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On stabilization of small solutions in the nonlinear Dirac equation with a trapping potential

Scipio Cuccagna [a,*], Mirko Tarulli [b]

[a] Department of Mathematics and Geosciences, University of Trieste, via Valerio 12/1, Trieste 34127, Italy

[b] Department of Mathematics, University of Pisa, Largo Bruno Pontecorvo 5, Pisa 56127, Italy

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ABSTRACT

We consider a Dirac operator with short range potential and with eigenvalues. We add a nonlinear term and we show that the small standing waves of the corresponding nonlinear Dirac equation (NLD) are attractors for small solutions of the NLD. This extends to the NLD results already known for the Nonlinear Schrödinger Equation (NLS).

1. Introduction

We consider

$$\begin{cases} \mathrm{i}u_t - Hu + g(u\overline{u})\beta u = 0, \text{ with } (t,x) \in \mathbb{R}\times\mathbb{R}^3, \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

where for $\mathscr{M} > 0$ we have for a potential $V(x)$,

$$H = D_{\mathscr{M}} + V, \tag{1.2}$$

where $D_{\mathscr{M}} = -\mathrm{i}\sum_{j=1}^{3} \alpha_j \partial_{x_j} + \mathscr{M}\beta$, with for $j = 1, 2, 3$,

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}, \beta = \begin{pmatrix} I_{\mathbb{C}^2} & 0 \\ 0 & -I_{\mathbb{C}^2} \end{pmatrix}, \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

* Corresponding author.
E-mail addresses: scuccagna@units.it (S. Cuccagna), tarulli@mail.dm.unipi.it (M. Tarulli).

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The unknown u is $\mathbb{C}^4$-valued. Given two vectors of $\mathbb{C}^4$, $uv := u \cdot v$ is the inner product in $\mathbb{C}^4$, v^* is the complex conjugate, $u \cdot v^*$ is the hermitian product in $\mathbb{C}^4$, which we write as $uv^* = u \cdot v^*$. We set $\bar{u} := \beta u^*$, so that $u\bar{u} = u \cdot \beta u^*$.

We introduce the Japanese bracket $\langle x \rangle := \sqrt{1+|x|^2}$ and the spaces defined by the following norms:

$L^{p,s}(\mathbb{R}^3, \mathbb{C}^4)$ defined with $\|u\|_{L^{p,s}(\mathbb{R}^3,\mathbb{C}^4)} := \|\langle x \rangle^s u\|_{L^p(\mathbb{R}^3,\mathbb{C}^4)}$;

$H^k(\mathbb{R}^3, \mathbb{C}^4)$ defined with $\|u\|_{H^k(\mathbb{R}^3,\mathbb{C}^4)} := \|\langle x \rangle^k \mathcal{F}(u)\|_{L^2(\mathbb{R}^3,\mathbb{C}^4)}$, where $\mathcal{F}$ is the classical Fourier transform (see for instance [25]);

$H^{k,s}(\mathbb{R}^3, \mathbb{C}^4)$ defined with $\|u\|_{H^{k,s}(\mathbb{R}^3,\mathbb{C}^4)} := \|\langle x \rangle^s u\|_{H^k(\mathbb{R}^3,\mathbb{C}^4)}$;

$$\Sigma_k := L^{2,k}(\mathbb{R}^3, \mathbb{C}^4) \cap H^k(\mathbb{R}^3, \mathbb{C}^4) \text{ with } \|u\|^2_{\Sigma_k} = \|u\|^2_{H^k(\mathbb{R}^3,\mathbb{C}^4)} + \|u\|^2_{L^{2,k}(\mathbb{R}^3,\mathbb{C}^4)}. \tag{1.3}$$

For $\mathbf{f}, \mathbf{g} \in L^2(\mathbb{R}^3, \mathbb{C}^4)$ consider the bilinear map

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int_{\mathbb{R}^3} \mathbf{f}(x)\mathbf{g}(x)dx = \int_{\mathbb{R}^3} \mathbf{f}(x) \cdot \mathbf{g}(x)dx. \tag{1.4}$$

We assume the following.

(H1) $g(0) = 0$, $g \in C^\infty(\mathbb{R}, \mathbb{R})$.

(H2) $V \in \mathcal{S}(\mathbb{R}^3, S_4(\mathbb{C}))$ with $S_4(\mathbb{C})$ the set of self-adjoint 4×4 matrices and $\mathcal{S}(\mathbb{R}^3, \mathbb{E})$ the space of Schwartz functions from $\mathbb{R}^3$ to $\mathbb{E}$, with the latter a Banach space on $\mathbb{C}$.

(H3) $\sigma_p(H) = \{e_1 < e_2 < e_3 \cdots < e_n\} \subset (-\mathscr{M}, \mathscr{M})$. Here we assume that all the eigenvalues have multiplicity 1. Each point $\tau = \pm\mathscr{M}$ is neither an eigenvalue nor a resonance (that is, if $(D_{\mathscr{M}} + V)u = \tau u$ with $u \in C^\infty$ and $|u(x)| \le C|x|^{-1}$ for a fixed C, then $u = 0$).

(H4) There is an $N \in \mathbb{N}$ with $N > (\mathscr{M} + |e_1|)(\min\{e_i - e_j : i > j\})^{-1}$ such that if $\mu \cdot \mathbf{e} := \mu_1 e_1 + \cdots + \mu_n e_n$ then

$$\mu \in \mathbb{Z}^n \text{ with } |\mu| \le 4N + 6 \ \Rightarrow |\mu \cdot \mathbf{e}| \ne \mathscr{M}, \tag{1.5}$$

$$(\mu - \nu) \cdot \mathbf{e} = 0 \text{ and } |\mu| = |\nu| \le 2N + 3 \Rightarrow \mu = \nu. \tag{1.6}$$

(H5) Consider the set M_{min} defined in (2.5) and for any $(\mu, \nu) \in M_{min}$ we consider the function $G_{\mu\nu}(x)$ (see the proof of Lemma 5.11 or also later in the introduction the effective hamiltonian), $\widehat{G}_{\mu\nu}(\xi)$ the distorted Fourier transform associated to H (see (5.64) and Appendix A for more details) of $G_{\mu\nu}(x)$ and the sphere $S_{\mu\nu} = \{\xi \in \mathbb{R}^3 : |\xi|^2 + \mathscr{M}^2 = |(\nu - \mu) \cdot \mathbf{e}|^2\}$. Then we assume that for any $(\mu, \nu) \in M_{min}$ the restriction of $\widehat{G}_{\mu\nu}$ on the sphere $S_{\mu\nu}$ is $\widehat{G}_{\mu\nu}|_{S_{\mu\nu}} \ne 0$.

To each e_j we associate an eigenfunction ϕ_j. We choose them such that $\mathrm{Re}\langle \phi_j, \phi_k^* \rangle = \delta_{jk}$. To each ϕ_j we associate nonlinear bound states.

Proposition 1.1 *(Bound states). Fix $j \in \{1, \cdots, n\}$. Then $\exists a_0 > 0$ such that $\forall z_j \in B_{\mathbb{C}}(0, a_0)$, there is a unique $Q_{jz_j} \in \mathcal{S}(\mathbb{R}^3, \mathbb{C}^4) := \cap_{t \ge 0} \Sigma_t(\mathbb{R}^3, \mathbb{C}^4)$, such that*

$$\begin{aligned} &HQ_{jz_j} + g(Q_{jz_j}\overline{Q}_{jz_j})\beta Q_{jz_j} = E_{jz_j} Q_{jz_j}, \\ &Q_{jz_j} = z_j \phi_j + q_{jz_j}, \ \langle q_{jz_j}, \phi_j^* \rangle = 0, \end{aligned} \tag{1.7}$$

and such that we have for any $r \in \mathbb{N}$:

(1) $(q_{jz_j}, E_{jz_j}) \in C^\infty(B_{\mathbb{C}}(0,a_0), \Sigma_r \times \mathbb{R})$; *we have* $q_{jz_j} = z_j \widehat{q}_j(|z_j|^2)$, *with* $\widehat{q}_j(t^2) = t^2 \widetilde{q}_j(t^2)$, $\widetilde{q}_j(t) \in C^\infty((-{a_0}^2, {a_0}^2), \Sigma_r(\mathbb{R}^3, \mathbb{C}^4))$ *and* $E_{jz_j} = E_j(|z_j|^2)$ *with* $E_j(t) \in C^\infty((-{a_0}^2, {a_0}^2), \mathbb{R})$;
(2) $\exists\, C > 0$ *such that* $\|q_{jz_j}\|_{\Sigma_r} \le C|z_j|^3$, $|E_{jz_j} - e_j| < C|z_j|^2$.

For the Σ_r see (1.3). The only non-elementary point in Proposition 1.1 is the independence of a_0 with respect of r (which strictly speaking is not necessary in this paper), which can be proved with routine arguments as in the Appendix of [21].

Definition 1.2. Let $b_0 > 0$ be sufficiently small so that for $z_j \in B_{\mathbb{C}^n}(0,b_0)$, Q_{jz_j} exists for all $j \in \{1,\cdots,n\}$. Set $z_j = z_{jR} + \mathrm{i} z_{jI}$, for $z_{jR}, z_{jI} \in \mathbb{R}$ and $D_{jA} := \frac{\partial}{\partial z_{jA}}$, for $A = R, I$. We set

$$\begin{aligned} \mathcal{H}_c[z] &= \mathcal{H}_c[z_1 \cdots, z_n] \text{ where } z := (z_1, \dots, z_n) \\ &:= \left\{\eta \in L^2(\mathbb{R}^3, \mathbb{C}^4);\ \mathrm{Re}\left\langle \mathrm{i}\,\eta^*, D_{jR} Q_{jz_j}\right\rangle = \mathrm{Re}\left\langle \mathrm{i}\,\eta^*, D_{jI} Q_{jz_j}\right\rangle = 0 \text{ for all } j\right\}. \end{aligned} \tag{1.8}$$

In particular we have

$$\mathcal{H}_c[0] = \left\{\eta \in L^2(\mathbb{R}^3, \mathbb{C}^4);\ \langle \eta^*, \phi_j\rangle = 0 \text{ for all } j\right\}. \tag{1.9}$$

We denote by P_c the orthogonal projection of $L^2(\mathbb{R}^3, \mathbb{C}^4)$ onto $\mathcal{H}_c[0]$.

We will prove the following theorem.

Theorem 1.3. *Assume* (H1)–(H5). *Then there exist* $\epsilon_0 > 0$ *and* $C > 0$ *such that for* $\epsilon = \|u(0)\|_{H^4} < \epsilon_0$ *then the solution* $u(t)$ *of* (1.1) *exists for all times and can be written uniquely for all times as*

$$u(t) = \sum_{j=1}^{n} Q_{jz_j(t)} + \eta(t), \ \text{ with } \eta(t) \in \mathcal{H}_c[z(t)], \tag{1.10}$$

such that there exist a unique j_0, *a* $\rho_+ \in [0,\infty)^n$ *with* $\rho_{+j} = 0$ *for* $j \ne j_0$ *such that* $|\rho_+| \le C\|u(0)\|_{H^4}$, *an* $\eta_+ \in H^4$ *with* $\|\eta_+\|_{L^\infty} \le C\|u(0)\|_{H^4}$ *and such that we have*

$$\begin{aligned} &\lim_{t\to+\infty} \|\eta(t) - e^{-\mathrm{i}tD_{\mathscr{M}}}\eta_+\|_{H^4(\mathbb{R}^3,\mathbb{C}^4)} = 0, \\ &\lim_{t\to+\infty} |z_j(t)| = \rho_{+j}. \end{aligned} \tag{1.11}$$

Furthermore we have $\eta = \widetilde{\eta} + A(t,x)$ *such that,*

$$\text{for preassigned } p_0 > 2, \text{ for all } p \ge p_0 \text{ and for } \frac{2}{p} = \frac{3}{2}\left(1 - \frac{2}{q}\right), \tag{1.12}$$

we have

$$\begin{aligned} &\|z\|_{L^\infty_t(\mathbb{R}_+)} + \|\widetilde{\eta}\|_{L^p_t([0,\infty), B^{4-\frac{2}{p}}_{q,2}(\mathbb{R}^3,\mathbb{C}^4))} \le C\|u(0)\|_{H^4(\mathbb{R}^3,\mathbb{C}^4)}\ , \\ &\|\dot{z}_j + \mathrm{i} e_j z_j\|_{L^\infty_t(\mathbb{R}_+)} \le C\|u(0)\|^2_{H^4(\mathbb{R}^3,\mathbb{C}^4)} \end{aligned} \tag{1.13}$$

(for the Besov spaces $B^k_{p,q}$ *see Sect.* 2*) and such that* $A(t,\cdot) \in \Sigma_4$ *for all* $t \ge 0$, *with*

$$\lim_{t\to+\infty} \|A(t,\cdot)\|_{\Sigma_4} = 0. \tag{1.14}$$

Remark 1.4. In $H^4(\mathbb{R}^3, \mathbb{C}^4)$ the functional $u \to g(u\bar{u})\beta u$ is locally Lipschitz and (1.1) is locally well posed, see pp. 293–294 in volume III [51].

Remark 1.5. There is no attempt here to get a sharp result with a minimum amount of regularity on the initial datum u_0. The need of H^4 comes up (5.39).

Theorem 1.3 states that small solutions of the NLD are asymptotically equal to exactly one standing wave (possibly the vacuum) up to radiation which scatters and up to a phase factor. As we explain below, this happens because of the natural tendency of radiation to scatter because of linear scattering, and of energy to leak out of all discrete modes except at most for one because of nonlinear interaction with the continuous modes which produces a form of friction on most discrete modes. Theorem 1.3 does not say if some of the standing waves are stable or unstable (this remains an open problem).

The asymptotic behavior of solutions which start close to equilibria represents a fundamental problem for nonlinear systems. Here we are interested to hamiltonian systems which are perturbations of linear systems admitting a mixture of discrete and continuous components. Decay phenomena imposed by the perturbation on the discrete modes and decay of *metastable* states are a classical topic of physical relevance in the study of radiation matter interaction, [15]. There is a substantial literature focused on the case of linear systems, see [17] for a recent reference and therein for some of the literature.

The NLS has been explored in papers such as [30,31,42,43,45–47,53–56]. An analogue of Theorem 1.3 for the NLS is proved in [21]. The case of the nonlinear Klein Gordon equation (NLKG), in the context of real valued solutions, where all small solutions scatter to 0, has been initiated in [47] in special case and to a large degree solved in a general way in [2]. For complex valued solutions of the NLKG see [22].

In this paper we focus on the NLD. There are many papers on global well posedness and dispersion for solutions belonging to much larger spaces than what considered here, for example see [3,11,12,24,27,34,59] and references therein. However, these papers do not address the asymptotic behavior in the presence of discrete modes. For a survey on the 1D case we refer also to [39]. The essential indefiniteness of the Dirac operators (1.2) forbids the use of the stability theory of standing waves developed in [13,29,57], which involves conditionally positive Lyapunov functionals. See [50] for an attempt to apply this theory to the Dirac equation, in combination to numerical computations. A successful use of rather elaborate Lyapunov functionals in the special case of an integrable NLD is in [40]. Since energy methods cannot be used to prove stability then another option is to prove stability by means of linear dispersion. This is a problem arising also in other settings, see for example [38]. Thus on a strictly technical standpoint the NLD is a rather interesting class of hamiltonian systems, especially because it has been explored much less than systems like the NLS.

We turn to the specific problem considered in this paper. In the case of the linear Dirac equation $\mathrm{i}u_t = Hu$ we know that there are invariant complex lines formed by standing waves in correspondence to the eigenstates of H. Proposition 1.1 describes the well known fact that the NLD (1.1), at least at small energies, has invariant disks formed by standing waves. The question arises then on what should be the behavior of other initially small solutions of the NLD. In the case of the NLS the answer given in [21] is that the union of these invariant disks is an attractor for all small solutions. This is by no means obvious in view of the fact that in the linear equation the situation is different. In fact in the linear case discrete modes and the continuous part are all independent from each other so that solutions of the linear equation contain a quasi periodic component. There are examples of nonlinear systems where these linear like patterns persist. Notably the NLS where $x \in \mathbb{Z}$, which has small quasi-periodic solutions, see for example in [35] and references therein. There are also equations in $\mathbb{R}^n$ which admit families of quasiperiodic solutions containing solitary waves as a special case, an example being exactly the (1.1) with potential $V = 0$ in (1.2) where solitary waves are special cases of more general quasiperiodic solutions. This is related to the existence of special eigenvalues of the linearization uncoupled to radiation, as the eigenvalues in claim 4 in Lemma 6.1 in [4]. Notice also that the stabilization displayed in [21] is a very slow phenomenon, non-detectable easily numerically: see [33] for comments on a somewhat different but related context. So results such as those in [21], from the earlier papers [37,47,48,53–55] which treat special cases, to the solutions in generic setting contained in [2,21], are far from obvious.

Here we transpose partially to the NLD the result proved for the NLS in [21]. No particular deep new insight is needed since the proof for the NLD is similar to that for the NLS, with some difference related to the dispersion theory of radiation. However, results such as Theorem 1.3 are important given that the literature gives a very fragmentary picture of the asymptotic analysis of solutions of the NLD.

While prior to [21] there had been a large body of work for the NLS, very little has been written for the NLD about our problem, that is the analysis of the NLD with a mixture of discrete and continuous components. It is worth comparing the known results in the literature with Theorem 1.3.

One study is [6]. [6] contains a number of useful results on the dispersion for the group $e^{\mathrm{i}tH}$ which are used here. In terms of analysis of small solutions of the NLD with a linear potential, [6] considers the case of $\sigma_p(H) = \{e_1 < e_2\}$ with $e_2 - e_1 < \min(m - e_1, e_1 + m)$ and then proves the existence of a hypersurface of initial data perpendicular at the origin to the eigenspace of e_2, and whose corresponding solutions of (1.1) converge to the 1st family of standing waves Q_{1z_1}.

Another study is [41]. [41] proves the asymptotic stability of the standing waves Q_{1z_1} when $\sigma_p(H) = \{e_1\}$, that is an analogue of [42,46]. [41], like [42,46], does not examine a whole neighborhood of 0 but proves that if an initial datum starts very close to a Q_{1z_1} then, in a sense similar to Theorem 1.3, it will converge to a nearby Q_{1z_1}. Notice that while here we don't prove that the Q_{1z_1} are stable, under the hypothesis $\sigma_p(H) = \{e_1\}$ it can be proved to be stable also in our 3 D set up. In fact this is much simpler to prove than what we do here.

In [8] is discussed the asymptotic stability of standing waves of the NLD in a different context, which is somewhat closer in spirit to Theorem 1.3 since the emphasis is on the case when many discrete modes are present. For another result, see also [16].

Here we give more comprehensive conclusions than in [6,41] because we treat all small solutions of (1.1). Maybe we could have analyzed the stability of specific standing waves, applying the theory in [8] which provides some tools characterize standing waves of the (1.1) through an analysis of the linearization of (1.1) at the standing wave, but we don't do this here. So unfortunately we do not identify stable standing waves and we don't produce a criterion to define ground states for (1.1). Recall that in [21] it is proved for the NLS that while the Q_{1z_1} are stable (which was well known since [43]), the Q_{jz_j} for $j \geq 2$ are unstable. No similar analysis is done here.

After adding to the Dirac equation a nonlinearity such as in (1.1) as we have already mentioned the lines of standing waves persist, only in the form of topological disks. Here we briefly explain how the asymptotics claimed in Theorem 1.3 comes about. The mechanism is the same of the NLS and to be seen requires the identification of an *effective* hamiltonian. In an appropriate system of coordinates, this turns out to be, heuristically, of the form

$$\mathcal{H}(z,\eta) = \sum_{j=1}^{n} e_j |z_j|^2 + \langle H\eta, \eta^* \rangle + \sum_{|(\mu-\nu)\cdot \mathbf{e}| > \mathscr{M}} (z^\mu \bar{z}^\nu \langle G_{\mu\nu}, \eta\rangle + \bar{z}^\mu z^\nu \langle G^*_{\mu\nu}, \eta^*\rangle), \text{ with } \mathbf{e} = (e_1, \dots, e_n),$$

where the 2nd summation is on an appropriate finite set of multi-indexes. The coordinates (z, η), with $z = (z_1, \dots, z_n)$, representing the discrete modes and $\eta \in \mathcal{H}_c[0]$ representing the radiation, are canonical (or Darboux) coordinates (but not so the initial coordinates in Lemma 2.9). This in particular means that the equation for η is

$$\mathrm{i}\dot{\eta} = H\eta + \sum \bar{z}^\mu z^\nu G^*_{\mu\nu}. \tag{1.15}$$

Then, succinctly,

$$\|\eta\|_{\text{Strichartz}} \le \epsilon + \sum \|z^{\mu+\nu}\|_{L^2}, \tag{1.16}$$

for appropriate Strichartz norms of η. These estimates are less well known than the corresponding ones for the NLS, but nonetheless are known, see Sect. 5.1 where they are proven using [6–8]. There are various other sets of Strichartz like estimates for Dirac potentials in the literature, see for example [9,11,23], but they do not apply to our case since they involve Dirac operators without eigenvalues. The equations for the discrete modes are, heuristically,

$$\mathrm{i}\dot{z}_j = e_j z_j + \sum_{|(\mu-\nu)\cdot\mathbf{e}|>\mathscr{M}} \nu_j \frac{z^\mu \bar{z}^\nu}{\bar{z}_j}\langle \eta, G_{\mu\nu}\rangle + \sum_{|(\mu-\nu)\cdot\mathbf{e}|>\mathscr{M}} \mu_j \frac{z^\nu \bar{z}^\mu}{\bar{z}_j}\langle \eta^*, G^*_{\mu\nu}\rangle. \tag{1.17}$$

By an argument introduced in [10,48] we write

$$\eta = g - Y, \text{ where } Y := \sum_{|(\alpha-\beta)\cdot\mathbf{e}|>\mathscr{M}} \bar{z}^\alpha z^\beta R_H^+(\mathbf{e}\cdot(\beta-\alpha))G^*_{\alpha\beta}. \tag{1.18}$$

We observe that (1.18) is a decomposition of η into $-Y$, which is the part of η that mostly affects the z's, and g which is small. Substituting in (1.17) and ignoring the very small g we get an autonomous system in z. Ignoring smaller terms, we have

$$\frac{d}{dt}\sum_j |z_j|^2 = -2\pi \sum_{|(\mu-\nu)\cdot\mathbf{e}|>\mathscr{M}} |z^\mu \bar{z}^\nu|^2 \langle \delta(H-(\nu-\mu)\cdot\mathbf{e})G^*_{\mu\nu}, G_{\mu\nu}\rangle. \tag{1.19}$$

The r.h.s. is negative, see Lemma A.1 in the Appendix. Notice that in [10,47] the structure of the r.h.s. is as clear as (1.19) only under very restrictive hypotheses on the discrete spectrum.

