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DOTTORATO DI RICERCA IN FISICA
XXXIII CICLO

THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

Revisiting Gravity and Cosmology through Quantum continuous measurements

Settore scientifico-disciplinare FIS/02

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Elena Garro
Los recuerdos del porvenir (1963)

Y quién sabe qué más sucederá, porque ¿dónde termina lo posible, cuando empezamos a vivir cosas que creíamos imposibles?

Luis Pescetti
Frin (1999)

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Introduction

This thesis explores possible applications of Quantum continuous measurements in Gravitation and Cosmology.

In the first part of the thesis, we focus on the study of models implementing a hybrid classical-quantum dynamics through the addition of a classical noise. For this purpose, we review the basic formalism of the theory of quantum continuous measurements. In particular, we focus on the stochastic Schrödinger equation that describes the evolution of a system subjected to a continuous measurement of a given observable, and on the Markovian feedback stochastic master equation introduced by Wiseman and Milburn [1].

We review two models implementing Newtonian gravity through a continuous measurement and feedback mechanism. These are the Kafri-Taylor-Milburn (KTM) [2] and the Tilloy-Diósi (TD) [3] models. We study in detail the regularization mechanism proposed in the original work of the TD model, and discuss the conditions that lead to its implementation. We compare the KTM and the TD models in an appropriate limit and explicitly show that the associated decoherence effects are different. Moreover, by analysing the conceptual differences between the two models, we propose a classification of models which implement gravity through a continuous measurement and feedback mechanism based on two main criteria: the type of observable measured and the type of interaction implemented through the feedback. We argue that within these scenarios, the TD model is the only framework implementing a full Newtonian gravitational interaction physically consistent.

In the second part of this thesis, we study how dynamical collapse models have been implemented within a cosmological inflationary context. Dynamical collapse models are phenomenological models that introduce nonlinear and stochastic modifications to the Schrödinger equation. The master equations of collapse models can be understood in terms of quantum continuous measurements of an observable which acts as a collapse operator. We provide a brief review of the main motivations that led to the proposal of dynamical models as an alternative theory to standard Quantum Mechanics, focusing on the so-called measurement problem. In particular, we describe the main properties of the Continuous Spontaneous Localization (CSL) model [4].

We briefly review previous works that analyse the effects of collapse models on scalar cosmological perturbations during inflation. Up to present date, there is not a consistent generalization of the CSL model to the relativistic regime. We discuss relevant aspects for such a generalization, as the choice of the collapse operator and its properties. We propose an approach to treat the dynamical collapse model contribution to the evolution of a system as a perturbation to the standard

cosmological scenario. We analyse the modifications to the power spectrum associated to scalar cosmological perturbations during inflation and the radiation-dominated era.

Chapter 1

Quantum continuous measurements and feedback

Among the frameworks which have been proposed as alternatives to the quantization of gravity, one can find those models implementing the gravitational interaction through a continuous measurement and feedback mechanism. In this chapter, we will review the main properties characterizing the continuous measurement of an observable, and one of the simplest protocols to implement the feedback, namely, the Wiseman-Milburn feedback. This discussion follows mainly from Refs. [5–7].

1.1 Gaussian quantum continuous measurements

In Quantum Mechanics, it is usually assumed that measurements take place instantaneously. However, in some situations, we are interested in describing measurement processes taking an appreciable time, and in which one continuously extracts information from a system. These kind of measurements are referred to as continuous measurements [5, 7].

We focus on Gaussian measurements of a continuous observable $\hat{a}$, with associated eigenstates $\{|a\rangle\}$ that satisfy the eigenvalue equation $\hat{a}\,|a\rangle = a\,|a\rangle$. We will now construct a measurement record which contains information about the measurement of $\hat{a}$. Let us proceed by dividing the time in which the measurement takes place into intervals of length Δt, which are assumed to be infinitesimal. For each of these infinitesimal intervals, one can construct a parametrized sum of projectors onto the eigenstates of the quantum operator $\hat{a}$ in the following way:

$$\hat{A}(r) = \left(\frac{\gamma\Delta t}{2\pi\hbar^2}\right)\int_{-\infty}^{\infty} \mathrm{d}a\, \exp\left[-\frac{\gamma\Delta t}{4\hbar^2}(\hat{a}-r)^2\right]|a\rangle\langle a|\,, \tag{1.1}$$

where the parameter γ encodes the strength of the measurement. The operator of Eq (1.1) constitutes a Gaussian-weighted sum of projections onto the eigenstates of the operator $\hat{a}$ [7]. The continuous index r labels the continuum of measurement results.

From the operator in Eq. (1.1), one can construct the probability density $P(r)$ of the measurement result r as

$$P(r) = \mathrm{Tr}\left[\hat{A}^{\dagger}(r)\hat{A}(r)\,|\psi\rangle\langle\psi|\right], \tag{1.2}$$

where $|\psi\rangle$ denotes the initial state of the system. The operator $\hat{A}^{\dagger}(r)\hat{A}(r)$ is the positive operator associated with the result r. Therefore, the map in Eq. (1.2) associates a positive operator with every result r, and thus constitutes a positive-operator valued measure (POVM) [5, 8]. Writing $|\psi\rangle$ in the basis of the operator $\hat{a}$ as $|\psi\rangle = \int \mathrm{d}a\, \psi(a)\, |a\rangle$, one can relate the mean and the variance of the parameter r with the mean and the variance of the quantum operator $\hat{a}$. Indeed, it can be shown that the mean value $\langle r\rangle$ of r is given by

$$\langle r\rangle = \int_{-\infty}^{\infty} rP(r)\mathrm{d}r = \int_{-\infty}^{\infty} a|\psi(a)|^2\mathrm{d}a = \langle\hat{a}\rangle\,, \tag{1.3}$$

and the variance σ_r^2 reads

$$\sigma_r^2 = \langle r^2\rangle - \langle r\rangle^2 = \sigma_{\hat{a}}^2 + \frac{\hbar^2}{\gamma\Delta t}. \tag{1.4}$$

By assuming that the intervals Δt are infinitesimal, the probability density $P(r)$ can be approximated as

$$P(r) \approx \frac{1}{\hbar}\sqrt{\frac{\gamma\Delta t}{2\pi}}\exp\left[-\frac{\gamma\Delta t}{2\hbar^2}(r - \langle\hat{a}\rangle)^2\right]. \tag{1.5}$$

From the results of Eq. (1.3), Eq. (1.4) and Eq. (1.5), the measurement result r can be understood as a stochastic quantity

$$r = \langle\hat{a}\rangle + \frac{\hbar}{\sqrt{\gamma}}\frac{\Delta W_t}{\Delta t}, \tag{1.6}$$

where ΔW_t is a Gaussian random variable with vanishing mean and variance Δt.

We are now interested in arriving at a stochastic differential equation which describes how the quantum state of the system $|\psi\rangle$ at a time t changes when performing a continuous measurement of the observable $\hat{a}$. The stochastic equation for $|\psi\rangle$ can be obtained by applying the operator of Eq. (1.1) to the state $|\psi\rangle$, and taking the limit in which $\Delta t \to 0^+$. The dynamical equation for the state $|\psi\rangle$ that preserves the norm is given by

$$\mathrm{d}\,|\psi\rangle = \left\{-\frac{\gamma}{8\hbar^2}(\hat{a} - \langle\hat{a}\rangle)^2\mathrm{d}t + \frac{\sqrt{\gamma}}{2\hbar}(\hat{a} - \langle\hat{a}\rangle)\mathrm{d}W_t\right\}|\psi\rangle\,, \tag{1.7}$$

where $\mathrm{d}W_t = \lim\limits_{\Delta t\to 0^+}\Delta W_t$. The above equation for the continuous measurement of the observable $\hat{a}$ is valid in the Itô sense.

The dynamical evolution of Eq. (1.7) is a stochastic Schrödinger equation [5]. The appearance of the expectation value $\langle\hat{a}\rangle = \langle\psi|\,\hat{a}\,|\psi\rangle$ in Eq. (1.7) implies that the dynamical evolution is nonlinear in the state $|\psi\rangle$. This is in contrast with the standard Schrödinger equation, which is linear, and therefore implies that the addition of measurements to the dynamics of a quantum system leads to a nonlinear evolution.

The corresponding stochastic master equation for the stochastic density operator $\hat{\rho}_s = |\psi\rangle \langle\psi|$ is given by

$$\mathrm{d}\hat{\rho}_s = -\frac{\gamma}{8\hbar^2}\left[\hat{a},\left[\hat{a},\hat{\rho}_s\right]\right]\mathrm{d}t + \frac{\sqrt{\gamma}}{2\hbar}\left(\{\hat{a},\hat{\rho}_s\} - 2\langle\hat{a}\rangle\hat{\rho}_s\right)\mathrm{d}W_t, \tag{1.8}$$

which explicitly depends on the value that $\mathrm{d}W_t$ takes at each time-step [5]. The evolution of $\hat{\rho}_s$ over the corresponding time period is called a quantum trajectory [5], and depends on the particular realization of the noise, which can be defined as a given sequence of sample values of $\mathrm{d}W_t$ for a given time period. By taking the stochastic average $\mathbb{E}[\cdot]$ over all the realizations, one defines the density operator to be given by $\hat{\rho} = \mathbb{E}\left[|\psi\rangle\langle\psi|\right]$. From Eq. (1.8), we thus obtain the master equation

$$\mathrm{d}\hat{\rho} = -\frac{\gamma}{8\hbar^2}\left[\hat{a},\left[\hat{a},\hat{\rho}\right]\right]\mathrm{d}t. \tag{1.9}$$

We can interpret the parameter γ as the rate at which the information is extracted by the measurement.

Thus far, we have only considered the case in which the system is only subjected to the continuous measurement of $\hat{a}$, and there is not an additional evolution while being measured. If we also consider an additional evolution due to a Hamiltonian $\hat{H}$, the stochastic master equation for the density operator is obtained by simply adding the standard evolution due to the Hamiltonian and the contribution due to the continuous measurement. Therefore, we obtain

$$\mathrm{d}\hat{\rho}_s = -\frac{i}{\hbar}\left[\hat{H},\hat{\rho}_s\right]\mathrm{d}t - \frac{\gamma}{8\hbar^2}\left[\hat{a},\left[\hat{a},\hat{\rho}_s\right]\right]\mathrm{d}t + \frac{\sqrt{\gamma}}{2\hbar}\left(\{\hat{a},\hat{\rho}_s\} - 2\langle\hat{a}\rangle\hat{\rho}_s\right)\mathrm{d}W_t. \tag{1.10}$$

The fact that Eq. (1.10) holds independently of whether the operators $\hat{H}$ and $\hat{a}$ commute is a consequence of the fact that Eq. (1.10) is a first-order equation in $\mathrm{d}t$ [5].

A similar argument allows to consider the continuous measurement of more than one observable, and to add each of the contributions to the dynamical evolution of the system. Indeed, let us consider a family of observables $\{\hat{a}_\lambda\}_{\lambda=1}^{\mathcal{M}}$, where each of the measurement results is given as in the continuous limit of Eq. (1.6), i.e.

$$r_\lambda = \langle\hat{a}_\lambda\rangle + \frac{\hbar}{\sqrt{\gamma_\lambda}}\frac{\mathrm{d}W_{\lambda,t}}{\mathrm{d}t}, \tag{1.11}$$

where one assumes that $\mathrm{d}W_{\lambda,t}$ and $\mathrm{d}W_{\lambda',t}$ are independent Wiener noises. At the level of the wave function, the dynamical evolution of the system due to the continuous measurements of the observables $\{\hat{a}_\lambda\}_{\lambda=1}^{\mathcal{M}}$ is given by

$$(\mathrm{d}|\psi\rangle)_{\mathrm{m}} = \sum_{\lambda=1}^{\mathcal{M}}\left\{-\frac{\gamma_\lambda}{8\hbar^2}\left(\hat{a}_\lambda - \langle\hat{a}_\lambda\rangle\right)^2\mathrm{d}t + \frac{\sqrt{\gamma_\lambda}}{2\hbar}(\hat{a}_\lambda - \langle\hat{a}_\lambda\rangle)\mathrm{d}W_{\lambda,t}\right\}|\psi\rangle\,, \tag{1.12}$$

where the label m in the left-hand side in the equation indicates the measurement contribution. We notice that the result in Eq. (1.12) holds independently of whether the operators $\{\hat{a}_\lambda\}_{\lambda=1}^{\mathcal{M}}$ commute or not. This is because at an infinitesimal level, any contribution coming from the commutator of any two operators $\hat{a}_\lambda$ and $\hat{a}_{\lambda'}$ is of second order in dt. Therefore, it does not contribute to Eq. (1.12), which is of first order in dt. The corresponding stochastic master equation reads

$$\mathrm{d}\hat{\rho}_s = -\frac{i}{\hbar}\left[\hat{H}, \hat{\rho}_s\right]\mathrm{d}t - \sum_{\lambda=1}^{\mathcal{M}} \frac{\gamma_\lambda}{8\hbar^2}\left[\hat{a}_\lambda, \left[\hat{a}_\lambda, \hat{\rho}_s\right]\right]\mathrm{d}t + \sum_{\lambda=1}^{\mathcal{M}} \frac{\sqrt{\gamma_\lambda}}{2\hbar}\left(\{\hat{a}_\lambda, \hat{\rho}_s\} - 2\langle\hat{a}_\lambda\rangle\hat{\rho}_s\right)\mathrm{d}W_{\lambda,t}, \tag{1.13}$$

which is a straightforward generalization of Eq. (1.10).

1.2 Quantum feedback

The quantum continuous measurement theory described in the previous section is essential to understand the concept of quantum feedback control. Let us consider a detector continuously measuring the system and correspondingly producing an output. Such an output can be exploited to control the system at later times. This process is called feedback control [6]. The system to be controlled is referred to be the primary system, whereas the system that acts as a feedback controller is denominated the auxiliary system [5]. If at least one of the systems is quantum, then one refers to quantum feedback control.

One of the ways to implement feedback control is through the use of continuous measurements. The idea is to take the continuum limit of the process in which one performs a sequence of measurements on a system, and in response to the result obtained from each of the measurements performed, one modifies the Hamiltonian. Therefore, in the continuum limit, the sequence of measurements turns into a continuous measurement, and we obtain also a continuous modification of the Hamiltonian of the system.

A single cycle of the feedback control process can be described by a continuous measurement performed over an infinitesimal time interval dt, followed by an unitary operation. If $\hat{\rho}(t)$ is the density operator at a given time t, then for a time $t + \mathrm{d}t$, we have [5]

$$\hat{\rho}_\lambda(t+\mathrm{d}t) = \frac{\hat{U}_\lambda \hat{A}_\lambda \hat{\rho}(t) \hat{A}_\lambda^\dagger \hat{U}_\lambda^\dagger}{\mathrm{tr}[\hat{A}_\lambda^\dagger \hat{A}_\lambda]}, \tag{1.14}$$

where the index λ labels the measurement result, the set $\left\{\hat{A}_\lambda\right\}$ are the measurement operators for a continuous measurement over dt, and $\hat{U}_\lambda = \exp\left(-\frac{i}{\hbar}\hat{H}_\lambda \mathrm{d}t/\hbar\right)$, where $\hat{H}_\lambda$ is the control Hamiltonian. When implementing the feedback using continuous measurements, the Hamiltonian $\hat{H}_{\mathrm{fb}}$ becomes a functional of the measurement record [5]. Usually, we have $\hat{H}_{\mathrm{fb}} = \sum_n \mu_n \hat{H}_n$, where some among the $\hat{H}_n$ operators can be modified by the controller. The parameters μ_n, which are real quantities, are called control parameters.

The corresponding stochastic master equation for the system, which includes feedback from a continuous measurement of an operator $\hat{a}$, is

$$\mathrm{d}\hat{\rho}_s = -\frac{i}{\hbar}\left[\hat{H}_0 + \sum_n \mu_n(\hat{\rho}_s)\hat{H}_n, \hat{\rho}_s\right]\mathrm{d}t - \frac{\gamma}{8\hbar^2}[\hat{a},[\hat{a},\hat{\rho}_s]]\,\mathrm{d}t + \frac{\sqrt{\gamma}}{2\hbar}\left(\{\hat{a},\hat{\rho}_s\} - 2\langle\hat{a}\rangle\hat{\rho}_s\right)\mathrm{d}W_t, \tag{1.15}$$

where we also included the action of the free Hamiltonian $\hat{H}_0$.

As the amount of computational resources required to process the measurement output and to update the density operator $\hat{\rho}$ at each time-step of the dynamical evolution may be prohibitive, there is an interest in developing feedback protocols where a complete knowledge of the density operator $\hat{\rho}$ is not required [5]. Among the proposed protocols, Wiseman and Milburn introduced a quantum treatment which consists in feeding the measurement record directly back to control the system, and therefore, performing no processing of the signal [1, 6, 9, 10]. The Wiseman-Milburn Markovian feedback stochastic master equation is obtained by fixing a single control Hamiltonian and letting the corresponding control parameter μ in Eq. (1.15) be directly proportional to the measurement record r [cf. Eq. (1.6)]. Thus, the feedback Hamiltonian $\hat{H}_{\text{fb}}$ reads

$$\hat{H}_{\text{fb}} = r\hat{b}, \tag{1.16}$$

with $\hat{b}$ being a Hermitian operator.

In order to derive the master equation, let us first calculate, at the level of the wave function, the evolution of the system due to the Hamiltonian $\hat{H}_{\text{fb}}$. The feedback evolution can be obtained by unitarily evolving the state of the system [11, 12], yielding

$$\exp\left[-\frac{i}{\hbar}\hat{H}_{\text{fb}}\mathrm{d}t\right]|\psi\rangle = |\psi\rangle + (\mathrm{d}\,|\psi\rangle)_{\text{fb}}, \tag{1.17}$$

where the label fb indicates the feedback contribution to the dynamics, which reads

$$(\mathrm{d}\,|\psi\rangle)_{\text{fb}} = \left\{\left[-\frac{i}{\hbar}\langle\hat{a}\rangle\hat{b} - \frac{1}{2\gamma}\hat{b}^2\right]\mathrm{d}t - \frac{i}{\sqrt{\gamma}}\hat{b}\,\mathrm{d}W_t\right\}|\psi\rangle\,. \tag{1.18}$$

We notice the presence of a term which is quadratic in the Hermitian operator $\hat{b}$. This is a consequence of the fact that in the expansion of the exponential in Eq. (1.17), there is a contribution that comes from the second order term in $\hat{H}_{\text{fb}}$, specifically from the noise term in the measurement record r of Eq. (1.6).

The wave function of the system undergoing the continuous measurement of the observable $\hat{a}$ and the subsequent feedback control driven by $\hat{H}_{\text{fb}}$ reads

$$|\psi(t+\mathrm{d}t)\rangle = |\psi\rangle + (\mathrm{d}\,|\psi\rangle)_{\text{m}} + (\mathrm{d}\,|\psi\rangle)_{\text{fb}} + (\mathrm{d}\,|\psi\rangle)_{\text{fb}}(\mathrm{d}\,|\psi\rangle)_{\text{m}}. \tag{1.19}$$

From the above result, and using the expressions of Eq. (1.7) and Eq. (1.18), we have that the stochastic differential equation for the wave function $|\psi\rangle$ reads

$$\mathrm{d}\,|\psi\rangle = \Bigg\{-\frac{\gamma}{8\hbar^2}\left(\hat{a}-\langle\hat{a}\rangle\right)^2\mathrm{d}t + \frac{\sqrt{\gamma}}{2\hbar}\left(\hat{a}-\langle\hat{a}\rangle\right)\mathrm{d}W_t + \left[-\frac{i}{\hbar}\langle\hat{a}\rangle\,\hat{b} - \frac{1}{2\gamma}\hat{b}^2\right]\mathrm{d}t - \frac{i}{\sqrt{\gamma}}\hat{b}\,\mathrm{d}W_t \\ - \frac{i}{2\hbar}\hat{b}(\hat{a}-\langle\hat{a}\rangle)\mathrm{d}t\Bigg\}\,|\psi\rangle\,, \tag{1.20}$$

where we see that the last term comes from the noise term of the feedback $(\mathrm{d}\,|\psi\rangle)_{\mathrm{fb}}$ applied to the state $(\mathrm{d}\,|\psi\rangle)_{\mathrm{m}}$ resulting of the continuous measurement.

In order to obtain the master equation for the density operator, $\hat{\rho} = \mathbb{E}\left[|\Psi\rangle\langle\Psi|\right]$, we consider that

$$\mathrm{d}\hat{\rho} = \mathrm{d}\left(\mathbb{E}[|\psi\rangle\langle\psi|]\right) = \mathbb{E}\left[(\mathrm{d}\,|\psi\rangle)\langle\psi| + |\psi\rangle(\mathrm{d}\,|\psi\rangle) + (\mathrm{d}\,|\psi\rangle)(\mathrm{d}\,|\psi\rangle)\right]. \tag{1.21}$$

Therefore, we can use the result of Eq. (1.20), and obtain the master equation

$$\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = -\frac{i}{2\hbar}\left[\hat{b},\{\hat{a},\hat{\rho}\}\right] - \frac{\gamma}{8\hbar^2}\left[\hat{a},\left[\hat{a},\hat{\rho}\right]\right] - \frac{1}{2\gamma}\left[\hat{b},\left[\hat{b},\hat{\rho}\right]\right]. \tag{1.22}$$

Let us now consider the continuous measurement of the family of observables $\{\hat{a}\}_{\lambda=1}^{N}$, with associated measurement records as in Eq. (1.11), and implement a feedback Hamiltonian of the form

$$\hat{H}_{\mathrm{fb}} = \sum_{\lambda=1}^{\mathcal{M}} r_\lambda \hat{b}_\lambda, \tag{1.23}$$

with $\{\hat{b}_\lambda\}_{\lambda=1}^{\mathcal{M}}$ a family of Hermitian operators. Then, the stochastic differential equation for the feedback is given by

$$(\mathrm{d}\,|\psi\rangle)_{\mathrm{fb}} = \sum_{\lambda=1}^{\mathcal{M}}\left\{\left[-\frac{i}{\hbar}\langle\hat{a}_\lambda\rangle\,\hat{b}_\lambda - \frac{1}{2\gamma_\lambda}\hat{b}_\lambda^2\right]\mathrm{d}t - \frac{i}{\sqrt{\gamma_\lambda}}\hat{b}_\lambda\mathrm{d}W_\lambda\right\}|\psi\rangle\,. \tag{1.24}$$

From Eq. (1.19), we can determine the stochastic differential equation for the wave function, and using Eq. (1.21), we can derive the master equation. After straightforward calculations, we obtain

$$\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = \sum_{\lambda=1}^{\mathcal{M}}\left(-\frac{i}{\hbar}\left[\hat{b}_\lambda,\{\hat{a}_\lambda,\hat{\rho}\}\right] - \frac{\gamma_\lambda}{8\hbar^2}\left[\hat{a}_\lambda,\left[\hat{a}_\lambda,\hat{\rho}\right]\right] - \frac{1}{2\gamma_\lambda}\left[\hat{b}_\lambda,\left[\hat{b}_\lambda,\hat{\rho}\right]\right]\right). \tag{1.25}$$

We can go one step further and generalize this master equation to describe the continuous measurement of an observable $\hat{a}(\boldsymbol{x})$ defined at each point in space. In this case, the measurement record reads

$$a(\boldsymbol{x}) = \langle \hat{a}(\boldsymbol{x}) \rangle + \hbar \int \mathrm{d}\boldsymbol{y} \gamma^{-1}(\boldsymbol{x}, \boldsymbol{y}) \xi_{a,t}(\boldsymbol{y}), \tag{1.26}$$

where the noise $\xi_{a,t}(\boldsymbol{x})$ is defined through the relations

$$\begin{aligned} \mathbb{E}\left[\xi_{a,t}(\boldsymbol{x})\right] &= 0, \\ \mathbb{E}\left[\xi_{a,t}(\boldsymbol{x})\xi_{a,t'}(\boldsymbol{y})\right] &= \gamma(\boldsymbol{x}, \boldsymbol{y})\delta(t - t'). \end{aligned} \tag{1.27}$$

