An antinorm theory for sets of matrices: bounds and approximations to the lower spectral radius

Nicola Guglielmi [a] Marino Zennaro [b]

[a] *Gran Sasso Science Institute, via Crispi 7, I-67100 L'Aquila (Italy)*
`nicola.guglielmi@gssi.it`

[b] *Dipartimento di Matematica e Geoscienze, Università di Trieste, via A. Valerio 12/1, 34127 Trieste, Italy* `zennaro@units.it`

Abstract

For the computation of the lower spectral radius of a finite family of matrices that shares an invariant cone, two recent papers by Guglielmi and Protasov [GP] and Guglielmi and Zennaro [GZ] make use of so-called antinorms. Antinorms are continuous, nonnegative, positively homogeneous and superadditive functions defined on the cone and turn out to be related to the lower spectral radius of the family in a similar way as norms are related to the joint spectral radius. In this paper, we revisit the theory of antinorms in a systematic way, filling in some theoretical holes, correcting a common mistake present in the literature and adding some new properties and results. In particular, we prove that, under suitable assumptions, the lower spectral radius is characterized by a Gelfand type limit computed on an antinorm.

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1 Introduction

In this paper we discuss some techniques for approximating and bounding (from below) the *lower spectral radius (LSR)* of a finite family of $d \times d$-matrices that share an invariant cone. Such techniques involve the so-called *antinorm* of a vector/matrix, whose role is analogous to that of a standard vector/matrix norm when studying the *joint spectral radius*.

In Section 2 we recall the main definitions and results relevant to the LSR. More precisely, starting from the original definition given by Gurvits [Gu], we take the opportunity to review in a systematic way the existing links among the various quantities involved in the characterization of the LSR that have been considered in the literature.

In Section 3 we recall the definition of cone of $\mathbb{R}^d$ and some of the most important spectral properties of those matrices that have an invariant cone.

In Section 4 we recall the definition of vector/matrix *antinorm* and discuss the main aspects of such type of functionals. In doing this, we take the opportunity to fill in some theoretical holes and correct a common mistake present in the literature that regards the continuity of the *dual antinorm* on the boundary of its domain. We also prove a certain number of new properties and results. To the best of our knowledge, the concept of antinorm was introduced by Merikoski [Mk1,Mk2], who also developed the basis of a related theory, which is parallel to the well-known one for vector/matrix norms (see also the subsequent paper [MO]). Much later the antinorms have been employed by Protasov [Pr], Guglielmi and Protasov [GP] and Guglielmi and Zennaro [GZ] in order to give some lower bounds to the LSR (although being unaware of the already existing papers by Merikoski). Moszynska and Richter [MR] also considered antinorms independently, but their definition was somehow different. Finally, some other authors used the term antinorm giving it a quite different meaning (see the introductory discussion in [MR]).

In Section 5 we give some new definitions regarding matrix antinorms and restate in this framework some known relationships between the operator antinorms and the LSR of a family $\mathcal{F}$ of matrices that share an invariant cone. Then we provide the main results of the paper, the most important of which is the characterization of the LSR as a Gelfand type limit, under suitable assumptions on the employed antinorm and on the spectral properties of the underlying family $\mathcal{F}$.

Finally, in Section 6 we construct a specific three-dimensional example of dual antinorm that is not continuous on the boundary of its domain. Given its length and complexity, such an example is not inserted in Section 4, which should be its natural environment, not to interrupt the logical flow of the paper.

2 Lower spectral radius

In this paper we deal with a finite family of matrices, say $\mathcal{F} = \{A_1, \ldots, A_m\}$, $A_i \in \mathbb{R}^{d,d}$, $i \in \{1, \ldots, m\}$. However, a large part of the definitions and results that follow are still valid for or could be adapted to infinite bounded families as well.

For $k = 1, 2, \ldots$, we consider the set of all products of *length* (or *degree*) k

$$\Sigma_k(\mathcal{F}) = \{A_{i_k} \cdots A_{i_1} \ \mid \ i_1, \ldots, i_k \in \{1, \ldots, m\}\}.$$

Then, following Gurvits [Gu, p. 69], for a given norm $\|\cdot\|$ of $\mathbb{R}^d$ and corresponding matrix norm, we define

$$\check{\rho}_k(\mathcal{F}) := \min_{P \in \Sigma_k(\mathcal{F})} \|P\|^{1/k}.$$

Definition 2.1 *The number*

$$\check{\rho}(\mathcal{F}) := \inf_{k \geq 1} \check{\rho}_k(\mathcal{F}) \tag{1}$$

is called the lower spectral radius (LSR) *of the family* $\mathcal{F}$*.*

It is known that the LSR satisfies the equality

$$\check{\rho}(\mathcal{F}) = \lim_{k \to \infty} \check{\rho}_k(\mathcal{F}). \tag{2}$$

(see, e.g., Bochi and Morris [BM] and Czornik [Cz]) and that it does not depend on the particular norm. Indeed, it can be also expressed via the standard spectral radius. In fact, by setting

$$\bar{\rho}_k(\mathcal{F}) := \min_{P \in \Sigma_k(\mathcal{F})} \rho(P)^{1/k}, \tag{3}$$

Gurvits [Gu] proved that

$$\check{\rho}(\mathcal{F}) = \inf_{k \geq 1} \bar{\rho}_k(\mathcal{F}).$$

The above equality clearly implies that

$$\check{\rho}(\mathcal{F}) \leq \bar{\rho}_k(\mathcal{F}) \leq \check{\rho}_k(\mathcal{F}), \qquad k \geq 1, \tag{4}$$

which, in turn, along with (2) immediately yields

$$\check{\rho}(\mathcal{F}) = \lim_{k \to \infty} \bar{\rho}_k(\mathcal{F}). \tag{5}$$

The LSR turns out to be the exponent of *minimal asymptotic growth* for the products of matrices from the family $\mathcal{F}$.

3 Matrices having an invariant cone

The case of families $\mathcal{F}$ that share an invariant cone is of particular interest.

The notion of *proper cone* is pretty standard in the literature (see, e.g., Rodman, Seyalioglu and Spitkovsky [RSS], Schneider and Tam [ST] and Tam [Ta]).

Definition 3.1 *A* proper cone *of* $\mathbb{R}^d$ *is a nonempty closed and convex set* K *such that:*

(i) $\mathbb{R}_+ K \subseteq K$ *(i.e.,* K *is* positively homogeneous*);*
(ii) $K \cap -K = \{0\}$ *(i.e.,* K *is* salient*);*
(iii) $\mathrm{span}(K) = \mathbb{R}^d$ *(i.e.,* K *is* full *or* solid*).*

Following Schneider and Vidyasagar [SV], in this paper we shall refer to a proper cone by simply using the term *cone*.

We also recall that, if K is a cone of $\mathbb{R}^d$, its *dual* is defined as

$$K^* := \{y \in \mathbb{R}^d \mid y^T x \geq 0 \quad \forall x \in K\}.$$

It is a cone as well and, besides the basic properties of the "geometric duality", it also fulfills the relations

$$\mathrm{int}(K^*) = \{y \in \mathbb{R}^d \mid y^T x > 0 \quad \forall x \in K \setminus \{0\}\} \tag{6}$$

and

$$K^* \setminus \{0\} = \{y \in \mathbb{R}^d \mid y^T x > 0 \quad \forall x \in \mathrm{int}(K)\}.$$

Definition 3.2 *A cone* K *of* $\mathbb{R}^d$ *is said to be* invariant *for a matrix* $A \in \mathbb{R}^{d,d}$ *if* $A(K) \subseteq K$*. Furthermore, it is said to be* strictly invariant *if* $A(K \setminus \{0\}) \subseteq \mathrm{int}(K)$*.*

Definition 3.3 *We say that a cone* K *is* invariant *(*strictly invariant*) for the family of matrices* $\mathcal{F} = \{A_1, \ldots, A_m\}$ *if it is so for each matrix* A_i*,* $i = 1, \ldots, m$*.*

The following statement is actually a well known result for single matrices, which trivially extends to families.

Proposition 3.1 *A cone* K *is invariant (strictly invariant) for the family of matrices* $\mathcal{F} = \{A_1, \ldots, A_m\}$ *if and only if the dual cone* K^* *is invariant (strictly invariant) for the transpose family* $\mathcal{F}^T = \{A_1^T, \ldots, A_m^T\}$*.*

Now we reconsider the notation used by Brundu and Zennaro [BZ1,BZ2,BZ3], extending temporarily the action of the matrix A to the complex space $\mathbb{C}^d$.

If $\lambda \in \mathbb{C}$ is an eigenvalue of A of algebraic multiplicity k, we denote the associated complex eigenspace by $\tilde{V}_\lambda := \ker(A - \lambda I)$ and the associated complex generalized eigenspace by $\tilde{W}_\lambda := \ker((A - \lambda I)^k)$. Clearly, $\tilde{V}_\lambda \subseteq \tilde{W}_\lambda$ and both of them are invariant subspaces of $\mathbb{C}^d$ under the action of A.