We assume that for an appropriate set of pairs (μ,ν) in (1.19) we have $\langle \delta(H-(\nu-\mu)\cdot\mathbf{e})G^*_{\mu\nu}, G_{\mu\nu}\rangle > 0$. This is the content of hypothesis (H5) and is an exact analogue of inequality (1.8) in [47]. Then integrating (1.19) we obtain

$$\sum_j |z_j(t)|^2 + \sum_{|(\mu-\nu)\cdot\mathbf{e}|>\mathscr{M}} \int_0^t |z^\mu \bar{z}^\nu|^2 \le C\sum_j |z_j(0)|^2 \le C\epsilon^2, \tag{1.20}$$

for a fixed C. This allows to close (1.16) and to prove that η scatters. From (1.20) and the fact that the $\dot{z}(t)$ is uniformly bounded, we get that the $\lim_{t\to+\infty} z^{\mu+\nu}(t) = 0$ for the (μ,ν) in (1.20). This allows to conclude that $\lim_{t\to+\infty} z_j(t) = 0$ for all except for at most one j, because the pairs of multi-indexes (μ,ν) involve cases such as $z_j^{\mu_j}\bar{z}_k^{\nu_k}$ for any pair $j\neq k$ (while cases of the form $z_j^{\mu_j}\bar{z}_j^{\nu_j}$ are not allowed).

What has happened is that when we have substituted (1.18) inside (1.17) and ignored g, we got a system on the z's which has some degree of friction, ultimately due to the coupling of the discrete modes with radiation. This can be proved thanks to the square structure of the r.h.s. of (1.19) where what is crucial is the fact that each $G_{\mu\nu}$ is paired with its complex conjugate $G^*_{\mu\nu}$. This comes about and can be seen transparently because we have a hamiltonian system in Darboux coordinates, which tells us that the $G^*_{\mu\nu}$'s in the r.h.s. of (1.15) are the same of the coefficients in the (1.17).

To be able to implement the above intuition gets some work because, as we mentioned, the most natural coordinates are not Darboux coordinates, that is the symplectic form is complicated when expressed in these coordinates (this would make the search of the effective hamiltonian very difficult in these initial coordinates) and the equations are not as simple as $\mathrm{i}\dot{z}_j = \partial_{\bar{z}_j}\mathcal{H}$ and $\mathrm{i}\dot{\eta} = \partial_{\eta^*}\mathcal{H}$.

There are various papers, such as [18,20], that discuss how to first produce Darboux coordinates and how to find the effective hamiltonian in an appropriate way that guarantees that the system remains a semilinear Dirac equation. In fact this part of the proof is here the same of [21], since in terms of the hamiltonian formalism, the NLS and the NLD are the same. So Darboux coordinates and search of the effective hamiltonian by means of Birkhoff normal forms are accomplished in [21].

The only part of the proof where there is difference between NLD and NLS is in the study of dispersion. The NLD requires its own set of technical machinery about linear dispersion theory, Strichartz and smoothing estimates, which is not the same of the NLS and which is somewhat tricker and less well understood. Nonetheless, all the linear theory we need here has been already developed in the literature, in particular in [6–8].

Since the main ideas on the hamiltonian structure and coordinate changes come from [21], we will refer to [21] for an extensive discussion of the main issues. The rest of the proof consists in proving dispersion of the continuous component of the solution (and here we use the technology of Dirac operators in [6,7]), i.e. (1.16), and the so called Nonlinear Fermi Golden Rule, i.e. (1.19). This latter part of the proof is similar to [21], but requires some modification in the spirit of [19] because of the possible presence of pairs of eigenvalues with different signs.

Here as in [21] we do not prove, as done in [2] in an easier setting, that hypothesis (H5) holds for generic pairs $(V, g(u\overline{u}))$. While in [21] what was missing to repeat the argument in [2] was a meaningful mass term, here we have mass $\mathscr{M}$ but the dependence of the linear operator H on $\mathscr{M}$ requires new ideas.

2. Further notation and coordinates

2.1. Notation

For $k \in \mathbb{R}$ and $1 \leq p, q \leq \infty$, the Besov space $B^k_{p,q}(\mathbb{R}^3, \mathbb{C}^d)$ is the space of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^3, \mathbb{C}^d)$ such that

$$\|f\|_{B^k_{p,q}} = (\sum_{j\in\mathbb{N}} 2^{jkq} \|\varphi_j * f\|_p^q)^{\frac{1}{q}} < +\infty,$$

with $\mathcal{F}(\varphi) \in \mathcal{C}_0^\infty(\mathbb{R}^n \setminus \{0\})$ such that $\sum_{j\in\mathbb{Z}} \mathcal{F}(\varphi)(2^{-j}\xi) = 1$ for all $\xi \in \mathbb{R}^3 \setminus \{0\}$, $\mathcal{F}(\varphi_j)(\xi) = \mathcal{F}(\varphi)(2^{-j}\xi)$ for all $j \in \mathbb{N}^*$ and for all $\xi \in \mathbb{R}^3$, and $\mathcal{F}(\varphi_0) = 1 - \sum_{j\in\mathbb{N}^*} \mathcal{F}(\varphi_j)$. It is endowed with the norm $\|f\|_{B^k_{p,q}}$.

- We denote by $\mathbb{N} = \{1, 2, \dots\}$ the set of natural numbers and set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.
- Given a Banach space X, $v \in X$ and $\delta > 0$ we set

$$B_X(v, \delta) := \{x \in X \mid \|v - x\|_X < \delta\}.$$

- We denote $z = (z_1, \dots, z_n)$, $|z| := \sqrt{\sum_{j=1}^n |z_j|^2}$.
- We set $\partial_l := \partial_{z_l}$ and $\partial_{\bar{l}} := \partial_{\bar{z}_l}$. Here as customary $\partial_{z_l} = \frac{1}{2}(D_{lR} - \mathrm{i}D_{lI})$ and $\partial_{\bar{z}_l} = \frac{1}{2}(D_{lR} + \mathrm{i}D_{lI})$.
- Occasionally we use a single index $\ell = j, \bar{j}$. To define $\bar{\ell}$ we use the convention $\bar{\bar{j}} = j$. We will also write $z_{\bar{j}} = \bar{z}_j$.
- We will consider vectors $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ and for vectors $\mu, \nu \in (\mathbb{N} \cup \{0\})^n$ we set $z^\mu \bar{z}^\nu := z_1^{\mu_1} \dots z_n^{\mu_n} \bar{z}_1^{\nu_1} \dots \bar{z}_n^{\nu_n}$. We will set $|\mu| = \sum_j \mu_j$.
- We have $dz_j = dz_{jR} + \mathrm{i}dz_{jI}$, $d\bar{z}_j = dz_{jR} - \mathrm{i}dz_{jI}$.
- We consider the vector $\mathbf{e} = (e_1, \dots, e_n)$ whose entries are the eigenvalues of H.

Remark 2.1. We draw the attention of the reader to the fact that the complex conjugate of $v \in \mathbb{C}^4$ is v^* with $\bar{v} = \beta v^*$ while for $\zeta \in \mathbb{C}$ the complex conjugate is $\bar{\zeta}$ which is more convenient notation in some later formulas than writing ζ^*.

2.2. Coordinates

The first thing we need is the following well standard ansatz, see for example in Lemma 2.6 [21].

Lemma 2.2. *There exists $c_0 > 0$ such that there exists a $C > 0$ such that for all $u \in H^1$ with $\|u\|_{H^1} < c_0$, there exists a unique pair $(z, \Theta) \in \mathbb{C}^n \times (H^1 \cap H_c[z])$ such that*

$$u = \sum_{j=1}^{n} Q_{jz_j} + \Theta \text{ with } |z| + \|\Theta\|_{H^4} \le C\|u\|_{H^4}. \tag{2.1}$$

Finally, the map $u \to (z, \Theta)$ is $C^\infty(B_{H^4}(0, c_0), \mathbb{C}^n \times H^4)$ and satisfies the gauge property

$$z(e^{i\vartheta}u) = e^{i\vartheta}z(u) \text{ and } \Theta(e^{i\vartheta}u) = e^{i\vartheta}\Theta(u). \qquad \square \tag{2.2}$$

We now recall from [21] the following definitions.

Definition 2.3. Given $z \in \mathbb{C}^n$, we denote by $\widehat{Z}$ the vector with entries $(z_i\bar{z}_j)$ with $i, j \in [1, n]$ ordered in lexicographic order (that is we write $z_i\bar{z}_j$ before $z_{i'}\bar{z}_{j'}$ if either $i < j$ or, when $i = j$, if $j < j'$). We denote by $\mathbf{Z}$ the vector with entries $(z_i\bar{z}_j)$ with $i, j \in [1, n]$ ordered in lexicographic order but only with pairs of indexes with $i \neq j$. Here $\mathbf{Z} \in L$ with L the subspace of $\mathbb{C}^{n_0} = \{(a_{i,j})_{i,j=1,\dots,n} : i \neq j\}$ where $n_0 = n(n-1)$, with $(a_{i,j}) \in L$ iff $a_{i,j} = \bar{a}_{j,i}$ for all i, j. For a multi index $\mathbf{m} = \{m_{ij} \in \mathbb{N}_0 : i \neq j\}$ we set $\mathbf{Z}^{\mathbf{m}} = \prod(z_i\bar{z}_j)^{m_{ij}}$ and $|\mathbf{m}| := \sum_{i,j} m_{ij}$.

Definition 2.4. Consider the set of multiindexes $\mathbf{m}$ as in Definition 2.3. Consider for any $k \in \{1, \dots, n\}$ the set

$$\begin{aligned} \mathcal{M}_k(r) &= \{\mathbf{m} : \left|\sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}(e_i - e_j) - e_k\right| > \mathscr{M} \text{ and } |\mathbf{m}| \le r\}, \\ \mathcal{M}_0(r) &= \{\mathbf{m} : \sum_{i=1}^{n}\sum_{j=1}^{n} m_{ij}(e_i - e_j) = 0 \text{ and } |\mathbf{m}| \le r\}. \end{aligned} \tag{2.3}$$

Set now

$$\begin{aligned} &M_k(r) = \{(\mu, \nu) \in \mathbb{N}_0^n \times \mathbb{N}_0^n : \exists \mathbf{m} \in \mathcal{M}_k(r) \text{ such that } z^\mu \bar{z}^\nu = \bar{z}_k \mathbf{Z}^{\mathbf{m}}\}, \\ &M(r) = \cup_{k=1}^{n} M_k(r). \end{aligned} \tag{2.4}$$

We also set $M = M(2N + 4)$ and

$$M_{min} = \{(\mu, \nu) \in M : (\alpha, \beta) \in M \text{ with } \alpha_j \le \mu_j \text{ and } \beta_j \le \nu_j \ \forall\, j \Rightarrow (\alpha, \beta) = (\mu, \nu)\}. \tag{2.5}$$

The following simple lemma is used in Lemma 5.17.

Lemma 2.5. *The following facts hold.*

(1) *If* $(\mu,\nu)\in M_{min}$ *then for any* j *we have* $\mu_j\nu_j=0$.
(2) *Suppose that* $(\mu,\nu)\in M_{min}$, $(\alpha,\beta)\in M_{min}$ *and* $(\mu-\nu)\cdot\mathbf{e}=(\alpha-\beta)\cdot\mathbf{e}$. *Then* $(\mu,\nu)=(\alpha,\beta)$.

Proof. First of all it is easy to show that $(\mu,\nu)\in M(r)$ if and only if $|\nu|=|\mu|+1$, $|\mu|\le r$ and $|(\mu-\nu)\cdot\mathbf{e}|>\mathscr{M}$.

Suppose that $\mu_j\ge 1$ and $\nu_j\ge 1$. Then consider (α,β)

$$\alpha_k=\begin{cases}\mu_k & \text{for } k\neq j,\\ \mu_j-1 & \text{for } k=j\end{cases}\qquad \beta_k=\begin{cases}\nu_k & \text{for } k\neq j,\\ \nu_j-1 & \text{for } k=j.\end{cases}$$

Since $|\nu|=|\mu|+1$ we have $|\beta|=|\alpha|+1$. Furthermore $(\mu-\nu)\cdot\mathbf{e}=(\alpha-\beta)\cdot\mathbf{e}$. This implies $(\alpha,\beta)\in M$ and so $(\mu,\nu)\notin M_{min}$. This proves claim (1).

By $(\mu-\nu)\cdot\mathbf{e}=(\alpha-\beta)\cdot\mathbf{e}$ and (H4) we obtain $\mu-\nu=\alpha-\beta$, which by claim (1) yields $(\mu,\nu)=(\alpha,\beta)$. □

Lemma 2.6. *Assuming* (*H*4) *then the following properties are fulfilled.*

(1) *For* $\mathbf{Z}^{\mathbf{m}}=z^\mu\bar z^\nu$, *then* $\mathbf{m}\in\mathcal{M}_0(2N+4)$ *implies* $\mu=\nu$. *In particular* $\mathbf{m}\in\mathcal{M}_0(2N+4)$ *implies* $\mathbf{Z}^{\mathbf{m}}=|z_1|^{2l_1}\dots|z_n|^{2l_n}$ *for some* $(l_1,\dots,l_n)\in\mathbb{N}_0^n$.
(2) *For* $|\mathbf{m}|\le 2N+3$ *and any* j *we have* $\sum_{a,b}(e_a-e_b)m_{ab}-e_j\neq 0$.

Proof. The following proof is in [21]. If $\mu=\nu$ then $z^\mu\bar z^\nu=|z_1|^{2\mu_1}...|z_n|^{2\mu_n}$. So the first sentence in claim (1) implies the second sentence in claim (1). We have

$$\mathbf{Z}^{\mathbf{m}}=\prod_{i,l=1}^n(z_i\bar z_l)^{m_{il}}=\prod_{i=1}^n z_i^{\sum_{l=1}^n m_{il}}\bar z_i^{\sum_{l=1}^n m_{li}}=z^\mu\bar z^\nu.$$

The pair (μ,ν) satisfies $|\mu|=|\nu|\le 2N+4$ by

$$|\mu|=\sum_l\mu_l=\sum_{i,l}m_{il}=|\nu|.$$

We have $(\mu-\nu)\cdot\mathbf{e}=0$ by $\mathbf{m}\in\mathcal{M}_0(2N+4)$ and

$$\sum_i\mu_ie_i-\sum_l\nu_le_l=\sum_{i,l}m_{il}(e_i-e_l)=0.$$

We conclude by (H4) that $\mu-\nu=0$. This proves the 1st sentence of claim (1).

The proof of claim (2) is similar. Set

$$\mathbf{Z}^{\mathbf{m}}\bar z_j=\prod_{i,l=1}^n(z_i\bar z_l)^{m_{il}}\bar z_j=\prod_{i=1}^n z_i^{\sum_{l=1}^n m_{il}}\bar z_i^{\sum_{l=1}^n m_{li}}\bar z_j=z^\mu\bar z^\nu.$$

We have

$$(\mu-\nu)\cdot\mathbf{e}=\sum_i\mu_ie_i-\sum_l\nu_le_l=\sum_{i,l}m_{il}(e_i-e_l)-e_j,$$

in addition we have also

$$|\mu|=\sum_l\mu_l=\sum_{i,l}m_{il}=|\nu|-1.\tag{2.6}$$

If $(\mu - \nu) \cdot \mathbf{e} = 0$ then by $|\mu - \nu| \le 4N + 5$ and by (H4) we would have $\mu = \nu$, impossible by (2.6). $\square$

The following elementary lemma is very similar to Lemma 2.4 [21] and is used to bound the 2nd and 3rd term in the 1st line of (5.2) in terms of the $z^\mu \bar{z}^\nu$ with $(\mu, \nu) \in M_{min}$ and η.

Lemma 2.7. *We have the following facts.*

(1) *Consider a vector* $\mathbf{m} = (m_{ij}) \in \mathbb{N}_0^{n_0}$ *such that* $\sum_{i<j} m_{ij} > N$ *for* $N > \mathscr{M}(\min\{e_j - e_i : j > i\})^{-1}$, *see (H4). Then for any eigenvalue* e_k *we have*

$$\sum_{i<j} m_{ij}(e_i - e_j) - e_k < -\mathscr{M}. \tag{2.7}$$

(2) *Consider* $\mathbf{m} \in \mathbb{N}_0^{n_0}$ *and the monomial* $z_j \mathbf{Z}^{\mathbf{m}}$. *Suppose* $|\mathbf{m}| \ge 2N + 3$. *Then there are* $\mathbf{a}, \mathbf{b} \in \mathbb{N}_0^{n_0}$ *such that we have*

$$\begin{aligned} &\sum_{i<j} a_{ij} = N + 1 = \sum_{i<j} b_{ij}, \\ &a_{ij} = b_{ij} = 0 \text{ for all } i > j, \\ &a_{ij} + b_{ij} \le m_{ij} + m_{ji} \text{ for all } (i,j) \end{aligned} \tag{2.8}$$

and moreover there are two indexes (k, l) *such that*

$$\sum_{i<j} a_{ij}(e_i - e_j) - e_k < -\mathscr{M}, \qquad \sum_{i<j} b_{ij}(e_i - e_j) - e_l < -\mathscr{M} \tag{2.9}$$

and such that for $|z| \le 1$

$$|z_j \mathbf{Z}^{\mathbf{m}}| \le |z_j| \; |z_k \mathbf{Z}^{\mathbf{a}}| \; |z_l \mathbf{Z}^{\mathbf{b}}|. \tag{2.10}$$

(3) *For* $\mathbf{m}$ *with* $|\mathbf{m}| \ge 2N + 3$ *there exist* (k, l) *and* $\mathbf{a} \in \mathcal{M}_k$ *and* $\mathbf{b} \in \mathcal{M}_l$ *such that* (2.10) *holds.*

Proof. The proof is very similar to Lemma 2.4 in [21]. For example, (2.7) follows immediately from

$$\sum_{i<j} m_{ij}(e_i - e_j) - e_k \le -\min\{e_j - e_i : j > i\}N - e_1 < -\mathscr{M},$$

with the latter inequality due to the definition of N. All the other claims can be proved like the rest of Lemma 2.4 in [21].

Given $\mathbf{a}, \mathbf{b} \in \mathbb{N}_0^{n_0}$ satisfying (2.8), by claim (1) they satisfy (2.9) for any pair of indexes (k, l). Consider now the monomial $z_j \mathbf{Z}^{\mathbf{m}}$. Since $|\mathbf{m}| \ge 2N + 3$, there are vectors $\mathbf{c}, \mathbf{d} \in \mathbb{N}_0^{n_0}$ such that $|\mathbf{c}| = |\mathbf{d}| = N + 1$ with $c_{ij} + d_{ij} \le m_{ij}$ for all (i, j). Furthermore we have

$$z_j \mathbf{Z}^{\mathbf{m}} = z_j z^\mu \bar{z}^\nu \mathbf{Z}^{\mathbf{c}} \mathbf{Z}^{\mathbf{d}} \text{ with } |\mu| > 0 \text{ and } |\nu| > 0. \tag{2.11}$$

So, for z_k a factor of z^μ and $\bar{z}_l$ a factor of $\bar{z}^\nu$, and for

$$a_{ij} = \begin{cases} c_{ij} + c_{ji} & \text{for } i < j \\ 0 & \text{for } i > j, \end{cases} \qquad b_{ij} = \begin{cases} d_{ij} + d_{ji} & \text{for } i < j \\ 0 & \text{for } i > j, \end{cases} \tag{2.12}$$

for $|z| \le 1$ we have from (2.11)

$$|z_j \mathbf{Z}^{\mathbf{m}}| \le |z_j| \, |z_k \mathbf{Z}^{\mathbf{c}}| \, |z_l \mathbf{Z}^{\mathbf{d}}| = |z_j| \, |z_k \mathbf{Z}^{\mathbf{a}}| \, |z_l \mathbf{Z}^{\mathbf{b}}|.$$

Furthermore, (2.8) is satisfied.

Since our $(\mathbf{a}, \mathbf{b})$ satisfy $\mathbf{a} \in \mathcal{M}_k$ and $\mathbf{b} \in \mathcal{M}_l$, claim (3) is a consequence of claim (2). □

Since (z, Θ) in (2.1) are not a system of independent coordinates we need the following, see Lemma 2.5 [21].

