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ON ASYMPTOTIC STABILITY OF GROUND STATES OF SOME SYSTEMS OF NONLINEAR SCHRÖDINGER EQUATIONS

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Abstract. We extend to a specific class of systems of nonlinear Schrödinger equations (NLS) the theory of asymptotic stability of ground states already proved for the scalar NLS. Here the key point is the choice of an adequate system of modulation coordinates and the novelty, compared to the scalar NLS, is the fact that the group of symmetries of the system is non-commutative.

1. **Introduction.** In this article we will consider the system of coupled nonlinear Schrödinger equations,

$$\begin{cases} \mathrm{i}\sigma_3\dot{u} + \Delta u - \beta(|u|^2)u = 0, \\ u(0,x) = u_0(x) \in \mathbb{C}^2, \qquad x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

where i is the imaginary unit and the Pauli matrices are given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.2}$$

We assume that the function β satisfies the following two hypotheses, which guarantee the local well-posedness of (1.1) in $H^1(\mathbb{R}^3, \mathbb{C}^2)$:

(H1) $\beta(0) = 0$, $\beta \in C^\infty(\mathbb{R}, \mathbb{R})$;

(H2) there exists $\alpha \in (1, 5)$ such that for every $k \in \mathbb{N}_0$ there is a fixed C_k with

$$\left| \frac{d^k}{dv^k} \beta(v^2) \right| \leqslant C_k |v|^{\alpha-k-1} \qquad \text{for} \quad v \in \mathbb{R}, \quad |v| \geqslant 1.$$

We recall that under further hypotheses, there exist ground state solutions of the scalar NLS

$$\mathrm{i}\dot{u} + \Delta u - \beta(|u|^2)u = 0\,, \qquad u(t,x)|_{t=0} = u_0(x) \in \mathbb{C}, \qquad x \in \mathbb{R}^3 \tag{1.3}$$

in $H^1(\mathbb{R}^3, \mathbb{C})$ which are of the form $e^{\mathrm{i}\omega t}\phi(x)$ with $\omega > 0$ and $\phi(x) > 0$. Here we assume:

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(H3) there is an open interval $\mathcal{O} \subseteq (0, \infty)$ and a C^∞-family

$$\mathcal{O} \ni \omega \mapsto \phi_\omega \in \bigcap_{n \in \mathbb{N}} \Sigma_n(\mathbb{R}^3, \mathbb{C}),$$

with $\Sigma_n(\mathbb{R}^3, \mathbb{C})$ defined in (2.1), such that ϕ_ω is a positive radial solution of

$$-\Delta u + \omega u + \beta(|u|^2)u = 0 \qquad \text{for } x \in \mathbb{R}^3; \tag{1.4}$$

(H4) we have $\frac{d}{d\omega}\|\phi_\omega\|^2_{L^2} > 0$ for $\omega \in \mathcal{O}$;

(H5) for $L_+ := -\Delta + \omega + \beta(\phi_\omega^2) + 2\beta'(\phi_\omega^2)\phi_\omega^2$ with the domain $H^2(\mathbb{R}^3, \mathbb{C})$, L_+ has one negative eigenvalue and $\ker L_+ = \operatorname{Span}\{\partial_{x_j}\phi_\omega : \ j = 1, 2, 3\}$.

The above hypotheses guarantee that the ground states are orbitally stable solutions of the scalar NLS (1.3); see [25, 37]. In [16, 18] it has been proved that, under some additional hypotheses, the solitary waves are also asymptotically stable, in a sense that will be clarified later. This paper shows that some solitary waves of (1.1) are asymptotically stable. To state the result, we denote by $\boldsymbol{K} : \ \mathbb{C}^n \to \mathbb{C}^n$ the operator of complex conjugation in $\mathbb{C}^n$ and by $\mathbf{SU}(2)$ the group

$$\mathbf{SU}(2) = \left\{ \begin{bmatrix} a & b \\ -\boldsymbol{K}b & \boldsymbol{K}a \end{bmatrix} : \ (a, b) \in \mathbb{C}^2 \text{ such that } |a|^2 + |b|^2 = 1 \right\}. \tag{1.5}$$

We consider the group

$$\mathbf{G} = \mathbb{R}^3 \times \mathbb{T} \times \mathbf{SU}(2). \tag{1.6}$$

There is a natural representation of $\mathbf{G}$ on $H^1(\mathbb{R}^3, \mathbb{C}^2)$, with $\vartheta \in \mathbb{T}$ acting on u_0 like $e^{\mathrm{i}\vartheta}u_0$, $x_0 \in \mathbb{R}^3$ acting like a translation operator, and with an element of $\mathbf{SU}(2)$ acting on u_0 by transforming it into $(a + b\sigma_2\boldsymbol{K})u_0$. System (1.1) admits solitary waves of the form

$$\psi_{\omega,v}(t) = e^{\mathrm{i}t\left(\omega + \frac{v^2}{4}\right)} e^{\frac{\mathrm{i}}{2} v \cdot (x - tv)} \phi_\omega(x - tv) \, \vec{\boldsymbol{e}}_1, \qquad \vec{\boldsymbol{e}}_1 := \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{1.7}$$

We will show later that, along with mass, which we will denote by Π_4, linear momenta, which we will denote by $\Pi_i|_{i=1}^3$, and energy, system (1.1) admits three further invariants related to $\mathbf{SU}(2)$ which we will denote by $\Pi_i|_{i=5}^7$. By Π we will denote the vector $\Pi_i|_{i=1}^7$. We will see later that acting with $\mathbf{G}$ on $\psi_{\omega,v}$ we can generalize the solitary waves. We will have solitary waves Φ_p characterized by $\Pi(\Phi_p) = p$. We will prove the following theorem.