The inverse kernel $\gamma^{-1}(\boldsymbol{x}, \boldsymbol{y})$ appearing in the stochastic term of the measurement record in Eq. (1.26) is related to the correlation kernel $\gamma(\boldsymbol{x}, \boldsymbol{y})$ appearing in Eq. (1.27) by [13]

$$(\gamma \circ \gamma^{-1})(\boldsymbol{x}, \boldsymbol{y}) = \int \mathrm{d}\boldsymbol{r} \gamma(\boldsymbol{x}, \boldsymbol{r}) \gamma^{-1}(\boldsymbol{r}, \boldsymbol{y}) = \delta(\boldsymbol{x} - \boldsymbol{y}). \tag{1.28}$$

At the level of the wavefunction, the contribution to the dynamic evolution due to the continuous measurement of the observable $\hat{a}(\boldsymbol{x})$ is given by

$$\begin{aligned} (\mathrm{d}\left|\psi\right\rangle)_{\mathrm{m}} = \Bigg\{ &-\frac{1}{8\hbar^2} \int \mathrm{d}\boldsymbol{x} \mathrm{d}\boldsymbol{y} \gamma(\boldsymbol{x}, \boldsymbol{y}) \left(\hat{a}(\boldsymbol{x}) - \langle \hat{a}(\boldsymbol{x}) \rangle\right) \left(\hat{a}(\boldsymbol{y}) - \langle \hat{a}(\boldsymbol{y}) \rangle\right) \mathrm{d}t \\ &+ \frac{1}{2\hbar} \int \mathrm{d}\boldsymbol{x} \left(\hat{a}(\boldsymbol{x}) - \langle \hat{a}(\boldsymbol{x}) \rangle\right) \xi_{a,t}(\boldsymbol{x}) \mathrm{d}t \Bigg\} \left|\psi\right\rangle . \end{aligned} \tag{1.29}$$

Let us consider now a feedback Hamiltonian of the form

$$\hat{H}_{\mathrm{fb}} = \int \mathrm{d}\boldsymbol{x} a(\boldsymbol{x}) \hat{b}(\boldsymbol{x}), \tag{1.30}$$

with $\hat{b}(\boldsymbol{x})$ a Hermitian operator, which is also defined at each point of space. The feedback contribution can be calculated using Eq. (1.17). Explicit calculation yields

$$(\mathrm{d}\left|\psi\right\rangle)_{\mathrm{fb}} = \left\{ -\frac{i}{\hbar} \int \mathrm{d}\boldsymbol{x} a(\boldsymbol{x}) \hat{b}(\boldsymbol{x}) \mathrm{d}t - \frac{1}{2} \int \mathrm{d}\boldsymbol{x} \mathrm{d}\boldsymbol{y} \gamma^{-1}(\boldsymbol{x}, \boldsymbol{y}) \hat{b}(\boldsymbol{x}) \hat{b}(\boldsymbol{y}) \mathrm{d}t \right\} \left|\psi\right\rangle . \tag{1.31}$$

Therefore, the full evolution of the wave function is obtained by merging the contributions of the continuous measurement in Eq. (1.29) and of the feedback in

Eq. (1.31). We obtain

$$
\begin{aligned}
\mathrm{d}\left|\psi\right\rangle = \Bigg\{ & -\frac{1}{8\hbar^2}\int \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\gamma(\boldsymbol{x},\boldsymbol{y})\left(\hat{a}(\boldsymbol{x})-\langle\hat{a}(\boldsymbol{x})\rangle\right)\left(\hat{a}(\boldsymbol{y})-\langle\hat{a}(\boldsymbol{y})\rangle\right)\mathrm{d}t \\
& + \frac{1}{2\hbar}\int \mathrm{d}\boldsymbol{x}\left(\hat{a}(\boldsymbol{x})-\langle\hat{a}(\boldsymbol{x})\rangle\right)\xi_{a,t}(\boldsymbol{x})\mathrm{d}t \\
& - \frac{i}{\hbar}\int \mathrm{d}\boldsymbol{x}a(\boldsymbol{x})\hat{b}(\boldsymbol{x})\mathrm{d}t - \frac{1}{2}\int \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\gamma^{-1}(\boldsymbol{x},\boldsymbol{y})\hat{b}(\boldsymbol{x})\hat{b}(\boldsymbol{y})\mathrm{d}t \\
& - \frac{i}{2\hbar}\int \mathrm{d}\boldsymbol{x}\hat{b}(\boldsymbol{x})\left(\hat{a}(\boldsymbol{x})-\langle\hat{a}(\boldsymbol{x})\rangle\right)\mathrm{d}t\Bigg\}\left|\psi\right\rangle,
\end{aligned}
\tag{1.32}
$$

and from this, the master equation is obtained through a straightforward application of the Itô calculus. Thus, one obtains

$$
\begin{aligned}
\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = & -\frac{i}{2\hbar}\int \mathrm{d}\boldsymbol{x}\left[\hat{b}(\boldsymbol{x}),\{\hat{a}(\boldsymbol{x}),\hat{\rho}\}\right] - \frac{1}{8\hbar^2}\int \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\gamma(\boldsymbol{x},\boldsymbol{y})\left[\hat{a}(\boldsymbol{x}),\left[\hat{a}(\boldsymbol{y}),\hat{\rho}\right]\right] \\
& - \frac{1}{2}\int \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\gamma^{-1}(\boldsymbol{x},\boldsymbol{y})\left[\hat{b}(\boldsymbol{x}),\left[\hat{b}(\boldsymbol{y}),\hat{\rho}\right]\right].
\end{aligned}
\tag{1.33}
$$

These results represent the basic ingredients behind the implementation of the models proposed by Kafri, Taylor and Milburn [2], and by Tilloy and Diósi [3]. We will also see how one of the most robust dynamical collapse models, the continuous spontaneous localization (CSL) model [4], can be understood within the quantum continuous measurement theory.

Chapter 2

Is Gravity necessarily Quantum?

2.1 The efforts to merge Quantum Mechanics and Gravity

The characterization of the gravitational interaction started with the works of Galileo and Newton, which date back to the 16th century. In the 20th century, the theory of General Relativity by Einstein provided a more robust description of gravity as an effect of the space-time deformation due to the presence of a mass [14]. Some years after General Relativity appeared, the works of Schrödinger and Heisenberg, among others, provided the theory of Quantum Mechanics, which constitutes a general framework for describing interactions between microscopic particles [15]. Up to date, there is no experimental evidence conflicting with either General Relativity or Quantum Mechanics [16].

All the forces have been successfully incorporated into the framework of Quantum Mechanics; but the gravitational one. Although there is no observation requiring a quantum theory of gravity for its explanation [16], efforts to quantize general relativity date back to the early 1930s [17]. However, up to the present day, there is no quantum theory of gravity which is both fundamental and consistent. Indeed, theories which result from the quantization of gravity by the standard procedures, are non-renormalizable. Thus, they are not fundamental and, in the best scenario, they can be taken only as effective theories [18]. In this regard, it was shown in 1986 that conventional quantum field theory techniques fail [19].

The key reason for the failure of quantization of gravity could be the fundamental difference between the frameworks of the two theories. On one hand, Quantum Mechanics indicates that all dynamical fields have quantum properties, while, on the other hand, in General Relativity the gravitational field is assumed to be a classical deterministic dynamical field. Moreover, Quantum Field Theory relies on the existence of a non-dynamical background spacetime metric, but General Relativity indicates that such metric simply does not exist in Nature [20]. This shows that the conceptual foundations of both theories contradict each other, which makes it difficult to reconcile them.

Some of the main motivations to find a single framework for both theories include the following situations. In the first place, that although quantum particles can exist in superposition states of different positions, we do not know which would be the corresponding gravitational field, as the latter does not exist in superpositions [18]. Secondly, General Relativity leads to the existence of singularities, instances of infinite energy density and gravitational forces [18]. The unphysical character of these singularities shows that General Relativity cannot be true at the most fundamental level [15]. In addition, the loss of information from the initial configuration of a black hole through thermal radiation until its evaporation is incompatible with Quantum Mechanics. This is known as the black hole information loss problem or black hole information paradox [18]. It is expected that quantum gravity will provide a fundamental understanding of both the early Universe and the final stages of black-hole evolution [15].

We notice that the problems which General Relativity and Quantum Mechanics address typically arise at very different length and energy scales [19]. It is widely expected that the scale at which the effects of quantum gravity become relevant is the Planck scale. This means that the classical description of spacetime provided by General Relativity is expected to hold at scales larger than the Planck length $l_p = \sqrt{\hbar G/c^3} \sim 10^{-33}$ cm. In contrast, once one approaches the Planck scale, the full structure of a "quantum spacetime" should become relevant and General Relativity should break down [20].

Another aspect that is expected to be resolved with a fundamental theory of quantum gravity would be the role of time, as the concepts of time (and space-time) in Quantum Mechanics and General Relativity are incompatible [15]. In standard Quantum Mechanics, time is treated as an external parameter, which is not described by an operator. In General Relativity, the coordinate time is a gauge variable, which is not observable [20], and in general, space-time is a dynamical object. Some scenarios suggest that the fundamental equations of Quantum Gravity might not be written as evolution equations in terms of an observable time variable [20], and that the metric has to be turned into an operator [15].

2.2 A semiclassical approach

From a theoretical point of view, the semiclassical Einstein equations have constituted a key approach to study the theory of quantum fields on an external spacetime. In order to introduce them, following Ref. [15], let us recall the Einstein-Hilbert action

$$S_{\text{EH}} = \frac{c^4}{16\pi G}\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{-g}\,(R-2\Lambda) - \frac{c^4}{8\pi G}\int_{\partial\mathcal{M}} \mathrm{d}^3x\sqrt{h}\mathcal{K}, \tag{2.1}$$

where the first integral covers a region $\mathcal{M}$ of the space-time manifold. In this integral, R stands for the Ricci scalar, Λ is the cosmological constant, and g is the determinant of the metric with components $g_{\mu\nu}$. The second integral is added to obtain a consistent variational principle that leads to Einstein field equations, and is carried out on the boundary $\partial\mathcal{M}$ of the region $\mathcal{M}$. The boundary is assumed to be spacelike [15]. In this surface term, h is the determinant of the induced three-dimensional metric on the boundary, and $\mathcal{K}$ is the trace of the extrinsic curvature of $\partial\mathcal{M}$ [21]. The action in Eq. (2.1) defines the formalism of General Relativity, along with the action for non-gravitational fields S_{m} (the so-called 'matter action'), which in turn leads to the energy-momentum tensor

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\partial S_m}{\partial g^{\mu\nu}}, \tag{2.2}$$

that constitutes a source of the gravitational field. The variation of the sum of the actions S_{EH} of Eq. (2.1) and S_{m} leads to the Einstein field equations

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}, \qquad \text{where} \qquad G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R. \tag{2.3}$$

In the above relation, the Einstein tensor $G_{\mu\nu}$ is defined in terms of the Ricci scalar R and the Ricci tensor $R_{\mu\nu}$. If we replace the energy-momentum tensor of Eq. (2.3) with the expectation value of the energy-momentum operator $T_{\mu\nu}$ with respect to some quantum state $|\Psi\rangle$, we obtain the semiclassical Einstein equations

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} \langle\Psi| \hat{T}_{\mu\nu} |\Psi\rangle \,. \tag{2.4}$$

These equations give the back reaction of the matter fields on the space-time [22]. In semiclassical gravity, the metric $g_{\mu\nu}$ is treated classically, while all the other forms of matter are assumed to be quantized [22, 23]. This semiclassical model implies a nonlinearity in Quantum Mechanics, as the Schrödinger equation for the wave function $|\Psi\rangle$ depends on the metric, which in turn depends on the wave function $|\Psi\rangle$ through Eq. (2.4) [19]. As the quantum field that defines the stress-energy tensor is an operator-valued distribution, the operator $\hat{T}_{\mu\nu}$ is ill-defined and the corresponding expectation value $\langle\Psi| \hat{T}_{\mu\nu} |\Psi\rangle$ is formally infinite [24]. This leads to the implementation of renormalization techniques [25]. Under very general assumptions, the expectation value $\left\langle \hat{T}_{\mu\nu} \right\rangle$ is independent of the details of the regularization and renormalization procedures employed [26].

It should be remarked out that several issues with semiclassical gravity have been pointed out. Among these issues, one finds inequivalent quantizations of a given classical theory as result of field redefinition ambiguities [17]. Furthermore, when the quantum state $|\Psi\rangle$ is in a superposition of two macroscopically distinct states, the measured gravitational field is given by the actual measured value of the energy-momentum, and not by its average in the state $|\Psi\rangle$ [23]. If one introduces a time-dependence to tackle this issue, one is led to a violation of the conservation equation $\nabla^\mu \left\langle \hat{T}_{\mu\nu} \right\rangle = 0$, which leads to an inconsistency, as the Einstein tensor is conserved [17, 23].

Given the high degree of complexity of the semiclassical Einstein equations shown in Eq. (2.4), some authors [27, 28] consider their Newtonian limit restriction, which leads to the so-called *Schrödinger-Newton equation*

$$i\hbar\frac{\mathrm{d}\Psi}{\mathrm{d}t} = -\frac{\hbar^2}{2m}\nabla^2\Psi - m\Phi\Psi, \qquad \text{where} \qquad \nabla^2\Phi = 4\pi Gm|\Psi|^2. \tag{2.5}$$

In the model of Eq. (2.5), the expectation value of the mass density sources the classical Newtonian potential Φ, which describes gravity [19]. However, in analogy to the violation of local energy-momentum conservation in the semiclassical Einstein equations [cf. Eq. 2.4], taking the Schrödinger-Newton equation together with the standard collapse postulate leads to superluminal effects [29].

In the case of semi-classical quantum theories of gravity, it is also important to remark that they may or may not include quantum backreactions. The latter are defined as the quantum fluctuations induced by the gravitational field by its coupling to the quantum fields and matter [30]. The possibility of a semi-classical quantum theory of gravity with quantum backreactions seems to be precluded in Ref. [31].

In general, the different approaches followed to merge both Quantum Mechanics and General Relativity into a fundamental Quantum Gravity theory show the need of the new theory to predict all phenomena at the low-energy regime. In order to classify the approaches to construct a consistent framework for Quantum Gravity, a distinction between primary and secondary theories has been proposed (see Ref. [15] and references within). According to this classification, in the primary theories, heuristic quantization rules are applied to a given classical theory. If at the classical level, there is a split of space-time into space and time, then the approach is said to be canonical, whereas if four-dimensional covariance is preserved at each step, the approach is referred to as covariant. In contrast with primary theories, in secondary theories one tries to derive General Relativity starting with a fundamental quantum framework which describes all interactions.

2.3 Hybrid classical and quantum dynamics

Although there are diverse arguments that support the idea of a gravitational field which is of quantum nature at a fundamental level, the difficulties to deal with Quantum Mechanics and General Relativity within a single framework have led to ask if the quantization of the gravitational field is necessary at all. Moreover, it is argued that various results demonstrating that the gravitational field must be quantized in reality show the inconsistency of particular theories where quantum-mechanical matter interacts with a classical gravitational field [32]. In general, the question of the quantization of the gravity field tackles the issue of constructing a consistent hybrid dynamics which couples quantum with classical systems. These alternative approaches also have difficulties of their own. For example, some authors [33] have studied the particular case of the coupling between a classical gravitational wave and a quantum system, pointing the inconsistencies that arise if the quantum nature of the measuring apparatus is not taken into account. Namely, if gravitational measurements do not cause wave function collapse, then the gravitational interactions with quantum matter could be used to transmit observable signals faster than light; and if a measurement by a gravitational wave leads to a wave function collapse, then the uncertainty relations are incompatible with the momentum conservation [19]. However, it has been argued that even if the proposed *gedankenexperiments* of Ref. [33] could be performed (there is already a discussion about the possibility of performing them), the need for a quantized gravity does not follow from these experiments. From an experimental perspective, some works have discussed the possibility of detecting a single graviton, concluding that, although there are no basic principles ruling out a detection, standard physics experiments in four dimensions do not seem to favour the detection, both because the characteristics of eventual detectors are well-beyond the current technological developments, and because of the presence of background noise, due to neutrinos and cosmic rays [34].

In standard quantum theory, the influence of classical macro-systems upon quantized micro-systems is taken into account as external forces, and the corresponding back-reaction of the micro-systems upon the macro-systems is usually ignored [35]. This is true only under the Markov approximation. Some proposals to construct a hybrid dynamics [36] introduce a Hamiltonian in the form

$$\hat{H}(x,p) = \hat{H}_Q + H_C(x,p)\hat{1} + \hat{H}_I(x,p) \tag{2.6}$$

where $\hat{H}_Q$ is the Hamilton operator appearing in the von Neumann equation describing the evolution of the density operator $\hat{\rho}_Q$, $H_C(x,p)$ is the Hamilton function corresponding to the Liouville equation of motion of the phase-space density $\rho_C(x,p)$ of a classical canonical system, $\hat{1}$ is the identity operator, and the interaction term $\hat{H}_I(x,p)$ is a Hermitian operator for the quantum system, which depends on the phase-space coordinates (x,p) of the classical system.

Other works [37] have proposed a formalism which, in the case of no interaction between quantum and classical systems, reduces to standard classical and quantum mechanics, but that leads to a violation of the correspondence principle in the presence of an interaction. The emergence of the classical behaviour from the underlying quantum structure has also been studied in Gaussian quantum systems, which behave classically in some respects [38].

Some other proposals offering quantum theories of gravity, do not involve a quantization of the gravitational field, as they regard gravity as an induced rather than a fundamental force [30]. Some work in this line of research include recovering gravity from Quantum Field Theory [39] by assuming a Lorentzian manifold as a background on which performing Quantum Field Theory [30], or recovering Einstein equations from black hole thermodynamics [40] inversely to the usual approach. In addition, other works [41] have constructed states of a quantum particle, called Ehrenfest monopole wavefunctions, for which there is an emergence, at an effective level, of a classical particle following a Newtonian trajectory in space-time.

From the previous discussion, one sees that the formulation of a consistent hybrid classical-quantum theories has followed several attempts. In Ref. [42], the following classification was suggested: (a) Theories where the description of the quantum sector and trajectories for the classical sector are implemented by the use of quantum states; (b) theories that formulate the quantum sector as a classical theory and then work with a completely classical system; (c) theories that formulate the classical sector as a quantum theory and then work with a completely quantum system; (d) theories that describe both quantum and classical sectors in a common formalism and then extend it to include interactions. It is argued that the proposals of Diósi [43] and Penrose [44], as well as that of Ghirardi *et al.* [45] find additional justification in light of these models. In what follows, we briefly discuss the Diósi-Penrose model.

2.3.1 Diósi-Penrose model

The works of Diósi and Penrose modify ordinary Quantum Mechanics with the inclusion of a gravitational contribution in order to cure some of the problems of the macroscopic quantum theory [43, 44]. By working in the Newtonian limit, and assuming the gravitational potential $\phi(\boldsymbol{r}, t)$ to be a stochastic variable defined through

$$\begin{aligned} \mathbb{E}\left[\phi(\boldsymbol{r}, t)\right] &= 0, \\ \mathbb{E}\left[\phi(\boldsymbol{r}, t)\phi(\boldsymbol{r}', t')\right] &= \hbar \frac{G}{|\boldsymbol{r} - \boldsymbol{r}'|}\delta(t - t'), \end{aligned} \tag{2.7}$$

one imposes the following equation for the state vector $|\psi\rangle$ [43]

$$\mathrm{d}\,|\psi(t)\rangle = -i\hbar\left(\hat{H}_0 + \int \mathrm{d}\boldsymbol{r}\phi(\boldsymbol{r}, t)\hat{\mu}(\boldsymbol{r})\right)\mathrm{d}t\,|\psi(t)\rangle\,, \tag{2.8}$$

where $\hat{H}_0$ is the Hamiltonian of the system without gravity, and $\hat{\mu}(\boldsymbol{r})$ stands for the mass density of the system. A straightforward application of stochastic calculus rules leads to the following equation

$$\frac{\mathrm{d}\hat{\rho}(t)}{\mathrm{d}t} = -\frac{i}{\hbar}\left[\hat{H}_0, \hat{\rho}\right] - \frac{G}{2\hbar}\int \frac{\mathrm{d}\boldsymbol{r}\mathrm{d}\boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|}\left[\hat{\mu}(\boldsymbol{r}), \left[\hat{\mu}(\boldsymbol{r}'), \hat{\rho}\right]\right]. \tag{2.9}$$

The second term in the above Markovian master equation modifies the standard unitary evolution of the system. Following a state-reduction proposal, according to which superposed gravitational fields are unstable, Penrose determined the deviations from the standard Schrödinger unitary evolution due to gravitational effects [44], obtaining results compatible with the proposal of Diósi. Regarding the short-distance behaviour of the Newtonian potential, both authors acknowledged the necessity to implement a regularization mechanism to avoid divergences. One of these mechanisms consists in the introduction of a coarse-grained mass density [46], such as Gaussianly smeared mass density [47]. The evolution for $\hat{\rho}$ in Eq. (2.9), together with the regularization mechanism, constitutes the

Diósi-Penrose model [43, 44, 48]. We point out that the model by Ghirardi *et al.* in Ref. [45], constructed from considerations involving dynamical collapse models, is encoded within the Diósi-Penrose model. Indeed, it suffices to take

$$\hat{\mu}(\boldsymbol{r}) = \sum_i m_i \left(\frac{\alpha_i}{2\pi}\right)^{3/2} \sum_{s_i} \int \mathrm{d}\boldsymbol{r}' e^{-(\alpha_i/2)(\boldsymbol{r}'-\boldsymbol{r})^2} \hat{a}^\dagger(\boldsymbol{r}', s_i)\hat{a}(\boldsymbol{r}', s_i), \tag{2.10}$$

where $\hat{a}^\dagger(\boldsymbol{r}', s_i)$ and $\hat{a}(\boldsymbol{r}', s_i)$ are the creation and annihilation operators of a particle at point $\boldsymbol{r}'$ with spin component s_i [45], and α_i are parameters of the model. The index i runs over the different species of particles.

The Diósi-Penrose model in Eq. (2.9) model turns out to be encoded within the formalism of the Tilloy-Diósi model [3], which we describe in the following chapter.

Chapter 3

Gravity through a continuous measurement and feedback mechanism

Among the proposals which merge Quantum Mechanics and Gravity without requiring the latter to be quantized one can find some models in which gravity is treated as classical and matter as quantum [43, 44, 49–58] . Among these we focus on two implementing the gravitational interaction through a continuous measurement and feedback mechanism, which were proposed respectively by Kafri, Taylor and Milburn (KTM) [2] and by Tilloy and Diósi (TD) [3]. In what follows, we study them in detail, and describe their relation, with the aim of studying viable scenarios for an implementation of a full Newtonian interaction through a feedback mechanism. The results in this section follow from our work in Ref. [59].

3.1 The Kafri-Taylor-Milburn Model

The KTM model considers gravity as a classical interaction and implements it through a continuous measurement and feedback mechanism. In the first step of this process, one implements a weak continuous measurement of the positions $\hat{x}$ of the masses constituting the system. After this measurement, one effectively implements the gravitational interaction by feeding the measurement record of the position of one particle to the other. The continuous measurement and feedback mechanism induces decoherence effects along-side the desired effective gravitational interaction between the constituents of the system. Concretely, KTM considered a system composed of a pair of point-like particles of masses m_1 and m_2 respectively. These particles are harmonically suspended at an initial distance d as it is shown in Fig. (3.1). The existence of only two masses allows to study the problem in one dimension. If one assumes that the initial distance d is greater than the fluctuations q_1, q_2 around the initial positions of the two masses, one can Taylor expand the gravitational interaction to second order in the relative displacement. Therefore, we have

$$V(q_1 - q_2) = -Gm_1m_2\left(\frac{1}{d} + \frac{1}{d^2}(q_1 - q_2) + \frac{1}{d^3}(q_1 - q_2)^2\right). \tag{3.1}$$

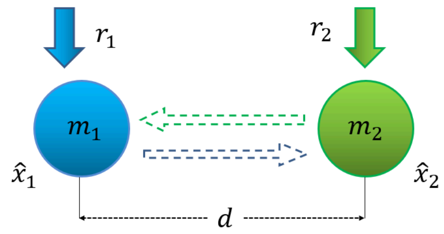

Figure 3.1: Figure adapted from Ref. [59]. Two particles harmonically trapped are separated by an initial distance d. The measurement records r_1 and r_2 are used to implement a Newtonian gravitational interaction between them.