Lemma 2.8. *There exists $d_0 > 0$ such that for all $z \in \mathbb{C}$ with $|z| < d_0$ there exists $R[z] : \mathcal{H}_c[0] \to \mathcal{H}_c[z]$ such that $P_c|_{\mathcal{H}_c[z]} = R[z]^{-1}$, with P_c the orthogonal projection of L^2 onto $\mathcal{H}_c[0]$, see Definition 1.2. Furthermore, for $|z| < d_0$ and $\eta \in \mathcal{H}_c[0]$, we have the following properties.*

1. *$R[z] \in C^\infty(B_{\mathbb{C}^n}(0, \delta_0), B(H^r, H^r))$, for any $r \in \mathbb{R}$.*
2. *For any $r > 0$, we have $\|(R[z] - 1)\eta\|_{\Sigma_r} \le c_r |z|^2 \|\eta\|_{\Sigma_{-r}}$ for a fixed c_r.*
3. *We have the covariance property $R[e^{i\vartheta} z] = e^{i\vartheta} R[z] e^{-i\vartheta}$.*
4. *We have, summing on repeated indexes,*

$$R[z]\eta = \eta + (\alpha_j[z]\eta)\phi_j, \text{ with } \alpha_j[z]\eta = \langle B_j(z), \eta \rangle + \langle C_j(z), \eta^* \rangle, \tag{2.13}$$

   where, for $\widehat{Z}$ as in Definition 2.3, we have $B_j(z) = \widehat{B}_j(\widehat{Z})$ and $C_j(z) = z_i z_\ell \widehat{C}_{i\ell j}(\widehat{Z})$, for $\widehat{B}_j$ and $\widehat{C}_{i\ell j}$ smooth in $\widehat{Z}$ with values in Σ_r.

5. *We have, for $r \in \mathbb{R}$, with $\mathbf{Z}$ as in Definition 2.3*

$$\|B_j(z) + \partial_{\bar{z}_j} q^*_{j z_j}\|_{\Sigma_r} + \|C_j(z) - \partial_{\bar{z}_j} q_{j z_j}\|_{\Sigma_r} \le c_r |\mathbf{Z}|^2. \qquad \square \tag{2.14}$$

Then Lemma 2.6 gives us a system of coordinates near the origin in H^4. The simple proof is the same of Lemma 2.6 [21].

Lemma 2.9. *For the $d_0 > 0$ of Lemma 2.8 the map $(z, \eta) \to u$ defined by*

$$u = \sum_{j=1}^{n} Q_{j z_j} + R[z]\eta, \text{ for } (z, \eta) \in B_{\mathbb{C}^n}(0, d_0) \times (H^4 \cap \mathcal{H}_c[0]), \tag{2.15}$$

is with values in H^4 and is C^∞. Furthermore, there is a $d_1 > 0$ such that for $(z, \eta) \in B_{\mathbb{C}^n}(0, d_1) \times (B_{H^4}(0, d_1) \cap \mathcal{H}_c[0])$, the above map is a diffeomorphism and

$$|z| + \|\eta\|_{H^4} \sim \|u\|_{H^4}. \tag{2.16}$$

Finally, we have the gauge properties $u(e^{i\vartheta} z, e^{i\vartheta} \eta) = e^{i\vartheta} u(z, \eta)$ and

$$z(e^{i\vartheta} u) = e^{i\vartheta} z(u) \text{ and } \eta(e^{i\vartheta} u) = e^{i\vartheta} \eta(u). \qquad \square \tag{2.17}$$

We end this section exploiting the notation introduced in claim (5) of Lemma 2.8 to introduce two classes of functions. First of all notice that the linear maps $\eta \to \langle \eta, \phi_j^* \rangle$ extend into bounded linear maps $\Sigma_r \to \mathbb{R}$ for any $r \in \mathbb{R}$. We set

$$\Sigma_r^c := \left\{ \eta \in \Sigma_r : \ \langle \eta, \phi_j^* \rangle = 0, \ j = 1, \cdots, n \right\}. \tag{2.18}$$

The following two classes of functions will be used in the rest of the paper. Recall that in Definition 2.3 we introduced $\mathbf{Z} \in L$ with $\dim L = n(n-1)$.

Definition 2.10. We will say that $F(t,z,Z,\eta) \in C^M(I \times \mathcal{A}, \mathbb{R})$, with I a neighborhood of 0 in $\mathbb{R}$ and $\mathcal{A}$ a neighborhood of 0 in $\mathbb{C}^n \times L \times \Sigma^c_{-K}$ is $F = \mathcal{R}^{i,j}_{K,M}(t,z,\mathbf{Z},\eta)$, if there exist a $C > 0$ and a smaller neighborhood $\mathcal{A}'$ of 0 such that

$$|F(t,z,\mathbf{Z},\eta)| \le C(\|\eta\|_{\Sigma_{-K}} + |\mathbf{Z}|)^j(\|\eta\|_{\Sigma_{-K}} + |\mathbf{Z}| + |z|)^i \text{ in } I \times \mathcal{A}'. \tag{2.19}$$

We will specify $F = \mathcal{R}^{i,j}_{K,M}(t,z,\mathbf{Z})$ if

$$|F(t,z,\mathbf{Z},\eta)| \le C|\mathbf{Z}|^j|z|^i \tag{2.20}$$

and $F = \mathcal{R}^{i,j}_{K,M}(t,z,\eta)$ if

$$|F(t,z,\mathbf{Z},\eta)| \le C\|\eta\|^j_{\Sigma_{-K}}(\|\eta\|_{\Sigma_{-K}} + |z|)^i. \tag{2.21}$$

We will omit t if there is no dependence on such variable. We write $F = \mathcal{R}^{i,j}_{K,\infty}$ if $F = \mathcal{R}^{i,j}_{K,m}$ for all $m \ge M$. We write $F = \mathcal{R}^{i,j}_{\infty,M}$ if for all $k \ge K$ the above F is the restriction of an $F(t,z,\eta) \in C^M(I \times \mathcal{A}_k, \mathbb{R})$ with $\mathcal{A}_k$ a neighborhood of 0 in $\mathbb{C}^n \times L \times \Sigma^c_{-k}$ and which is $F = \mathcal{R}^{i,j}_{k,M}$. Finally we write $F = \mathcal{R}^{i,j}_{\infty,\infty}$ if $F = \mathcal{R}^{i,j}_{k,\infty}$ for all k.

Definition 2.11. We will say that an $T(t,z,\eta) \in C^M(I \times \mathcal{A}, \Sigma_K(\mathbb{R}^3,\mathbb{C}))$, with the above notation, is $T = \mathbf{S}^{i,j}_{K,M}(t,z,\mathbf{Z},\eta)$, if there exists a $C > 0$ and a smaller neighborhood $\mathcal{A}'$ of 0 such that

$$\|T(t,z,\mathbf{Z},\eta)\|_{\Sigma_K} \le C(\|\eta\|_{\Sigma_{-K}} + |\mathbf{Z}|)^j(\|\eta\|_{\Sigma_{-K}} + |\mathbf{Z}| + |z|)^i \text{ in } I \times \mathcal{A}'. \tag{2.22}$$

We use notations $\mathbf{S}^{i,j}_{K,M}(t,z,\mathbf{Z})$, $\mathbf{S}^{i,j}_{K,M}(t,z,\eta)$ etc. as above.

Remark 2.12. For given functions $F(t,z,\eta)$ and $T(t,z,\eta)$ we write $F(t,z,\eta) = \mathcal{R}^{i,j}_{K,M}(t,z,\mathbf{Z},\eta)$ and $T(t,z,\eta) = \mathbf{S}^{i,j}_{K,M}(t,z,\mathbf{Z},\eta)$ when they are restrictions to the set of vectors $\mathbf{Z} \in \{(z_i\bar{z}_j)_{i,j=1,\dots,n} : i \ne j\}$ of functions satisfying the two above definitions.

Furthermore later, when we write $\mathcal{R}^{i,j}_{K,M}$ and $\mathbf{S}^{i,j}_{K,M}$, we mean $\mathcal{R}^{i,j}_{K,M}(z,\mathbf{Z},\eta)$ and $\mathbf{S}^{i,j}_{K,M}(z,\mathbf{Z},\eta)$.

Notice that $F = \mathcal{R}^{i,j}_{K,M}(z,\mathbf{Z})$ or $S = \mathbf{S}^{i,j}_{K,M}(z,\mathbf{Z})$ do not mean independence by the variable η.

3. Invariants

Equation (1.1) admits the energy and mass invariants, defined as follows for $G(0) = 0$ and $G'(s) = g(s)$:

$$\begin{aligned} &E(u) := E_K(u) + E_P(u), \text{ where } E_K(u) := \langle D_{\mathscr{M}}u, u^*\rangle \text{ and} \\ &E_P(u) = \frac{1}{2}\int_{\mathbb{R}^3} G(u\bar{u})dx; \quad Q(u) := \langle u, u^*\rangle. \end{aligned} \tag{3.1}$$

We have $E \in C^\infty(H^4(\mathbb{R}^3,\mathbb{C}),\mathbb{R})$ and $Q \in C^\infty(L^2(\mathbb{R}^3,\mathbb{C}),\mathbb{R})$. We denote by dE the Frechét derivative of E. We define $\nabla E \in C^\infty(H^4(\mathbb{R}^3,\mathbb{C}), H^4(\mathbb{R}^3,\mathbb{C}))$ by $dE(X) = \mathrm{Re}\langle\nabla E, X^*\rangle$, for any $X \in H^4$. We define also $\nabla_u E$ and $\nabla_{u^*}E$ by

$$dEX = \langle\nabla_u E, X\rangle + \langle\nabla_{u^*}E, X^*\rangle, \text{ that is } \nabla_u E = 2^{-1}(\nabla E)^* \text{ and } \nabla_{u^*}E = 2^{-1}\nabla E.$$

Notice that $\nabla E = 2Hu + 2g(u\bar{u})\beta u$. Then equation (1.1) can be interpreted as

$$\mathrm{i}\dot{u} = \nabla_{u^*} E(u). \tag{3.2}$$

We recall that normal forms arguments consist in making Taylor expansions of the hamiltonian and in the cancellation of the *non-resonant* terms of the expansion. The following proposition identifies the kind of expansion we have in mind. We should think of (3.3) as an expansion in the variables z, η and the auxiliary variable $\mathbf{Z}$. Eventually the effective hamiltonian will contain terms in the r.h.s. such as the 1st and 2nd in the 1st line and the terms of the 2nd line. The cancellations will occur later in the 2nd line.

Proposition 3.1. *We have the following expansion of the energy for any preassigned $r_0 \in \mathbb{N}$:*

$$\begin{aligned} E(u) = &\sum_{j=1}^{n} E(Q_{jz_j}) + \langle H\eta, \eta^*\rangle + \mathcal{R}^{1,2}_{r_0,\infty}(z,\eta) + \mathcal{R}^{0,2N+5}_{r_0,\infty}(z,\mathbf{Z}) \\ &+ \sum_{j=1}^{n}\sum_{l=0}^{2N+3}\sum_{|\mathbf{m}|=l+1} \mathbf{Z}^{\mathbf{m}} a_{j\mathbf{m}}(|z_j|^2) + \sum_{j,k=1}^{n}\sum_{l=0}^{2N+3}\sum_{|\mathbf{m}|=l} (\bar{z}_j \mathbf{Z}^{\mathbf{m}} \langle G_{jk\mathbf{m}}(|z_k|^2), \eta\rangle + c.c.) \\ &+ \mathrm{Re}\langle \mathbf{S}^{0,2N+4}_{r_0,\infty}(z,\mathbf{Z}), \eta^*\rangle + \sum_{i+j=2}\sum_{|\mathbf{m}|\le 1} \mathbf{Z}^{\mathbf{m}} \langle G_{2\mathbf{m}ij}(z), \eta^{\otimes i} \otimes (\eta^*)^{\otimes j}\rangle \\ &+ \sum_{d=2}^{3}\sum_{i=1}^{d} \mathcal{R}^{0,3-d}_{r_0,\infty}(z,\eta) \int_{\mathbb{R}^3} G_{di}(x,z,\eta,\eta(x)) \eta^{\otimes i}(x) \otimes (\eta^*(x))^{\otimes (d-i)} dx + E_P(\eta), \end{aligned} \tag{3.3}$$

where c.c. is the complex conjugate of the term right before in the ( ) and where:

1. $(a_{j\mathbf{m}}, G_{jk\mathbf{m}}, G_{2\mathbf{m}ij}) \in C^\infty(B_{\mathbb{R}}(0,d_0), \mathbb{C} \times \Sigma_{r_0}(\mathbb{R}^3,\mathbb{C}) \times \Sigma_{r_0}(\mathbb{R}^3, B^{i+j}(\mathbb{C}^4,\mathbb{C})))$;
2. $G_{di}(\cdot, z, \eta, \zeta) \in C^\infty(B_{\mathbb{C}^n}(0,d_0) \times \Sigma_{-r_0}(\mathbb{R}^3,\mathbb{C}) \times \mathbb{C}^4, \Sigma_{r_0}(\mathbb{R}^3, B^d(\mathbb{C}^4,\mathbb{C}))))$;
3. *for $|\mathbf{m}| = 0$ we have $G_{\mathbf{20}ij}(0) = 0$ and*

$$\begin{aligned} \sum_{i+j=2} \langle G_{\mathbf{20}ij}(z), \eta^{\otimes i}(\eta^*)^{\otimes j}\rangle = &\, 2^{-1}\sum_{j=1}^{n} \langle g(Q_{jz_j}\overline{Q}_{jz_j})\eta, \eta^*\rangle \\ &+ \sum_{j=1}^{n} \mathrm{Re}\langle g'(Q_{jz_j}\overline{Q}_{jz_j})\,\mathrm{Re}(Q_{jz_j}\bar{\eta})\beta Q_{jz_j}, \eta^*\rangle. \end{aligned} \tag{3.4}$$

In order to prove Proposition 3.1 we set

$$K(z,\eta) := E(\sum_{j=1}^{n} Q_{jz_j} + R[z]\eta) = K_K(z,\eta) + K_P(z,\eta), \text{ with}$$

$$K_K(z,\eta) := E_K(\sum_{j=1}^{n} Q_{jz_j} + R[z]\eta), \quad K_P(z,\eta) := E_P(\sum_{j=1}^{n} Q_{jz_j} + R[z]\eta).$$

By Taylor expansion, we write

$$K(z,\eta) = K(z,0) + \mathrm{Re}\left\langle \partial_\eta K(z,0), \eta^*\right\rangle + \frac{1}{2}\mathrm{Re}\left\langle \partial^2_\eta K(z,0)\eta, \eta^*\right\rangle + K_3(z,\eta), \tag{3.5}$$

$$K_3(z,\eta) := \frac{1}{2}\int_0^1 (1-t)^2\,\mathrm{Re}\left\langle \partial^3_\eta K(z,t\eta)\eta^2, \eta^*\right\rangle dt. \tag{3.6}$$

We expand

$$K_3(z,\eta) = K_3(0,\eta) + R_P(z,\eta), \text{ where } K_3(0,\eta) = K_P(0,\eta) = E_P(\eta) \tag{3.7}$$

and

$$\begin{aligned} R_P(z,\eta) &= \int_0^1 \partial_z K_3(tz,\eta) z \, dt := \int_0^1 \sum_{j=1}^n \sum_{A=R,I} D_{jA} K_3(tz,\eta) z_{jA} \, dt \\ &= \sum_{j=1}^n \int_0^1 \left(\partial_j K_3(tz,\eta) z_j + \partial_{\bar{j}} K_3(tz,\eta) \bar{z}_j \right) \, dt. \end{aligned} \tag{3.8}$$

To prove Proposition 3.1 we compute the terms of

$$K(z,\eta) = K(z,0) + \mathrm{Re}\,\langle \partial_\eta K(z,0), \eta^* \rangle + \frac{1}{2}\,\mathrm{Re}\,\langle \partial_\eta^2 K(z,0)\eta, \eta^* \rangle + E_P(\eta) + R_P(z,\eta), \tag{3.9}$$

starting with $\partial_\eta K$, $\partial_\eta^2 K$ and $\partial_\eta^3 K$.

Lemma 3.2. *Set $u = u(z,\eta) = \sum_{j=1}^n Q_{jz_j} + R[z]\eta$. We have the following equalities:*

$$\begin{aligned} &\partial_\eta K_K(z,\eta) = 2R[z]^* H u, \\ &\partial_\eta^2 K_K(z,\eta) = 2R[z]^* H R[z], \quad \partial_\eta^3 K_K(z,\eta) = 0; \end{aligned} \tag{3.10}$$

$$\begin{aligned} &\partial_\eta K_P(z,\eta) = R[z]^* \left(g(u\bar{u})\beta u \right), \\ &\partial_\eta^2 K_P(z,\eta)\nu = R[z]^* g(u\bar{u})\beta R[z]\nu + 2R[z]^* g'(u\bar{u})\mathrm{Re}\left(u \; \overline{R[z]\nu} \right) \beta u, \\ &\left(\partial_\eta^3 K_P(z,\eta)\nu \right) \nu = 6R[z]^* g'(u\bar{u})\mathrm{Re}\left(u\overline{R[z]\nu} \right) \beta R[z]\nu + 4R[z]^* g''(u\bar{u}) \left(\mathrm{Re}\left(u\overline{R[z]\nu} \right) \right)^2 \beta u. \end{aligned} \tag{3.11}$$

In particular we have

$$\begin{aligned} &\partial_\eta K_K(z,0) = 2R[z]^* H \sum_{j=1}^n Q_{jz_j}, \quad \partial_\eta^2 K_K(z,0) = 2R[z]^* H R[z], \\ &\partial_\eta K_P(z,0) = g(\sum_{j,k} Q_{jz_j} \overline{Q}_{kz_k}) R[z]^* \beta \sum_{j=1}^n Q_{jz_j}, \\ &\partial_\eta^2 K_P(z,0) = g(\sum_{j,k} Q_{jz_j} \overline{Q}_{kz_k}) R[z]^* \beta R[z] \\ &\qquad + 2g'(\sum_{j,k} Q_{jz_j} \overline{Q}_{kz_k}) \mathrm{Re} \left(\sum_{j=1}^n Q_{jz_j} \overline{R[z] \,\cdot\,} \right) R[z]^* \beta \sum_{j=1}^n Q_{jz_j}. \end{aligned}$$

Proof. We get (3.10) by

$$\begin{aligned} K_K(z,\eta+\varepsilon\nu) &= \mathrm{Re}\,\langle H(u(z,\eta) + \varepsilon R[z]\nu), (u(z,\eta) + \varepsilon R[z]\nu)^* \rangle \\ &= K_K(z,\eta) + 2\varepsilon\,\mathrm{Re}\,\langle Hu(z,\eta), (R[z]\nu)^* \rangle + \varepsilon^2\,\mathrm{Re}\,\langle HR[z]\nu, (R[z]\nu)^* \rangle \,. \end{aligned}$$

Moreover we arrive at (3.11) by

$$K_P(z,\eta+\varepsilon\nu)=2^{-1}\int G((u(z,\eta)+\varepsilon R[z]\nu)(\overline{u(z,\eta)+\varepsilon R[z]\nu}))\,dx,$$

and by

$$\begin{aligned}
&G((u(z,\eta)+\varepsilon R[z]\nu)(\overline{u(z,\eta)+\varepsilon R[z]\nu}))\\
&\quad=G\left(u(z,\eta)\overline{u(z,\eta)}+2\varepsilon\mathrm{Re}\left(u(z,\eta)\overline{R[z]\nu}\right)+\varepsilon^2R[z]\nu\overline{R[z]\nu}\right)\\
&\quad=o(\varepsilon^3)+G(u(z,\eta)\overline{u(z,\eta)})+2\varepsilon g(u(z,\eta)\overline{u(z,\eta)})\mathrm{Re}\,(\beta u(z,\eta)(R[z]\nu)^*)\\
&\qquad+\varepsilon^2\left[g(u(z,\eta)\overline{u(z,\eta)})R[z]\nu\overline{R[z]\nu}+2g'(u(z,\eta)\overline{u(z,\eta)})\left(\mathrm{Re}\left(u(z,\eta)\overline{R[z]\nu}\right)\right)^2\right]\\
&\qquad+\varepsilon^3\left[2g'(u(z,\eta)\overline{u(z,\eta)})\mathrm{Re}\left(u(z,\eta)\overline{R[z]\nu}\right)R[z]\nu\overline{R[z]\nu}+\frac{4}{3}g''(u(z,\eta)\overline{u(z,\eta)})\,\mathrm{Re}\left(u(z,\eta)\overline{R[z]\nu}\right)^3\right].\qquad\square
\end{aligned}$$

We now examine the r.h.s. of (3.9).

Lemma 3.3. *Consider the first two terms in the r.h.s. of (3.9). We then have*

$$K(z,0)=\sum_{j=1}^n E(Q_{jz_j})+\sum_{j=1}^n\sum_{l=0}^{2N+3}\sum_{|\mathbf{m}|=l+1}\mathbf{Z}^{\mathbf{m}}a_{j\mathbf{m}}(|z_j|^2)+\mathcal{R}^{0,2N+5}_{\infty,\infty}(z,\mathbf{Z}),\tag{3.12}$$

$$\mathrm{Re}\,\langle\partial_\eta K(z,0),\eta^*\rangle=\sum_{j,k=1}^n\sum_{l=0}^{2N+3}\sum_{|\mathbf{m}|=l}(\bar z_j\mathbf{Z}^{\mathbf{m}}\langle G_{jk\mathbf{m}}(|z_k|^2),\eta\rangle+c.c.)+\mathrm{Re}\langle\mathbf{S}^{0,2N+4}_{\infty,\infty}(z,\mathbf{Z}),\eta^*\rangle,\tag{3.13}$$

where the coefficients in the r.h.s.'s have the properties listed in claim (1) in Proposition 3.1.