Theorem 1.1. *Assume (H1)–(H5) stated above, (H6)–(H8) stated in Section 7, and (H9) stated in Sect. 11. Pick $\omega^1 \in \mathcal{O}$. Then there exist $\epsilon_0 = \epsilon_0(\omega^1) > 0$ and $C = C(\omega^1) > 0$ such that if u solves (1.3) with $u|_{t=0} = u_0$ and if*

$$\epsilon := \inf_{g \in \mathbf{G}} \|u_0 - T(g)\psi_{\omega^1, 0}(0)\|_{H^1(\mathbb{R}^3, \mathbb{C}^2)} < \epsilon_0, \tag{1.8}$$

then there exist a solitary wave ψ_{ω^+, v^+}, a function $g \in C^1(\mathbb{R}_+, \mathbf{G})$ and an element $h_+ \in H^1(\mathbb{R}^3, \mathbb{C}^2)$ with $\|h_+\|_{H^1(\mathbb{R}^3, \mathbb{C}^2)} + |\omega_+ - \omega^1| + |v^+| \leqslant C\epsilon$ such that

$$\lim_{t \to +\infty} \|u(t) - T(g(t))\psi_{\omega^+, v^+}(t) - e^{-\mathrm{i}\sigma_3 \Delta t} h_+\|_{H^1(\mathbb{R}^3, \mathbb{C}^2)} = 0. \tag{1.9}$$

Remark 1.2. Noncommutative symmetry groups which involve the complex conjugation, of the type considered in this article, are interesting in particular in view of the $\mathbf{SU}(1, 1)$ symmetry group which appears in the nonlinear Dirac equation with scalar-type self-interaction (the Soler model) and in the Dirac–Klein–Gordon model; see [24, 31]. Such symmetry groups result in the emergence of two-frequency solitary waves [8, 10] (see also the monograph [9]). As a consequence, the asymptotic stability of standard (one-frequency) solitary waves could only make sense if one takes into account the convergence of perturbed solutions to both one- and two-frequency solitary waves, which creates additional difficulties on the way to treating the asymptotic stability. Let us mention that this difficulty was avoided in the proof of asymptotic stability in the Soler model in [6, 29, 7, 14] by restricting the class of perturbations so that the convergence to a bi-frequency solitary wave was prohibited by symmetry considerations.

Theorem 1.1 is a transposition to a system of the result proved for scalar equations in [16, 18, 19]; see also [2]. We are not aware of previous similar results for systems of PDE's. For the orbital stability of systems of NLS we refer to Grillakis et al. [25], De Bièvre and Rota Nodari [22]; see also [5] and references therein.

The proof of Theorem 1.1 goes along the lines of the proof for the scalar NLS. If we look at the analogous classical problem of the asymptotic stability of the equilibrium 0 for a system $\dot{r} = Ar + g(r)$, where $g(r) = o(r)$ at $r = 0$ and with a matrix-valued operator A, of key importance is the location of the spectrum $\sigma(A)$. Stability requires that if $\varsigma \in \sigma(A)$ then $\mathrm{Re}\,\varsigma \leqslant 0$. Isolated eigenvalues on the imaginary axis correspond to central directions whose contribution to stability or instability can be ascertained only analyzing the nonlinear system, and not just the linearization $\dot{r} = Ar$. This classical framework is also used for Theorem 1.1. First of all, an appropriate expansion of u at the ground states (see Lemma 3.1 below) gives us the variable r. The analogue of A is given by (2.24). In our case the spectrum is all contained in the imaginary axis, but the continuous spectrum plays the same role of the stable spectrum of A, thanks to dispersion and along the lines described in pp. 36–37 of Strauss's introduction to nonlinear wave equations [36]. The discrete spectrum of (2.24) plays the role of central directions. The nonlinear mechanism acting on the corresponding discrete modes and responsible for the stabilization indicated in (1.9) has been termed *Nonlinear Fermi Golden Rule* in [34] and was explored initially in [12, 35]. A detailed description, by means of some elementary examples, is in [21, Introduction]; see also [38]. The same mechanisms, described in [21] and used in [2, 3, 12, 16, 18, 19, 35] and in a number of other papers referenced therein, are applied here to prove Theorem 1.1. A novel difficulty occurs with the choice of modulation. Here the the idea is to use the representation (2.19). The rest of the paper is not very different from [16, 17, 18, 19]. In the course of the proof there are some difficulties related to the fact that the Lie algebra of $\mathbf{G}$ is not commutative, and correspondingly, the Poisson brackets $\{\Pi_j, \Pi_l\}$ are not identically zero like in the earlier papers. This is solved quite naturally by exploiting conservation laws and considering the reduced manifold; see [28, Ch. 6]. Thanks to an appropriate uniformity with respect to the conserved quantities of the coordinate changes, we obtain the desired result.

