mathbb{W}_{1d}(\nu, \bar{\nu}, \tau, \bar{\tau})} \mathcal{I}(\nu, \tau), \quad (2.25)$$

for any $\mathcal{N} = (0, 2)$ supersymmetric theory, with \mathbb{W}_{1d} a function to be determined below. Note that \mathbb{W}_{1d} cannot be holomorphic in ν if (2.23) holds true. In the case of a GLSM, it is easy to check that imposing (2.23) *a priori* completely determines the real part of \mathbb{W}_{1d} to be:¹³

$$\text{Re} \left[\mathbb{W}_{1d}(\nu, \bar{\nu}, \tau, \bar{\tau}) \right] = \pi\tau_2 \mathcal{A}^{\alpha\beta} a_{x,\alpha} a_{x,\beta}, \quad (2.26)$$

up to a pure constant. The explicit result (2.24) is a special case of the relation (2.26).

We see that gauge invariance imposes that the T^2 partition function is not fully holomorphic in the flavor parameter ν (and possibly in τ , as well). Naively, this is in contradiction with supersymmetry. As we will review below, the anti-holomorphic parameter $\bar{\nu}$ couples to a Q -exact operator, and therefore we would expect to have:

$$\frac{\partial}{\partial \bar{\nu}} Z_{T^2}(\nu, \tau) = 0, \quad (2.27)$$

as a supersymmetric Ward identity. The presence of a non-trivial quantum correction to the supersymmetric Ward identity [15] resolves this puzzle, as we explain in the next subsection.

Behavior under modular transformations. Let us also briefly discuss the behavior of the elliptic genus under large diffeomorphisms of the torus, which span the modular group $\text{SL}(2, \mathbb{Z})$. Let k_g denote the gravitational anomaly coefficient, defined as:

$$k_g = \sum_i \dim(\mathcal{R}_i) - \sum_I \dim(\mathcal{R}_I) - \dim(\mathbf{G}), \quad (2.28)$$

for a GLSM. Using the properties of the θ -function summarized in appendix B, one can check that:

$$\mathcal{I} \left(\frac{\nu}{\tau}, -\frac{1}{\tau} \right) = e^{\frac{\pi i}{2} k_g} e^{\frac{\pi i}{\tau} \mathcal{A}^{\alpha\beta} \nu_\alpha \nu_\beta} \mathcal{I}(\nu, \tau), \quad \mathcal{I}(\nu, \tau + 1) = e^{-\frac{\pi i}{6} k_g} \mathcal{I}(\nu, \tau), \quad (2.29)$$

under the S and T generators of $\text{SL}(2, \mathbb{Z})$, respectively. This behavior presents physical puzzles similar to the case of the behavior under large gauge transformations, and their resolutions will be similar.

¹³Recall the definitions $\nu = \tau a_x - a_y$ and $\tau = \tau_1 + i\tau_2$.

2.3 Non-holomorphy of the supersymmetric partition function Z_{T^2}

We will now derive the correction to (2.27) in the “gauge-invariant” scheme. To this end, we first review the structure of the conserved current $(0, 2)$ multiplet, its coupling to a background vector multiplet, and the supersymmetry algebra in WZ gauge.

Consider a 2d $\mathcal{N} = (0, 2)$ supersymmetric theory with $G_F = U(1)$, for simplicity. We can preserve the two supercharges, Q_+ and \tilde{Q}_+ , on the flat torus. Let us denote the supersymmetry variations by:

$$\delta_\zeta = -i\zeta_- Q_+, \quad \delta_{\tilde{\zeta}} = -i\tilde{\zeta}_- \tilde{Q}_+, \quad (2.30)$$

where ζ_- and $\tilde{\zeta}_-$ are constant Weyl spinors,¹⁴ which we take to be bosonic, so that the supersymmetry variations are fermionic.

Coupling to a background vector multiplet. The $U(1)$ flavor symmetry current, j^μ , sits in a 2d $\mathcal{N} = (0, 2)$ current multiplet, \mathcal{J} , with component operators:

$$\mathcal{J} = (J, j_+, \tilde{j}_+, j^\mu), \quad \partial_\mu j^\mu = 0, \quad (2.31)$$

where j_+ and \tilde{j}_+ are fermionic operators. Let us denote by $(j_w, j_{\bar{w}})$ the components of j_μ in complex coordinates; in particular, we have $\partial_w j_{\bar{w}} + \partial_{\bar{w}} j_w = 0$. The $\mathcal{N} = (0, 2)$ supersymmetry variations of the current multiplet read:¹⁵

$$\begin{aligned} \delta_\zeta J &= -i\zeta_- j_+, & \delta_{\tilde{\zeta}} J &= -i\tilde{\zeta}_- \tilde{j}_+, \\ \delta_\zeta j_+ &= 0, & \delta_{\tilde{\zeta}} j_+ &= -2i\tilde{\zeta}_- (j_{\bar{w}} + i\partial_{\bar{w}} J), \\ \delta_\zeta \tilde{j}_+ &= 2i\zeta_- (j_{\bar{w}} - i\partial_{\bar{w}} J), & \delta_{\tilde{\zeta}} \tilde{j}_+ &= 0, \\ \delta_\zeta j_w &= -\zeta_- \partial_w j_+, & \delta_{\tilde{\zeta}} j_w &= \tilde{\zeta}_- \partial_w \tilde{j}_+, \\ \delta_\zeta j_{\bar{w}} &= \zeta_- \partial_{\bar{w}} j_+, & \delta_{\tilde{\zeta}} j_{\bar{w}} &= -\tilde{\zeta}_- \partial_{\bar{w}} \tilde{j}_+. \end{aligned} \quad (2.32)$$

We couple the current multiplet to an abelian background vector multiplet in the Wess-Zumino (WZ) gauge, which consists of a gauge field $a_\mu = (a_w, a_{\bar{w}})$ and its superpartners:

$$\mathcal{V}_F = (a_\mu, \lambda_-, \tilde{\lambda}_-, D). \quad (2.33)$$

Its supersymmetry variations read:

$$\begin{aligned} \delta_\zeta a_w &= -i\zeta_- \tilde{\lambda}_-, & \delta_{\tilde{\zeta}} a_w &= -i\tilde{\zeta}_- \lambda_-, \\ \delta_\zeta a_{\bar{w}} &= 0, & \delta_{\tilde{\zeta}} a_{\bar{w}} &= 0, \\ \delta_\zeta \lambda_- &= i\zeta_- (D + 2if_{w\bar{w}}), & \delta_{\tilde{\zeta}} \lambda_- &= 0, \\ \delta_\zeta \tilde{\lambda}_- &= 0, & \delta_{\tilde{\zeta}} \tilde{\lambda}_- &= -i\tilde{\zeta}_- (D - 2if_{w\bar{w}}), \\ \delta_\zeta D &= 2\zeta_- \partial_{\bar{w}} \tilde{\lambda}_-, & \delta_{\tilde{\zeta}} D &= -2\tilde{\zeta}_- \partial_w \lambda_-, \end{aligned} \quad (2.34)$$

¹⁴Our conventions for two-dimensional spinors are summarized in appendix A.3.

¹⁵In this subsection, we are setting $\beta_1 = 1$ to avoid clutter, so that the torus metric is simply $ds^2 = dw d\bar{w}$; equivalently, we are absorbing β_1 into the definition of w, \bar{w} .

where $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$ is the field strength. The supersymmetry algebra in the WZ gauge takes the form:

$$(\delta_\zeta)^2 \Phi_{(q)} = 0, \quad (\delta_{\tilde{\zeta}})^2 \Phi_{(q)} = 0, \quad \{\delta_\zeta, \delta_{\tilde{\zeta}}\} \Phi_{(q)} = -4i\zeta_- \tilde{\zeta}_- (\partial_{\bar{w}} - iqa_{\bar{w}}) \Phi_{(q)}, \quad (2.35)$$

on any field $\Phi_{(q)}$ of charge q under the U(1) flavor symmetry; note that it is gauge-covariant. We can also write the last anti-commutator as:

$$\{\delta_\zeta, \delta_{\tilde{\zeta}}\} = -4i\zeta_- \tilde{\zeta}_- \partial_{\bar{w}} + \delta_{\alpha(a)}, \quad \alpha(a) \equiv 4i\zeta_- \tilde{\zeta}_- a_{\bar{w}}. \quad (2.36)$$

The second term on the right-hand-side can be understood as an a_μ -dependent gauge transformation, δ_α . Then, (2.36) gives the supersymmetry algebra acting on any fields, including the gauge field itself (with $\delta_\alpha a_\mu = \partial_\mu \alpha$).

The supersymmetric minimal coupling between the current and the background gauge field takes the form:

$$\mathcal{L}_{\mathcal{V}\mathcal{J}} = 2a_w j_{\bar{w}} + 2a_{\bar{w}} j_w + DJ + \lambda_- j_+ - \tilde{\lambda}_- \tilde{j}_+. \quad (2.37)$$

In particular, we can consider supersymmetric background values for \mathcal{V}_F , which are such that the gauginos λ_- , $\tilde{\lambda}_-$ and their variations vanish — that is:

$$D = 0, \quad f_{w\bar{w}} = 0, \quad (2.38)$$

and the gauge field a_μ must be flat. For future purpose, we will consider small variations $(a_\mu, \lambda_-, \tilde{\lambda}_+, \Delta D)$ around that locus, while keeping the gauge field flat. Then, using (2.38), one can write (2.37) as:

$$\mathcal{L}_{\mathcal{V}\mathcal{J}} = 2a_w(j_{\bar{w}} - i\partial_{\bar{w}}J) + 2a_{\bar{w}}(j_w + i\partial_wJ) + \Delta DJ + \lambda_- j_+ - \tilde{\lambda}_- \tilde{j}_+. \quad (2.39)$$

Looking at (2.32), we see that a_w couples to a δ_ζ -exact operator,¹⁶ which should not affect supersymmetric observables. This is why we expect the T^2 partition function to be holomorphic in the flat connection $a_{\bar{w}}$, which is related to the flavor parameter ν as in (2.9).

Supersymmetry variation of the effective action. Let W denote the effective action on the torus, in the presence of an arbitrary background vector multiplet:

$$W[a_\mu, \lambda, \tilde{\lambda}, D] = -\log Z_{T^2}[a_\mu, \lambda, \tilde{\lambda}, D]. \quad (2.40)$$

The U(1) symmetry generally suffers from a quadratic 't Hooft anomaly, with coefficient $\mathcal{A}_{qq} = \text{tr}(\gamma^3 Q_F^2)$, as in (2.19). Under a gauge transformation $\delta_\alpha a_\mu = \partial_\mu \alpha$, the effective action must then transform as:

$$\delta_\alpha W = \frac{i\mathcal{A}_{qq}}{4\pi} \int_{T^2} \alpha f, \quad (2.41)$$

where we used the form notation, with $f \equiv da = \frac{1}{2} f_{\mu\nu} dx^\mu \wedge dx^\nu$. Since the gauge transformations commute with supersymmetry (in the WZ gauge), we should have:

$$[\delta_\zeta, \delta_\alpha]W = 0, \quad [\delta_{\tilde{\zeta}}, \delta_\alpha]W = 0. \quad (2.42)$$

¹⁶One can write it as a $\delta_{\tilde{\zeta}}$ -exact term as well, by integration by part.

The supersymmetry variations of (2.41) give:

$$\begin{aligned}\delta_\zeta \delta_\alpha W &= \frac{i \mathcal{A}_{qq}}{4\pi} \int \alpha d(-i\zeta_- \tilde{\lambda}_- dw) = -\frac{\mathcal{A}_{qq}}{4\pi} \int d\alpha \wedge (\zeta_- \tilde{\lambda}_- dw) \\ &= \delta_\alpha \left(\frac{\mathcal{A}_{qq}}{4\pi} \int a_{\bar{w}} \zeta_- \tilde{\lambda}_- dw \wedge d\bar{w} \right) = \delta_\alpha \delta_\zeta W,\end{aligned}\tag{2.43}$$

where we used (2.42) in the last equality, and similarly:

$$\delta_\alpha \delta_{\tilde{\zeta}} W = \delta_\alpha \left(\frac{\mathcal{A}_{qq}}{4\pi} \int a_{\bar{w}} \tilde{\zeta}_- \lambda_- dw \wedge d\bar{w} \right).\tag{2.44}$$

This implies the presence of a non-zero supersymmetry variation, $\delta_\zeta W \neq 0$ [15]. The equations (2.43)–(2.44) determine $\delta_\zeta W$ up to the addition of a gauge-invariant term. To fix this ambiguity, we use the WZ condition coming from the anti-commutators (2.36) of two supersymmetry transformations, namely:¹⁷

$$\{\delta_\zeta, \delta_{\tilde{\zeta}}\} W = \delta_{\alpha(a)} W, \quad \alpha(a) \equiv 4i\zeta_- \tilde{\zeta}_- a_{\bar{w}}.\tag{2.45}$$

Taking into account this additional constraint, the supersymmetry variation of W is uniquely fixed to be:

$$\delta_\zeta W = \frac{\mathcal{A}_{qq}}{4\pi} \int_{T^2} \zeta_- \tilde{\lambda}_- a_{\bar{w}} dw \wedge d\bar{w}, \quad \delta_{\tilde{\zeta}} W = \frac{\mathcal{A}_{qq}}{4\pi} \int_{T^2} \tilde{\zeta}_- \lambda_- a_{\bar{w}} dw \wedge d\bar{w}.\tag{2.46}$$

Let us now discuss how these supersymmetric Ward identities allow us to constrain the form of the partition function, $Z_{T^2}(\nu, \tau)$.

Holomorphy anomaly of $Z_{T^2}(\nu, \tau)$ in ν . On a flat torus, we can take the supersymmetric gauge field components $(a_w, a_{\bar{w}})$ to be constant, in which case the flavor parameters are given by:

$$\nu = 2i\tau_2 a_{\bar{w}}, \quad \bar{\nu} = -2i\tau_2 a_w.\tag{2.47}$$

An infinitesimal variation of $\bar{\nu}$ then simply corresponds to an insertion of the component $j_{\bar{w}}$ of the conserved current, which is itself Q -exact:

$$\frac{\partial}{\partial \bar{\nu}} W = \frac{i}{2\tau_2} \frac{\partial}{\partial a_w} W = \frac{i}{2\tau_2} \left\langle \int_{T^2} d^2x \sqrt{g} 2j_{\bar{w}} \right\rangle = \frac{1}{2\tau_2 \zeta_-} \left\langle \int_{T^2} d^2x \sqrt{g} \delta_\zeta \tilde{j}_+ \right\rangle.\tag{2.48}$$

Due to the non-trivial supersymmetry variations (2.46), the last expression in (2.48) does not vanish. Instead, since the operator \tilde{j}_+ couples to the gaugino $\tilde{\lambda}_-$ as in (2.39), we obtain:

$$\frac{\partial}{\partial \bar{\nu}} W = -\frac{1}{2\tau_2 \zeta_-} \frac{\partial}{\partial \tilde{\lambda}_-} (\delta_\zeta W) \Big|_{\lambda=\tilde{\lambda}=0} = -\frac{\mathcal{A}_{qq}}{8\pi\tau_2} a_{\bar{w}} \int_{T^2} dw \wedge d\bar{w}.\tag{2.49}$$

¹⁷The first term on the right-hand-side of the anti-commutator, $-4i\zeta_- \tilde{\zeta}_- \partial_{\bar{w}}$, acts on fields as a translation; for simplicity, we may assume that the theory is free from gravitational anomalies, so this transformation leaves W invariant. The inclusion of the gravitational anomaly in the present discussion is left for future work.

This gives us a simple anomalous Ward identity for the supersymmetric T^2 partition function:

$$\frac{\partial}{\partial \bar{\nu}} W(\nu, \bar{\nu}) = \frac{\pi A_{qq}}{2\tau_2} \nu, \tag{2.50}$$

which entirely fixes its dependence on the anti-holomorphic parameter $\bar{\nu}$ — we must have:

$$Z_{T^2}(\nu) = e^{-\frac{\pi A_{qq}}{2\tau_2} |\nu|^2} Z_{\text{holo}}(\nu), \tag{2.51}$$

with $Z_{\text{holo}}(\nu)$ a locally holomorphic function of ν . To relate $Z_{\text{holo}}(\nu)$ to the holomorphic partition function $\mathcal{I}(\nu)$ defined above, we will need additional information. We can already note, however, that if we had:

$$Z_{\text{holo}}(\nu) = e^{\frac{\pi A_{qq}}{2\tau_2} \nu^2} \mathcal{I}(\nu), \tag{2.52}$$

then the expectation (2.25)–(2.26) would hold true, and the absolute value of the T^2 partition function Z_{T^2} would be properly gauge invariant. In the rest of this section, we will argue that (2.52) precisely holds.

2.4 The small- β_1 limit of $Z_{T^2}(\nu, \tau)$

We can attain a complementary understanding of the non-holomorphy of the partition function by considering the reduction of the 2d $\mathcal{N} = (0, 2)$ theory to a one-dimensional $\mathcal{N} = 2$ supersymmetric theory (as studied e.g. in [53]) in the limit when we take one of the circle of T^2 to be very small. Using the metric (2.8), we consider $\beta_1 \rightarrow 0$. This correspond to the limit:

$$\tau \rightarrow i\infty, \quad \nu = \text{fixed}, \tag{2.53}$$

on the T^2 partition function. The basic idea is that, in this limit, all the anomalies should be reproduced by *local terms* in the sources of the dimensionally-reduced theory, because generically the effective one-dimensional theory will be gapped [32, 33]. Upon evaluating these local terms, we find that they depend on $\bar{\nu}$ exactly as expected from (2.50). This procedure will also allow us to fix the entire ratio between Z_{T^2} and the holomorphic partition function, \mathcal{I}_{T^2} , including the holomorphic term.

In the small- β_1 limit (2.53), we study the expansion of the effective action:

$$W \underset{\beta_1 \rightarrow 0}{\sim} \frac{1}{\beta_1} W_{1d}^{(-1)} + W_{1d}^{(0)} + \beta_1 W_{1d}^{(1)} + \dots, \tag{2.54}$$

as a functional of background vector multiplets. The divergent term, of order $\frac{1}{\beta_1}$, is proportional to the gravitational anomaly coefficient, k_g ; the functional $W_{1d}^{(-1)}$ is supersymmetric and gauge-invariant.¹⁸ The finite term, $W_{1d}^{(0)}$, is essentially the 1d index of the dimensionally reduced theory [53]. Here, we are interested in the term linear in β_1 . Let us define:

$$\mathbb{W}_{1d}[\mathcal{V}_F] \equiv \beta_1 W_{1d}^{(1)}[\mathcal{V}_F]. \tag{2.55}$$

¹⁸This is similar to the supersymmetric Cardy formula in 4d $\mathcal{N} = 1$ theories [33]. This point would deserve further discussion.

It is a local functional of a one-dimensional background vector multiplet, which is strongly constrained by the two-dimensional anomalies. This is because each of the two-dimensional anomalies, being local, reduces to a term of order β_1 [32], which must then be exactly reproduced by the variation of the local functional \mathbb{W}_{1d} .

Dimensional reduction and 1d $\mathcal{N} = 2$ background vector multiplet. Let $X^\mu = (x, y)$ denote the 2d coordinate, with y the 1d coordinate in the $\beta_1 \rightarrow 0$ limit. The torus metric:

$$ds^2(T^2) = \beta_1^2(dx + \tau_1 dy)^2 + \beta_2^2 dy^2, \quad (2.56)$$

is already in the ‘‘Kaluza-Klein’’ form, with $ds^2(S^1) = \beta_2^2 dy^2$ the 1d metric and $\tau_1 dy$ a flat graviphoton. The two-dimensional gauge field, a_μ , reduces to a one-dimensional gauge field $A_{\mathbf{y}}$ and a real scalar σ , which are related to a_μ by:

$$a_\mu dX^\mu = \sigma \beta_1(dx + \tau_1 dy) + A_{\mathbf{y}} d\mathbf{y}. \quad (2.57)$$

Here, for convenience, we introduced the one-dimensional coordinate $\mathbf{y} \equiv \beta_2 y$, with $\mathbf{y} \sim \mathbf{y} + 2\pi\beta_2$. The 1d vector multiplet consists of the fields:

$$\mathcal{V}_F^{(1d)} = (A_{\mathbf{y}}, \sigma, \lambda, \tilde{\lambda}, D), \quad (2.58)$$

with the supersymmetry variations:

$$\begin{aligned} \delta_\zeta A_{\mathbf{y}} &= \zeta \tilde{\lambda}, & \delta_{\tilde{\zeta}} A_{\mathbf{y}} &= \tilde{\zeta} \lambda, \\ \delta_\zeta \sigma &= -i\zeta \tilde{\lambda}, & \delta_{\tilde{\zeta}} \sigma &= -i\tilde{\zeta} \lambda, \\ \delta_\zeta \lambda &= i\zeta(D + \partial_{\mathbf{y}}\sigma), & \delta_{\tilde{\zeta}} \lambda &= 0, \\ \delta_\zeta \tilde{\lambda} &= 0, & \delta_{\tilde{\zeta}} \tilde{\lambda} &= -i\tilde{\zeta}(D - \partial_{\mathbf{y}}\sigma), \\ \delta_\zeta D &= \zeta \partial_{\mathbf{y}} \tilde{\lambda}, & \delta_{\tilde{\zeta}} D &= -2\tilde{\zeta} \partial_{\mathbf{y}} \lambda. \end{aligned} \quad (2.59)$$

This directly follows from (2.34).

One-dimensional local functional. The functional \mathbb{W}_{1d} is constrained by the requirement that its gauge variation (under $\delta_\alpha A_{\mathbf{y}} = \partial_{\mathbf{y}}\alpha$) should reproduce the dimensional reduction of the U(1) anomaly:

$$\delta_\alpha \mathbb{W}_{1d} = \frac{i\mathcal{A}_{qq}\beta_1}{2} \int d\mathbf{y} \alpha (-\partial_{\mathbf{y}}\sigma) = \delta_\alpha \left(\frac{i\mathcal{A}_{qq}\beta_1}{2} \int d\mathbf{y} A_{\mathbf{y}} \sigma \right). \quad (2.60)$$

It should also reproduce the 1d reduction of the supersymmetry variations (2.46), namely:

$$\delta_\zeta \mathbb{W}_{1d} = -\frac{i\mathcal{A}_{qq}\beta_1}{2} \int \partial_{\mathbf{y}} \zeta \tilde{\lambda} (\sigma + iA_{\mathbf{y}}), \quad \delta_{\tilde{\zeta}} \mathbb{W}_{1d} = -\frac{i\mathcal{A}_{qq}\beta_1}{2} \int \partial_{\mathbf{y}} \tilde{\zeta} \lambda (\sigma + iA_{\mathbf{y}}). \quad (2.61)$$

This fixes \mathbb{W}_{1d} to be:

$$\mathbb{W}_{1d} = \frac{i\mathcal{A}_{qq}\beta_1}{2} \int d\mathbf{y} (A_{\mathbf{y}} \sigma - i\sigma^2), \quad (2.62)$$

up to terms of dimension two that are gauge-invariant and supersymmetric. Such terms vanish on the supersymmetric locus ($D = \partial_{\mathbf{y}}\sigma = 0$). We can therefore conclude that, in the limit (2.53), the order- β_1 term in the expansion of the T^2 supersymmetric partition function is fully determined by (2.62).

2.5 The supersymmetric T^2 partition function, revisited

Consider the elliptic genus of a GLSM, as given explicitly in (2.15)–(2.16). We can consider its small- β_1 limit explicitly. One finds:

$$-\log \mathcal{I}(\nu, \tau) \underset{\tau \rightarrow i\infty}{\sim} \tau \frac{\pi i k_g}{6} + W_{1d}^{(0)} + \mathcal{O}(e^{2\pi i \tau}). \quad (2.63)$$

In other words, there is no order- β_1 contribution appearing in the small- β_1 expansion of the “holomorphic partition function.” This was also observed recently in [54].

By contrast, the order- β_1 term in the expansion of the partition function Z_{T^2} is fully captured by (2.62), which evaluates to:¹⁹

$$\mathbb{W}_{1d} = \pi \mathcal{A}_{qq} \beta_1 \beta_2 \sigma(\sigma + iA_y) = \pi \mathcal{A}_{qq} \frac{\nu \bar{\nu} - \nu^2}{2\tau_2}. \quad (2.64)$$

Of course, this satisfies the quantum Ward identity (2.50). Incidentally, we note that \mathbb{W}_{1d} is holomorphic in τ when expressed in terms of the 2d variables a_x and a_y :

$$\mathbb{W}_{1d} = -\pi i \mathcal{A}_{qq} a_x (\tau a_x - a_y). \quad (2.65)$$

Based on these considerations, we make a conjecture for the exact form of the “gauge invariant” supersymmetric T^2 partition function, for any 2d $\mathcal{N} = (0, 2)$ theory. It is simply given by:

$$Z_{T^2}(a_x, a_y, \tau) = e^{-\mathbb{W}_{1d}(a_x, a_y, \tau)} \mathcal{I}(\nu, \tau), \quad (2.66)$$

with \mathcal{I} the elliptic genus, which is holomorphic in $\nu = \tau a_x - a_y$ and $\tau = \tau_1 + i\tau_2$, and with the prefactor \mathbb{W}_{1d} given by:

$$\mathbb{W}_{1d}(a_x, a_y, \tau) = -\frac{\pi i}{2} \mathcal{A}^{\alpha\beta} (2\tau a_{x,\alpha} a_{x,\beta} - a_{x,\alpha} a_{y,\beta} - a_{y,\alpha} a_{x,\beta}). \quad (2.67)$$

By construction, this is fully consistent with gauge invariance and supersymmetry, with the ’t Hooft anomalies properly taken into account. Note that its real part is given by (2.26), as expected.

Large gauge transformations. Under large gauge transformations of the background gauge fields on T^2 :

$$(a_{x,\alpha}, a_{y,\alpha}) \sim (a_{x,\alpha} + m_\alpha, a_{y,\alpha} - n_\alpha), \quad n_\alpha, m_\alpha \in \mathbb{Z}, \quad (2.68)$$

we obtain:

$$\begin{aligned} Z_{T^2}(a_x + m, a_y - n, \tau) &= (-1)^{\mathcal{A}_q^\alpha (n_\alpha + m_\alpha)} \\ &\times e^{\pi i \mathcal{A}^{\alpha\beta} (n_\alpha a_{x,\beta} + m_\alpha a_{y,\beta} + n_\alpha m_\beta)} Z_{T^2}(a_x, a_y, \tau), \end{aligned} \quad (2.69)$$

to be compared with (2.22). In particular, the absolute value of Z_{T^2} is gauge invariant. This behavior agrees with general expectations for gauge anomalies on the torus. As explained in appendix D.1, general considerations determine the behavior of any flavored T^2 partition function under large gauge transformations in terms of the ’t Hooft anomaly coefficients, modulo various ambiguities that depend on the renormalization scheme. These general considerations are consistent with the explicit result (2.69).

¹⁹Note that $\nu = i\beta_2(\sigma + iA_y)$ in the 1d variables.