Proof. First of all, both l.h.s.'s of (3.12)–(3.13) are gauge invariant. Then (3.13) is an immediate consequence of claim (3) of Lemma 3.4 below.

We have

$$\begin{aligned}
K(z,0)&=E(\sum_{j=1}^n Q_{jz_j})=E(Q_{1z_1})+E(\sum_{j>1}Q_{jz_j})+\int_{[0,1]^2}\frac{\partial^2}{\partial s\partial t}E(sQ_{1z_1}+t\sum_{j>1}Q_{jz_j})dtds\\
&=\sum_{k=1}^n E(Q_{kz_k})+\alpha_k(z)\text{ with }\alpha_k(z):=\sum_{k=1}^n\int_{[0,1]^2}\frac{\partial^2}{\partial s\partial t}E(sQ_{kz_k}+t\sum_{j>k}Q_{jz_j})dtds.
\end{aligned}$$

The $\alpha_k(z)$ are gauge invariant, so that we can apply to them claim (2) of Lemma 3.4 below. Furthermore, since $\alpha_k(z)=O(|\mathbf{Z}|)$ we conclude that in the expansion (3.14) for $\alpha_k(z)$ we have equalities $b_{j\mathbf{0}}(|z_j|^2)=0$. $\square$

Lemma 3.4. *The following facts hold:*

(1) *For $a(\zeta)$ smooth from $B_{\mathbb{C}}(0,\delta)$ to $\mathbb{R}$ such that $a(e^{i\theta}\zeta)=a(\zeta)$ for any $\theta\in\mathbb{R}$ there exists $\alpha\in\mathbb{C}^\infty([0,\delta^2);\mathbb{R})$ such that $\alpha(|\zeta|^2)=a(\zeta)$.*

(2) *Let $a\in C^\infty(B_{\mathbb{C}^n}(0,\delta),\mathbb{R})$ satisfy $a(e^{i\theta}z_1,\cdots,e^{i\theta}z_n)=a(z_1,\cdots,z_n)$ for all $\theta\in\mathbb{R}$ and $a(0)=0$. Then for any $M>0$ there exist smooth $b_{j\mathbf{0}}$ such that $b_{j\mathbf{0}}(|z_j|^2)=a(0,\cdots,0,z_j,0,\cdots,0)$ and*

$$a(z_1,\cdots,z_n)=\sum_{|\mathbf{m}|\le M-1}\mathbf{Z}^{\mathbf{m}}b_{j\mathbf{m}}(|z_j|^2)+\mathcal{R}^{0,M}_{\infty,\infty}(z,\mathbf{Z}).\tag{3.14}$$

(3) *Let* $a \in C^\infty(B_{\mathbb{C}^n}(0,\delta),\Sigma_r)$ $\forall r \in \mathbb{R}$ *such that* $a(e^{i\theta}z_1,\cdots,e^{i\theta}z_n) = e^{i\theta}a(z_1,\cdots,z_n)$. *Then for any* $M > 0$ $\exists G_{j\mathbf{m}}$ *such that* $G_{j\mathbf{m}} \in C^\infty(B_{\mathbb{C}^n}(0,\delta),\Sigma_r)$ $\forall r$, $z_jG_{j\mathbf{0}}(|z_j|^2) = a(0,\cdots,0,z_j,0,\cdots,0)$ *and*

$$a(z_1,\cdots,z_n) = \sum_{j=1}^{n}\sum_{|\mathbf{m}|\le M-1} z_j\mathbf{Z}^{\mathbf{m}}G_{j\mathbf{m}}(|z_j|^2) + \mathcal{S}^{1,M}_{\infty,\infty}(z,\mathbf{Z}). \tag{3.15}$$

Proof. This elementary lemma is proved in [21]. $\square$

The 3rd term in the r.h.s. of (3.9) is dealt by the following lemma.

Lemma 3.5. *There exist* $G_{2\mathbf{m}i(2-i)}(z)$ *as in the statement of Proposition 3.1 such that (3.4) holds and*

$$\operatorname{Re}\left\langle \partial_\eta^2 K(z,0)\eta,\eta^*\right\rangle = \langle H\eta,\eta^*\rangle + \mathcal{R}^{1,2}_{\infty,\infty}(z,\eta) + \sum_{i=0}^{2}\sum_{|\mathbf{m}|\le 1}\mathbf{Z}^{\mathbf{m}}\langle G_{2\mathbf{m}i(2-i)}(z),\eta^{\otimes i}(\eta^*)^{\otimes(2-i)}\rangle. \tag{3.16}$$

Proof. By Lemma 3.2 and by claim (2) in Lemma 2.8 we have

$$\begin{aligned}
\frac{1}{2}\operatorname{Re}\left\langle \partial_\eta^2 K(z,0)\eta,\eta\right\rangle &= \langle HR[z]\eta,(R[z]\eta)^*\rangle + \langle g(\sum_{j,k} Q_{jz_j}\overline{Q}_{jz_j})R[z]\eta,(R[z]\eta)^*\rangle \\
&\quad + 2\operatorname{Re}\langle g'(\sum_{j,k} Q_{jz_j}\overline{Q}_{jz_j})\operatorname{Re}\left(\sum_{j=1}^{n} Q_{jz_j}\overline{R[z]\eta}\right)\sum_{j=1}^{n} Q_{jz_j},(R[z]\eta)^*\rangle \\
&= \langle H\eta,\eta^*\rangle + \mathcal{R}^{1,2}_{\infty,\infty}(z,\eta) \\
&\quad + \langle g(\sum_{j,k} Q_{jz_j}\overline{Q}_{kz_k})\eta,\eta^*\rangle + 2\operatorname{Re}\langle g'(\sum_{j,k} Q_{jz_j}\overline{Q}_{kz_k})\operatorname{Re}\left(\sum_{j=1}^{n} Q_{jz_j}\overline{\eta}\right)\sum_{j=1}^{n} Q_{jz_j},\eta^*\rangle.
\end{aligned}$$

The last line yields the last term of (3.16) and it is straightforward to see that it has the required properties. $\square$

To finish the discussion of the r.h.s. of (3.9) we need to compute R_P. We consider first a preparatory lemma.

Lemma 3.6. *Let* $l = 1,\cdots,n$ *and* $A = R, I$. *Then we have* $(D_{lA}R[z])\eta = \sum_{j=1}^{n}\mathcal{R}^{1,1}_{\infty,\infty}(z,\eta)\phi_j$.

Proof. One has $R[z]\eta = \eta + \sum_{j=1}^{n}(\alpha_j[z]\eta)\phi_j$. So $D_{lA}R[z]\eta = \sum_{j=1}^{n}\left((D_{lA}\alpha_j[z])\eta\right)\phi_j$. Now, $D_{lA}\alpha[z]\eta = \int D_{lA}B_j(z)\eta\,dx + \int D_{lA}C_j(z)\eta^*\,dx$, where B and C are given in Lemma 2.8 and we have $D_{lA}B_j(z) = \mathcal{S}^{1,0}_{\infty,\infty}$ and $D_{lA}C_j(z) = \mathcal{S}^{1,0}_{\infty,\infty}$. This yields the lemma. $\square$

We set $u := \sum_{j=1}^{n} Q_{jz_j} + R[z]\eta$. Then, we have $D_{lA}u = D_{lA}Q_{lz_l} + (D_{lA}R[z])\eta$. For $l = 1,\cdots,n$ and $A = R, I$ we have

$$\begin{aligned}
&D_{lA}\operatorname{Re}\left\langle \left(\partial_\eta^3 K_P(z,\eta)\eta\right)\eta,\eta^*\right\rangle = 6\mathscr{A} + 4\mathscr{B} \\
&\mathscr{A} := D_{lA}\operatorname{Re}\langle g'(u\bar{u})\operatorname{Re}\left(\overline{R[z]\eta}\right)\beta R[z]\eta,(R[z]\eta)^*\rangle, \\
&\mathscr{B} := D_{lA}\operatorname{Re}\langle g''(u\bar{u})\left(\operatorname{Re}\left(u\overline{R[z]\eta}\right)\right)^2\beta u,(R[z]\eta)^*\rangle.
\end{aligned}$$

We have

$$
\begin{aligned}
\mathscr{A} &= 2\langle g''(u\bar{u})\mathrm{Re}\left(\overline{D_{lA}u}\right)\mathrm{Re}\left(u\overline{R[z]\eta}\right)\beta R[z]\eta,(R[z]\eta)^*\rangle \\
&\quad + \langle g'(u\bar{u})\mathrm{Re}\left(D_{lA}u\overline{R[z]\eta}\right)\beta R[z]\eta,(R[z]\eta)^*\rangle + \langle g'(u\bar{u})\mathrm{Re}\left(u\overline{(D_{lA}R[z])\eta}\right)\beta R[z]\eta,(R[z]\eta)^*\rangle \\
&\quad + 2\langle g'(u\bar{u})\mathrm{Re}\left(u\overline{R[z]\eta}\right)\beta(D_{lA}R[z])\eta,(R[z]\eta)^*\rangle, \\
\mathscr{B} &= 2\langle g'''(u\bar{u})\mathrm{Re}(u\overline{D_{lA}u})\left(\mathrm{Re}\left(u\overline{R[z]\eta}\right)\right)^2\beta u,(R[z]\eta)^*\rangle \\
&\quad + 2\langle g''(u\bar{u})\mathrm{Re}\left(u\overline{R[z]\eta}\right)\mathrm{Re}\left(D_{lA}u\overline{R[z]\eta}\right)\beta u,(R[z]\eta)^*\rangle \\
&\quad + 2\langle g''(u\bar{u})\mathrm{Re}\left(u\overline{R[z]\eta}\right)\mathrm{Re}\left(u\overline{(D_{lA}R[z])\eta}\right)u,(R[z]\eta)^*\rangle \\
&\quad + \langle g''(u\bar{u})\left(\mathrm{Re}\left(u\overline{R[z]\eta}\right)\right)^2\beta D_{lA}u,(R[z]\eta)^*\rangle + \langle g''(u\bar{u})\left(\mathrm{Re}\left(u\overline{R[z]\eta}\right)\right)^2\beta u,((D_{lA}R[z])\eta)^*\rangle.
\end{aligned}
$$

All the terms in the formulas for $\mathscr{A}$ and $\mathscr{B}$ can be expressed as

$$
\sum_{d=2,3}\sum_{i+j=d}\langle G_{dij}(z,\eta),\eta^{\otimes i}\otimes(\eta^*)^{\otimes j}\rangle\mathcal{R}^{2,3-d}_{\infty,\infty}(z,\eta)+\mathcal{R}^{2,2}_{\infty,\infty}(z,\eta). \tag{3.17}
$$

Therefore R_P admits an expansion of the form (3.17), which is absorbed in terms of the r.h.s. of (3.3).

Proof of Proposition 3.1. We have just seen that R_P is absorbed in the r.h.s. of (3.3). The other terms of the r.h.s. of (3.9) are treated by Lemmas 3.3 and 3.5. □

4. Effective Hamiltonian

In this section we apply the theory of Sect. 4 and 5 in [21] which yields an effective Hamiltonian in an appropriate coordinate system, which in turn will be used to prove Theorem 5.1 which yields Theorem 1.3. Finding an effective Hamiltonian entails canceling as many terms as possible from the 2nd line of (3.3) through appropriate changes of variables. This process is called Birkhoff normal form argument and is done by means of a recursive procedure where each time we need to cancel a term from the hamiltonian we find an appropriate coordinate change by first solving an equation, the *homological* equation. It is easier to implement this procedure using coordinates which are Darboux. In a finite dimensional setting this would mean that the symplectic form is equal to a simple model, like $\omega_0 := \sum_j \mathrm{i}dz_j\wedge d\bar{z}_j$. This corresponds to diagonalizing the homological equations. Furthermore, it is important that the new coordinates remain Darboux. This means that the change of coordinates should leave ω_0 invariant. One way to do this is to make changes of coordinates using flows of hamiltonian vector fields. See for example Sect. 1.8 [32] for a general introduction to the subject.

The system (3.2) is Hamiltonian with respect to the symplectic form

$$
\Omega(X,Y):=-2\,\mathrm{Im}\langle X,Y^*\rangle. \tag{4.1}
$$

The first thing to notice is that the coordinates in Lemma 2.9, initially the most natural coordinates in our problem, do not form a system of Darboux coordinates for (4.1) in any reasonable sense. Indeed Ω is rather complicated in this coordinate system.

We consider as a local model the symplectic form

$$
\begin{aligned}
&\Omega_0 := \mathrm{i}\sum_{j=1}^n(1+\gamma_j(|z_j|^2))dz_j\wedge d\bar{z}_j+\mathrm{i}\,\langle d\eta,d\eta^*\rangle-\mathrm{i}\,\langle d\eta^*,d\eta\rangle\,, \\
&\quad\text{where }\gamma_j(|z_j|^2):=-\left\langle\widehat{q}_j(|z_j|^2),\widehat{q}^{\,*}_j(|z_j|^2)\right\rangle+2|z_j|^2\,\mathrm{Re}\left\langle\widehat{q}^{\,*}_j(|z_j|^2),\widehat{q}^{\,\prime}_j(|z_j|^2)\right\rangle,
\end{aligned} \tag{4.2}
$$

with $\widehat{q}^{\,\prime}_j(t)=\frac{d}{dt}\widehat{q}_j(t)$. By Proposition 1.1 and Definition 2.10 we have $\gamma_j(|z_j|^2)=\mathcal{R}^{2,0}_{\infty,\infty}(|z_j|^2)$.

Remark 4.1. Ω_0 is the same local model symplectic form of [21]. We do not know if Proposition 4.2 below holds when choosing $\gamma_j \equiv 0$ because we do not know if in (4.6) below we would still have $S_j = \mathcal{R}^{1,1}_{r,\infty}$ and $S_\eta = \mathbf{S}^{1,1}_{r,\infty}$, which is crucial. Notice, incidentally, that the Darboux Theorem is an abstract result. But in [21] it is proved with an *ad hoc* argument exactly because we want this change of coordinates, as well as all the other coordinates changes in the paper, to have this crucial property. The fact that all coordinate changes have this property guarantees that the limits (1.10) in one coordinate system imply the same limit in any other coordinate system.

While Ω_0 would be simpler if $\gamma_j \equiv 0$, nonetheless it is simple enough for a normal form argument.

In Sect. 4 [21] the following proposition is proved.

Proposition 4.2 *(Darboux Theorem). Fix any $r \in \mathbb{N}$. There exists a $\delta_0 \in (0, d_0)$ such that the following facts hold.*

(1) *There exists a gauge invariant 1-form $\Gamma = \Gamma_{jA} dz_{jA} + \langle \Gamma_\eta, d\eta \rangle + \langle \Gamma_{\eta^*}, d\eta^* \rangle$, with*

$$\Gamma_{jA} = \mathcal{R}^{1,1}_{\infty,\infty}(z, \mathbf{Z}, \eta) \text{ and } \Gamma_\xi = \mathbf{S}^{1,1}_{\infty,\infty}(z, \mathbf{Z}, \eta) \text{ for } \xi = \eta, \eta^*, \tag{4.3}$$

such that $d\Gamma = \Omega - \Omega_0$.

(2) *For any $(t, z, \eta) \in (-4, 4) \times B_{\mathbb{C}^n}(0, \delta_0) \times B_{\Sigma^c_{-r}}(0, \delta_0)$ there exists exactly one solution $\mathcal{X}^t(z, \eta) \in L^2$ of the equation $i_{\mathcal{X}^t} \Omega_t = -\Gamma$. Furthermore, $\mathcal{X}^t(z, \eta)$ is gauge invariant, $\mathcal{X}^t(z, \eta) \in \Sigma_r$ and if we set $\mathcal{X}^t_{jA}(z, \eta) = dz_{jA} \mathcal{X}^t(z, \eta)$ and $\mathcal{X}^t_\eta(z, \eta) = d\eta \mathcal{X}^t(z, \eta)$, we have $\mathcal{X}^t_{jA}(z, \eta) = \mathcal{R}^{1,1}_{r,\infty}(t, z, \mathbf{Z}, \eta)$ and $\mathcal{X}^t_\eta(z, \eta) = \mathbf{S}^{1,1}_{r,\infty}(t, z, \mathbf{Z}, \eta)$.*

(3) *Consider the following system in $(t, z, \eta) \in (-4, 4) \times B_{\mathbb{C}^n}(0, \delta_0) \times B_{\Sigma^c_k}(0, \delta_0)$ for all $k \in \mathbb{Z} \cap [-r, r]$:*

$$\dot{z}_j = \mathcal{X}^t_j(z, \eta) \text{ and } \dot{\eta} = \mathcal{X}^t_\eta(z, \eta). \tag{4.4}$$

Then the following facts hold for the corresponding flow $\mathfrak{F}^t$.

(3.1) *For $\delta_1 \in (0, \delta_0)$ sufficiently small we have*

$$\begin{aligned} &\mathfrak{F}^t \in C^\infty((-2, 2) \times B_{\mathbb{C}^n}(0, \delta_1) \times B_{\Sigma^c_k}(0, \delta_1), B_{\mathbb{C}^n}(0, \delta_0) \times B_{\Sigma^c_k}(0, \delta_0)) \text{ for all } k \in \mathbb{Z} \cap [-r, r] \\ &\mathfrak{F}^t \in C^\infty((-2, 2) \times B_{\mathbb{C}^n}(0, \delta_1) \times B_{H^1 \cap \mathcal{H}_c[0]}(0, \delta_1), B_{\mathbb{C}^n}(0, \delta_0) \times B_{H^1 \cap \mathcal{H}_c[0]}(0, \delta_0)). \end{aligned} \tag{4.5}$$

In particular we have

$$z^t_j = z_j + S_j(t, z, \eta) \text{ and } \eta^t = \eta + S_\eta(t, z, \eta), \tag{4.6}$$

with $S_j(t, z, \eta) = \mathcal{R}^{1,1}_{r,\infty}(t, z, \mathbf{Z}, \eta)$ and $S_\eta(t, z, \eta) = \mathbf{S}^{1,1}_{r,\infty}(t, z, \mathbf{Z}, \eta)$.

(3.2) *The map $\mathfrak{F} = \mathfrak{F}^1$ is a local diffeomorphism of H^1 into itself near the origin, and we have $\mathfrak{F}^* \Omega = \Omega_0$.*

(3.3) *We have $S_j(t, e^{i\vartheta} z, e^{i\vartheta} \eta) = e^{i\vartheta} S_j(t, z, \eta)$, $S_\eta(t, e^{i\vartheta} z, e^{i\vartheta} \eta) = e^{i\vartheta} S_\eta(t, z, \eta)$.* $\square$

We now consider the pullback $K := E \circ \mathfrak{F}$.

Lemma 4.3. *Consider the $\delta_1 > 0$ and $\delta_0 > 0$ of Proposition 4.2 and set $r_0 = r$ with r the index in Proposition 3.1. Then we have*

$$\mathfrak{F}(B_{\mathbb{C}^n}(0, \delta_1) \times (B_{H^1}(0, \delta_1) \cap \mathcal{H}_c[0])) \subset B_{\mathbb{C}^n}(0, \delta_0) \times (B_{H^1}(0, \delta_0) \cap \mathcal{H}_c[0]) \tag{4.7}$$

and $\mathfrak{F}|_{B_{\mathbb{C}^n}(0,\delta_1) \times (B_{H^1}(0,\delta_1) \cap \mathcal{H}_c[0])}$ is a diffeomorphism between domain and an open neighborhood of the origin in $\mathbb{C}^n \times (H^1 \cap \mathcal{H}_c[0])$ and furthermore the functional K admits an expansion for $r_1 = r_0 - 2$

$$
\begin{aligned}
K(z,\eta) &= H_2(z,\eta) + \sum_{j=1,\dots,n} \lambda_j(|z_j|^2) \\
&+ \sum_{l=1}^{2N+4} \sum_{|\mathbf{m}|=l+1} \mathbf{Z}^{\mathbf{m}} a^{(1)}_{\mathbf{m}}(|z_1|^2,\dots,|z_n|^2) + \sum_{j=1}^{n} \sum_{l=1}^{2N+3} \sum_{|\mathbf{m}|=l} (\bar z_j \mathbf{Z}^{\mathbf{m}} \langle G^{(1)}_{j\mathbf{m}}(|z_j|^2),\eta\rangle + c.c.) \\
&+ \mathcal{R}^{1,2}_{r_1,\infty}(z,\eta) + \mathcal{R}^{0,2N+5}_{r_1,\infty}(z,\mathbf{Z},\eta) + \mathrm{Re}\langle \mathbf{S}^{0,2N+4}_{r_1,\infty}(z,\mathbf{Z},\eta),\eta^*\rangle \\
&+ \sum_{d=2}^{3}\sum_{i=1}^{d} \mathcal{R}^{0,3-d}_{r_0,\infty}(z,\eta) \int_{\mathbb{R}^3} G^{(1)}_{di}(x,z,\eta,\eta(x))\eta^{\otimes i}(x)\otimes(\eta^*(x))^{\otimes(d-i)}dx \\
&+ \sum_{i+j=2}\sum_{|\mathbf{m}|\le 1} \mathbf{Z}^{\mathbf{m}}\langle G^{(1)}_{2\mathbf{m}ij}(z),\eta^{\otimes i}\otimes(\eta^*)^{\otimes j}\rangle + E_P(\eta),
\end{aligned} \tag{4.8}
$$

where

$$
H_2(z,\eta) = \sum_{j=1}^{n} e_j |z_j|^2 + \langle H\eta,\eta^*\rangle
$$

and where: $G^{(1)}_{j\mathbf{m}}$, $G^{(1)}_{2\mathbf{m}ij}$ *are* $\mathbf{S}^{0,0}_{r_1,\infty}$*;* $a^{(1)}_{\mathbf{m}}(|z_1|^2,\dots,|z_n|^2) = \mathcal{R}^{0,0}_{\infty,\infty}(z)$*; c.c. means complex conjugate;* $\lambda_j(|z_j|^2) = \mathcal{R}^{2,0}_{\infty,\infty}(|z_j|^2)$*;*

$$
G^{(1)}_{di}(\cdot,z,\eta,\zeta) \in C^\infty(B_{\mathbb{C}^n}(0,\delta_1)\times\Sigma_{-r_1}(\mathbb{R}^3,\mathbb{C})\times\mathbb{C}^4,\Sigma_{r_1}(\mathbb{R}^3,B^d(\mathbb{C}^4,\mathbb{C}))).
$$

For $|\mathbf{m}|=0$, $G^{(1)}_{2\mathbf{m}ij}(z,\eta) = G_{2\mathbf{m}ij}(z)$ *is the same of (3.4). Finally, we have the invariance* $\mathcal{R}^{1,2}_{r_1,\infty}(e^{\mathrm{i}\vartheta}z, e^{\mathrm{i}\vartheta}\eta) \equiv \mathcal{R}^{1,2}_{r_1,\infty}(z,\eta)$.