In particular, if $\lambda \in \mathbb{R}$, then $V_\lambda := \tilde{V}_\lambda \cap \mathbb{R}^d$ and $W_\lambda := \tilde{W}_\lambda \cap \mathbb{R}^d$ are linear subspaces of $\mathbb{R}^d$, still called associated eigenspace and associated generalized eigenspace respectively, which are invariant under the action of A. Otherwise, if $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then the conjugate $\bar{\lambda}$ is an eigenvalue of A as well and, hence, we define $U_{\mathbb{R}}(\lambda, \bar{\lambda}) := (\tilde{W}_\lambda \oplus \tilde{W}_{\bar{\lambda}}) \cap \mathbb{R}^d$, which is a subspace of $\mathbb{R}^d$, still invariant under the action of A.

Now we assume that $\rho(A) > 0$ and that A has a real *leading eigenvalue* λ_1 such that $|\lambda_1| = \rho(A)$. Therefore, denoting by $V_A := V_{\lambda_1}$ and $W_A := W_{\lambda_1}$ the associated *leading eigenspace* and *leading generalized eigenspace* respectively, if $\lambda_2, \ldots, \lambda_r \in \mathbb{R}$ and $\mu_1, \bar{\mu}_1 \ldots, \mu_s, \bar{\mu}_s \in \mathbb{C} \setminus \mathbb{R}$ are the remaining distinct roots of the characteristic polynomial, then it holds that

$$\mathbb{R}^d = W_A \oplus H_A,$$

where

$$H_A = \bigoplus_{i=2}^{r} W_{\lambda_i} \oplus \bigoplus_{i=1}^{s} U_{\mathbb{R}}(\mu_i, \bar{\mu}_i).$$

It is easy to prove that (see also [BZ1] for a proof of (a2)):

(a1) the linear space H_A is invariant for A;
(a2) $x \in \mathbb{R}^d$, $Ax \in H_A \quad \Longrightarrow \quad x \in H_A$.

In particular, we consider the class of so called *asymptotically rank-one* matrices (see again [BZ1,BZ2,BZ3]).

Definition 3.4 *A matrix A is said to be* asymptotically rank-one *if it satisfies the following properties:*

(i) $\rho(A) > 0$*;*
(ii) either $\rho(A)$ or $-\rho(A)$ is a simple eigenvalue of A (denoted by λ_A);
(iii) $|\lambda| < \rho(A)$ for any other eigenvalue λ of A.

For an asymptotically rank-one matrix A the generalized leading eigenspace W_A reduces to the one-dimensional leading eigenspace V_A and is called the *leading invariant line* of A. Furthermore, the supplementary subspace H_A, whose dimension is $d-1$, is called the *secondary invariant hyperplane* of A.

It is immediate to see that a matrix A is asymptotically rank-one if and only if the transpose matrix A^T is so. Moreover, if we denote by v_A the (unique up to scalar factors) leading eigenvector of A belonging to K and by h_A the (unique up to scalar factors) leading eigenvector of A^T belonging to K^*, we easily have that

$$H_A = \{h_A\}^\perp \qquad \text{and} \qquad H_{A^T} = \{v_A\}^\perp. \tag{7}$$

The following result includes the well known Perron-Frobenius theorem (which may be found, for instance, in Vandergraft [Va]) and an additional property, whose proof may be found, for instance, in [BZ1].

Theorem 3.1 *Let a cone K be invariant for a nonzero matrix A. Then the following facts hold:*

(i) *the spectral radius $\rho(A)$ is an eigenvalue of A;*
(ii) *the cone K contains an eigenvector $v_A \in V_A$ corresponding to $\rho(A)$;*
(iii) $\mathrm{int}(K) \cap H_A = \emptyset$.

In particular, if K is strictly invariant for A, then it also holds that

(iv) *the matrix A is asymptotically rank-one;*
(v) *the (unique) leading eigenvector $v_A \in \mathrm{int}(K)$;*
(vi) $K \cap H_A = \emptyset$.

4 A revisited theory of antinorms

In order to be able to find a lower bound to $\check{\rho}(\mathcal{F})$, now we assume that the family $\mathcal{F}$ shares an invariant cone K and make use of an *antinorm* defined on K.

In what follows all the definitions and all the results that are given without proof are taken from Merikoski [Mk2], unless differently specified. Anyway, some slight variations in the terminology may occur.

Definition 4.1 *Given a cone K, an* antinorm $a(\cdot)$ *is a nontrivial (not identically zero) continuous function defined on K such that*

(b1) $a(x) \geq 0$ *for all* $x \in K$ *(nonnegativity);*
(b2) $a(\lambda x) = \lambda a(x)$ *for all* $\lambda \geq 0$ *and* $x \in K$ *(positive homogeneity);*
(b3) $a(x+y) \geq a(x) + a(y)$ *for all* $x, y \in K$ *(superadditivity).*

Remark 4.1 Since $a(\cdot)$ is not identically zero, properties (b1), (b2) and (b3) easily assure that
$$a(x) > 0 \qquad \forall x \in \mathrm{int}(K).$$

Remark 4.2 Note that properties (b2) and (b3) imply concavity and, consequently, as is well known, continuity in $\mathrm{int}(K)$. Therefore, the continuity condition for $a(\cdot)$ is an actual requirement on the boundary ∂K only.

Remark 4.3 One could remove the requirement for continuity and consider an antinorm that is not continuous on ∂K. In this case, we should speak of a *discontinuous antinorm.*

Proposition 4.1 *Any antinorm* $a(\cdot)$ *is a* monotone *functional on* K *(with respect to the partial order defined by* K *itself), i.e.,*

$$x, y \in K \quad \text{with} \quad x - y \in K \quad \Longrightarrow \quad a(x) \geq a(y).$$

Definition 4.2 *Let* $a(\cdot)$ *be an antinorm on a cone* K*. Then the set*

$$\mathcal{A} := \{x \in K \,\big|\, a(x) \geq 1\}$$

is the corresponding unit antiball*. Moreover, the set*

$$\mathcal{A}' := \{x \in K \,\big|\, a(x) = 1\}$$

is the corresponding unit antisphere.

Remark 4.4 Due to concavity (see Remark 4.2), the unit antiball $\mathcal{A}$ is always convex, even if the antinorm $a(\cdot)$ is discontinuous on ∂K.

Remark 4.5 Note that, in general, the unit antisphere $\mathcal{A}'$ is included in, but not necessarily equal to, the boundary $\partial \mathcal{A}$ of the unit antiball $\mathcal{A}$ intended as a subset of the whole space $\mathbb{R}^d$. Indeed, it is immediate to see that

$$\mathcal{A}' = \partial \mathcal{A} \iff \mathcal{A}' \cap \partial K = \emptyset \iff a(x) = 0 \quad \forall x \in \partial K$$

(see Section 4.4 for an example).

The proof of the next proposition is easy and, therefore, omitted.

Proposition 4.2 *Let* $a(\cdot)$ *be an antinorm on a cone* K*, possibly discontinuous on* ∂K*. Then the unit antiball* $\mathcal{A}$ *is closed if and only if* $a(\cdot)$ *is upper semi-continuous on* ∂K*.*

The following result was proved in [GP], but here we give a different proof that works even in the discontinuous case.

Proposition 4.3 *Let* $a(\cdot)$ *be an antinorm on a cone* K*, possibly discontinuous on* ∂K*, and let* $\|\cdot\|$ *be a norm on* $\mathbb{R}^d$*. Then there exists* $\beta > 0$ *such that*

$$a(x) \leq \beta \|x\| \qquad \forall x \in K. \tag{8}$$

Proof. Assume by contradiction that there exists a sequence $\{x_k\} \subset K$ such that $\|x_k\| = 1$ and

$$a(x_k) \longrightarrow +\infty. \tag{9}$$

It is not restrictive to assume that $x_k \longrightarrow \bar{x} \in K$ with $\|\bar{x}\| = 1$. Therefore, if we choose $\hat{x} \in \text{int}(K)$, we obtain $x_k + \hat{x} \longrightarrow \bar{x} + \hat{x} \in \text{int}(K)$ and hence, since $a(\cdot)$ is continuous on $\text{int}(K)$ (see Remark 4.2), also

$$a(x_k + \hat{x}) \longrightarrow a(\bar{x} + \hat{x}). \tag{10}$$

On the other hand, the superadditivity of $a(\cdot)$ and (9) imply

$$a(x_k + \hat{x}) \geq a(x_k) + a(\hat{x}) \longrightarrow +\infty,$$

contradicting (10). We conclude that (8) necessarily holds. □

Definition 4.3 *An antinorm $a(\cdot)$ is said to be* positive *if $a(x) > 0$ for all $x \in K \setminus \{0\}$ (i.e., for all $x \in \partial K \setminus \{0\}$).*

Proposition 4.4 *Let $a(\cdot)$ be an antinorm on a cone K and $\| \cdot \|$ be a norm on $\mathbb{R}^d$. Then the following statements are equivalent:*

(i) $a(\cdot)$ is positive;
(ii) there exists $\gamma > 0$ such that

$$\|x\| \leq \gamma\, a(x) \qquad \forall x \in K; \tag{11}$$

(iii) the unit antisphere $\mathcal{A}'$ is compact.

Proof.