Let us define the following change of coordinates [60]

$$\begin{aligned} x_1 &= q_1 + \frac{Gm_2\Omega_2^2 d}{2G(m_2\Omega_2^2 + m_1\Omega_1^2) - \Omega_2^2\Omega_1^2 d^3}, \\ x_2 &= q_2 + \frac{Gm_1\Omega_1^2 d}{\Omega_2^2\Omega_1^2 d^3 - 2G(m_2\Omega_2^2 + m_1\Omega_1^2)}, \end{aligned} \tag{3.2}$$

where Ω_1 and Ω_2 are the frequencies of the harmonic motion of the two particles. After quantizing the system, the Hamiltonian now reads

$$\hat{H} = \hat{H}_0 + \hat{H}_{\text{grav}}, \tag{3.3}$$

where $\hat{H}_0$ is the Hamiltonian of a pair of uncoupled harmonic oscillators

$$\hat{H}_0 = \sum_{\alpha=1}^{2} \frac{\hat{p}_\alpha^2}{2m_\alpha} + \frac{1}{2} m_\alpha \omega_\alpha^2 \hat{x}_\alpha^2, \qquad \text{with} \qquad \omega_\alpha^2 = \Omega_\alpha^2 - \frac{2Gm_\alpha}{d^3}, \tag{3.4}$$

and $\hat{H}_{\text{grav}}$ encodes the linearized gravitational interaction between the masses:

$$\hat{H}_{\text{grav}} = K\hat{x}_1\hat{x}_2, \qquad \text{with} \qquad K = \frac{2Gm_1m_2}{d^3}, \tag{3.5}$$

where G is the gravitational constant. The proposal of KTM consists in replacing the Hamiltonian operator $\hat{H}_{\text{grav}}$ with an appropriate mechanism that effectively provides the gravitational interaction due to this term. In what follows, we describe the two steps that constitute this mechanism.

The first step consists in a weak continuous measurement of the positions of the two masses. Following the result of Eq. (1.12), we have that the stochastic differential equation that encodes this process is

$$(\mathrm{d}\,|\psi\rangle)_{\mathrm{m}} = \sum_{\alpha=1}^{2} \left(-\frac{\gamma_\alpha}{8\hbar^2} \left(\hat{x}_\alpha - \langle \hat{x}_\alpha \rangle\right)^2 \mathrm{d}t + \frac{\sqrt{\gamma_\alpha}}{2\hbar} \left(\hat{x}_\alpha - \langle \hat{x}_\alpha \rangle\right) \mathrm{d}W_{\alpha,t} \right) |\psi\rangle\,. \tag{3.6}$$

After the continuous measurements, the system undergoes the feedback evolution. In this step, one replaces the Hamiltonian $\hat{H}_{\text{grav}}$ with the following feedback

Hamiltonian

$$\hat{H}_{\text{fb}} = \chi_{12} r_1 \hat{x}_2 + \chi_{21} r_2 \hat{x}_1, \tag{3.7}$$

where χ_{12} and χ_{21} are parameters and the measurement records r_α are defined as

$$r_\alpha = \langle \hat{x}_\alpha \rangle + \frac{\hbar}{\sqrt{\gamma_\alpha}} \frac{\mathrm{d}W_{\alpha,t}}{\mathrm{d}t}. \tag{3.8}$$

which follows Eq. (1.11). The Wiener processes $W_{\alpha,t}$ of the measurement records in Eq. (3.8) satisfy

$$\begin{aligned} \mathbb{E}\left[\mathrm{d}W_{\alpha,t}\right] &= 0, \\ \mathbb{E}\left[\mathrm{d}W_{\alpha,t}\mathrm{d}W_{\beta,t}\right] &= \delta_{\alpha\beta}\mathrm{d}t. \end{aligned} \tag{3.9}$$

We notice that the feedback Hamiltonian $\hat{H}_{\text{fb}}$ in Eq. (3.7) is of the form of Eq. (1.23). Therefore, according to Eq. (1.24), the corresponding stochastic differential equation describing the evolution due to the feedback is

$$(\mathrm{d}\,|\psi\rangle)_{\text{fb}} = -\sum_{\substack{\alpha,\beta=1 \\ \beta\neq\alpha}}^{2} \left[\frac{i}{\hbar} r_\alpha + \frac{\chi_{\alpha\beta}\hat{x}_\beta}{2\gamma_\alpha}\right] \chi_{\alpha\beta}\hat{x}_\beta \mathrm{d}t\,|\psi\rangle\,. \tag{3.10}$$

From Eq. (3.6) and Eq. (3.10), the full evolution of the system is given by

$$\begin{aligned} \mathrm{d}\,|\psi\rangle = \Bigg\{ &-\sum_{\substack{\alpha,\beta=1 \\ \beta\neq\alpha}}^{2} \left[\frac{i}{\hbar} r_\alpha + \frac{\chi_{\alpha\beta}\hat{x}_\beta}{2\gamma_\alpha}\right] \chi_{\alpha\beta}\hat{x}_\beta \mathrm{d}t + \sum_{\alpha=1}^{2} \Bigg[-\frac{\gamma_\alpha}{8\hbar^2}\left(\hat{x}_\alpha - \langle \hat{x}_\alpha\rangle\right)^2 \mathrm{d}t \\ &+ \frac{\sqrt{\gamma_\alpha}}{2\hbar}(\hat{x}_\alpha - \langle \hat{x}_\alpha\rangle)\mathrm{d}W_{\alpha,t}\Bigg] - \frac{i}{2\hbar}\sum_{\substack{\alpha,\beta=1 \\ \beta\neq\alpha}}^{2} \chi_{\alpha\beta}\hat{x}_\beta(\hat{x}_\alpha - \langle \hat{x}_\alpha\rangle)\mathrm{d}t \Bigg\}\,|\psi\rangle\,. \end{aligned} \tag{3.11}$$

From the above result, one obtains the corresponding master equation for the density operator by averaging over the Wiener processes

$$\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = -\frac{i}{\hbar}\left[\hat{H}_0, \hat{\rho}\right] - \frac{i}{2\hbar}\sum_{\substack{\alpha,\beta=1 \\ \beta\neq\alpha}}^{2} \chi_{\alpha\beta}\left[\hat{x}_\beta, \{\hat{x}_\alpha, \hat{\rho}\}\right] - \sum_{\alpha=1}^{2}\left(\frac{\gamma_\alpha}{8\hbar^2} + \sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^{2} \frac{\chi_{\beta\alpha}^2}{2\gamma_\beta}\right)\left[\hat{x}_\alpha, \left[\hat{x}_\alpha, \hat{\rho}\right]\right]. \tag{3.12}$$

In this equation, which agrees with the general result of Eq. (1.25), we also included the evolution due to $\hat{H}_0$. The last term in Eq. (3.12) leads to decoherence effects as a result of the noise part in the dynamics of the system. The second term will allow, under the particular choices of the parameters $\chi_{\alpha\beta} = K$, to effectively mimic the gravitational interaction. In this case, Eq. (3.12) becomes

$$\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = -\frac{i}{\hbar}\left[\hat{H}_0 + K\hat{x}_1\hat{x}_2, \hat{\rho}\right] - \sum_{\alpha=1}^{2}\left(\frac{\gamma_\alpha}{8\hbar^2} + \sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{2}\frac{K^2}{2\gamma_\beta}\right)\left[\hat{x}_\alpha, \left[\hat{x}_\alpha, \hat{\rho}\right]\right]. \tag{3.13}$$

Hence, we see that we recover the gravitational interaction defined by $\hat{H}_{\text{grav}}$ in Eq. (3.5), and thus the KTM model indeed retrieves the gravitational interaction through a continuous measurement and feedback mechanism. This constitutes an example of a local operation and classical communication (LOCC) dynamics [61–64]. In this case, the local operations are the continuous measurements and the feedback dynamics works as a classical communication. This scheme allows to simulate the action of a gravitational field. However, this particular mechanism also includes the appearance of terms which lead to decoherence effects, whose strength is quantified by γ_α. For equal masses, it is reasonable to assume that the measurement processes have the same rate [2], so one can set $\gamma_1 = \gamma_2 = \gamma$. Although the value of γ can be obtained only through experiments, one can theoretically determine the value of γ that minimizes such decoherence effects. The structure of the decoherence terms in Eq. (3.13) allows a minimization procedure with respect to γ. Under this procedure, Eq. (3.13) becomes

$$\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = -\frac{i}{\hbar}\left[\hat{H}_0 + \hat{H}_{\text{grav}}\right] - \frac{K}{2\hbar}\sum_{\alpha=1}^{2}\left[\hat{x}_\alpha, \left[\hat{x}_\alpha, \hat{\rho}\right]\right]. \tag{3.14}$$

This result corresponds to an information gain rate equal to $\gamma_{\text{min}} = 2\hbar K$ [2].

3.2 Generalizations of the KTM model

The KTM model describes the gravitational interaction of two particles and provides a one-dimensional master equation [cf. Eq. (3.13)]. It is natural to ask whether one may generalize this model to describe an arbitrary finite number of particles in three dimensions, within a linearized Newtonian interaction context. In what follows, we consider two possible generalizations of the KTM model. The difference between them is given by the way the measurement is performed.

3.2.1 Pairwise KTM model

We define a pairwise model to be a framework in which the interaction between the constituents of the systems is established in pairs. Namely, we define measurement records for each pair of particles. Consequently, the information about a given particle depends also on the particle receiving such an information. A pairwise generalization of the KTM model was first proposed by Altamirano *et al.* [55]. In their work, which is based on a quantum collisional approach [55, 65] rather than a continuous measurement and feedback formalism, they describe the interaction between two bodies of N_1 and N_2 constituents of the system. In their

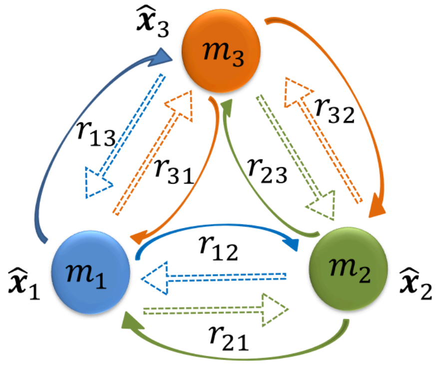

Figure 3.2: Figure adapted from Ref. [59]. In the pairwise generalization of KTM, the measurement records $r_{\alpha\beta}$ (solid arrows) link the particles in pairs. The position of each particle is measured by all the others, and allows to implement Newtonian gravity through the feedback interaction (dashed arrows).

approach, the system interacts with a Markovian environment in a suitably chosen parameter regime [65]. The particular system studied in their work allowed an effective one-dimensional description of the system dynamics. The master equation of Altamirano *et al.* [55] reads

$$\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = -\frac{i}{\hbar}\left[\hat{H}_0 + \frac{1}{2}\sum_{\substack{\alpha,\beta=1\\ \beta\neq\alpha}}^{N_1+N_2} \hat{V}_{\alpha\beta}, \hat{\rho}\right] - \frac{1}{2}\sum_{\substack{\alpha,\beta=1\\ \beta\neq\alpha}}^{N_1+N_2} \Gamma_{\alpha\beta}\left([\hat{x}_\alpha,[\hat{x}_\alpha,\hat{\rho}]] + [\hat{x}_\beta,[\hat{x}_\beta,\hat{\rho}]]\right), \tag{3.15}$$

where $\hat{V}_{\alpha\beta}$ stands for the gravitational potential expanded up to the second order in the positions $\hat{x}_\alpha$, and the constants $\Gamma_{\alpha\beta}$ are the decoherence rates.

Now, we proceed to lift the one-dimensional effective restriction and generalize the KTM model to describe the gravitational interaction for N particles in three dimensions. A scheme of this pairwise generalization of the KTM model is shown in Fig. (3.2). For any two particles, the gravitational potential $V(\boldsymbol{x}_\alpha, \boldsymbol{x}_\beta)$ can be approximated in the following form

$$\begin{aligned} V(\boldsymbol{x}_\alpha, \boldsymbol{x}_\beta) &= -\frac{Gm_\alpha m_\beta}{|\mathbf{d}_{\alpha\beta} - \boldsymbol{x}_\alpha + \boldsymbol{x}_\beta|} \\ &\approx -Gm_\alpha m_\beta\left(\frac{1}{|\mathbf{d}_{\alpha\beta}|} - \frac{1}{|\mathbf{d}_{\alpha\beta}|}\mathbf{d}_{\alpha\beta}\cdot(-\boldsymbol{x}_\alpha + \boldsymbol{x}_\beta)\right. \\ &\quad \left. - \frac{1}{2}\frac{1}{|\mathbf{d}_{\alpha\beta}|^3}(-\boldsymbol{x}_\alpha + \boldsymbol{x}_\beta)^2 + \frac{3}{2}\frac{1}{|\mathbf{d}_{\alpha\beta}|^5}\left(\mathbf{d}_{\alpha\beta}\cdot(-\boldsymbol{x}_\alpha + \boldsymbol{x}_\beta)\right)^2\right), \end{aligned} \tag{3.16}$$

where the vector $\mathbf{d}_{\alpha\beta}$ joins the positions of the two particles of masses m_α and m_β respectively. One can redefine the coordinates $\boldsymbol{x}_\alpha$ in such a way that, after quantization, the potential $\hat{V}(\hat{\boldsymbol{x}}_\alpha, \hat{\boldsymbol{x}}_\beta)$ reads

$$\hat{V}(\hat{\boldsymbol{x}}_\alpha, \hat{\boldsymbol{x}}_\beta) \approx \sum_{\alpha=1}^{N} \hat{Y}_\alpha + \frac{1}{2} \sum_{\substack{\alpha,\beta=1 \\ \beta\neq\alpha}}^{N} \sum_{l,j=1}^{3} K_{\alpha\beta lj} \hat{x}_{\alpha l} \hat{x}_{\beta j}, \tag{3.17}$$

where the Latin indices denote the directions in space. In the above equations, $\hat{Y}_\alpha$ denotes a second-order polynomial in the components of the position operator $\hat{\boldsymbol{x}}_\alpha$. The constants $K_{\alpha\beta lj}$ are defined as

$$K_{\alpha\beta lj} = G m_\alpha m_\beta \left[\frac{3 d_{\alpha\beta l} d_{\alpha\beta j}}{|\mathbf{d}_{\alpha\beta}|^5} - \frac{\delta_{lj}}{|\mathbf{d}_{\alpha\beta}|^3} \right]. \tag{3.18}$$

The above expression is the generalization of the constant K defined in Eq. (3.5).

From this expansion of the gravitational potential, one can follow in a straightforward way the original approach of KTM. The stochastic differential equation for the wave function that describes the continuous measurements of the positions $\hat{x}_{\alpha l}$ is given by

$$(\mathrm{d}\,|\psi\rangle)_{\mathrm{m}} = \sum_{\substack{\alpha,\beta=1 \\ \beta\neq\alpha}}^{N} \sum_{l,j=1}^{3} \left(-\frac{\gamma_{\alpha\beta lj}}{8\hbar^2} \left(\hat{x}_{\alpha l} - \langle \hat{x}_{\alpha l} \rangle\right)^2 \mathrm{d}t + \frac{\sqrt{\gamma_{\alpha\beta lj}}}{2\hbar} \left(\hat{x}_{\alpha l} - \langle \hat{x}_{\alpha l} \rangle\right) \mathrm{d}W_{\alpha\beta lj,t} \right) |\psi\rangle\,, \tag{3.19}$$

where the parameters $\gamma_{\alpha\beta lj}$ encode the information gain rate of the measurements. The standard Wiener processes $W_{\alpha\beta lj,t}$ in the above equation are defined through the following relations

$$\begin{aligned} \mathbb{E}\left[\mathrm{d}W_{\alpha\beta lj,t}\right] &= 0, \\ \mathbb{E}\left[\mathrm{d}W_{\alpha\beta lj,t} \mathrm{d}W_{\alpha'\beta' l' j',t}\right] &= \delta_{\alpha\alpha'}\delta_{\beta\beta'}\delta_{ll'}\delta_{jj'}\mathrm{d}t. \end{aligned} \tag{3.20}$$

The measurement records now read

$$r_{\alpha\beta lj} = \langle \hat{x}_{\alpha l} \rangle + \frac{\hbar}{\sqrt{\gamma_{\alpha\beta lj}}} \frac{\mathrm{d}W_{\alpha\beta lj,t}}{\mathrm{d}t}, \tag{3.21}$$

and manifestly show that the information about the position of the particle α along the l-th direction influences the dynamics of the particle β along the j-th direction. Moreover, they describe a pairwise implementation of the continuous measurement and feedback mechanism. As mentioned before, the measurement record links the particles in pairs: those whose position is measured (identified with the index α in Eq. (3.21)), and those receiving the information encoded in the measurement record. For a fixed particle, the other particles receive the information about its position through a different measurement record.

From the measurement records in Eq. (3.21), we can define the following feedback Hamiltonian

$$\hat{H}_{\text{fb}} = \sum_{\substack{\alpha,\beta=1\\ \beta\neq\alpha}}^{N} \sum_{l,j=1}^{3} \chi_{\alpha\beta lj} r_{\alpha\beta lj} \hat{x}_{\beta j}, \tag{3.22}$$

where we assume that the symmetries $\chi_{\alpha\beta lj} = \chi_{\beta\alpha lj} = \chi_{\alpha\beta jl}$ hold. The above Hamiltonian indicates that, for a given value of α, the measurement record $r_{\alpha\beta lj}$ influences the evolution of the positions of the rest of the masses constituting the system. From Eq. (3.22), one obtains the following evolution due to the feedback

$$(\mathrm{d}\left|\psi\right\rangle)_{\text{fb}} = -\sum_{\substack{\alpha,\beta=1\\ \beta\neq\alpha}}^{N} \sum_{l,j=1}^{3} \left[\frac{i}{\hbar} r_{\alpha\beta lj} + \frac{\chi_{\alpha\beta lj}\hat{x}_{\beta j}}{2\gamma_{\alpha\beta lj}}\right] \chi_{\alpha\beta lj}\hat{x}_{\beta j} \mathrm{d}t \left|\psi\right\rangle. \tag{3.23}$$

From Eq. (3.19) and Eq. (3.22), one obtains

$$\begin{aligned}\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = &-\frac{i}{\hbar}\left[\hat{H}_0, \hat{\rho}\right] - \frac{i}{2\hbar} \sum_{\substack{\alpha,\beta=1\\ \beta\neq\alpha}}^{N} \sum_{l,j=1}^{3} K_{\alpha\beta lj}\left[\hat{x}_{\alpha l}\hat{x}_{\beta j}, \hat{\rho}\right] \\ &- \sum_{\substack{\alpha,\beta=1\\ \beta\neq\alpha}}^{N} \sum_{l,j=1}^{3} \left(\frac{\gamma_{\alpha\beta lj}}{8\hbar^2} + \frac{1}{2}\frac{K^2_{\alpha\beta lj}}{\gamma_{\alpha\beta lj}}\right)\left[\hat{x}_{\alpha l}, \left[\hat{x}_{\alpha l}, \hat{\rho}\right]\right],\end{aligned} \tag{3.24}$$

where we assumed that $\gamma_{\alpha\beta lj} = \gamma_{\beta\alpha lj} = \gamma_{\alpha\beta jl}$, and the operators $\hat{Y}_\alpha$ were absorbed in the Hamiltonian $\hat{H}_0$.

Although the model is mathematically consistent, it was ruled out through comparison with results from experiment [55].

3.2.2 Universal KTM model

In contrast with the pairwise approach that leads to Eq. (3.24), we can consider the scenario in which, once a given particle is measured, the remaining constituents of the system receive the same information about the measurement. We will refer to this approach as a universal one. The situation is depicted in Fig. (3.3). For simplicity, one can consider a one-dimensional scenario. In this case, the measurement record is given as in Eq. (3.8), and the continuous measurement contribution to the evolution of the system is described in Eq. (3.6). One implements the gravitational interaction through the following feedback Hamiltonian

$$\hat{H}_{\text{fb}} = \sum_{\substack{\alpha,\beta=1\\ \beta\neq\alpha}}^{N} \chi_{\alpha\beta} r_\alpha \hat{x}_\beta. \tag{3.25}$$

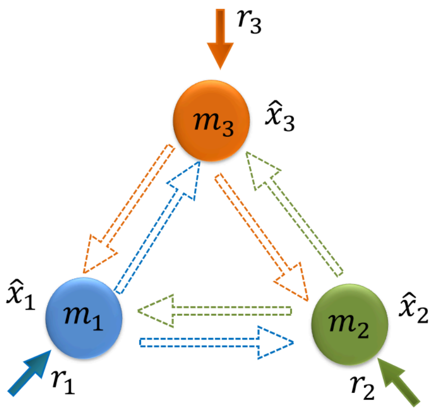

Figure 3.3: Figure adapted from Ref. [59]. In the universal generalization of the KTM model, there is a single measurement record r_α (solid arrows) of the position of each particle. Newtonian gravity is implemented through the feedback interaction (dashed arrows).

This leads to the master equation

$$\begin{aligned}\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = &-\frac{i}{\hbar}\left[\hat{H}_0, \hat{\rho}\right] - \frac{i}{2\hbar}\sum_{\substack{\alpha,\beta=1\\ \beta\neq\alpha}}^{N} \chi_{\alpha\beta}\left[\hat{x}_\beta, \{\hat{x}_\alpha, \hat{\rho}\}\right] - \sum_{\alpha=1}^{N}\frac{\gamma_\alpha}{8\hbar^2}\left[\hat{x}_\alpha, [\hat{x}_\alpha, \hat{\rho}]\right] \\ &- \sum_{\substack{\alpha,\beta,\epsilon=1\\ \beta,\epsilon\neq\alpha}}^{N} \frac{\chi_{\alpha\beta}\chi_{\alpha\epsilon}}{2\gamma_\alpha}\left[\hat{x}_\beta, [\hat{x}_\epsilon, \hat{\rho}]\right].\end{aligned} \tag{3.26}$$

Although this model reduces to the KTM model for the case of $N = 2$ particles, one can show that if one has two bodies of N_1 and N_2 constituents respectively, an considers them as two single objects, with center of mass position operators $\hat{x}_1$ and $\hat{x}_2$, inconsistencies arise at the level of the master equation. Namely, after tracing the relative degrees of freedom of the two bodies, one does not recover the KTM master equation for two particles, as one gets extra contributions proportional to $[\hat{x}_1, [\hat{x}_2, \hat{\rho}]]$. Therefore, this approach for implementing the gravitational interaction is also discarded.