2. **Notation and preliminaries.**

We start with some notation. For $\varsigma \in \mathbb{C}^n$ we use the Japanese Bracket notation $\langle \varsigma \rangle = \sqrt{1 + |\varsigma|^2}$.

Given two Banach spaces $\mathbb{X}$ and $\mathbb{Y}$, we denote by $B(\mathbb{X}, \mathbb{Y})$ the Banach space of bounded linear transformations from $\mathbb{X}$ to $\mathbb{Y}$.

Let m, k, $s \in \mathbb{R}$. Given a Banach space $\mathbb{E}$ and functions $\mathbb{R}^3 \to \mathbb{E}$, we denote by $\Sigma_m(\mathbb{R}^3, \mathbb{E})$ and $H^{k,s}(\mathbb{R}^3, \mathbb{E})$ the Banach spaces with the norms

$$\|u\|^2_{\Sigma_m} := \|\big\langle \sqrt{-\Delta + |x|^2}\, \big\rangle^m u\|^2_{L^2(\mathbb{R}^3,\mathbb{E})}, \tag{2.1}$$

$$\|f\|_{H^{k,s}(\mathbb{R}^3,\mathbb{E})} := \|\langle x \rangle^s \big\langle \sqrt{-\Delta}\, \big\rangle^k f\|_{L^2(\mathbb{R}^3,\mathbb{E})}, \tag{2.2}$$

where we will use mostly $\mathbb{E} = \mathbb{C}^2$. We also consider

$$\text{the space of Schwartz functions } \mathcal{S}(\mathbb{R}^3, \mathbb{E}) := \cap_{m\in\mathbb{R}} \Sigma_m(\mathbb{R}^3, \mathbb{E}); \tag{2.3}$$

$$\text{the space of tempered distributions } \mathcal{S}'(\mathbb{R}^3, \mathbb{E}) := \cup_{m\in\mathbb{R}} \Sigma_m(\mathbb{R}^3, \mathbb{E}). \tag{2.4}$$

We denote by ${}^t v$ the transpose of $v \in \mathbb{C}^n$, so that the hermitian conjugate of $v \in \mathbb{C}^n$ is given by ${}^t(\boldsymbol{K}v)$, where $\boldsymbol{K} : \mathbb{C}^n \to \mathbb{C}^n$ is the complex conjugation in $\mathbb{C}^n$. For $u, v \in \mathbb{C}^n$ we set $|v|^2 = {}^t(\boldsymbol{K}v)v$. We denote the hermitian form in $L^2(\mathbb{R}^3, \mathbb{C}^2)$ by

$$\langle u, v\rangle = \operatorname{Re} \int_{\mathbb{R}^3} {}^t(\boldsymbol{K}u(x))\, v(x)\, dx, \qquad u,\, v \in L^2(\mathbb{R}^3, \mathbb{C}^2), \tag{2.5}$$

and we consider the symplectic form

$$\Omega(X, Y) := \langle \mathrm{i}\sigma_3 X, Y\rangle, \qquad X,\, Y \in L^2(\mathbb{R}^3, \mathbb{C}^2). \tag{2.6}$$

Definition 2.1. Given a differentiable function F, its Hamiltonian vector field with respect to a *strong* symplectic form Ω is the field X_F such that $\Omega(X_F, Y) = dF(Y)$ for any tangent vector Y, with dF the Fréchet derivative. For differentiable functions F and G, their Poisson bracket is $\{F, G\} := dF(X_G)$ if G is scalar-valued and F is either scalar-valued or takes values in a Banach space $\mathbb{E}$.

Notice that since $X \mapsto \langle \mathrm{i}\sigma_3 X, \ \rangle$ defines an isomorphism of $L^2(\mathbb{R}^3, \mathbb{C}^2)$, or of $H^1(\mathbb{R}^3, \mathbb{C}^2)$, into itself, our symplectic form (2.6) is strong. For $u \in H^1(\mathbb{R}^3, \mathbb{C}^2)$ we have the following functionals (the linear momenta and mass) which are conserved in time by (1.1):

$$\Pi_a(u) = 2^{-1}\langle \Diamond_a u, u\rangle, \quad \Diamond_a := -\mathrm{i}\sigma_3 \partial_{x_a} \ \text{ for } a = 1, 2, 3; \tag{2.7}$$

$$\Pi_4(u) = 2^{-1}\langle \Diamond_4 u, u\rangle, \quad \Diamond_4 := \ \mathbb{1}(= \text{identity operator}); \tag{2.8}$$

see [25, (2.6) and p. 343] for (2.7). We also consider the following functionals Π_j, $j = 5, 6, 7$:

$$\Pi_j(u) := 2^{-1}\langle \Diamond_j u, u\rangle \text{ with } \Diamond_j := \begin{cases} \sigma_3\sigma_2\boldsymbol{K}, & j = 5, \\ \mathrm{i}\sigma_3\sigma_2\boldsymbol{K}, & j = 6, \\ \sigma_3, & j = 7. \end{cases} \tag{2.9}$$