(i) $\Longleftrightarrow$ *(ii)* Since $a(\cdot)$ is continuous, it attains a minimum μ on the compact set

$$\mathcal{B} := \{x \in K \;\big|\; \|x\| = 1\}.$$

If $a(\cdot)$ is positive, we have that $\mu > 0$. Therefore, thanks to property (b2), inequality (11) is proved with $\gamma = \mu^{-1}$. Conversely, $a(x) = 0$ in (11) clearly implies $x = 0$ and, thus, positivity is proved.

(ii) $\Longleftrightarrow$ *(iii)* Inequality (11) clearly implies the boundedness of $\mathcal{A}'$. Furthermore, since $a(\cdot)$ is continuous and K is closed, also $\mathcal{A}'$ turns out to be closed and, consequently, compact. Conversely, the compactness of $\mathcal{A}'$ yields the existence of a maximum $\gamma > 0$ for $\| \cdot \|$ on $\mathcal{A}'$ and, consequently, again by property (b2), we get (11). □

4.1 The dual antinorm

The following definition is taken from [GZ] and slightly differs from the one given in [Mk2]. Moreover, we understand that it is also extended to discontinuous antinorms.

Definition 4.4 *Given an antinorm* $a(\cdot)$ *on a cone* K*, the* dual antinorm *on the dual cone* K^* *is defined as*

$$a^*(y) := \inf_{x \in \mathcal{A}'} y^T x. \tag{12}$$

Properties (b1), (b2) and (b3) of Definition 4.1 are immediate to verify, even if $a(\cdot)$ is discontinuous on ∂K. As for the continuity on ∂K^*, it was always understood to hold, without giving a proof, in all of the previous papers where this subject had already been treated.

In the sequel, by means of a suitable counterexample (see Theorem 4.3 and Section 6), we instead show that, if the unit antisphere $\mathcal{A}'$ is unbounded (which may well happen, see Proposition 4.4), in general the dual antinorm $a^*(\cdot)$ is not necessarily continuous on ∂K^*.

However, the next well known general theorem (see, e.g., Aliprantis and Border [AB]) allows us to prove at least upper semi-continuity.

Theorem 4.1 *Given a cone* C *of* $\mathbb{R}^d$*, let* $\{f_i(\cdot)\}_{i \in \mathcal{I}}$ *an arbitrary collection of nonnegative continuous functions on* C*. Then the function*

$$f(y) := \inf_{i \in \mathcal{I}} f_i(y) \tag{13}$$

is upper semi-continuous on C*.*

Remark 4.6 If the collection $\{f_i(\cdot)\}_{i \in \mathcal{I}}$ is finite, the resulting function $f(\cdot)$ is obviously continuous on C.

Proposition 4.5 *Let* $a(\cdot)$ *be an antinorm on a cone* K*, possibly discontinuous on* ∂K*. Then the dual antinorm* $a^*(\cdot)$ *is upper semi-continuous on* ∂K^**.*

Proof. Since the scalar product $y^T x$ is nonnegative and continuous with respect to y, Theorem 4.1 assures upper semi-continuity of $a^*(\cdot)$ on ∂K^*.

Moreover, we observe that, given the arbitrariness of the index set $\mathcal{I}$ in (13), the continuity with respect to x is not involved and, hence, the result holds even if $a(\cdot)$ is discontinuous on ∂K. □

Proposition 4.2 clearly yields the following corollary.

Corollary 4.1 *Let* $a(\cdot)$ *be an antinorm on a cone* K*, possibly discontinuous on* ∂K*. Then the unit antiball* $\mathcal{A}^*$ *of the dual antinorm* $a^*(\cdot)$ *is closed.*

Now we show that it is not restrictive to confine ourselves to $x \in \mathrm{int}(K)$ in (12), obtaining the same definition of dual antinorm given in [Mk2].

Proposition 4.6 *Given an antinorm $a(\cdot)$ on a cone K, possibly discontinuous on ∂K, the dual antinorm $a^*(\cdot)$, defined on the dual cone K^*, is such that*

$$a^*(y) = \inf_{x \in \mathrm{int}(K) \cap \mathcal{A}'} y^T x.$$

Proof. If $\mathcal{A}' \cap \partial K = \emptyset$, the thesis is obvious. Otherwise, it is sufficient to prove that for all $\bar{x} \in \mathcal{A}' \cap \partial K$ we have

$$\inf_{x \in \mathrm{int}(K) \cap \mathcal{A}'} y^T x \le y^T \bar{x}. \tag{14}$$

To this purpose, choose $\hat{x} \in \mathrm{int}(K)$ such that $a(\hat{x}) = 1$ (such a point $\hat{x}$ exists because of Remark 4.1 and property (b2)) and consider the sequence $\{x_k\} \subset \mathrm{int}(K)$, where $x_k := \frac{k-1}{k}\bar{x} + \frac{1}{k}\hat{x}$. Clearly, property (b3) implies that $a(x_k) \ge 1$ for all $k \ge 1$. Furthermore, since $x_k \longrightarrow \bar{x}$ and since the scalar product $y^T x$ is continuous with respect to x, it holds that $y^T x_k \longrightarrow y^T \bar{x}$, which, in turn, implies

$$\limsup_{k \longrightarrow \infty} \frac{y^T x_k}{a(x_k)} \le y^T \bar{x}$$

and, a fortiori, inequality (14). Moreover, since the continuity of $a(\cdot)$ has not been used, we conclude that the result holds even if $a(\cdot)$ is discontinuous on ∂K. $\square$

Now we prove continuity of $a^*(\cdot)$ on ∂K^* in some particular cases. To this aim, we first state two preliminary lemmas and recall a definition from [GZ] (see also [GP]).

Lemma 4.1 *Let $f(\cdot)$ be an antinorm a cone C of $\mathbb{R}^2$, possibly discontinuous on ∂C. Then, in any case, it is lower semi-continuous on ∂C.*

Proof. In line with Definition 4.2, we consider the unit antiball and the unit antisphere

$$\mathcal{U} := \{x \in C \mid f(x) \ge 1\} \quad \text{and} \quad \mathcal{U}' := \{x \in C \mid f(x) = 1\}, \tag{15}$$

respectively, and their subsets

$$\mathcal{U}_{int} := \mathcal{U} \cap \mathrm{int}(C) \quad \text{and} \quad \mathcal{U}'_{int} := \mathcal{U}' \cap \mathrm{int}(C).$$

Now we show that

$$\mathcal{U} \subseteq \mathrm{clos}(\mathcal{U}_{int}). \tag{16}$$

In fact, since $f(\cdot)$ is continuous in $\mathrm{int}(C)$ (see Remark 4.2), it turns out that

$$\mathrm{clos}(\mathcal{U}_{int}) \cap \mathrm{int}(C) = \mathcal{U}_{int} \tag{17}$$

and

$$\mathcal{U}'_{int} \subseteq \partial \mathrm{clos}(\mathcal{U}_{int}). \tag{18}$$

Therefore, $\mathcal{U}$ and $\text{clos}(\mathcal{U}_{int})$ may differ from each other on ∂C only. Hence, if (16) were not true, there would exist $\bar{x} \in \partial C \setminus \{0\}$ such that $\bar{x} \in \mathcal{U} \setminus \text{clos}(\mathcal{U}_{int})$. On the other hand, the convexity of $\mathcal{U}$ (see Remark 4.4) clearly yields also the convexity of $\mathcal{U}_{int}$ and, in turn, of $\text{clos}(\mathcal{U}_{int})$. Consequently, by (17) and (18), there exists $\hat{x} \in \mathcal{U}'_{int}$ such that

$$x' := \frac{1}{2}(\hat{x} + \bar{x}) \in \text{int}(C) \setminus \text{clos}(\mathcal{U}_{int}) = \text{int}(C) \setminus \mathcal{U}$$

and, by concavity of $f(\cdot)$, at the same time

$$f(x') \geq \frac{1}{2}(f(\hat{x}) + f(\bar{x})) \geq 1,$$

which is an absurd.

Now we observe that a cone C of $\mathbb{R}^2$ is necessarily of polyhedral type and has precisely two one-dimensional faces, say $\gamma_1 = \text{span}(\{x^{(1)}\}) \cap C$ and $\gamma_2 = \text{span}(\{x^{(2)}\}) \cap C$ for some $x^{(1)}, x^{(2)} \neq 0$, such that $\partial C = \gamma_1 \cup \gamma_2$.

Therefore, thanks to property (b2), we are only required to prove lower semi-continuity at the points $x^{(i)}$, $i = 1, 2$. Moreover, because of property (b1), lower semi-continuity obviously holds at $x^{(i)}$ if $f(x^{(i)}) = 0$. So we can confine ourselves to the case $f(x^{(i)}) > 0$ which, in turn, implies $\mathcal{U} \cap \gamma_i \neq \emptyset$.