Proof. The proof of the above statement with possibly nonzero terms also corresponding to $l=0$ in both summations in the 2nd line of (4.8) is elementary, see Lemma 4.10 in [21] and Lemma 4.3 [20].

The key fact that in the 2nd line of (4.8) both summations start from $l=1$ and there are no $l=0$ is proved in the Cancellation Lemma 4.11 in [21]. □

Consider now the symplectic form Ω_0 in (4.2). We introduce an index $\ell = j,\bar j$, for $\bar{\bar j} = j$ with $j=1,\dots,n$. We write $\partial_j = \partial_{z_j}$ and $\partial_{\bar j} = \partial_{\bar z_j}$, $z_{\bar j} = \bar z_j$. Given $F\in C^1(U,\mathbb{C})$ with U an open subset of $\mathbb{C}^n\times\Sigma^c_r$, its Hamiltonian vector field X_F is defined by $i_{X_F}\Omega_0 = dF$. We have summing on j

$$
\begin{aligned}
i_{X_F}\Omega_0 &= \mathrm{i}(1+\gamma_j(|z_j|^2))((X_F)_j d\bar z_j - (X_F)_{\bar j} dz_j) + \mathrm{i}\,\langle (X_F)_\eta, d\bar\eta\rangle - \mathrm{i}\,\langle (X_F)_{\bar\eta}, d\eta\rangle \\
&= \partial_j F dz_j + \partial_{\bar j} F d\bar z_j + \langle \nabla_\eta F, d\eta\rangle + \langle \nabla_{\eta^*}F, d\eta^*\rangle\,,
\end{aligned} \tag{4.9}
$$

where (4.9) is used also to define $\nabla_\xi F$ for $\xi = \eta,\eta^*$.

Comparing the components of the two sides of (4.9) we get for $1+\varpi_j(|z_j|^2) = (1+\gamma_j(|z_j|^2))^{-1}$ where $\varpi_j(|z_j|^2) = \mathcal{R}^{2,0}_{\infty,\infty}(|z_j|^2)$:

$$
\begin{aligned}
&(X_F)_j = -\mathrm{i}(1+\varpi_j(|z_j|^2))\partial_{\bar j}F\,, \quad (X_F)_{\bar j} = \mathrm{i}(1+\varpi_j(|z_j|^2))\partial_j F, \\
&(X_F)_\eta = -\mathrm{i}\nabla_{\eta^*}F\,, \quad (X_F)_{\eta^*} = \mathrm{i}\nabla_\eta F.
\end{aligned} \tag{4.10}
$$

Given $G\in C^1(U,\mathbb{C})$ and $F\in C^1(U,\mathbf{E})$, with $\mathbf{E}$ a Banach space, we set $\{F,G\} := dF X_G$.

Definition 4.4 *(Normal Forms)*. Recall Definition 2.4 and (2.3). Fix $r \in \mathbb{N}_0$. A real valued function $Z(z, \eta)$ is in normal form if $Z = Z_0 + Z_1$ with Z_0 and Z_1 finite sums of the following type, for $\mathbf{l} \geq 1$: for $G_{j\mathbf{m}}(|z_j|^2) = S^{0,0}_{r,\infty}(|z_j|^2)$, c.c. the complex conjugate and $a_{\mathbf{m}}(|z_1|^2, \ldots, |z_n|^2) = \mathcal{R}^{0,0}_{r,\infty}(|z_1|^2, \ldots, |z_n|^2)$,

$$Z_1(z, \mathbf{Z}, \eta) = \sum_{j=1}^{n} \sum_{\substack{|\mathbf{m}|=\mathbf{l} \\ \mathbf{m}\in\mathcal{M}_j(\mathbf{l})}} \left(\bar{z}_j \mathbf{Z}^{\mathbf{m}} \langle G_{j\mathbf{m}}(|z_j|^2), \eta\rangle + \text{c.c.}\right), \tag{4.11}$$

$$Z_0(z, \mathbf{Z}, \eta) = \sum_{\substack{|\mathbf{m}|=\mathbf{l}+1 \\ \mathbf{m}\in\mathcal{M}_0(\mathbf{l}+1)}} \mathbf{Z}^{\mathbf{m}} a_{\mathbf{m}}(|z_1|^2, \ldots, |z_n|^2). \tag{4.12}$$

Remark 4.5. By Hypothesis (H4), in particular by (1.6), for any $\mathbf{m} \in \mathcal{M}_0(2N + 4)$ we have $\mathbf{Z}^{\mathbf{m}} = |z_1|^{2m_1} \ldots |z_n|^{2m_n}$ for an $m \in \mathbb{N}_0^n$ with $2|m| = |\mathbf{m}|$. Similarly by (H4), in particular by (1.5), for $|\mathbf{m}| \leq 2N + 4$ we have $|\sum_{a,b}(e_a - e_b)m_{ab} - e_j| \neq M$.

For $\mathbf{l} \leq 2N+4$ we will consider flows associated to Hamiltonian vector fields X_χ with real valued functions χ of the following form, with $b_{\mathbf{m}} = \mathcal{R}^{0,0}_{\mathbf{r},\infty}(|z_1|^2, \ldots, |z_n|^2)$ and $B_{j\mathbf{m}} = S^{0,0}_{\mathbf{r},\infty}(|z_j|^2)$ for some $\mathbf{r} \in \mathbb{N}$ defined in $B_{\mathbb{C}^n}(0, \mathbf{d})$ for some $\mathbf{d} > 0$:

$$\chi = \sum_{\substack{|\mathbf{m}|=\mathbf{l}+1 \\ \mathbf{m}\notin\mathcal{M}_0(\mathbf{l}+1)}} \mathbf{Z}^{\mathbf{m}} b_{\mathbf{m}}(|z_1|^2, \ldots, |z_n|^2) + \sum_{j=1}^{n} \sum_{\substack{|\mathbf{m}|=\mathbf{l} \\ \mathbf{m}\notin\mathcal{M}_j(\mathbf{l})}} (\bar{z}_j \mathbf{Z}^{\mathbf{m}} \langle B_{j\mathbf{m}}(|z_j|^2), \eta\rangle + \text{c.c.}). \tag{4.13}$$

The following result is proved in [21].

Proposition 4.6 *(Birkhoff normal forms). For any $\iota \in \mathbb{N} \cap [2, 2N + 4]$ there are a $\delta_\iota > 0$, a polynomial χ_ι as in (4.13) with $\mathbf{l} = \iota$, $\mathbf{d} = \delta_\iota$ and $\mathbf{r} = r_\iota = r_0 - 2(\iota + 1)$ such that for all $k \in \mathbb{Z} \cap [-r(\iota), r(\iota)]$ we have for each χ_ι a flow (for $\delta_1 > 0$ the constant in Proposition 4.2)*

$$\begin{aligned} &\phi^t_\iota \in C^\infty((-2, 2) \times B_{\mathbb{C}^n}(0, \delta_\iota) \times B_{\Sigma^c_k}(0, \delta_\iota), B_{\mathbb{C}^n}(0, \delta_{\iota-1}) \times B_{\Sigma^c_k}(0, \delta_{\iota-1})), \\ &\phi^t_\iota \in C^\infty((-2, 2) \times B_{\mathbb{C}^n}(0, \delta_\iota) \times B_{H^1\cap\mathcal{H}_c[0]}(0, \delta_\iota), B_{\mathbb{C}^n}(0, \delta_{\iota-1}) \times B_{H^1\cap\mathcal{H}_c[0]}(0, \delta_{\iota-1})) \end{aligned} \tag{4.14}$$

and such that, if we set $\mathfrak{F}^{(\iota)} := \mathfrak{F} \circ \phi_2 \circ \cdots \circ \phi_\iota$, with $\mathfrak{F}$ the transformation in Proposition 4.2 and the $\phi_j = \phi^1_\iota$, then for $(z, \eta) \in B_{\mathbb{C}^n}(0, \delta_\iota) \times (B_{H^1}(0, \delta_\iota) \cap \mathcal{H}_c[0])$ we have the following expansion

$$\begin{aligned} H^{(\iota)}(z, \eta) &:= E \circ \mathfrak{F}^{(\iota)}(z, \eta) = H_2(z, \eta) + \sum_{j=1}^{n} \lambda_j(|z_j|^2) + Z^{(\iota)}(z, \mathbf{Z}, \eta) \\ &+ \sum_{l=\iota}^{2N+3} \sum_{|\mathbf{m}|=l+1} \mathbf{Z}^{\mathbf{m}} a^{(\iota)}_{\mathbf{m}}(|z_1|^2, \ldots, |z_n|^2) + \sum_{j=1}^{n} \sum_{l=\iota}^{2N+3} \sum_{|\mathbf{m}|=l} (\bar{z}_j \mathbf{Z}^{\mathbf{m}} \langle G^{(\iota)}_{j\mathbf{m}}(|z_j|^2), \eta\rangle + c.c.) \\ &+ \mathcal{R}^{1,2}_{r_\iota,\infty}(z, \eta) + \mathcal{R}^{0,2N+5}_{r_\iota,\infty}(z, \mathbf{Z}, \eta) + \mathrm{Re}\langle \mathbf{S}^{0,2N+4}_{r_\iota,\infty}(z, \mathbf{Z}, \eta), \eta^*\rangle \\ &+ \sum_{d=2}^{3} \sum_{i=1}^{d} \mathcal{R}^{0,3-d}_{r_0,\infty}(z, \eta) \int_{\mathbb{R}^3} G^{(\iota)}_{di}(x, z, \eta, \eta(x)) \eta^{\otimes i}(x) \otimes (\eta^*(x))^{\otimes(d-i)} dx \\ &+ \sum_{i+j=2} \sum_{|\mathbf{m}|\leq 1} \mathbf{Z}^{\mathbf{m}} \langle G^{(\iota)}_{2\mathbf{m}ij}(z), \eta^{\otimes i} \otimes (\eta^*)^{\otimes j}\rangle + E_P(\eta), \end{aligned} \tag{4.15}$$

where, for coefficients like in Definition 4.4 for $(r, m) = (r_\iota, \infty)$,

$$Z^{(\iota)} = \sum_{\mathbf{m}\in\mathcal{M}_0(\iota)} \mathbf{Z}^{\mathbf{m}} a_{\mathbf{m}}(|z_1|^2,\dots,|z_n|^2) + \sum_{j=1}^{n}\Big(\sum_{\mathbf{m}\in\mathcal{M}_j(\iota-1)} \bar{z}_j \mathbf{Z}^{\mathbf{m}} \langle G_{j\mathbf{m}}(|z_j|^2),\eta\rangle + c.c.\Big). \tag{4.16}$$

We have $\mathcal{R}^{1,2}_{r_\iota,\infty} = \mathcal{R}^{1,2}_{r_2,\infty}$ *and* $\mathcal{R}^{1,2}_{r_2,\infty}(e^{i\vartheta}z, e^{i\vartheta}\eta) \equiv \mathcal{R}^{1,2}_{r_2,\infty}(z,\eta)$.

In particular we have for $\delta_f := \delta_{2N+4}$ *and for the* δ_0 *in Proposition 4.2,*

$$\mathcal{F}^{(2N+4)}(B_{\mathbb{C}^n}(0,\delta_f)\times(B_{H^1}(0,\delta_f)\cap\mathcal{H}_c[0])) \subset B_{\mathbb{C}^n}(0,\delta_0)\times(B_{H^1}(0,\delta_0)\cap\mathcal{H}_c[0]) \tag{4.17}$$

with $\mathcal{F}|_{B_{\mathbb{C}^n}(0,\delta_f)\times(B_{H^1}(0,\delta_f)\cap\mathcal{H}_c[0])}$ *a diffeomorphism between its domain and an open neighborhood of the origin in* $\mathbb{C}^n\times(H^1\cap\mathcal{H}_c[0])$.

Furthermore, for $r = r_0 - 4N - 10$ *there is a pair* $\mathcal{R}^{1,1}_{r,\infty}$ *and* $\mathbf{S}^{1,1}_{r,\infty}$ *such that for* $(z',\eta') = \mathcal{F}^{(2N+4)}(z,\eta)$ *we have*

$$z' = z + \mathcal{R}^{1,1}_{r,\infty}(z,\mathbf{Z},\eta), \quad \eta' = \eta + \mathbf{S}^{1,1}_{r,\infty}(z,\mathbf{Z},\eta). \tag{4.18}$$

Furthermore, by taking all the $\delta_\iota > 0$ *sufficiently small, we can assume that all the symbols in the proof, i.e. the symbols in (4.18) and the symbols in the expansions (4.15), satisfy the estimates of Definitions 2.10 and 2.11 for* $|z| < \delta_\iota$ *and* $\|\eta\|_{\Sigma_{r(\iota)}} < \delta_\iota$ *for their respective* ι*'s.* $\square$

5. Dispersion

We apply Proposition 4.6, set $\mathcal{H} = H^{(2N+4)}$ so that for some $r\in\mathbb{N}$ which we can take arbitrarily large,

$$\mathcal{H}(z,\eta) = H_2(z,\eta) + \sum_{j=1}^{n}\lambda_j(|z_j|^2) + \mathcal{Z}(z,\mathbf{Z},\eta) + \mathcal{R}, \tag{5.1}$$

with $\mathcal{Z}(z,\mathbf{Z},\eta) = \mathcal{Z}^{(2N+4)}(z,\mathbf{Z},\eta)$ and

$$\begin{aligned}\mathcal{R} &= \mathcal{R}^{1,2}_{r_\iota,\infty}(z,\eta) + \mathcal{R}^{0,2N+5}_{r_\iota,\infty}(z,\mathbf{Z},\eta) + \mathrm{Re}\langle \mathbf{S}^{0,2N+4}_{r_\iota,\infty}(z,\mathbf{Z},\eta),\eta^*\rangle \\ &+ \sum_{d=2}^{3}\sum_{i=1}^{d}\mathcal{R}^{0,3-d}_{r_0,\infty}(z,\eta)\int_{\mathbb{R}^3} G^{(\iota)}_{di}(x,z,\eta,\eta(x))\eta^{\otimes i}(x)\otimes(\eta^*(x))^{\otimes(d-i)}dx \\ &+ \sum_{i+j=2}\sum_{|\mathbf{m}|\le 1}\mathbf{Z}^{\mathbf{m}}\langle G^{(\iota)}_{2\mathbf{m}ij}(z),\eta^{\otimes i}\otimes(\eta^*)^{\otimes j}\rangle + E_P(\eta).\end{aligned} \tag{5.2}$$

Our ambient space is $H^4(\mathbb{R}^3,\mathbb{C}^4)$. So under (H1) the functional $u\to g(u\bar{u})\beta u$ is locally Lipschitz and (1.1), (3.2) and the equivalent system with Hamiltonian $\mathcal{H}(z,\eta)$ and symplectic form Ω_0, are locally well posed, see pp. 293–294 in volume III [51].

By standard arguments, see [21], Theorem 5.1 below implies Theorem 1.3.

Theorem 5.1 *(Main Estimates). Consider the* ϵ *of Theorem 1.3. Then there exist* $\epsilon_0 > 0$ *and a* $C_0 > 0$ *such that if* $\epsilon < \epsilon_0$ *then for* $I = [0,\infty)$ *we have the following inequalities:*

$$\|\eta\|_{L^p_t(I,B^{4-\frac{2}{p}}_{q,2}(\mathbb{R}^3,\mathbb{C}^4))} \le C\epsilon, \text{ for all the pairs } (p,q) \text{ as of (1.12)}, \tag{5.3}$$

$$\|\eta\|_{L^2_t(I,H^{4,-10}(\mathbb{R}^3,\mathbb{C}^4))} \le C\epsilon, \tag{5.4}$$

$$\|\eta\|_{L^2_t(I,L^\infty(\mathbb{R}^3,\mathbb{C}^4))} \le C\epsilon, \tag{5.5}$$

$$\|z_j\mathbf{Z}^{\mathbf{m}}\|_{L^2_t(I)} \le C\epsilon, \text{ for all } (j,\mathbf{m}) \text{ with } \mathbf{m}\in\mathcal{M}_j(2N+4), \tag{5.6}$$

$$\|z_j\|_{W^{1,\infty}_t(I)} \le C\epsilon, \text{ for all } j\in\{1,\dots,n\}. \tag{5.7}$$

Furthermore, there exists $\rho_+ \in [0,\infty)^n$ such that there exists a j_0 with $\rho_{+j} = 0$ for $j \neq j_0$ and there exists $\eta_+ \in L^\infty$ such that $|\rho_+| \le C\epsilon$ and $\|\eta_+\|_{L^\infty} \le C\epsilon$ such that

$$\lim_{t\to+\infty} \|\eta(t,x) - e^{-\mathrm{i}tD_{\mathscr{M}}}\eta_+(x)\|_{L^\infty_x(\mathbb{R}^3,\mathbb{C}^4)} = 0, \tag{5.8}$$

$$\lim_{t\to+\infty} |z_j(t)| = \rho_{+j}. \tag{5.9}$$

By an elementary continuation argument (see [8,21] or [49], end of the proof of Theorem 2.1, Sect. II), the estimates (5.3)–(5.7) for $I = [0,\infty)$ are a consequence of the following proposition.

Proposition 5.2. *There exists a constant $c_0 > 0$ such that for any $C_0 > c_0$ there is a value $\epsilon_0 = \epsilon_0(C_0)$ such that if the inequalities (5.3)–(5.7) hold for $I = [0,T]$ for some $T > 0$, for $C = C_0$ and for $0 < \epsilon < \epsilon_0$, then in fact for $I = [0,T]$ the inequalities (5.3)–(5.7) hold for $C = C_0/2$.*

Proof. By Lemma 5.11, there exists a fixed $c_1 > 0$ such that given any C_0 if $\epsilon_0 > 0$ is small enough we have for all admissible pairs (p,q), for the M of Definition 2.4 and for a preassigned $\tau_0 > 1$,

$$\|\eta\|_{L^p_t([0,T],B^{4-\frac{2}{p}}_{q,2})\cap L^2_t([0,T],H^{4,-\tau_0}_x)\cap L^2_t([0,T],L^\infty_x)} \le c_1\epsilon + c_1 \sum_{(\mu,\nu)\in M} |z^\mu \bar{z}^\nu|_{L^2_t(0,T)}. \tag{5.10}$$

The aim of Sect. 5.1 is to prove that there exists a fixed $c_2 > 0$ such that if $\epsilon_0 > 0$ for any given any C_0 if $\epsilon_0 > 0$ is small enough we have

$$\sum_j \|z_j\|^2_{L^\infty_t(0,T)} + \sum_{(\mu,\nu)\in M} \|z^{\mu+\nu}\|^2_{L^2(0,T)} \le c_2\epsilon^2 + c_2 C_0 \epsilon^2. \tag{5.11}$$

This implies that we can replace C_0 with C_1 with $C_1 = \sqrt{c_2(1+C_0)} \le 2\sqrt{c_2 C_0}$. We have $C_1 \le C_0/2$ if $C_0 \ge c_0 := 16c_2$. The proof is now completed. □

5.1. Completion of the proof of Proposition 5.2

5.1.1. Bounds on the continuous modes

We start this section by listing some results known in the literature, then we prove some auxiliary tools required for the proof of Proposition 5.2. The following theorem is Theorem 1.1 [6].

Theorem 5.3. *Under hypotheses (H2)–(H3) for $s > 5/2$ and any $k \in \mathbb{R}$ we have*

$$\|e^{\mathrm{i}Ht}P_c u_0\|_{H^{k,-s}(\mathbb{R}^3,\mathbb{C}^4)} \le C_{s,k}\langle t\rangle^{-\frac{3}{2}} \|P_c u_0\|_{H^{k,s}(\mathbb{R}^3,\mathbb{C}^4)}. \qquad \square \tag{5.12}$$

The subsequent is Theorem 1.1 in [7].