The energy is defined as follows: for $B(0) = 0$ and $B' = \beta$ we write

$$\begin{aligned} &E(u) := E_K(u) + E_P(u), \\ &E_K(u) := \frac{1}{2}\langle -\Delta u, u\rangle, \qquad E_P(u) := -\frac{1}{2}\int_{\mathbb{R}^3} B(|u|^2)\, dx. \end{aligned} \tag{2.10}$$

It is a standard fact which can be proved like for the scalar equation (1.3) (for the latter, see [13]) that (H1)–(H2) imply local well-posedness of (1.1) in $H^1(\mathbb{R}^3, \mathbb{C}^2)$.

We denote by dE the Fréchet derivative of the energy E; see (2.10). We define ∇E by $dEX = \langle \nabla E, X\rangle$. Notice that $\nabla E \in C^1(H^1(\mathbb{R}^3, \mathbb{C}^2), H^{-1}(\mathbb{R}^3, \mathbb{C}^2))$, that $\nabla E(u) = -\Delta u + \beta(|u|^2)u$ and henceforth that (1.1) can be written as

$$\dot{u} = -\mathrm{i}\sigma_3 \nabla E(u) = X_E(u), \tag{2.11}$$

that is, as a hamiltonian system with hamiltonian E. Notice that $\nabla\Pi_j(u) = \Diamond_j u$ for $j = 1 \leqslant j \leqslant 7$.

By (2.7) and (H4),

$$(\omega, v) \mapsto (\Pi_j(e^{\sigma_3 \frac{\mathrm{i}}{2} v\cdot x}\phi_\omega\ \vec{\boldsymbol{e}}_1))_{j=1}^4, \qquad \vec{\boldsymbol{e}}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

is a diffeomorphism into an open subset of $\mathbb{R}_+ \times \mathbb{R}^3$. We introduce

$$p = p(\omega, v) \in \mathbb{R}^7 \text{ with } p_j(\omega, v) := \begin{cases} \Pi_j(e^{\sigma_3 \frac{\mathrm{i}}{2} v\cdot x}\phi_\omega \ \vec{\boldsymbol{e}}_1), & 1 \leqslant j \leqslant 4; \\ 0, & j = 5, 6; \\ \Pi_j(e^{\sigma_3 \frac{\mathrm{i}}{2} v\cdot x}\phi_\omega \ \vec{\boldsymbol{e}}_1) = p_4(\omega, v), & j = 7. \end{cases} \tag{2.12}$$

Notice that $\Pi_j(e^{\sigma_3 \frac{\mathrm{i}}{2} v\cdot x}\phi_\omega \ \vec{\boldsymbol{e}}_1) = 0$ for $j = 5, 6$. We denote by $\mathcal{P}$ the subset of $\mathbb{R}^7$ defined by

$$\mathcal{P} = \{p(\omega, v);\, \omega \in \mathcal{O},\ v \in \mathbb{R}^3\}. \tag{2.13}$$

For $p = p(\omega, v) \in \mathcal{P}$, we set

$$\Phi_p(x) := e^{\frac{\mathrm{i}}{2} v\cdot x}\phi_\omega(x) \ \vec{\boldsymbol{e}}_1 \ . \tag{2.14}$$

Obviously $\Phi_{p(\omega,v)} = \psi_{\omega,v}(0)$; see (1.7). We will set $\Phi_{p_1} = \psi_{\omega^1,0}(0)$ for the function in Theorem 1.1. We have $\Pi_j(\Phi_{p_1}) = 0$ for $j = 1, 2, 3, 5, 6$. It is not restrictive to pick the initial datum such that

$$\Pi_j(u_0) = 0 \quad \text{for } j = 1, 2, 3, 5, 6. \tag{2.15}$$

Indeed, by continuity, Π_j for $j = 1, 2, 3, 5, 6$ take values close to 0 in a neighborhood of Φ_{p_1}. By boosts and Lemma 5.1, one can act on u_0 changing it into another nearby initial datum which satisfies (2.15); we skip the elementary details. We introduce

$$\lambda(p) = (\lambda_1(p), \dots, \lambda_7(p)) \in \mathbb{R}^7 \text{ defined by } \lambda_j(p) := \begin{cases} -v_j, & 1 \leqslant j \leqslant 3; \\ -\omega - \frac{v^2}{4}, & j = 4; \\ 0, & 5 \leqslant j \leqslant 7. \end{cases} \tag{2.16}$$