Modular transformations. One can also check that the partition function (2.66) transforms naturally under $SL(2, \mathbb{Z})$. Using (2.29), one finds the simple result:

$$Z_{T^2}(a_x, a_y + a_x, \tau + 1) = e^{-\frac{\pi i}{6} k_g} Z_{T^2}(a_x, a_y, \tau), \quad (2.70)$$

for the T transformation, and:

$$Z_{T^2}\left(a_y, -a_x, -\frac{1}{\tau}\right) = e^{\frac{\pi i}{2} k_g} Z_{T^2}(a_x, a_y, \tau), \quad (2.71)$$

for the S transformation. We therefore see that the anomalous behavior of Z_{T^2} under large diffeomorphisms is determined by the gravitational anomaly only, as one would have naively expected on physical ground; when $k_g = 0$, the supersymmetric partition function is modular invariant. This has to be contrasted with the behavior of the holomorphic elliptic genus under $SL(2, \mathbb{Z})$, which is given by (2.29). In fact, we again find perfect agreement with general constraints on Z_{T^2} imposed by the anomalies, as explained in appendix D.1.

Local counterterm and holomorphy. We have seen that the non-holomorphic dependence of the T^2 supersymmetric partition function is dictated by the 't Hooft anomalies, as in (2.50). Namely, we have:

$$\frac{\partial}{\partial \bar{\nu}_\alpha} W(\nu, \bar{\nu}) = \frac{\pi}{2\tau_2} \mathcal{A}_{qq}^{\alpha\beta} \nu_\beta, \quad (2.72)$$

for an arbitrary abelian flavor symmetry, $G_F = \prod_\alpha U(1)_\alpha$, with 't Hooft anomaly coefficients $\mathcal{A}_{qq}^{\alpha\beta}$. Given the explicit form of the supersymmetric Ward identity (2.46), it is clear that one could cancel this anomaly — and, therefore, restore the holomorphy of the supersymmetric partition function in ν — by adding the following local term to the effective action:

$$W_{\text{ct}} = -\frac{\mathcal{A}_{qq}}{8\pi} \int d^2x \sqrt{g} a_\mu a^\mu. \quad (2.73)$$

Thus, there certainly exists a scheme where the holomorphy in ν is manifest, corresponding to the partition function (2.52).²⁰ The counterterm (2.73) is not gauge invariant, however. Note also that, since:

$$\delta_\zeta(W + W_{\text{ct}}) = 0, \quad \delta_{\bar{\zeta}}(W + W_{\text{ct}}) = 0, \quad (2.74)$$

it follows from (2.45) that $\delta_{\alpha(a)}(W + W_{\text{ct}}) = 0$. This does not mean that we have cancelled the gauge anomaly, however, only that this counterterm also cancelled the $U(1)$ anomaly under a *particular* gauge transformation with parameter $\alpha = \alpha(a)$.

In conclusion, the holomorphy of the T^2 supersymmetric partition function in ν can only be maintained, by adding the counterterm (2.73), at the cost of violating gauge invariance for background gauge fields. (Note that (2.73) violates explicitly the gauge-invariance of the *real part* of W .)

²⁰In this scheme, we would still not have holomorphy in τ . To recover the fully holomorphic elliptic genus $\mathcal{I}(\nu, \tau)$, we should add an additional counterterm corresponding to the prefactor in (2.52), which breaks diffeomorphism invariance explicitly. We leave a better understanding of this point for future work.

2.6 Anomalous supersymmetry variation from WZ gauge-fixing

As explained in the introduction, we can understand the anomalous supersymmetry variation, $\delta_\zeta W \neq 0$, as a simple consequence of fixing the WZ gauge. Let us show this in more detail. Here, we will use a superspace notation, for convenience.

WZ gauge-fixing. The $\mathcal{N} = (0, 2)$ abelian background vector multiplet can be written in terms of two superfields:

$$\begin{aligned} \mathcal{V} &= C + i\theta^+ \chi_+ + i\tilde{\theta}^+ \tilde{\chi}_+ + 2\theta^+ \tilde{\theta}^+ a_{\bar{z}}, \\ \mathcal{V}_w &= a_w + i\theta^+ (\tilde{\lambda}_- - i\partial_w \chi_+) + i\tilde{\theta}^+ (\lambda_- + i\partial_w \tilde{\chi}_+) - \theta^+ \tilde{\theta}^+ (D + 2\partial_w \partial_{\bar{w}} C). \end{aligned} \quad (2.75)$$

The U(1) gauge symmetry, $\delta_\alpha a_\mu = \partial_\mu \alpha$, can then be supersymmetrized by:

$$\delta_\Omega \mathcal{V} = \frac{i}{2} (\Omega - \tilde{\Omega}), \quad \delta_\Omega \mathcal{V}_w = \frac{1}{2} \partial_w (\Omega + \tilde{\Omega}), \quad (2.76)$$

where the gauge parameters sit in the chiral and anti-chiral superfields:

$$\begin{aligned} \Omega &= \omega + \sqrt{2}\theta^+ \psi_+^\Omega - 2i\theta^+ \tilde{\theta}^+ \partial_{\bar{w}} \omega + \dots, \\ \tilde{\Omega} &= \tilde{\omega} - \sqrt{2}\tilde{\theta}^+ \tilde{\psi}_+^\Omega + 2i\theta^+ \tilde{\theta}^+ \partial_{\bar{w}} \tilde{\omega} + \dots \end{aligned} \quad (2.77)$$

Using this larger gauge freedom, one can fix the WZ gauge:

$$C = \chi_+ = \tilde{\chi}_+ = 0. \quad (2.78)$$

The WZ gauge (2.78) is not invariant under supersymmetry, because:

$$\delta \chi_+ = 2i\tilde{\zeta}_- a_{\bar{z}}, \quad \delta \tilde{\chi}_+ = -2i\zeta_- a_{\bar{z}}. \quad (2.79)$$

Nonetheless, one can define consistent supersymmetry transformations, which we denote by δ_ζ , $\delta_{\tilde{\zeta}}$, by performing a compensating gauge transformation that restores the WZ gauge:

$$\delta_\zeta = \delta_\zeta^{(0)} + \delta_{\Omega(\zeta)}, \quad \delta_{\tilde{\zeta}} = \delta_{\tilde{\zeta}}^{(0)} + \delta_{\tilde{\Omega}(\tilde{\zeta})}. \quad (2.80)$$

Here, $\delta_{\zeta, \tilde{\zeta}}^{(0)}$ is the “bare” supersymmetry transformation before gauge fixing, and $\delta_{\Omega(\zeta)}$, $\delta_{\tilde{\Omega}(\tilde{\zeta})}$ are compensating gauge transformations with field-dependent gauge parameters:

$$\begin{aligned} \Omega(\zeta) : \quad \omega &= \psi_+^\Omega = 0, & \tilde{\omega} &= 0, \quad \tilde{\psi}_+^\Omega = 2\sqrt{2}i\zeta_- a_{\bar{z}}, \\ \Omega(\tilde{\zeta}) : \quad \omega &= 0, \quad \psi_+^\Omega = -2\sqrt{2}i\tilde{\zeta}_- a_{\bar{z}}, & \tilde{\omega} &= \tilde{\psi}_+^\Omega = 0, \end{aligned} \quad (2.81)$$

respectively.

't Hooft anomaly and supersymmetry. The 't Hooft anomaly for the background U(1) symmetry (2.41) can be supersymmetrized as:

$$\delta_\Omega W = \frac{\mathcal{A}_{qq}}{8\pi} \int d^2x \sqrt{g} \left[\int d\theta^+ \Omega \mathcal{Y}_- + \int d\tilde{\theta}^+ \tilde{\Omega} \tilde{\mathcal{Y}}_- \right], \quad (2.82)$$

where \mathcal{Y}_- and $\tilde{\mathcal{Y}}_-$ are the gauge-invariant gaugino multiplets:²¹

$$\mathcal{Y}_- = \tilde{D}_+(\partial_w \mathcal{V} + i\mathcal{V}_w), \quad \tilde{\mathcal{Y}}_- = D_+(\partial_w \mathcal{V} - i\mathcal{V}_w). \quad (2.83)$$

In components, this reads:

$$\delta_\Omega W = \frac{\mathcal{A}_{qq}}{8\pi} \int d^2x \sqrt{g} \left(2(\omega + \tilde{\omega}) f_{w\bar{w}} - i(\omega - \tilde{\omega}) D + \sqrt{2}(\psi_+^\Omega \lambda_- - \tilde{\psi}_+^\Omega \tilde{\lambda}_-) \right), \quad (2.84)$$

where we identify $\alpha = \frac{1}{2}(\omega + \tilde{\omega})$ as the ordinary U(1) gauge parameter. The expression (2.82) is fully supersymmetric with respect to the “bare” supersymmetry variation:

$$\delta_\zeta^{(0)} \left(\delta_\Omega W \right) = 0. \quad (2.85)$$

On the other hand, we have the WZ consistency condition:

$$\left[\delta_\zeta^{(0)}, \delta_\Omega \right] W = 0. \quad (2.86)$$

From the last two equations, we have $\delta_\Omega \delta_\zeta^{(0)} W = 0$, which implies that:

$$\delta_\zeta^{(0)} W = 0. \quad (2.87)$$

In other words, the U(1) ’t Hooft anomaly is compatible with supersymmetry and there is no genuine “supersymmetry anomaly.” Then, by its definition (2.80), the supersymmetry *in the Wess-Zumino gauge* does have an anomalous variation, namely:

$$\delta_\zeta W = \delta_{\Omega(\zeta)} W, \quad \delta_{\tilde{\zeta}} W = \delta_{\Omega(\tilde{\zeta})} W. \quad (2.88)$$

Plugging the values (2.81) inside (2.84), one finds:

$$\delta_\zeta W = -\frac{i\mathcal{A}_{qq}}{2\pi} \int d^2x \zeta_- \tilde{\lambda}_- a_{\bar{w}}, \quad \delta_{\tilde{\zeta}} W = -\frac{i\mathcal{A}_{qq}}{2\pi} \int d^2x \tilde{\zeta}_- \lambda_- a_{\bar{w}}. \quad (2.89)$$

This indeed reproduces the anomalous supersymmetry variations (2.46).

3 ’t Hooft anomalies and 4d $\mathcal{N} = 1$ partition functions

Let us now consider 4d $\mathcal{N} = 1$ supersymmetric theories. In this section, after reviewing the curved-space rigid supersymmetry formalism [25, 26] and the main result of [6] on the holomorphic dependence of supersymmetric partition functions on continuous parameters, we derive the supersymmetric Ward identities on a general half-BPS background. This determines the non-holomorphic dependence of the supersymmetric partition function, in our “gauge invariant” scheme for background gauge fields. In the case of half-BPS four-manifolds of topology $\mathcal{M}_3 \times S^1$, with S^1 a circle of radius β_{S^1} , we also discuss how the ’t Hooft anomaly together with supersymmetry determine the order- β_{S^1} term in the small- β_{S^1} expansion of the partition function. Our four-dimensional conventions are summarized in appendix A.1.

²¹The SUSY-covariant derivatives are given by $D_+ = \partial_+ - 2i\tilde{\theta}^+ \partial_{\bar{w}}$ and $\tilde{D}_+ = -\tilde{\partial}_+ + 2i\theta^+ \partial_{\bar{w}}$.

3.1 Curved-space supersymmetry for 4d $\mathcal{N} = 1$ theories

Consider a four-dimensional $\mathcal{N} = 1$ supersymmetric theory with a $U(1)_R$ symmetry. By assumption, there exists an \mathcal{R} -multiplet, which contains the R -symmetry current j_R^μ together with the energy-momentum tensor $T^{\mu\nu}$, and which can be consistently coupled to four-dimensional new-minimal supergravity [55] — see e.g. [56, 57]. At the linearized level:

$$\Delta\mathcal{L} = -\frac{1}{2}\Delta g_{\mu\nu}T^{\mu\nu} + A_\mu^{(R)}j_R^\mu + \Psi_\mu^\alpha S_\alpha^\mu + \tilde{\Psi}_{\dot{\alpha}\mu}\tilde{S}^{\dot{\alpha}\mu} + \frac{i}{4}\epsilon^{\mu\nu\rho\lambda}B_{\mu\nu}\mathcal{F}_{\rho\lambda}. \quad (3.1)$$

We consider a curved-space rigid supersymmetric background on a compact four-manifold \mathcal{M}_4 , which can be obtained as a rigid limit of new-minimal supergravity, with the background fields:

$$g_{\mu\nu}, \quad A_\mu^{(R)}, \quad V^\mu = \frac{i}{2}\epsilon^{\mu\nu\rho\lambda}\partial_\nu B_{\rho\lambda}, \quad (3.2)$$

such that the gravitino and its supersymmetry variations vanish [25]:

$$\Psi_\mu = \tilde{\Psi}_\mu = 0, \quad \delta\Psi_\mu = \delta\tilde{\Psi}_\mu = 0. \quad (3.3)$$

This means that there exists spinors ζ_α and/or $\tilde{\zeta}^{\dot{\alpha}}$ on the Riemannian manifold $(\mathcal{M}_4, g_{\mu\nu})$ that satisfy the generalized Killing equations:

$$(\nabla_\mu - iA_\mu^{(R)})\zeta = -\frac{i}{2}V^\nu\sigma_\mu\tilde{\sigma}_\nu\zeta, \quad (\nabla_\mu + iA_\mu^{(R)})\tilde{\zeta} = \frac{i}{2}V^\nu\tilde{\sigma}_\mu\sigma_\nu\tilde{\zeta}. \quad (3.4)$$

The Killing spinors are globally defined and nowhere-vanishing. Their existence allows us to define curved-space rigid supersymmetries, with the supersymmetry variations on fields written as:

$$\delta_\zeta = i\zeta\mathcal{Q}, \quad \delta_{\tilde{\zeta}} = i\tilde{\zeta}\tilde{\mathcal{Q}}, \quad (3.5)$$

where \mathcal{Q}_α and $\tilde{\mathcal{Q}}^{\dot{\alpha}}$ denote the “curved-space supercharges” (one for each Killing spinor). Note that the Killing spinors are bosonic and of R -charge ± 1 while the supercharges have R -charge ∓ 1 , so that δ is a fermionic operation of vanishing R -charge.

3.1.1 Complex structure and background supergravity fields

Given a Killing spinor ζ , the real bilinear tensor:

$$J^\mu{}_\nu = -2i\frac{\zeta^\dagger\sigma^\mu{}_\nu\zeta}{|\zeta|^2}, \quad (3.6)$$

defines an integrable complex structure on \mathcal{M}_4 [26, 27]. One can preserve a single supercharge on any Hermitian four-manifold. Given an Hermitian manifold,

$$(\mathcal{M}_4, g_{\mu\nu}, J^\mu{}_\nu), \quad (3.7)$$

one can explicitly solve for ζ and for the remaining supergravity background fields:

$$V_\mu = \frac{1}{2}\nabla_\nu J^\nu{}_\mu + U_\mu, \quad A_\mu^{(R)} = \hat{A}_\mu^{(R)} + \frac{i}{4}J_\mu{}^\nu\nabla_\rho J^\rho{}_\nu, \quad (3.8)$$

with U_μ satisfying $U_\nu J^\nu{}_\mu = iU_\mu$ and $\nabla_\mu U^\mu = 0$. Here, for later reference, we defined:

$$\hat{A}_\mu^{(R)} = A_\mu^c - \frac{1}{2}\nabla_\nu J^\nu{}_\mu, \quad A_\mu^c \equiv \frac{1}{4}(\partial_\nu \log \sqrt{g})J^\nu{}_\mu + \partial_\mu s. \quad (3.9)$$

The expression for A_μ^c is only valid in complex coordinates adapted to the complex structure; the function s in (3.9) is a $U(1)_R$ gauge transformation parameter. We will use the notation X^μ ($\mu = 1, \dots, 4$) for the four-dimensional real coordinates, and:

$$w = w(X), \quad z = z(X), \quad (3.10)$$

for the complex coordinates w, z on \mathcal{M}_4 .

3.1.2 Half-BPS Hermitian geometry

In the following, our main interest will be in half-BPS supersymmetric backgrounds which preserve two Killing spinors of opposite chiralities, ζ and $\tilde{\zeta}$. The second Killing spinor, $\tilde{\zeta}$, defines a second complex structure:

$$\tilde{J}^\mu{}_\nu = -2i \frac{\tilde{\zeta}^\dagger \tilde{\sigma}^\mu{}_\nu \tilde{\zeta}}{|\tilde{\zeta}|^2}, \quad (3.11)$$

and a complex Killing vector:

$$K^\mu = \zeta \sigma^\mu \tilde{\zeta}, \quad \nabla_\mu K_\nu + \nabla_\nu K_\mu = 0. \quad (3.12)$$

We will assume that K commutes with its complex conjugate, $[K, \bar{K}] = 0$. Then, the background geometry admits two commuting complex structures, $[J, \tilde{J}] = 0$, and K is anti-holomorphic with respect to both:

$$J^\mu{}_\nu K^\nu = -iK^\mu, \quad \tilde{J}^\mu{}_\nu K^\nu = -iK^\mu. \quad (3.13)$$

We choose the holomorphic coordinates (3.10) such that:

$$K = \frac{2}{\beta_1} \partial_{\bar{w}}, \quad (3.14)$$

with $\beta_1 > 0$ a real positive constant introduced for later convenience. The holomorphic coordinates adapted to the complex structure \tilde{J} are (w, \bar{z}) . The background geometry is locally a T^2 fibration,²² and the Hermitian metric (with respect to J) can be written in the canonical form [26]:

$$ds^2(\mathcal{M}_4) = \Omega(z, \bar{z})(dw + h(z, \bar{z})dz)(d\bar{w} + \bar{h}(z, \bar{z})d\bar{z}) + 2g_{z\bar{z}}(z, \bar{z})dzd\bar{z}. \quad (3.15)$$

The factor $\Omega(z, \bar{z})$ is related to the norm of the Killing vector, and of the Killing spinors, as:

$$|K|^2 \equiv K_\mu \bar{K}^\mu = 2|\zeta|^2 |\tilde{\zeta}|^2 = 2\beta_1^{-2} \Omega^2. \quad (3.16)$$

²²This is also true globally for \mathcal{M}_4 as a topological manifold, but the T^2 fibers need not coincide with the orbits of K^μ , as the latter need not close. When they do, we have an elliptic fibration over an Riemann surface orbifold. (See e.g. [58] for a thorough discussion of the 3d analogue.)

For simplicity (and without much loss of generality), we will restrict ourselves to the case $\Omega = \beta_1$, so that $|K|^2 = 2$.

The supergravity background fields V_μ and $A_\mu^{(R)}$ are still given by (3.8), with the additional constraint:

$$U_\mu = \kappa K_\mu, \quad K^\mu \partial_\mu \kappa = 0. \quad (3.17)$$

The parameter κ is part of the definition of the supersymmetric background, and it can be chosen arbitrarily. In the following, we will mostly be interested in the case of a product four-manifold $\mathcal{M}_4 \cong \mathcal{M}_3 \times S^1$, and we will require that the 4d $\mathcal{N} = 1$ supersymmetric background admits a consistent reduction to a 3d $\mathcal{N} = 2$ supersymmetric background on \mathcal{M}_3 [42]. This will fix κ uniquely.

Topological-holomorphic twist. In the following adapted complex frame:²³

$$\begin{aligned} e^1 &= \beta_1(dw + h(z, \bar{z})dz), & e^2 &= \sqrt{2g_{z\bar{z}}(z, \bar{z})}dz, \\ e^{\bar{1}} &= \beta_1(d\bar{w} + \bar{h}(z, \bar{z})d\bar{z}), & e^{\bar{2}} &= \sqrt{2g_{z\bar{z}}(z, \bar{z})}d\bar{z}, \end{aligned} \quad (3.18)$$

the Killing spinors take the particularly simple form:

$$\zeta_\alpha = e^{is} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \tilde{\zeta}^{\dot{\alpha}} = e^{-is} \begin{pmatrix} -1 \\ 0 \end{pmatrix}. \quad (3.19)$$

Let us define the projectors onto holomorphic vectors in either complex structure:

$$\Pi^\mu{}_\nu = \frac{1}{2}(\delta^\mu{}_\nu - iJ^\mu{}_\nu), \quad \tilde{\Pi}^\mu{}_\nu = \frac{1}{2}(\delta^\mu{}_\nu - i\tilde{J}^\mu{}_\nu), \quad (3.20)$$

and similarly for the anti-holomorphic projectors:

$$\bar{\Pi}^\mu{}_\nu = \frac{1}{2}(\delta^\mu{}_\nu + iJ^\mu{}_\nu), \quad \tilde{\bar{\Pi}}^\mu{}_\nu = \frac{1}{2}(\delta^\mu{}_\nu + i\tilde{J}^\mu{}_\nu). \quad (3.21)$$

Since the two complex structures commute, we can project any vector X^μ into its four components $X^w, X^{\bar{w}}, X^z, X^{\bar{z}}$, respectively, and similarly for any tensor. One can also define an adapted connection, $\hat{\nabla}$, that preserves *both* complex structures:

$$\hat{\nabla}_\mu g_{\nu\rho} = 0, \quad \hat{\nabla}_\mu J^\nu{}_\rho = 0, \quad \hat{\nabla}_\mu \tilde{J}^\nu{}_\rho = 0. \quad (3.22)$$

It is defined by:

$$\hat{\Gamma}^\mu{}_{\nu\rho} = \Gamma^\mu{}_{\nu\rho} + K^\mu{}_{\nu\rho} \quad (3.23)$$

with $\Gamma^\mu{}_{\nu\rho}$ the Levi-Civita connection, and:

$$K_{\mu\nu\rho} = \frac{1}{2}J_\mu{}^\alpha J_\nu{}^\beta J_\rho{}^\gamma (dJ)_{\alpha\beta\gamma}, \quad (3.24)$$

the contorsion tensor, with $(dJ)_{\mu\nu\rho} \equiv \nabla_\mu J_{\nu\rho} + \nabla_\nu J_{\rho\mu} + \nabla_\rho J_{\mu\nu}$.²⁴ This connection has non-trivial torsion:

$$T^\mu{}_{\nu\rho} = K^\mu{}_{\nu\rho} - K^\mu{}_{\rho\nu} = \epsilon^\mu{}_{\nu\rho\lambda} J^\lambda{}_\sigma \nabla_\kappa J^{\kappa\sigma}. \quad (3.25)$$

²³Our conventions for spinors are given in appendix A.1. Here and henceforth, we set $\Omega = \beta_1$.

²⁴The expression for $K_{\mu\nu\rho}$ takes the same form when written in terms of $\tilde{J}^\mu{}_\nu$.

In terms of the adapted connection, the Killing spinor equations take the simple form:

$$(\hat{\nabla}_\mu - i\hat{A}_\mu^{(R)})\zeta = 0, \quad (\hat{\nabla}_\mu + i\hat{A}_\mu^{(R)})\tilde{\zeta} = 0, \quad (3.26)$$

with $\hat{A}_\mu^{(R)}$ defined in (3.9). The connection $\hat{\nabla}$ has a U(1) holonomy which can be “twisted” by the $U(1)_R$ gauge field. Whenever the orbits of K^μ close, we have a proper elliptic fibration over a two-dimensional complex space Σ and the 4d $\mathcal{N} = 1$ supersymmetric background is precisely the four-dimensional uplift of the topological A -twist on Σ [26, 59]. The half-BPS supersymmetric background (3.8)–(3.15) can then be understood as a topological-holomorphic twist — that is, holomorphic along the fiber direction [13] (with coordinate w) and topological along the base (with coordinate z).

3.2 Background vector multiplet and minimal coupling

Consider an $\mathcal{N} = 1$ theory with a flavor symmetry group G_F , and let us introduce a background vector multiplet \mathcal{V}_F for that symmetry. Its component fields are:

$$\mathcal{V}_F = (a_\mu, \lambda_\alpha, \tilde{\lambda}^{\dot{\alpha}}, D), \quad (3.27)$$

with $\lambda, \tilde{\lambda}$ a “background” gaugino. Note again that we fixed the Wess-Zumino gauge for \mathcal{V}_F , and work only in terms of the remaining “physical” components (3.27). The supersymmetry variations in WZ gauge [23] are famously modified by including a compensating gauge transformation into the supersymmetry transformation, to preserve the WZ gauge. This implies that the supersymmetry algebra itself become gauge-covariant under G_F .

Let us then consider the background vector multiplet (3.27) in addition to the supersymmetric curved-space background on \mathcal{M}_4 . Note that, while we require the supergravity background fields to be fixed to definite values so that (3.3) holds, as described above, we do not impose any requirement on \mathcal{V}_F at this point — we will need to consider arbitrary background values of the fields (3.27), including for the gaugino.

Given two Killing spinors ζ and $\tilde{\zeta}$, we have two independent supersymmetries on \mathcal{M}_4 , denoted by δ_ζ and $\delta_{\tilde{\zeta}}$, respectively, satisfying the following curved-space supersymmetry algebra [13]:

$$\begin{aligned} \delta_\zeta^2 \Phi_{(s,r)} &= \delta_{\tilde{\zeta}}^2 \Phi_{(s,r)} = 0, \\ \{\delta_\zeta, \delta_{\tilde{\zeta}}\} \Phi_{(s,r)} &= 2i \left(\mathcal{L}_K - irK^\mu \left(A_\mu^{(R)} + \frac{3}{2}V_\mu \right) - iK^\mu a_\mu \right) \Phi_{(s,r)}, \end{aligned} \quad (3.28)$$

on any field $\Phi_{(s,r)}$ of spin s , R -charge r and arbitrary flavor charges. Here, \mathcal{L}_K denotes the Lie derivative along the Killing vector K^μ , and the background gauge field a_μ acts in the appropriate G_F representation.

The curved-space rigid supersymmetry transformations for the vector multiplet (3.27) are given by:

$$\begin{aligned} \delta_\zeta a_\mu &= i\zeta \sigma_\mu \tilde{\lambda}, & \delta_{\tilde{\zeta}} a_\mu &= i\tilde{\zeta} \tilde{\sigma}_\mu \lambda, \\ \delta_\zeta \lambda &= i\zeta D + \sigma^{\mu\nu} \zeta f_{\mu\nu}, & \delta_{\tilde{\zeta}} \lambda &= 0, \\ \delta_\zeta \tilde{\lambda} &= 0, & \delta_{\tilde{\zeta}} \tilde{\lambda} &= -i\tilde{\zeta} D + \tilde{\sigma}^{\mu\nu} \tilde{\zeta} f_{\mu\nu}, \\ \delta_\zeta D &= -D_\mu (\zeta \sigma^\mu \tilde{\lambda}) + 2iV_\mu \zeta \sigma^\mu \tilde{\lambda}, & \delta_{\tilde{\zeta}} D &= D_\mu (\tilde{\zeta} \tilde{\sigma}^\mu \lambda) + 2iV_\mu \tilde{\zeta} \tilde{\sigma}^\mu \lambda, \end{aligned} \quad (3.29)$$

in the WZ gauge. Here, we defined the field strength

$$f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu - i[a_\mu, a_\nu], \quad (3.30)$$

and the covariant derivatives D_μ are understood to be appropriately gauge covariant, including under $U(1)_R$ — for instance, $D_\mu \lambda = \nabla_\mu \lambda - iA_\mu^{(R)} \lambda - i[a_\mu, \lambda]$.