3.3 TD model

The Tilloy-Diósi (TD) model retakes the KTM idea of implementing gravity through a continuous measurement and feedback mechanism to the full Newtonian interaction. Instead of the measurements of the position, they focus on those of the mass density $\hat{\mu}(\boldsymbol{x})$ of the system. The Hamiltonian that describes the gravitational interaction of the system is

$$\hat{H}_{\text{grav}} = \frac{1}{2}\int \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\mathcal{V}(\boldsymbol{x}-\boldsymbol{y})\hat{\mu}(\boldsymbol{x})\hat{\mu}(\boldsymbol{y}), \qquad \text{where} \qquad \mathcal{V}(\boldsymbol{x}-\boldsymbol{y}) = -\frac{G}{|\boldsymbol{x}-\boldsymbol{y}|}. \tag{3.27}$$

The TD model implements a continuous measurement and feedback mechanism to replace $\hat{H}_{\text{grav}}$ in Eq. (3.27), and effectively recover the gravitational interaction at the level of the master equation. The contribution due to the continuous measurement of the mass density is

$$\begin{aligned}(\mathrm{d}\,|\psi\rangle)_{\mathrm{m}} = \Bigg[&-\frac{1}{8\hbar^2}\int \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\gamma(\boldsymbol{x},\boldsymbol{y})\left(\hat{\mu}(\boldsymbol{x}) - \langle\hat{\mu}(\boldsymbol{x})\rangle\right)\left(\hat{\mu}(\boldsymbol{y}) - \langle\hat{\mu}(\boldsymbol{y})\rangle\right)\mathrm{d}t \\ &+\frac{1}{2\hbar}\int \mathrm{d}\boldsymbol{x}\left(\hat{\mu}(\boldsymbol{x}) - \langle\hat{\mu}(\boldsymbol{x})\rangle\right)\xi_{\mu,t}(\boldsymbol{x})\mathrm{d}t\Bigg]|\psi\rangle\,.\end{aligned} \tag{3.28}$$

In the above equation, the noise $\xi_{\mu,t}(\boldsymbol{x})$ is defined through the relations [cf. Eq. (1.27)]

$$\begin{aligned}\mathbb{E}\left[\xi_{\mu,t}(\boldsymbol{x})\right] &= 0, \\ \mathbb{E}\left[\xi_{\mu,t}(\boldsymbol{x})\xi_{\mu,t'}(\boldsymbol{y})\right] &= \gamma(\boldsymbol{x},\boldsymbol{y})\delta(t-t').\end{aligned} \tag{3.29}$$

In this case, the measurement record at each point of space is defined as

$$\mu(\boldsymbol{x}) = \langle\hat{\mu}(\boldsymbol{x})\rangle + \hbar\int \mathrm{d}\boldsymbol{y}\gamma^{-1}(\boldsymbol{x},\boldsymbol{y})\xi_{\mu,t}(\boldsymbol{y}), \tag{3.30}$$

where $\gamma^{-1}(\boldsymbol{x},\boldsymbol{y})$ is the inverse kernel of $\gamma(\boldsymbol{x},\boldsymbol{y})$ [cf. Eq. (1.28)]. From the measurement record, we can define the corresponding feedback Hamiltonian

$$\hat{H}_{\text{fb}} = \int \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\mathcal{V}(\boldsymbol{x}-\boldsymbol{y})\hat{\mu}(\boldsymbol{x})\mu(\boldsymbol{y}). \tag{3.31}$$

The Hamiltonian in Eq. (3.31) constitutes a universal implementation of the full Newtonian interaction, as all the constituents receive the same information about the mass density at a given point. This Hamiltonian leads to the following contribution to the total dynamics of the system

$$(\mathrm{d}\,|\psi\rangle)_{\text{fb}} = -\int \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\left\{\frac{i}{\hbar}\mathcal{V}(\boldsymbol{x}-\boldsymbol{y})\mu(\boldsymbol{y}) + \frac{1}{2}(\mathcal{V}\circ\gamma^{-1}\circ\mathcal{V})(\boldsymbol{x},\boldsymbol{y})\hat{\mu}(\boldsymbol{y})\right\}\hat{\mu}(\boldsymbol{x})\mathrm{d}t\,|\psi\rangle\,. \tag{3.32}$$

We obtain the full stochastic differential equation for the wave function from the results of Eq. (3.28) and Eq. (3.32). Straightforward calculations lead to

$$\begin{aligned}\mathrm{d}\,|\psi\rangle = \Bigg(&-\frac{i}{\hbar}\int \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\mathcal{V}(\boldsymbol{x}-\boldsymbol{y})\hat{\mu}(\boldsymbol{x})\mu(\boldsymbol{y})\mathrm{d}t - \frac{1}{2}\int \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\left(\mathcal{V}\circ\gamma^{-1}\circ\mathcal{V}\right)(\boldsymbol{x},\boldsymbol{y})\hat{\mu}(\boldsymbol{x})\hat{\mu}(\boldsymbol{y})\mathrm{d}t \\ &-\frac{1}{8\hbar^2}\int \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\left(\hat{\mu}(\boldsymbol{x}) - \langle\hat{\mu}(\boldsymbol{x})\rangle\right)\left(\hat{\mu}(\boldsymbol{y}) - \langle\hat{\mu}(\boldsymbol{y})\rangle\right)\mathrm{d}t + \frac{1}{2\hbar}\int \mathrm{d}\boldsymbol{x}\left(\hat{\mu}(\boldsymbol{x}) - \langle\hat{\mu}(\boldsymbol{x})\rangle\right)\xi_{\mu,t}(\boldsymbol{x})\mathrm{d}t \\ &-\frac{i}{2\hbar}\int \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\mathcal{V}(\boldsymbol{x}-\boldsymbol{y})\hat{\mu}(\boldsymbol{x})\left(\hat{\mu}(\boldsymbol{y}) - \langle\hat{\mu}(\boldsymbol{y})\rangle\right)\mathrm{d}t\Bigg)|\psi\rangle\,.\end{aligned} \tag{3.33}$$

The above evolution leads to the following master equation

$$\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = -\frac{i}{\hbar}\left[\hat{H}_0 + \hat{H}_{\mathrm{grav}}, \hat{\rho}\right] - \int \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} D(\boldsymbol{x},\boldsymbol{y})\left[\hat{\mu}(\boldsymbol{x}),\left[\hat{\mu}(\boldsymbol{y}),\hat{\rho}\right]\right], \tag{3.34}$$

where we included the free Hamiltonian $\hat{H}_0$. The decoherence kernel $D(\boldsymbol{x},\boldsymbol{y})$ appearing in Eq. (3.34) is defined as

$$D(\boldsymbol{x},\boldsymbol{y}) = \left[\frac{\gamma}{8\hbar^2} + \frac{1}{2}\left(\mathcal{V}\circ\gamma^{-1}\circ\mathcal{V}\right)\right](\boldsymbol{x},\boldsymbol{y}). \tag{3.35}$$

3.3.1 The divergences in the decoherence term

Thus far, we have not specified the correlation kernel $\gamma(\boldsymbol{x},\boldsymbol{y})$ in Eq. (3.35). Under the assumption of invariance under translations ($D(\boldsymbol{x},\boldsymbol{y}) = D(\boldsymbol{x}-\boldsymbol{y})$), one can show that any choice of γ leads to divergences in the decoherence term in Eq. (3.34). In order to proceed, we consider a system constituted of point-like particles. Therefore, the mass density reads

$$\hat{\mu}(\boldsymbol{x}) = \sum_{\alpha=1}^{N} m_\alpha \delta(\boldsymbol{x}-\hat{\boldsymbol{x}}_\alpha). \tag{3.36}$$

In Fourier space, the decoherence term of Eq. (3.34) with the mass density of Eq. (3.36) is given by

$$\int \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} D(\boldsymbol{x},\boldsymbol{y})\left[\hat{\mu}(\boldsymbol{x}),\left[\hat{\mu}(\boldsymbol{y}),\hat{\rho}\right]\right] = \sum_{\alpha,\beta=1}^{N} m_\alpha m_\beta \int \frac{\mathrm{d}\boldsymbol{k}\widetilde{D}(\boldsymbol{k})}{(2\pi\hbar)^{3/2}}\left[e^{-\frac{i}{\hbar}\boldsymbol{k}\cdot\hat{\boldsymbol{x}}_\alpha},\left[e^{\frac{i}{\hbar}\boldsymbol{k}\cdot\hat{\boldsymbol{x}}_\beta},\hat{\rho}\right]\right], \tag{3.37}$$

where $\widetilde{D}(\boldsymbol{k})$ is the Fourier transform of the decoherence kernel $D(\boldsymbol{x}-\boldsymbol{y})$. The divergences in the decoherence term come from the contributions in the above sum corresponding to the same particle ($\alpha=\beta$). They are proportional to

$$\int \mathrm{d}\boldsymbol{k}\widetilde{D}(\boldsymbol{k})\left(2\hat{\rho} - e^{-\frac{i}{\hbar}\boldsymbol{k}\cdot\hat{\boldsymbol{x}}_\alpha}\hat{\rho}e^{\frac{i}{\hbar}\boldsymbol{k}\cdot\hat{\boldsymbol{x}}_\alpha} - e^{\frac{i}{\hbar}\boldsymbol{k}\cdot\hat{\boldsymbol{x}}_\alpha}\hat{\rho}e^{-\frac{i}{\hbar}\boldsymbol{k}\cdot\hat{\boldsymbol{x}}_\alpha}\right). \tag{3.38}$$

Let us focus on the first term of the above expression. Explicit calculations show that

$$\int \mathrm{d}\boldsymbol{k}\widetilde{D}(\boldsymbol{k}) = \int \mathrm{d}\boldsymbol{k}\left(\frac{\widetilde{\gamma}(\boldsymbol{k})}{8\hbar^2} + \frac{\hbar G^2}{\pi}\frac{1}{k^4\widetilde{\gamma}(\boldsymbol{k})}\right), \tag{3.39}$$

where we used the property $\widetilde{\gamma}(\boldsymbol{k})\widetilde{\gamma^{-1}}(\boldsymbol{k}) = (2\pi\hbar)^{-3}$. Tilloy and Diósi proceeded to minimize the decoherence kernel $\widetilde{D}(\boldsymbol{k})$ with respect to $\widetilde{\gamma}(\boldsymbol{k})$. The minimum is achieved for

$$\widetilde{\gamma}(\boldsymbol{k}) = G(2\pi\hbar)^{3/2}/(\pi^2k^2) \Rightarrow \gamma(\boldsymbol{x},\boldsymbol{y}) = -2\hbar\mathcal{V}(\boldsymbol{x}-\boldsymbol{y}). \tag{3.40}$$

The above correlation kernel leads to the decoherence rate of the Diósi-Penrose model [43, 44], and in particular to

$$\int \mathrm{d}\boldsymbol{k}\widetilde{D}(\boldsymbol{k}) = \frac{2(2\pi\hbar)^{1/2}G}{\hbar}\int_0^\infty \mathrm{d}k \to \infty, \tag{3.41}$$

which is divergent and has been regularized by smearing the mass density [48]. Therefore, without a regularization mechanism, such as the one proposed by Tilloy and Diósi, which we describe in the following subsection, any choice of $\gamma(\boldsymbol{x},\boldsymbol{y})$ in the decoherence kernel in Eq. (3.35) leads to divergences in the corresponding master equation.

3.3.2 Regularization in the TD formalism

The regularization mechanism for the master equation [cf. Eq. (3.34)] must include both the correlation kernel $\gamma(\boldsymbol{x},\boldsymbol{y})$ as well as the Newtonian interaction $\mathcal{V}(\boldsymbol{x},\boldsymbol{y})$. Indeed, a regularization of the correlation kernel $\gamma(\boldsymbol{x},\boldsymbol{y})$ alone would result in another correlation which would still lead to divergences, as seen in the previous subsection. On the other hand, if one regularizes only the Newtonian potential $\mathcal{V}(\boldsymbol{x},\boldsymbol{y})$, one may eventually remove the divergences on the contribution in the decoherence term due to the feedback, but not those due to the measurement. These arguments lead to consider a regularization mechanism which involves both the correlation kernel $\gamma(\boldsymbol{x},\boldsymbol{y})$ and the Newtonian potential $\mathcal{V}(\boldsymbol{x},\boldsymbol{y})$.

A possible regularization mechanism consists in smearing the mass density operator [3]. Under this approach, one replaces the mass density $\hat{\mu}(\boldsymbol{x})$ with the smeared one. The latter is given by

$$\hat{\nu}(\boldsymbol{r}) = \int \mathrm{d}\boldsymbol{x} g(\boldsymbol{x}-\boldsymbol{r})\hat{\mu}(\boldsymbol{x}), \tag{3.42}$$

where $g(\boldsymbol{x}-\boldsymbol{r})$ is a smearing function. Explicit substitution of the above expression in Eq. (3.34) shows that this approach is formally equivalent to the regularization of the Newtonian potential $\mathcal{V}(\boldsymbol{x}-\boldsymbol{y})$ and the correlation kernel $\gamma(\boldsymbol{x},\boldsymbol{y})$ with the smearing function $g(\boldsymbol{x},\boldsymbol{y})$. One implements this approach through the replacements

$$\gamma \to g\circ\gamma\circ g, \qquad \text{and} \qquad \mathcal{V} \to g\circ\mathcal{V}\circ g, \tag{3.43}$$

where the smearing function $g(\boldsymbol{x},\boldsymbol{y})$ should be chosen to avoid all the divergences in Eq. (3.34) for any choice of the mass density and the correlation kernel $\gamma(\boldsymbol{x},\boldsymbol{y})$. The Hamiltonian $\hat{H}_{\text{grav}}$ becomes

$$\hat{H}'_{\text{grav}} = \frac{1}{2}\int \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\,(g\circ\mathcal{V}\circ g)\,(\boldsymbol{x},\boldsymbol{y})\hat{\mu}(\boldsymbol{x})\hat{\mu}(\boldsymbol{y}), \tag{3.44}$$

and the decoherence kernel $D(\boldsymbol{x},\boldsymbol{y})$ in Eq. (3.35) now reads

$$D'(\boldsymbol{x},\boldsymbol{y}) = \left[\frac{g\circ\gamma\circ g}{8\hbar^2} + \frac{1}{2}g\circ\left(\mathcal{V}\circ\gamma^{-1}\circ\mathcal{V}\right)\circ g\right](\boldsymbol{x},\boldsymbol{y}). \tag{3.45}$$

Therefore, the master equation (3.34) now reads

$$\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = -\frac{i}{\hbar}\left[\hat{H}_0 + \hat{H}'_{\mathrm{grav}}\right] - \int \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} D'(\boldsymbol{x},\boldsymbol{y})\left[\hat{\mu}(\boldsymbol{x}),[\hat{\mu}(\boldsymbol{y}),\hat{\rho}]\right]. \tag{3.46}$$

From now on, we refer to Eq. (3.46) as the master equation of the Tilloy-Diósi formalism. We remark that the choice $\gamma(\boldsymbol{x},\boldsymbol{y}) = -2\hbar\mathcal{V}(\boldsymbol{x},\boldsymbol{y})$, together with the regularization mechanism described above, completely specifies the TD model.

3.3.3 A model within the TD formalism

In Ref. [51], Kafri, Milburn and Taylor study the dynamical evolution of N particles of mass m forming a lattice. In order to avoid confusion with the model described in Section 3.1, we refer to their model as the KTM2 model. The mass density operator is taken as

$$\hat{\mu}(\boldsymbol{x}) = m\sum_{\alpha}\hat{n}_\alpha\delta(\boldsymbol{x}-\boldsymbol{x}_\alpha), \tag{3.47}$$

where $\hat{n}_\alpha$ stands for the number density of the α-th lattice site at the position $\boldsymbol{x}_\alpha$. They considered a smeared Newtonian potential $\hat{V}_{\alpha\beta} = \chi_{\alpha\beta}\hat{n}_\alpha\hat{n}_\beta$, where the coefficients $\chi_{\alpha\beta}$ are defined as $\chi_{\alpha\beta} = -Gm^2/\left[2\left(|\boldsymbol{x}_\alpha - \boldsymbol{x}_\beta| + a\right)\right]$, with a representing a length cutoff. By defining an ancillary field that stores a weak measurement result of the mass $m\hat{n}_\alpha$ at a given location, they implemented a measurement and feedback approach [51]. Therefore, as in the TD model, the KTM2 model implements the Newtonian interaction between the constituents of the system through the measurement of the mass density; in this case, through the use of the number operator. The corresponding master equation of the KTM2 model reads

$$\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = -\frac{i}{\hbar}\left[\hat{H}_0 + \frac{1}{2}\sum_{\substack{\alpha,\beta=1\\ \beta\neq\alpha}}^{N}\hat{V}_{\alpha\beta},\hat{\rho}\right] - \frac{\gamma}{2}\sum_{\alpha=1}^{N}[\hat{n}_\alpha,[\hat{n}_\alpha,\hat{\rho}]] - \frac{1}{2\gamma}\sum_{\substack{\alpha,\beta,\epsilon=1\\ \beta,\epsilon\neq\alpha}}^{N}\chi_{\alpha\beta}\chi_{\alpha\epsilon}[\hat{n}_\beta,[\hat{n}_\epsilon,\hat{\rho}]], \tag{3.48}$$

where we also included the evolution due to the free Hamiltonian $\hat{H}_0$, and the parameter γ stands for the information rate gained by the measurement. The KTM2 model in Eq. (3.48) can be derived from the TD master equation [cf. Eq. (3.46)]. In order to see this, let us work first with the decoherence terms in both models, and denote them by $\mathcal{L}(\hat{\rho})$. For the KTM2 model, we have

$$\mathcal{L}(\hat{\rho}) = -\sum_{\alpha=1}^{N}\left(\frac{\gamma}{2} + \frac{1}{2\gamma}\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{N}\chi^2_{\beta\alpha}\right)[\hat{n}_\alpha,[\hat{n}_\alpha,\hat{\rho}]] - \frac{1}{2\gamma}\sum_{\alpha=1}^{N}\sum_{\substack{\beta=1\\ \beta\neq\alpha}}^{N}\sum_{\substack{\epsilon=1\\ \beta\neq\alpha,\beta}}^{N}\chi_{\epsilon\alpha}\chi_{\epsilon\beta}[\hat{n}_\alpha,[\hat{n}_\beta,\hat{\rho}]], \tag{3.49}$$

where we are separating the double commutators terms that involve the same particle and those that mix different particles. On the other hand, by substituting the mass density in Eq. (3.47) in the master equation of the TD model [cf. Eq. (3.46)], one obtains

$$\mathcal{L}(\hat{\rho}) = -\sum_{\alpha=1}^{N} m^2 D'(\mathbf{0}) \left[\hat{n}_\alpha, [\hat{n}_\alpha, \hat{\rho}]\right] - \sum_{\alpha=1}^{N} \sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^{N} m^2 D'(\boldsymbol{x}_\alpha - \boldsymbol{x}_\beta) \left[\hat{n}_\alpha, [\hat{n}_\beta, \hat{\rho}]\right]. \tag{3.50}$$

From the results of Eq. (3.49) and Eq. (3.50), we can establish the following map between the KTM2 and the TD models

$$\begin{aligned} m^2 D'(\mathbf{0}) &= \frac{\gamma}{2} + \frac{1}{2\gamma} \sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^{N} \chi_{\beta\alpha}^2, \\ m^2 D'(\boldsymbol{x}_\alpha - \boldsymbol{x}_\beta) &= \frac{1}{2\gamma} \sum_{\substack{\epsilon=1 \\ \epsilon\neq\alpha,\beta}}^{N} \chi_{\epsilon\alpha}\chi_{\epsilon\beta}. \end{aligned} \tag{3.51}$$

Regarding the unitary part of both models, we will denote it by $U(\hat{\rho})$. For the KTM2 model, this term is given by

$$U(\hat{\rho}) = -\frac{i}{2\hbar} \sum_{\substack{\alpha,\beta=1 \\ \beta\neq\alpha}}^{N} \chi_{\alpha\beta} \left[\hat{n}_\alpha \hat{n}_\beta, \hat{\rho}\right], \tag{3.52}$$

whereas for the TD model we have

$$U(\hat{\rho}) = -\frac{i}{2\hbar} m^2 \sum_{\alpha=1}^{N} \mathcal{V}'(\mathbf{0}) \left[\hat{n}_\alpha^2, \hat{\rho}\right] - \frac{i}{2\hbar} m^2 \sum_{\substack{\alpha,\beta=1 \\ \beta\neq\alpha}}^{N} \mathcal{V}'(\boldsymbol{x}_\alpha - \boldsymbol{x}_\beta) \left[\hat{n}_\alpha \hat{n}_\beta, \hat{\rho}\right], \tag{3.53}$$

where $\mathcal{V}' = g \circ \mathcal{V} \circ g$. Comparing the results of Eq. (3.52) and Eq. (3.53), we obtain the following relation

$$m^2 \mathcal{V}'(\boldsymbol{x}_\alpha - \boldsymbol{x}_\beta) = \chi_{\alpha\beta}. \tag{3.54}$$

The results of Eq. (3.51) and Eq. (3.54) show that the KTM2 model can be understood within the TD formalism. The self-interaction terms that appear in the TD master equation are removed by construction in the implementation of the KTM2 model as in the definition of the feedback operator in Ref. [51], they are explicitly excluded.

3.4 Comparison between the KTM and TD models

Let us compare the master equation of the TD model [cf. Eq. (3.46)] and that of the KTM model [cf. Eq. (3.13)]. In order to do this, let us reduce the TD model to a linearized Newtonian potential regime. We consider the position operator of each particle to be given as

$$\hat{\boldsymbol{x}}_\alpha = \boldsymbol{x}_\alpha^{(0)} + \Delta\hat{\boldsymbol{x}}_\alpha, \tag{3.55}$$

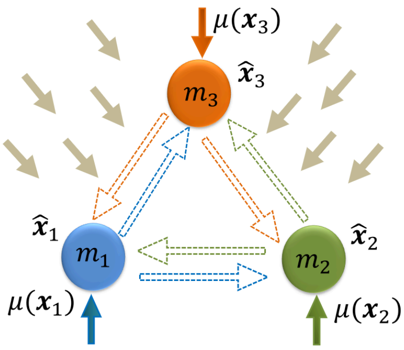

Figure 3.4: Figure adapted from Ref. [59]. In the TD model, one measures the mass density at each point of space (solid arrows). Each particle receives the same same information about the mass density of the constituents. A full Newtonian-gravity is implemented through a feedback interaction (dashed arrows).

where $\Delta\hat{\boldsymbol{x}}_\alpha$ stands for the quantum displacement from the initial position $\boldsymbol{x}_\alpha^{(0)}$. Working with the Fourier representation of the mass density of the system [cf. Eq. (3.36)], one can directly substitute the position operator $\hat{\boldsymbol{x}}_\alpha$ with the expression in Eq. (3.55). In the case of small displacements, the TD model in Eq. (3.46) can be approximated as

$$\begin{aligned}\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = &-\frac{i}{\hbar}\left[\hat{H}_0,\hat{\rho}\right] + \frac{2i\pi G}{\hbar}\sum_{\substack{\alpha,\beta=1\\ \beta\neq\alpha}}^{N}\sum_{l,j=1}^{3} m_\alpha m_\beta \eta_{\alpha\beta 2lj}\left[\hat{x}_{\alpha l}\hat{x}_{\beta j},\hat{\rho}\right] \\ &-\sum_{\alpha,\beta=1}^{N}\sum_{l,j=1}^{3} m_\alpha m_\beta \eta_{\alpha\beta lj}\left[\hat{x}_{\alpha l},\left[\hat{x}_{\beta j},\hat{\rho}\right]\right],\end{aligned} \tag{3.56}$$

where $\hat{x}_{\alpha l}$ is the component in the l direction of $\Delta\hat{\boldsymbol{x}}_\alpha$. We show a representation of the TD model in Fig. (3.4). The terms corresponding to the same particle $(\alpha = \beta)$ in $\hat{H}'_{\text{grav}}$ were absorbed in the Hamiltonian $\hat{H}_0$. The parameters $\eta_{\alpha\beta lj}$ are defined as

$$\eta_{\alpha\beta lj} = \left(\frac{\pi^3}{8\hbar^5}\right)^{1/2}\eta_{\alpha\beta 0lj} + (8\pi\hbar)^{1/2}G^2\eta_{\alpha\beta 4lj}, \tag{3.57}$$

where we set the coefficients $\eta_{\alpha\beta nlj}$ to be given by

$$\begin{aligned}\eta_{\alpha\beta 0lj} &= \int \mathrm{d}\boldsymbol{k}\, \widetilde{g}^2(\boldsymbol{k})\widetilde{\gamma}(\boldsymbol{k})k_l k_j e^{-\frac{i}{\hbar}\boldsymbol{k}\cdot(\boldsymbol{x}_\alpha^{(0)}-\boldsymbol{x}_\beta^{(0)})}, \\ \eta_{\alpha\beta 2lj} &= \int \frac{\mathrm{d}\boldsymbol{k}}{k^2}\,\widetilde{g}(\boldsymbol{k})k_l k_j e^{-\frac{i}{\hbar}\boldsymbol{k}\cdot(\boldsymbol{x}_\alpha^{(0)}-\boldsymbol{x}_\beta^{(0)})}, \\ \eta_{\alpha\beta 4lj} &= \int \frac{\mathrm{d}\boldsymbol{k}}{k^4}\,\frac{\widetilde{g}^2(\boldsymbol{k})}{\widetilde{\gamma}(\boldsymbol{k})}k_l k_j e^{-\frac{i}{\hbar}\boldsymbol{k}\cdot(\boldsymbol{x}_\alpha^{(0)}-\boldsymbol{x}_\beta^{(0)})},\end{aligned} \tag{3.58}$$

and we are assuming that both the smearing function $g(\boldsymbol{x},\boldsymbol{y})$ and the correlation kernel $\gamma(\boldsymbol{x},\boldsymbol{y})$ are both given, in order to completely characterize the coefficients $\eta_{\alpha\beta nlj}$ in the above equation.