Theorem 5.4 *(Smoothness estimates). For any $\tau > 1$ and $k \in \mathbb{R}$ $\exists C$ such that*

$$\|e^{-\mathrm{i}tH}P_c\psi\|_{L^2_t(\mathbb{R},H^{k,-\tau}(\mathbb{R}^3,\mathbb{C}^4))} \le C\|P_c\psi\|_{H^k(\mathbb{R}^3,\mathbb{C}^4)}, \tag{5.13}$$

$$\|\int_{\mathbb{R}} e^{\mathrm{i}tH}P_c F(t)\,dt\|_{H^k} \le C\|P_c F\|_{L^2_t(\mathbb{R},H^{k,\tau}(\mathbb{R}^3,\mathbb{C}^4))}, \tag{5.14}$$

$$\|\int_{t'<t} e^{-\mathrm{i}(t-t')H}P_c F(t')\,dt'\|_{L^2_t(\mathbb{R},H^{k,-\tau}(\mathbb{R}^3,\mathbb{C}^4))} \le C\|P_c F\|_{L^2_t(\mathbb{R},H^{k,\tau}(\mathbb{R}^3,\mathbb{C}^4))}. \tag{5.15}$$

The following is Theorem 3.1 in [6].

Theorem 5.5. *For $p \in [1,2]$, $\theta \in [0,1]$, with $k - k' \geq (2+\theta)(\frac{2}{p}-1)$ and $q \in [1,\infty]$, there is a constant C such that for $p' = \frac{p}{p-1}$,*

$$\|e^{\mathrm{i}tD_{\mathscr{M}}}\|_{B^k_{p,q} \to B^{k'}_{p',q}} \leq C(K(t))^{\frac{2}{p}-1}, \text{ where } K(t) := \begin{cases} |t|^{-1+\theta/2} & \text{if } |t| \leq 1 \\ |t|^{-1-\theta/2} & \text{if } |t| \geq 1. \end{cases} \qquad \square$$

The following is Theorem 1.2 in [7].

Theorem 5.6 *(Strichartz estimates). For any $2 \leq p, q \leq \infty$, $\theta \in [0,1]$, with $(1-\frac{2}{q})(1 \pm \frac{\theta}{2}) = \frac{2}{p}$ and $(p,\theta) \neq (2,0)$, and for any reals k, k' with $k' - k \geq \alpha(q)$, where $\alpha(q) = (1+\frac{\theta}{2})(1-\frac{2}{q})$, there exists a positive constant C such that*

$$\left\|e^{-\mathrm{i}tH} P_c \psi\right\|_{L^p_t(\mathbb{R}, B^k_{q,2}(\mathbb{R}^3,\mathbb{C}^4))} \leq C \left\|P_c\psi\right\|_{H^{k'}(\mathbb{R}^3,\mathbb{C}^4)}, \tag{5.16}$$

$$\left\|\int_{\mathbb{R}} e^{\mathrm{i}tH} P_c F(t)\, dt\right\|_{H^k(\mathbb{R}^3,\mathbb{C}^4)} \leq C \left\|P_c F\right\|_{L^{p'}_t(\mathbb{R}, B^{k'}_{q',2}(\mathbb{R}^3,\mathbb{C}^4))}, \tag{5.17}$$

$$\left\|\int_{t'<t} e^{-\mathrm{i}(t-t')H} P_c F(t')\, dt'\right\|_{L^p_t(\mathbb{R}, B^k_{q,2}(\mathbb{R}^3,\mathbb{C}^4))} \leq C \left\|P_c F\right\|_{L^{a'}_t(\mathbb{R}, B^h_{b',2}(\mathbb{R}^3,\mathbb{C}^4))}, \tag{5.18}$$

for any (a,b) chosen like (p,q) and $h - k \geq \alpha(q) + \alpha(b)$. $\square$

We have the following facts concerning the resolvent of the operator $D_{\mathscr{M}}$, see [6–8].

Lemma 5.7. *The following facts are true.*

(1) *For $z \notin \sigma(D_{\mathscr{M}})$ for the integral kernel we have $R_{D_{\mathscr{M}}}(x,y,z) = R_{D_{\mathscr{M}}}(x-y,z)$ with*

$$R_{D_{\mathscr{M}}}(x,z) = \begin{pmatrix} (z+\mathscr{M})I_2 & \mathrm{i}\sqrt{\mathscr{M}^2-z^2}\sigma\cdot\widehat{x} \\ \mathrm{i}\sqrt{\mathscr{M}^2-z^2}\sigma\cdot\widehat{x} & (z-\mathscr{M})I_2 \end{pmatrix} \frac{e^{-\sqrt{\mathscr{M}^2-z^2}|x|}}{4\pi|x|} + \mathrm{i}\frac{\alpha\cdot\widehat{x}}{4\pi|x|^2} e^{-\sqrt{\mathscr{M}^2-z^2}|x|}, \tag{5.19}$$

where $\widehat{x} = x/|x|$ and where for $\zeta = e^{\mathrm{i}\vartheta}r$ with $r \geq 0$ and $\vartheta \in (-\pi,\pi)$ we set $\sqrt{\zeta} = e^{\mathrm{i}\vartheta/2}\sqrt{r}$.

(2) *For any $\tau > 1$ there exists C such that $\|R_{D_{\mathscr{M}}}(z)\psi\|_{L^{2,-\tau}(\mathbb{R}^3,\mathbb{C}^4)} \leq C\|\psi\|_{L^{2,\tau}(\mathbb{R}^3,\mathbb{C}^4)}$, for all $z \notin \mathbb{R}$.*

(3) *For any $\tau > 1$ the following limits exist in $B(H^{1,\tau}(\mathbb{R}^3,\mathbb{C}^4), L^{2,-\tau}(\mathbb{R}^3,\mathbb{C}^4))$,*

$$R^{+}_{D_{\mathscr{M}}}(\lambda) = \lim_{\varepsilon \searrow 0} R_{D_{\mathscr{M}}}(\lambda \pm \mathrm{i}\varepsilon) \text{ for } \lambda \in \mathbb{R}\backslash(-\mathscr{M},\mathscr{M}) \tag{5.20}$$

and the convergence is uniform for λ in compact subsets of $\mathbb{R}\backslash(-\mathscr{M},\mathscr{M})$.

(4) *$R^{+}_{D_{\mathscr{M}}}(x,z)$ for $z > \mathscr{M}$ (resp. $z < -\mathscr{M}$) is obtained substituting $\sqrt{\mathscr{M}^2-z^2}$ in (5.19) with $-\mathrm{i}\sqrt{z^2-\mathscr{M}^2} = \lim_{\varepsilon\searrow 0}\sqrt{\mathscr{M}^2-(z+\mathrm{i}\varepsilon)^2}$ (resp. $\mathrm{i}\sqrt{z^2-\mathscr{M}^2} = \lim_{\varepsilon\searrow 0}\sqrt{\mathscr{M}^2-(z+\mathrm{i}\varepsilon)^2}$).*

(5) *We have*

$$R^{\pm}_{D_{\mathscr{M}}}(\lambda) = R^{\pm}_{-\Delta+\mathscr{M}^2}(\lambda^2)\mathcal{A}(\lambda,\nabla) \text{ with } \mathcal{A}(\lambda,\nabla) := \begin{pmatrix} \lambda+\mathscr{M} & -\mathrm{i}\sigma\cdot\nabla \\ -\mathrm{i}\sigma\cdot\nabla & \lambda-\mathscr{M} \end{pmatrix}. \tag{5.21}$$

By Lemma 5.7 above we are able to deal with the resolvent of the perturbed Dirac operator H.

Lemma 5.8. *For any preassigned $\tau > 1$ the following facts hold.*

(1) *The limits $R_H^{\pm}(\lambda) = R_H(\lambda \pm \mathrm{i}0) := \lim_{\varepsilon \searrow 0} R_H(\lambda \pm \mathrm{i}\varepsilon)$, for $\lambda \in (-\infty, -\mathscr{M}) \cup (\mathscr{M}, \infty)$, exist in $B(L^{2,\tau}, L^{2,-\tau})$ and the convergence is uniform in compact subsets of $(-\infty, -\mathscr{M}) \cup (\mathscr{M}, \infty)$.*

(2) *There exists a constant $C_1 = C_1(\tau)$ such that for any $u_0 \in L^2(\mathbb{R}^3, \mathbb{C}^4)$ and any $\varepsilon \geq 0$ we have*

$$\|\langle x\rangle^{-\tau} R_H(\lambda \pm \mathrm{i}\varepsilon) P_c u_0\|_{L^2_\lambda(\mathbb{R}, L^2_x(\mathbb{R}^3))} \leq C_1 \|P_c u_0\|_{L^2(\mathbb{R}^3)}. \tag{5.22}$$

(3) *Let Λ be a compact subset of $(-\infty, -\mathscr{M}) \cup (\mathscr{M}, \infty)$. There exists a constant $C_1 = C_1(\tau, \Lambda)$ such that*

$$\|\langle x\rangle^{-\tau} R_H^{\pm}(\lambda) P_c u_0\|_{L^\infty_\lambda(\Lambda, L^2_x(\mathbb{R}^3))} \leq C_1 \|P_c u_0\|_{L^2(\mathbb{R}^3)}. \tag{5.23}$$

Proof. Claim (1) is an immediate consequence of Theorem 5.3 in the case $\tau > 5/2$, as observed on p. 783 [6]. The extension to the case $\tau > 1$ follows by the proof of Proposition 3.10 [6]. (5.22) is equivalent to (5.13) for $k = 0$.

We prove now (5.23). Let $u_0 = P_c u_0$, $A(x) = \langle x\rangle^{-\tau}$ and $B(x) \in \mathcal{S}(\mathbb{R}^3, S_4(\mathbb{C}))$ such that $B^* A = V$. Then

$$AR_H(z)u_0 = (1 + AR_{D_\mathscr{M}}(z)B^*)^{-1} AR_{D_\mathscr{M}}(z)u_0 \text{ for } z \in \mathbb{C}\backslash\mathbb{R}.$$

This equality continues to hold on $\mathbb{R} \pm \mathrm{i}0$ by Lemmas 5.7 and 5.8. We then have

$$\|R_H^+(\lambda)P_c\|_{B(L^{2,\tau}_x, L^{2,-\tau}_x)} \leq \|(1 + AR^+_{D_\mathscr{M}}(\lambda)B^*)^{-1}\|_{B(L^{2,\tau}_x, L^{2,\tau}_x)} \|R^+_{D_\mathscr{M}}(\lambda)\|_{B(L^{2,\tau}_x, L^{2,-\tau}_x)}. \tag{5.24}$$

By [1] there is a $C'(\tau) > 0$ such that for all $\lambda \in \mathbb{R}$

$$\|\lambda R^+_{-\Delta}(\lambda^2)\|_{B(L^{2,\tau}_x, L^{2,-\tau}_x)} + \|\nabla R^+_{-\Delta}(\lambda^2)\|_{B(L^{2,\tau}_x, L^{2,-\tau}_x)} \leq C'(\tau).$$

Then by (5.21) we have $\|R^+_{D_\mathscr{M}}(\lambda)\|_{B(L^{2,\tau}_x, L^{2,-\tau}_x)} \leq C(\tau)$ for all $\lambda \in \mathbb{R}$.

We obtain (5.23) from

$$\sup_{\lambda \in \Lambda} \|(1 + AR^+_{D_\mathscr{M}}(\lambda)B^*)^{-1}\|_{B(L^{2,\tau}_x, L^{2,\tau}_x)} < \infty, \tag{5.25}$$

which follows from the analytic Fredholm alternative. □

Remark 5.9. Notice that (5.25) is in fact true for $\Lambda = \mathbb{R}$ by (H3) and, for large λ, by [26], see Appendix A [8].

The next lemma is proved by an argument of [36] reviewed in Lemma 5.7 [8].

Lemma 5.10. *Consider pairs (p, q) as in Theorem 5.6 with $p > 2$, $k \in \mathbb{R}$ arbitrary and $k' - k \geq \alpha(q)$. Then for any $\tau > 1$ there is a constant $C_0 = C_0(\tau, k, p, q)$ such that*

$$\left\| \int_0^t e^{\mathrm{i}H(t'-t)} P_c F(t') dt' \right\|_{L^p_t B^k_{q,2}} \leq C_0 \|P_c F\|_{L^2_t H^{k',\tau}}. \tag{5.26}$$

Proof. For $F(t, x) \in C_0^\infty(\mathbb{R} \times \mathbb{R}^3)$ set

$$TF(t) := \int_0^{+\infty} e^{\mathrm{i}(t'-t)H} P_c F(t') dt' \, , \quad f := \int_0^{+\infty} e^{\mathrm{i}t' H} P_c F(t') dt'.$$

Theorem 5.6 implies $\|TF\|_{L^p_t B^k_{q,2}} \le \|f\|_{H^{k'}}$ for $k' - k = \alpha(q)$. By Theorem 5.4 we have $\|f\|_{H^{k'}} \le C\|F\|_{L^2_t H^{k',\tau}}$. By $p > 2$ a lemma by Christ and Kiselev [14], see Lemma 3.1 [44], yields Lemma 5.10. $\square$

Lemma 5.11. *Assume the hypotheses of Proposition 5.2 and recall the definition of M in Definition 2.4. Let $\tau_0 > 1$. Then there is a fixed c_1 such that for all admissible pairs (p, q) inequality (5.10) holds.*

Proof. By picking $\epsilon_0 > 0$ sufficiently small and $\epsilon = \|u(0)\|_{H^4} < \epsilon_0$, for a fixed $c_1 > 0$ for the final coordinates $(z(0), \eta(0))$ of $u(0)$ we have

$$|z(0)| + \|\eta(0)\|_{H^4} \le c_1 \epsilon. \tag{5.27}$$

We have for $G^*_{j\mathbf{m}} = G^*_{j\mathbf{m}}(0)$

$$\mathrm{i}\dot{\eta} = \mathrm{i}\{\eta, \mathcal{H}\} = H\eta + \sum_{j=1}^{n} \sum_{l=1}^{2N+3} \sum_{|\mathbf{m}|=l} z_j \overline{\mathbf{Z}}^{\mathbf{m}} G^*_{j\mathbf{m}} + \mathbb{A}, \text{ where}$$

$$\mathbb{A} := \sum_{j=1}^{n} \sum_{l=1}^{2N+3} \sum_{|\mathbf{m}|=l} z_j \overline{\mathbf{Z}}^{\mathbf{m}} [G^*_{j\mathbf{m}}(|z_j|^2) - G^*_{j\mathbf{m}}] + \nabla_{\eta^*} \mathcal{R}. \tag{5.28}$$

We rewrite

$$\sum_{j=1}^{n} \sum_{l=1}^{2N+3} \sum_{|\mathbf{m}|=l} z_j \overline{\mathbf{Z}}^{\mathbf{m}} G^*_{j\mathbf{m}} = \sum_{(\mu,\nu)\in M} \bar{z}^{\mu} z^{\nu} G^*_{\mu\nu}. \tag{5.29}$$

Notice that (5.6) is the same as

$$\|z^{\mu} \bar{z}^{\nu}\|_{L^2_t(I)} \le C\epsilon \text{ for all } (\mu, \nu) \in M. \tag{5.30}$$

The proof of Lemma 5.11 is a consequence of Lemmas 5.12, 5.13 and 5.14 below. $\square$

Lemma 5.12. *For $I_T := [0, T]$ and for $S \in \mathbb{R}$ and $\epsilon_0 > 0$ small enough then for a constant $C(S, C_0)$ independent from T and ϵ we have*

$$\|\mathbb{A}\|_{L^2(I_T, H^{4,S}) + L^1(I_T, H^4)} \le C(S, C_0)\epsilon^2. \tag{5.31}$$

Proof. We have $r - 1 \ge S$,

$$\begin{aligned} &\|z_j \overline{\mathbf{Z}}^{\mathbf{m}} [G^*_{j\mathbf{m}}(|z_j|^2) - G^*_{j\mathbf{m}}]\|_{L^2(I_T, H^{4,S})} \le \|z_j \overline{\mathbf{Z}}^{\mathbf{m}}\|_{L^2(I_T, \mathbb{C})} \|G_{j\mathbf{m}}(|z_j|^2) - G_{j\mathbf{m}}\|_{L^\infty(I_T, H^{4,S})} \\ &\quad \le C_0 \epsilon \sup\{\|G'_{j\mathbf{m}}(|z_j|^2)\|_{\Sigma_r} : |z_j| \le \delta_0\} \|z_j^2\|_{L^\infty(I_T, \mathbb{C})} \le C C_0^3 \epsilon^3. \end{aligned} \tag{5.32}$$

Furthermore we get for a fixed $c_1 > 0$

$$\|\nabla_{\eta^*} E_P(\eta)\|_{L^1(I_T, H^4)} = 2\|g(\eta\bar{\eta})\eta\|_{L^1(I_T, H^4)} \le c_1 \|\eta\|_{L^\infty(I_T, H^4)} \|\eta\|^2_{L^2(I_T, L^\infty)} \le c_1 C_0^3 \epsilon^3. \tag{5.33}$$

The rest of Lemma 5.12 follows by the fact that for arbitrarily preassigned $S > 2$,

$$\|R_1\|_{L^2(I_T,H^{4,S})} \le C(S,C_0)\epsilon^2 \text{ for } R_1 = \nabla_{\eta^*}(\mathcal{R} - E_P(\eta)). \tag{5.34}$$

This inequality is proved in [21] (for $H^{4,S}$ replaced by $H^{1,S}$, but the proof is the same). Then (5.32)–(5.34) imply (5.31). □

Lemma 5.13. *Consider* $\mathrm{i}\dot\psi - H\psi = F$ *where* P_c *and* $\psi = P_c\psi$. *Let* $k \in \mathbb{Z}$ *with* $k \ge 0$ *and* $\tau_0 > 1$. *Then for* (p,q) *as in* (1.12) *and* $\tau_0 > 1$ *for a constant* $C = C(p,q,k,\tau_0)$ *we have*

$$\|\psi\|_{L_t^p([0,T],B_{q,2}^{k-\frac{2}{p}})\cap L_t^2([0,T],H_x^{k,-\tau_0})} \le C\|\psi(0)\|_{H^k} + C\|F\|_{L_t^1([0,T],H^k)+L_t^2([0,T],H^{k,\tau_0})}. \tag{5.35}$$

Proof. We split $F = F_1 + F_2$ with $F_1 \in L^1([0,T],H^k)$ and $F_2 \in L^2([0,T],H^{k,\tau_0})$ and we write

$$\psi(t) = e^{-\mathrm{i}tH}\psi(0) - \mathrm{i}\sum_{j=1}^{2}\int_0^t e^{-\mathrm{i}(t-s)H}F_j(s)ds. \tag{5.36}$$

Estimate (5.35) in the special case $F = 0$ is a consequence of (5.16) and (5.13). The case $\psi_0 = 0$, $F_2 = 0$ follows by (5.18) and (5.13). Finally, the case $\psi_0 = 0$, $F_1 = 0$ follows by (5.26) and (5.15). □

Lemma 5.14. *Using the notation of Lemma 5.13, but this time picking* $\tau_0 > 3/2$, *we have*

$$\|\psi\|_{L_t^2([0,T],L^\infty)} \le C\|\psi(0)\|_{H^4} + C\|F\|_{L_t^1([0,T],H_x^4)+L_t^2([0,T],H_x^{4,\tau_0})} \tag{5.37}$$

Proof. The argument is the same of Lemma 10.5 [8]. We consider

$$\psi(t) = e^{-\mathrm{i}tD_{\mathscr{M}}}\psi(0) + \mathrm{i}\int_0^t e^{-\mathrm{i}(t-s)D_{\mathscr{M}}}V\psi(s)ds - \mathrm{i}\sum_{j=1}^{2}\int_0^t e^{-\mathrm{i}(t-s)D_{\mathscr{M}}}F_j(s)ds. \tag{5.38}$$

We have for $k \in (1/2,3]$

$$\|e^{-\mathrm{i}D_{\mathscr{M}}t}\psi(0)\|_{L_t^2L_x^\infty} \le C\|e^{-\mathrm{i}D_{\mathscr{M}}t}\psi(0)\|_{L_t^2B_{6,2}^k} \le C'\|\psi(0)\|_{H^{k+1}} \le C'\|\psi(0)\|_{H^4}$$

by the flat version of (5.16), which holds by [6]. Similarly we have

$$\begin{aligned}\|\int_0^t e^{\mathrm{i}D_{\mathscr{M}}(t'-t)}F_1(t')dt'\|_{L_t^2L_x^\infty} &\le C\|\int_0^t e^{\mathrm{i}D_{\mathscr{M}}(t'-t)}F_1(t')dt'\|_{L_t^2B_{6,2}^k} \\ &\le C'\|F_1\|_{L_t^1H^{k+1}} \le C'\|F_1\|_{L_t^1H^4}.\end{aligned}$$

Using $B_{\infty,2}^k \subset L^\infty$ for $k > 1/2$ and picking $k < 1$, by Theorem 5.5 we have

$$\begin{aligned}\|\int_0^t e^{\mathrm{i}D_{\mathscr{M}}(t'-t)}F_2(t')dt'\|_{L_t^2L_x^\infty} &\le C\left\|\int_0^t \min\{|t-t'|^{-\frac{1}{2}},|t-t'|^{-\frac{3}{2}}\}\|F_2(t')\|_{B_{1,2}^{k+3}}dt'\right\|_{L_t^2} \\ &\le C'\|F_2\|_{L_t^2B_{1,2}^4} \le C''\|\langle x\rangle^{\tau_0}F_2\|_{L_t^2B_{2,2}^4} = C''\|F_2\|_{L_t^2H^{4,\tau_0}},\end{aligned} \tag{5.39}$$

where we have used $\|\varphi_j * F_2\|_{L_x^1} \le \|\langle x\rangle^{-\tau_0}\|_{L_x^2}\|\langle x\rangle^{\tau_0}\varphi_j * F_2\|_{L_x^2} \le C'''\|\varphi_j * (\langle\cdot\rangle^{\tau_0}F_2)\|_{L_x^2}$ for fixed $C''' > 0$ and fixed $\tau_0 > 3/2$. With F_2 replaced by $V\psi$ we get a similar estimate. This yields inequality (5.37). □

Setting $M = M(2N+4)$, see Definition 2.4, we now introduce a new variable g setting

$$g = \eta + Y \text{ with } Y := \sum_{(\alpha,\beta)\in M} \bar{z}^{\alpha} z^{\beta} R_H^{+}(\mathbf{e}\cdot(\beta-\alpha))G^{*}_{\alpha\beta}. \tag{5.40}$$

This can be traced in [10,47] and has the following meaning. When we write $\eta = -Y + g$ the term $-Y$ is the part of η which has the most significant effect on the variables z_j. Substituting η by $-Y + g$ in the equations for the z_j, these equations reduce to equations dependent only on z, up to a perturbation.