Coupling to an abelian conserved current. Let us focus on the case of an abelian G_F , for simplicity. In a supersymmetric theory, any $U(1)$ conserved current operator j^μ (with the conservation equation $\nabla_\mu j^\mu = 0$ on \mathcal{M}_4), sits in a linear multiplet:

$$\mathcal{J} = (J, j^\mu, j_\alpha, \tilde{j}^{\dot{\alpha}}). \quad (3.31)$$

The supersymmetry variations are:

$$\begin{aligned} \delta_\zeta J &= i\zeta j, & \delta_{\tilde{\zeta}} J &= i\tilde{\zeta} \tilde{j}, \\ \delta_\zeta j_\alpha &= 0, & \delta_{\tilde{\zeta}} j_\alpha &= (\sigma^\mu \tilde{\zeta})_\alpha (\partial_\mu J - ij_\mu - 2iV_\mu J), \\ \delta_\zeta \tilde{j}^{\dot{\alpha}} &= -(\tilde{\sigma}^\mu \zeta)^{\dot{\alpha}} (\partial_\mu J + ij_\mu + 2iV_\mu J), & \delta_{\tilde{\zeta}} \tilde{j}^{\dot{\alpha}} &= 0, \\ \delta_\zeta j_\mu &= -2\nabla^\nu (\zeta \sigma_{\mu\nu} j), & \delta_{\tilde{\zeta}} j_\mu &= 2\nabla^\nu (\tilde{\zeta} \tilde{\sigma}_{\mu\nu} \tilde{j}). \end{aligned} \quad (3.32)$$

We then have the supersymmetric minimal coupling:

$$S_{\mathcal{V}\mathcal{J}}[\mathcal{V}_F] = \int d^4x \sqrt{g} \mathcal{L}_{\mathcal{V}\mathcal{J}}[\mathcal{V}_F], \quad \mathcal{L}_{\mathcal{V}\mathcal{J}}[\mathcal{V}_F] \equiv a_\mu j^\mu + DJ - \lambda j - \tilde{\lambda} \tilde{j}. \quad (3.33)$$

In the following, we will consider small variations of the background vector multiplet around a fixed value, $\mathcal{V}_F^{(0)}$, denoted by $\Delta\mathcal{V}_F$:

$$\mathcal{V} = \mathcal{V}_F^{(0)} + \Delta\mathcal{V}_F. \quad (3.34)$$

On any supersymmetric background \mathcal{M}_4 , the first-order correction to the effective action upon a small change of the sources is captured by the expectation value of the minimal coupling (3.33).

Supersymmetric background. A supersymmetric background for the vector multiplet is a bosonic configuration (a_μ, D) such that the gaugino and its variations vanish:

$$\lambda = \tilde{\lambda} = 0, \quad \delta_\zeta \lambda = 0, \quad \delta_{\tilde{\zeta}} \tilde{\lambda} = 0. \quad (3.35)$$

On our half-BPS geometry, this corresponds to:

$$f_{w\bar{w}} = 0, \quad f_{\bar{w}\bar{z}} = 0, \quad f_{\bar{w}z} = 0, \quad D = -\frac{1}{2} J^{\mu\nu} f_{\mu\nu}. \quad (3.36)$$

We will consider *real* background gauge fields a_μ on $\mathcal{M}_{g,p} \times S^1$, in which case the only non-vanishing gauge-field curvature component is $f_{z\bar{z}}$. The background gauge field is the connection of a holomorphic line bundle over \mathcal{M}_4 [6], which we denote by \mathbf{L}_F .

3.3 \mathcal{M}_4 supersymmetric partition functions and holomorphy

Consider any 4d $\mathcal{N} = 1$ supersymmetric theory with a $U(1)_R$ symmetry. Its supersymmetric partition function on \mathcal{M}_4 , denoted by $Z_{\mathcal{M}_4}$, is the path integral on a fixed supersymmetric background for the geometry and for the vector multiplet \mathcal{V}_F :

$$\mathbf{L}_F \rightarrow \mathcal{M}_4 : (g_{\mu\nu}, A_\mu^{(R)}, V_\mu; a_\mu, D). \tag{3.37}$$

Naively, the partition function may depend on all the background fields shown in (3.37). However, it was argued in [6] that, given an Hermitian four-manifold \mathcal{M}_4 and a choice of flavor line bundle \mathbf{L}_F , the partition function $Z_{\mathcal{M}_4}$ only depends on two types of *continuous* complex parameters:

$$Z_{\mathcal{M}_4}(\nu, \tau). \tag{3.38}$$

Here, we have:²⁵

- (i) the “geometric” parameters τ , which are *complex structure moduli* of the complex manifold \mathcal{M}_4 ;
- (ii) the flavor parameters ν , which are *holomorphic line bundle moduli* of the line bundles \mathbf{L}_F . (That is, for abelian symmetries; they are holomorphic vector bundle moduli in general.)

In particular, for any fixed complex structure on \mathcal{M}_4 , the supersymmetric partition function is independent of the choice of Hermitian metric. Moreover, the partition function was found to be *holomorphic* in all of its continuous parameters:

$$\frac{\partial}{\partial \bar{\nu}} Z_{\mathcal{M}_4}(\nu, \tau) = 0, \quad \frac{\partial}{\partial \bar{\tau}} Z_{\mathcal{M}_4}(\nu, \tau) = 0. \tag{3.39}$$

These results seemingly holds for any supersymmetric background (3.37) preserving at least one supercharge. For half-BPS geometries, we have further restrictions, and the theory becomes fully topological along the z, \bar{z} directions [6].

These results were derived by a standard argument. Consider the effective action $W[b]$, defined as minus the logarithm of the partition function in the presence of the supersymmetric background fields b_n :

$$W[b] = -\log Z_{\mathcal{M}_4}[b], \tag{3.40}$$

and consider an infinitesimal variation:

$$b_n = b_n^{(0)} + \Delta b_n, \tag{3.41}$$

of the background fields b_n around some particular supersymmetric value, $b_n^{(0)}$, *while keeping b_n supersymmetric*. Each independent variation Δb_n couples to a particular bosonic operator \mathcal{O}_n contained in the \mathcal{R} -multiplet or in the linear multiplet \mathcal{J} of the quantum field theory coupled to \mathcal{M}_4 , so that:

$$\frac{\partial}{\partial \Delta b_n} W[b^{(0)}] = \left\langle \int d^4x \sqrt{g} \mathcal{O}_n \right\rangle_{\mathcal{M}_4}, \tag{3.42}$$

²⁵Here we are glossing over some of the needed geometric data [6], for simplicity.

at first order in Δb_n . Whenever the insertion is Q -exact:

$$\int d^4x \sqrt{g} \mathcal{O}_n = \left\{ Q, \int d^4x \sqrt{g} \psi_n \right\}, \quad (3.43)$$

for ψ_n some fermionic operator, the expression (3.42) vanishes and the partition function is independent of that particular parameter b_n . This result relies on the fact that the expectation value of any Q -exact operator vanishes:

$$\langle \mathcal{O} \rangle_{\mathcal{M}_4} = \langle \{Q, \psi\} \rangle_{\mathcal{M}_4} = 0. \quad (3.44)$$

As in 2d, the existence of flavor 't Hooft anomalies will amend this simple picture, and imply an anomalous dependence of the partition function on some Q -exact couplings [14, 22].

3.4 Supersymmetry variation of the effective action

Let $W = W[\mathcal{V}_F]$ denote the effective action in the presence of *arbitrary* flavor-symmetry sources (3.27), including the background gauginos:

$$W[\mathcal{V}_F] = W[a_\mu, \lambda, \tilde{\lambda}, D], \quad (3.45)$$

on a fixed background \mathcal{M}_4 ; we now consider $G_F = \text{U}(1)$, for simplicity. Since the theory is supersymmetric, we would expect that:

$$\delta_\zeta W \stackrel{?}{=} 0, \quad \delta_{\tilde{\zeta}} W \stackrel{?}{=} 0, \quad (3.46)$$

under the supersymmetry transformations (3.29) of the background fields. This is inconsistent with $\text{U}(1)$ gauge invariance and diff-invariance whenever the $\text{U}(1)$ symmetry suffers from 't Hooft anomalies [14], in the following sense. Let δ_α denote a gauge transformation with gauge parameter α , with:

$$\delta_\alpha a_\mu = \partial_\mu \alpha, \quad \delta_\alpha \lambda = \delta_\alpha \tilde{\lambda} = \delta_\alpha D = 0. \quad (3.47)$$

The consistent anomalous variation of the effective action takes the form:

$$\delta_\alpha W = \frac{i\mathcal{A}_{qqq}}{96\pi^2} \int d^4x \sqrt{g} \alpha \epsilon^{\mu\nu\rho\sigma} f_{\mu\nu} f_{\rho\sigma} + \frac{i\mathcal{A}_q}{384\pi^2} \int d^4x \sqrt{g} \alpha \mathcal{P} + \dots, \quad (3.48)$$

with:

$$f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu, \quad \mathcal{P} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} R_{\mu\nu\lambda\kappa} R_{\rho\sigma}{}^{\lambda\kappa}, \quad (3.49)$$

the $\text{U}(1)$ field strength and the Pontryagin density, respectively, with $R_{\mu\nu\rho\lambda}$ the Riemann tensor of the metric $g_{\mu\nu}$. The ellipsis in (3.48) denotes additional contributions from mixed $\text{U}(1)_R$ - $\text{U}(1)$ anomalies, to be discussed in subsection 3.8 below. The cubic and mixed $\text{U}(1)$ -gravitational 't Hooft anomalies coefficients are normalized such that $\mathcal{A}_{qqq} = q^3$ and $\mathcal{A}_q = q$ for a single chiral fermion ψ_α of charge q . In a Lagrangian theory, we have:

$$\mathcal{A}_{qqq} = \text{tr}(\gamma^5 Q_F^3), \quad \mathcal{A}_q = \text{tr}(\gamma^5 Q_F), \quad (3.50)$$

where the trace is a sum over all the chiral fermions charged under U(1). The supersymmetry variations of the U(1) anomaly (3.48) are non-trivial. We have:

$$\delta_\zeta(\delta_\alpha W) = \frac{\mathcal{A}_{qqq}}{24\pi^2} \int d^4x \sqrt{g} \partial_\mu \alpha \epsilon^{\mu\nu\rho\lambda} \zeta \sigma_\nu \tilde{\lambda} f_{\rho\lambda}, \quad (3.51)$$

and similarly for $\delta_{\tilde{\zeta}}(\delta_\alpha W)$. Since the supersymmetry and gauge transformations commute, we necessarily have $\delta_\alpha(\delta_\zeta W) \neq 0$, and therefore there must be a non-trivial supersymmetry variation of the quantum effective action, $\delta_\zeta W \neq 0$. More generally, the supersymmetry variations must satisfy the following Wess-Zumino consistency conditions [14, 22]:

$$\begin{aligned} [\delta_\alpha, \delta_\zeta]W &= [\delta_\alpha, \delta_{\tilde{\zeta}}]W = 0, \\ \{\delta_\zeta, \delta_{\tilde{\zeta}}\}W &= \delta_{\alpha(a)}W, \quad \text{with } \alpha(a) = -2iK^\mu a_\mu. \end{aligned} \quad (3.52)$$

Note the a_μ -dependent gauge transformation on the second line, which is a consequence of working in the WZ gauge for the vector multiplet. The consistency conditions (3.52) determines the anomalous supersymmetry variations, $\delta_\zeta W$ and $\delta_{\tilde{\zeta}}W$, on a fixed curved-space supersymmetric background.

At this point, an important comment is in order. The curved-space supersymmetry algebra (3.28) takes the form:

$$\{\delta_\zeta, \delta_{\tilde{\zeta}}\} = \delta_{\xi(K)} + \delta_{\alpha(a)}, \quad (3.53)$$

with $\delta_{\xi(K)}$ a diffeomorphism along the Killing vector K . In four dimensions, we can always choose a scheme which preserves diffeomorphism invariance, and therefore $\delta_\xi W = 0$ for any ξ . This leads to the WZ conditions (3.52). The results of this paper are then only valid in such a diff-invariant scheme. The condition (3.52) has the important implication that, in any diff-invariant scheme, the variations $\delta_\zeta W$ and $\delta_{\tilde{\zeta}}W$ cannot vanish, because their commutator gives a gauge-transformation, and the latter is non-zero in any scheme by the assumption that a flavor 't Hooft anomaly exists [16].²⁶

Let us decompose the supersymmetry variations into contributions from the cubic and linear flavor 't Hooft anomalies:

$$\delta_\zeta W = \delta_\zeta W|_{q^3} + \delta_\zeta W|_q + \dots, \quad \delta_{\tilde{\zeta}}W = \delta_{\tilde{\zeta}}W|_{q^3} + \delta_{\tilde{\zeta}}W|_q + \dots, \quad (3.54)$$

where the ellipsis denotes the contribution from other potential anomalies, to be discussed later on. One finds:

$$\begin{aligned} \delta_\zeta W|_{q^3} &= -\frac{\mathcal{A}_{qqq}}{24\pi^2} \int d^4x \sqrt{g} \left(\epsilon^{\mu\nu\rho\sigma} \zeta \sigma_\mu \tilde{\lambda} a_\nu f_{\rho\sigma} - 3i\zeta \lambda \tilde{\lambda} \tilde{\lambda} \right), \\ \delta_{\tilde{\zeta}}W|_{q^3} &= -\frac{\mathcal{A}_{qqq}}{24\pi^2} \int d^4x \sqrt{g} \left(\epsilon^{\mu\nu\rho\sigma} \tilde{\zeta} \tilde{\sigma}_\mu \lambda a_\nu f_{\rho\sigma} + 3i\tilde{\zeta} \tilde{\lambda} \lambda \lambda \right), \end{aligned} \quad (3.55)$$

²⁶Note, however, that this argument holds only when the background supergravity fields are allowed to vary arbitrarily. On a fixed half-BPS geometry — that is, for *fixed Killing spinors* —, there might well exist a diff-invariant local term that cancels $\delta_\zeta W$, and therefore cancels the gauge anomaly for the particular gauge parameter $\alpha(a)$ in (3.52), as we discussed at the end of section 2.5. In any case, this appears not to be the case in 4d [14].

for the cubic-anomaly contribution. The term linear in the gaugino obviously follows from (3.51); the second term, which is gauge-invariant and cubic in the gaugino, is fixed by the second line of (3.52). This result was first obtained in flat space in [14], and much more recently on curved space but at first order in the gauginos in [22]; here we give the full expression valid on any curved-space $\mathcal{N} = 1$ new-minimal rigid-supersymmetric background.

The mixed U(1)-gravitational anomaly contribution to $\delta_\zeta W$ is harder to study on a fixed supergravity background. On any half-BPS four-manifold, the Pontryagin density turns out to be a total derivative:

$$\mathcal{P} = -\nabla_\mu \mathcal{P}^\mu, \tag{3.56}$$

with the vector \mathcal{P}^μ a local expression of dimension 3 in the supergravity fields [60]. Then, one can easily check that the following terms solve the Wess-Zumino consistency conditions:

$$\delta_\zeta W|_q = -\frac{\mathcal{A}_q}{384\pi^2} \int d^4x \sqrt{g} \zeta \sigma_\mu \tilde{\lambda} \mathcal{P}^\mu, \quad \delta_{\tilde{\zeta}} W|_q = -\frac{\mathcal{A}_q}{384\pi^2} \int d^4x \sqrt{g} \tilde{\zeta} \tilde{\sigma}_\mu \lambda \mathcal{P}^\mu.$$

This solves (3.52), as one can see using (3.56) and the fact that \mathcal{P}^μ is invariant under the isometry K of the background. Note that the vector \mathcal{P}^μ is only defined up to a divergenceless quantities; to completely fix it, one should consider the full background new-minimal supergravity away from the rigid-supersymmetry limit. One actually finds [22]:

$$\delta_\zeta W|_q = 0, \quad \delta_{\tilde{\zeta}} W|_q = 0, \tag{3.57}$$

on any half-BPS supersymmetric background. We will give another argument to that effect in section 3.9.

3.5 Non-holomorphy of the supersymmetric partition function $Z_{\mathcal{M}_4}$

Even though the supersymmetry variation (3.54) vanishes when we set the background gauginos to zero, it has some important consequence for supersymmetric partition functions, as we now explain.

On curved space with two Killing spinors ζ and $\tilde{\zeta}$, it will be convenient to introduce the following twisted notations for the gauginos in the background vector multiplet:

$$\tilde{\Lambda}_\mu = \zeta \sigma_\mu \tilde{\lambda}, \quad \Lambda_\mu = \tilde{\zeta} \tilde{\sigma}_\mu \lambda. \tag{3.58}$$

The minimal coupling (3.33) to the linear multiplet then reads:

$$\mathcal{L}_{\mathcal{V}\mathcal{J}} = a_\mu j^\mu + DJ + \Lambda_\mu \mathbf{j}^\mu + \tilde{\Lambda}_\mu \tilde{\mathbf{j}}^\mu, \tag{3.59}$$

where we introduced the twisted fermionic operators:

$$\tilde{\mathbf{j}}^\mu = \frac{1}{2} \frac{\zeta^\dagger \sigma^\mu \tilde{j}}{|\zeta|^2}, \quad \mathbf{j}^\mu = \frac{1}{2} \frac{\tilde{\zeta}^\dagger \tilde{\sigma}^\mu j}{|\tilde{\zeta}|^2}. \tag{3.60}$$

It follows from their definition that $\tilde{\Lambda}$ and $\tilde{\mathbf{j}}$ are holomorphic one-forms and vectors, respectively, with respect to the complex structure $J^\mu{}_\nu$; similarly, Λ and \mathbf{j} are holomorphic with respect to $\tilde{J}^\mu{}_\nu$.

The case of a single Killing spinor ζ . Let us first look at the case of a single curved-space supercharge, with Killing spinor ζ . We consider small variations of the background gauge field near the supersymmetric locus:

$$f_{\bar{w}\bar{z}} = 0, \quad D = -\frac{1}{2} J^{\mu\nu} f_{\mu\nu}. \quad (3.61)$$

In particular, we allow a small variation ΔD of the background field D around its supersymmetric value. We also turn on background gauginos:²⁷

$$\lambda_\alpha, \quad \tilde{\Lambda}_\mu = \zeta \sigma_\mu \tilde{\lambda}. \quad (3.62)$$

Then, the effective action $W = W[a_\mu, \lambda_\alpha, \tilde{\Lambda}_\mu, \Delta D]$ has a nilpotent supersymmetry:

$$\delta_\zeta a_\mu = i \tilde{\Lambda}_\mu, \quad \delta_\zeta \lambda_\alpha = i \zeta_\alpha \Delta D, \quad \delta_\zeta \tilde{\Lambda}_\mu = 0, \quad \delta_\zeta \Delta D = 0, \quad (3.63)$$

with $\delta_\zeta W$ determined by the 't Hooft anomalies as discussed above. Let us define the operator:

$$\mathcal{J}^\mu \equiv j^\mu - i \partial^\mu J + 2V^\mu J, \quad (3.64)$$

in terms of the linear multiplet operators (3.31). It follows from (3.32) that:

$$\delta_\zeta \tilde{\mathbf{j}}^\mu = i \Pi^\mu{}_\nu \mathcal{J}^\nu, \quad (3.65)$$

where $\Pi^\mu{}_\nu$ is the projector on the holomorphic indices, as defined in (3.20). Using the complex coordinate indices $X^\mu = (z^i, \bar{z}^{\bar{i}})$, we can write the minimal coupling as:

$$\mathcal{L}_{\mathcal{V}\mathcal{J}} = a_i \mathcal{J}^i + a_{\bar{i}} (j^{\bar{i}} + i \partial^{\bar{i}} J + 2(V^{\bar{i}} - U^{\bar{i}}) J) + J \Delta D - \lambda_j + \tilde{\Lambda}_\mu \tilde{\mathbf{j}}^\mu. \quad (3.66)$$

We see that the *holomorphic* gauge field components, a_i , couple to a Q -exact operator. We have:

$$\frac{\delta W}{\delta a_i} = \langle \mathcal{J}^i \rangle_{\mathcal{M}_4} = -i \langle \delta_\zeta \tilde{\mathbf{j}}^i \rangle_{\mathcal{M}_4}. \quad (3.67)$$

Assuming that Q -exact operators decouple, we would conclude that the partition function is independent of a_i , and only depends on the background gauge field a_μ through $a_{\bar{i}}$; in fact, one can then easily show that the dependence is only through the *holomorphic moduli* of the line bundle \mathbf{L}_F ; they are in one-to-one correspondence with the Dolbeault cohomology classes:

$$[\Delta a_{\bar{i}}] \in H^{0,1}(\mathcal{M}_4). \quad (3.68)$$

Due to the quantum correction to the supersymmetric Ward identity, however, the right-hand-side of (3.67) does not vanish. Instead, since $\tilde{\mathbf{j}}^i$ couples to $\tilde{\Lambda}_i$ as shown in (3.67), we have:

$$\frac{\delta W}{\delta a_i} = -i \frac{\delta}{\delta \tilde{\Lambda}_i} (\delta_\zeta W) \Big|_{\lambda=\tilde{\lambda}=0}. \quad (3.69)$$

This anomalous Ward identity gives rise to a *holomorphy anomaly* in the flavor-parameter dependence of supersymmetric partition functions. Since the supersymmetry variation (3.55) has a term linear in $\tilde{\Lambda}$, the holomorphy anomaly (3.69) will generally be non-vanishing.

²⁷When considering the effect of a single supercharge, we cannot use $\tilde{\zeta}$ to “twist” λ_α into a one-form as in (3.58); instead, we could write it in terms of two scalars $\gamma = \zeta \lambda$ and $\eta = \frac{\zeta^\dagger \lambda}{|\zeta|^2}$, but this will not be necessary for our discussion.

The case of a two Killing spinors, ζ and $\tilde{\zeta}$. Consider now the case of a half-BPS background. We define the linear-multiplet operators:

$$\tilde{\mathcal{J}}^\mu \equiv j^\mu + i\partial^\mu J + 2V^\mu J, \tag{3.70}$$

in addition to (3.64). We now have the relations:

$$\delta_\zeta \tilde{\mathbf{j}}^\mu = i\Pi^\mu{}_\nu \mathcal{J}^\nu, \quad \delta_{\tilde{\zeta}} \mathbf{j}^\mu = i\tilde{\Pi}^\mu{}_\nu \tilde{\mathcal{J}}^\nu, \tag{3.71}$$

in terms of the projectors (3.20). This is equivalent to:

$$\delta_\zeta \tilde{\mathbf{j}}^w = i\mathcal{J}^w, \quad \delta_{\tilde{\zeta}} \tilde{\mathbf{j}}^z = i\mathcal{J}^z, \quad \delta_{\tilde{\zeta}} \mathbf{j}^w = i\tilde{\mathcal{J}}^w, \quad \delta_{\tilde{\zeta}} \mathbf{j}^{\bar{z}} = i\tilde{\mathcal{J}}^{\bar{z}}. \tag{3.72}$$

Expanding around the supersymmetry locus (3.36), the linear coupling (3.59) reads:

$$\begin{aligned} \mathcal{L}_{\mathcal{V}\mathcal{J}} = & a_w \mathcal{J}^w + a_{\bar{w}} (\tilde{\mathcal{J}}^{\bar{w}} - \kappa K^{\bar{w}}) + a_z \mathcal{J}^z + a_{\bar{z}} \mathcal{J}^{\bar{z}} + J\Delta D \\ & + \Lambda_w \mathbf{j}^w + \Lambda_z \mathbf{j}^z + \tilde{\Lambda}_w \tilde{\mathbf{j}}^w + \tilde{\Lambda}_{\bar{z}} \tilde{\mathbf{j}}^{\bar{z}}. \end{aligned} \tag{3.73}$$

In this case, only $a_{\bar{w}}$ does not couple to a \mathcal{Q} -exact operator, and therefore the partition function is expected to depend only on the corresponding line bundle modulus, which we denote by ν . Note also that the coupling $a_w \mathcal{J}^w$ in (3.73) is equivalent to $a_w \tilde{\mathcal{J}}^w$ by integration by part. We then find the anomalous Ward identities:

$$\begin{aligned} \frac{\delta W}{\delta a_w} &= -i \frac{\delta}{\delta \tilde{\Lambda}_w} (\delta_\zeta W) \Big|_{\lambda=\tilde{\lambda}=0} = -i \frac{\delta}{\delta \Lambda_w} (\delta_{\tilde{\zeta}} W) \Big|_{\lambda=\tilde{\lambda}=0}, \\ \frac{\delta W}{\delta a_z} &= -i \frac{\delta}{\delta \tilde{\Lambda}_z} (\delta_\zeta W) \Big|_{\lambda=\tilde{\lambda}=0}, \\ \frac{\delta W}{\delta a_{\bar{z}}} &= -i \frac{\delta}{\delta \tilde{\Lambda}_{\bar{z}}} (\delta_{\tilde{\zeta}} W) \Big|_{\lambda=\tilde{\lambda}=0}. \end{aligned} \tag{3.74}$$

In section 4, we will evaluate these identities explicitly, for a large class of half-BPS supersymmetric partition functions.

3.6 The small- β_{S^1} limit of $Z_{\mathcal{M}_3 \times S^1}(\nu, \tau)$

Whenever the four-dimensional background is of the form:

$$\mathcal{M}_4 \cong \mathcal{M}_3 \times S^1, \tag{3.75}$$

it is interesting to consider the small-circle limit, $\beta_{S^1} \rightarrow 0$, where β_{S^1} is the radius of the S^1 . Formally, this is a “high-temperature” limit of the supersymmetric partition function. At finite temperature — and also with supersymmetry-preserving boundary conditions on the Euclidean time circle, provided that generic fugacities are turned on —, it is expected that the effective action is *local* in the background gauge field and metric, order by order in β_{S^1} [32, 33]. For any set of background fields b , we have:

$$W[b] \underset{\beta_{S^1} \rightarrow 0}{\sim} \frac{1}{\beta_{S^1}} W_{3d}^{(-1)}[b] + W_{3d}^{(0)}[b] + \beta_{S^1} W_{3d}^{(1)}[b] + \dots \tag{3.76}$$

The $1/\beta_{S^1}$ term was studied in [33]; it is fully gauge invariant and supersymmetric. The finite term $W_{3d}^{(0)}$ roughly corresponds to the supersymmetric partition function of the 3d $\mathcal{N} = 2$ theory on \mathcal{M}_3 obtained from the 4d $\mathcal{N} = 1$ theory by dimensional reduction, up to important subtleties that will not affect our discussion — see [41, 61–63]. Here, we are interested in *the order- β_{S^1} functional* in (3.76). Let us introduce the notation:

$$\mathbb{W}_{3d}[b] \equiv \beta_{S^1} W_{3d}^{(1)}[b]. \tag{3.77}$$

This term should capture all the anomalies of the four-dimensional theory, including the anomalous variation $\delta_\zeta W$, which allows us to fix it entirely — modulo fully gauge-invariant and supersymmetric terms, which do not contribute on the supersymmetric locus.