Let us now consider the particular case of the linearized TD model in one dimension, for a system of two particles. From Eq. (3.56), we get

$$\begin{aligned}\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = &-\frac{i}{\hbar}\left[\hat{H}_0 - 2\pi G m_1 m_2 \left(\eta_{122}+\eta_{212}\right)\hat{x}_1\hat{x}_2, \hat{\rho}\right] - \sum_{\alpha=1}^{N} m_\alpha^2 \eta_{\alpha\alpha}\left[\hat{x}_\alpha,\left[\hat{x}_\alpha,\hat{\rho}\right]\right] \\ &- m_1 m_2\left(\eta_{12}\left[\hat{x}_1,\left[\hat{x}_2,\hat{\rho}\right]\right] + \eta_{21}\left[\hat{x}_2,\left[\hat{x}_1,\hat{\rho}\right]\right]\right).\end{aligned} \tag{3.59}$$

The above expression clearly differs from the KTM model in Eq. (3.13). In the KTM model, the double commutator terms contain only position operators corresponding to the same particle. In contrast, in the TD model, Eq. (3.59) mixes the position operators of different particles, as it is shown in the second line. Although both generalizations of the KTM model reduce to the result in Eq. (3.13) for two particles, the TD model and the KTM model are fundamentally different models [59].

Let us notice that one of the key aspects that explains the difference between these models relies in the definition of the noise in each model. In the KTM model, as well as its corresponding generalizations, the noises are intrinsically related to the positions of the particles, and follow them while they move in space. In the TD model, the particle feels different noises while moving in space, as by construction, there is one noise for each point of space. Moreover, the noise term in the measurement record in the KTM model [cf. Eq. (3.8)] monitors the fluctuations in the positions of a particle, whereas, in the TD model, the measurement record [cf. Eq. (3.30)] accounts for the fluctuations of the density in space without a direct reference to the particle that produces such mass density.

3.5 Full Newtonian gravity through a feedback mechanism

It is natural to ask if there are other possible ways to implement a full Newtonian gravity through a continuous measurement and feedback framework, as in the TD model. In order to address this issue, let us first consider the choice of the operator which is measured. For a system of point-like particles, it is morally the same to measure the position of the particles or the mass density of the system. However, from the physical point of view, the master equations are different as they correspond to different measurement schemes. If we measure the position as in the pairwise KTM model, then the noise is attached to the particle, while if we measure the mass density as in the TD model, then different noises act for a given particle, as the noise is defined at each point of space. Furthermore, in the case of a continuous measurement of the mass density, it is straightforward to apply the Wiseman-Milburn [1, 6] feedback formalism, as shown in Eq. (3.32). However, when one measures the position, it is not clear how to implement this prescription. Indeed, for a system of point-like particles, the Hamiltonian in Eq. (3.27) reduces to

$$\hat{H}_{\text{grav}} = -\frac{1}{2}\sum_{\substack{\alpha,\beta=1\\ \beta\neq\alpha}}^{N}\frac{Gm_\alpha m_\beta}{|\hat{\boldsymbol{x}}_\alpha - \hat{\boldsymbol{x}}_\beta|}, \tag{3.60}$$

once we remove the self-interactions. The application of the continuous measurement and feedback protocol would imply the replacement of the position operators $\hat{\boldsymbol{x}}_\alpha$ with the measurement records $\mathbf{r}_\alpha$. In this case, the feedback Hamiltonian would read

$$\hat{H}_{\text{fb}} = -\sum_{\substack{\alpha,\beta=1\\ \beta\neq\alpha}}^{N}\frac{Gm_\alpha m_\beta}{|\hat{\boldsymbol{x}}_\alpha - \mathbf{r}_\beta|}. \tag{3.61}$$

We notice that the measurement records appear in a nonlinear form which, in particular, does not allow to implement the Wiseman-Milburn prescription to obtain the feedback contribution to the dynamics. Thus, there is not a clear procedure to construct a stochastic equation that leads to a completely positive dynamics [66]. Then, there is no guarantee that the resulting master equation for the density operator will be of the Lindblad form. This may result in faster-than-light signalling, which is an unacceptable feature for any consistent theory. Therefore, within a Markovian feedback framework, we are led to consider the mass density measurement in order to implement a full Newtonian interaction.

The second point to address is the protocol to be used. In the TD model, the whole system is measured and the information about the measurement is used to drive the dynamics of the system. This is the universal approach described before. In what follows, we show that a pairwise approach does not lead to a consistent model. Indeed, let us consider a system of N point-like particles of mass density $\hat{\mu}_\alpha(\boldsymbol{x})$, and corresponding pairwise measurement records

$$\mu_{\alpha\beta}(\boldsymbol{x}) = \langle\hat{\mu}_\alpha(\boldsymbol{x})\rangle + \hbar\int \mathrm{d}\mathbf{z}\gamma^{-1}_{\alpha\beta}(\boldsymbol{x},\mathbf{z})\xi_{\mu,\alpha\beta,t}(\mathbf{z}), \tag{3.62}$$

where the noises are defined as

$$\begin{aligned}\mathbb{E}\left[\xi_{\mu,\alpha\beta,t}(\boldsymbol{x})\right] &= 0,\\ \mathbb{E}\left[\xi_{\mu,\alpha\beta,t}(\boldsymbol{x})\xi_{\mu,\alpha'\beta',t'}(\boldsymbol{y})\right] &= \delta_{\alpha\alpha'}\delta_{\beta\beta'}\gamma_{\alpha\beta}(\boldsymbol{x},\boldsymbol{y})\delta(t-t').\end{aligned} \tag{3.63}$$

If we consider the following feedback Hamiltonian

$$\hat{H}_{\text{fb}} = \sum_{\substack{\alpha,\beta=1\\ \beta\neq\alpha}}^{N}\int \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\mathcal{V}(\boldsymbol{x},\boldsymbol{y})\hat{\mu}_\beta(\boldsymbol{x})\mu_{\alpha\beta}(\boldsymbol{y}), \tag{3.64}$$

we arrive to the master equation

$$\begin{aligned}\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = &-\frac{i}{2\hbar}\sum_{\substack{\alpha,\beta=1\\ \beta\neq\alpha}}^{N}\int \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\mathcal{V}'(\boldsymbol{x},\boldsymbol{y})\left[\hat{\mu}_\alpha(\boldsymbol{x})\hat{\mu}_\beta(\boldsymbol{y}),\hat{\rho}\right]\\ &-\sum_{\substack{\alpha,\beta=1\\ \beta\neq\alpha}}^{N}\int \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}D'_{\alpha\beta}(\boldsymbol{x},\boldsymbol{y})\left[\hat{\mu}_\alpha(\boldsymbol{x}),\left[\hat{\mu}_\alpha(\boldsymbol{y}),\hat{\rho}\right]\right],\end{aligned} \tag{3.65}$$

with

$$\begin{aligned}\mathcal{V}'(\boldsymbol{x},\boldsymbol{y}) &= (g\circ\mathcal{V}\circ g)(\boldsymbol{x},\boldsymbol{y}),\\ D'_{\alpha\beta}(\boldsymbol{x},\boldsymbol{y}) &= \left(g\circ D_{\alpha\beta}\circ g\right)(\boldsymbol{x},\boldsymbol{y}),\end{aligned} \tag{3.66}$$

where the decoherence kernel $D_{\alpha\beta}(\boldsymbol{x},\boldsymbol{y})$ reads

$$D_{\alpha\beta}(\boldsymbol{x},\boldsymbol{y}) = \left[\frac{\gamma_{\alpha\beta}}{8\hbar^2}+\frac{1}{2}\left(\mathcal{V}\circ\gamma^{-1}_{\beta\alpha}\circ\mathcal{V}\right)\right](\boldsymbol{x},\boldsymbol{y}). \tag{3.67}$$

The model in Eq. (3.65) successfully recovers the quantum Hamiltonian dynamics due to the Newtonian interaction between the constituents of the system, but also leads to decoherence effects. If we assume that all the correlation kernels $\gamma_{\alpha\beta}(\boldsymbol{x},\boldsymbol{y})$ are equal, then we obtain that $D_{\alpha\beta}(\boldsymbol{x},\boldsymbol{y}) = D(\boldsymbol{x},\boldsymbol{y})$ for all α,β. One can perform a minimization procedure of the decoherence effects, which leads to $\gamma(\boldsymbol{x},\boldsymbol{y}) = -2\hbar\mathcal{V}(\boldsymbol{x},\boldsymbol{y})$, as in the TD model. Let us now consider the interaction between two systems, constituted by $N_1 = 1$ (with mass m_1) and N_2 particles, respectively, and a Gaussian smearing $g(\mathbf{z}) = (2\pi\sigma^2)^{-3/2}\exp\left(-\mathbf{z}^2/2\sigma^2\right)$. Working in Fourier space, tracing out over the degrees of freedom of the system with N_2 particles leads to

$$\begin{aligned}&\mathrm{Tr}_{N_2}\left(-\sum_{\substack{\alpha,\beta=1\\ \beta\neq\alpha}}^{N}\int \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}D'_{\alpha\beta}(\boldsymbol{x},\boldsymbol{y})\left[\hat{\mu}_\alpha(\boldsymbol{x}),\left[\hat{\mu}_\alpha(\boldsymbol{y}),\hat{\rho}\right]\right]\right)\\ &=\sum_{\beta=2}^{N}\frac{2Gm_1^2}{4\pi^2\hbar^2}\int\frac{\mathrm{d}\boldsymbol{k}}{k^2}e^{-\frac{\sigma^2}{\hbar^2}\boldsymbol{k}^2}\left(e^{\frac{i}{\hbar}\boldsymbol{k}\cdot\hat{\boldsymbol{x}}_1}\hat{\rho}_1 e^{-\frac{i}{\hbar}\boldsymbol{k}\cdot\hat{\boldsymbol{x}}_1}-\hat{\rho}_1\right),\end{aligned} \tag{3.68}$$

where we are setting $\hat{\rho}_1 = \int \mathrm{d}\boldsymbol{x}_2\cdots\mathrm{d}\boldsymbol{x}_N\,\langle\boldsymbol{x}_2|\otimes\cdots\otimes\langle\boldsymbol{x}_N|\,\hat{\rho}\,|\boldsymbol{x}_N\rangle\otimes\cdots\otimes|\boldsymbol{x}_2\rangle$. For a delocalized state such that the second term in Eq. (3.68) is dominant over the first one, the coherence decays with a rate Γ given by

$$\Gamma = \sum_{\beta=2}^{N}\frac{2Gm_1^2}{4\pi^2\hbar^2}\int\frac{\mathrm{d}\boldsymbol{k}}{k^2}e^{-\frac{\sigma^2}{\hbar^2}\boldsymbol{k}^2} = \frac{N_2Gm_1^2}{\sqrt{\pi}\hbar\sigma}. \tag{3.69}$$

The above result indicates that the rate depends on the second system through its number of constituents [59]. This result is unphysical, since one can consider the second system to be the whole Universe, and therefore, the above result would yield a vastly large decoherence rate. Therefore, unless one introduces major changes in the construction of the model, the pairwise approach is inconsistent. Let us notice that for the TD model, this inconsistency does not arise, as under the same approximations one finds that the coherence rate is given by

$$\Gamma_{\mathrm{TD}} = \frac{Gm_1^2}{\sqrt{\pi}\hbar\sigma}, \tag{3.70}$$

and therefore, depends only on the single particle of mass m_1 and not on the other system. These results indicate that, under the most natural constructions which implement gravity through a continuous measurement and Markovian feedback framework, the only physically consistent one is the Tilloy-Diósi model.

Chapter 4

Inflation and Cosmological Perturbations Theory

In this chapter we briefly review the basic aspects of inflation and the theory of cosmological perturbations, focusing on the scalar sector. The discussion mainly follows Refs. [67–69] Throughout the rest of this thesis, we work in reduced Planck units ($\hbar = 1, c = 1$).

4.1 Inflation: Basic Concepts

Let us recall that the Cosmological Principle on which standard Cosmology is constructed states that, at each epoch, the Universe is homogeneous and isotropic [70]. This principle is valid for scales larger than 100 Mpc. In the case of smaller scales, one has well developed models describing the inhomogeneous structure [68].

The metric encoding the geometry resulting from the Cosmological Principle, when expressed in comoving coordinates, is the Robertson-Walker metric [70]. This metric corresponds to the unperturbed Universe. The time coordinate t is chosen so that spacetime slices of fixed t are homogeneous and isotropic [69]. Let us consider the time-ordered sequence of three-dimensional space-like hypersurfaces which are the natural choice for surfaces of constant time and are homogeneous and isotropic. The concept of homogeneity implies that at every point of any given hypersurface the physical conditions are the same. In addition, the isotropy requirement implies that the physical conditions are identical in all directions when viewed from a given point on the hypersurface [68]. These two assumptions lead to the construction of the Robertson-Walker line element $\mathrm{d}s^2$, which, in terms of the system (x, θ, ϕ) of spatial coordinates, is described by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}x^2}{1-Kx^2} + x^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)\right], \tag{4.1}$$

where $a(t)$ is the dimensionless scale factor, and K is the curvature constant [71]. In inflationary cosmology the value $K = 0$, which corresponds to a flat (Euclidean) space, is favoured by observations [68]. In this case, we can use comoving Cartesian coordinates (x, y, z) and set the line element as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right). \tag{4.2}$$

In terms of conformal time η, which is defined through the relation

$$\mathrm{d}t = a\mathrm{d}\eta, \tag{4.3}$$

the Robertson-Walker metric is given by

$$\mathrm{d}s^2 = a^2(\eta)\left[-\mathrm{d}\eta^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right]. \tag{4.4}$$

The standard Friedmann-Lemaître-Robertson Walker cosmology is a very successful model which describes the evolution and the composition of the Universe. However, it faces some problematic aspects which can be explained by the cosmic inflation theory. Let us recall these problems [69]:

i) Flatness problem: The density parameter Ω is defined as

$$\Omega - 1 = \frac{K}{a^2H^2}, \tag{4.5}$$

where H is the Hubble parameter. At the present time, Ω is very close to unity, which implies an even closer value to 1 in the past. The present Universe requires the condition $|\Omega(t_{\text{nuc}}) - 1| \leq 10^{-16}$ to hold, where t_{nuc} is the time at nucleosynthesis. Such finely initial condition seems extremely unlikely.

This problem is also known as the *initial velocities* problem, as it can be equivalently be rephrased by saying that for a given energy density distribution, the initial Hubble velocities must be adjusted to an accuracy of 10^{-56} in order to allow the negative gravitational energy of the matter to be compensated by a positive kinetic energy (due to Hubble expansion) [68]. The importance of the initial conditions is that if this fine-tuning is not present, almost all initial conditions lead to basically two scenarios. In the first of them, the outcome is a closed universe with an almost immediate recollapse. In the other scenario, one is led to an open universe in which the density parameter Ω becomes smaller than what is compatible with observations.

ii) Horizon problem: The comoving distance that the radiation travels after decoupling is considerably larger than the comoving distance over which causal interactions can occur before the cosmic microwave background (CMB) is released. Furthermore, the Universe must be homogeneous on scales much larger than the horizon size at the time of nucleosynthesis [72]. In general, it is not clear how to explain that the observable Universe is so nearly homogeneous at early times.

Concretely, at the initial Planckian time, the size of the Universe exceeded the causality scale by 28 orders of magnitude. The energy density ϵ was distributed with a fractional variation not greater than $\delta\epsilon/\epsilon \sim 10^{-4}$ within 10^{84} causally disconnected regions. No causal physical process (i.e., those in which signals do not propagate faster than light) can explain such distribution. For this reason, this problem is also called the *homogeneity problem* [68]. The Big Bang model did not offer a satisfactory explanation of the observed near homogeneity in the temperature seen in different regions of the sky.

Inflation provided an explanation to these problems of the FLRW model. It is defined as an initial era during which the expansion rate $\dot{a}$ is accelerating [69]. Therefore, inflation requires

$$\ddot{a} > 0. \tag{4.6}$$

The above condition can be expressed as requiring that $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{Ha}\right) < 0$, which can be interpreted in a more straightforward way, as it indicates that the comoving Hubble length is decreasing with time.

Inflation solves both the flatness and the horizon problems. For the former, inflation allows to stretch any initial curvature of space to virtual flatness. For reasonable values of the density parameter Ω, inflation guarantees that it will evolve to unity to very high precision. Therefore, the flatness problem is solved by inflation, as long as the observable Universe starts well inside the horizon ($aH \ll H_0$). For the latter, inflation allows causal contact between two points further apart than the apparent horizon length [70].

The simplest scenario to implement cosmic inflation is through the consideration of a scalar field, which is denominated *inflaton* [68]. From the energy-momentum tensor of a classical field with potential $V(\varphi)$

$$T^{\alpha}_{\beta} = \varphi^{,\alpha}\varphi_{,\beta} - \left(\frac{1}{2}\varphi^{,\gamma}\varphi_{,\gamma} - V(\varphi)\right)\delta^{\alpha}_{\beta}, \tag{4.7}$$

we obtain the energy density

$$\varepsilon = \frac{1}{2}\dot{\varphi}^2 + V(\varphi), \tag{4.8}$$

and pressure

$$p = \frac{1}{2}\dot{\varphi}^2 - V(\varphi), \tag{4.9}$$

where the dot indicates a derivate with respect to cosmic time t. In order to successfully implement inflation, one requires to keep $\dot{\varphi}^2$ small compared to $V(\varphi)$ during a sufficiently long time interval, which must last at least 75 e-folds (i.e. 75 Hubble times). Under this condition, we have that $p \approx -\varepsilon$.

The inflaton, being a homogeneous classical scalar field, obeys the following equation of motion [68]

$$\ddot{\varphi} + 3H\dot{\varphi} + \frac{\partial V}{\partial \varphi} = 0, \tag{4.10}$$

which can be obtained from requiring that $T^{\alpha}_{\beta;\alpha} = 0$. The above equation, as well the Friedmann equation

$$H^2 = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\varphi}^2 + V(\varphi)\right), \tag{4.11}$$

both describe the evolution of the inflaton. Notice that in the above equation, we have already assumed a flat universe $K = 0$.

In the case of a massive scalar field ($V(\varphi) = \frac{1}{2}m^2\varphi^2$), the above equation coincides with the equation for a harmonic oscillator with a friction term which is proportional to the Hubble parameter H. For the cases in which the friction is large, the acceleration can be neglected, enforcing the so-called slow-roll regime.

For a generic potential V, one has that $H \propto \sqrt{V}$ and, in the particular case of large values of the potential, the friction term can also lead to a slow-roll stage [68]. Moreover, during inflation the slow-roll is practically always satisfied [69]. We can assume that inflation is almost-exponential, which leads to consider

$$\frac{|\dot{H}|}{H^2} \ll 1. \tag{4.12}$$

From Eq. (4.12) and Eq. (4.11), we get the following condition

$$3H\dot{\varphi} + \frac{\partial V}{\partial \varphi} \simeq 0, \tag{4.13}$$

which is equivalent to neglecting the acceleration term in Eq. (4.10), i.e. setting the inequality $|\ddot{\varphi}| \ll 3H|\dot{\varphi}|$. This is the condition required for slow-roll, and it indicates that $\dot{\varphi}$ does not change appreciably in one Hubble time [69]. Taking the time derivative of Eq. (4.13), we obtain

$$\ddot{\varphi} \simeq -\frac{\dot{H}}{H}\dot{\varphi} - \frac{\dot{\varphi}}{3H}\frac{\partial^2 V}{\partial \varphi^2}. \tag{4.14}$$

This condition, along with Eq. (4.12) and Eq. (4.13), constitute the slow-roll approximation [69], which in turn implies the so-called flatness conditions:

$$\begin{aligned}
&\epsilon_{\mathrm{f}}(\varphi) \ll 1 \qquad \text{where} \qquad \epsilon_{\mathrm{f}} \equiv \frac{M_{\mathrm{Pl}^2}}{2}\left(\frac{1}{V}\frac{\partial V}{\partial \varphi}\right)^2, \\
&|\eta_{\mathrm{f}}(\varphi)| \ll 1 \qquad \text{where} \qquad \eta_{\mathrm{f}} \simeq M_{\mathrm{Pl}}^2 \frac{1}{V}\frac{\partial^2 V}{\partial \varphi^2},
\end{aligned} \tag{4.15}$$

where $M_{\mathrm{Pl}} = (8\pi G)^{-1/2}$ is the reduced Planck mass. The importance of the flatness conditions relies on two facts. The first one is that inflation will not take place if these conditions are violated and that they are necessary for almost-exponential inflation. And the second one is that, for any reasonable initial value of $\dot{\varphi}$, the flatness conditions are sufficient for having a slow-roll inflation [69].

4.2 Cosmological perturbations

A more realistic representation of the Universe includes the presence of perturbations. If we denote by $g_{\mu\nu}^{(0)}$ the Robertson-Walker metric described in Eq. (4.1), which constitutes the background, then the full metric can be written as

$$g_{\mu\nu} = g_{\mu\nu}^{(0)} + \delta g_{\mu\nu}. \tag{4.16}$$

We will limit ourselves to consider only scalar perturbations, which in general, may lead to growing inhomogeneities that in turn have effects on the dynamical evolution of matter. For a spatially flat universe, scalar quantities may enter into δg_{ij} by taking ordinary partial derivatives of a scalar function. This is possible as, in the flat universe, the ordinary partial derivatives coincide with the covariant ones. In what follows, using the notation in Ref. [67], the (background) three-dimensional covariant derivative of a function f with respect to some coordinate i will be denoted as $f_{|i}$. The most general form of the line element for the background and scalar perturbations is given by

$$\mathrm{d}s^2 = a^2(\eta)\left\{(1+2\phi)\mathrm{d}\eta^2 - 2B_{|i}\mathrm{d}x^i\mathrm{d}\eta - \left[(1-2\psi)\gamma_{ij} + 2E_{|ij}\right]\mathrm{d}x^i\mathrm{d}x^j\right\}, \tag{4.17}$$

where ϕ, ψ, B and E are four scalar quantities, which depend both on the spatial and time coordinates, and γ_{ij} is the spatial background metric tensor. These four functions allow to construct gauge-invariant quantities. The simplest of them were introduced by Bardeen [73] and span the two-dimensional space of gauge-invariant variables. They read

$$\begin{aligned} \Phi &= \phi + \frac{1}{a}\left[(B-E')a\right]', \\ \Psi &= \psi + \frac{a'}{a}(B-E'), \end{aligned} \tag{4.18}$$

where the prime indicates a derivative with respect to conformal time η. In general, the freedom of gauge choice allows to impose two conditions on the four functions ϕ, ψ, B and E.

For a metric with small perturbations, the Einstein tensor $G^{\mu}{}_{\nu}$ can be written in the form

$$G^{\mu}{}_{\nu} = {}^{(0)}G^{\mu}{}_{\nu} + \delta G^{\mu}{}_{\nu}, \tag{4.19}$$

and similarly for the energy-momentum tensor $T^{\mu}{}_{\nu}$. For small perturbations linearized around the background metric, the equations of motion read

$$\delta G^{\mu}{}_{\nu} = \frac{1}{M_{\mathrm{Pl}}^2}\delta T^{\mu}{}_{\nu}. \tag{4.20}$$

Although both sides of the perturbed Einstein equations are not separately invariant under gauge transformations, one can construct gauge-invariant variables for both tensors through the gauge-invariant variables Φ and Ψ in Eq. (4.18).