The following lemma is an easier version of Lemma 10.7 [8] and so we give the proof in few lines.

Lemma 5.15. *Assume the hypotheses of Proposition 5.2 and fix $S > 9/2$. Then there is a $c_1(S) > 0$ such that for any C_0 there is a $\epsilon_0 = \epsilon_0(C_0, S) > 0$ such that for $\epsilon \in (0, \epsilon_0)$ in Theorem 1.3 we have*

$$\|g\|_{L^2([0,T],L^{2,-S})} \le c_1(S)\epsilon. \tag{5.41}$$

Proof. Substituting (5.40) in (5.28) and using (5.29) we obtain

$$\mathrm{i}\dot{g} = Hg + \mathrm{i}\dot{Y} - HY + \sum_{(\alpha,\beta)\in M} \bar{z}^{\alpha} z^{\beta} G^{*}_{\alpha\beta} + \mathbb{A}. \tag{5.42}$$

We then compute

$$\begin{aligned} \mathrm{i}\dot{Y} &= \sum_{(\alpha,\beta)\in M} \mathbf{e}\cdot(\beta-\alpha)\bar{z}^{\alpha} z^{\beta} R_H^{+}(\mathbf{e}\cdot(\beta-\alpha))G^{*}_{\alpha\beta} + \mathbf{T} \text{ where} \\ \mathbf{T} &:= \sum_j \left[\partial_{z_j} Y(\mathrm{i}\dot{z}_j - e_j z_j) + \partial_{\bar{z}_j} Y(\mathrm{i}\dot{\bar{z}}_j + e_j \bar{z}_j)\right]. \end{aligned} \tag{5.43}$$

Then in (5.42) we have the cancellation

$$\sum_{(\alpha,\beta)\in M} \mathbf{e}\cdot(\beta-\alpha)\bar{z}^{\alpha} z^{\beta} R_H^{+}(\mathbf{e}\cdot(\beta-\alpha))G^{*}_{\alpha\beta} - HY + \sum_{(\alpha,\beta)\in M} \bar{z}^{\alpha} z^{\beta} G^{*}_{\alpha\beta} = 0.$$

So (5.42) becomes

$$\mathrm{i}\dot{g} = Hg + \mathbb{A} + \mathbf{T}. \tag{5.44}$$

We then have

$$g(t) = e^{-\mathrm{i}Ht}\eta(0) + e^{-\mathrm{i}Ht}Y(0) - \mathrm{i}\int_0^t e^{-\mathrm{i}H(t-s)}(\mathbb{A}(s) + \mathbf{T}(s))ds. \tag{5.45}$$

We have $\|e^{-\mathrm{i}Ht}\eta(0)\|_{L^2(\mathbb{R},L^{2,-S})} \le c\|\eta(0)\|_{L^2} \le c\epsilon$ by (5.13). The rest of the proof of Lemma 5.15 is exactly the same of Lemma 6.4 [21], where the auxiliary Lemma 6.5 [21] needs to be replaced by Lemma 5.16 below. □

Following the proof of Lemma 5.8 [8] we obtain Lemma 5.16 below. Lemma 5.16 is standard ingredient in this type of proof. For example in [47] the analogous ingredient is Proposition 2.2, but versions appear in [10,53] (see also references therein) just to name a few.

Lemma 5.16. *Let Λ be a compact subset of $(-\infty,-\mathscr{M})\cup(\mathscr{M},\infty)$ and let $S>7/2$. Then there exists a fixed $c(S,\Lambda)$ such that for every $t\ge 0$ and $\lambda\in\Lambda$,*

$$\|e^{-\mathrm{i}Ht}R_H^+(\lambda)P_c v_0\|_{L^{2,-S}(\mathbb{R}^3)} \le c(S,\Lambda)\langle t\rangle^{-\frac{3}{2}}\|P_c v_0\|_{L^{2,S}(\mathbb{R}^3)} \text{ for all } v_0\in L^{2,S}(\mathbb{R}^3). \tag{5.46}$$

Proof. We expand $R_H^+(\lambda)=R_{D_\mathscr{M}}^+(\lambda)-R_{D_\mathscr{M}}^+(\lambda)VR_H^+(\lambda)$. For $\tau_1>5/2$, by Theorem 5.3 and by [5, Theorem 2] we have

$$\|e^{-\mathrm{i}tD_\mathscr{M}}R_{D_\mathscr{M}}^+(\lambda)\psi_0\|_{L^{2,-\tau_1}(\mathbb{R}^3)} \le C\langle t\rangle^{-\frac{3}{2}}\|R_{D_\mathscr{M}}^+(\lambda)\psi_0\|_{L^{2,\tau_1}(\mathbb{R}^3)} \le C_1\langle t\rangle^{-\frac{3}{2}}\|\psi_0\|_{L^{2,\tau_1+1}(\mathbb{R}^3)},$$

with C_1 locally bounded in λ and τ_1. Hence, by the rapid decay of V and by Lemma 5.8

$$\begin{aligned}&\|e^{-\mathrm{i}tD_\mathscr{M}}R_{D_\mathscr{M}}^+(\lambda)VR_H^+(\lambda)P_c\psi_0\|_{L^{2,\tau_1}}\\ &\le C_1\langle t\rangle^{-\frac{3}{2}}\|V\|_{B(L^{2,-\tau_1},L^{2,\tau_1+1})}\left\|R_H^+(\lambda)P_c\right\|_{B(L^{2,\tau_1},L^{2,-\tau_1})}\|\psi_0\|_{L^{2,\tau_1}}\le C'\langle t\rangle^{-\frac{3}{2}}. \qquad \square\end{aligned}$$

5.1.2. The analysis of the discrete modes

Let us turn now to the analysis of the Fermi Golden Rule (FGR). We have

$$\begin{aligned}\mathrm{i}\dot z_j &= (1+\varpi_j(|z_j|^2))(e_jz_j+\partial_{\bar z_j}\mathcal{Z}_0(|z_1|^2,\dots,|z_n|^2)+\partial_{\bar z_j}\mathcal{R})\\ &+(1+\varpi_j(|z_j|^2))[\sum_{(\mu,\nu)\in M}\nu_j\frac{z^\mu\bar z^\nu}{\bar z_j}\langle\eta,G_{\mu\nu}\rangle+\sum_{(\mu',\nu')\in M}\mu_j'\frac{z^{\nu'}\bar z^{\mu'}}{\bar z_j}\langle\eta^*,G^*{}_{\mu'\nu'}\rangle]\\ &+(1+\varpi_j(|z_j|^2))[\sum_{\mathbf{m}\in\mathcal{M}_j(2N+3)}|z_j|^2\mathbf{Z}^{\mathbf{m}}\left\langle G'_{j\mathbf{m}},\eta\right\rangle+z_j^2\overline{\mathbf{Z}}^{\mathbf{m}}\left\langle G'^*_{j\mathbf{m}},\eta^*\right\rangle].\end{aligned}\tag{5.47}$$

We use (5.40) to substitute an appropriately η in the equations above getting, for M_{min} defined as in (2.5),

$$\mathrm{i}\dot z_j-e_jz_j=\varpi_j(|z_j|^2))e_jz_j+(1+\varpi_j(|z_j|^2))\partial_{\bar z_j}\mathcal{Z}_0(|z_1|^2,\dots,|z_n|^2)+\mathcal{A}_j+\mathcal{B}_j+X_j(M_{min}), \tag{5.48}$$

where for $\widehat{M}\subset M$ we set

$$\begin{aligned}X_j(\widehat{M}) :=& -\sum_{\substack{(\mu,\nu)\in\widehat{M}\\(\alpha,\beta)\in\widehat{M}}}\nu_j\frac{z^{\mu+\beta}\bar z^{\nu+\alpha}}{\bar z_j}\langle R_H^+(\mathbf{e}\cdot(\beta-\alpha))G^*_{\alpha\beta},G_{\mu\nu}\rangle\\ &-\sum_{\substack{(\mu',\nu')\in\widehat{M}\\(\alpha',\beta')\in\widehat{M}}}\mu_j'\frac{z^{\nu'+\alpha'}\bar z^{\mu'+\beta'}}{\bar z_j}\langle R_H^-(\mathbf{e}\cdot(\beta'-\alpha'))G_{\alpha'\beta'},G^*_{\mu'\nu'}\rangle,\end{aligned}\tag{5.49}$$

with

$$\begin{aligned}\mathcal{A}_j =& (1+\varpi_j(|z_j|^2))\partial_{\bar z_j}\mathcal{R}+\varpi_j(|z_j|^2)[\sum_{(\mu,\nu)\in M}\nu_j\frac{z^\mu\bar z^\nu}{\bar z_j}\langle\eta,G_{\mu\nu}\rangle+\sum_{(\mu',\nu')\in M}\mu_j'\frac{z^{\nu'}\bar z^{\mu'}}{\bar z_j}\langle\eta^*,G^*_{\mu'\nu'}\rangle]\\ &+\sum_{(\mu,\nu)\in M}\nu_j\frac{z^\mu\bar z^\nu}{\bar z_j}\langle g,G_{\mu\nu}\rangle+\sum_{(\mu',\nu')\in M}\mu_j'\frac{z^{\nu'}\bar z^{\mu'}}{\bar z_j}\langle g^*,G^*_{\mu'\nu'}\rangle\\ &+(1+\varpi_j(|z_j|^2))[\sum_{\mathbf{m}\in\mathcal{M}_j(2N+3)}|z_j|^2\mathbf{Z}^{\mathbf{m}}\left\langle G'_{j\mathbf{m}},\eta\right\rangle+z_j^2\overline{\mathbf{Z}}^{\mathbf{m}}\left\langle G'^*_{j\mathbf{m}},\eta^*\right\rangle]\end{aligned}\tag{5.50}$$

and

$$\mathcal{B}_j = X_j(M) - X_j(M_{min}). \tag{5.51}$$

We notice that the r.h.s. of the identity (5.49) is well defined by Definition 2.4 in combination with Lemma 5.16. This observation allows also the introduction of the variable ζ defined by

$$\begin{aligned}
&\zeta_j = z_j + T_j(z) \text{ where} \\
&T_j(z) := \sum_{\substack{(\mu,\nu)\in M_{min} \\ (\alpha,\beta)\in M_{min}}} \frac{\nu_j z^{\mu+\beta}\bar{z}^{\nu+\alpha}}{((\mu-\nu)\cdot\mathbf{e}-(\alpha-\beta)\cdot\mathbf{e})\bar{z}_j}\langle R_H^+(\mathbf{e}\cdot(\beta-\alpha))G^*_{\alpha\beta}, G_{\mu\nu}\rangle \\
&\qquad + \sum_{\substack{(\mu',\nu')\in M_{min} \\ (\alpha',\beta')\in M_{min}}} \frac{\mu'_j z^{\nu'+\alpha'}\bar{z}^{\mu'+\beta'}}{((\alpha'-\beta')\cdot\mathbf{e}-(\mu'-\nu')\cdot\mathbf{e})\bar{z}_j}\langle R_H^-(\mathbf{e}\cdot(\beta'-\alpha'))G_{\alpha'\beta'}, G^*_{\mu'\nu'}\rangle,
\end{aligned} \tag{5.52}$$

with the summation performed over the pairs where the formula makes sense, that is $(\mu-\nu)\cdot\mathbf{e} \neq (\alpha-\beta)\cdot\mathbf{e}$. We have the following lemma (see also [21]).

Lemma 5.17. *Assume (5.30). We have*

$$\|\zeta - z\|_{L^2(0,T)} \le c(N, C_0)\epsilon^2 \text{ and } \|\zeta - z\|_{L^\infty(0,T)} \le c(N, C_0)\epsilon^2 \tag{5.53}$$

and ζ equations

$$\begin{aligned}
\mathrm{i}\dot{\zeta}_j &= (1+\varpi(|z_j|^2))(e_j\zeta_j + \lambda'_j(|z_j|^2)\zeta_j + \partial_{\bar{\zeta}_j}\mathcal{Z}_0(\zeta)) \\
&\quad - \sum_{(\mu,\nu)\in M_{min}} \nu_j \frac{|\zeta^\mu\bar{\zeta}^\nu|^2}{\bar{\zeta}_j}\langle R_H^+(\mathbf{e}\cdot(\nu-\mu))G^*_{\mu\nu}, G_{\mu\nu}\rangle \\
&\quad - \sum_{(\mu',\nu')\in M_{min}} \mu'_j \frac{|\zeta^{\nu'}\bar{\zeta}^{\mu'}|^2}{\bar{\zeta}_j}\langle R_H^-(\mathbf{e}\cdot(\nu'-\mu'))G_{\mu'\nu'}, G^*_{\mu'\nu'}\rangle + \mathcal{G}_j,
\end{aligned} \tag{5.54}$$

where there are fixed c_4 and $\epsilon_0 > 0$ such that for $T > 0$ and $\epsilon \in (0, \epsilon_0)$ we have

$$\|\mathcal{G}_j\bar{\zeta}_j\|_{L^1[0,T]} \le (1 + C_0)c_4\epsilon^2. \tag{5.55}$$

Proof. We have

$$\mathrm{i}\dot{\zeta}_j = \mathrm{i}\dot{T}_j - e_jT_j + e_j\zeta_j + \text{r.h.s. of (5.48)}.$$

By elementary computation we have

$$\begin{aligned}
&\mathrm{i}\dot{T}_j - e_jT_j = -X_j(M_{min}) + \mathbb{T}_j, \text{ where} \\
&\mathbb{T}_j := \sum_l \left[\partial_{z_l}T_j(\mathrm{i}\dot{z}_l - e_lz_l) + \partial_{\bar{z}_l}T_j(\mathrm{i}\dot{\bar{z}}_l + e_j\bar{z}_l)\right].
\end{aligned} \tag{5.56}$$

This leads to the cancellation of $X_j(M_{min})$ and, for a $\mathcal{G}_j$ which we write explicitly below, to

$$
\begin{aligned}
\mathrm{i}\dot{\zeta}_j &= (1+\varpi(|z_j|^2))(e_j\zeta_j + \partial_{\bar{j}}\mathcal{Z}_0(|\zeta_1|^2,\dots,|\zeta_n|^2)) \\
&\quad - \sum_{\substack{(\mu,\nu),(\alpha,\beta)\in M_{min} \\ (\alpha-\beta)\cdot\mathbf{e}=(\mu-\nu)\cdot\mathbf{e}}} \nu_j \frac{\zeta^{\mu+\beta}\bar{\zeta}^{\nu+\alpha}}{\bar{\zeta}_j}\langle R_H^+(\mathbf{e}\cdot(\beta-\alpha))G^*_{\alpha\beta}, G_{\mu\nu}\rangle \\
&\quad - \sum_{\substack{(\mu',\nu'),(\alpha',\beta')\in M_{min} \\ (\alpha'-\beta')\cdot\mathbf{e}=(\mu'-\nu')\cdot\mathbf{e}}} \mu'_j \frac{\zeta^{\nu'+\alpha'}\bar{\zeta}^{\mu'+\beta'}}{\bar{\zeta}_j}\langle R_H^-(\mathbf{e}\cdot(\beta'-\alpha'))G_{\alpha'\beta'}, G^*_{\mu'\nu'}\rangle + \mathcal{G}_j.
\end{aligned} \tag{5.57}
$$

By Lemma 2.5 we achieve that $(\alpha-\beta)\cdot\mathbf{e} = (\mu-\nu)\cdot\mathbf{e}$ implies $(\alpha,\beta)=(\mu,\nu)$. Similarly $(\alpha',\beta')=(\mu',\nu')$. Hence (5.57) can be written as

$$
\begin{aligned}
\mathrm{i}\dot{\zeta}_j &= (1+\varpi(|z_j|^2))(e_j\zeta_j + \partial_{\bar{\zeta}_j}\mathcal{Z}_0(|\zeta_1|^2,\dots,|\zeta_n|^2)) \\
&\quad - \sum_{(\mu,\nu)\in M_{min}} \nu_j \frac{|\zeta^{\mu}\bar{\zeta}^{\nu}|^2}{\bar{\zeta}_j}\langle R_H^+(\mathbf{e}\cdot(\nu-\mu))G^*_{\mu\nu}, G_{\mu\nu}\rangle \\
&\quad - \sum_{(\mu,\nu)\in M_{min}} \mu_j \frac{|\zeta^{\mu}\bar{\zeta}^{\nu}|^2}{\bar{\zeta}_j}\langle R_H^-(\mathbf{e}\cdot(\nu-\mu))G_{\mu\nu}, G^*_{\mu\nu}\rangle + \mathcal{G}_j,
\end{aligned}
$$

that is the equation (5.54), where, recalling $\mathcal{A}_j$ as in (5.50) and $\mathcal{B}_j$ as in (5.51),

$$
\begin{aligned}
&\mathcal{G}_j = \mathcal{B}_j + \mathcal{G}'_j, \text{ where} \\
&\mathcal{G}'_j := \mathcal{A}_j + (1+\varpi(|z_j|^2))[\partial_{\bar{j}}\mathcal{Z}_0(|z_1|^2,\dots,|z_n|^2) - \partial_{\bar{j}}\mathcal{Z}_0(|\zeta_1|^2,\dots,|\zeta_n|^2)] - e_j\varpi(|z_j|^2)T_j(z) + \mathbb{T}_j.
\end{aligned}
$$

The proof of (5.53), an easy consequence of the estimate (5.30) (or equivalently (5.6)), and the proof of $\|\mathcal{G}'_j\bar{\zeta}_j\|_{L^1[0,T]} \le (1+C_0)c_4\epsilon^2$ are in [21]. The proof of (5.55) is then a consequence of

$$
\|\mathcal{B}_j\zeta_j\|_{L^1_t} \le \|\mathcal{B}_j z_j\|_{L^1_t} + \|\mathcal{B}_j\|_{L^2_t}\|z_j-\zeta_j\|_{L^2_t} \le C(C_0)\epsilon^3. \tag{5.58}
$$

To prove (5.58) we use

$$
\|\mathcal{B}_j z_j\|_{L^1_t} \lesssim \sum_{\substack{(\mu,\nu)\in M\setminus M_{min} \\ (\alpha,\beta)\in M}} \|z^{\mu}\bar{z}^{\nu}\|_{L^2_t}\|z^{\alpha}\bar{z}^{\beta}\|_{L^2_t} \le C(C_0)\epsilon^3,
$$

by the definition of M_{min}. In an analogous way it is possible to prove the remaining estimates needed for the second term on the r.h.s. of (5.58). □

By multiplying the identity (5.54) above by $\bar{\zeta}_j$ and summing over the index j we achieve, in like manner as in [21],

$$
\begin{aligned}
2^{-1}\sum_j \frac{d}{dt}|\zeta_j|^2 &= -\sum_j \mathrm{Im}\Big[\sum_{(\mu,\nu)\in M_{min}} \nu_j\, |\zeta^{\mu}\bar{\zeta}^{\nu}|^2\langle R_H^+((\nu-\mu)\cdot\mathbf{e})G^*_{\mu\nu}, G_{\mu\nu}\rangle \\
&\quad + \sum_{(\mu,\nu)\in M_{min}} \mu_j\, |\zeta^{\mu'}\bar{\zeta}^{\nu}|^2\langle R_H^-((\nu-\mu)\cdot\mathbf{e})G_{\mu\nu}, G^*_{\mu\nu}\rangle\Big] + \sum_j \mathrm{Im}[\mathcal{G}_j\bar{\zeta}_j].
\end{aligned} \tag{5.59}
$$

Thus by using the substitution, for any $(\nu-\mu)\cdot\mathbf{e}\in\Lambda$ (see the formulas (A.7) and (A.9) in Lemma A.1 of Appendix A),

$$R_H^{\pm}((\nu-\mu)\cdot\mathbf{e}) = P.V.\frac{1}{H-(\nu-\mu)\cdot\mathbf{e}} \pm \mathrm{i}\pi\delta(H-(\nu-\mu)\cdot\mathbf{e}), \tag{5.60}$$

we can state the following lemma.