Still by (b2), we have that $\mathcal{U}' \cap \gamma_i$ consists of just one point and we can assume, without any restriction, that $\mathcal{U}' \cap \gamma_i = \{x^{(i)}\}$, i.e.,

$$f(x^{(i)}) = 1.$$

On the other hand, the inclusion (16) yields

$$x^{(i)} \in \text{clos}(\mathcal{U}_{int}).$$

Furthermore, the dimension of the space being $d = 2$ and, consequently, $\text{clos}(\mathcal{U}'_{int})$ being a (continuous) one-dimensional curve, (16) also assures that

$$\text{clos}(\mathcal{U}'_{int}) \cap \gamma_i = \{\tau x^{(i)}\} \quad \text{for some } \tau \in (0, 1].$$

Therefore, again by property (b2) and by the continuity of $f(\cdot)$ in $\text{int}(C)$, we obtain the existence of

$$\lim_{x \to x^{(i)},\ x \in \text{int}(C)} f(x) = \tau^{-1} \geq 1.$$

Eventually, since $\lim_{x \to x^{(i)},\ x \in \gamma_i} f(x) = 1$, we can conclude that

$$\liminf_{x \to x^{(i)}} f(x) \geq 1,$$

that is lower semi-continuity at $x^{(i)}$. $\square$

Remark 4.7 As is clear from the foregoing proof, the inclusion (16) holds in general, in any dimension d.

Remark 4.8 In the example of Section 6, which lives in dimension $d = 3$, the intersection $\mathrm{clos}(\mathcal{U}'_{int}) \cap \gamma_i$ is not limited to one point only, but consists of an entire segment where $f(x) > 1$. This fact does not allow us to get lower semi-continuity at $x^{(i)}$.

Lemma 4.2 *Let $a(\cdot)$ be an antinorm on a cone K, possibly discontinuous on ∂K. Then the dual antinorm $a^*(\cdot)$ is continuous at any point $\bar{y} \in \partial K^*$ such that $a^*(\bar{y}) = 0$.*

Proof. Since $a^*(\cdot)$ is nonnegative on K^*, it is obviously lower semi-continuous at any $\bar{y}$ where it vanishes. Therefore, Proposition 4.5 concludes the proof. □

Definition 4.5 *An antinorm $a(\cdot)$ on a cone K is said to be a* polytope antinorm *if its unit antiball $\mathcal{A}$ is a* positive infinite polytope *(with respect to K), i.e., there exists a minimal finite set $V := \{v_1, \ldots, v_n\} \subset K \setminus \{0\}$ of points, called the* vertices *of $\mathcal{A}$, with $a(v_i) = 1$, $i = 1, \ldots, n$, such that*

$$\mathcal{A} = \mathrm{conv}(V) + K = \{v + w \mid v \in \mathrm{conv}(V),\ w \in K\}, \tag{19}$$

where $\mathrm{conv}(V)$ *denotes the convex hull of V.*

Note that the positive infinite polytope $\mathcal{A}$ is not necessarily bounded by affine hyperplanes only. Indeed, this is true if and only if the cone K is itself of polyhedral type. In other words, the term *polytope* is used in a generalized sense, taking into account the fact that $\mathcal{A}$ lives inside the cone K, which may well have a non-polyhedral structure.

Theorem 4.2 *Given an antinorm $a(\cdot)$ on a cone K, possibly discontinuous on ∂K, the* dual antinorm *$a^*(\cdot)$ is continuous on ∂K^* if one of the following conditions holds:*

(i) the dimension of the space is $d = 2$;
(ii) the unit antisphere $\mathcal{A}'$ is bounded;
(iii) $a(\cdot)$ is a polytope antinorm.

Furthermore, in case (ii) we have that $a^*(\bar{y}) = 0$ for all $\bar{y} \in \partial K^*$.

Proof. (i) On one side, Proposition 4.5 assures that $a^*(\cdot)$ is upper semi-continuous on ∂K^* and, on the other side, the dimension being $d = 2$, Lemma 4.1 yields also lower semi-continuity, completing the proof.

(ii) Let $\bar{y} \in \partial K^* \setminus \{0\}$. Then the characterization (6) tells us that there exists $\bar{x} \in K \setminus \{0\}$ such that

$$\bar{y}^T \bar{x} = 0. \tag{20}$$

More precisely, since $K^{**} = K$, again (6) implies $\bar{x} \in \partial K$ as well.

Now consider a sequence $\{x_k\} \subset \mathrm{int}(K)$ such that $x_k \longrightarrow \bar{x}$ and the corresponding normalized sequence $\{\tilde{x}_k := x_k/a(x_k)\} \subset \mathrm{int}(K) \cap \mathcal{A}'$. Then the boundedness of $\mathcal{A}'$ allows us to state, possibly confining ourselves to a subsequence, that $\tilde{x}_k \longrightarrow \tilde{x}$, where $\tilde{x} = \tilde{\mu}\bar{x}$ for some $\tilde{\mu} > 0$. On the other hand, the continuity of the scalar product $y^T x$ with respect to x and relation (20) imply

$$\bar{y}^T \tilde{x}_k \longrightarrow \bar{y}^T \tilde{x} = \tilde{\mu} \bar{y}^T \bar{x} = 0$$

and, thus, Proposition 4.6 yields $a^*(\bar{y}) = 0$.

Eventually, Lemma 4.2 concludes the proof.

(iii) Given the form (19) of the unit antiball and the convexity of $\mathrm{conv}(V)$, for all $y \in K^*$ and $x \in \mathcal{A}$, we have

$$y^T x = y^T v + y^T w \geq y^T v \geq \min_{1 \leq i \leq n} y^T v_i,$$

which, along with (12), clearly implies $a^*(y) \geq \min_{1 \leq i \leq n} y^T v_i$.

On the other hand, being $v_i \in \mathcal{A}'$, $i = 1, \ldots, n$, (12) itself assures the opposite inequality as well. So we can conclude that

$$a^*(y) = \min_{1 \leq i \leq n} y^T v_i. \tag{21}$$

Therefore, Remark 4.6 concludes the proof. □

Definition 4.6 *An antinorm $a^*(\cdot)$ of the type (21) is said to be a* dual-polytope antinorm *and its unit antiball $\mathcal{A}^*$ is said to be a* positive infinite dual-polytope *(with respect to K^*).*

The following result is straightforward and, again, it still holds when $a(\cdot)$ and/or $b(\cdot)$ are discontinuous antinorms.

Proposition 4.7 *Let $a(\cdot)$ and $b(\cdot)$ be two antinorms on a cone K such that $a(x) \leq b(x)$ for all $x \in K$. Then the related dual antinorms $a^*(\cdot)$ and $b^*(\cdot)$ satisfy the opposite inequality, i.e., $a^*(y) \geq b^*(y)$ for all $y \in K^*$.*

4.2 The bidual antinorm

Definition 4.7 *Let $a(\cdot)$ be an antinorm on a cone K and $a^*(\cdot)$ be its dual antinorm (defined on K^*). Then the dual antinorm of $a^*(\cdot)$, which is defined on the original cone K (that is the dual of K^*), is called the* bidual antinorm *of $a(\cdot)$ and is denoted by $a^{**}(\cdot)$.*

Concerning the bidual antinorm, now we slightly generalize what is proved in [Mk2, Theorem 2A and Corollary 3A]. The proof is almost the same and, hence, we just give a sketch of it making a strong reference to that paper.

Proposition 4.8 *Let $a(\cdot)$ be an antinorm on a cone K, possibly discontinuous on ∂K. Then it holds that*

$$a^{**}(x) \geq a(x) \qquad \forall x \in K \tag{22}$$

and that

$$a^{**}(x) = a(x) \qquad \forall x \in K \tag{23}$$

if and only if $a(\cdot)$ is upper semi-continuous on ∂K.

Sketch of the proof. The inequality (22) is straightforward and has already been proved in [Mk2].

As for the equality (23), by looking deeper in the proof of in [Mk2, Theorem 2A], we can easily conclude that $\mathrm{conv}(\mathcal{A})$ (the convex hull of the unit antiball $\mathcal{A}$) equals the unit antiball $\mathcal{A}^{**}$ of the bidual antinorm $a^{**}(\cdot)$ if and only if $\mathrm{conv}(\mathcal{A})$ is closed. On the other hand, $\mathcal{A}$ is always convex (i.e., $\mathcal{A} = \mathrm{conv}(\mathcal{A})$) even if $a(\cdot)$ is discontinuous on ∂K (see Remark 4.4) and, moreover, it is closed if and only if $a(\cdot)$ is upper semi-continuous on ∂K (see Proposition 4.2). Therefore, we can conclude that $\mathcal{A}^{**} = \mathcal{A}$ if and only if $a(\cdot)$ is upper semi-continuous on ∂K. $\square$

4.3 Existence of discontinuous dual antinorms

Eventually, as an immediate corollary to Proposition 4.8, thanks to the example given in Section 6, we are in a position to state the existence of discontinuous dual antinorms $a^*(\cdot)$ on ∂K^*.

Theorem 4.3 *Let $f(\cdot)$ be an antinorm on a cone C which is upper semi-continuous on ∂C and such that the unit antisphere $\mathcal{U}'$ (see (15)) is bounded. Moreover, assume that $f(\cdot)$ is discontinuous at some point $\bar{y} \in \partial C$. Then the antinorm $a(\cdot) := f^*(\cdot)$, which is defined on the dual cone $K := C^*$, is continuous on ∂K but, at the same time, its dual $a^*(\cdot)$ equals $f(\cdot)$ on the entire cone $K^* = C$ and, thus, is not continuous at $\bar{y} \in \partial K^*$.*

Proof. Since $\mathcal{U}'$ is bounded, Theorem 4.2-(ii) assures that $a(\cdot) = f^*(\cdot)$ identically vanishes and is continuous on $\partial K = \partial C^*$.