For the case of a scalar field minimally coupled to gravity, the action reads

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2}\varphi^{;\alpha}\varphi_{;\alpha} - V(\varphi)\right]. \tag{4.21}$$

If one considers a homogeneous and isotropic universe with scalar perturbations as in Eq. (4.17), then the field $\varphi(\eta, \boldsymbol{x})$ can be decomposed as

$$\varphi(\eta, \boldsymbol{x}) = \varphi_0(\eta) + \delta\varphi(\eta, \boldsymbol{x}). \tag{4.22}$$

In this case, the gauge-invariant perturbations of the energy-momentum tensor can be written in terms of the Bardeen variable Φ [cf. Eq. (4.18)] and the gauge-invariant perturbation of the scalar field

$$\delta\varphi^{(\mathrm{gi})} = \delta\varphi + \varphi_0'(B - E'). \tag{4.23}$$

In what follows, we will work under the assumption $K = 0$ [cf. Eq. (4.1)].

If we now consider a theory with total action

$$S = -\frac{M_{\mathrm{Pl}}^2}{2}\int \mathrm{d}^4x\sqrt{-g}R + \int \mathrm{d}^4x\sqrt{-g}\left(\frac{1}{2}\varphi_{,\alpha}\varphi^{,\alpha} - V(\varphi)\right), \tag{4.24}$$

the corresponding background equations read

$$\mathcal{H}^2 = \frac{1}{3M_{\mathrm{Pl}}^2}\left(\frac{1}{2}{\varphi_0'}^2 + V(\varphi_0)a^2\right), \tag{4.25}$$

$$2\mathcal{H}' + \mathcal{H}^2 = \frac{1}{M_{\mathrm{Pl}}^2}\left(-\frac{1}{2}{\varphi_0'}^2 + V(\varphi_0)a^2\right), \tag{4.26}$$

where $\mathcal{H}$ is the Hubble parameter in conformal time. From the results of Eq. (4.25) and Eq. (4.26), one can derive the following relation

$$\mathcal{H}^2 - \mathcal{H}' = \frac{1}{2M_{\mathrm{Pl}}^2}{\varphi_0'}^2. \tag{4.27}$$

Let us now define the first Hubble flow parameter

$$\epsilon = 1 - \frac{\mathcal{H}'}{\mathcal{H}^2}. \tag{4.28}$$

From Eq. (4.27), ϵ can be recast in the following form

$$\epsilon = \frac{{\varphi_0'}^2}{2M_{\mathrm{Pl}}^2\mathcal{H}^2}. \tag{4.29}$$

Let us introduce the gauge-invariant potential

$$v = a\left[\delta\varphi + \left(\frac{\varphi_0'}{\mathcal{H}}\right)\psi\right] = a\left[\delta\varphi^{(\mathrm{gi})} + \frac{\varphi_0'}{\mathcal{H}}\Psi\right], \tag{4.30}$$

where the gauge-invariant scalar field variation $\delta\varphi^{(\mathrm{gi})}$ is given in Eq. (4.23). The gauge invariant quantity in Eq. (4.30) is called Mukhanov-Sasaki variable [67, 74]. In terms of this quantity, the expansion of the action in Eq. (4.24) up to the second order in the perturbations reads

$$\delta S = \frac{1}{2}\int \mathrm{d}^4x\left(v'^2 - v_{,i}v_{,i} + \frac{z''}{z}v^2\right). \tag{4.31}$$

In the above expression, we defined

$$z = aM_{\rm Pl}\sqrt{2\epsilon}/c_s = \frac{a\varphi_0'}{\mathcal{H}}, \tag{4.32}$$

where c_s is the speed of sound ($c_s = 1$ during inflation and $c_s = 1/\sqrt{3}$ during the radiation-dominated era). We remark that the first equality in the above equation is a general expression, valid at any stage of the evolution of the Universe, and the last equality is valid in the inflationary stage. The action in Eq. (4.31) is a particular case of the more general expression

$$\delta S = \int \mathcal{L} \mathrm{d}^4x = \frac{1}{2}\int \sqrt{\gamma}\mathrm{d}^4x \left(v'^2 - c_s^2\gamma^{ij}v_{,i}v_{,j} + \frac{z''}{z}v^2\right), \tag{4.33}$$

where γ^{ik} is the metric on the background hypersurfaces of constant conformal time η, γ the corresponding determinant, and we assume that the parameter c_s^2 is time-independent. Moreover, for the case of a flat universe, we have that $\gamma^{ik} = \delta^{ik}$, and therefore $\sqrt{\gamma} = 1$. In order to quantize the action in Eq. (4.33), one needs to define the momentum π canonically conjugate to v, which is given by $\partial\mathcal{L}/\partial v'$. From the expression for the Lagrangian in Eq. (4.33), one obtains

$$\pi(\eta, \boldsymbol{x}) = v'(\eta, \boldsymbol{x}). \tag{4.34}$$

From Eq. (4.34) and Eq. (4.30), in the case of a flat universe, the Hamiltonian reads

$$H = \frac{1}{2}\int \mathrm{d}\mathbf{x}\sqrt{\gamma}\left(\pi^2 + c_s^2\delta^{ij}v_{,i}v_{,j} - \frac{z''}{z}v^2\right). \tag{4.35}$$

In order to quantize the Mukhanov-Sasaki variable v and the conjugate momentum π, one requires the associated operators $\hat{v}$ and $\hat{\pi}$ to satisfy the following equal time commutation relations

$$\begin{aligned} &[\hat{v}(\eta, \boldsymbol{x}), \hat{v}(\eta, \boldsymbol{x}')] = 0, \\ &[\hat{\pi}(\eta, \boldsymbol{x}), \hat{\pi}(\eta, \boldsymbol{x}')] = 0, \\ &[\hat{v}(\eta, \boldsymbol{x}), \hat{\pi}(\eta, \boldsymbol{x}')] = i\delta(\boldsymbol{x} - \boldsymbol{x}'). \end{aligned} \tag{4.36}$$

From the variation of the action Eq. (4.33) with respect to the variable v, one gets the field equation for the operator $\hat{v}$, which reads

$$\hat{v}'' - c_s^2\nabla^2\hat{v} - \frac{z''}{z}\hat{v} = 0. \tag{4.37}$$

It is convenient to work with the Fourier mode decomposition of the operator $\hat{v}$, namely:

$$\hat{v}(\eta, \boldsymbol{x}) = \frac{1}{\sqrt{2}}\int \frac{\mathrm{d}\boldsymbol{k}}{(2\pi)^{3/2}}\left[v_{\boldsymbol{k}}^*(\eta)e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\hat{a}_{\boldsymbol{k}} + v_{\boldsymbol{k}}(\eta)e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\hat{a}_{\boldsymbol{k}}^\dagger\right], \tag{4.38}$$

where the creation ($\hat{a}^\dagger_{\boldsymbol{k}}$) and annihilation operators ($\hat{a}_{\boldsymbol{k}}$) obey the following bosonic commutation relations

$$\begin{aligned} [\hat{a}_{\boldsymbol{k}}, \hat{a}_{\boldsymbol{k}'}] &= 0, \\ \left[\hat{a}^\dagger_{\boldsymbol{k}}, \hat{a}^\dagger_{\boldsymbol{k}'}\right] &= 0, \\ \left[\hat{a}_{\boldsymbol{k}}, \hat{a}^\dagger_{\boldsymbol{k}'}\right] &= \delta(\boldsymbol{k} - \boldsymbol{k}''). \end{aligned} \tag{4.39}$$

Here, we defined the mode functions $v_{\boldsymbol{k}}(\eta)$, which satisfy the relation

$$v''_{\boldsymbol{k}}(\eta) + \omega_k^2(\eta) v_{\boldsymbol{k}}(\eta) = 0, \qquad \text{where} \qquad \omega_k^2(\eta) \equiv c_s^2 k^2 - \frac{z''}{z}, \tag{4.40}$$

and we see the explicit time dependence in the frequency $\omega_k(\eta)$. The commutation relations defined for the canonical variable $\hat{v}$ and its conjugate momentum $\hat{\pi}$ in Eq. (4.36) are consistent with the commutation relations for the creation and annihilation operators in Eq. (4.39) as long as the modes $v_{\boldsymbol{k}}(\eta)$ obey the condition

$$v'_{\boldsymbol{k}}(\eta) v^*_{\boldsymbol{k}}(\eta) - v_{\boldsymbol{k}}(\eta) v^{*\prime}_{\boldsymbol{k}}(\eta) = 2i. \tag{4.41}$$

To fully characterize the mode solution $v_{\boldsymbol{k}}(\eta)$, one needs to set initial conditions for the mode as well as for its derivative, for the initial time $\eta = \eta_0$. In addition, it is necessary to define the vacuum state $|0\rangle$, which is the quantum state that satisfies

$$\hat{a}_{\boldsymbol{k}} |0\rangle = 0, \qquad \forall \boldsymbol{k}. \tag{4.42}$$

During inflation one has $c_s^2 = 1$. Moreover, if we consider a constant Hubble parameter H_{inf}, the scale factor reads

$$a(\eta) = -\frac{1}{\eta H_{\text{inf}}}, \tag{4.43}$$

and therefore the solution to Eq. (4.40) with the initial condition

$$v_{\boldsymbol{k}}(\eta) = \frac{1}{\sqrt{2k}} e^{-ik\eta}, \tag{4.44}$$

is given by

$$v_{\boldsymbol{k}}(\eta) = \frac{1}{\sqrt{2k}} e^{-ik\eta} \left(1 - \frac{i}{k\eta}\right). \tag{4.45}$$

We are interested in calculating the curvature perturbation $\hat{\mathcal{R}}$ [75], which is related to the variable $\hat{v}$ as

$$\hat{\mathcal{R}} = -\frac{\hat{v}}{z}, \tag{4.46}$$

in the comoving gauge. The comoving curvature perturbation $\hat{\mathcal{R}}$ represents the gravitational potential on constant -φ surfaces. The corresponding power spectrum $\mathcal{P}_{\hat{\mathcal{R}}}(k)$ is related to the expectation value of the operator $\hat{\mathcal{R}}^2$ as

$$\langle 0| \hat{\mathcal{R}}^2(\eta, \boldsymbol{x}) |0\rangle = \langle 0| \frac{\hat{v}^2(\boldsymbol{x}, \eta)}{z^2} |0\rangle = \int \mathrm{d} \ln k \; \mathcal{P}_{\hat{\mathcal{R}}}(k), \tag{4.47}$$

where $k = |\mathbf{k}|$. These results will allow us to calculate the modifications to the power spectrum of $\hat{\mathcal{R}}$ due to the inclusion of dynamical collapse models.

Chapter 5

Dynamical Collapse Models

Dynamical collapse models are phenomenological models which propose the addition of nonlinear and stochastic terms to the standard Schrödinger evolution, in order to account for the collapse of the wave function of the system without imposing it through a postulate. In this chapter, we briefly review the basic concepts which define this alternative theory to Quantum Mechanics. We also discuss their connection with Quantum Measurement Theory and, in particular, with Quantum continuous measurements. This discussion follows mainly from Ref. [76].

5.1 An alternative to Quantum Mechanics

Quantum Mechanics is undoubtedly one of the most successful theories in Physics, as it has been able to explain a wide spectrum of phenomena, and is not yet disproved by experimental evidence. However, at the level of its foundations, there are issues for which the standard Copenhagen interpretation does not provide full and incontrovertible explanations. Among these issues, one has the lack of a mass scale that divides microscopic from macroscopic objects [76]. This is important because the superposition principle holds for microscopic objects, but there are no observed superpositions in different position states of macroscopic systems. Although at the pragmatical level, one can implement without problems the postulates of Quantum Mechanics and compare the predictions of the theory with the experimental evidence, at a theoretical level the separation between micro and macro is artificial and ill-defined.

Moreover, the Copenhagen interpretation postulates that the evolution of a system takes place in two different forms, depending if it is being measured or not [76, 77]. During a measurement, the evolution is nondeterministic and probabilistic. At any other time, the evolution is deterministic, and it follows the Schrödinger equation. This framework does not solve the so-called quantum measurement problem [78].

The above issues have given rise to either alternative interpretations of standard Quantum Mechanics, or to the proposal of alternative theories [79–81]. In particular, we will focus on the works in Refs. [4, 82, 83], which have pointed out that there must be a universal dynamics to which both Quantum and Classical Mechanics are approximations in appropriate limits. They proposed that this universal dynamics should be stochastic and nonlinear. On one hand, the nonlinearity allows the breakdown of superpositions during a measurement. On the other hand, the stochasticity of the dynamics indicates that the outcomes of the measurement occur randomly. It is necessary in order to avoid faster-than-light communication due to the presence of the nonlinear terms. Both of these characteristics should be negligible for microscopic systems, in order to reproduce the experimentally observed superposition in quantum systems, and to preserve the deterministic Schrödinger evolution [76].

The acknowledgement of the need to introduce a nonlinear equation to describe the quantum evolution of a system [84], instead of the standard Schrödinger equation, led to the Quantum Mechanics with Spontaneous Localization (QMSL) model. This model, which is also referred to as Ghirardi-Rimini-Weber (GRW) model, constructs a dynamical reduction model based on two main lines [82]. The first one is to guarantee the definite position in space to macroscopic objects, through the choice of a preferred basis in which the reduction takes place. The other one is the existence of an amplification mechanism that appears when going from a microscopic to a macroscopic level. This mechanism allows to recover the classical-like behaviour of macroscopic objects, as the wave function of an object collapses with a rate which is proportional to the size of the system [76]. The GRW model relies on two assumptions, which are applied to a system of N distinguishable particles. The first one is that each of these particles experiences a sudden spontaneous localization process. The second one is that between successive spontaneous processes, the system evolves following the Schrödinger equation.

In the context of dynamical collapse models one should also refer to the Diósi-Penrose model [43, 44], which proposes a different dynamics, derived from gravitational considerations, in place of that described by the Schrödinger equation. Moreover, to explain the lack of universal macroscopic quantum fluctuations, the Quantum Mechanics with Universal Density Localization (QMUDL) model was proposed. The latter is based on density localization processes whose strength is proportional to the gravitational constant [4, 49] . This construction resembles the localization processes implemented in the Quantum Mechanics with Universal Position Localization (QMUPL) proposal, which in turn presents slight differences with respect to the GRW model [49].

The ideas of the GRW model were retaken and refined in the Continuous Spontaneous Localization (CSL) model [4, 83]. In this proposal, one implements the collapse towards one of the spatially localized eigenstates of the particle number density operator of a quantum system, through the assumption of the existence of a randomly fluctuating field that couples with the particle number density operator. In its mass proportional version, this model is defined at the level of the wave function through the following stochastic differential equation [76]

$$\begin{aligned}\mathrm{d}\left|\psi\right\rangle = \Bigg[&-\frac{i}{\hbar}\hat{H}\mathrm{d}t + \frac{\sqrt{\gamma}}{m_0}\int \mathrm{d}\boldsymbol{x}\left[\hat{M}(\boldsymbol{x}) - \left\langle \hat{M}(\boldsymbol{x})\right\rangle\right]\mathrm{d}W_t(\boldsymbol{x}) \\ &-\frac{\gamma}{2m_0^2}\int \mathrm{d}\boldsymbol{x}\left[\hat{M}(\boldsymbol{x}) - \left\langle \hat{M}(\boldsymbol{x})\right\rangle\right]^2\mathrm{d}t\Bigg]\left|\psi\right\rangle ,\end{aligned} \tag{5.1}$$

where $\hat{H}$ is the standard quantum Hamiltonian of the system and the parameter

γ is a coupling constant that encodes the strength of the collapse process. In addition, $W_t(\boldsymbol{x})$ denotes an ensemble of independent Wiener processes, one defined at each space point $\boldsymbol{x}$. They are characterized through the correlation

$$\mathbb{E}[\xi_t(\boldsymbol{x})\xi_{t'}(\boldsymbol{y})] = \delta(\boldsymbol{x}-\boldsymbol{y})\delta(t-t'), \tag{5.2}$$

where $\xi(t,\boldsymbol{x}) = \mathrm{d}W_t(\boldsymbol{x})/\mathrm{d}t$ is the associated white-noise field, and $\mathbb{E}[\cdot]$ denotes the stochastic average, as before. Finally, the operator $\hat{M}(\boldsymbol{x})$ in Eq. (5.1) is a smeared mass density operator

$$\hat{M}(\boldsymbol{x}) = \sum_j m_j \hat{N}_j(\boldsymbol{x}), \qquad \hat{N}_j(\boldsymbol{x}) = \frac{1}{(\sqrt{2\pi}r_c)^3}\int \mathrm{d}\boldsymbol{y} e^{-(\boldsymbol{y}-\boldsymbol{x})^2/2r_c^2}\hat{a}_j^\dagger(\boldsymbol{y})\hat{a}_j(\boldsymbol{y}), \tag{5.3}$$

where $\hat{a}_j^\dagger(\boldsymbol{y})$ and $\hat{a}_j(\boldsymbol{y})$ are the creation and annihilation operators of a particle of type j in the space point $\boldsymbol{y}$, and r_c is a second phenomenological parameter of the model. The collapse operators are the density number operators $\hat{a}_j^\dagger(\boldsymbol{y})\hat{a}_j(\boldsymbol{y})$. This leads to the suppression of superpositions containing different number of particles in different points of space.

An equivalent form of writing the stochastic differential equation for the wave function of the CSL model is the following

$$\begin{aligned}\mathrm{d}\left|\psi\right\rangle = \Bigg[&-\frac{i}{\hbar}\hat{H}\mathrm{d}t + \frac{\sqrt{\gamma}}{m_0}\int \mathrm{d}\boldsymbol{x}\left[\hat{\mathcal{M}}(\boldsymbol{x}) - \left\langle\hat{\mathcal{M}}(\boldsymbol{x})\right\rangle\right]\mathrm{d}\overline{W}_t(\boldsymbol{x}) \\ &-\frac{\gamma}{2m_0^2}\int \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\left[\hat{\mathcal{M}}(\boldsymbol{x}) - \left\langle\hat{\mathcal{M}}(\boldsymbol{x})\right\rangle\right]G(\boldsymbol{x}-\boldsymbol{y})\left[\hat{\mathcal{M}}(\boldsymbol{y}) - \left\langle\hat{\mathcal{M}}(\boldsymbol{y})\right\rangle\right]\mathrm{d}t\Bigg]\left|\psi\right\rangle,\end{aligned} \tag{5.4}$$

where now the mass density operator $\hat{\mathcal{M}}(\boldsymbol{x})$ is not smeared. Namely, it has the form

$$\hat{\mathcal{M}}(\boldsymbol{x}) = \sum_j m_j \hat{\mathcal{N}}_j(\boldsymbol{x}) = \sum_j m_j \hat{a}_j^\dagger(\boldsymbol{x})\hat{a}_j(\boldsymbol{x}), \tag{5.5}$$

and

$$G(\boldsymbol{x}) = \frac{1}{(4\pi r_c^2)^{3/2}}e^{-\boldsymbol{x}^2/4r_c^2}. \tag{5.6}$$

The noises $\overline{W}_t(\boldsymbol{x})$ are now correlated as

$$\mathbb{E}[\overline{\xi}_t(\boldsymbol{x})\overline{\xi}_{t'}(\boldsymbol{y})] = G(\boldsymbol{x}-\boldsymbol{y})\delta(t-t'). \tag{5.7}$$

The importance of this equivalent form of expressing the stochastic differential equation of the CSL model is that one can provide a physical interpretation to the noise assumed in the model. Namely, it can be interpreted as a classical random field filling space, which is Gaussian correlated in space.

To see the connection between the CSL model as described in Eq. (5.1) and Quantum Measurement Theory, let us consider once again the stochastic differential equation that describes the continuous measurement of a quantum observable, namely Eq. (1.29), and set

$$\hat{a}(\boldsymbol{x}) = \hat{M}(\boldsymbol{x}), \qquad \delta a_t(\boldsymbol{x}) = \frac{2\hbar\sqrt{\gamma}}{m_0}\xi_t(\boldsymbol{x}) \Rightarrow \gamma(\boldsymbol{x},\boldsymbol{y}) = \frac{4\hbar^2\gamma}{m_0^2}\delta(\boldsymbol{x}-\boldsymbol{y}). \tag{5.8}$$

In this way, we arrive at the stochastic differential equation of the CSL model [cf. Eq. (5.1)]. This shows that formally, the CSL model is indistinguishable from a Quantum continuous measurement of the smeared mass density operator defined in Eq. (5.3), and therefore, we can use the concepts of Quantum Measurement Theory in order to describe the properties of the CSL model. The relation between spontaneous collapse models and continuous measurements was already pointed out in Ref [85]. From Eq. (1.33), we have that the CSL master equation reads

$$\frac{\mathrm{d}\hat{\rho}}{\mathrm{d}t} = -\frac{\gamma}{2m_0^2}\int \mathrm{d}\boldsymbol{x}\left[\hat{M}(\boldsymbol{x}),\left[\hat{M}(\boldsymbol{x}),\hat{\rho}\right]\right]. \tag{5.9}$$

For completeness, let us notice that the alternative stochastic differential equation of the CSL model [cf. Eq. (5.4)] can be obtained from Eq. (1.29) by setting

$$\hat{a}(\boldsymbol{x}) = \hat{\mathcal{M}}(\boldsymbol{x}), \qquad \delta a_t(\boldsymbol{x}) = \frac{2\hbar\sqrt{\gamma}}{m_0}\bar{\xi}_t(\boldsymbol{x}) \Rightarrow \gamma(\boldsymbol{x},\boldsymbol{y}) = \frac{4\hbar^2\gamma}{m_0^2}G(\boldsymbol{x}-\boldsymbol{y}), \tag{5.10}$$

which makes manifest the fact that in the equivalent form the correlation is redefined as a Gaussian one, and therefore the mass density operator $\hat{\mathcal{M}}(\boldsymbol{x})$ is just the standard one.

Let us finally notice the following property about the expectation values of operators. For an arbitrary operator $\hat{O}$, the expectation value $\bar{O} = \mathbb{E}\left[\langle\psi|\,\hat{O}\,|\psi\rangle\right]$ can be calculated in terms of the density operator $\hat{\rho}$ as $\bar{O} = \mathrm{Tr}\left[\hat{O}\hat{\rho}\right]$. This allows to use any unravelling that yields the same master equation of the CSL model in order to calculate the expectation values of the observables. In particular, let us consider the following unravelling, which is equivalent to that in Eq. (5.4):

$$\begin{aligned}\mathrm{d}\,|\psi\rangle = \Bigg[&-\frac{i}{\hbar}\hat{H}\mathrm{d}t - i\frac{\sqrt{\gamma}}{m_0}\int \mathrm{dx}\hat{\mathcal{M}}(\boldsymbol{x})\mathrm{d}\overline{W}_t(\boldsymbol{x}) \\ &-\frac{\gamma}{2m_0^2}\int \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}G(\boldsymbol{y}-\boldsymbol{x})\hat{\mathcal{M}}(\boldsymbol{x})\hat{\mathcal{M}}(\boldsymbol{y})\mathrm{d}t\Bigg]|\psi\rangle\,.\end{aligned} \tag{5.11}$$

Thus far, the stochastic differential equations for the wave function have been expressed in the Itô formalism. Let us now consider the corresponding Stratonovich form of the unravelling in Eq. (5.11). We have

$$\frac{\mathrm{d}\,|\psi\rangle}{\mathrm{d}t} = -\frac{i}{\hbar}\left[\hat{H} + \frac{\hbar\sqrt{\gamma}}{m_0}\int \mathrm{d}\boldsymbol{x}\hat{\mathcal{M}}(\boldsymbol{x})\bar{\xi}_t(\boldsymbol{x})\right]|\psi\rangle\,. \tag{5.12}$$

From the above equation we can interpret the CSL model to be a modification of the standard Schrödinger evolution by the addition of the following stochastic

Hamiltonian

$$\hat{H}_{\mathrm{CSL}} = \frac{\hbar\sqrt{\gamma}}{m_0} \int \mathrm{d}\boldsymbol{x} \hat{\mathcal{M}}(\boldsymbol{x}) \bar{\xi}_t(\boldsymbol{x}). \tag{5.13}$$

The use of $\hat{H}_{\mathrm{CSL}}$ in order to calculate the expectation value of a quantum operator $\bar{O}$ is known as the *noise trick* [86], and will prove to be a key element in the implementation of dynamical collapse models in a Cosmological scenario that is presented in Chapter 6.