3.6.1 Dimensional reduction and 3d $\mathcal{N} = 2$ supersymmetry

Before discussing the anomalies, we need to recall some formalism to deal with supersymmetry on \mathcal{M}_3 . Here, we assume that the background is half-BPS, with two Killing spinors ζ and $\tilde{\zeta}$.

3d $\mathcal{N} = 2$ supergravity background fields. We couple the 4d $\mathcal{N} = 1$ theory to a fixed $\mathcal{N} = 1$ new-minimal supergravity background on $\mathcal{M}_3 \times S^1$, and similarly the three-dimensional reduced theory couples to a 3d $\mathcal{N} = 2$ supersymmetric background on \mathcal{M}_3 , of the type studied in [42]. The three-dimensional bosonic supergravity fields are denoted by:

$$g_{\mu\nu}^{(3d)}, \quad \mathbf{A}_\mu^{(R)}, \quad \mathbf{H}, \quad \mathbf{V}_\mu. \tag{3.78}$$

The three-dimensional Killing spinor equations read:

$$\begin{aligned} (\nabla_\mu - i\mathbf{A}_\mu^{(R)})\zeta &= -\frac{1}{2}\mathbf{H}\gamma_\mu\zeta + \frac{i}{2}\mathbf{V}^\nu\gamma_\mu\gamma_\nu\zeta, \\ (\nabla_\mu + i\mathbf{A}_\mu^{(R)})\tilde{\zeta} &= -\frac{1}{2}\mathbf{H}\gamma_\mu\tilde{\zeta} - \frac{i}{2}\mathbf{V}^\nu\gamma_\mu\gamma_\nu\tilde{\zeta}. \end{aligned} \tag{3.79}$$

Our conventions for the 3d γ -matrices are given in appendix A.2. We choose the following Kaluza-Klein reduction ansatz for the metric:

$$ds^2(\mathcal{M}_4) = (\beta_{S^1} dt + c_\mu dx^\mu)^2 + ds^2(\mathcal{M}_3). \tag{3.80}$$

We will focus on the case of a topologically-trivial circle bundle over \mathcal{M}_3 , in which case the graviphoton c_μ is actually a well-defined one-form on \mathcal{M}_3 . The three-dimensional background field \mathbf{V}_μ should be given in terms of the 3d graviphoton, as:

$$\mathbf{V}^\mu = -i\epsilon^{\mu\nu\rho}\partial_\nu c_\rho. \tag{3.81}$$

Let us also define the unit-norm one-form:

$$\hat{e}^0 = \beta_{S^1} dt + c_\mu dx^\mu. \tag{3.82}$$

Then, the 3d and 4d supergravity fields are related as [42]:

$$V_\mu dX^\mu = \mathbf{H}\hat{e}^0 + \frac{1}{2}\mathbf{V}_\mu dx^\mu, \quad A_\mu^{(R)} dX^\mu = -\frac{1}{2}\mathbf{H}\hat{e}^0 + \left(\mathbf{A}_\mu^{(R)} + \frac{1}{4}\mathbf{V}_\mu\right) dx^\mu, \tag{3.83}$$

where X^μ and x^μ denote the coordinates on \mathcal{M}_4 and \mathcal{M}_3 , respectively.

Transversely holomorphic structure (THF) on \mathcal{M}_3 . The 4d Killing spinors descend to 3d Killing spinors. In terms of the latter, one can define the following real one-form on \mathcal{M}_3 :

$$\eta_\mu \equiv \frac{\zeta^\dagger \gamma_\mu \zeta}{|\zeta|^2} = -\frac{\tilde{\zeta}^\dagger \gamma_\mu \tilde{\zeta}}{|\tilde{\zeta}|^2}, \quad (3.84)$$

with the last equality being a property of the half-BPS backgrounds under consideration [42]. We also define the three-dimensional tensor:

$$\Phi^\mu{}_\nu = -\epsilon^\mu{}_{\nu\rho} \eta^\rho. \quad (3.85)$$

These objects define a metric-compatible THF, which is obtained from the complex structure on \mathcal{M}_4 by dimensional reduction. In the conventions of appendix A.2, we have:

$$\eta_\mu = J^0{}_\mu = \hat{e}_\nu^0 J^\nu{}_\mu, \quad \Phi^\mu{}_\nu = J^\mu{}_\nu|_{3d}. \quad (3.86)$$

The one-form $\eta = \eta_\mu dx^\mu$ has unit norm and defines the foliation. The THF-adapted metric on \mathcal{M}_3 takes the canonical form:

$$ds^2(\mathcal{M}_3) = \eta^2 + 2g_{z\bar{z}} dz d\bar{z}, \quad (3.87)$$

with z, \bar{z} the transverse holomorphic coordinates [42].

Dimensional reduction of the background gauge field. The 4d $\mathcal{N} = 1$ vector multiplet reduces to a 3d $\mathcal{N} = 2$ vector multiplet:²⁸

$$\mathcal{V}_F^{(3d)} = (A_\mu, \sigma, \lambda_\alpha, \tilde{\lambda}_\alpha, D). \quad (3.88)$$

The three-dimensional gauge field A_μ and scalar σ are related to the four-dimensional gauge field a_μ by:

$$a_\mu dX^\mu = \sigma \hat{e}^0 + A_\mu dx^\mu. \quad (3.89)$$

The three-dimensional supersymmetry transformations on \mathcal{M}_3 are given by:

$$\begin{aligned} \delta_\zeta A_\mu &= -i\zeta \gamma_\mu \tilde{\lambda}, \\ \delta_\zeta \sigma &= -\zeta \tilde{\lambda}, \\ \delta_\zeta \lambda_\alpha &= i\zeta_\alpha (D + \sigma \mathbf{H}) - i(\gamma^\mu \zeta)_\alpha \left(\frac{1}{2} \epsilon_{\mu\nu\rho} F^{\nu\rho} + \partial_\mu \sigma + i\mathbf{V}_\mu \sigma \right), \\ \delta_\zeta \tilde{\lambda}_\alpha &= 0, \\ \delta_\zeta D &= \nabla_\mu (\zeta \gamma^\mu \tilde{\lambda}) - i\mathbf{V}_\mu \zeta \gamma^\mu \tilde{\lambda} - \mathbf{H} \zeta \tilde{\lambda}, \end{aligned} \quad (3.90)$$

and:

$$\begin{aligned} \delta_{\tilde{\zeta}} A_\mu &= -i\tilde{\zeta} \gamma_\mu \lambda, \\ \delta_{\tilde{\zeta}} \sigma &= \tilde{\zeta} \lambda, \\ \delta_{\tilde{\zeta}} \lambda_\alpha &= 0, \\ \delta_{\tilde{\zeta}} \tilde{\lambda}_\alpha &= -i\tilde{\zeta}_\alpha (D + \sigma \mathbf{H}) - i(\gamma^\mu \tilde{\zeta})_\alpha \left(\frac{1}{2} \epsilon_{\mu\nu\rho} F^{\nu\rho} - \partial_\mu \sigma + i\mathbf{V}_\mu \sigma \right), \\ \delta_{\tilde{\zeta}} D &= -\nabla_\mu (\tilde{\zeta} \gamma^\mu \lambda) - i\mathbf{V}_\mu \tilde{\zeta} \gamma^\mu \lambda + \mathbf{H} \tilde{\zeta} \lambda, \end{aligned} \quad (3.91)$$

²⁸The scalar D is defined with a shift with respect to its 4d counterpart, $D_{3d} = D_{4d} - \sigma H$ [42].

with $F_{\mu\nu}$ the field strength of A_μ . Here, we again focus on the case of an abelian vector multiplet, for simplicity. On the supersymmetric locus, we have:

$$D = -\sigma\mathbf{H} + \frac{1}{2}\epsilon^{\mu\nu\rho}\eta_\mu F_{\nu\rho} + i\eta^\mu\mathbf{V}_\mu\sigma, \quad \partial_\mu\sigma = 0. \quad (3.92)$$

Note also that the combination $\eta^\mu A_\mu + i\sigma$ is invariant under supersymmetry.

3.6.2 't Hooft anomalies and three-dimensional local functional

Let us now consider the reduction of the 4d anomalies to three dimensions. Since the 3d effective action should be local, in the small- β_{S^1} expansion, the 3d reduction of the anomalies should be the variation of a 3d local term of order β_{S^1} , which we denoted by \mathbb{W}_{3d} in (3.77).

Reduction of the U(1) anomaly. Consider first the U(1) 't Hooft anomaly (3.48). Its three-dimensional reduction is given by:

$$\begin{aligned} \delta_\alpha W|_{3d, q^3} &= -\frac{i\mathcal{A}_{qqq}\beta_{S^1}}{12\pi} \int d^3x \sqrt{g} \alpha \left(-\epsilon^{\mu\nu\rho} \partial_\mu \sigma F_{\nu\rho} - 2i\mathbf{V}^\mu \sigma \partial_\mu \sigma \right) \\ &= -\frac{i\mathcal{A}_{qqq}\beta_{S^1}}{12\pi} \int d^3x \sqrt{g} \partial_\mu \alpha \left(\sigma \epsilon^{\mu\nu\rho} F_{\nu\rho} + i\mathbf{V}^\mu \sigma^2 \right), \end{aligned} \quad (3.93)$$

for the cubic anomaly, and:

$$\delta_\alpha W|_{3d, q} = \frac{i\mathcal{A}_q\beta_{S^1}}{192\pi} \int d^3x \sqrt{g} \partial_\mu \alpha \mathbf{N}^\mu, \quad (3.94)$$

for the mixed U(1)-gravitational anomaly, formally.²⁹ On any four-manifold of the form $\mathcal{M}_3 \times S^1$, the Pontryagin density vanishes, and therefore so does (3.94), namely:

$$\delta_\alpha W|_{3d, q} = 0. \quad (3.95)$$

Reduction of the supersymmetry variations. Similarly, the three-dimensional reduction of $\delta_\zeta W$ and $\delta_{\tilde{\zeta}} W$ is given by:

$$\begin{aligned} \delta_\zeta W|_{3d, q^3} &= \frac{\mathcal{A}_{qqq}\beta}{12\pi} \int d^3x \sqrt{g} \left\{ \epsilon^{\mu\nu\rho} \left(i\zeta \tilde{\lambda} A_\mu F_{\nu\rho} + 2\zeta \gamma_\mu \tilde{\lambda} A_\nu \partial_\rho \sigma \right) \right. \\ &\quad \left. - 2\zeta \tilde{\lambda} \sigma A_\mu \mathbf{V}^\mu + \epsilon^{\mu\nu\rho} \zeta \gamma_\mu \tilde{\lambda} \sigma F_{\nu\rho} + 2i\zeta \gamma_\mu \tilde{\lambda} \sigma^2 \mathbf{V}^\mu - 3i\zeta \lambda \tilde{\lambda} \tilde{\lambda} \right\}, \\ \delta_{\tilde{\zeta}} W|_{3d, q^3} &= \frac{\mathcal{A}_{qqq}\beta}{12\pi} \int d^3x \sqrt{g} \left\{ \epsilon^{\mu\nu\rho} \left(-i\tilde{\zeta} \lambda A_\mu F_{\nu\rho} + 2\tilde{\zeta} \gamma_\mu \lambda A_\nu \partial_\rho \sigma \right) \right. \\ &\quad \left. + 2\tilde{\zeta} \lambda \sigma A_\mu \mathbf{V}^\mu + \epsilon^{\mu\nu\rho} \tilde{\zeta} \gamma_\mu \lambda \sigma F_{\nu\rho} + 2i\tilde{\zeta} \gamma_\mu \lambda \sigma^2 \mathbf{V}^\mu + 3i\tilde{\zeta} \tilde{\lambda} \lambda \lambda \right\}, \end{aligned} \quad (3.96)$$

²⁹Here, we defined the vector:

$$\mathbf{N}^\mu = 4i\mathbf{G}^{\mu\nu}\mathbf{V}_\nu - 2i\mathbf{V}^\mu\mathbf{V}^\nu\mathbf{V}_\nu, \quad \mathbf{G}_{\mu\nu} \equiv \mathbf{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathbf{R},$$

with $\mathbf{G}_{\mu\nu}$ the Einstein tensor of the three-dimensional metric, for \mathcal{M}_4 a circle bundle over \mathcal{M}_3 , with \mathbf{V}^μ defined as in (3.81). Then, the Pontryagin density of the 4d metric (3.80) can always be written as $\mathcal{P} = -\nabla_\mu \mathbf{N}^\mu$. Whenever the S^1 is trivially fibered, $\mathcal{P} = 0$.

for the cubic anomaly contribution (3.55), and by:

$$\delta_\zeta W|_{3d,q} = \delta_{\tilde{\zeta}} W|_{3d,q} = 0, \tag{3.97}$$

for the mixed U(1)-gravitational anomaly contribution, according to (3.57).

Explicit formula for \mathbb{W}_{3d} . As we explained, there should exist a three-dimensional local functional, \mathbb{W}_{3d} , of the vector multiplet, in the fixed 3d $\mathcal{N} = 2$ supergravity background on \mathcal{M}_3 , such that:

$$\delta_\alpha \mathbb{W}_{3d} = \delta_\alpha W|_{3d}, \quad \delta_\zeta \mathbb{W}_{3d} = \delta_\zeta W|_{3d}, \quad \delta_{\tilde{\zeta}} \mathbb{W}_{3d} = \delta_{\tilde{\zeta}} W|_{3d}. \tag{3.98}$$

Up to any three-dimensional action that is gauge-invariant and supersymmetric under (3.90)–(3.91), the functional \mathbb{W}_{3d} must take the form:³⁰

$$\mathbb{W}_{3d} = \mathbb{W}_{3d}|_{q^3} + \mathbb{W}_{3d}|_q. \tag{3.99}$$

The mixed U(1)-gravitational anomaly contribution must vanish, $\mathbb{W}_{3d}|_q = 0$. By a straightforward computation, one can check that there is a unique possibility for the cubic-anomaly contribution, given by:

$$\begin{aligned} \mathbb{W}_{3d}|_{q^3} = \frac{\mathcal{A}_{qqq}\beta_{S^1}}{12\pi} \int d^3x \sqrt{g} & \left(-i\sigma\epsilon^{\mu\nu\rho} A_\mu F_{\nu\rho} + 3\sigma^2 D - 6i\sigma\tilde{\lambda}\lambda \right. \\ & \left. + \sigma^2 A_\mu \mathbf{V}^\mu + \sigma^3 \mathbf{H} \right). \end{aligned} \tag{3.100}$$

The first line in (3.100) is the flat-space answer, while the second line adds additional couplings to the curved-space supersymmetric background.

3.7 Generalization to any abelian flavor symmetry

The above discussion can be generalized straightforwardly to any abelian flavor group:

$$G_F = \prod_\alpha U(1)_\alpha. \tag{3.101}$$

(For a non-abelian G_F , we focus on the maximal torus, which is what is generally done when discussing supersymmetric partition functions.) The G_F anomaly can be obtained from the anomaly polynomial:

$$P_6 = \frac{1}{24\pi^2} \sum_{\alpha,\beta,\gamma} \mathcal{A}_{qqq}^{\alpha\beta\gamma} f_\alpha \wedge f_\beta \wedge f_\gamma - \frac{1}{384\pi^2} \sum_\alpha \mathcal{A}^\alpha f_\alpha \wedge \text{tr}(R \wedge R), \tag{3.102}$$

by the descent relations, $P_6 = dQ_5$ and $\delta_\alpha Q_5 = d(\alpha\omega_4)$. Then, the cubic 't Hooft anomaly reads:

$$\delta_\alpha W|_{q^3} = \sum_{\alpha,\beta,\gamma} \frac{i\mathcal{A}_{qqq}^{\alpha\beta\gamma}}{24\pi^2} \int_{\mathcal{M}_4} \alpha_\alpha f_\beta \wedge f_\gamma, \tag{3.103}$$

³⁰Here we take only into account the contribution from the U(1) flavor anomaly. There can be additional contributions from mixed U(1)-U(1)_R anomalies, which we will consider in section 3.8 below.

with α denoting an arbitrary abelian gauge parameter along (3.101), and with $f_\alpha = da_\alpha$ for the background gauge fields $a_{\alpha,\mu}$. The sum over repeated flavor indices will be left implicit, in the following. Note that the 't Hooft anomaly coefficient $\mathcal{A}_{qqq}^{\alpha\beta\gamma}$ is fully symmetric in the three flavor indices α, β, γ .

The expression (3.103) is valid up to ambiguities that correspond to the gauge-variations of local terms — those, in turn, correspond to ambiguities in the descent relations. For instance, given two abelian symmetries $U(1)_\alpha$ and $U(1)_\beta$ with $\alpha \neq \beta$, we could always add the local term:³¹

$$W_{\text{loc}} = \frac{i}{24\pi^2} \int_{\mathcal{M}_4} \left(s \mathcal{A}_{qqq}^{\alpha\alpha\beta} a_\beta \wedge a_\alpha \wedge f_\alpha + t \mathcal{A}_{qqq}^{\alpha\beta\beta} a_\alpha \wedge a_\beta \wedge f_\beta \right), \quad (3.104)$$

with $s, t \in \mathbb{R}$ some arbitrary coefficients. The addition of such counterterms to the effective action can “move around” the mixed anomalies; for instance, setting $s = -2$ and $t = 1$ would ensure that the variation $\delta_{\alpha_\alpha} W$ only depends on the gauge field a_α , at the price of pushing all the mixed anomalies into $\delta_{\alpha_\beta} W$. Here and in the following, we always work in the “symmetric” scheme which treats all the $U(1)_\alpha$ factors equally, in which case (3.103) holds.

The supersymmetry variations (3.55) are then generalized similarly, in the fully symmetric scheme:³²

$$\begin{aligned} \delta_\zeta W|_{q^3} &= -\frac{\mathcal{A}_{qqq}^{\alpha\beta\gamma}}{24\pi^2} \int d^4x \sqrt{g} \left(\epsilon^{\mu\nu\rho\sigma} \zeta \sigma_\mu \tilde{\lambda}_\alpha a_{\beta,\nu} f_{\gamma,\rho\sigma} - 3i \zeta \lambda_\alpha \tilde{\lambda}_\beta \tilde{\lambda}_\gamma \right), \\ \delta_{\tilde{\zeta}} W|_{q^3} &= -\frac{\mathcal{A}_{qqq}^{\alpha\beta\gamma}}{24\pi^2} \int d^4x \sqrt{g} \left(\epsilon^{\mu\nu\rho\sigma} \tilde{\zeta} \tilde{\sigma}_\mu \lambda_\alpha a_{\beta,\nu} f_{\gamma,\rho\sigma} + 3i \tilde{\zeta} \tilde{\lambda}_\alpha \lambda_\beta \lambda_\gamma \right), \end{aligned} \quad (3.105)$$

The 3d local term that reproduces the 3d reduction of the anomalies (3.103) and (3.105) takes the form:

$$\begin{aligned} \mathbb{W}_{3d}|_{q^3} &= \frac{\mathcal{A}_{qqq}^{\alpha\beta\gamma} \beta_{S^1}}{12\pi} \int d^3x \sqrt{g} \left(-i \epsilon^{\mu\nu\rho} \sigma_\alpha A_{\beta,\mu} F_{\gamma,\nu\rho} + 3\sigma_\alpha \sigma_\beta D_\gamma \right. \\ &\quad \left. + \sigma_\alpha \sigma_\beta A_{\gamma,\mu} \mathbf{V}^\mu + \sigma_\alpha \sigma_\beta \sigma_\gamma \mathbf{H} - 6i \sigma_\alpha \tilde{\lambda}_\beta \lambda_\gamma \right), \end{aligned} \quad (3.106)$$

generalizing (3.100).

3.8 R-symmetry contributions to $\delta_\zeta W$

Finally, we should discuss the effect of the R -symmetry background gauge field. Consider again the case $G_F = U(1)$, to avoid clutter. The anomalous $U(1)$ variation of the effective action has contributions from mixed 't Hooft anomalies $U(1)^2$ - $U(1)_R$ and $U(1)$ - $U(1)_R^2$, which read:

$$\delta_\alpha W|_R = \frac{i \mathcal{A}_{qqR}}{48\pi^2} \int d^4x \sqrt{g} \alpha \epsilon^{\mu\nu\rho\sigma} f_{\mu\nu} f_{\rho\sigma}^{(R)} + \frac{i \mathcal{A}_{qRR}}{96\pi^2} \int d^4x \sqrt{g} \alpha \epsilon^{\mu\nu\rho\sigma} f_{\mu\nu}^{(R)} f_{\rho\sigma}^{(R)}. \quad (3.107)$$

³¹In this equation, there is no sum over repeated flavor indices.

³²Here, the indices α, β, \dots are flavor indices, while the spinor indices are left implicit.

This is the contribution that was denoted by an ellipsis in (3.48). Here, $f^{(R)} = da^{(R)}$ is the field strength of the $U(1)_R$ gauge field:

$$a_\mu^{(R)} \equiv A_\mu^{(R)} + \frac{3}{2}V_\mu. \tag{3.108}$$

The second term in (3.107) vanishes on the half-BPS background, because:

$$\epsilon^{\mu\nu\rho\sigma} f_{\mu\nu}^{(R)} f_{\rho\sigma}^{(R)} = 0, \tag{3.109}$$

in that case. The mixed 't Hooft anomalies (3.107) then contribute to right-hand-side of the supersymmetry variation of W as:

$$\begin{aligned} \delta_\zeta W|_R = & -\frac{\mathcal{A}_{qqR}}{24\pi^2} \int d^4x \sqrt{g} \epsilon^{\mu\nu\rho\sigma} \zeta \sigma_\mu \tilde{\lambda} \left(a_\nu f_{\rho\sigma}^{(R)} + a_\nu^{(R)} f_{\rho\sigma} \right) \\ & -\frac{\mathcal{A}_{qRR}}{24\pi^2} \int d^4x \sqrt{g} \epsilon^{\mu\nu\rho\sigma} \zeta \sigma_\mu \tilde{\lambda} a_\nu^{(R)} f_{\rho\sigma}^{(R)}, \end{aligned} \tag{3.110}$$

and similarly for $\delta_{\bar{\zeta}}W$. Note again that, in this paper, we focus on the flavor anomaly on a fixed half-BPS background. The more general form of the mixed anomalies on arbitrary supergravity backgrounds, including the contributions from the gravitino, were recently discussed in [22].

4d $U(1)_R$ vector multiplet. An important property of the 4d $\mathcal{N} = 1$ supergravity multiplet is that it contains an ordinary *vector multiplet for the R -symmetry*, \mathcal{V}_R , as a sub-multiplet of supersymmetry [64]. The gaugino $\lambda^{(R)}$ inside \mathcal{V}_R is a contraction of the gravitino Ψ_μ . On the half-BPS background \mathcal{M}_4 , with $\Psi_\mu = \delta_\zeta \Psi_\mu = \delta_{\bar{\zeta}} \Psi_\mu = 0$, we are left with the bosonic fields:

$$\mathcal{V}_R = \left(a_\mu^{(R)}, D_{4d}^{(R)} \right), \tag{3.111}$$

with $a_\mu^{(R)}$ given in terms of the supergravity fields by (3.108), and:

$$D_{4d}^{(R)} = \frac{1}{4} (R + 6V_\mu V^\mu), \tag{3.112}$$

with R the four-dimensional Ricci scalar. One can verify that these combinations of background fields satisfy the conditions (3.36) for an half-BPS vector multiplet, namely:

$$f_{w\bar{w}}^{(R)} = f_{\bar{w}z}^{(R)} = f_{\bar{w}z}^{(R)} = 0, \quad D_{4d}^{(R)} = -\frac{1}{2} J^{\mu\nu} f_{\mu\nu}^{(R)}. \tag{3.113}$$

Note that the background gauge field $a_\mu^{(R)}$ is not real, unlike the background flavor gauge fields that we consider in this paper; supersymmetry generally imposes that $a_\mu^{(R)}$ be complex.

The $U(1)_R$ vector multiplet controls the R -charge dependence of supersymmetric partition functions [13]. Consider any mixing of the R -charge with abelian flavor symmetries $U(1)_\alpha$:

$$R \rightarrow R + \sum_\alpha t_\alpha Q_F^\alpha, \tag{3.114}$$

with $t_\alpha \in \mathbb{R}$ some mixing parameters. The effect of this shift on the supersymmetric partition function, computed with background vector multiplets $\mathcal{V}_{F,\alpha}$, is simply to shift the latter by the $U(1)_R$ vector multiplet itself:

$$\mathcal{V}_{F,\alpha} \rightarrow \mathcal{V}_{F,\alpha} + t_\alpha \mathcal{V}_R. \tag{3.115}$$

This allows us to treat the R -charge dependence systematically once we understand the “flavor sector.” For instance, as a typical example of this approach, let us consider a free chiral multiplet. The mixed flavor- R ’t Hooft anomaly coefficients for a chiral multiplet of $U(1)$ charge q and R -charge r are:

$$\mathcal{A}_{qqR} = q^2(r - 1), \quad \mathcal{A}_{qRR} = q(r - 1)^2, \tag{3.116}$$

and therefore we can ignore the effect of those anomalies if we choose $r = 1$. Then, the general case can be obtained by starting from the “reference” R -charge R_0 with $r = 1$ and shifting the partition function of the free chiral to $R = R_0 + (r - 1)Q$ using (3.115).

Similarly, the anomalous variations (3.107)–(3.110) can be obtained from (3.103) and (3.105), simply by replacing one flavor index “ α ” by “ R .” This again corresponds to a fully symmetric scheme, treating $a_\mu^{(R)}$ and $a_{\alpha,\mu}$ on the same footing.

Dimensional reduction and 3d $U(1)_R$ vector multiplet. Finally, let us briefly discuss the 3d reduction of the $U(1)_R$ vector multiplet:

$$\mathcal{V}_R^{(3d)} = (\mathcal{A}_\mu^{(R)}, \sigma^{(R)}, D_{3d}^{(R)}). \tag{3.117}$$

Using (3.83) and (3.89), one can check that [13]:

$$\mathcal{A}_\mu^{(R)} = \mathbf{A}_\mu + \mathbf{V}_\mu, \quad \sigma^{(R)} = \mathbf{H}, \quad D_{3d}^{(R)} = \frac{1}{4}(\mathbf{R} + 2\mathbf{H}^2 + 2\mathbf{V}_\mu \mathbf{V}^\mu), \tag{3.118}$$

where \mathbf{R} is the 3d Ricci scalar, which is related to the 4d Ricci by $R = \mathbf{R} + \frac{1}{2}\mathbf{V}_\mu \mathbf{V}^\mu$. On any half-BPS background \mathcal{M}_3 , we have the analogue of the condition (3.92) for a vector multiplet on its supersymmetric locus, namely:

$$e^{\mu\nu\rho} \eta_\mu \partial_\nu (\mathbf{A}_\rho + \mathbf{V}_\rho) + i \eta_\mu \mathbf{V}^\mu \mathbf{H} = \frac{1}{4} \mathbf{R} + \frac{3}{2} \mathbf{H}^2 + \frac{1}{2} \mathbf{V}_\mu \mathbf{V}^\mu, \quad \partial_\mu \mathbf{H} = 0. \tag{3.119}$$

One can then find 3d local terms that reproduce the mixed $U(1)$ - $U(1)_R$ ’t Hooft anomalies and are consistent with supersymmetry, by an appropriate generalization of (3.100).