Anyway, since $f(\cdot)$ is upper semi-continuous on ∂C, Proposition 4.8 implies that $a^*(\cdot) = f^{**}(\cdot) = f(\cdot)$ on the entire cone $K^* = C$, which includes the discontinuity point $\bar{y} \in \partial K^*$. $\square$

Remark 4.9 Since $a(\cdot) = f^*(\cdot)$ identically vanishes and is continuous on $\partial K = \partial C^*$, its unit antisphere $\mathcal{A}'$ is necessarily unbounded, as it can be immediately seen. Alternatively, we can easily arrive at the same conclusion by using Theorem 4.2-(ii) and reasoning by contradiction.

4.4 Some examples of antinorms

In this section we mention a particular class of antinorms that are defined on the cone $K = \mathbb{R}^d_+$ of the nonnegative vectors (see Merikoski [Mk1]). More precisely, we consider *p-antinorms* defined by

$$a_p(x) := \left(\sum_{i=1}^{d} x_i^p \right)^{1/p} \tag{24}$$

for $p \leq 1$, $p \neq 0$, and, letting $p \longrightarrow -\infty$, by

$$a_{-\infty}(x) := \min_{1 \leq i \leq d} x_i.$$

Note that, when $p \geq 1$, formula (24) defines the usual p-norm (restricted to $K = \mathbb{R}^d_+$). In particular, $a_1(\cdot)$ is both a norm and an antinorm (in fact, it is a linear functional on K).

Like in the case of p-norms, it is easy to prove that

$$a_p^*(\cdot) = a_q(\cdot) \qquad \text{with } q = \frac{p}{p-1}$$

when $p < 1$, $p \neq 0, -\infty$, and

$$a_1^*(\cdot) = a_{-\infty}(\cdot) \qquad \text{and} \qquad a_{-\infty}^*(\cdot) = a_1(\cdot).$$

Moreover, $a_p(\cdot)$ is positive if $p > 0$ and $a_p(x) = 0$ for all $x \in \partial K$ if $p < 0$. Accordingly to Proposition 4.4, we have that the unit antisphere $\mathcal{A}'_p$ is bounded if and only if $p > 0$.

We illustrate the extremal cases, dual to each other, obtained for $p = 1$ (see Figure 1 (left)) and $p = -\infty$ (see Figure 1 (right)) in dimension $d = 2$.

4.5 A note on pre-antinorms

Merikoski [Mk2] introduced the notion of *pre-norm* on a cone K of $\mathbb{R}^d$, which we call *pre-antinorm* here.

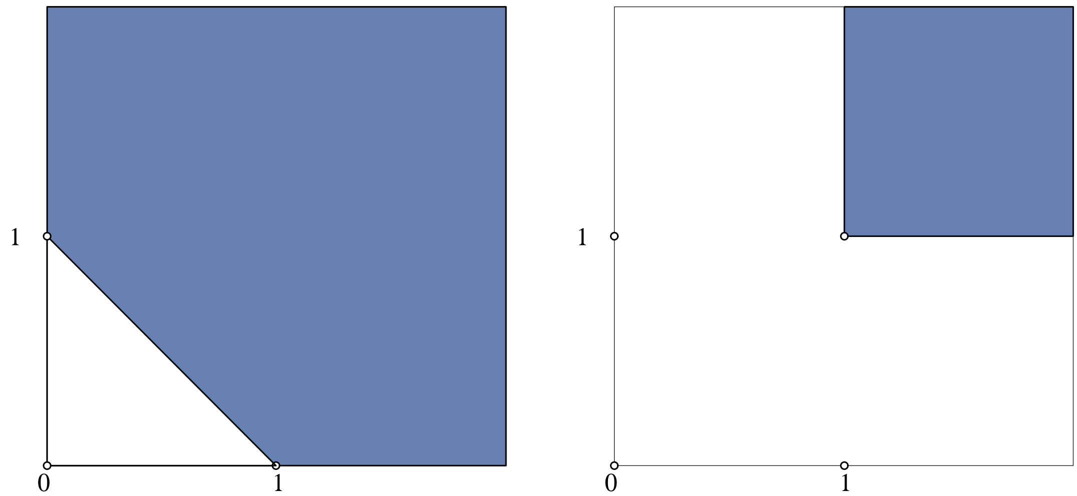

Fig. 1. The unit antiball $\mathcal{A}_1$ in $K = \mathbb{R}^2_+$ (left) and the unit antiball $\mathcal{A}_{-\infty}$ in $K = \mathbb{R}^2_+$ (right)

Definition 4.8 *Given a cone K, a* pre-antinorm $p(\cdot)$ *is a nontrivial (not identically zero) continuous function defined on K such that*

- *(b1)* $p(x) \geq 0$ *for all $x \in K$ (nonnegativity);*
- *(b1')* $p(x) > 0$ *for all $x \in \operatorname{int}(K)$ (weak positivity);*
- *(b2)* $p(\lambda x) = \lambda p(x)$ *for all $\lambda \geq 0$ and $x \in K$ (positive homogeneity).*

In other words, a pre-antinorm $p(\cdot)$ is an antinorm which is not necessarily superadditive. This means that, in general, it is not concave and that the set

$$\mathcal{P} := \{x \in K \mid p(x) \geq 1\},$$

which is the analogue of the unit antiball, is not necessarily convex. Also, continuity of $p(\cdot)$ in $\operatorname{int}(K)$ is not granted automatically by the hypotheses (b1), (b1') and (b2) and, a fortiori, it is not guaranteed on ∂K. Therefore, as it happens for antinorms, we can also consider pre-antinorms that are not continuous somewhere in ∂K and, in such a case, we talk of a *discontinuous pre-antinorm*.

Moreover, in line with Definition 4.3, we say that the pre-antinorm is *positive* if $p(x) > 0$ for all $x \in K \setminus \{0\}$. Note that, in [Mk2], the term "positive" is used instead for the weaker property (b1').

It is easy to see that a significant number of the previous definitions and results regarding antinorms, whose proofs do not involve property (b3), are valid for pre-antinorms, too. More precisely, this applies to:

- Proposition 4.2, where the unit antiball $\mathcal{A}$ has to be substituted by $\mathcal{P}$, even if the pre-antinorm $p(\cdot)$ is discontinuous;
- Proposition 4.4, where the unit antisphere $\mathcal{A}'$ has to be substituted by the set $\mathcal{P}' := \{x \in K \mid p(x) = 1\}$;

- Definition 4.4, where the dual functional

$$p^*(y) := \inf_{x \in \mathcal{P}'} y^T x$$

  is still an antinorm (and not only a pre-antinorm), even if the pre-antinorm $p(\cdot)$ is discontinuous;
- Proposition 4.5, Corollary 4.1, Lemma 4.2 and Proposition 4.7, even if the pre-antinorm $p(\cdot)$ is discontinuous;
- Definition 4.7;
- inequality (22) in Proposition 4.8, which shows a canonical way to transform a pre-antinorm into an antinorm, at least upper semi-continuous on ∂K.

However, we do not consider pre-antinorms any more in this paper.

4.6 Matrix antinorms

Given a cone K, now we consider the set $\mathcal{L}(K)$ of all the matrices $A \in \mathbb{R}^{d,d}$ for which K is invariant. As is easy to see, $\mathcal{L}(K)$ is itself a cone of the vector space $\mathbb{R}^{d,d}$ whose boundary $\partial\mathcal{L}(K)$ contains the identity matrix I. Moreover, $\mathrm{int}(\mathcal{L}(K))$ consists of those matrices for which K is strictly invariant.

Similarly to the well-known notion of operator norm of a matrix, we have the following definition.

Definition 4.9 *Let $a(\cdot)$ be an antinorm on a cone K and let $A \in \mathcal{L}(K)$. Then the number*

$$a(A) := \inf_{x \in \mathcal{A}'} a(Ax) \tag{25}$$

is called the (operator) antinorm *of A induced by the vector antinorm defined on K.*

In analogy to Proposition 4.6, we obtain the following result by a similar proof. However, this time we need to assume continuity of $a(\cdot)$ on the whole image $A(K)$ (which could possibly include ∂K or part of it).

Proposition 4.9 *Let $a(\cdot)$ be an antinorm on a cone K and let $A \in \mathcal{L}(K)$. Then it holds that*

$$a(A) = \inf_{x \in \mathrm{int}(K) \cap \mathcal{A}'} a(Ax).$$

Proof. We proceed in the same way we did to prove Proposition 4.6, except that now we use the continuity of the functional $a(A\,\cdot)$ (in place of the scalar product) in order to be sure that $a(Ax_k) \longrightarrow a(A\bar{x})$. □

Note that, if K is strictly invariant for A, then continuity on $A(K)$ is automatically assured even if $a(\cdot)$ is discontinuous on ∂K (see Definition 3.2 and

Remark 4.2).