Lemma 5.18. *For any $(\mu,\nu)\in M_{min}$, we have*

$$\sum_j \mathrm{Im}[\sum_{(\mu,\nu)\in M_{min}} \nu_j \ |\zeta^\mu\bar{\zeta}^\nu|^2\langle P.V.\frac{1}{H-(\nu-\mu)\cdot\mathbf{e}}G^*_{\mu\nu}, G_{\mu\nu}\rangle$$
$$+\sum_{(\mu,\nu)\in M_{min}} \mu_j \ |\zeta^\mu\bar{\zeta}^\nu|^2\langle P.V.\frac{1}{H-(\nu-\mu)\cdot\mathbf{e}}G_{\mu\nu}, G^*_{\mu\nu}\rangle] = 0. \tag{5.61}$$

Proof. By formula (A.9) below the terms $\langle P.V.\frac{1}{H-(\nu-\mu)\cdot\mathbf{e}}f, f^*\rangle$, for $f = G_{\mu\nu}, G^*_{\mu\nu}$, are real valued. □

We have also the lemma below.

Lemma 5.19. *For any $(\mu,\nu)\in M_{min}$ we have*

$$\pi\sum_j \mathrm{Im}[\mathrm{i}\sum_{(\mu,\nu)\in M_{min}} \nu_j\,|\zeta^\mu\bar{\zeta}^\nu|^2\langle\delta(H-(\nu-\mu)\cdot\mathbf{e})G^*_{\mu\nu}, G_{\mu\nu}\rangle$$
$$-\mathrm{i}\sum_{(\mu,\nu)\in M_{min}} \mu_j\,|\zeta^\mu\bar{\zeta}^\nu|^2\langle\delta(H-(\nu-\mu)\cdot\mathbf{e})G_{\mu\nu}, G^*_{\mu\nu}\rangle]$$
$$=\pi\sum_{(\mu,\nu)\in M_{min}} |\zeta^\mu\bar{\zeta}^\nu|^2\langle\delta(H-(\nu-\mu)\cdot\mathbf{e})G^*_{\mu\nu}, G_{\mu\nu}\rangle \geq 0. \tag{5.62}$$

Proof. By (A.8) we have $\langle\delta(H-(\nu-\mu)\cdot\mathbf{e})G_{\mu\nu}, G^*_{\mu\nu}\rangle = \langle\delta(H-(\nu-\mu)\cdot\mathbf{e})G^*_{\mu\nu}, G_{\mu\nu}\rangle \geq 0$. Then, commuting the summations and by $|\nu|-|\mu|=1$ we get

$$\text{l.h.s. (5.62)} = \pi\sum_j \mathrm{Re}[\sum_{(\mu,\nu)\in M_{min}} (\nu_j-\mu_j)\,|\zeta^\mu\bar{\zeta}^\nu|^2\langle\delta(H-(\nu-\mu)\cdot\mathbf{e})G^*_{\mu\nu}, G_{\mu\nu}\rangle]$$
$$=\pi\sum_{(\mu,\nu)\in M_{min}} \langle\delta(H-(\nu-\mu)\cdot\mathbf{e})G^*_{\mu\nu}, G_{\mu\nu}\rangle \geq 0. \qquad \square$$

By an application of Lemmas 5.18 and 5.19 to the identity (5.59) we arrive as in [21] at the following.

Corollary 5.20. *We have*

$$2^{-1}\frac{d}{dt}\sum_j |\zeta_j|^2 = \sum_j \mathrm{Im}[\mathcal{G}_j\bar{\zeta}_j] - \pi\sum_{(\mu,\nu)\in M_{min}} |\zeta^\mu\bar{\zeta}^\nu|^2\langle\delta(H-(\nu-\mu)\cdot\mathbf{e})G^*_{\mu\nu}, G_{\mu\nu}\rangle. \qquad \square \tag{5.63}$$

By Lemma A.1 in Appendix A we have

$$\langle\delta(H-(\nu-\mu)\cdot\mathbf{e})G^*_{\mu\nu}, G_{\mu\nu}\rangle = \frac{|(\nu-\mu)\cdot\mathbf{e}|}{\sqrt{((\nu-\mu)\cdot\mathbf{e})^2-\mathscr{M}^2}}\int_{S_{\mu\nu}} |\widehat{G}_{\mu\nu}(\xi)|^2 dS(\xi), \tag{5.64}$$

for $S_{\mu\nu} = \{\xi\in\mathbb{R}^3 : |\xi|^2+\mathscr{M}^2 = |(\nu-\mu)\cdot\mathbf{e}|^2\}$ and for $\widehat{G}_{\mu\nu}(\xi) = \mathcal{F}_{V,+}(G_{\mu\nu})(\xi)$ the distorted Fourier transform in (A.6). The $\widehat{G}_{\mu\nu}$ are continuous functions by the fact that the $G_{\mu\nu}$ are rather regular and quite rapidly decreasing. Since by (H5) we have $\widehat{G}_{\mu\nu}|_{S_{\mu\nu}} \neq 0$ for all $(\mu, \nu) \in M_{min}$, it follows that there exists a $\Gamma > 0$ such that

$$\langle \delta(H - (\nu - \mu) \cdot \mathbf{e}) G^*_{\mu\nu}, G_{\mu\nu} \rangle > \Gamma > 0, \text{ for all } (\mu, \nu) \in M_{min}. \tag{5.65}$$

Now we complete the proof of Proposition 5.2. By the inequalities (5.3)–(5.7) and (5.55) in Lemma 5.17, integrating (5.63) and using (5.65) we get for, any $t \in [0, T]$ and for a fixed c_2,

$$\sum_j |z_j(t)|^2 + \sum_{(\mu,\nu)\in M} \|z^{\mu+\nu}\|^2_{L^2(0,t)} \le c_2(\epsilon^2 + C_0\epsilon^2). \tag{5.66}$$

Here we have used the inequalities (5.53) in Lemma 5.17, which allow to switch from inequalities on ζ to inequalities on z. The function $|\dot{z}_j(t)|$ can be estimated by using (5.48).

In particular (5.66) yields (5.11). This completes the proof of Proposition 5.2.

5.2. *Proof of the asymptotics* (5.8)

We write (5.28) in the form $\mathrm{i}\dot{\eta} = D_{\mathscr{M}}\eta + V\eta + \mathbb{B}$ with, see (5.28),

$$\mathbb{B} = \sum_{(\mu,\nu)\in M} \bar{z}^\mu z^\nu G^*_{\mu\nu} + \mathbb{A}.$$

Then $\partial_t(e^{\mathrm{i}D_{\mathscr{M}}t}\eta) = -\mathrm{i}e^{\mathrm{i}D_{\mathscr{M}}t}(V\eta + \mathbb{B})$ and so

$$e^{\mathrm{i}D_{\mathscr{M}}t_2}\eta(t_2) - e^{\mathrm{i}D_{\mathscr{M}}t_1}\eta(t_1) = -\mathrm{i}\int_{t_1}^{t_2} e^{\mathrm{i}D_{\mathscr{M}}t}(V\eta(t) + \mathbb{B}(t))dt \text{ for } t_1 < t_2.$$

Then for a fixed c_2 by Lemma 5.13, and specifically (5.35) for $k = 4$, $p = \infty$, $q = 2$ and using $B^k_{2,2} = H^4$ we have

$$\|e^{\mathrm{i}Ht_2}\eta(t_2) - e^{\mathrm{i}Ht_1}\eta(t_1)\|_{H^4} \le c_2\|V\eta(t) + \mathbb{B}(t)\|_{L^1([t_1,t_2],H^4)+L^2([t_1,t_2],H^{4,10}))}. \tag{5.67}$$

By (5.3), valid now in $[0, \infty)$ and for a fixed C, we have

$$\|V\eta(t)\|_{L^1([t_1,t_2],H^{4,S})} \le c_1\|\eta\|_{L^1(\mathbb{R}_+,H^{4,-10})} \le C\epsilon.$$

By Proposition 4.6 and (5.30) we similarly have

$$\|\sum_{(\mu,\nu)\in M} \bar{z}^\mu z^\nu G^*_{\mu\nu}\|_{L^2(\mathbb{R}_+,H^{4,10})} \le C' \sum_{(\mu,\nu)\in M} \|\bar{z}^\mu z^\nu\|_{L^2(\mathbb{R}_+,\mathbb{C})} \le C\epsilon.$$

We also have (5.31) for $I_T = \mathbb{R}_+$. We then conclude that there exists an $\eta_+ \in H^4 \cap \mathcal{H}_c[0]$ with

$$\lim_{t\nearrow\infty} e^{\mathrm{i}D_{\mathscr{M}}t}\eta(t) = \eta_+ \text{ in } H^4 \text{ and with } \|\eta_+\|_{H^4(\mathbb{R}^3)} \le C\epsilon.$$

Now we prove the existence of ρ_+ and the facts about it in Theorem 5.1. First of all we have

$$\frac{1}{2}\sum_j \frac{d}{dt}|z_j|^2 = \sum_j \mathrm{Im}\left[\partial_{\bar{j}}\mathcal{R}\bar{z}_j + \sum_{(\mu,\nu)\in M} \nu_j z^\mu \bar{z}^\nu \langle \eta, G_{\mu\nu}\rangle + \sum_{(\mu',\nu')\in M} \mu'_j z^{\nu'} \bar{z}^{\mu'} \langle \eta^*, G^*_{\mu'\nu'}\rangle\right].$$

Since the r.h.s. has $L^1(0,\infty)$ norm bounded by $C\epsilon^2$ for a fixed C, we conclude that the limit

$$\lim_{t\nearrow\infty}(|z_1(t)|,\dots,|z_n(t)|)=(\rho_{1+},\dots,\rho_{n+})$$

exists, with $|\rho_+|\le C\|u(0)\|_{H^4}$. By $\lim_{t\nearrow\infty} z^\mu\bar{z}^\nu(t)=0$ for all $(\mu,\nu)\in M$, we can conclude that all but at most one of the ρ_{j+} are equal to 0.

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Appendix A. Proof of the formula (5.60)

This section is devoted to prove the Plemelj formula (5.60) associated to the resolvent of the operator (1.2). With this aim we need to rely now on the following facts. Borrowed by [6] we first introduce the matrix functions $\psi_0(x,\xi)\in M_4(\mathbb{C})$ with vector column given by

$$\psi_0^j(x,\xi)=e^{\mathrm{i}x\cdot\xi}v(\xi)e_j,\quad j=1,\dots 4, \tag{A.1}$$

with $L(\xi)=\sqrt{|\xi|^2+\mathscr{M}^2}$,

$$v(\xi)=\frac{(L(\xi)+\mathscr{M})I_4-\beta\alpha\cdot\xi}{\sqrt{2L(\xi)((L(\xi)+\mathscr{M}))}}, \tag{A.2}$$

is a unitary matrix and e_j are vectors of the canonical basis of $\mathbb{C}^4$ (for more details see [52], Sect. 1). We recall that the transformation

$$v(\xi)\mathcal{F}(u)(\xi)=\frac{1}{(2\pi)^{\frac{3}{2}}}\int\limits_{\mathbb{R}^3}\psi_0(x,\xi)^*u(x)dx,$$

diagonalizes the free Dirac operator $D_{\mathscr{M}}$ as follows $v^*(\xi)\mathcal{F}D_{\mathscr{M}}\mathcal{F}^*v(\xi)=L(\xi)\beta$ (we recall that $\mathcal{F}$ denotes the classical Fourier transform with inverse $\mathcal{F}^*$). Consequently we can define the distorted plane wave functions $\psi_V^{\pm}(x,\xi)\in M_4(\mathbb{C})$ associated to the continuous spectrum of H (see for instance [1] and [2] for the cases of Schröndinger and Klein–Gordon) as follows,

$$\psi_V^{j,\pm}(x,\xi)=\psi_0^j(x,\xi)-\Lambda_V^{j,\pm}(x,\xi),\quad j=1,\dots 4, \tag{A.3}$$

where

$$\Lambda_V^{j,\pm}(x,\xi)=\begin{cases}\lim_{\varepsilon\searrow 0}R_H(L(\xi)\pm\mathrm{i}\varepsilon)V\psi_0^j(x,\xi) & \text{for } j\in\{1,2\},\\ \lim_{\varepsilon\searrow 0}R_H(-L(\xi)\pm\mathrm{i}\varepsilon)V\psi_0^j(x,\xi) & \text{for } j\in\{3,4\}.\end{cases} \tag{A.4}$$

We recall also that the distorted plane wave associated to the perturbed Dirac operator (1.2), here denoted by $\psi_V^+(x,\xi)$, satisfies the equation

$$H\psi_V^+(x,\xi)=\pm L(\xi)\psi_V^+(x,\xi) \tag{A.5}$$

(the same equation holds for $\psi_V^-(x,\xi)$). By this, for any $g \in \mathcal{S}(\mathbb{R}^3,\mathbb{C}^4)$, we have the distorted Fourier transform associated to H (in the sense of Sect. 3.3 in [6]).

$$\mathcal{F}_{V,\pm}(g)(\xi) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbb{R}^3} \psi_V^{\pm}(x,\xi)^* g(x)dx, \tag{A.6}$$

is a bounded linear operator from $\mathcal{H}_c[0]$ in $L^2(\mathbb{R}^3,\mathbb{C}^4)$ and with inverse $\mathcal{F}_{V,\pm}^*$ defined as in Theorem 3.2 of [6]. Notice that we have also the relation $\mathcal{F}(g)(\xi) = v^*(\xi)\mathcal{F}_{0,\pm}(g)(\xi)$. Motivated by this we state the following lemma.

Lemma A.1. *Let be $\lambda \in \mathbb{R}\backslash[-\mathscr{M},\mathscr{M}]$, then we have the following representation of the resolvent $R_H^{\pm}(\lambda)$ of the perturbed Dirac operator H defined as in Lemma 5.8,*

$$R_H^{\pm}(\lambda) = P.V.\frac{1}{H-\lambda} \pm \mathrm{i}\pi\delta(H-\lambda), \tag{A.7}$$

characterized by

$$\begin{aligned}\pi\mathrm{i}\langle\delta(H-\lambda)f,f^*\rangle &= \frac{1}{2}\langle R_H^+(\lambda) - R_H^-(\lambda)f,f^*\rangle \\ &= \frac{\pi\mathrm{i}|\lambda|}{\sqrt{\lambda^2-\mathscr{M}^2}} \int_{|\xi|=\sqrt{\lambda^2-\mathscr{M}^2}} |\mathcal{F}_{V,+}(f)|^2 d\xi,\end{aligned} \tag{A.8}$$

and

$$\begin{aligned}\langle P.V.\frac{1}{H-\lambda}f,f^*\rangle &= \frac{1}{2}\langle R_H^+(\lambda) + R_H^-(\lambda)f,f^*\rangle \\ &= \lim_{\epsilon\searrow 0} \int_{|\,|\xi|-\sqrt{\lambda^2-\mathscr{M}^2}|\geq\epsilon} \frac{L(\xi)\beta+\lambda I_{\mathbb{C}^4}}{|\xi|^2+\mathscr{M}^2-\lambda^2}|\mathcal{F}_{V,+}(f)|^2 d\xi,\end{aligned} \tag{A.9}$$

for any function $f \in \mathcal{S}(\mathbb{R}^3,\mathbb{C}^4) \cap \mathcal{H}[0]$.

Proof. We deal with the proof of the formulas (A.8) and (A.9) for $R_H^+(\lambda)$ because the one for $R_H^-(\lambda)$ is similar. Select a $f \in \mathcal{S}(\mathbb{R}^3,\mathbb{C}^4)$, then by transposing the arguments of [1] and following [58] one can see that, for any $z \in \mathbb{C}\backslash\sigma(H)$, the following identity is fulfilled

$$\mathcal{F}(R_H(z)f)(\xi) = \mathcal{F}\left(\frac{1}{H-z}f\right)(\xi) = v(\xi)\frac{L(\xi)\beta+zI_{\mathbb{C}^4}}{|\xi|^2+\mathscr{M}^2-z^2}f(\xi,z), \tag{A.10}$$

where we set

$$\frac{(L(\xi)\beta+zI_{\mathbb{C}^4})}{|\xi|^2+\mathscr{M}^2-z^2} = \begin{pmatrix} \frac{1}{L(\xi)-z}I_{\mathbb{C}^2} & 0 \\ 0 & -\frac{1}{L(\xi)+z}I_{\mathbb{C}^2}\end{pmatrix},$$

(see once again [52]) and with the vector valued function $f(\xi,z)$ having the form

$$f(\xi,z) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3} (\psi_0(x,\xi) - R_H(z^*)V\psi_0(x,\xi))^* f(x)dx. \tag{A.11}$$

Moreover by a use of $R_H(z) - R_H(z^*) = 2\mathrm{i}R_H(z)R_H(z^*)\,\mathrm{Im}\,z$ and $R_H(z^*) = (R_H(z))^*$ for any $z \in \mathbb{C}\backslash\mathbb{R}$, in connection with the Parseval identity and with the fact that $v(\xi)v^*(\xi) = v^*(\xi)v(\xi) = I_{\mathbb{C}^4}$, one gets

$$\frac{1}{2}\langle [R_H(z) - R_H(z^*)]f, f^*\rangle = \mathrm{i} \int_{\mathbb{R}^3} \mathrm{Im}\, \frac{(\lambda(\xi)\beta + zI_{\mathbb{C}^4})}{|\xi|^2 + \mathscr{M}^2 - z^2} f(\xi, z) f(\xi, z)^* d\xi, \tag{A.12}$$

with the matrix

$$\mathrm{Im}\, \frac{(L(\xi)\beta + zI_{\mathbb{C}^4})}{|\xi|^2 + \mathscr{M}^2 - z^2} = \begin{pmatrix} \frac{\mathrm{Im}\, z}{(L(\xi)-z)(L(\xi)-z^*)} I_{\mathbb{C}^2} & 0 \\ 0 & \frac{\mathrm{Im}\, z}{(L(\xi)+z)(L(\xi)+z^*)} I_{\mathbb{C}^2} \end{pmatrix}.$$

Pick now $z = \lambda + \mathrm{i}\varepsilon$, then we are allowed by the trace Lemma 5.8 to take the limit $\varepsilon \searrow 0$ (see also [1]). Combining this step with an application of the Plemelj formula $\frac{1}{x \mp i0} = PV\frac{1}{x} \pm \mathrm{i}\pi\delta(x)$, we obtain from (A.12) the identity

$$\frac{1}{2}\langle [R_H^+(\lambda) - R_H^-(\lambda)]f, f^*\rangle = \pi\mathrm{i} \int_{\mathbb{R}^3} \varXi(L(\xi), \lambda) \mathcal{F}_{V,+}(f) \mathcal{F}_{V,+}(f)^* d\xi, \tag{A.13}$$

with

$$\varXi(L(\xi), \lambda) = \begin{pmatrix} \delta(L(\xi) - \lambda) I_{\mathbb{C}^2} & 0 \\ 0 & \delta(-L(\xi) - \lambda) I_{\mathbb{C}^2} \end{pmatrix}.$$

At this point, an application of the identities (in $\mathcal{S}'(\mathbb{R}^3, \mathbb{C}^4)$, see for example [28], Chap. II, Sec. 2.5 or Chap. III for a more general theory)

$$\delta(L(\xi) - \lambda) = \frac{\lambda}{\sqrt{\lambda^2 - \mathscr{M}^2}} \delta(|\xi| - \sqrt{\lambda^2 - \mathscr{M}^2}), \quad \text{for } \lambda > \mathscr{M}, \tag{A.14}$$

$$\delta(-L(\xi) - \lambda) = \frac{-\lambda}{\sqrt{\lambda^2 - \mathscr{M}^2}} \delta(|\xi| - \sqrt{\lambda^2 - \mathscr{M}^2}), \quad \text{for } \lambda < -\mathscr{M}, \tag{A.15}$$

shows that the r.h.s. of identity (A.13) is equal to

$$\begin{cases} \dfrac{\pi\mathrm{i}\lambda}{\sqrt{\lambda^2 - \mathscr{M}^2}} \displaystyle\int_{|\xi| = \sqrt{\lambda^2 - \mathscr{M}^2}} \sum_{i=1}^{2} |\mathcal{F}_{V,+}^i(f)|^2 d\xi, & \text{for } \lambda > \mathscr{M}, \quad \text{(a)} \\ \dfrac{-\pi\mathrm{i}\lambda}{\sqrt{\lambda^2 - \mathscr{M}^2}} \displaystyle\int_{|\xi| = \sqrt{\lambda^2 - \mathscr{M}^2}} \sum_{i=3}^{4} |\mathcal{F}_{V,+}^i(f)|^2 d\xi, & \text{for } \lambda < -\mathscr{M}, \quad \text{(b)} \end{cases} \tag{A.16}$$

which in turn implies the identity (A.8). A similar discussion (actually easier) yields

$$\frac{1}{2}\langle R_H^+(\lambda) + R_H^-(\lambda) f, f^*\rangle = P.V. \int \frac{L(\xi)\beta + \lambda I_{\mathbb{C}^4}}{|\xi|^2 + \mathscr{M}^2 - \lambda^2} |\mathcal{F}_{V,+}(f)|^2 d\xi,$$

that is the identity (A.9). This completes the proof of the lemma. □

Remark A.2. All the convergence arguments are well defined because of Lemma 5.8. Moreover in the proof we used functions in $f \in \mathcal{S}(\mathbb{R}^3, \mathbb{C}^4)$. One can easily extend to $f \in H^{k,s}(\mathbb{R}^3, \mathbb{C}^4) \cap \mathcal{H}[0]$ for $k \geq 0$ and $s > 1/2$ (which implies $\mathcal{F}_{V,\pm} f \in H^s_{loc}(\mathbb{R}^3, \mathbb{C}^4)$ so that the restriction to spheres makes sense) by a density argument.

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