They are Lagrange multipliers, and an elementary computation shows that

$$e^{-\mathrm{i}\sigma_3 t \lambda(p)\cdot\Diamond}\Phi_p = \psi_{\omega,v}(t) \tag{2.17}$$

and that Φ_p is a constrained critical value for the energy satisfying

$$\nabla E(\Phi_p) - \sum_{j=1,\dots,7} \lambda_j(p)\Diamond_j \Phi_p = 0. \tag{2.18}$$

We consider the representation $T : \mathbf{G} \to B(H^1(\mathbb{R}^3, \mathbb{C}^2), H^1(\mathbb{R}^3, \mathbb{C}^2))$ defined by

$$T(g)u_0 := e^{\mathrm{i}\sigma_3 \tau\cdot\Diamond}(a + b\sigma_2 \boldsymbol{K})u_0 \quad \text{for } g = \left(\tau, \begin{bmatrix} a & b \\ -\boldsymbol{K}b & \boldsymbol{K}a \end{bmatrix}\right), \tag{2.19}$$
$$\text{where } \tau = (\tau_1, \tau_2, \tau_3, \tau_4) \in \mathbb{R}^3 \times \mathbb{T} \text{ and } \tau \cdot \Diamond := \sum_{j=1,\dots,4} \tau_j \Diamond_j.$$

An elementary but very important fact to us is the following lemma.

Lemma 2.2. *We have the following facts:*

1. *The action of* $\mathbf{G}$ *given by* (2.19) *preserves the symplectic form* Ω *defined in* (2.6)*;*
2. *The action* (2.19) *preserves the invariants* Π_j *for* $1 \leqslant j \leqslant 4$ *and* E*;*
3. *The functionals* Π_j*,* $1 \leqslant j \leqslant 7$*, and* E *are conserved by the flow of* (1.1) *in* $H^1(\mathbb{R}^3, \mathbb{C}^2)$.

Proof. (1) follows from the commutation $[\mathbb{i}\sigma_3, a + b\sigma_2\boldsymbol{K}] = 0$. (2) is a consequence of

$$
\begin{aligned}
|(a + b\sigma_2\boldsymbol{K})u|^2 &= \operatorname{Re}{}^t(\boldsymbol{K}u)((\boldsymbol{K}a) + \boldsymbol{K}\sigma_2(\boldsymbol{K}b))(a + b\sigma_2\boldsymbol{K})u \\
&= (|a|^2 + |b|^2)|u|^2 + \operatorname{Re}{}^t(\boldsymbol{K}u)((\boldsymbol{K}a)b\sigma_2\boldsymbol{K} + \boldsymbol{K}\sigma_2 a\boldsymbol{K}b)u = |u|^2.
\end{aligned}
$$

The fact that the functionals Π_j, $1 \leqslant j \leqslant 4$, and the energy E are preserved by the flow of (1.1) is standard. To deal with the cases $j = 5, 6, 7$, we first recall that the Lie algebra of $\mathbf{SU}(2)$ can be written as $\mathfrak{su}(2) = \operatorname{Span}(\mathbb{i}\sigma_i,\ 1 \leqslant i \leqslant 3)$. We have

$$
\frac{d}{dt} T(e^{-\mathbb{i}t\sigma_i})\big|_{t=0} = \begin{cases} \frac{d}{dt}\,(\cos(t) - \mathbb{i}\sin(t)\sigma_2\boldsymbol{K})|_{t=0} = -\mathbb{i}\sigma_2\boldsymbol{K}, & i = 1; \\ \frac{d}{dt}\,(\cos(t) + \sin(t)\sigma_2\boldsymbol{K})|_{t=0} = \sigma_2\boldsymbol{K}, & i = 2; \\ \frac{d}{dt}\, e^{-\mathbb{i}t}\big|_{t=0} = -\mathbb{i}, & i = 3. \end{cases} \tag{2.20}
$$

Like in [25, line 5 p. 313],

$$
\begin{aligned}
&\frac{d}{dt}\Pi_{4+i}(u) = \langle \Diamond_{4+i}u, -\mathbb{i}\sigma_3\nabla E(u)\rangle = \langle \mathbb{i}\sigma_3\Diamond_{4+i}u, \nabla E(u)\rangle \\
&= \frac{d}{ds}\left\langle T(e^{\mathbb{i}s\sigma_i})u, \nabla E(u)\right\rangle\Big|_{s=0} = \frac{d}{ds}E(T(e^{\mathbb{i}s\sigma_i})u)\Big|_{s=0} = 0,
\end{aligned}
$$

where the first equality holds for sufficiently regular solutions, while the last one follows from (2). By a density argument and well-posedness of (1.1), we obtain claim (3). □

Lemma 2.3. *The following 10 vectors are linearly independent over* $\mathbb{R}$*:*

$$
\partial_{p_1}\Phi_p,\ \partial_{p_2}\Phi_p,\ \partial_{p_3}\Phi_p,\ \partial_{p_4}\Phi_p,\ \partial_{x_1}\Phi_p,\ \partial_{x_2}\Phi_p,\ \partial_{x_3}\Phi_p,\ \mathbb{i}\sigma_2\boldsymbol{K}\Phi_p,\ \sigma_2\boldsymbol{K}\Phi_p,\ \mathbb{i}\Phi_p. \tag{2.21}
$$