3.9 Anomalous supersymmetry variation from WZ gauge-fixing

As in 2d, we can understand the 4d “anomalous supersymmetry variation,” $\delta_\zeta W \neq 0$, as a direct consequence of the Wess-Zumino gauge-fixing procedure. This provides a conceptually and technically simpler derivation of those anomalous variations, and demonstrates that they are simple consequences of the ordinary ’t Hooft anomalies, when working in the WZ gauge.

For simplicity of presentation, in this subsection, let us work in flat-space, \mathbb{R}^4 , so that we can use the 4d $\mathcal{N} = 1$ superfield notation. (The generalization to rigid supersymmetry on \mathcal{M}_4 is completely straightforward.)

WZ gauge-fixing. Let us consider a theory coupled to a U(1) background vector multiplet,

$$\mathcal{V}_F = \left(C, \chi, \tilde{\chi}, M, \tilde{M}, a_\mu, \lambda, \tilde{\lambda}, D \right). \quad (3.120)$$

This corresponds to the real superfield:

$$\begin{aligned} \mathcal{V}_F = & C + i\theta\chi - i\tilde{\theta}\tilde{\chi} + \frac{i}{2}\theta\theta M - \frac{i}{2}\tilde{\theta}\tilde{\theta}\tilde{M} - \theta\sigma^\mu\tilde{\theta}a_\mu(x) \\ & + i\theta\tilde{\theta}\tilde{\theta}\left(\tilde{\lambda} + \frac{i}{2}\tilde{\sigma}^\mu\partial_\mu\chi\right) - i\tilde{\theta}\tilde{\theta}\theta\left(\lambda + \frac{i}{2}\sigma^\mu\partial_\mu\tilde{\chi}\right) + \frac{1}{2}\theta\theta\tilde{\theta}\tilde{\theta}\left(D + \frac{1}{2}\partial_\mu\partial^\mu C\right). \end{aligned} \quad (3.121)$$

The gauge freedom of the gauge field, $\delta_\alpha a_\mu = \partial_\mu\alpha$, is supersymmetrized to:

$$\delta_\Omega\mathcal{V}_F = \frac{i}{2}(\Omega - \tilde{\Omega}), \quad (3.122)$$

with the gauge parameters valued in a pair of chiral and anti-chiral multiplets, Ω and $\tilde{\Omega}$ — in terms of superfields, we have:

$$\Omega = \omega + \sqrt{2}\theta\psi^\Omega + \theta\theta F^\Omega + \dots, \quad \tilde{\Omega} = \tilde{\omega} + \sqrt{2}\tilde{\theta}\tilde{\psi}^\Omega + \tilde{\theta}\tilde{\theta}\tilde{F}^\Omega + \dots. \quad (3.123)$$

Let us define the gauge parameters α and γ by:

$$\omega = \alpha + i\gamma, \quad \tilde{\omega} = \alpha - i\gamma. \quad (3.124)$$

In components, (3.122) is equivalent to:

$$\begin{aligned} \delta_\Omega C = -\gamma, \quad \delta_\Omega\chi = \frac{1}{\sqrt{2}}\psi^\Omega, \quad \delta_\Omega\tilde{\chi} = \frac{1}{\sqrt{2}}\tilde{\psi}^\Omega, \\ \delta_\Omega a_\mu = \partial_\mu\alpha, \quad \delta_\Omega M = F^\Omega, \quad \delta_\Omega\tilde{M} = \tilde{F}^\Omega, \end{aligned} \quad (3.125)$$

while $\lambda, \tilde{\lambda}$ and D are gauge-invariant. Using this gauge-freedom, we can fix the so-called WZ gauge [23]:

$$C = \chi = \tilde{\chi} = M = \tilde{M} = 0. \quad (3.126)$$

This leaves $\delta_\alpha\mathcal{V}_F$ as a residual U(1) gauge transformation. Obviously, the WZ gauge does not commute with supersymmetry. Nonetheless, one can define a consistent set of supersymmetry transformations by accompanying any “bare” supersymmetry transformation, denoted by $\delta_\zeta^{(0)}$ or $\delta_{\tilde{\zeta}}^{(0)}$, with a gauge transformation to restore the WZ gauge:

$$\delta_\zeta = \delta_\zeta^{(0)} + \delta_{\Omega(\zeta)}, \quad \delta_{\tilde{\zeta}} = \delta_{\tilde{\zeta}}^{(0)} + \delta_{\Omega(\tilde{\zeta})}. \quad (3.127)$$

The compensating gauge transformations are given by specializing (3.125) to the field-dependent gauge parameters:

$$\Omega(\zeta) : \omega = \psi_\alpha^\Omega = F^\Omega = \tilde{\omega} = 0, \quad \tilde{\psi}_\alpha^\Omega = i\sqrt{2}(\zeta\sigma^\mu)_\alpha a_\mu, \quad \tilde{F}^\Omega = -2\zeta\lambda, \quad (3.128)$$

or:

$$\Omega(\tilde{\zeta}) : \psi_\alpha^\Omega = -i\sqrt{2}(\sigma^\mu\tilde{\zeta})_\alpha a_\mu, \quad F^\Omega = -2\tilde{\zeta}\tilde{\lambda}, \quad \omega = \tilde{\omega} = \tilde{\psi}_\alpha^\Omega = \tilde{F}^\Omega = 0, \quad (3.129)$$

respectively.³³

³³In flat space, ζ and $\tilde{\zeta}$ are arbitrary constant spinors. They become our non-trivial Killing spinors ζ and $\tilde{\zeta}$ on \mathcal{M}_4 once we generalize this discussion to curved space.

't Hooft anomaly and supersymmetry. Let us first consider the cubic 't Hooft anomaly:

$$\delta_\alpha W = \frac{i\mathcal{A}_{qqq}}{96\pi^2} \int d^4x \alpha \epsilon^{\mu\nu\rho\sigma} f_{\mu\nu} f_{\rho\sigma}. \quad (3.130)$$

Its supersymmetric completion takes the form [65]:

$$\delta_\Omega W = -\frac{i\mathcal{A}_{qqq}}{48\pi^2} \int d^4x \left[\int d^2\theta \Omega \mathcal{W}^\alpha \mathcal{W}_\alpha - \int d^2\tilde{\theta} \tilde{\Omega} \tilde{\mathcal{W}}_{\dot{\alpha}} \tilde{\mathcal{W}}^{\dot{\alpha}} \right], \quad (3.131)$$

possibly up to the δ_Ω -variation of a local term. Here, we defined the standard field-strength chiral and anti-chiral multiplets:³⁴

$$\mathcal{W}_\alpha = -\frac{1}{4} \tilde{D} \tilde{D} D_\alpha V, \quad \tilde{\mathcal{W}}_{\dot{\alpha}} = -\frac{1}{4} D D \tilde{D}_{\dot{\alpha}} V. \quad (3.132)$$

Expanding in components, (3.131) gives:

$$\begin{aligned} \delta_\Omega W = \frac{i\mathcal{A}_{qqq}}{48\pi^2} \int d^4x \left\{ \alpha \left(\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} f_{\mu\nu} f_{\rho\sigma} + 2i \partial_\mu (\lambda \sigma^\mu \tilde{\lambda}) \right) \right. \\ + i\gamma \left(f_{\mu\nu} f^{\mu\nu} - 2D^2 + 2i\lambda \sigma^\mu \partial_\mu \tilde{\lambda} + 2i\tilde{\lambda} \tilde{\sigma}^\mu \partial_\mu \lambda \right) + F^\Omega \lambda \lambda - \tilde{F}^\Omega \tilde{\lambda} \tilde{\lambda} \\ \left. + \sqrt{2} \left(\psi^\Omega \sigma^{\mu\nu} \lambda - \tilde{\psi}^\Omega \tilde{\sigma}^{\mu\nu} \tilde{\lambda} \right) f_{\mu\nu} - i\sqrt{2} \left(\psi^\Omega \lambda + \tilde{\psi}^\Omega \tilde{\lambda} \right) D \right\}. \end{aligned} \quad (3.133)$$

Specializing to the WZ gauge, we see that (3.133) reproduces (3.130) up to the variation of a local term — namely, the cubic 't Hooft anomaly $\delta_\alpha W$ in (3.130) is given by:

$$\delta_\alpha W = \delta_\Omega W \Big|_{\text{WZ}} + \delta_\alpha W_{\text{loc}}, \quad (3.134)$$

with the simple local term:

$$W_{\text{loc}} \equiv -\frac{\mathcal{A}_{qqq}}{24\pi^2} \int d^4x a_\mu \lambda \sigma^\mu \tilde{\lambda}. \quad (3.135)$$

The gauge variation of (3.135) in (3.134) cancels the second term on the first line of (3.133).

Anomalous supersymmetry variation in WZ gauge. Before fixing the Wess-Zumino gauge, the “bare” supersymmetry transformations are non-anomalous [66]:

$$\delta_\zeta^{(0)} W = 0, \quad \delta_{\tilde{\zeta}}^{(0)} W = 0. \quad (3.136)$$

This is the statement that, on the one hand, there is no genuine supersymmetry anomaly; and, on the other hand, the 't Hooft anomaly (3.131) does not contribute any quantum correction to (3.136). The latter statement directly follows from the WZ consistency condition:

$$[\delta_\zeta^{(0)}, \delta_\Omega] W = 0, \quad (3.137)$$

³⁴With the usual definitions: $D_\alpha = \partial_\alpha + i(\sigma^\mu \tilde{\theta})_\alpha \partial_\mu$ and $\tilde{D}_{\dot{\alpha}} = -\tilde{\partial}_{\dot{\alpha}} - i(\theta \sigma^\mu)_{\dot{\alpha}} \partial_\mu$.

and the fact that (3.131) is supersymmetric, $\delta_\zeta^{(0)}\delta_\Omega W = 0$. The WZ gauge fixing modifies the supersymmetry algebra, introducing an explicit dependence on the gauge fields due to (3.128)–(3.129). The compensating gauge transformation also introduces non-vanishing WZ-gauge supersymmetry variations, since:

$$\begin{aligned}\delta_\zeta W &= \left(\delta_\zeta^{(0)} + \delta_{\Omega(\zeta)}\right)W = \delta_{\Omega(\zeta)}W, \\ \delta_{\tilde{\zeta}} W &= \left(\delta_{\tilde{\zeta}}^{(0)} + \delta_{\Omega(\tilde{\zeta})}\right)W = \delta_{\Omega(\tilde{\zeta})}W,\end{aligned}\tag{3.138}$$

and $\delta_\Omega W \neq 0$ due to the U(1) 't Hooft anomalies. This gives an alternative way to compute the anomalous variations $\delta_\zeta W$ in WZ gauge. To compare to the results from the previous subsections, we should take into account the fact that we are working in a scheme in which the cubic anomaly is given by (3.130) exactly. This corresponds to (3.134), including the contribution from the local term (3.135). We thus have:

$$\begin{aligned}\delta_\zeta W &= \delta_{\Omega(\zeta)}W + \delta_\zeta W_{\text{loc}}, \\ \delta_{\tilde{\zeta}} W &= \delta_{\Omega(\tilde{\zeta})}W + \delta_{\tilde{\zeta}} W_{\text{loc}}.\end{aligned}\tag{3.139}$$

Here, the gauge variation (3.133) is specialized to $\Omega = \Omega(\zeta)$ or $\Omega = \Omega(\tilde{\zeta})$, and the second term is the (WZ-gauge) supersymmetry variation of (3.135). One finds:

$$\begin{aligned}\delta_{\Omega(\zeta)}W &= \frac{A_{qqq}}{24\pi^2} \int d^4x \left(\zeta \sigma^\mu \tilde{\sigma}^{\nu\rho} \tilde{\lambda} a_\mu f_{\nu\rho} + i\zeta \sigma^\mu \tilde{\lambda} a_\mu D + i\zeta \lambda \tilde{\lambda} \tilde{\lambda} \right), \\ \delta_{\Omega(\tilde{\zeta})}W &= \frac{A_{qqq}}{24\pi^2} \int d^4x \left(-\tilde{\zeta} \tilde{\sigma}^\mu \sigma^{\nu\rho} \lambda a_\mu f_{\nu\rho} + i\tilde{\zeta} \tilde{\sigma}^\mu \lambda a_\mu D - i\tilde{\zeta} \lambda \lambda \lambda \right),\end{aligned}\tag{3.140}$$

and:

$$\begin{aligned}\delta_\zeta W_{\text{loc}} &= \frac{A_{qqq}}{24\pi^2} \int d^4x \left(\zeta \sigma^{\nu\rho} \sigma^\mu \tilde{\lambda} a_\mu f_{\nu\rho} - i\zeta \sigma^\mu \tilde{\lambda} a_\mu D + 2i\zeta \lambda \tilde{\lambda} \tilde{\lambda} \right), \\ \delta_{\tilde{\zeta}} W_{\text{loc}} &= \frac{A_{qqq}}{24\pi^2} \int d^4x \left(-\tilde{\zeta} \tilde{\sigma}^{\nu\rho} \tilde{\sigma}^\mu \lambda a_\mu f_{\nu\rho} - i\tilde{\zeta} \tilde{\sigma}^\mu \lambda a_\mu D - 2i\tilde{\zeta} \lambda \lambda \lambda \right).\end{aligned}\tag{3.141}$$

It follows that (3.139) exactly reproduces the result (3.55), which was obtained by solving the WZ consistency conditions for supersymmetry and gauge transformations directly in WZ gauge.

Note also that, even if we worked in WZ gauge in the scheme defined by (3.133) with $\Omega = \tilde{\Omega} = \alpha$, in which case the anomalous supersymmetry variation takes the more cumbersome form (3.140), we would still obtain the same results for supersymmetric partition functions, because the local term (3.135) vanishes on the supersymmetric locus for the background vector multiplet.

One can similarly discuss the mixed U(1)-gravitational anomaly in this language, generalized to supergravity. The supersymmetrization of the anomaly is similar to (3.131), with \mathcal{W}_α replaced by a chiral multiplet, $\mathcal{T}_{\mu\nu\alpha}$, that contains the Riemann tensor in its θ -component [60]. The bottom and $\theta\theta$ components of $\mathcal{T}_{\mu\nu\alpha}$ vanish once we set the gravitino to zero and therefore, using (3.138), we find that the mixed U(1)-gravitational anomaly does not contribute to $\delta_\zeta W$ on a rigid-supersymmetric background.

4 4d $\mathcal{N} = 1$ supersymmetric partition functions, revisited

In this section, as an application of the above formalism, we study how the supersymmetric Ward identities constrain supersymmetric partition functions on some specific half-BPS backgrounds. This leads us to discuss the explicit form of the partition functions $Z(\nu, \tau)$ on $\mathcal{M}_3 \times S^1$, which differ from the well-known holomorphic results, $\mathcal{I}(\nu, \tau)$, in the Casimir-like prefactor.

4.1 The $\mathcal{M}_{g,p} \times S^1$ background

We will focus on the special case of \mathcal{M}_4 a principal elliptic fiber bundle over a closed genus- g Riemann surface:

$$T^2 \longrightarrow \mathcal{M}_4 \xrightarrow{\pi} \Sigma_g. \quad (4.1)$$

In that case, the four-manifold necessarily takes the form:

$$\mathcal{M}_4 \cong \mathcal{M}_{g,p} \times S^1, \quad S^1 \xrightarrow{p} \mathcal{M}_{g,p} \xrightarrow{\pi} \Sigma_g, \quad (4.2)$$

where the three-manifold $\mathcal{M}_{g,p}$ is a degree- p principal circle bundle over Σ_g . As a special case, this includes $S^3 \times S^1$ (for $p = 1, g = 0$) and $\Sigma_g \times T^2$ (for $p = 0$). The corresponding supersymmetric backgrounds were studied e.g. in [5, 10, 11, 25–27].³⁵ The metric on $\mathcal{M}_{g,p} \times S^1$ is chosen to be:

$$ds^2(\mathcal{M}_{g,p} \times S^1) = \beta_1^2 (d\psi + \tau_1 dt + \mathcal{C}(z, \bar{z}))^2 + \beta_2^2 dt^2 + 2g_{z\bar{z}} dz d\bar{z}, \quad (4.3)$$

with the angular coordinates $t \in [0, 2\pi)$ and $\psi \in [0, 2\pi)$, with t the coordinate along the circle direction. In (4.3), $g_{z\bar{z}}$ is the Hermitian metric on Σ_g , which we normalize such that $\text{vol}(\Sigma_g) = \pi$; the one-form \mathcal{C} is the connection of a principal circle bundle over Σ , with first Chern class p , which satisfies:

$$\partial_z \mathcal{C}_{\bar{z}} - \partial_{\bar{z}} \mathcal{C}_z = p 2i g_{z\bar{z}}, \quad \mathcal{C} = \pi^*(\mathcal{C}_\Sigma), \quad \frac{1}{2\pi} \int_{\Sigma_g} d\mathcal{C}_\Sigma = p \in \mathbb{Z}, \quad (4.4)$$

with \mathcal{C}_Σ a two-dimensional gauge field on Σ_g which pulls back to \mathcal{C} . The complex coordinates w along the T^2 fiber is given explicitly by:

$$w = \psi + \tau t + f(z, \bar{z}), \quad \mathcal{C}_z = \partial_z \bar{f}, \quad \mathcal{C}_{\bar{z}} = \partial_{\bar{z}} f, \quad (4.5)$$

where the complex function $f(z, \bar{z})$ is related to the one-form \mathcal{C} as shown, and we introduced the complex parameter:

$$\tau = \tau_1 + i\tau_2, \quad \tau_2 \equiv \frac{\beta_2}{\beta_1}. \quad (4.6)$$

This τ is a *complex structure modulus* of the Hermitian manifold $\mathcal{M}_4 \cong \mathcal{M}_{g,p} \times S^1$ — of course, it is also the complex structure of the T^2 fiber itself; note that it is kept constant over the base, Σ_g . The Killing vector K^μ is given by:

$$K = 2\beta_1^{-1} \partial_{\bar{w}} = -i\beta_2^{-1} (\tau \partial_\psi - \partial_t). \quad (4.7)$$

³⁵We closely follow the discussion in [5], with only slightly different conventions; in particular, the sign of the complex structure here differs to the one in [5] by a sign.

The metric (4.3) is equivalent to the canonical expression (3.15)–(3.18), with:

$$e^1 \equiv e^{(1)} + ie^{(2)}, \quad e^{(1)} = \beta_1(d\psi + \tau_1 dt + \mathcal{C}(z, \bar{z})), \quad e^{(2)} = \beta_2 dt. \quad (4.8)$$

and $e^2 \equiv e^{(3)} + ie^{(4)}$ as given in (3.18). The other new-minimal supergravity background fields, V_μ and $A_\mu^{(R)}$, can be computed from the general formula (3.8) with (3.17). This gives:

$$V_\mu dX^\mu = p\beta_1 e^{(1)} + \kappa K_\mu dX^\mu, \quad A_\mu^{(R)} dX^\mu = -p\beta_1 e^{(1)} + \frac{ip\beta_1}{2} e^{(2)} + A_\mu^c dX^\mu, \quad (4.9)$$

with A_μ^c defined in (3.9). The gauge field A_μ^c is the pull-back of a non-trivial two-dimensional R -symmetry gauge field with $1 - g$ units of flux on Σ_g :

$$A^c = \pi^*(A_\Sigma), \quad \frac{1}{2\pi} \int_{\Sigma_g} dA_\Sigma = 1 - g. \quad (4.10)$$

The free parameter κ in (4.9) can be fixed by requiring the existence of a consistent dimensional reduction of this 4d $\mathcal{N} = 1$ supergravity background to a corresponding three-dimensional $\mathcal{N} = 2$ supergravity background on $\mathcal{M}_{g,p}$ [42]. This imposes:

$$\kappa = p\beta_1 \frac{\bar{\tau}}{\tau}. \quad (4.11)$$

This is an important point for the computations that follow, since only with this particular choice, (4.11), do we have a consistent reduction of the 4d $\mathcal{N} = 1$ half-BPS background to a 3d $\mathcal{N} = 2$ half-BPS background on the three-manifold $\mathcal{M}_{g,p}$.

Kaluza-Klein form of the metric. In order to discuss the 3d reduction, it is useful to work with a slightly different frame, obtained from (4.8) by an $\text{SO}(2)$ rotation:

$$\begin{pmatrix} \eta \\ \hat{e}^0 \end{pmatrix} \equiv \frac{1}{|\tau|} \begin{pmatrix} \tau_2 & -\tau_1 \\ \tau_1 & \tau_2 \end{pmatrix} \begin{pmatrix} e^{(1)} \\ e^{(2)} \end{pmatrix}. \quad (4.12)$$

Let us define the radii:

$$\beta_{S^1} \equiv \beta_1 |\tau|, \quad \beta_{3d} \equiv \frac{\beta_2}{|\tau|}. \quad (4.13)$$

In term of these, (4.12) gives us the one-forms:

$$\eta = \beta_{3d}(d\psi + \mathcal{C}), \quad \hat{e}^0 = \beta_{S^1} dt + c_\mu dx^\mu, \quad (4.14)$$

with c_μ , the so-called graviphoton, given by:

$$c_\mu = \frac{\tau_1}{\tau_2} \eta_\mu. \quad (4.15)$$

This brings the metric into the Kaluza-Klein (KK) form:

$$ds^2(\mathcal{M}_{g,p} \times S^1) = (\beta_{S^1} dt + c_\mu dx^\mu)^2 + ds^2(\mathcal{M}_{g,p}). \quad (4.16)$$

Note that, in our conventions, we have the volumes:

$$\text{vol}(\mathcal{M}_{g,p}) = 2\pi^2 \beta_{3d}, \quad \text{vol}(\mathcal{M}_{g,p} \times S^1) = 4\pi^3 \beta_{S^1} \beta_{3d} = 4\pi^3 \beta_1 \beta_2. \quad (4.17)$$

Supersymmetric background on $\mathcal{M}_{g,p}$. In the following, we will also need the explicit form of the corresponding 3d $\mathcal{N} = 2$ supersymmetric background on $\mathcal{M}_{g,p}$, as discussed in general terms in section 3.6.1. The three-dimensional metric on $\mathcal{M}_{g,p}$ takes the simple form:

$$ds^2(\mathcal{M}_{g,p}) = \eta^2 + ds^2(\Sigma_g), \tag{4.18}$$

in the coordinates (ψ, z, \bar{z}) adapted to the THF. Here, η is defined as in (4.14), and $ds^2(\Sigma_g) = 2g_{z\bar{z}} dzd\bar{z}$ is the metric on Σ_g . This manifold is also a Seifert manifold,³⁶ with ψ the coordinate on the Seifert fiber of constant radius β_{3d} . The other three-dimensional supergravity fields are given by:

$$\mathbf{H} = -ip\beta_{3d} + 2p\beta_{3d}\frac{\tau_1}{\tau_2}, \quad \mathbf{V}_\mu = -2ip\beta_{3d}\frac{\tau_1}{\tau_2}\eta_\mu, \quad \mathbf{A}_\mu^{(R)} = -p\beta_{3d}\eta_\mu + A_\mu^c, \tag{4.19}$$

with A_μ^c defined as in (4.10). The three-dimensional background field \mathbf{V}_μ is the dual field strength of a *real* graviphoton c_μ , with:³⁷

$$\mathbf{V}^\mu = -i\epsilon^{\mu\nu\rho}\partial_\nu c_\rho, \quad c_\mu = \frac{\tau_1}{\tau_2}\eta_\mu. \tag{4.20}$$

This is as expected for a 3d background obtained by dimensional reduction from 4d, with the KK metric (4.16). Requiring the existence of this supersymmetric reduction to 3d uniquely fixes the parameter κ in the 4d background to (4.11).

4.2 Supersymmetric Ward identity on $\mathcal{M}_{g,p} \times S^1$

Let us evaluate $\delta_\zeta W$ on the $\mathcal{M}_{g,p} \times S^1$ supergravity background (focussing on the cubic 't Hooft anomaly). We may write the anomaly in terms of the gaugino one-forms defined in (3.58). In the complex frame basis, they are given by:

$$\tilde{\Lambda} = \tilde{\Lambda}_1 e^1 + \tilde{\Lambda}_2 e^2, \quad \Lambda = \Lambda_1 e^1 + \Lambda_2 e^{\bar{2}}. \tag{4.21}$$

We will choose $\tilde{\Lambda}_1, \tilde{\Lambda}_2$ constant, for our purposes. Then, the cubic-anomaly contribution (3.55) takes the simple form:

$$\delta_\zeta W|_{q^3} = -\frac{A_{qqq}}{12\pi^2} \int \tilde{\Lambda} \wedge a \wedge da, \tag{4.22}$$

at first order in the gaugino.³⁸ The 4d background U(1) gauge field takes the form:

$$a = \hat{a}_1 e^1 + \hat{a}_{\bar{1}} e^{\bar{1}} + \mathbf{a}^{(m)}, \quad \hat{a}_1 = \frac{\hat{a}_w}{\beta_1}, \quad \hat{a}_{\bar{1}} = \frac{\hat{a}_{\bar{w}}}{\beta_1}, \tag{4.23}$$

in terms of the complex frame, with $\hat{a}_{\bar{w}} = (\hat{a}_w)^* \in \mathbb{C}$ an arbitrary complex parameter. The term $\mathbf{a}^{(m)}$ in (4.23) denotes a topologically non-trivial gauge field with flux $\mathbf{m} \in \mathbb{Z}$. If $p = 0$, we have $\mathcal{M}_4 \cong \Sigma_g \times T^2$ and $-\mathbf{m}$ units of flux through the Riemann surface Σ_g .³⁹

$$\frac{1}{2\pi} \int_{\Sigma_g} d\mathbf{a}^{(m)} = -\mathbf{m}, \quad \text{if } p = 0. \tag{4.24}$$

³⁶See [58] for a systematic study of 3d $\mathcal{N} = 2$ supersymmetric backgrounds on Seifert manifolds.

³⁷We have the useful relation: $\epsilon^{\mu\nu\rho}\partial_\nu\eta_\rho = 2p\beta_{3d}\eta^\mu$ in this geometry.

³⁸The higher-order terms in λ can be ignored for our purpose, since they do not affect the anomalous Ward identities (3.74).

³⁹The choice of $-\mathbf{m}$ instead of $+\mathbf{m}$ here is to conform to the definition of the ‘‘flux operator’’ in [5], where the orientation convention for Σ was the opposite as it is here.