5.2 Dynamical Collapse models and Cosmology

One of the open problems of dynamical collapse models is to derive a consistent generalization that is valid within a relativistic scenario [87–92]. However, different implementations of dynamical collapse models (or of the need of describing a collapse of the wavefunction), and in particular of the CSL model, have been proposed within a cosmological context [93–114]. The choice to work with the CSL model follows from its robustness and the fact that it is regarded as the most advanced collapse model [76]. We recall that the Cosmological Principle leads to consider the existence of a privileged frame, namely the frame comoving with the cosmic fluid, in which the Robertson-Walker metric is defined. This indicates that, within a cosmological context, there is a natural notion of time [115]. Moreover, let us recall that one of the main motivations which led in the first place to the construction of dynamical collapse models was to find solutions to the quantum measurement problem [78]. In this regard, it is argued that, in a cosmological context, this issue becomes more manifest and, in addition, concepts such as “observers" and “detectors" do not have precise definitions within a primordial Universe [112]. The fact that dynamical collapse models do not depend on the existence of an observer in order to describe the dynamical evolution of a system constitutes an appealing feature for considering them within a cosmological context, where the origin of the noise which is assumed in the model might be explained [116]. In addition, collapse models are falsifiable models, and therefore the theoretical predictions can be verified with observations.

As mentioned before, the CSL model leaves two parameters to be determined, namely λ and r_c. Among the constraints of these parameters, some have considered Cosmology related phenomena to compare the theoretical predictions due to the CSL model with observations. These works obtained bounds from a consideration of the heating of the intergalactic medium [94, 95]. In addition, the importance of the collapse of the state vector has been acknowledged in works implementing the construction of chronogenesis and cosmogenesis models [102]. Moreover, collapse models have been considered as a candidate to implement an effective cosmological constant [117]. Here, one exploits the fact that the narrowing of the wavefunction amounts to violations of energy conservation [76]. However, most of the work in this area is related to possible effects of collapse models during inflation and the emergence of the cosmic structure in the Universe. Regarding the latter, several works remark the necessity of a collapse mechanism to provide a satisfactory explanation of the origin of the observed structure [93, 98, 99, 103, 104]. As for the former, the work is focused on the study of the modifications to the spectra of primordial perturbations, either at a scalar or at a tensorial level [105–107, 109, 111, 112].

Despite the research that has already been developed, several doubts have arisen about how to implement dynamical collapse models within a cosmological context. The origin of these problems lies, as mentioned before, in the lack of a consistent and universal generalization of dynamical collapse models, and in particular of the CSL model. Concretely, the choice of what should be the proper collapse operator that generalizes the non-relativistic collapse models is unclear [113, 114]. We took into account these problems when proposing an implementation of collapse model within an inflationary context, as described in the following chapter.

Chapter 6

Dynamical Collapse Models effects on the Evolution of the Comoving Curvature Perturbation

In this chapter we present a framework to account for the effects of dynamical collapse models on the power spectrum of the comoving curvature perturbation $\hat{\mathcal{R}}$. We apply such a framework to the inflationary stage of the evolution of the Universe, as well as to the radiation-dominated era. The results in this section follow from Ref. [118].

6.1 An interaction picture approach

Following previous works which discuss the effects of the CSL model on the power spectrum of the comoving curvature perturbation $\hat{\mathcal{R}}$ during inflation [112–114], we tackle the same problem. In order to do so, let us consider that the total Hamiltonian describing the system is given by

$$\hat{H}_{\text{total}} = \hat{H} + \hat{H}_{\text{DC}}, \tag{6.1}$$

where $\hat{H}$ is the Hamiltonian of the scalar perturbations in standard Cosmology, i.e. in absence of dynamical collapse model effects; while the Hamiltonian $\hat{H}_{\text{DC}}$ encodes the contribution to dynamical collapse models to the evolution of the perturbations.

Let us denote with $\hat{U}$ and $\hat{U}_{\text{DC}}$ the evolution operators due to the Hamiltonians $\hat{H}$ and $\hat{H}_{\text{DC}}$, respectively. The evolution operators are explicitly given by

$$\hat{U} = \mathcal{T}\left\{\exp\left[-i\int_{\eta_0}^{\eta} \mathrm{d}\eta' \hat{H}(\eta')\right]\right\}, \tag{6.2}$$

$$\hat{U}_{\text{DC}} = \mathcal{T}\left\{\exp\left[-i\int_{\eta_0}^{\eta} \mathrm{d}\eta' \hat{H}_{\text{DC}}(\eta')\right]\right\}, \tag{6.3}$$

where η_0 is some initial (conformal) time.

In general, the interaction picture form of an arbitrary operator $\hat{O}^I(\eta)$ and state $|\psi^I(\eta)\rangle$ is respectively given by [119]

$$\hat{O}^I(\eta) = \hat{U}^{-1}(\eta,\eta_0)\hat{O}\hat{U}(\eta,\eta_0), \tag{6.4}$$

$$|\psi^I(\eta)\rangle = \hat{U}_{\rm DC}(\eta,\eta_0)\,|\psi(\eta_0)\rangle\,. \tag{6.5}$$

Thus far, we have not specified the form of the Hamiltonian $\hat{H}_{\rm DC}$. Following the *noise trick*, described in the previous chapter, we consider in analogy to Eq. (5.13), that $\hat{H}_{\rm DC}$ in the Schrödinger picture is defined as

$$\hat{H}_{\rm DC} = \frac{\sqrt{\gamma}}{m_0}\int {\rm d}\boldsymbol{x}\xi_\eta(\boldsymbol{x})\hat{\mathcal{H}}_{\rm DC}(\eta,\boldsymbol{x}). \tag{6.6}$$

In the above equation, the Hamiltonian density $\hat{\mathcal{H}}_{\rm DC}$ is a non-stochastic operator. Furthermore, the noise $\xi_\eta(\boldsymbol{x})$ in Eq. (6.6) is characterized through the following properties

$$\begin{aligned}\mathbb{E}\left[\xi_\eta(\boldsymbol{x})\right] &= 0,\\ \mathbb{E}\left[\xi_\eta(\boldsymbol{x})\xi_{\eta'}(\boldsymbol{y})\right] &= \frac{\delta(\eta-\eta')}{a(\eta)}\frac{1}{(4\pi r_c^2)^{3/2}}e^{-a^2(\eta)(\boldsymbol{x}-\boldsymbol{y})^2/(4r_c^2)}.\end{aligned} \tag{6.7}$$

The correlation in Eq. (6.7) is defined in such a way that when performing the change of coordinates to cosmic time t and physical coordinates $\boldsymbol{x_p}$, the correlation coincides with that of the standard CSL model. From Eq. (6.6) and Eq. (6.4), we have that in the interaction picture

$$\hat{H}^I_{\rm DC}(\eta) = \frac{\sqrt{\gamma}}{m_0}\int {\rm d}\boldsymbol{x}\xi_\eta(\boldsymbol{x})\hat{\mathcal{H}}^I_{\rm DC}(\eta,\boldsymbol{x}). \tag{6.8}$$

For an arbitrary operator $\hat{O}$, let us consider the expansion of the expectation value $\langle\psi|\,\hat{O}\,|\psi\rangle = \langle\psi^I(\eta)|\,\hat{O}^I(\eta)\,|\psi^I(\eta)\rangle$. We have

$$\begin{aligned}\langle\psi|\,\hat{O}\,|\psi\rangle \approx{}& \langle\psi(\eta_0)|\,\hat{O}^I(\eta)\,|\psi(\eta_0)\rangle - i\int_{\eta_0}^{\eta}{\rm d}\eta'\,\langle\psi(\eta_0)|\left[\hat{O}^I(\eta),\hat{H}^I_{\rm DC}(\eta')\right]|\psi(\eta_0)\rangle\\ &-\int_{\eta_0}^{\eta}\int_{\eta_0}^{\eta'}{\rm d}\eta'{\rm d}\eta''\,\langle\psi(\eta_0)|\left[\hat{H}^I_{\rm DC}(\eta''),\left[\hat{H}^I_{\rm DC}(\eta'),\hat{O}^I(\eta)\right]\right]|\psi(\eta_0)\rangle\,.\end{aligned} \tag{6.9}$$

Taking the stochastic average over all the realizations, and using the expression for the Hamiltonian in Eq. (6.8), we find that

$$
\begin{aligned}
\bar{O} &= \mathbb{E}\left[\langle\psi|\,\hat{O}\,|\psi\rangle\right] \approx \mathbb{E}\left[\langle\psi(\eta_0)|\,\hat{O}^{I}\,|\psi(\eta_0)\rangle\right] \\
&- \frac{i\sqrt{\gamma}}{m_0}\int_{\eta_0}^{\eta} \mathrm{d}\eta'\mathrm{d}\boldsymbol{x}'\mathbb{E}[\xi_\eta(\boldsymbol{x}')]\,\langle\psi(\eta_0)|\left[\hat{O}^{I}(\eta),\hat{\mathcal{H}}^{I}_{\mathrm{DC}}(\eta',\boldsymbol{x}')\right]|\psi(\eta_0)\rangle \\
&- \frac{\gamma}{m_0^2}\int_{\eta_0}^{\eta}\int_{\eta_0}^{\eta'} \mathrm{d}\eta'\mathrm{d}\eta''\mathrm{d}\boldsymbol{x}'\mathrm{d}\boldsymbol{x}''\mathbb{E}\left[\xi_{\eta''}(\boldsymbol{x}'')\xi_{\eta'}(\boldsymbol{x}')\right] \\
&\times \langle\psi(\eta_0)|\left[\hat{\mathcal{H}}_{\mathrm{DC}}(\eta'',\boldsymbol{x}''),\left[\hat{\mathcal{H}}_{\mathrm{DC}}(\eta',\boldsymbol{x}'),\hat{O}^{I}(\eta)\right]\right]|\psi(\eta_0)\rangle\,,
\end{aligned}
\tag{6.10}
$$

By applying the expressions in Eq. (6.7) and integrating over one of the conformal times, we obtain

$$
\begin{aligned}
\bar{O} &= \langle 0|\,\hat{O}^{I}(\eta)\,|0\rangle \\
&- \frac{\lambda}{2m_0^2}\int_{\eta_0}^{\eta}\frac{\mathrm{d}\eta'}{a(\eta')}\mathrm{d}\boldsymbol{x}'\mathrm{d}\boldsymbol{x}''e^{-\frac{a^2(\eta')(\boldsymbol{x}''-\boldsymbol{x}')^2}{4r_c^2}}\,\langle 0|\left[\hat{\mathcal{H}}^{I}_{\mathrm{DC}}(\eta',\boldsymbol{x}''),\left[\hat{\mathcal{H}}^{I}_{\mathrm{DC}}(\eta',\boldsymbol{x}'),\hat{O}^{I}(\eta)\right]\right]|0\rangle\,,
\end{aligned}
\tag{6.11}
$$

where the collapse rate λ is defined as $\lambda \equiv \gamma/(4\pi r_c^2)^{3/2}$, and we considered that the initial state of the system is the vacuum state $|0\rangle$.

In what follows, we are interested in calculating the expectation of the operator

$$
\hat{O}^{I}(\eta) = \hat{\mathcal{R}}^2 = \frac{\hat{v}^2(\eta,\boldsymbol{x})}{z^2} = \frac{c_s^2\hat{v}^2(\eta,\boldsymbol{x})}{2\epsilon M_{\mathrm{Pl}}^2 a^2(\eta)}, \tag{6.12}
$$

where we recall that $\hat{\mathcal{R}}$ is the comoving curvature perturbation, described in terms of the Mukhanov-Sasaki variable $\hat{v}(\eta,\boldsymbol{x})$ [cf. Eq. (4.46)]. We are thus considering the same problem as previous works in the literature [105–107, 109, 111, 112, 115], namely, to calculate the corrections to the power spectrum of the comoving curvature perturbation due to the incorporation of collapse models.

6.2 The choice of collapse operator

As seen in Chapter 4, in linear perturbation theory, the dynamical evolution of $\hat{\mathcal{R}}$ is usually computed within a Fourier space description. Indeed, linear perturbation theory allows a description in which the Fourier modes evolve independently. This particular feature of the evolution of the comoving curvature perturbation has been kept in previous works which incorporate dynamical collapse models within a cosmological scenario. These works consider a system of independent stochastic differential equations, one for each Fourier mode. This is possible under appropriate choices of the collapse operator. For example, in Ref. [111], the collapse operator is chosen to be the Mukhanov-Sasaki variable itself. From Eq. (4.38), we see that this operator does not mix different Fourier modes because it is linear in the creation and annihilation operators. Similarly, in Ref. [112], the collapse operator is chosen to be a smeared density contrast $\hat{\mu}(\boldsymbol{x})$, which is subsequently linearized with respect to $\hat{v}$ and its conjugate momentum $\hat{p}$. In cosmic time t, $\hat{\mu}(\boldsymbol{x})$ is given by

$$\hat{\mu}(\boldsymbol{x}) = 3M_{\mathrm{Pl}}H^2\left(\frac{a}{r_c}\right)^3 \int \mathrm{d}\boldsymbol{y}\frac{\delta\hat{\rho}}{\bar{\rho}}(\boldsymbol{x}+\boldsymbol{y})e^{-\frac{|\boldsymbol{y}|^2a^2}{2r_c^2}}, \tag{6.13}$$

where $\delta\hat{\rho}$ is the density fluctuation and $\bar{\rho}$ is the homogeneous component of the energy density. Both this choice, as well as the one in Ref. [111], are such that, in Fourier space, the collapse operator $\hat{C}(\boldsymbol{k})$ is of the form

$$\hat{C}(\boldsymbol{k}) = \alpha_{\boldsymbol{k}}\hat{v}(\eta,\boldsymbol{k})(\eta) + \beta_{\boldsymbol{k}}\hat{p}(\eta,\boldsymbol{k}), \tag{6.14}$$

where $\alpha_{\boldsymbol{k}}$ and $\beta_{\boldsymbol{k}}$ are suitable functions of $\boldsymbol{k}$. Therefore, these choices of collapse operator are linear with respect to the Mukhanov-Sasaki variable $\hat{v}(\eta,\boldsymbol{k})$ and its conjugate momentum $\hat{p}(\eta,\boldsymbol{k})$ in Fourier space. This allows to write down a stochastic evolution for the wave function that is independent for each Fourier mode.

In order to motivate our choice of collapse operator, let us consider once again the standard CSL model. In this case, the collapse operator [cf. Eq. (5.5)] is quadratic in the creation and annihilation operators. Therefore, when working in Fourier space, this operator couples different Fourier modes, as shown in Ref. [120]. Thus, if already at a standard level the collapse operator mixes different Fourier modes, one expects that this property also holds when considering its extension within a cosmological setting. We notice that the collapse operator choices of previous works do not mix different Fourier modes comes only as a consequence of choosing a linearized version of the operators.

These considerations lead us to choose the collapse operator $\hat{\mathcal{H}}_{\mathrm{DC}}(\eta,\boldsymbol{x})$ in Eq. (6.6) to the Hamiltonian density of the rescaled variable $\hat{v}(\eta,\boldsymbol{x})$ of the standard inflationary cosmological scenario. This operator encodes the energy of the system, which is a suitable generalization within a cosmological context of the mass density in the standard CSL model, which is non-relativistic. Moreover, as we will see below, this operator is quadratic in the creation and annihilation operators, in analogy to the mass density in the standard CSL model. As we will see, this particular choice does not allow an independent Fourier-mode description of the dynamical evolution of the comoving curvature perturbation $\hat{\mathcal{R}}$. We also notice that, from the definition of the unitary operator $\hat{U}$ in Eq. (6.2), we have that the interaction picture operators $\hat{O}^I$ in our framework, coincide with the Heisenberg picture operators $\hat{O}^H$ in the standard cosmological scenario. This will allow a straightforward identification of the explicit form of the collapse operator.

6.3 Dynamical collapse effects during inflation

From the expression for the Hamiltonian density in Eq. (4.35), the collapse operator during the inflationary stage of the Universe evolution reads

$$\hat{\mathcal{H}}^{I}_{\text{DC}}(\eta, \boldsymbol{x}) = \frac{1}{2}\left(\left(\frac{\partial \hat{v}(\eta, \boldsymbol{x})}{\partial \eta}\right)^2 + (\nabla \hat{v}(\eta, \boldsymbol{x}))^2 - \frac{2}{\eta^2}\hat{v}^2(\eta, \boldsymbol{x})\right). \tag{6.15}$$

We notice that in the above equation, we are approximating the metric to correspond to that of a de Sitter universe [69]. From the definition of the rescaled variable $\hat{v}(\eta, \boldsymbol{x})$ in Eq. (4.38), we have that

$$\begin{aligned}\hat{\mathcal{H}}^{I}_{\text{DC}}(\eta, \boldsymbol{x}) = \frac{1}{2}\frac{1}{(2\pi)^3}\int \mathrm{d}\boldsymbol{q}\mathrm{d}\boldsymbol{p} e^{i(\boldsymbol{p}+\boldsymbol{q})\cdot \boldsymbol{x}} \Big(& f_2(\eta, \boldsymbol{p}, \boldsymbol{q})\hat{a}_{\boldsymbol{p}}\hat{a}_{\boldsymbol{q}} + g_2(\eta, \boldsymbol{p}, \boldsymbol{q})\hat{a}_{\boldsymbol{p}}\hat{a}^{\dagger}_{-\boldsymbol{q}} \\ & + g_2^*(\eta, \boldsymbol{p}, \boldsymbol{q})\hat{a}^{\dagger}_{-\boldsymbol{p}}\hat{a}_{\boldsymbol{q}} + f_2^*(\eta, \boldsymbol{p}, \boldsymbol{q})\hat{a}^{\dagger}_{-\boldsymbol{p}}\hat{a}^{\dagger}_{-\boldsymbol{q}}\Big).\end{aligned} \tag{6.16}$$

In the above equations, the functions $f_2(\eta, \boldsymbol{p}, \boldsymbol{q})$ and $g_2(\eta, \boldsymbol{p}, \boldsymbol{q})$ are defined

$$\begin{aligned} f_2(\eta, \boldsymbol{p}, \boldsymbol{q}) &= j(\eta, \boldsymbol{p}, \boldsymbol{q}) - \left[(\boldsymbol{p}\cdot\boldsymbol{q}) + \frac{2}{\eta^2}\right] f(\eta, \boldsymbol{p}, \boldsymbol{q}), \\ g_2(\eta, \boldsymbol{p}, \boldsymbol{q}) &= l(\eta, \boldsymbol{p}, \boldsymbol{q}) - \left[(\boldsymbol{p}\cdot\boldsymbol{q}) + \frac{2}{\eta^2}\right] g(\eta, \boldsymbol{p}, \boldsymbol{q}). \end{aligned} \tag{6.17}$$

Here, we set the functions $f(\eta, \boldsymbol{p}, \boldsymbol{q})$ and $g(\eta, \boldsymbol{p}, \boldsymbol{q})$ to be

$$\begin{aligned} f(\eta, \boldsymbol{p}, \boldsymbol{q}) &= v_{\boldsymbol{p}}(\eta)v_{\boldsymbol{q}}(\eta), \\ g(\eta, \boldsymbol{p}, \boldsymbol{q}) &= v_{\boldsymbol{p}}(\eta)v^*_{\boldsymbol{q}}(\eta), \end{aligned} \tag{6.18}$$

and, in terms of $v'_{\boldsymbol{p}}(\eta)$, we define the functions $j(\eta, \boldsymbol{p}, \boldsymbol{q})$ and $l(\eta, \boldsymbol{p}, \boldsymbol{q})$ as

$$\begin{aligned} j(\eta, \boldsymbol{p}, \boldsymbol{q}) &= v'_{\boldsymbol{p}}(\eta)v'_{\boldsymbol{q}}(\eta), \\ l(\eta, \boldsymbol{p}, \boldsymbol{q}) &= v'_{\boldsymbol{p}}(\eta)v'^*_{\boldsymbol{q}}(\eta). \end{aligned} \tag{6.19}$$

By substituting the collapse operator defined in Eq. (6.15) in the expression for the expectation value of an arbitrary operator $\hat{O}$ in Eq. (6.11), we have that in general,

$$\begin{aligned} \bar{O} &= \langle 0|\, \hat{O}^{I}(\eta)\, |0\rangle \\ &- \frac{\lambda}{2m_0^2}\int_{\eta_0}^{\eta} \frac{\mathrm{d}\eta'}{a(\eta')}\mathrm{d}\boldsymbol{x}'\mathrm{d}\boldsymbol{x}'' e^{-\frac{a^2(\eta')(\boldsymbol{x}''-\boldsymbol{x}')^2}{4r_C^2}} \langle 0| \left[\hat{\mathcal{H}}^{I}_{\text{DC}}(\eta', \boldsymbol{x}''), \left[\hat{\mathcal{H}}^{I}_{\text{DC}}(\eta', \boldsymbol{x}'), \hat{O}^{I}(\eta)\right]\right] |0\rangle \\ &= \left\langle \hat{O}^{I}\right\rangle + \Delta\left\langle \hat{O}^{I}\right\rangle, \end{aligned} \tag{6.20}$$

where $\left\langle \hat{O}^I \right\rangle = \langle 0| \hat{O}^I(\eta) |0\rangle$. From Eq. (6.20), and in the case of $\hat{O}^I(\eta) = \hat{\mathcal{R}}^2$, the first term in Eq. (6.20) yields the standard power spectrum of the comoving curvature perturbation, and the second term yields the correction due to the dynamical collapse model. Therefore, we have

$$\Delta\left\langle \hat{\mathcal{R}}^2 \right\rangle = -\frac{\lambda}{2m_0^2}\int_{\eta_0}^{\eta}\frac{\mathrm{d}\eta'}{a(\eta')}\mathrm{d}\boldsymbol{x}'\mathrm{d}\boldsymbol{x}''e^{-\frac{a^2(\eta')(\boldsymbol{x}''-\boldsymbol{x}')^2}{4r_c^2}}\langle 0|\left[\hat{\mathcal{H}}^I_{\mathrm{DC}}(\eta',\boldsymbol{x}''),\left[\hat{\mathcal{H}}^I_{\mathrm{DC}}(\eta',\boldsymbol{x}'),\hat{\mathcal{R}}^2\right]\right]|0\rangle . \tag{6.21}$$

By taking into consideration the explicit form of $\hat{\mathcal{H}}^I_{\mathrm{DC}}(\eta,\boldsymbol{x})$ in Eq. (6.16), we can simply compute the commutators. Thus, we obtain

$$\begin{aligned}\Delta\left\langle \hat{\mathcal{R}}^2 \right\rangle = &-\frac{\lambda r_c^3}{8\epsilon_{\mathrm{inf}}M_{\mathrm{Pl}}^2 m_0^2 a^2(\eta)\pi^{9/2}}\int_{\eta_0}^{\eta}\frac{\mathrm{d}\eta'}{a^4(\eta')}\int \mathrm{d}\boldsymbol{q}\mathrm{d}\boldsymbol{p}e^{-\frac{r_c^2}{a^2(\eta')}(\boldsymbol{q}+\boldsymbol{p})^2}\\ &\times \mathfrak{Re}\left[f_2(\eta',\boldsymbol{q},\boldsymbol{p})\left(g_2(\eta',-\boldsymbol{q},-\boldsymbol{p})f^*(\eta,\boldsymbol{q},-\boldsymbol{q}) - f_2^*(\eta',-q,-\boldsymbol{p})g(\eta,\boldsymbol{q},-\boldsymbol{q})\right)\right] .\end{aligned} \tag{6.22}$$