The proof is elementary. □

We consider now the "solitary manifold"

$$
\mathcal{M} := \left\{ e^{\mathbb{i}\sigma_3\tau\cdot\Diamond}(a + b\sigma_2\mathbf{K})\Phi_p(x) :\ \tau \in \mathbb{R}^3 \times \mathbb{T},\ \begin{bmatrix} a & b \\ -\boldsymbol{K}b & \boldsymbol{K}a \end{bmatrix} \in \mathbf{SU}(2),\ p \in \mathcal{P} \right\}. \tag{2.22}
$$

The vectors in (2.21) are obtained computing the partial derivatives in $(0, p, 0)$ of the function in $C^\infty(\mathbb{D}_{\mathbb{C}}(0, \varepsilon_0) \times \mathcal{P} \times \mathbb{T} \times \mathbb{R}^3, \Sigma_k(\mathbb{R}^3, \mathbb{C}^2))$ given by

$$
(b, p, \tau) \mapsto e^{\mathbb{i}\sigma_3\tau\cdot\Diamond}\mathfrak{s}(b)\Phi_p, \text{ where } \mathfrak{s}(b) := \sqrt{1 - |b|^2} + b\sigma_2\boldsymbol{K}. \tag{2.23}
$$

Then Lemma 2.3 implies that for any $k > 0$ there is $\varepsilon_0 >$ such that (2.23) is an embedding and $\mathcal{M}$ is a manifold. The $\mathbb{R}$-vector space generated by vectors in Lemma 2.3 is the tangent space $T_{\Phi_p}\mathcal{M}$.

Consider the *linearized* operator $\mathcal{H}_p := -\mathbb{i}\sigma_3(\nabla^2 E(\Phi_p) - \lambda(p)\cdot\Diamond)$. By $\lambda(p(\omega, 0))\cdot\Diamond = -\omega$ we have

$$
\begin{aligned}
&\mathcal{H}_{p(\omega,0)}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = -\begin{pmatrix} \mathbb{i}\mathfrak{L}_\omega^{(1)}u_1 \\ -\mathbb{i}\mathfrak{L}_\omega^{(2)}u_2 \end{pmatrix}, \quad \text{where} \\
&\mathfrak{L}_\omega^{(1)}u_1 = -\Delta u_1 + \beta(\phi_\omega^2)u_1 + 2\beta'(\phi_\omega^2)\operatorname{Re}(u_1) + \omega u_1, \\
&\mathfrak{L}_\omega^{(2)}u_2 = -\Delta u_2 + \beta(\phi_\omega^2)u_2 + \omega u_2.
\end{aligned} \tag{2.24}
$$

It is well-known that $\mathcal{H}_p$ is $\mathbb{R}$-linear but not $\mathbb{C}$-linear; see [11, 15]. For this reason we interpret $H^1(\mathbb{R}^3, \mathbb{C}^2)$ as a vector space over $\mathbb{R}$. Later, in Section 7, we perform a complexification. Recall the generalized kernel $N_g(\mathcal{H}_p) := \cup_{j=1}^\infty \ker(\mathcal{H}_p)^j$. The following lemma is very important.

Lemma 2.4. *We have* $N_g(\mathcal{H}_{p(\omega,0)}) = T_{\Phi_{p(\omega,0)}}\mathcal{M}$.

Proof. First of all, $\mathfrak{L}_\omega^{(i)}$ for $i = 1, 2$ are decoupled, so that it is enough to consider them separately. We have the following, which is a well-known fact about ground states (see, for example, [32, Sect.XIII.12]):

$$\ker(\mathbb{i}\mathfrak{L}_\omega^{(2)}) = N_g(\mathbb{i}\mathfrak{L}_\omega^{(2)}) = \mathrm{Span}\{\mathbb{i}\phi_\omega, \phi_\omega\}.$$

The following well-known consequence of (H4)–(H5), derived in [37], completes the proof:

$$\begin{aligned}&\ker(\mathbb{i}\mathfrak{L}_\omega^{(1)}) = \mathrm{Span}\{\mathbb{i}\phi_\omega,\, \partial_{x_a}\phi_\omega|_{a=1}^3\},\\ &N_g(\mathfrak{L}_\omega^{(1)}) = \ker(\mathbb{i}\mathfrak{L}_\omega^{(1)})^2 = (\mathbb{i}\ker\mathfrak{L}_\omega^{(1)}) \oplus \mathrm{Span}\{\partial_{p_j} e^{\frac{\mathbb{i}}{2}v\cdot x}\phi_\omega|_{j=1}^4\}.\end{aligned}$$

□

System (1.1) is an interesting example for the stability theory in the classical paper by Grillakis *et al.* [25] because all the examples of systems of NLS's in Sect. 9 in [25] for $x \in \mathbb{R}^3$ and $u(t,x) \in \mathbb{R}^4$ have 4-dimensional centralizers, while for (1.1) dimension is 6; see the following two remarks.