If $p \neq 0$, on the other hand, the flux \mathbf{m} is a torsion flux valued in \mathbb{Z}_p (more precisely, we have a large gauge transformation that identifies $(\nu, \mathbf{m}) \sim (\nu + \tau, \mathbf{m} + p)$), and $\mathbf{a}^{(\mathbf{m})}$ is a flat connection with non-trivial holonomy along the Seifert fiber of the three-manifold $\mathcal{M}_{g,p}$:

$$e^{i \int_\gamma \mathbf{a}^{(\mathbf{m})}} = e^{2\pi i \frac{\mathbf{m}}{p}}, \quad \text{if } p \neq 0. \quad (4.25)$$

While the two cases, $p = 0$ or $p \neq 0$, must be treated separately, the final result is valid for every $\mathcal{M}_{g,p} \times S^1$ background. The gauge field component $a_{\bar{w}}$ are related to the above parameters as:

$$a_{\bar{w}} = \begin{cases} \hat{a}_{\bar{w}} & \text{if } p = 0 \\ \hat{a}_{\bar{w}} - \frac{i\tau}{2\tau_2} \frac{\mathbf{m}}{p}. & \text{if } p \neq 0 \end{cases}, \quad (4.26)$$

and similarly for the complex conjugate a_w . The flavor parameters $\nu, \bar{\nu}$ are then defined as:

$$\nu = 2i\tau_2 a_{\bar{w}} \equiv \tau a_\psi - a_t, \quad \bar{\nu} = -2i\tau_2 a_w = \bar{\tau} a_\psi - a_t. \quad (4.27)$$

Here, for future reference, we also gave these expressions in terms of the real gauge-field components a_ψ and a_t .

The case $p \neq 0, \mathbf{m} = 0$. Consider first the case without flux. When $\mathbf{m} = 0$, the gauge-field (4.23) is a well-defined one-form, and the integral (4.22) can be evaluated straightforwardly. One finds:

$$\delta_\zeta W|_{q^3, \mathbf{m}=0} = -\frac{\pi i p \mathcal{A}_{qqq} \tilde{\Lambda}_w \nu (\nu - \bar{\nu})}{3\tau_2}. \quad (4.28)$$

The general case. In the general case with $\mathbf{m} \neq 0$, the integral (4.22) is not well-defined, strictly speaking. Nonetheless, we claim that the general result should be:

$$\delta_\zeta W|_{q^3} = -\pi \mathcal{A}_{qqq} \tilde{\Lambda}_w \left(\frac{ip}{3\tau_2} \nu (\nu - \bar{\nu}) + \mathbf{m}\nu \right). \quad (4.29)$$

The term linear in \mathbf{m} can be deduced by requiring consistency with explicit results for one-loop determinants and with the 3d reduction, as we will discuss below. A completely rigorous derivation of (4.29) would require an *a-priori* definition of (4.22) for topologically non-trivial gauge fields, possibly akin to the definition of the 3d Chern-Simons functional in terms of a bounding four-manifold. We leave this as an interesting question for future work.

4.2.1 The holomorphy anomaly on $Z_{\mathcal{M}_{g,p} \times S^1}$

Given the above discussion, the holomorphy anomaly of the supersymmetric partition function (in the flavor sector) follows from (3.74). We have:

$$\frac{\partial W}{\partial \bar{\nu}} = \frac{1}{2\tau_2} \frac{\partial}{\partial \tilde{\Lambda}_w} \delta_\zeta W. \quad (4.30)$$

We also see that, since the supersymmetry variation (4.29) is independent of $\tilde{\Lambda}_z$ (and, similarly, $\delta_\zeta W$ is independent of $\Lambda_{\bar{z}}$), there is no dependence of the partition function on background flat connections on the Riemann surface base, Σ_g .

The supersymmetry variation (4.29) contributes the following holomorphy anomaly in the flavor parameters:

$$\frac{\partial W}{\partial \bar{\nu}} \Big|_{q^3} = -\pi \mathcal{A}_{qqq} \left(\frac{ip}{6\tau_2^2} \nu(\nu - \bar{\nu}) + \frac{\mathbf{m}}{2\tau_2} \nu \right). \quad (4.31)$$

The generalization to any abelian symmetry is straightforward.

4.2.2 The small- β_{S^1} limit and the 3d functional \mathbb{W}_{3d}

Let us now compute \mathbb{W}_{3d} , as defined in section 3.6.2. It is obtained in the limit $\tau \rightarrow 0$ with ν/τ kept finite. We again focus on the cubic 't Hooft anomaly contribution. We should then evaluate the local action:

$$\mathbb{W}_{3d}|_{q^3} = \frac{\mathcal{A}_{qqq}\beta_{S^1}}{12\pi} \int d^3x \sqrt{g} \left(-i\sigma \epsilon^{\mu\nu\rho} A_\mu F_{\nu\rho} + 3\sigma^2 D + \sigma^2 A_\mu \mathbf{V}^\mu + \sigma^3 \mathbf{H} \right), \quad (4.32)$$

on the 3d $\mathcal{N} = 2$ supergravity background (4.18)–(4.19). We also consider the background vector multiplet on its supersymmetric locus, (3.92). Then, the three-dimensional gauge field takes the form:

$$A_\mu = \hat{a}_0 \eta_\mu + \mathbf{A}^{(\mathbf{m})}, \quad (4.33)$$

with $\hat{a}_0 \in \mathbb{R}$ a constant, and with the non-trivial gauge field $\mathbf{A}^{(\mathbf{m})}$ defined exactly as $\mathbf{a}^{(\mathbf{m})}$ in (4.24) or (4.25). Let us also introduce the real parameter:

$$a_0 \equiv A_\mu \eta^\mu = \begin{cases} \hat{a}_0 & \text{if } p = 0 \\ \hat{a}_0 + \frac{1}{\beta_{3d}} \frac{\mathbf{m}}{p} & \text{if } p \neq 0 \end{cases}. \quad (4.34)$$

The three-dimensional flavor parameters σ and a_0 are related to the 4d flavor parameters (4.27) by:

$$\frac{\nu}{\tau} = \beta_{3d}(a_0 + i\sigma), \quad \frac{\bar{\nu}}{\tau} = \beta_{3d}(a_0 - i\sigma). \quad (4.35)$$

Equivalently, we have:

$$\beta_{3d}\sigma = \frac{\tau_2 a_t}{|\tau|^2}, \quad \beta_{3d}a_0 = a_\psi - \frac{\tau_1 a_t}{|\tau|^2}, \quad (4.36)$$

in terms of the 4d gauge-field components a_t and a_ψ . Similarly to the 4d expression (4.22) for $\delta_\zeta W$, the action (4.32) is only well-defined for $\mathbf{m} = 0$. We claim that the complete expression for (4.32) on this supersymmetric background is:

$$\mathbb{W}_{3d}|_{q^3} = \pi \mathcal{A}_{qqq} \beta_{S^1} \beta_{3d} \left\{ p \beta_{3d} \left(-\frac{2i}{3} \sigma a_0^2 + \left(1 - i \frac{\tau_1}{3\tau_2}\right) \sigma^2 a_0 + \frac{\tau_1 + i\tau_2}{3\tau_2} \sigma^3 \right) + \mathbf{m} (i\sigma a_0 - \sigma^2) \right\}. \quad (4.37)$$

For $\mathbf{m} = 0$, this is obtained by a straightforward computation using the supergravity and flavor background fields. In the general case with $\mathbf{m} \neq 0$, this is essentially a conjecture

— assuming there exists a first-principle definition of the 3d action for non-trivial gauge fields. Using the relation (4.35), one can write (4.37) as:

$$\mathbb{W}_{3d}|_{q^3} = \frac{i\pi p \mathcal{A}_{qqq}}{\tau_2^2} \left(\frac{\nu \bar{\nu}^2}{12} - \frac{\nu^2 \bar{\nu}}{6} + \frac{\bar{\tau}(2\tau - \bar{\tau})\nu^3}{12\tau^2} \right) - \frac{\pi \mathbf{m} \mathcal{A}_{qqq}}{2\tau_2} \left(\nu \bar{\nu} - \frac{\bar{\tau}\nu^2}{\tau} \right). \quad (4.38)$$

In particular, we see that \mathbb{W}_{3d} reproduces the holomorphy anomaly (4.31) — as it should, by construction. It will also be useful to give (4.38) in terms of the real parameters a_ψ and a_t ; we have:

$$\mathbb{W}_{3d}(a_\psi, a_t, \tau)|_{q^3} = \frac{i\pi \mathcal{A}_{qqq}}{3\tau^2} a_t(\tau a_\psi - a_t)(p(a_t - 2\tau a_\psi) + 3\mathbf{m}\tau). \quad (4.39)$$

Incidentally, $\mathbb{W}_{3d}|_{q^3}$ is holomorphic in τ when viewed as a function of (a_ψ, a_t) and τ . Finally, for future reference, let us also give the explicit expression for the real part:

$$\text{Re} \left[\mathbb{W}_{3d}|_{q^3} \right] = \frac{\pi \mathcal{A}_{qqq}}{|\tau|^2} \left(\tau_2(p a_\psi - \mathbf{m}) a_t^2 - \frac{2p\tau_1\tau_2}{3|\tau|^2} a_t^3 \right). \quad (4.40)$$

4.3 The holomorphic supersymmetric partition function

In the rest of this section, we would like to discuss the “gauge invariant” supersymmetric partition functions $Z_{\mathcal{M}_{g,p} \times S^1}$, in light of our general results for the supersymmetric Ward identities in WZ gauge. Before proceeding, let us first review the standard expressions for the holomorphic partition functions.

4.3.1 The $S^3 \times S^1$ partition function

The most studied 4d $\mathcal{N} = 1$ supersymmetric partition function is certainly the $S^3 \times S^1$ partition function, which computes the 4d $\mathcal{N} = 1$ supersymmetric index:

$$\mathcal{I}_{S^3 \times S^1}(\mathbf{p}, \mathbf{q}, y) = \text{Tr}_{S^3} \left((-1)^F \mathbf{p}^{J_3 + J'_3 + \frac{1}{2}R} \mathbf{q}^{J_3 - J'_3 + \frac{1}{2}R} \prod_{\alpha} y_{\alpha}^{Q_{\alpha}^F} \right). \quad (4.41)$$

The parameters \mathbf{p} and \mathbf{q} are complex structure parameters on $S^3 \times S^1$ seen as a Hopf surface [6], and $y_{\alpha} = e^{2\pi i \nu_{\alpha}}$ are the flavor parameters. In this work, we will only discuss the special case:

$$\mathbf{p} = \mathbf{q} = q = e^{2\pi i \tau}, \quad (4.42)$$

which corresponds to $(g, p) = (0, 1)$ for the $\mathcal{M}_{g,p} \times S^1$ background defined above [5].

The S^3 index is easily computed for a free theory. For a free chiral multiplet, Φ , of unit charge under a background $U(1)$ and of R -charge $r \in \mathbb{R}$, we have:

$$\mathcal{I}_{S^3 \times S^1}^{\Phi}(\nu, \tau) = \mathcal{F}_1^{\Phi}(\nu + \tau(r - 1), \tau), \quad (4.43)$$

in terms of the function [5]:

$$\mathcal{F}_1^{\Phi}(\nu, \tau) \equiv e^{2\pi i \tau \left(\frac{\nu^3}{6\tau^3} - \frac{\nu}{12\tau} \right)} \Gamma_0(\nu, \tau). \quad (4.44)$$

Here, the “restricted elliptic Γ -function,” Γ_0 , is the $\mathbf{p} = \mathbf{q}$ limit of the elliptic- Γ function, defined as:⁴⁰

$$\Gamma_0(\nu, \tau) = \prod_{k=0}^{\infty} \left(\frac{1 - e^{2\pi i(-\nu+(k+1)\tau)}}{1 - e^{2\pi i(\nu+(k+1)\tau)}} \right)^{k+1}, \quad (4.45)$$

for τ on the upper-half plane. The prefactor in (4.43) is the contribution from the supersymmetric Casimir energy for a free chiral multiplet [10, 67]. More generally, there exists an explicit formula for any 4d $\mathcal{N} = 1$ gauge theory, with gauge group \mathbf{G} and chiral multiplets Φ_i in representations \mathfrak{R} of the gauge algebra \mathfrak{g} . It is given by [10, 30]:

$$\mathcal{I}_{S^3 \times S^1}(\nu, \tau) = e^{2\pi i \tau \mathcal{E}(\nu, \tau)} \mathbf{I}_{S^3}(\nu, \tau). \quad (4.46)$$

Here, \mathbf{I}_{S^3} denotes the “ordinary” 4d $\mathcal{N} = 1$ index; for SCFTs, it starts at 1 when expanded in powers of q [3, 4, 68, 69]:

$$\mathbf{I}_{S^3} \equiv \frac{(q; q)_{\infty}^{2\text{rk}(\mathbf{G})}}{|W_{\mathbf{G}}|} \oint_{\prod_a \mathbb{T}_{x_a}} \prod_{a=1}^{\text{rk}(\mathbf{G})} \frac{dx_a}{2\pi i x_a} \frac{\prod_i \prod_{\rho_i} \Gamma_0(\rho_i(u) + \nu_i + \tau(r_i - 1), \tau)}{\prod_{\alpha \in \mathfrak{g}} \Gamma_0(\alpha(u) - \tau, \tau)}, \quad (4.47)$$

with $x_a \equiv e^{2\pi i u_a}$. The supersymmetric Casimir energy contribution in (4.46) is given entirely in terms of the ’t Hooft anomalies [30, 31, 70]:

$$\mathcal{E}(\nu, \tau) = \frac{1}{6\tau^3} \mathcal{A}^{\alpha\beta\gamma} \nu_{\alpha} \nu_{\beta} \nu_{\gamma} - \frac{1}{12\tau} \mathcal{A}^{\alpha} \nu_{\alpha} + \dots, \quad (4.48)$$

where the ellipsis denotes contributions from the ’t Hooft anomaly coefficients involving the R -symmetry.

Behavior under large gauge transformations. On $S^3 \times S^1$, one has the equivalence $\nu \sim \nu + 1$ for the flavor parameter, corresponding to the large gauge transformation $a_t \sim a_t - 1$ along the S^1 . While the “pure index” (4.47) is clearly gauge invariant, the holomorphic partition function (4.46) is not. Due to the supersymmetric Casimir energy contribution, we have:⁴¹

$$\mathcal{I}_{S^3 \times S^1}(\nu + n, \tau) = (-1)^{\mathcal{A}_q^{\alpha} n_{\alpha}} e^{\frac{\pi i}{\tau^2} \mathcal{A}_{qq}^{\alpha\beta\gamma} (n_{\alpha} \nu_{\beta} \nu_{\gamma} + n_{\alpha} n_{\beta} \nu_{\gamma} + \frac{1}{3} n_{\alpha} n_{\beta} n_{\gamma})} \mathcal{I}_{S^3 \times S^1}(\nu, \tau), \quad (4.49)$$

under the transformation $\nu_{\alpha} \rightarrow \nu_{\alpha} + n_{\alpha}$ for the flavor parameters. Similarly to the case of the elliptic genus discussed in section 2, the puzzling aspect of the transformation (4.49) is that the absolute value of $\mathcal{I}_{S^3 \times S^1}$ is not gauge-invariant. As we discussed, it cannot possibly be gauge invariant whenever the partition function is holomorphic.

The three-dimensional limit. Consider the small- β_{S^1} limit of the holomorphic partition function, namely:

$$\tau \rightarrow 0, \quad \nu \rightarrow 0, \quad \frac{\nu}{\tau} = \text{finite}. \quad (4.50)$$

⁴⁰The definition of Γ_0 corresponds to the limit $\mathbf{q} = \mathbf{p}$ of the elliptic Γ -function with a shifted argument $\nu + \tau$. See appendix B.

⁴¹Here we ignored the effect of the mixed flavor- $U(1)_R$ anomalies, for simplicity.

In this limit, we claim that:

$$-\log \mathcal{I}_{S^3 \times S^1}(\nu, \tau) \underset{\beta_{S^1} \rightarrow 0}{\sim} \frac{1}{\beta_{S^1}} W_{3d}^{(-1)} + W_{3d}^{(0)} + \mathcal{O}\left(e^{-\frac{1}{\beta_{S^1}}}\right). \quad (4.51)$$

Here, $W_{3d}^{(-1)}$ is given by the Cardy-like formula discussed in [33, 62], and $W_{3d}^{(0)}$ is essentially the partition function of the dimensionally-reduced theory. The important point, for our purpose, is that there is no order- β_{S^1} term in the expansion (4.51). This requires the additional assumption that the S^3 partition function of the dimensionally reduced theory is finite [43]. The resulting expansion is analogous to the expansion (2.63) for the 2d elliptic genus.

Note that the supersymmetric Casimir energy in (4.48) precisely survives in the limit (4.50). Therefore, the property (4.51) is equivalent to the statement that the order- β_{S^1} term in the expansion of the ordinary index is governed by the supersymmetric Casimir energy:

$$-\log \mathbf{I}_{S^3}(\nu, \tau) \underset{\beta_{S^1} \rightarrow 0}{\sim} \frac{1}{\beta_{S^1}} W_{3d}^{(-1)} + W_{3d}^{(0)} + 2\pi i \tau \mathcal{E}(\nu, \tau) + \mathcal{O}\left(e^{-\frac{1}{\beta_{S^1}}}\right). \quad (4.52)$$

For general gauge theories, using the explicit localisation formula (4.47), this expansion follows from the estimate:

$$-\log \Gamma_0(\tau \tilde{\nu}, \tau) \underset{\tau \rightarrow 0}{\sim} \frac{2\pi i}{\tau} \frac{\tilde{\nu}}{12} + \mathbf{w}^{(0)}(\tilde{\nu}) + 2\pi i \tau \left(\frac{\tilde{\nu}^3}{6} - \frac{\tilde{\nu}}{12} \right) + \mathcal{O}\left(e^{-\frac{1}{\beta_{S^1}}}\right), \quad (4.53)$$

for the restricted elliptic Γ -function (4.45), with $\tilde{\nu}$ held fixed (and modulo $2\pi i$, corresponding to a choice of branch cut for the log). Here, the finite term is given by:

$$\begin{aligned} \mathbf{w}^{(0)}(\tilde{\nu}) &= -\log \left(e^{-\frac{\pi i}{2} \tilde{\nu}^2 + \frac{\pi i}{12}} f_{\Phi}(\tilde{\nu}) \right), \\ f_{\Phi}(\tilde{\nu}) &\equiv \exp \left(\frac{1}{2\pi i} \text{Li}_2 \left(e^{2\pi i \tilde{\nu}} \right) + \tilde{\nu} \log \left(1 - e^{2\pi i \tilde{\nu}} \right) \right). \end{aligned} \quad (4.54)$$

corresponding to a free chiral multiplet on the round S^3 [71], with $\tilde{\nu} \equiv \nu/\tau$ the 3d flavor parameter (4.35). The expansion (4.53) was proven in [43].

4.3.2 The $\mathcal{M}_{g,p} \times S^1$ partition function

The $\mathcal{M}_{g,p} \times S^1$ holomorphic partition functions were computed in [5]. They can be written as generalized indices:

$$\mathcal{I}_{\mathcal{M}_{g,p} \times S^1}(\tau, \nu)_{\mathbf{m}} = \text{Tr}_{\mathcal{M}_{g,p}} \left((-1)^F q^{2J+R} \prod_{\alpha} y_{\alpha}^{Q_{\alpha}^F} \right), \quad (4.55)$$

with $q = e^{2\pi i \tau}$ and $y = e^{2\pi i \nu}$ as above. For $p = 0$, we have the topologically-twisted index on $\Sigma_g \times S^1$ [29, 72, 73]; for $p = 1, g = 0$, we recover the S^3 index discussed above. The parameter \mathbf{m} in (4.55) denotes a background flux for the flavor symmetry on $\mathcal{M}_{g,p}$, as discussed in subsection 4.2 above.

For our purpose, we only need to know that the $\mathcal{M}_{g,p} \times S^1$ holomorphic partition function for a 4d $\mathcal{N} = 1$ gauge theory may be written as:

$$\mathcal{I}_{\mathcal{M}_{g,p} \times S^1}(\tau, \nu)_{\mathbf{m}} = \sum_{\hat{u} \in \mathcal{S}_{\text{BE}}} \mathcal{F}_1(\hat{u}, \nu, \tau)^p \mathcal{H}(\hat{u}, \nu, \tau)^{g-1} \prod_{\alpha} \Pi_{\alpha}(\hat{u}, \nu, \tau)^{m_{\alpha}}, \quad (4.56)$$

where the sum is over the solutions to the “4d Bethe equations,” with \hat{u} a gauge parameter evaluated onto the so-called Bethe vacua [72]. The objects \mathcal{F}_1 , \mathcal{H} and Π_{α} are called the fibering operator, the handle-gluing operator, and the flavor flux operators, respectively.

For a free chiral multiplet, the (holomorphic) fibering operator is given by (4.44). The flavor flux operator, on the other hand, takes the form:

$$\Pi^{\Phi}(\nu, \tau) = \frac{e^{-\frac{\pi i \nu^2}{\tau}}}{\theta(\nu, \tau)}, \quad (4.57)$$

with $\theta(\nu, \tau)$ defined in appendix B. The handle-gluing operator of a free chiral multiplet of R -charge r (with $r \in \mathbb{Z}$, on a generic $\mathcal{M}_{g,p} \times S^1$) takes the form $\mathcal{H}^{\Phi} = (\Pi^{\Phi})^{r-1}$ [5].

Large gauge transformations. On $\mathcal{M}_{g,p} \times S^1$, there are two types of large gauge transformations for the gauge parameters. In addition to $\nu \sim \nu + 1$, corresponding to the large gauge transformation $a_t \sim a_t - 1$ along the circle, we also have $\nu \sim \nu + \tau$ together with $\mathbf{m} \sim \mathbf{m} + p$, a large gauge transformation along the S^1 fiber of $\mathcal{M}_{g,p}$.

Consider a 4d $\mathcal{N} = 1$ gauge theory whose partition function is computed as in (4.55). For generic values of g, p , the R -charges of the matter fields should be integer-quantized on $\mathcal{M}_{g,p} \times S^1$. Then, the entire R -charge dependence of $\mathcal{I}_{\mathcal{M}_{g,p}}$ is through the handle-gluing operator, \mathcal{H} , which drops out for $g = 1$. Here we focus on the contributions from the fibering and flux operator — schematically:

$$\mathcal{I}(\nu, \tau)_{\mathbf{m}} = \mathcal{F}_1(\nu, \tau)^p \Pi(\nu, \tau)^{\mathbf{m}}. \quad (4.58)$$

Under the large gauge transformation:

$$(\nu_{\alpha}, \mathbf{m}_{\alpha}) \sim (\nu_{\alpha} + n_{\alpha} + m_{\alpha} \tau, \mathbf{m}_{\alpha} + m_{\alpha} p), \quad n_{\alpha}, m_{\alpha} \in \mathbb{Z}, \quad (4.59)$$

for the flavor parameters, the fibering and the flux operator transform as:

$$\begin{aligned} \mathcal{F}_1(\nu + n + m\tau, \tau) &= e^{\varphi_{n,m}^{\mathcal{F}_1}} \Pi(\nu, \tau)^{-m} \mathcal{F}_1(\nu, \tau), \\ \Pi_{\alpha}(\nu + n + m\tau, \tau) &= e^{\varphi_{n,m}^{\Pi_{\alpha}}} \Pi_{\alpha}(\nu, \tau), \end{aligned} \quad (4.60)$$

with the prefactors:⁴²

$$\begin{aligned} e^{\varphi_{n,m}^{\mathcal{F}_1}} &\equiv e^{-\frac{\pi i}{2} \mathcal{A}_{qq}^{\alpha\beta} m_{\alpha} m_{\beta}} e^{-\frac{\pi i}{6} \mathcal{A}_q^{\alpha} (n_{\alpha} + 3m_{\alpha})} e^{\pi i \mathcal{A}_{qqq}^{\alpha\beta\gamma} \left(n_{\alpha} m_{\beta} m_{\gamma} + \frac{n_{\alpha} m_{\beta}}{\tau} (n_{\gamma} + 2\nu_{\gamma}) + \frac{n_{\alpha}}{3\tau^2} (n_{\beta} n_{\gamma} + 3n_{\beta} \nu_{\gamma} + 3\nu_{\beta} \nu_{\gamma}) \right)}, \\ e^{\varphi_{n,m}^{\Pi_{\alpha}}} &\equiv (-1)^{\mathcal{A}_{qq}^{\alpha\beta} (n_{\beta} + m_{\beta})} e^{-\frac{\pi i}{\tau} \mathcal{A}_{qqq}^{\alpha\beta\gamma} (n_{\beta} n_{\gamma} + 2m_{\beta} \nu_{\gamma})}. \end{aligned}$$

⁴²Here, $\mathcal{A}_{qq}^{\alpha\beta}$ denotes some quadratic “pseudo-anomalies” in 4d, similar to the ones discussed in footnote 12. We leave a proper understanding of those terms for future work.

Then, the holomorphic partition function (4.58) transforms as:

$$\mathcal{I}(\nu + n + m\tau, \tau)_{\mathfrak{m}+m\mathfrak{p}} = e^{\varphi_{n,m}^{\mathcal{I}}} \mathcal{I}(\nu, \tau)_{\mathfrak{m}}, \quad e^{\varphi_{n,m}^{\mathcal{I}}} \equiv e^{p\varphi_{n,m}^{\mathcal{F}_1}} \prod_{\alpha} e^{(m_{\alpha}+m_{\alpha}p)\varphi_{n,m}^{\Pi_{\alpha}}}. \quad (4.61)$$

One can easily generalize this expression to include the effect of the handle-gluing operators [5].

The three-dimensional limit. As in the special case of the S^3 index, we again claim that there is no order- β_{S^1} term to the supersymmetric index (4.55). More precisely, one can again write the index as a “pure index” contribution times the supersymmetric Casimir energy contribution:

$$\mathcal{I}_{\mathcal{M}_{g,p} \times S^1}(\nu, \tau)_{\mathfrak{m}} = e^{2\pi i \tau \mathcal{E}_{\mathcal{M}_{g,p}}(\nu, \tau)_{\mathfrak{m}}} \mathbf{I}_{\mathcal{M}_{g,p}}(\nu, \tau)_{\mathfrak{m}}. \quad (4.62)$$

Explicit expressions for the Casimir energy were given in [5]. Then, we claim that the analogue of (4.52) holds for any $\mathcal{M}_{g,p}$. In gauge-theories, this can be shown easily for the handle-gluing operator and for the flux operators, which are given in terms of θ -functions. For the fibering operator, that small- β_{S^1} property again follows from the estimate (4.53).

4.4 The “gauge-invariant” supersymmetric partition function

As we have reviewed, there are various exact results for 4d $\mathcal{N} = 1$ supersymmetric indices in the literature, and those formulae are locally holomorphic in all the parameters. While this holomorphy agrees with the known classical supersymmetric Ward identities on curved space [6], it clashes with simpler requirements that stem from gauge invariance for real background gauge fields coupling to the flavor symmetry. For instance, there should exist a scheme in which the absolute value of the supersymmetric partition function is gauge invariant.

The presence of a non-zero contribution from the flavor ’t Hooft anomaly to the supersymmetric Ward identity, in the “gauge invariant scheme,” resolves this apparent puzzle — it is then possible to have a supersymmetric partition function which respects all the quantum Ward identities at the same time. The only price to pay is that the partition function is not fully holomorphic in the flavor parameters.