In order to simplify the above result, we can make use of the properties of the functions defined in Eq. (6.17) and Eq. (6.18). The exponential term in the above integrand, coming from the Gaussian correlation of the noise, is invariant under the interchange of integration variables $\boldsymbol{p}$ and $\boldsymbol{q}$. Thus, we can rewrite the above equation in the following form

$$\Delta\left\langle \hat{\mathcal{R}}^2 \right\rangle = -\frac{\lambda r_c^3}{8\epsilon_{\mathrm{inf}}M_{\mathrm{Pl}}^2 m_0^2 a^2(\eta)\pi^{9/2}}\int_{\eta_0}^{\eta}\frac{\mathrm{d}\eta'}{a^4(\eta')}\mathrm{d}\boldsymbol{q}\mathrm{d}\boldsymbol{p}e^{-\frac{r_c^2}{a^2(\eta')}(\boldsymbol{q}+\boldsymbol{p})^2}\mathcal{F}(\eta',\boldsymbol{p},\boldsymbol{q}), \tag{6.23}$$

where

$$\mathcal{F}(\eta',\boldsymbol{p},\boldsymbol{q}) = \mathfrak{Re}\left(f_2(\eta',\boldsymbol{p},\boldsymbol{q})g_2(\eta',\boldsymbol{q},\boldsymbol{p})f^*(\eta,\boldsymbol{q},\boldsymbol{q}) - f_2(\eta',\boldsymbol{p},\boldsymbol{q})f_2^*(\eta',\boldsymbol{q},\boldsymbol{p})g(\eta,\boldsymbol{p},\boldsymbol{p})\right) . \tag{6.24}$$

During the inflationary stage, and in the perfect de Sitter limit [69], the mode $v_{\boldsymbol{k}}(\eta)$ is given by Eq. (4.45). Therefore, the substitution of the corresponding expression in Eq. (6.24) yields

$$\begin{aligned}&\mathcal{F}(\eta',\boldsymbol{p},\boldsymbol{q}) =\\ &\mathfrak{Re}\left[\frac{1}{8\eta'^8p^3q^4}\left(1+\frac{i}{\eta_e q}\right)^2 e^{-2iq(\eta'-\eta_e)}\left[\left(\left(-\eta'^2p^2+i\eta' p+1\right)\right.\right.\right.\\ &\times\left(\eta'^2q^2-i\eta' q-1\right)-(\eta' p-i)(\eta' q-i)\left(\eta'^2(\boldsymbol{p}\cdot\boldsymbol{q})+2\right)\Big)\\ &\Big(\left(\eta'^2p^2+i\eta' p-1\right)\left(\eta'^2q^2-i\eta' q-1\right)-(\eta' p+i)(\eta' q-i)\left(\eta'^2(\boldsymbol{p}\cdot\boldsymbol{q})+2\right)\Big)\Big]\\ &-\frac{1}{\eta'^8p^4q^3}\left(1-\frac{i}{\eta_e p}\right)\left(1+\frac{i}{\eta_e p}\right)\Big[\Big(\left(-\eta'^2p^2+i\eta' p+1\right)\\ &\times\left(\eta'^2q^2-i\eta' q-1\right)-(\eta' p-i)(\eta' q-i)\left(\eta'^2(\boldsymbol{p}\cdot\boldsymbol{q})+2\right)\Big)\\ &\left.\left.\Big(-\left(\eta'^2p^2+i\eta' p-1\right)\left(\eta'^2q^2+i\eta' q-1\right)-(\eta' p+i)(\eta' q+i)\left(\eta'^2(\boldsymbol{p}\cdot\boldsymbol{q})+2\right)\Big)\right]\right] .\end{aligned}$$

During the inflationary stage, the scale factor a is inversely proportional to the conformal time. Namely, we have that $a(\eta) \approx -\frac{1}{H_{\text{inf}}\eta}$. Therefore, the exponential appearing in Eq. (6.23) becomes $\exp\left(-r_c^2 H_{\text{inf}}^2 \eta'^2 (\boldsymbol{p}+\boldsymbol{q})^2\right)$. We assume that the parameter r_c of the model is such that it is safe to assume that the condition $r_c H_{\text{inf}} \gg 1$ holds. For illustrative purposes, if we take the GRW value of $r_c \sim 10^{27} M_{\text{Pl}}^{-1}$, and one has $H_{\text{inf}} \sim 10^{-5} M_{\text{Pl}}$, then the condition $r_c H \gg 1$ holds. This condition allows to suppress all the contributions in the integrand of Eq. (6.23) for which the condition $q\eta' \ll 1$ is violated. Notice that the latter condition is satisfied for the modes of cosmological interest at the end of inflation. Therefore, we can, in a first approximation, expand $\mathcal{F}(\eta', \boldsymbol{p}, \boldsymbol{q})$ in powers of $q\eta'$. To leading order, we obtain

$$\mathcal{F}(\eta', \boldsymbol{p}, \boldsymbol{q}) = \frac{1}{8p^3 q^4 \eta'^8}\left(-\frac{2q^4\eta_e^4}{9} + \frac{16q^4\eta_e\eta'^3}{9} - \frac{4p^3 q\eta'^6}{\eta_e^2} - \frac{32q^4\eta'^6}{9\eta_e^2}\right). \tag{6.25}$$

From this result, and by taking into account the condition $\eta_e < \eta'$, we retain only the last two terms in the right-hand side of the above equation. From this, we obtain the leading-order contribution

$$\Delta\left\langle\hat{\mathcal{R}}^2\right\rangle \approx \frac{17}{36}\frac{\lambda H_{\text{inf}}^3}{\pi^2\epsilon_{\text{inf}} M_{\text{Pl}}^2 m_0^2}\int_{\eta_0}^{\eta_e} \mathrm{d}\ln\eta \, \mathrm{d}\ln k. \tag{6.26}$$

Therefore, comparing this result with the expression in Eq. (4.47), we find that the leading order correction to the power spectrum $\mathcal{P}_{\hat{\mathcal{R}}}$ of the curvature perturbation reads

$$\Delta\mathcal{P}_{\hat{\mathcal{R}}} \approx \frac{17}{36}\frac{\lambda H_{\text{inf}}^3}{\pi^2\epsilon_{\text{inf}} M_{\text{Pl}}^2 m_0^2}\ln\left(\frac{\eta_e}{\eta_0}\right). \tag{6.27}$$

The above results depend only on the collapse rate λ and not on r_c. However, in general, we expect the correction to depend on both of the phenomenological parameters of the model, λ and r_c. The above expression can be used to set bounds on these parameters. Once again, let us consider the typical value of λ in the GRW model, $\lambda = 10^{-16}\,\text{s}^{-1}$ [76, 82]. In this case, we find that the correction $\Delta\mathcal{P}_{\hat{\mathcal{R}}}$ to the power spectrum of $\hat{\mathcal{R}}$ is of order $\mathcal{O}[10^{-34}]$. This is 24 orders of magnitude lower than the power spectrum, which is of order $\mathcal{O}[10^{-10}]$. This result indicates that, for proper choices of the phenomenological parameter of the dynamical collapse model defined through Eq. (6.6), the corrections to the power spectrum of $\hat{\mathcal{R}}$ predicted by incorporating the dynamical collapse model are negligible with respect to the standard cosmological scenario.

6.4 Dynamical collapse effects during the radiation dominated era

In standard Cosmology, the power spectrum of the comoving curvature perturbation is frozen after the inflationary stage for the modes of cosmological interest [68, 69]. There is no need to study the evolution of the modes during the radiation dominated era that follows inflation. However, if we now incorporate the action of dynamical collapse models into the description of the evolution of the modes, this may no longer hold [112]. As a first approximation, we do not consider the reheating period after inflation and study the effects of dynamical collapse models during the radiation dominated era.

From Eq. (4.40), during the radiation dominated era, the modes $v_{\boldsymbol{k}}(\eta)$ satisfy

$$v''_{\boldsymbol{k}}(\eta) + \frac{1}{3}k^2 v_{\boldsymbol{k}}(\eta) = 0. \tag{6.28}$$

In order to determine the general solution of the above equation, one matches the curvature perturbation and its derivative at $\eta = \eta_e$, i.e. at the end of inflation, with those corresponding to the inflationary stage [112]. A straightforward calculation yields

$$\begin{aligned} v_{\boldsymbol{k}}(\eta) = \frac{\sqrt{3}}{2\eta_e^2\sqrt{\epsilon_{\text{inf}}}k^{5/2}} e^{-ik\eta_e} \Big\{ &\left[(1+\sqrt{3})(k\eta_e)^2 - \sqrt{3} - i(1+\sqrt{3})k\eta_e\right] e^{-ik\frac{\eta-\eta_e}{\sqrt{3}}} \\ &+ \left[(1-\sqrt{3})(k\eta_e)^2 + \sqrt{3} - i(1-\sqrt{3})k\eta_e\right] e^{ik\frac{\eta-\eta_e}{\sqrt{3}}} \Big\}. \end{aligned} \tag{6.29}$$

From the expression for the Hamiltonian density in Eq. (4.35), and taking into consideration that, during the radiation dominated era, the scale factor a reads

$$a(\eta) = \frac{1}{H_{\text{inf}}\eta_e^2}(\eta - 2\eta_e), \tag{6.30}$$

we find that at this stage of the evolution of the Universe, the Hamiltonian density is given by

$$\hat{\mathcal{H}}^I_{\text{DC}}(\eta, \boldsymbol{x}) = \frac{1}{2}\left(\frac{\partial^2 \hat{v}(\eta, \boldsymbol{x})}{\partial \eta^2} + \frac{1}{3}\left(\nabla \hat{v}(\eta, \boldsymbol{x})\right)^2\right). \tag{6.31}$$

Now, if we substitute the Fourier decomposition of the rescaled variable $\hat{v}(\eta, \boldsymbol{x})$ of Eq. (4.38) in the above equation, we obtain that the Hamiltonian density $\hat{\mathcal{H}}^I_{\text{DC}}(\eta, \boldsymbol{x})$ has the same structure as in Eq. (6.16), where now the functions $f_2(\eta, \boldsymbol{q}, \boldsymbol{p})$ and $g_2(\eta, \boldsymbol{q}, \boldsymbol{p})$ read

$$\begin{aligned} f_2(\eta, \boldsymbol{q}, \boldsymbol{p}) &= j(\eta, \boldsymbol{q}, \boldsymbol{p}) - \frac{1}{3}(\boldsymbol{q}\cdot\boldsymbol{p}) f(\eta, \boldsymbol{q}, \boldsymbol{p}), \\ g_2(\eta, \boldsymbol{q}, \boldsymbol{p}) &= l(\eta, \boldsymbol{q}, \boldsymbol{p}) - \frac{1}{3}(\boldsymbol{q}\cdot\boldsymbol{p}) g(\eta, \boldsymbol{q}, \boldsymbol{p}), \end{aligned} \tag{6.32}$$

and all the functions in the above equations are given as in Eq. (6.18) and Eq. (6.19), where $v_{\boldsymbol{k}}(\eta)$ is that in Eq. (6.29). During the radiation dominated era, the (squared) comoving curvature perturbation operator reads

$$\hat{\mathcal{R}}^2(\eta, \boldsymbol{x}) = \frac{\hat{v}^2(\eta, \boldsymbol{x})}{12 M_{\text{Pl}}^2 a^2(\eta)}, \tag{6.33}$$

where the scale factor $a(\eta)$ is given in Eq. (6.30).

Let us denote the conformal time at the end of the radiation dominated era by η_r. Then, we have that the correction to the power spectrum of the comoving curvature perturbation $\hat{\mathcal{R}}$ during the radiation dominated era reads

$$\Delta\left\langle \hat{\mathcal{R}}^2 \right\rangle = -\frac{\lambda}{2m_0^2}\int_{\eta_e}^{\eta_r}\frac{\mathrm{d}\eta'}{a(\eta')}\int \mathrm{d}\boldsymbol{x}'\mathrm{d}\boldsymbol{x}'' e^{-\frac{a^2(\eta')(\boldsymbol{x}''-\boldsymbol{x}')^2}{4r_C^2}} \langle 0|\left[\hat{\mathcal{H}}^I_{\mathrm{DC}}(\eta',\boldsymbol{x}''),\left[\hat{\mathcal{H}}^I_{\mathrm{DC}}(\eta',\boldsymbol{x}'),\hat{\mathcal{R}}^2(\eta)\right]\right]|0\rangle\,. \tag{6.34}$$

By merging the latter with Eq. (6.31) and Eq. (6.33), we we obtain

$$\Delta\left\langle \hat{\mathcal{R}}^2 \right\rangle = -\frac{\lambda r_c^3}{48M_{\mathrm{Pl}}^2 m_0^2 a^2(\eta_r)\pi^{9/2}}\int_{\eta_e}^{\eta_r}\frac{\mathrm{d}\eta'}{a^4(\eta')}\mathrm{d}\boldsymbol{q}\mathrm{d}\boldsymbol{p}e^{-\frac{r_c^2}{a^2(\eta')}(\boldsymbol{q}+\boldsymbol{p})^2}\mathcal{F}(\eta',\boldsymbol{p},\boldsymbol{q}). \tag{6.35}$$

Here, the function $\mathcal{F}(\eta',\boldsymbol{p},\boldsymbol{q})$ has the same structure as in Eq. (6.24). By imposing the forms of Eq. (6.32) and Eq. (6.18), we obtain the following expression

$$\begin{aligned}
&\mathcal{F}(\eta',\boldsymbol{p},\boldsymbol{q}) = -\frac{1}{8p^5q^5\epsilon^3\eta_e^4}9e^{-\frac{2i(p(\eta'-\eta_e)+q(\eta'+\eta_r-2\eta_e))}{\sqrt{3}}}\\
&\times\left(4e^{\frac{2i(p+q)(\eta'-\eta_e)}{\sqrt{3}}}(\boldsymbol{p}\cdot\boldsymbol{q})\left(-2q\eta_e\left(q^3\eta_e^3-2q\eta_e+\sqrt{3}i\right)-3\right)p^5\right.\\
&+4e^{\frac{2i(p(\eta'-\eta_e)+q(\eta'+2\eta_r-3\eta_e))}{\sqrt{3}}}(\boldsymbol{p}\cdot\boldsymbol{q})\left(2q\eta_e\left(-q^3\eta_e^3+2q\eta_e+\sqrt{3}i\right)-3\right)p^5\\
&+2e^{\frac{2i(p+2q)(\eta'-\eta_e)}{\sqrt{3}}}q^3\left(p^2q^2-(\boldsymbol{p}\cdot\boldsymbol{q})^2\right)\left(4p^4\eta_e^4-2p^2\eta_e^2+3\right)\\
&+2e^{\frac{2i(p\eta'+2q\eta_r-(p+2q)\eta_e)}{\sqrt{3}}}q^3\left(p^2q^2-(\boldsymbol{p}\cdot\boldsymbol{q})^2\right)\left(4p^4\eta_e^4-2p^2\eta_e^2+3\right)\\
&+4e^{\frac{2i(p(\eta'-\eta_e)+q(\eta'+\eta_r-2\eta_e))}{\sqrt{3}}}\left(4p^4q^3(pq+\boldsymbol{p}\cdot\boldsymbol{q})^2\eta_e^4-2p^2q^2\left(2(\boldsymbol{p}\cdot\boldsymbol{q})p^3+q^3p^2+q(\boldsymbol{p}\cdot\boldsymbol{q})^2\right)\eta_e^2\right.\\
&\left.+3\left(2(\boldsymbol{p}\cdot\boldsymbol{q})p^5+q^5p^2+q^3(\boldsymbol{p}\cdot\boldsymbol{q})^2\right)\right)\\
&+e^{\frac{4i(p\eta'+q\eta_r-(p+q)\eta_e)}{\sqrt{3}}}q^3(-pq+\boldsymbol{p}\cdot\boldsymbol{q})^2\left(2p\eta_e\left(p^3\eta_e^3-2p\eta_e-\sqrt{3}i\right)+3\right)\\
&+e^{\frac{4i(p+q)(\eta'-\eta_e)}{\sqrt{3}}}q^3(pq+\boldsymbol{p}\cdot\boldsymbol{q})^2\left(2p\eta_e\left(p^3\eta_e^3-2p\eta_e-\sqrt{3}i\right)+3\right)\\
&+2e^{\frac{2i(2p\eta'+q\eta'+q\eta_r-2(p+q)\eta_e)}{\sqrt{3}}}q^3\left(p^2q^2-(\boldsymbol{p}\cdot\boldsymbol{q})^2\right)\left(2p\eta_e\left(p^3\eta_e^3-2p\eta_e-\sqrt{3}i\right)+3\right)\\
&+e^{\frac{4iq(\eta'-\eta_e)}{\sqrt{3}}}q^3(-pq+\boldsymbol{p}\cdot\boldsymbol{q})^2\left(2p\eta_e\left(p^3\eta_e^3-2p\eta_e+\sqrt{3}i\right)+3\right)\\
&+e^{\frac{4iq(\eta_r-\eta_e)}{\sqrt{3}}}q^3(pq+\boldsymbol{p}\cdot\boldsymbol{q})^2\left(2p\eta_e\left(p^3\eta_e^3-2p\eta_e+\sqrt{3}i\right)+3\right)\\
&\left.+2e^{\frac{2iq(\eta'+\eta_r-2\eta_e)}{\sqrt{3}}}q^3\left(p^2q^2-(\boldsymbol{p}\cdot\boldsymbol{q})^2\right)\left(2p\eta_e\left(p^3\eta_e^3-2p\eta_e+\sqrt{3}i\right)+3\right)\right).
\end{aligned} \tag{6.36}$$

Now, the scales of cosmological interest, which satisfy the condition $q\eta' \ll 1$ during inflation, already crossed the horizon by the end of that stage, and remain outside the horizon until they re-enter. This may happen during the radiation dominated era or at a later time. Moreover, from Eq. (6.30), we have that the ratio between the scale factor at the end of the radiation dominated era and that at the end of inflation is given by $a(\eta_r)/a(\eta_e) = 3 \times 10^{26}$ [121]. Thus, the scale factor is an increasing function of time, and the contributions to the integrand become suppressed as the time increases in magnitude. Therefore, the leading order contributions will come from times close to the end of inflation, as the condition $|\eta_e| \ll |\eta_r|$ is satisfied. Thus, as a first approximation, we proceed $\mathcal{F}(\eta', \boldsymbol{p}, \boldsymbol{q})$ in powers of $q\eta'$, similarly as done for the inflationary stage. In this case, the leading order contribution reads

$$\mathcal{F}(\eta', \boldsymbol{p}, \boldsymbol{q}) \approx -\frac{54}{p^3 \epsilon_{\text{inf}}^3 \eta_e^4} \,. \tag{6.37}$$

Substituting this result in Eq. (6.35), we have that the leading order term in the correction to the expectation value of the operator $\hat{\mathcal{R}}^2$ reads

$$\Delta\langle\hat{\mathcal{R}}^2\rangle \approx \frac{9\lambda H_{\text{inf}}^3 \eta_e^2}{2M_{\text{Pl}}^2 \epsilon_{\text{inf}}^3 (\eta_r - 2\eta_e)^2 \pi^2 m_0^2} \int_{\eta_e}^{\eta_r} \mathrm{d}\ln(\eta' - 2\eta_e) \int \mathrm{d}\ln q. \tag{6.38}$$

Thus, the leading order term in the correction to the power spectrum $\mathcal{P}_{\hat{\mathcal{R}}}$ of the comoving curvature perturbation is given by

$$\Delta\mathcal{P}_{\hat{\mathcal{R}}} \approx \frac{9\lambda H_{\text{inf}}^3 \eta_e^2}{2M_{\text{Pl}}^2 \epsilon_{\text{inf}}^3 (\eta_r - 2\eta_e)^2 \pi^2 m_0^2} \ln\left(\frac{2\eta_e - 2\eta_r}{\eta_e}\right). \tag{6.39}$$

As in the case of the leading order correction during the inflationary stage [cf. Eq. (6.27)], we see that the expression for the correction $\Delta\mathcal{P}_{\hat{\mathcal{R}}}$ during the radiation dominated era depends only on the collapse rate parameter λ and not on r_c. Nevertheless, taking into consideration the full expression for $\mathcal{F}(\eta', \boldsymbol{p}, \boldsymbol{q})$ in Eq. (6.36), one expects that the correction will depend on both phenomenological parameters of the model. Let us take once again the value for λ of the GRW model. For this case, the correction $\Delta\mathcal{P}_{\hat{\mathcal{R}}}$ is of order $\mathcal{O}[10^{-81}]$. From this result, we see that the correction to the power spectrum of the comoving curvature perturbation $\hat{\mathcal{R}}$ is negligible with respect to the predicted spectrum in the standard scenario. They differ by 71 orders of magnitude. Moreover, when we compare the corrections at each stage of the evolution of the Universe, we see that the correction during the radiation dominated era [cf. Eq. (6.39)] is negligible with respect to that due to inflation [cf. Eq. (6.27)], as it is 47 orders of magnitude lower. This indicates that in a first approximation, one can restrict the study of the effects of dynamical collapse models to the inflationary stage only. Moreover, both results validate the expansion of the operator $\hat{U}_{\text{DC}}$ when evaluating the expectation value of $\hat{\mathcal{R}}^2$.

Chapter 7

Conclusions

In this thesis, we saw how Quantum continuous measurements can be successfully implemented within Gravity and Cosmology in a wide variety of contexts. As a first step, we reviewed the basic formalism necessary to describe the stochastic differential equation resulting from the Quantum continuous measurement of a given observable, as well as the corresponding master equation for the density operator. We also described the assumptions made in the Wiseman-Milburn feedback protocol, which leads to a Markovian evolution of the system under study.

In the first part of the thesis, we discussed about the problems which arise when trying to merge Quantum Mechanics and General Relativity. Although it is generally expected that a final theory of quantum gravity will involve a quantization of gravity through an appropriate procedure, in this work we devoted particular attention to explore the possibility of alternative scenarios to the quantization of gravity.

In this line of thought, we described in detail how Quantum continuous measurements have been used to implement the Newtonian gravitational interaction through a Markovian feedback mechanism, by analysing the properties of the Kafri-Taylor-Milburn (KTM) and the Tilloy-Diósi (TD) models. Regarding the KTM model, we explored the viability of a pair of generalizations of the model, and showed that one of them is theoretically discarded as it yields inconsistent results when considering composite systems as effective single particles. As for the TD formalism, we reviewed in detail the reasons which motivated the introduction of an appropriate regularization mechanism of the master equation. Moreover, we provided a concrete example of the robustness of the TD formalism, by studying the KTM2 model and showing that a map can be established between the two models.

We compared the KTM and the TD models in the appropriate limit and showed that they are fundamentally different models, although both of them implement Newtonian gravity. Moreover, by considering the main characteristics that define each of the two models, we explored if, in addition to the TD model, there are other possibilities of implementing a full Newtonian interaction within a Markovian feedback context. We argued that the measurement of the positions of the particles is not viable, and showed that a pairwise measurement of the mass densities yields unphysical coherence rates when considering the interaction between composite systems. Other possible implementations of Newtonian gravity would seem to require more complicated feedback mechanisms than the prescription of Wiseman and Milburn.

In the second part of this thesis, we reviewed the basic notions of inflation, starting with the historical motivations that led to the construction of this theory, namely, some problems in standard FLRW cosmology. We also described the basic aspects of the theory of scalar cosmological perturbations, devoting particular attention to the definition of the comoving curvature perturbation, and the corresponding power spectrum.

In addition, we briefly described the fundamental aspects of dynamical collapse models, which were introduced as an alternative theory to Quantum mechanics which explains the collapse of the wave function without the need of introducing a postulate. We described how one of the most robust dynamical collapse modes, the Continuous Spontaneous Localization (CSL) model, yields the same master equation as the one resulting from the Quantum continuous measurement of a Gaussian-smeared mass density.

In this work, we determined possible corrections due to dynamical collapse models to the power spectrum of the comoving curvature perturbation. Although there is not a fully consistent generalization of the CSL model within a relativistic context, we chose a well-motivated form of the collapse operator that could lead to a CSL generalization to the relativistic regime. We proposed an interaction picture framework in order to account for the effects of dynamical collapse models on the comoving curvature perturbation and we determined the leading order corrections to the associated power spectrum. For standard values of the collapse model parameters, we found out that the corrections are negligible with respect to the magnitude of the standard Cosmology values. Our results represent a step forward in the incorporation of dynamical collapse models within a cosmological context.

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