Remark 2.5. From the identification $\mathbb{C}^2 = \mathbb{R}^4$ there is a natural inclusion $\mathbf{SU}(2) \subseteq \mathrm{SO}(4)$. By the identification implicit in (1.5) of $\mathbf{a} \in \mathbf{SU}(2)$ and an element in the unit sphere $\tilde{\mathbf{a}} \in \mathbb{S}^3 \subset \mathbb{R}^4$, the action of $\mathbf{a} \in \mathbf{SU}(2)$ on $\mathbf{v} \in \mathbb{R}^4$ is nothing else but the product of quaternions, $\mathbf{v}\tilde{\mathbf{a}}$. Similarly, by elementary computations, it is possible to see that $(a + b\sigma_2\boldsymbol{K})\mathbf{v} = \hat{\mathbf{a}}\mathbf{v}$ (on the r.h.s. multiplication of two quaternions) for all $\mathbf{v} \in \mathbb{R}^4$ and for an appropriate $\hat{\mathbf{a}} \in \mathbb{S}^3$. In the framework of [25] when applied to (1.1), a key role is played by the centralizer of the group $\{e^{\tau_4\mathbb{i}\sigma_3};\, \tau_4 \in \mathbb{R}\}$ inside $\mathbb{R}^3 \times \mathrm{SO}(4)$. Using [39, p. 111], it can be shown that $\mathbf{G}$, acting as in (2.19), is a connected component of this centralizer.

Remark 2.6. The key hypothesis in [25] is Assumption 3 on p. 314, stating $Z = \ker(\mathcal{H}_{p(\omega,0)})$ for

$$Z := \left\{\partial_t\, \widetilde{T}(e^{t\varpi})\Phi_{p(\omega,0)}\Big|_{t=0} :\ \varpi \in \mathbb{R}^3 \times \mathfrak{so}(4) \text{ commutes in } \mathbb{R}^3 \times \mathfrak{so}(4) \text{ with } \mathbb{i}\sigma_3\right\},$$

where for $\varpi \in \mathbb{R}^3$ we have $\widetilde{T}(e^{t\varpi}) = T(e^{t\varpi})$ and for $\varpi \in \mathfrak{so}(4)$ we set $\widetilde{T}(e^{t\varpi})w = e^{t\varpi}w$ for any $w \in \mathbb{R}^4$, with the usual product row column $\mathrm{SO}(4) \times \mathbb{R}^4 \to \mathbb{R}^4$.

Always $Z \subseteq \ker(\mathcal{H}_{p(\omega,0)})$; see [25, Lemma 2.2]. Lemma 2.4 yields the equality. Assumption 1, i.e. local well-posedness, is true and Assumption 2, about bound states, is true under our hypothesis (H3). Other hypotheses needed in [25], such as that the centralizer, or at least its connected component containing the unit element in $\mathbb{R}^3 \times \mathrm{SO}(4)$, acts by symplectomorphisms which leave the energy invariant, follow from Lemma 2.2. So by [25] the bound states (2.17) are $\mathbf{G}$-orbitally stable.

3. **Modulation.** The manifold $\mathcal{M}$ introduced in (2.22) is a symplectic submanifold of $L^2(\mathbb{R}^3, \mathbb{C}^2)$. This follows from

$$\begin{aligned}&\Omega(\mathbb{i}\sigma_2\boldsymbol{K}\Phi_p, \sigma_2\boldsymbol{K}\Phi_p) = p_4\,, \quad \Omega(\partial_{p_4}\Phi_p, \mathbb{i}\Phi_p) = 2^{-1}\partial_{p_4}\langle\mathbb{i}\sigma_3\Phi_p, \mathbb{i}\Phi_p\rangle = \partial_{p_4}p_4 = 1,\\ &\Omega(\partial_{p_a}\Phi_p, \partial_{x_a}\Phi_p) = 2^{-1}\partial_{p_a}\langle\Phi_p, \Diamond_a\Phi_p\rangle = \partial_{p_a}p_a = 1 \quad \text{for } a = 1, 2, 3,\end{aligned}$$

and from symplectic orthogonality of all other pairs of vectors in (2.21). We obtain a bilinear form

$$\Omega:\ \mathcal{S}(\mathbb{R}^3, \mathbb{C}^2) \times \mathcal{S}'(\mathbb{R}^3, \mathbb{C}^2) \to \mathbb{R}.$$

Since $T_{\Phi_p}\mathcal{M} \subseteq \mathcal{S}(\mathbb{R}^3, \mathbb{C}^2)$, we can define the subspace $T^{\perp\Omega}_{\Phi_p}\mathcal{M} \subseteq \mathcal{S}'(\mathbb{R}^3, \mathbb{C}^2)$. Ω also defines a pairing

$$\Omega : \ \Sigma_n(\mathbb{R}^3, \mathbb{C}^2) \times \Sigma_{-n}(\mathbb{R}^3, \mathbb{C}^2) \to \mathbb{R}.$$