Given this understanding, it is now straightforward to revisit previous localisation computations to derive the “gauge invariant” partition functions. As explained before, the localisation argument still goes through. The only modification comes from a more careful computation of the various one-loop determinants on the supersymmetric locus. Since the new non-holomorphic terms depend on ’t Hooft anomalies, they factor out of various “localization integrals” in gauge theories, and only contribute to a Casimir-like overall pre-factor.

One-loop determinant and holomorphy. Let us focus on the contribution from a free chiral multiplet, Φ . This is a free theory, whose path integral can be computed on any $\mathcal{M}_{g,p} \times S^1$. In fact, we can factorize this one-loop determinant in terms of the fibering and

flux operators [5]:⁴³

$$Z_{\mathcal{M}_{g,p} \times S^1}^\Phi(\nu, \tau) = \hat{\mathcal{F}}_1^\Phi(\nu, \tau)^p \hat{\Pi}^\Phi(\nu, \tau)^m. \quad (4.63)$$

Here, the hatted symbols denote the “gauge invariant” objects, in contrast to the holomorphic fibering and flux operators discussed above. In particular, the $S^3 \times S^1$ partition function is given by:

$$Z_{S^3 \times S^1}^\Phi(\nu, \tau) = \hat{\mathcal{F}}_1^\Phi(\nu, \tau). \quad (4.64)$$

As always with perturbative anomalies, the difficulty lies in treating the chiral fermion, ψ_α , whose one-loop determinant is not well-defined. On the other hand, we can consider a theory consisting of the chiral multiplet Φ together with a “shadow chiral multiplet” $\bar{\Phi}$ of opposite chirality and gauge charge, whose kinetic operator is the Hermitian conjugate to the one of Φ .⁴⁴ Then, the partition function of the pair Φ and $\bar{\Phi}$ unambiguously defines the *absolute value* of Z^Φ .

In appendix C.2, we give a derivation of this absolute value by an explicit heat-kernel computation, generalizing the discussion of the Quillen anomaly in section 2.1. One finds:

$$|\hat{\mathcal{F}}_1^\Phi(\nu, \tau)| = e^{-\frac{\pi\tau_2}{3}(a_\psi^3 - \frac{1}{2}a_\psi)} |\Gamma_0(\nu, \tau)|, \quad (4.65)$$

for the fibering operator, and:

$$|\hat{\Pi}^\Phi(\nu, \tau)| = e^{\pi\tau_2(a_\psi^2 - a_\psi + \frac{1}{6})} \frac{1}{|\theta_0(\nu, \tau)|} \quad (4.66)$$

for the flavor flux operator, with $\nu = \tau a_\psi - a_t$. Moreover, from our general discussion in subsection 4.2.2, we know the explicit form of the chiral-multiplet supersymmetric effective action at order β_{S^1} in the small-circle limit; it is given by (4.39) with $\mathcal{A}_{qqq} = 1$, namely:

$$\mathbb{W}_{3d}^\Phi(a_\psi, a_t, \tau) = \frac{i\pi}{3\tau^2} a_t(\tau a_\psi - a_t)(p(a_t - 2\tau a_\psi) + 3m\tau). \quad (4.67)$$

We then propose that the supersymmetric partition function of the free chiral multiplet is given by:

$$Z_{\mathcal{M}_{g,p} \times S^1}^\Phi(a_\psi, a_t, \tau)_m = e^{-\mathbb{W}_{3d}^\Phi(a_\psi, a_t, \tau)} \mathcal{I}_{\mathcal{M}_{g,p} \times S^1}^\Phi(\nu, \tau)_m, \quad (4.68)$$

with $\mathcal{I}_{\mathcal{M}_{g,p} \times S^1}^\Phi$ the holomorphic partition function, including the supersymmetric Casimir energy contributions:

$$\mathcal{I}_{\mathcal{M}_{g,p} \times S^1}^\Phi(\nu, \tau)_m = \mathcal{F}_1(\nu, \tau)^p \Pi(\nu, \tau)^m. \quad (4.69)$$

One can check that (4.68) reproduces (4.65)–(4.66) — that is, minus the real part of \mathbb{W}_{3d} , as given in (4.40), plus the real part of $2\pi i \mathcal{E}_{\mathcal{M}_{g,p}}$ (the supersymmetric Casimir energy) exactly reproduces the factors in the exponents in (4.65)–(4.66). In particular, the absolute value of (4.68), $|Z_{\mathcal{M}_{g,p} \times S^1}^\Phi|$, is fully gauge invariant.

⁴³Here, without loss of generality, we pick the R -charge $r = 1$.

⁴⁴This is not a supersymmetric setup, but this is not an issue. We can think of the shadow chiral as coupling to the “complex conjugate” background geometry, so that its partition function is the complex conjugate of the one for Φ .

The “gauge-invariant” supersymmetric partition function. Building the partition function of any “Lagrangian” gauge theory from the chiral and vector multiplet building blocks, in the usual manner, one then obtains a simple proposal for the complete answer for the supersymmetric partition function in the gauge- and diff-invariant scheme.

We may state this proposal as a conjecture for any supersymmetric partition function on a product four-manifold $\mathcal{M}_3 \times S^1$, with geometric parameters τ and flavor parameters ν . Let $\mathcal{I}_{\mathcal{M}_3 \times S^1}$ denote the holomorphic partition function (or “holomorphic \mathcal{M}_3 index”), defined such that there is no order- β_{S^1} contribution to $\mathcal{I}_{\mathcal{M}_3 \times S^1}$ in the 3d limit — as in (4.51) for $\mathcal{M}_3 = S^3$.

Conjecture. The “gauge-invariant” supersymmetric partition function $Z_{\mathcal{M}_3 \times S^1}$ is related to the “holomorphic partition function,” $\mathcal{I}_{\mathcal{M}_3 \times S^1}$, by the 3d local functional \mathbb{W}_{3d} , evaluated on the supersymmetric background and viewed as a function of the 4d parameters, according to:

$$Z_{\mathcal{M}_3 \times S^1}(a_\psi, a_t, \tau, \bar{\tau}) = e^{-\mathbb{W}_{3d}(a_\psi, a_t, \tau, \bar{\tau})} \mathcal{I}_{\mathcal{M}_3 \times S^1}(\nu, \tau). \quad (4.70)$$

By construction, this supersymmetric partition function is properly gauge-invariant and supersymmetric, in the sense that it is compatible with all the anomalous Ward identities.

In this work, we focussed on the dependence of the partition function on the flavor parameters. The form (4.70) of the supersymmetric partition function should also account for more subtle anomalous Ward identities in the “geometric sector,” which should also contribute to \mathbb{W}_{3d} , and whose detailed study we leave for future work.

Behavior of $Z_{\mathcal{M}_{g,p} \times S^1}$ under large gauge transformations. Finally, we can discuss the behavior of the partition function under large gauge transformations. We again focus on the contributions from the fibering and flux operators in the $\mathcal{M}_{g,p} \times S^1$ background. Then, under a large gauge transformations across the S^1 :

$$a_{t,\alpha} \sim a_{t,\alpha} + n_\alpha, \quad (4.71)$$

we have:

$$\begin{aligned} Z_{\mathcal{M}_{g,p} \times S^1}(a_\psi, a_t - n, \tau, \bar{\tau})_{\mathfrak{m}} &= e^{\pi i \mathcal{A}_{qq}^{\alpha\beta} \mathfrak{m}_\alpha n_\beta} e^{-\frac{\pi i}{6} p \mathcal{A}^\alpha n_\alpha} \\ &\times e^{\pi i \mathcal{A}^{\alpha\beta\gamma} (\frac{p}{3} a_{\psi,\alpha} a_{\psi,\beta} n_\gamma - \mathfrak{m}_\alpha a_{\psi,\beta} n_\gamma)} Z_{\mathcal{M}_{g,p} \times S^1}(a_\psi, a_t, \tau, \bar{\tau})_{\mathfrak{m}}. \end{aligned} \quad (4.72)$$

We can also consider the large gauge transformation along the $\mathcal{M}_{g,p}$ Seifert fiber:

$$a_{\psi,\alpha} \sim a_{\psi,\alpha} + m_\alpha, \quad \mathfrak{m}_\alpha \sim \mathfrak{m}_\alpha + m_\alpha p, \quad (4.73)$$

which gives:

$$\begin{aligned} Z_{\mathcal{M}_{g,p} \times S^1}(a_\psi + m, a_t, \tau, \bar{\tau})_{\mathfrak{m} + mp} &= e^{\pi i \mathcal{A}_{qq}^{\alpha\beta} (\mathfrak{m}_\alpha m_\beta + \frac{p}{2} m_\alpha m_\beta)} e^{-\frac{\pi i}{2} p \mathcal{A}_q^\alpha m_\alpha} \\ &\times e^{\pi i \mathcal{A}_{qqq}^{\alpha\beta\gamma} (\frac{p}{3} (a_{t,\alpha} a_{\psi,\beta} m_\gamma - a_{t,\alpha} m_\beta m_\gamma) - \mathfrak{m}_\alpha a_{t,\beta} m_\gamma)} Z_{\mathcal{M}_{g,p} \times S^1}(a_\psi, a_t, \tau, \bar{\tau})_{\mathfrak{m}}. \end{aligned} \quad (4.74)$$

For $p = 0$, we find the behavior of the supersymmetric $\Sigma_g \times T^2$ partition function under large gauge transformations, which nicely agrees with the general expectations discussed in appendix D.2.

In the case $p = 1$, $\mathbf{m} = 0$, we also find a very simple behavior for the $S^3 \times S^1$ supersymmetric partition function under large gauge transformations across the circle, namely:

$$Z_{S^3 \times S^1}(\nu + n, \tau) = e^{-\frac{\pi i}{6} \mathcal{A}^\alpha n_\alpha} e^{\frac{\pi i p}{3} \mathcal{A}^{\alpha\beta\gamma} a_{\psi,\alpha} a_{\psi,\beta} n_\gamma} Z_{S^3 \times S^1}(\nu, \tau), \quad (4.75)$$

for any $n \in \mathbb{Z}$. This is the result up to terms that depend on the R -charge; the full result can be obtained from (4.75) by simply allowing the flavor indices α, β, \dots in the exponent to run over the value “ R ” as well, at least formally. A more complete account of the R -symmetry dependence of partition functions, which is tied to the “geometric sector,” will be given elsewhere.

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A Conventions in various dimensions

In this appendix, we briefly summarize our conventions in various dimensions. In 4d and 3d, our conventions are the same as in [6].

A.1 Four-dimensional conventions

We denote by X^μ the real coordinates, with $\mu = 1, \dots, 4$. We denote the real coframe one-forms by $e^{(a)}$, with the frame indices $a = 1, \dots, 4$, and:

$$ds^2(\mathcal{M}_4) = \sum_{a=1}^4 (e^{(a)})^2, \quad e_\mu^{(a)} e_\nu^{(b)} g^{\mu\nu} = \delta^{ab}. \quad (A.1)$$

Then, the complex coframe one-forms is given by:

$$e^1 \equiv e^{(1)} + ie^{(2)}, \quad e^{\bar{1}} \equiv e^{(1)} - ie^{(2)}, \quad e^2 \equiv e^{(3)} + ie^{(4)}, \quad e^{\bar{2}} \equiv e^{(3)} - ie^{(4)}. \quad (A.2)$$

Four-dimensional spinors are written as Weyl spinors, denoted by $\psi = \psi_\alpha$ and $\tilde{\psi} = \tilde{\psi}^{\dot{\alpha}}$ for positive- and negative-chirality spinors, respectively, with the indices in their natural positions (in the standard Wess-and-Bagger conventions [74]). We use the Euclidean σ^μ -matrices (in the real frame basis):

$$(\sigma_{\alpha\dot{\beta}}^a) = (\sigma^1, \sigma^2, \sigma^3, -i\mathbf{1}), \quad (\tilde{\sigma}^{a\dot{\alpha}\beta}) = (-\sigma^1, -\sigma^2, -\sigma^3, -i\mathbf{1}), \quad (A.3)$$

for $a = 1, \dots, 4$, respectively, with σ^i the Pauli matrices and $\mathbf{1}$ the 2×2 identity matrix (and, of course, $\sigma^\mu \equiv e_a^\mu \sigma^a$ in terms of the inverse vierbein). We also define the antisymmetric combinations:

$$\sigma^{\mu\nu} = \frac{1}{4}(\sigma^\mu \tilde{\sigma}^\nu - \sigma^\nu \tilde{\sigma}^\mu), \quad \tilde{\sigma}^{\mu\nu} = \frac{1}{4}(\tilde{\sigma}^\mu \sigma^\nu - \tilde{\sigma}^\nu \sigma^\mu), \quad (\text{A.4})$$

which are proportional to the Lorentz symmetry generators in the Weyl spinor representations; they are self-dual and anti-self dual, respectively:

$$\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma} = \sigma^{\mu\nu}, \quad \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\tilde{\sigma}_{\rho\sigma} = -\tilde{\sigma}^{\mu\nu}. \quad (\text{A.5})$$

Note the following useful identities:

$$\begin{aligned} \sigma^\mu \tilde{\sigma}^\nu + \sigma^\nu \tilde{\sigma}^\mu &= -2g^{\mu\nu}, & \tilde{\sigma}^\mu \sigma^\nu + \tilde{\sigma}^\nu \sigma^\mu &= -2g^{\mu\nu}, \\ \sigma^\mu \tilde{\sigma}^{\nu\rho} + \sigma^{\nu\rho} \tilde{\sigma}^\mu &= \epsilon^{\mu\nu\rho\lambda} \sigma_\lambda, & \tilde{\sigma}^\mu \sigma^{\nu\rho} + \tilde{\sigma}^{\nu\rho} \tilde{\sigma}^\mu &= -\epsilon^{\mu\nu\rho\lambda} \tilde{\sigma}_\lambda. \end{aligned} \quad (\text{A.6})$$

Spinor indices of either chirality are raised and lowered with $\epsilon^{\alpha\beta}$, $\epsilon_{\alpha\beta}$ or $\epsilon^{\dot{\alpha}\dot{\beta}}$, $\epsilon_{\dot{\alpha}\dot{\beta}}$, with $\epsilon^{12} = \epsilon_{21} = 1$. Implicit spinor indices are contracted in the usual conventions, $\eta\psi \equiv \eta^\alpha \psi_\alpha$ and $\tilde{\eta}\tilde{\psi} \equiv \tilde{\eta}_{\dot{\alpha}} \tilde{\psi}^{\dot{\alpha}}$. For any two spinors η_α and $\tilde{\eta}_{\dot{\alpha}}$ (which are independent from each other, in Euclidean signature), we have:

$$\eta^\alpha \eta^\beta = -\frac{1}{2}\epsilon^{\alpha\beta} \eta\eta, \quad \tilde{\eta}^{\dot{\alpha}} \tilde{\eta}^{\dot{\beta}} = \frac{1}{2}\epsilon^{\dot{\alpha}\dot{\beta}} \tilde{\eta}\tilde{\eta}, \quad \tilde{\eta}^{\dot{\alpha}} \eta^\alpha = \frac{1}{2}\tilde{\sigma}^{\mu\dot{\alpha}\alpha} \eta\sigma_\mu \tilde{\eta}. \quad (\text{A.7})$$

Various other Fierz identities can be derived straightforwardly.⁴⁵

A.2 Three-dimensional conventions and dimensional reduction from 4d

We denote by x^μ ($\mu = 1, 2, 3$) the three-dimensional real coordinates. The three-dimensional real frame is denoted by:⁴⁶

$$\hat{e}^a = (\hat{e}^1, \hat{e}^2, \hat{e}^3) = (\eta, e^{(3)}, e^{(4)}), \quad a = 1, 2, 3. \quad (\text{A.8})$$

This is obtained by dimensional reduction from 4d along the “ $e^{(2)}$ direction,” in our conventions. We denote by:

$$\hat{e}^{(0)} \equiv e^{(2)} \quad (\text{A.9})$$

the frame covector in that direction. Note that the 4d and 3d ϵ -tensor are related by $\epsilon^{abc} = -\epsilon^{0abc}$, in the frame basis. (The dimensional reduction is discussed in detail in [42], which we follow.) In terms of local coordinates, we will have:

$$x^\mu = (x^1, x^2, x^3) = (X^1, X^3, X^4). \quad (\text{A.10})$$

⁴⁵One can directly adapt various identities from [74] by replacing $\eta^{\mu\nu}$ and $\epsilon^{\mu\nu\rho\sigma}$ there with $g^{\mu\nu}$ and $-i\epsilon^{\mu\nu\rho\sigma}$, respectively, to account for the change in space-time signature.

⁴⁶Note that we use coordinate and frame indices μ, ν and a, b, \dots , respectively, both for the four-dimensional and three-dimensional quantities. This should not lead to any confusion.

The three-dimensional spinors are two-component Dirac spinors, denoted by either ψ_α or $\tilde{\psi}_\alpha$. The reduction of the 4d Weyl spinors give 3d spinors, with the index position given as:

$$\psi_\alpha|_{3d} = \psi_\alpha, \quad \tilde{\psi}^{\dot{\alpha}}|_{3d} = \tilde{\psi}^{\dot{\alpha}}. \quad (\text{A.11})$$

The three-dimensional γ -matrices are (note the default index positions):

$$(\gamma^a)_\alpha{}^\beta = (\sigma^3, -\sigma^1, -\sigma^2). \quad (\text{A.12})$$

They satisfy $\gamma^\mu\gamma^\nu = g^{\mu\nu} + i\epsilon^{\mu\nu\rho}\gamma_\rho$. The 3d γ^μ -matrices are obtained from the 4d σ^μ matrices according to:

$$\sigma_{\alpha\dot{\beta}}^a = \gamma_{\alpha\beta}^a, \quad \sigma_{\alpha\dot{\beta}}^0 = i\varepsilon_{\alpha\beta}, \quad (\tilde{\sigma}^a)^{\dot{\beta}\alpha} = (\gamma^a)^{\beta\alpha}, \quad (\tilde{\sigma}^0)^{\dot{\beta}\alpha} = i\varepsilon^{\beta\alpha}. \quad (\text{A.13})$$

One has to be careful with the signs when reducing from 4d to 3d, due in particular to the fact that the 4d “dotted indices” become “undotted” in 3d; for instance:

$$\tilde{\zeta}\tilde{\lambda}|_{3d} = \tilde{\zeta}_{\dot{\alpha}}\tilde{\lambda}^{\dot{\alpha}}|_{3d} = -\tilde{\zeta}^{\dot{\alpha}}\tilde{\lambda}_{\dot{\alpha}} = -\tilde{\zeta}\tilde{\lambda}. \quad (\text{A.14})$$

A.3 Two-dimensional and one-dimensional conventions

In two dimensions, we denote the real coordinates by $X^\mu = (x, y)$ and we pick a complex coordinate $w = x + \tau y$. The γ -matrices can be chosen as:

$$(\gamma^\mu) = (-\sigma^1, -\sigma^2), \quad \gamma^3 = \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (\text{A.15})$$

The 2d Dirac spinors decomposes into Weyl spinors of positive and negative chiralities, which we denote by ψ_- and ψ_+ , respectively — namely:

$$\psi_\alpha = \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix}. \quad (\text{A.16})$$

They have 2d spin $\frac{1}{2}$ and $-\frac{1}{2}$, respectively. We then reduce to one dimension along the coordinate x . The 1d fermions do not carry any space-time index, and are simply denoted by:

$$\psi_-|_{1d} = \psi, \quad \eta_+|_{1d} = \eta. \quad (\text{A.17})$$

B θ -function and elliptic Γ -function

In this appendix, we collect some useful definitions and properties for some θ - and Γ -functions used in the main text. We will use the standard notation:

$$q = e^{2\pi i\tau}, \quad y = e^{2\pi i\nu}. \quad (\text{B.1})$$

The function $\theta(\nu, \tau)$. Let us define the “reduced” theta-function:

$$\theta_0(\nu, \tau) \equiv \prod_{k=0}^{\infty} (1 - q^k y)(1 - q^{k+1} y^{-1}), \quad (\text{B.2})$$

with $\nu \in \mathbb{C}$ and τ valued on the upper-half-plane. The function (B.2) has simple zeros at $\nu = n + m\tau$, with $n, m \in \mathbb{Z}$. Then, the “ordinary” θ -function is defined as:

$$\theta(\nu, \tau) \equiv e^{\frac{\pi i \tau}{6}} e^{-\pi i \nu} \prod_{k=0}^{\infty} (1 - q^k y)(1 - q^{k+1} y^{-1}). \quad (\text{B.3})$$

It satisfies:

$$\theta(\nu + n + m\tau, \tau) = (-1)^{n+m} e^{-2\pi i m \nu - \pi i m^2 \tau} \theta(\nu, \tau), \quad (\text{B.4})$$

for any $n, m \in \mathbb{Z}$, and $\theta(-\nu, \tau) = -\theta(\nu, \tau)$. Under modular transformations, we have:

$$\theta(\nu; \tau + 1) = e^{\frac{\pi i}{6}} \theta(\nu; \tau), \quad \theta\left(\frac{\nu}{\tau}; -\frac{1}{\tau}\right) = e^{2\pi i \left(\frac{\nu^2}{2\tau} - \frac{1}{4}\right)} \theta(\nu, \tau), \quad (\text{B.5})$$

for the generator T and S of $\text{SL}(2, \mathbb{Z})$, respectively. Let us also define the η -function: we have:

$$\eta(\tau) = e^{\frac{\pi i \tau}{12}} \prod_{k=1}^{\infty} (1 - q^k). \quad (\text{B.6})$$

It transforms as:

$$\eta(\tau + 1) = e^{\frac{\pi i}{12}} \eta(\tau), \quad \eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau} \eta(\tau), \quad (\text{B.7})$$

under $\text{SL}(2, \mathbb{Z})$.

Restricted elliptic gamma function, $\Gamma_0(\nu, \tau)$. The elliptic gamma function $\Gamma_e(\nu, \tau, \sigma)$ can be defined as:

$$\Gamma_e(\nu, \tau, \sigma) = \prod_{j,k=0}^{\infty} \frac{1 - y^{-1} \mathbf{p}^{j+1} \mathbf{q}^{k+1}}{1 - y \mathbf{p}^j \mathbf{q}^k}, \quad (\text{B.8})$$

where $\mathbf{p} = e^{2\pi i \sigma}$ and $\mathbf{q} = e^{2\pi i \tau}$. See e.g. [75] for a discussion of some of its properties. In this paper, we focus on the following “restricted value” of the elliptic gamma function, for $\mathbf{p} = \mathbf{q} = q$:

$$\Gamma_0(\nu, \tau) \equiv \Gamma_e(\nu + \tau, \tau, \tau) = \prod_{k=0}^{\infty} \left(\frac{1 - y^{-1} q^{k+1}}{1 - y q^{k+1}} \right)^{k+1}. \quad (\text{B.9})$$

It satisfies:

$$\Gamma_0(\nu + n + m\tau, \tau) = (-y)^{-\frac{m(m+1)}{2}} q^{-\frac{1}{6} m(m^2-1)} \theta_0(\nu, \tau)^m \Gamma_0(\nu, \tau), \quad (\text{B.10})$$

for any $n, m \in \mathbb{Z}$.

C One-loop determinants and ζ -function regularization

In this appendix, we compute explicitly the (gauge-invariant) absolute value of one-loop determinants for chiral fermions and their superpartners. We first review the computation of the determinant of the Dirac operator $D_{\bar{w}}$ on T^2 , following the computation of Ray and Singer in [76]. We then generalize that procedure in order to compute the absolute value of the one-loop determinant of a 4d $\mathcal{N} = 1$ chiral multiplet Φ on $\mathcal{M}_{g,p} \times S^1$.