Theorem 4.4 *Let $a(\cdot)$ be an antinorm on a cone K. Then the functional defined by* (25) *is an antinorm on the cone $\mathcal{L}(K)$, possibly discontinuous on $\partial\mathcal{L}(K)$. Moreover, for all matrices $A, B \in \mathcal{L}(K)$ it holds that*

$$a(AB) \geq a(A)a(B). \tag{26}$$

Remark that, similarly to the case of the dual antinorm $a^*(\cdot)$, continuity on $\partial\mathcal{L}(K)$ is not guaranteed. In fact, we have the following negative result.

Theorem 4.5 *Let $a(\cdot)$ be an antinorm on a cone K such that the dual $a^*(\cdot)$ is discontinuous at some point $\bar{y} \in \partial K^*$. Then, for any $z \in K$ such that $a(z) > 0$, the operator antinorm defined on $\mathcal{L}(K)$ is discontinuous at the rank-one matrix $z\bar{y}^T$, which belongs to $\partial\mathcal{L}(K)$.*

Proof. Choose $z \in K$ such that $a(z) > 0$ and, for each $y \in K^*$, consider the rank-one matrix zy^T. Clearly, we have that $zy^T(K) \subseteq \mathrm{span}(z) \cap K$, implying $zy^T \in \mathcal{L}(K)$. Now it holds that

$$a(zy^T) = \inf_{x \in \mathcal{A}'} a(zy^T x) = a(z) \inf_{x \in \mathcal{A}'} y^T x = a(z)a^*(y).$$

Therefore, since $a(z) > 0$, the assumed discontinuity of $a^*(\cdot)$ at $\bar{y}$ is inherited by the operator antinorm $a(\cdot)$ at the rank-one matrix $z\bar{y}^T$.

Finally, since $\bar{y} \in \partial K^*$, there exists $\bar{x} \in \partial K$, $\bar{x} \neq 0$, such that $\bar{y}^T\bar{x} = 0$. So $z\bar{y}^T\bar{x} = 0$, meaning that K is not strictly invariant for $z\bar{y}^T$, that is, $z\bar{y}^T \in \partial\mathcal{L}(K)$. □

Proposition 3.1 reveals that $A \in \mathcal{L}(K)$ if and only if $A^T \in \mathcal{L}(K^*)$. The following result holds.

Proposition 4.10 *Let $a(\cdot)$ be an antinorm on a cone K and let $a^*(\cdot)$ be the dual antinorm (defined on K^*). Then, for all $A \in \mathcal{L}(K)$ and $A^T \in \mathcal{L}(K^*)$, the corresponding operator antinorms are such that*

$$a^*(A^T) = a(A).$$

5 Antinorms, lower spectral radius and Gelfand's limit

In this section we first report some of the main relationships between the operator antinorms and the lower spectral radius of a family of matrices $\mathcal{F}$ that has a common invariant cone. Then we provide the main results of this paper, the most important of which is a Gelfand type limit under suitable assumptions.

Definition 5.1 *Let a family $\mathcal{F} = \{A_1, \ldots, A_m\}$ have a common invariant cone K. For any antinorm $a(\cdot)$ on K, we define the* antinorm of $\mathcal{F}$ *as*

$$a(\mathcal{F}) := \min_{1 \le i \le m} a(A_i).$$

The following result, which gives the lower bounds to the lower spectral radius $\check{\rho}(\mathcal{F})$ missing in (4), was proved by Guglielmi and Protasov [GP, Proposition 6]. Here, we express it by using the just introduced notion of matrix antinorm.

Proposition 5.1 *Let a family $\mathcal{F} = \{A_1, \ldots, A_m\}$ have a common invariant cone K and let $a(\cdot)$ be an antinorm defined on K. Then it holds that*

$$a(\mathcal{F}) \le \check{\rho}(\mathcal{F}). \tag{27}$$

In the particular case of a singleton family $\mathcal{F} = \{A\}$, inequality (27) reduces to

$$a(A) \le \rho(A)$$

(see also [Mk2]).

Definition 5.2 *An antinorm $a(\cdot)$ is said to be* extremal *for the family $\mathcal{F}$ if equality holds in (27), i.e., if*

$$a(\mathcal{F}) = \check{\rho}(\mathcal{F}).$$

In the particular case of a singleton family $\mathcal{F} = \{A\}$, an antinorm $a(\cdot)$ defined on K is extremal if

$$a(A) = \rho(A).$$

Guglielmi and Protasov [GP, Theorem 5] also proved the following existence result.

Theorem 5.1 *Let a family $\mathcal{F} = \{A_1, \ldots, A_m\}$ have a common invariant cone K. Then there exists an extremal antinorm $a(\cdot)$ on K for the family $\mathcal{F}$.*

With the terminology of Definition 5.2, the above theorem can be rephrased as follows.

Corollary 5.1 *Let a family $\mathcal{F}$ have a common invariant cone K. Then*

$$\check{\rho}(\mathcal{F}) = \max_{a(\cdot) \in \mathcal{A}nt(K)} a(\mathcal{F}),$$

where $\mathcal{A}nt(K)$ stands for the set of all antinorms defined on K.

Although the proof of Theorem 5.1 is of constructive type, it may be of little practical use.

With reference to Definition 3.4, now we give an example of extremal antinorm for a singleton family $\mathcal{F} = \{A\}$ that has an invariant cone K.

Proposition 5.2 *Let a matrix A be asymptotically rank-one and have an invariant cone K. Moreover, let the leading eigenvectors v_A and h_A of A and A^T, respectively, be such that $h_A^T v_A = 1$. Then the linear functional*

$$a(x) := h_A^T x$$

defines an extremal dual-polytope antinorm on K (see Definition 4.6).

Proof. Since $h_A \in K^*$, property (b1) of Definition 4.1 is guaranteed. Also continuity and properties (b2) and (b3) are obvious as $a(\cdot)$ is a linear functional. In order to prove also extremality, we consider $x \in \mathrm{int}(K)$ (this is not restrictive by Proposition 4.9) and rewrite it in the form

$$x = \gamma v_A + u,$$

where $u \in H_A$ and, by Theorem 3.1-(iii), $\gamma \neq 0$. Therefore, (7) yields

$$a\,(x) = \gamma + h_A^T u = \gamma \neq 0.$$

On the other hand, since H_A is invariant for A (see (a1)), again by (7) we also have that

$$a(Ax) = \gamma \lambda_A h_A^T v_A + h_A^T A u = \gamma \lambda_A = \rho(A) a(x),$$

concluding the proof. □

Now, given a cone K, invariant for a finite family of matrices $\mathcal{F}$, and an antinorm $a(\cdot)$ defined on K, for each $k \geq 1$ we introduce the quantity

$$\alpha_k(\mathcal{F}) := \min_{P \in \Sigma_k(\mathcal{F})} a(P)^{1/k}. \tag{28}$$

Note that $\alpha_1(\mathcal{F}) = a(\mathcal{F})$, the antinorm of $\mathcal{F}$ (see Definition 5.1).

Lemma 5.1 *For all $m, n \geq 1$ it holds that*

$$\alpha_{m+n}(\mathcal{F})^{m+n} \geq \alpha_m(\mathcal{F})^m \alpha_n(\mathcal{F})^n. \tag{29}$$

Proof. Let $P = A_{i_1} \cdots A_{i_m} A_{i_{m+1}} \cdots A_{i_{m+n}} \in \Sigma_{m+n}(\mathcal{F})$. By (26) and (28), we have that

$$a(P) \geq a(A_{i_1} \cdots A_{i_m}) a(A_{i_{m+1}} \cdots A_{i_{m+n}}) \geq \alpha_m(\mathcal{F})^m \alpha_n(\mathcal{F})^n$$

and, thus, the arbitrariness of P concludes the proof. $\square$

Now we prove a preliminary result, whose first part follows from and includes Proposition 5.1.

Theorem 5.2 *Let the cone K be invariant for a finite family of matrices $\mathcal{F} = \{A_1, \ldots, A_m\}$ and let $a(\cdot)$ be an antinorm on K. Then it holds that*

$$\alpha_k(\mathcal{F}) \leq \check{\rho}\,(\mathcal{F}), \qquad k \geq 1, \tag{30}$$

and, if either $\check{\rho}\,(\mathcal{F}) = 0$ or $\alpha_1(\mathcal{F}) = a(\mathcal{F}) > 0$, there exists

$$\alpha(\mathcal{F}) := \lim_{k \to \infty} \alpha_k(\mathcal{F}) = \sup_{k \geq 1} \alpha_k(\mathcal{F}). \tag{31}$$

Proof. For any $k \geq 1$, we consider the family of matrices $\mathcal{G}_k := \Sigma_k(\mathcal{F})$. It is immediate to see that $\check{\rho}_n(\mathcal{G}_k) = \check{\rho}_{nk}(\mathcal{F})^k$ and, consequently, by using (2) applied to both $\mathcal{G}_k$ and $\mathcal{F}$, we get

$$\check{\rho}(\mathcal{G}_k) = \lim_{n \to \infty} \check{\rho}_n(\mathcal{G}_k) = \lim_{n \to \infty} \check{\rho}_{nk}(\mathcal{F})^k = \lim_{m \to \infty} \check{\rho}_m(\mathcal{F})^k = \check{\rho}(\mathcal{F})^k.$$

On the other hand, taking (28) into account, Proposition 5.1 applied to the family $\mathcal{G}_k$ leads to

$$\alpha_k(\mathcal{F})^k \leq \check{\rho}(\mathcal{G}_k),$$

concluding the proof of (30).

Now, if $\check{\rho}\,(\mathcal{F}) = 0$, inequality (30) clearly implies (31) with $\alpha(\mathcal{F}) = 0$.

Otherwise, if $\check{\rho}\,(\mathcal{F}) > 0$, we assume that $\alpha_1(\mathcal{F}) = a(\mathcal{F}) > 0$. Therefore, Lemma 5.1 implies that $\alpha_k(\mathcal{F}) > 0$ for all $k \geq 1$ and, consequently, we can apply the log function to both sides of (29) and get

$$(m+n) \log(\alpha_{m+n}(\mathcal{F})) \geq m \log(\alpha_m(\mathcal{F})) + n \log(\alpha_n(\mathcal{F})).$$

Then we apply the "reverse" (i.e., superadditive) Fekete lemma (see Fekete [Fe]) and obtain the existence of

$$\lim_{k \to \infty} \log(\alpha_k(\mathcal{F})) = \sup_{k \geq 1} \log(\alpha_k(\mathcal{F})).$$

Hence exponentiation yields (31). $\square$

Remark 5.1 Note that, as a corollary of (30), we get

$$\alpha(\mathcal{F}) \leq \check{\rho}\,(\mathcal{F})\,. \tag{32}$$

Furthermore, (31) represents the analogue of (2) for antinorms. However, differently from the case of norms, the limit $\alpha(\mathcal{F})$ is not independent of the particular antinorm $a(\cdot)$ and the above inequality might well be strict (see, e.g., the examples given at the end of this section).

In order to find sufficient conditions for (32) being an equality, we consider the *product semigroup*

$$\Sigma(\mathcal{F}) := \bigcup_{k\geq 1} \Sigma_k(\mathcal{F})$$

and, keeping in mind Definition 3.4, we recall the following definition from [BZ3].

Definition 5.3 *We say that a family of matrices $\mathcal{F}$ is* asymptotically rank-one *if all products $P \in \Sigma(\mathcal{F})$ are so.*

Remark that, in general, all the matrices A_i, $i = 1, \ldots, m$, being asymptotically rank-one does not guarantee that all products $P \in \Sigma(\mathcal{F})$ are so and that a family $\mathcal{F}$ is asymptotically rank-one if and only if the transpose family $\mathcal{F}^T$ is so.

For an asymptotically rank-one family $\mathcal{F}$, we then define the *leading set* as

$$\mathcal{V}(\mathcal{F}) = \bigcup_{P\in\Sigma(\mathcal{F})} V_P$$

and the *secondary set* as

$$\mathcal{H}(\mathcal{F}) = \bigcup_{P\in\Sigma(\mathcal{F})} H_P.$$

Note that both $\mathcal{V}(\mathcal{F})$ and $\mathcal{H}(\mathcal{F})$ are homogeneous and symmetric sets. Furthermore, if $\mathcal{F}$ has a common invariant cone K, by Theorem 3.1 it turns out that

$$\mathrm{clos}(\mathcal{V}(\mathcal{F})) \subseteq K \cup -K \qquad \text{and} \qquad \mathrm{clos}(\mathcal{H}(\mathcal{F})) \cap \mathrm{int}(K) = \emptyset$$

(see again [BZ3]).

Theorem 5.3 *Let $\mathcal{F}$ be an asymptotically rank-one family of matrices, let K be an invariant cone for $\mathcal{F}$ such that*

$$K \cap \mathrm{clos}\,(\mathcal{H}(\mathcal{F})) = \{0\} \tag{33}$$

and let $a(\cdot)$ be a positive antinorm on K. Then the Gelfand limit holds, that is,

$$\alpha(\mathcal{F}) := \lim_{k\to\infty} \alpha_k(\mathcal{F}) = \check{\rho}\,(\mathcal{F})\,.$$

Proof. Since $a(\cdot)$ is positive, Proposition 4.4-(iii) implies that, for all $P \in \Sigma(\mathcal{F})$, the infimum in (25) is attained by some $x \in \mathcal{A}'$. So, for each $k \geq 1$, let

$Q_k \in \Sigma_k(\mathcal{F})$ be such that

$$a\left(Q_k\right)^{1/k} = \alpha_k(\mathcal{F})$$

and let $x_k \in \mathcal{A}'$ (i.e., $a(x_k) = 1$) such that

$$a\left(Q_k\right) = a\left(Q_k x_k\right).$$

Furthermore, let v_k be the leading eigenvector of Q_k such that $a(v_k) = 1$ (note that $v_k \in K$, K being invariant for Q_k). It is clear that, for all $k \geq 1$, there exists $\delta_k > 0$ and $u_k \in H_{Q_k}$ such that

$$x_k = \delta_k v_k + u_k. \tag{34}$$

Now the proof proceeds by contradiction. Assume that there exists a vanishing subsequence $\{\delta_{k_n}\}$. Then, by Proposition 4.4-(ii), it would follow that

$$v_{k_n} - u_{k_n} = \delta_{k_n} w_{k_n} \longrightarrow 0$$

and, hence, using (33) and the fact that $-u_{k_n} \in \operatorname{clos}\left(\mathcal{H}(\mathcal{F})\right)$, we would infer that $v_{k_n} \longrightarrow 0$ (and $u_{k_n} \longrightarrow 0$). On the other hand, we also have that $a(v_{k_n}) = 1 \leq \beta \|v_{k_n}\|$ for some constant $\beta > 0$ (see Proposition 4.3), which is an absurd.

As a consequence, the sequence $\{\delta_k\}$ must be uniformly positive, that is, there exists $\delta_0 > 0$ such that

$$\delta_k \geq \delta_0 \qquad \forall k \geq 1. \tag{35}$$

Now observe that equality (34) yields

$$Q_k x_k = \delta_k Q_k v_k + Q_k u_k = \delta_k \rho(Q_k) v_k + Q_k u_k$$

and that, the family $\mathcal{F}$ being asymptotically rank-one, we have $\rho\left(P\right) > 0$ for all $P \in \Sigma(\mathcal{F})$. Therefore, we can write

$$a\left(Q_k\right) = a\left(Q_k x_k\right) = \rho\left(Q_k\right) a\left(\delta_k v_k + \rho\left(Q_k\right)^{-1} Q_k u_k\right). \tag{36}$$

Then we proceed again by contradiction and suppose that there exists a vanishing subsequence $\left\{a\left(\delta_{k_n} w_{k_n} + \rho\left(Q_{k_n}\right)^{-1} Q_{k_n} u_{k_n}\right)\right\}$. Then (11) would imply

$$\delta_{k_n} w_{k_n} + \rho\left(Q_{k_n}\right)^{-1} Q_{k_n} u_{k_n} \longrightarrow 0$$

and thus, since $w_{k_n} \in K$ and $Q_{k_n} u_{k_n} \in H_{Q_{k_n}} \subseteq \operatorname{clos}\left(\mathcal{H}(\mathcal{F})\right)$, like before, we would obtain $\delta_{k_n} w_{k_n} \longrightarrow 0$ (and $\rho\left(Q_{k_n}\right)^{-1} Q_{k_n} u_{k_n} \longrightarrow 0$). On the other hand, this is not possible since, as before, because of (35) it contradicts the fact that $a\left(w_{k_n}\right) = 1 \leq \beta \|w_{k_n}\|$ for some constant $\beta > 0$ (see Proposition 4.3).

Consequently, we conclude with the existence of $C > 0$ such that

$$a\left(\delta_k v_k + \rho\left(Q_k\right)^{-1} Q_k u_k\right) \geq C \qquad \forall k \geq 1$$

so that, in turn, (3) and (36) yield

$$\alpha_k(\mathcal{F}) \geq \rho\left(Q_k\right)^{1/k} C^{1/k} \geq \bar{\rho}_k(\mathcal{F}) C^{1/k}.$$

Finally, letting $k \longrightarrow \infty$, by (5) we get

$$\liminf_{k\to\infty} \alpha_k(\mathcal{F}) \geq \check{\rho}\left(\mathcal{F}\right)$$

which, together with (30), concludes the proof. □

Corollary 5.2 *Let A be an asymptotically rank-one matrix, K be an invariant cone for A such that*

$$K \cap H_A = \{0\}$$

and $a(\cdot)$ be a positive antinorm on K. Then

$$\lim_{k\to\infty} a\left(A^k\right)^{1/k} = \rho(A).$$

5.1 Some counterexamples

In this section we show that the hypotheses of Theorem 5.3 are sharp by giving two suitable simple counterexamples for singleton families.