C.1 Free fermions on T^2

Consider a free chiral fermion λ_- (of positive chirality) on T^2 , with complex coordinate $w = x + \tau y$, coupled to the background $U(1)$ flat connection parametrised by a constant:

$$\nu = a_x \tau - a_y. \tag{C.1}$$

We consider the determinant of the Dirac operator $D_{\bar{w}}$, twisted by the background connection:

$$D_{\bar{w}} = \partial_{\bar{w}} - i a_{\bar{w}} = \partial_{\bar{w}} - \frac{\nu}{2\tau_2}, \tag{C.2}$$

with the periodic boundary condition along two circles. We may then *define* the absolute value squared of the “determinant” of $D_{\bar{w}}$ as:

$$|\det(D_{\bar{w}})|^2 \equiv \det(D_w D_{\bar{w}}). \tag{C.3}$$

Physically, this corresponds to the partition function for a pair of fermions of opposite chiralities, which admits an obviously gauge-invariant regularization. The determinant of the Hermitian operator $D_w D_{\bar{w}}$ can be written as an infinite product:

$$\det(D_w D_{\bar{w}}) = \prod_{m,n \in \mathbb{Z}} \lambda_{m,n}, \quad \lambda_{m,n} = \frac{1}{\tau_2} |\nu + m - \tau n|^2. \tag{C.4}$$

Alternatively, one can consider the untwisted Laplacian $\Delta \equiv \partial_w \partial_{\bar{w}}$ with the twisted boundary condition along the two circles:

$$\psi(w + 2\pi) = e^{-2\pi i a_x} \psi(w), \quad \psi(w + 2\pi\tau) = e^{-2\pi i a_y} \psi(w). \tag{C.5}$$

We would like to compute the effective action, defined as

$$S_{\text{eff}}(\nu, \tau) = - \sum_{n,m \in \mathbb{Z}} \log \lambda_{n,m} = \frac{d}{ds} \zeta_{\Delta}(s) \Big|_{s=0}, \tag{C.6}$$

where we introduced the zeta function for the Laplacian:

$$\begin{aligned} \zeta_{\Delta}(s) &\equiv \sum_{m,n \in \mathbb{Z}} \lambda_{m,n}^{-s} \\ &= \frac{1}{\Gamma(s)} \int_0^{\infty} dt t^{s-1} \text{tr}(e^{-t\Delta}). \end{aligned} \tag{C.7}$$

In order to compute this, let us consider the heat kernel for the Laplacian:

$$G(w, z, t) = \langle w, \bar{w} | e^{-t\Delta} | z, \bar{z} \rangle, \quad (\text{C.8})$$

which satisfies

$$(\partial_t - \Delta_w) G(w, z, t) = \begin{cases} \delta^2(w - z) & \text{if } t = 0 \\ 0 & \text{if } t > 0 \end{cases}. \quad (\text{C.9})$$

On the complex plane, \mathbb{C} , the solution can be written as:

$$G_0(w, z, t) = \frac{1}{4\pi t} e^{-\frac{|w-z|^2}{4t}}. \quad (\text{C.10})$$

The heat kernel on T^2 can be easily obtained by implementing the twisted boundary condition (C.5). One can check that the following solution has the desired periodicities:

$$G(w, z, t) = \sum_{m, n \in \mathbb{Z}} \frac{1}{4\pi t} e^{2\pi i(na_x + ma_y)} \exp\left(-\frac{1}{4t} |w - z - 2\pi(n + m\tau)|^2\right). \quad (\text{C.11})$$

We can now write the trace in (C.7) as:

$$\text{tr}(e^{-t\Delta}) = \sum_{n, m \in \mathbb{Z}} \frac{\pi\tau_2}{t} e^{2\pi i(na_x + ma_y)} e^{-\frac{\pi^2 |n + m\tau|^2}{t}}. \quad (\text{C.12})$$

Removing the divergence from the term $m = n = 0$, we can perform the t integral when $\text{Re}(s) < 1$, to obtain:

$$\zeta_\Delta(s) = \frac{\pi\tau_2\Gamma(1-s)}{\Gamma(s)} \sum_{n^2 + m^2 > 0} e^{2\pi i(na_x + ma_y)} \left(\frac{1}{\pi^2 |m\tau + n|^2}\right)^{1-s}. \quad (\text{C.13})$$

One can show the summation uniformly converges when $\text{Re}(s) < \frac{1}{2}$, once we first sum over $n \in \mathbb{Z}$. Taking the derivative with respect to s , we have:

$$\zeta'_\Delta(0) = \frac{2\tau_2}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2\pi na_x)}{n^2} + \sum_{m \neq 0} \frac{1}{|m|} e^{2\pi ima_y} \sum_{n \in \mathbb{Z}} e^{2\pi ina_x} \frac{|m|\tau_2}{\pi |m\tau + n|^2}. \quad (\text{C.14})$$

The first term gives:⁴⁷

$$\begin{aligned} \frac{2\tau_2}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2\pi na_x)}{n^2} &= \frac{\tau_2}{\pi} \left(\text{Li}_2(e^{2\pi ia_x}) + \text{Li}_2(e^{-2\pi ia_x}) \right) \\ &= 2\pi\tau_2 \left(a_x^2 - a_x + \frac{1}{6} \right). \end{aligned} \quad (\text{C.15})$$

The second term in (C.14) can be computed using the Poisson summation formula:

$$\sum_{n \in \mathbb{Z}} f(\nu + n) = \sum_{k \in \mathbb{Z}} e^{-2\pi ik\nu} \hat{f}(k), \quad \hat{f}(k) = \int_{-\infty}^{\infty} f(\xi) e^{2\pi i\xi k} d\xi, \quad (\text{C.16})$$

⁴⁷On the second line here, we chose a particular branch for the dilogarithm, so that the final expression, in (C.18) below, is invariant under the large gauge transformation.

for any function $f(\nu)$ which is periodic under $\nu \rightarrow \nu + 1$. This gives:

$$\begin{aligned}
 & \sum_{m \neq 0} \frac{1}{|m|} e^{2\pi i m a_y} \sum_{n \in \mathbb{Z}} e^{2\pi i n a_x} \frac{|m| \tau_2}{\pi |m\tau + n|^2} \\
 &= \sum_{m \neq 0} \frac{1}{|m|} e^{2\pi i m a_y} \sum_{k \in \mathbb{Z}} e^{-2\pi (|m| \tau_2 |a_x + k| + i m \tau_1 (a_x + k))} \\
 &= - \sum_{k=0}^{\infty} \log |1 - q^k e^{2\pi i (-a_y + \tau a_x)}|^2 - \sum_{k=0}^{\infty} \log |1 - q^{k+1} e^{2\pi i (a_y - \tau a_x)}| \\
 &= - \log |\theta_0(\nu, \tau)|^2,
 \end{aligned} \tag{C.17}$$

with $\theta_0(\nu, \tau)$ defined as in (B.2). We then arrive at the expression:

$$e^{-S_{\text{eff}}(\nu, \tau)} = e^{-2\pi \tau_2 (a_x^2 - a_x + \frac{1}{6})} |\theta_0(\nu, \tau)|^2. \tag{C.18}$$

As discussed in section 2.1, an holomorphic square root of (C.18) does not exist. Instead, we find:

$$\det(D_{\bar{w}}) = e^{-2\pi \tau_2 \left(\frac{a_x^2}{2} - \frac{a_x}{2} + \frac{1}{12} \right)} \theta_0(\nu, \tau), \tag{C.19}$$

up to a phase factor, which should be compatible with the U(1) anomaly.

C.2 4d $\mathcal{N} = 1$ chiral multiplet on $\mathcal{M}_{g,p} \times S^1$

We can run a similar computation for a chiral multiplet (of R -charge $r = 1$) on $\mathcal{M}_{g,p} \times S^1$. As reviewed in section 4.4, the partition function on that space can be factorized into contributions from “fibering and flux operators.” The flux operator computation is essentially identical to the T^2 computation above. We then focus on the fibering operator contribution.

The fibering operator for the chiral multiplet, $\hat{\mathcal{F}}_1^\Phi$, is given formally by the infinite product [5]:

$$\hat{\mathcal{F}}_1^\Phi(\nu) \equiv \prod_{n, m \in \mathbb{Z}} \left[\frac{1}{\tau_2^2} (\nu + m - \tau n) \right]^n. \tag{C.20}$$

We will compute the absolute value of this expression, by a careful zeta-function regularization of the quantity:

$$\left| \hat{\mathcal{F}}_1^\Phi(\nu) \right|^2 \equiv \prod_{n, m \in \mathbb{Z}} (\lambda_{m,n})^n, \quad \text{where } \lambda_{m,n} = \frac{1}{\tau_2^2} |\nu + m - \tau n|^2. \tag{C.21}$$

We will write $\nu = \tau a_\psi - a_t$, to match the 4d notation in the main text. Following the 2d computation in the last section, we introduce the zeta function:

$$\zeta_F(s) = \sum_{m, n \in \mathbb{Z}} n \lambda_{m,n}^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} \sum_{m, n \in \mathbb{Z}} n e^{-t \lambda_{m,n}}, \tag{C.22}$$

so that we have:

$$\log |\hat{\mathcal{F}}_1^\Phi(u)|^2 = - \left. \frac{d}{ds} \zeta_F(s) \right|_{s=0}. \tag{C.23}$$

Let us define:

$$G(t) = \sum_{m,n \in \mathbb{Z}} n e^{-t\lambda_{m,n}}. \quad (\text{C.24})$$

In order to use the Poisson summation formula, we decompose $G(t)$ into a linear combination of two functions periodic in $\nu \rightarrow \nu + 1$ and $\nu \rightarrow \nu + \tau$:

$$G(t) = G^{(1)}(t) + a_\psi G^{(2)}(t), \quad (\text{C.25})$$

where:

$$G^{(1)}(t) = \sum_{m,n \in \mathbb{Z}} (-a_\psi + n) e^{-t\lambda_{m,n}}, \quad G^{(2)}(t) = \sum_{m,n \in \mathbb{Z}} e^{-t\lambda_{m,n}}. \quad (\text{C.26})$$

Note that $G^{(2)}(t)$ is equivalent to (C.12). On the other hand, $G^{(1)}(t)$ can be computed by applying the Poisson summation formula (C.16) twice. We find:

$$G^{(1)}(t) = \frac{i\pi^2 \tau_2}{t^2} \sum_{m,n \in \mathbb{Z}} (n + m\tau_1) e^{2\pi i(na_\psi + ma_t)} e^{-\pi^2 |n + m\tau|^2 / t}. \quad (\text{C.27})$$

We can also decompose the zeta function (C.22) accordingly:

$$\zeta_F(s) = \zeta_F^{(1)}(s) + a_\psi \zeta_F^{(2)}(s), \quad (\text{C.28})$$

with:

$$\zeta_F^{(i)}(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1} G^{(i)}(t), \quad i = 1, 2. \quad (\text{C.29})$$

The t integral for $\zeta_F^{(1)}(s)$ converges provided that $\text{Re}(s) < 2$. We have

$$\zeta_F^{(1)}(s) = \frac{i\pi^2 \tau_2 \Gamma(2-s)}{\Gamma(s)} \sum_{n^2 + m^2 > 0} (n + m\tau_1) e^{2\pi i(ma_t + na_\psi)} \left(\frac{1}{\pi^2 |n + m\tau|^2} \right)^{2-s}. \quad (\text{C.30})$$

As in the 2d case, one can show that the infinite sum uniformly converges when $\text{Re}(s) < \frac{3}{2}$, if we sum over $n \in \mathbb{Z}$ first with fixed m :

$$\begin{aligned} \zeta_F^{(1)}(s) &= \frac{i\pi^2 \tau_2 \Gamma(2-s)}{\Gamma(s)} \sum_{n \neq 0} n e^{2\pi i na_\psi} \left(\frac{1}{\pi^2 n^2} \right)^{2-s} \\ &+ \frac{i\pi^2 \tau_2 \Gamma(2-s)}{\Gamma(s)} \sum_{m \neq 0} e^{2\pi i ma_t} \sum_{n \in \mathbb{Z}} (n + m\tau_1) e^{2\pi i na_\psi} \left(\frac{1}{\pi^2 |n + m\tau|^2} \right)^{2-s}. \end{aligned}$$

We have:

$$\begin{aligned} \left. \frac{d}{ds} \zeta_F^{(1)}(s) \right|_{s=0} &= \frac{i\tau_2}{\pi^2} \sum_{n \neq 0} \frac{e^{2\pi i na_\psi}}{n^3} \\ &+ \frac{i\tau_2}{\pi^2} \sum_{m \neq 0} e^{2\pi i ma_t} \sum_{n \in \mathbb{Z}} e^{2\pi i na_\psi} \frac{(n + m\tau_1)}{|n + m\tau|^4}. \end{aligned} \quad (\text{C.31})$$

The first line evaluates to:

$$\begin{aligned}
\frac{i\tau_2}{\pi^2} \sum_{n \neq 0} \frac{e^{2\pi i n a_\psi}}{n^3} &= \frac{i\tau_2}{\pi^2} (\text{Li}_3(e^{2\pi i a_\psi}) - \text{Li}_3(e^{-2\pi i a_\psi})) \\
&= -\frac{i\tau_2}{\pi^2} \left(\frac{1}{6} (2\pi i a_\psi - i\pi)^3 + \frac{\pi^2}{6} (2\pi i a_\psi - i\pi) \right) \\
&= -2\pi\tau_2 \left(\frac{2}{3} a_\psi^3 - a_\psi^2 + \frac{1}{3} a_\psi \right).
\end{aligned} \tag{C.32}$$

and the second line in (C.31) can be again computed using the Poisson summation formula:

$$\begin{aligned}
\frac{i\tau_2}{\pi^2} \sum_{m \neq 0} e^{2\pi i m a t} \sum_{n=-\infty}^{\infty} e^{2\pi i n a_\psi} \frac{(n + m\tau_1)}{|n + m\tau|^4} \\
= \sum_{k \in \mathbb{Z}} \sum_{m \neq 0} \frac{e^{2\pi i m a t}}{|m|} (-a_\psi - k) e^{-2\pi [|m|\tau_2 |a_\psi + k| + i m \tau_1 (a_\psi + k)]} \\
= - \sum_{k=1}^{\infty} \log |1 - y^{-1} q^k|^{2k} + \sum_{k=1}^{\infty} \log |1 - y q^k|^{2k} + a_\psi \log |\theta_0(\nu, \tau)|^2 \\
= - \log |\Gamma_0(\nu, \tau)|^2 + a_\psi \log |\theta_0(\nu, \tau)|^2,
\end{aligned} \tag{C.33}$$

with $\Gamma_0(\nu, \tau)$ defined as in (B.9). Combining all the terms, we have:

$$\begin{aligned}
- \frac{d}{ds} \zeta_F(s) \Big|_{s=0} &= - \frac{d}{ds} \left(\zeta_F^{(1)}(s) + a_\psi \zeta_F^{(2)}(s) \right) \Big|_{s=0} \\
&= -2\pi\tau_2 \left(\frac{a_\psi^3}{3} - \frac{a_\psi}{6} \right) + \log |\Gamma_0(\nu, \tau)|^2.
\end{aligned} \tag{C.34}$$

Therefore, we found:

$$\left| \hat{\mathcal{F}}_1^\Phi(\nu, \tau) \right|^2 = \exp \left[-2\pi\tau_2 \left(\frac{a_\psi^3}{3} - \frac{a_\psi}{6} \right) \right] |\Gamma_0(\nu, \tau)|^2. \tag{C.35}$$

This expression does not admit a holomorphic square root. We have:

$$\hat{\mathcal{F}}_1^\Phi(\nu, \tau) = \exp \left[-\pi\tau_2 \left(\frac{a_\psi^3}{3} - \frac{a_\psi}{6} \right) \right] \Gamma_0(\nu, \tau), \tag{C.36}$$

up to a phase factor.

D Large-gauge transformations and the anomaly polynomial

In the presence of 't Hooft anomalies, the partition function $Z[a]$, viewed as a functional over the space of background gauge fields a_μ , \mathfrak{M} , is not a “function” over \mathfrak{M} but rather a non-trivial section of a line bundle:

$$\mathcal{L} \rightarrow \mathfrak{M}. \tag{D.1}$$

In the case of free fermions, the partition function is the “determinant” of a Dirac operator, and \mathcal{L} is then known as the determinant line bundle. The anomalies of the theory are encoded invariantly in terms of the non-trivial topology of \mathcal{L} [77].

In this appendix, we compute explicitly the first Chern class of the determinant line bundle in some simple cases, with $Z[a]$ depending only on background flat connections. The first Chern class $c_1(\mathcal{L})$ can be obtained by pushing-forward the anomaly polynomial onto space-time [77], and it is thus determined directly by the anomalies. In this approach, it is easy to see that, even as we restrict ourselves to a subspace of flat gauge fields a_μ , $\mathfrak{M}_0 \subset \mathfrak{M}$, the anomaly still implies that the partition function must transform non-trivially under large gauge transformations.

D.1 Determinant line bundles on $J(T^2)$ and torus partition function

Consider a 2d $\mathcal{N} = (0, 2)$ supersymmetric theory coupled to a U(1) background gauge multiplet \mathcal{V}_F . In the presence of a t’Hooft anomaly, the partition function $Z(\nu, \tau)$, with a fixed complex structure τ , can be thought of as a section of a holomorphic line bundle \mathcal{L} over the Jacobian torus:

$$\nu \in J(T^2) \cong T^2, \tag{D.2}$$

with $\nu = \tau a_x - a_y$ as in section 2. Let \mathcal{N} be the product space $J \times T^2$, with the projection map:⁴⁸

$$J \times T^2 \xrightarrow{\pi} J. \tag{D.3}$$

Then, \mathcal{L} can be obtained from the Poincaré line bundle \mathcal{L} over $J \times T^2$ of degree zero.⁴⁹ The first Chern class of the “determinant line bundle,” \mathcal{L} , is obtained by pushing-forward the 4-form anomaly polynomial on T^2 , according to [77]:

$$c_1(\mathcal{L}) = \int_{T^2} \text{ch}(\mathcal{L}) \wedge \hat{A}(TN) \Big|_{4\text{-form}}. \tag{D.4}$$

In particular, under the large gauge transformation:

$$\nu \rightarrow \nu + 1, \quad \nu \rightarrow \nu + \tau, \tag{D.5}$$

the T^2 partition function should pick up a phase determined by the curvature of the line bundle (D.4). Since the tangent bundle on $T^2 \times J$ is trivial, (D.4) can be written as:

$$c_1(\mathcal{L}) = \int_{T^2} \frac{1}{2} c_1(\mathcal{L})^2. \tag{D.6}$$

We can decompose $c_1(\mathcal{L})$ as:

$$c_1(\mathcal{L}) = \delta_{0,2} + \delta_{1,1} + \delta_{2,0}, \tag{D.7}$$

⁴⁸We denote by T^2 the physical spacetime, and by J the Jacobian.

⁴⁹The Poincaré line bundle \mathcal{L} for a curve Σ is defined by the property that, when we restrict to a point $L \in J$, it reduces to L on $\Sigma \sim \Sigma \times \{L\}$. \mathcal{L} is defined up to a pull-back of a line bundle \mathcal{R} on J , i.e., $\mathcal{L} \sim \mathcal{L}' \otimes \pi^* \mathcal{R}$. See [78] for a detailed explanation.

where $\delta_{p,q}$ is a p -form along T^2 and a q -form along J . Now, we have:

$$c_1(\mathcal{L}) = \frac{1}{2} \int_{T^2} \left(\delta_{1,1}^2 + 2\delta_{2,0}\delta_{0,2} \right). \tag{D.8}$$

Restricting ourselves to flat gauge fields on T^2 , $f = da = 0$, we are left with:

$$c_1(\mathcal{L}) = \frac{1}{2} \int_{T^2} (\delta_{1,1})^2. \tag{D.9}$$

This can be computed by explicitly constructing the locally holomorphic section of \mathcal{L} [79]. One can show that:

$$c_1(\mathcal{L}) = \mathcal{A}_{qq} \omega, \tag{D.10}$$

where \mathcal{A}_{qq} is the quadratic 't Hooft anomaly, and ω is the normalised Kähler form on J . For a theory coupled to a general abelian background gauge multiplet, the parameter space is a product of the Jacobian tori for each $U(1)_\alpha$, and the formula generalises to:

$$c_1(\mathcal{L}) = 2\pi \mathcal{A}^{\alpha\beta} da_{x,\alpha} da_{y,\beta}, \tag{D.11}$$

where $\nu_\alpha = a_{x,\alpha}\tau - a_{y,\alpha}$. In particular, the connection on the line bundle \mathcal{L} can be locally chosen as:

$$A = 2\pi \mathcal{A}^{\alpha\beta} (c \cdot a_{\alpha,x} da_{\beta,y} - (1-c) \cdot a_{\beta,y} da_{\alpha,x}), \tag{D.12}$$

for any $c \in \mathbb{R}$. This implies that a section $Z(\mathbf{n})$ of the line bundle \mathcal{L} picks up a phase under the large gauge transformation along a 1-cycle γ , as:

$$\phi_\gamma : Z(\nu) \rightarrow e^{i\Lambda_\gamma(a_x, a_y)} Z(\nu), \tag{D.13}$$

when the connection transforms as $\delta_\gamma A = d\Lambda_\gamma$. More explicitly, we have

$$\begin{aligned} Z(\nu_\alpha + 1, \tau) &= e^{2\pi i(1-c)\mathcal{A}^{\alpha\beta} a_{\beta,x}} Z(\nu, \tau), \\ Z(\nu_\alpha + \tau, \tau) &= e^{2\pi i c \mathcal{A}^{\alpha\beta} a_{\beta,y}} Z(\nu, \tau). \end{aligned} \tag{D.14}$$

For 2d $\mathcal{N} = (0, 2)$ supersymmetric theories, the consistency of the gauge anomaly with supersymmetry fixes the constant $c = \frac{1}{2}$, as explained in the main text.

Modular transformation. The relation between the gravitational anomaly and the behaviour of the partition function under the $SL(2, \mathbb{Z})$ modular transformation:

$$\begin{aligned} T &: (a_x, a_y, \tau) \rightarrow (a_x, a_y + a_x, \tau + 1), \\ S &: (a_x, a_y, \tau) \rightarrow (a_y, -a_x, -1/\tau), \end{aligned} \tag{D.15}$$

can be understood in a similar way, as explained in detail in [80]. We view the parameter space Y as a fibration:

$$J \rightarrow Y \rightarrow \mathbb{H}/SL(2, \mathbb{Z}), \tag{D.16}$$

where the fiber at each point $\tau \in \mathbb{H}/\text{SL}(2, \mathbb{Z})$ is the Jacobian J (with the complex structure induced by τ). Now let \mathcal{N} be the product $T^2 \times Y$. Again, the first chern class of \mathcal{L} over Y can be computed from the anomaly polynomial:

$$\begin{aligned} c_1(\mathcal{L}) &= \int_{T^2} \text{ch}(\mathcal{L}) \wedge \hat{A}(T\mathcal{N}) \Big|_{4\text{-form}} \\ &= \frac{1}{2} c_1(\mathcal{L})^2 + \frac{1}{24} \int_{T^2} p_1(T\mathcal{N}) \end{aligned} \tag{D.17}$$

In the absence of the flavour symmetry, the first Chern class is determined by the gravitational term:

$$c_1(\mathcal{L}) = \frac{k_g}{2} \lambda, \tag{D.18}$$

where λ is the first Chern class of the Hodge bundle over $\mathbb{H}/\text{SL}(2, \mathbb{Z})$. This implies that the partition function, $Z(\tau)$, transforms as the one-dimensional representation of $\text{SL}(2, \mathbb{Z})$:

$$Z(\tau + 1) = e^{-i\pi k_g/6} Z(\tau), \quad Z(-1/\tau) = e^{i\pi k_g/2} Z(\tau). \tag{D.19}$$

Now, let us couple the theory to the flat background gauge for the flavour symmetry, $\nu \in J$. The first term of (D.17) implies that the connection on the line bundle restricted to the fiber direction can be locally written as (D.12), which transforms under the T operation as:

$$\delta_T A = \pi \mathcal{A}^{\alpha\beta} (2c - 1) d(a_{\alpha,x} a_{\beta,x}), \tag{D.20}$$

and under S as:

$$\delta_S A = -2\pi \mathcal{A}^{\alpha\beta} (2c - 1) d(a_{\alpha,x} a_{\beta,y}). \tag{D.21}$$

Therefore the section $Z(\nu, \tau)$ of $\mathcal{L} \rightarrow Y$ transforms as:

$$\begin{aligned} Z(a_x, a_y + a_x, \tau + 1) &= e^{-i\pi k_g/6} e^{\pi i \mathcal{A}^{\alpha\beta} (2c-1) a_{\alpha,x} a_{\beta,x}} Z(a_x, a_y, \tau), \\ Z(a_y, -a_x, -1/\tau) &= e^{i\pi k_g/2} e^{-2\pi i \mathcal{A}^{\alpha\beta} (2c-1) a_{\alpha,x} a_{\beta,y}} Z(a_x, a_y, \tau), \end{aligned} \tag{D.22}$$

under the T and S transformations, respectively. In particular, given the ‘‘supersymmetric’’ value $c = \frac{1}{2}$, the flavour dependence drops out and we reproduce the simple formulae (2.70) and (2.71).

D.2 Large gauge transformations for the 4d partition function

Consider now the supersymmetric partition function on $\mathcal{M}_{g,p} \times S^1$. Now, a supersymmetric $U(1)$ background gauge field is characterized by an integer $\mathbf{m} \in \mathbb{Z}$ (if $p = 0$) or $\mathbf{m} \in \mathbb{Z}_p$ (if $p \neq 0$), and by a complex parameter ν , with the identifications:

$$(\nu, \mathbf{m}) \rightarrow (\nu + 1, \mathbf{m}), \quad (\nu, \mathbf{m}) \rightarrow (\nu + \tau, \mathbf{m} + p), \tag{D.23}$$

under large gauge transformations.

Here, we will only discuss the case $p = 0$, corresponding to $\Sigma_g \times T^2$. Then, ν is valued in the Jacobian torus $J(T^2)$, and the discussion above generalizes easily. The $\Sigma_g \times T^2$ partition function can be built out of the (non-holomorphic) flavor flux operator and handle-gluing

operator. As a special case of the discussion in the main text, the flux operator of a free chiral is given by:

$$\hat{\Pi}^\Phi(\nu, \tau) = \frac{e^{-\pi i(\tau a_\psi^2 - a_\psi a_t)}}{\theta(\nu, \tau)}, \tag{D.24}$$

with $\nu = \tau a_\psi - a_t$, and similarly for the handle-gluing operator. A general partition function can be constructed from those building blocks. By explicit computation, one can check that the non-holomorphic flux operators of a general 4d $\mathcal{N} = 1$ gauge theory transform as:

$$\begin{aligned} \hat{\Pi}_\alpha(\nu_\alpha + 1, \tau) &= (-1)^{\mathcal{A}^{\alpha\beta}} e^{-\pi i \mathcal{A}^{\alpha\beta\gamma} a_{\psi,\gamma}} \hat{\Pi}_\alpha(\nu, \tau), \\ \hat{\Pi}_\alpha(\nu_\beta + \tau, \tau) &= (-1)^{\mathcal{A}^{\alpha\beta}} e^{-\pi i \mathcal{A}^{\alpha\beta\gamma} a_{t,\gamma}} \hat{\Pi}_\alpha(\nu, \tau). \end{aligned} \tag{D.25}$$

These relations can be understood as before, on general grounds (except for the sign, which is more subtle).

For a fixed background gauge flux $-\mathbf{m}$ on the base Riemann surface Σ_g , following the discussion in section D.1, we consider the product space $\mathcal{N}_6 = J \times \Sigma_g \times T^2$ with the projection π onto J :

$$J \times \Sigma_g \times T^2 \xrightarrow{\pi} J. \tag{D.26}$$

The determinant line bundle \mathcal{L} on J can be obtained from the line bundle $\mathcal{L}_{-\mathbf{m}}$ on \mathcal{N}_6 :

$$\mathcal{L}_{-\mathbf{m}} = L_{-\mathbf{m}} \otimes \mathcal{L}, \tag{D.27}$$

where $L_{-\mathbf{m}}$ is the pull-back of the line bundle of degree $-\mathbf{m}$ on Σ_g , and \mathcal{L} is the pull-back of the Poincaré line bundle of degree zero for T^2 . The first Chern class of \mathcal{L} can be computed from the 6-form anomaly polynomial, according to the formula:

$$c_1(\mathcal{L}) = \int_{\Sigma_g \times T^2} \text{ch}(\mathcal{L}_{-\mathbf{m}}) \wedge \hat{A}(T\mathcal{N}_6) \Big|_{6\text{-form}}. \tag{D.28}$$

Since $p_1(T\mathcal{N}_6) = 0$, we have:

$$c_1(\mathcal{L}) = \frac{1}{6} \int_{\Sigma_g \times T^2} c_1(\mathcal{L}_{-\mathbf{m}})^3. \tag{D.29}$$

It is convenient to decompose the integrand into a linear combination of type $\delta_{p,q,r}$, the p -, q - and r -forms along the Σ_g , T^2 and J , respectively:

$$c_1(\mathcal{L}_{-\mathbf{m}}) = \delta_{2,0,0} + \delta_{0,2,0} + \delta_{0,0,2} + \delta_{1,1,0} + \delta_{0,1,1} + \delta_{1,0,1}. \tag{D.30}$$

Since $\delta_{0,2,0} = \delta_{1,1,0} = 0$ for our background, the expression reduces to:

$$c_1(\mathcal{L}) = \frac{1}{2} \int_{\Sigma_g \times T^2} \delta_{2,0,0} \delta_{0,1,1}^2. \tag{D.31}$$

The integration over Σ_g gives

$$\int_{\Sigma_g} \delta_{2,0,0} = -\mathcal{A}_{qqq} \mathbf{m} - \mathcal{A}_{qqR}(g-1), \tag{D.32}$$

in terms of the 't Hooft anomaly coefficients. The integration over T^2 can be done as in (D.9). We conclude that, for a theory coupled to a general abelian background gauge field, we obtain

$$c_1(\mathcal{L}) = -\mathcal{A}^{\alpha\beta\gamma} \mathbf{m}^\alpha da_{x,\beta} da_{y,\gamma} - \mathcal{A}^{R\alpha\beta} (g-1) da_{x,\alpha} da_{y,\beta}. \quad (\text{D.33})$$

Note that this is linear in \mathbf{m} . This then explains the large gauge transformations (D.25) for the flux operator, exactly as in section D.1. The second term in (D.33) similarly determines the transformations properties of the handle-gluing operator.

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