Necessity of condition (33)

Let $K = \mathbb{R}^2_+$ and consider the positive antinorm defined by

$$a_1(x) := x_1 + x_2 \qquad \forall x = [x_1, x_2]^T \in \mathbb{R}^2_+$$

(see Figure 1 (left) in Section 4.4) and the asymptotically rank-one matrix

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Denoting the canonical basis vectors by $e^{(1)}$ and $e^{(2)}$, we have $Ae^{(1)} = 2e^{(1)}$ and $Ae^{(2)} = e^{(2)}$. Clearly, K is invariant for A but, since $e^{(2)} \in K$, it turns out that $K \cap H_A \neq \{0\}$, i.e., (33) does not hold. Furthermore, it is easy to see that

$$a_1\left(A^k\right) = \inf_{a_1(x)=1} a_1\left(A^k x\right) = \min_{1\leq j\leq 2} \sum_{i=1}^{2} \left(A^k\right)_{ij} = 1 \qquad \forall k \geq 1,$$

from which we get

$$\lim_{k\to\infty} a_1\left(A^k\right)^{1/k} = 1 < \rho(A) = 2.$$

Necessity of positivity

Let $K = \mathbb{R}^2_+$ and consider the antinorm defined by

$$a_{-\infty}(x) := \min\{x_1, x_2\} \qquad \forall x = [x_1, x_2]^T \in \mathbb{R}^2_+$$

(see Figure 1 (right) in Section 4.4). Since

$$a_{-\infty}(x) = 0 \qquad \forall x \in \partial K = h_1 \cup h_2, \tag{37}$$

where $h_1 = \{x \in K \mid x_2 = 0\}$ and $h_2 = \{x \in K \mid x_1 = 0\}$, positivity does not hold. Then consider the asymptotically rank-one matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$$

which is such that $Ae^{(1)} = e^{(1)}$ and $A \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0$. Clearly, K is invariant for A and $K \cap H_A = \{0\}$, i.e., (33) holds true. On the other hand, since $A^k(K) = h_1$ for all $k \geq 1$, by (37) we obtain

$$a_{-\infty}\left(A^k\right) = \inf_{a_{-\infty}(x)=1} a_{-\infty}\left(A^k x\right) = 0$$

and, consequently,

$$\lim_{k\to\infty} a_{-\infty}\left(A^k\right)^{1/k} = 0 < \rho(A) = 1.$$

6 An example of discontinuous dual antinorm

In this last section, we construct a three-dimensional example of dual antinorm $a^*(\cdot)$ that is discontinuous somewhere on the boundary of the dual cone K^*.

To this purpose, with reference to Theorem 4.3, we consider the cone C of $\mathbb{R}^3$ defined by

$$\begin{aligned} C := \{(y_1, y_2, y_3) \in \mathbb{R}^3 \mid\ & y_3 \geq 0,\ -y_3 \leq y_1 \leq y_3,\ y_2 \leq y_3, \\ & -y_1^2 - 2y_1y_3 + 2\sqrt{3}y_2y_3 + (2\sqrt{3} - 1)y_3^2 \geq 0\}, \end{aligned}$$

whose intersection with the plane of equation $y_3 = 1$ is given by

$$S = \{(y_1, y_2, 1) \in \mathbb{R}^3 \mid -1 \leq y_1 \leq 1,\ y_2 \leq 1,$$
$$-y_1^2 - 2y_1 + 2\sqrt{3}y_2 + (2\sqrt{3} - 1) \geq 0\}$$

and is illustrated by Figure 2.

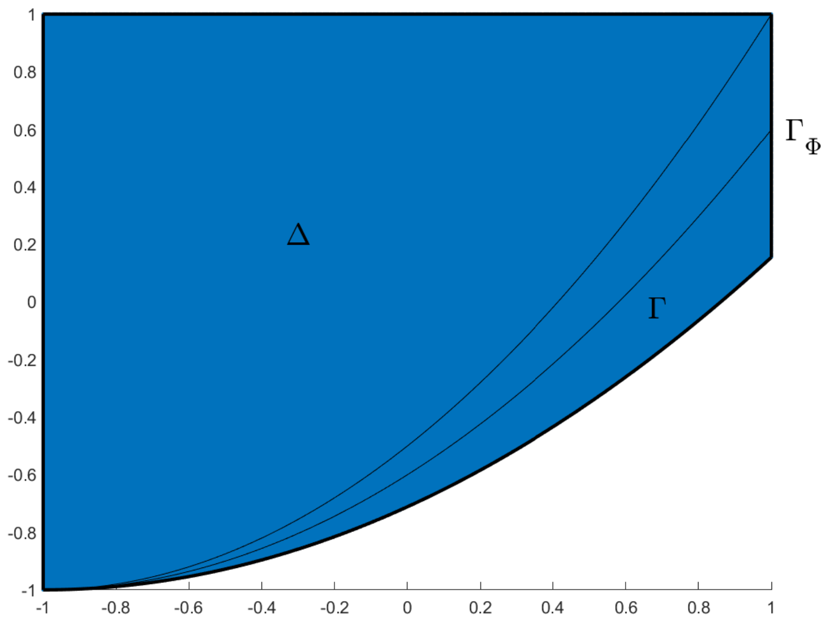

Fig. 2. The intersection S of the cone C with the plane of equation $y_3 = 1$

Note that the piece of ∂S of equation $-y_1^2 - 2y_1 + 2\sqrt{3}y_2 + (2\sqrt{3} - 1) = 0$ is a parabolic arc.

Then we define the following function on S:

$$\tilde{f}(y_1, y_2) := \begin{cases} g(y_1, y_2) & \text{if } (y_1, y_2) \in \Gamma, \\ 1 & \text{if } (y_1, y_2) \in \Delta, \end{cases}$$

where

$$\Gamma := \{(y_1, y_2, 1) \in S \mid -1 < y_1 \leq 1,\ \frac{1}{2\sqrt{3}}(y_1 + 1)^2 - 1 \leq y_2 \leq \frac{1}{2}(y_1 + 1)^2 - 1\},$$

$$\Delta := \{(y_1, y_2, 1) \in S \mid -1 \leq y_1 \leq 1,\ \frac{1}{2}(y_1 + 1)^2 - 1 \leq y_2 \leq 1\}$$

and

$$g(y_1, y_2) := \frac{4(y_1 + 1)^2(y_2 + 1)}{(y_1 + 1)^4 + 4(y_2 + 1)^2},$$

and then we prolong it by positive homogeneity to the entire cone C by setting

$$f(y_1, y_2, y_3) := \begin{cases} 0 & \text{if } y_3 = 0, \\ y_3 \tilde{f}(\frac{y_1}{y_3}, \frac{y_2}{y_3}) & \text{if } y_3 > 0. \end{cases}$$

It turns out that

$$\Gamma = \bigcup_{\frac{1}{\sqrt{3}} \le \phi \le 1} \Gamma_\phi,$$

where $\Gamma_\phi := \{(y_1, y_2, 1) \in S \mid -1 \le y_1 \le 1,\ y_2 = \frac{\phi}{2}(y_1+1)^2 - 1\}$ is a parabolic arc on which $\tilde{f}(\cdot)$ attains the constant value $\frac{2\phi}{\phi^2+1} \in [\frac{\sqrt{3}}{2}, 1]$ (see again Figure 2).

This fact implies that the function $\tilde{f}(\cdot)$ is not continuous, but only upper semi-continuous, at the point $(-1, -1) \in \partial S$. Indeed, it holds that

$$\liminf_{(y_1,y_2)\to(-1,-1)} \tilde{f}(y_1, y_2) = \frac{\sqrt{3}}{2} < 1$$

and

$$\limsup_{(y_1,y_2)\to(-1,-1)} \tilde{f}(y_1, y_2) = \tilde{f}(-1, -1) = 1. \tag{38}$$

On the other hand, $\tilde{f}(\cdot)$ is clearly continuous at any other point of its domain.

Now, some tedious calculations reveal that the Hessian matrix $\frac{\partial^2 \tilde{f}}{\partial(y_1,y_2)^2}$ is negative semi-definite in $\Gamma \setminus \{(-1, -1)\}$. Therefore, in the light of (38) and since $\tilde{f}(\cdot)$ attains the constant value 1 on Δ, it is easy to conclude that it is concave on S.

Summarizing, we have defined a function $f(\cdot)$ on the cone C that satisfies the properties of an antinorm, except that it is discontinuous on the half-line $\gamma := \mathrm{span}(\{\bar{y}\}) \cap C$, $\bar{y} := (-1, -1, 1) \in \partial C$, where it is just upper semi-continuous.

Moreover, since $f(y_1, y_2, 1) = \tilde{f}(y_1, y_2) \ge \frac{\sqrt{3}}{2}$ on S, it is also clear that the unit antisphere $\mathcal{U}'$ of $f(\cdot)$ is bounded. More precisely, we have that

$$(y_1, y_2, y_3) \in \mathcal{U}' \quad \Longrightarrow \quad 1 \le y_3 \le \frac{2}{\sqrt{3}}.$$

In conclusion, all the hypotheses of Theorem 4.3 are satisfied.

Remark 6.1 According to what was anticipated in Remark 4.8, the set $\mathcal{U}'_{int} :=$

$\mathcal{U}' \cap \text{int}(C)$ is such that

$$\text{clos}(\mathcal{U}'_{int}) \cap \gamma = \{\psi\bar{y} \;\Big|\; 1 \leq \psi \leq \frac{2}{\sqrt{3}}\} \quad \text{and} \quad f(\psi\bar{y}) = \psi.$$

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