This yields the decomposition

$$\Sigma_{-n}(\mathbb{R}^3, \mathbb{C}^2) = T_{\Phi_p}\mathcal{M} \oplus (T^{\perp\Omega}_{\Phi_p}\mathcal{M} \cap \Sigma_{-n}(\mathbb{R}^3, \mathbb{C}^2)). \tag{3.1}$$

We denote by $\widehat{P}_p$ and P_p the projections onto the first and second term of the direct sum, respectively:

$$\begin{aligned} \widehat{P}_p &: \ \Sigma_{-n}(\mathbb{R}^3, \mathbb{C}^2) \to T_{\Phi_p}\mathcal{M}, \\ P_p &: \ \Sigma_{-n}(\mathbb{R}^3, \mathbb{C}^2) \to T^{\perp\Omega}_{\Phi_p}\mathcal{M} \cap \Sigma_{-n}(\mathbb{R}^3, \mathbb{C}^2). \end{aligned} \tag{3.2}$$

A special case of (3.1) is

$$L^2(\mathbb{R}^3, \mathbb{C}^2) = T_{\Phi_p}\mathcal{M} \oplus (T^{\perp\Omega}_{\Phi_p}\mathcal{M} \cap L^2(\mathbb{R}^3, \mathbb{C}^2)). \tag{3.3}$$

It is easy to see that the map $p \mapsto \widehat{P}_p$ is in $C^\infty(\mathcal{P}, B(\Sigma_{-n}(\mathbb{R}^3, \mathbb{C}^2), \Sigma_n(\mathbb{R}^3, \mathbb{C}^2)))$ for any $n \in \mathbb{Z}$. The following about the $\mathfrak{s}(b)$ in (2.23) is consequence of elementary computations:

$$\begin{aligned} &(\mathfrak{s}(b))^{-1} = (\mathfrak{s}(b))^* = \mathfrak{s}(-b) \ ; \\ &\mathfrak{s}(b)\sigma_j = \sigma_j \mathfrak{s}(-b) \ \text{ for all } j = 1, 2, 3 \, ; \\ &\boldsymbol{K}\mathfrak{s}(b) = \mathfrak{s}\left(-\boldsymbol{K}b\right) \ , \quad \mathfrak{s}(b)\mathbb{i} = \mathbb{i}\mathfrak{s}(-b). \end{aligned} \tag{3.4}$$

Lemma 3.1 (Modulation)**.** *Fix $n_1 \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $p^1 \in \mathcal{P}$. Then there exists an open neighborhood $\mathcal{U}_{-n_1}$ of Φ_{p^1} in $\Sigma_{-n_1}(\mathbb{R}^3, \mathbb{C}^2)$ and functions $p \in C^\infty(\mathcal{U}_{-n_1}, \mathcal{P})$, $\tau \in C^\infty(\mathcal{U}_{-n_1}, \mathbb{R}^3 \times \mathbb{T})$ and $b \in C^\infty(\mathcal{U}_{-n_1}, \mathbb{C})$ such that $p(\Phi_{p^1}) = p^1$, $\tau(\Phi_{p^1}) = 0$, $b(\Phi_{p^1}) = 0$ and $\vartheta(\Phi_{p^1}) = 0$ so that for any $u \in \mathcal{U}_{-n_1}$,*

$$u = e^{-\mathbb{i}\sigma_3\tau(u)\cdot\Diamond}\mathfrak{s}(b(u))(\Phi_{p(u)} + R(u)), \ \text{with } R(u) \in T^{\perp\Omega}_{\Phi_{p(u)}}\mathcal{M} \cap \Sigma_{-n_1}(\mathbb{R}^3, \mathbb{C}^2). \tag{3.5}$$

Proof. The proof is standard. For $\mathbf{v}_\iota(p)$, $1 \leqslant \iota \leqslant 10$ varying among the 10 vectors in (2.21), set

$$F_\iota(u, p, \tau, b) := \Omega(e^{\mathbb{i}\sigma_3\tau\cdot\Diamond}\mathfrak{s}(-b)u - \Phi_p, \mathbf{v}_\iota(p)).$$

Next, setting $\overrightarrow{\boldsymbol{F}} = (F_1, ..., F_{10})$, we compute

$$\overrightarrow{\boldsymbol{F}}(u, p, \tau, b)\Big|_{u = e^{-\mathbb{i}\sigma_3\tau\cdot\Diamond}\Phi_p, \ b=0} = 0 \text{ and the Jacobian matrix is}$$

$$\frac{\partial \overrightarrow{\boldsymbol{F}}(u, p, \tau, b)}{\partial(p, \tau, b)}\Bigg|_{u = e^{-\mathbb{i}\sigma_3\tau\cdot\Diamond}\Phi_p, \ b=0} = \left[\varepsilon_{ij}\Omega(\mathbf{v}_i(p), \mathbf{v}_j(p))\right]_{i,j}, \qquad 1 \leqslant i, j \leqslant 10, \tag{3.6}$$

where the numbers ε_{ij} belong to $\{1, -1\}$. Since for each $\mathbf{v}_i(p)$ there is exactly one $\mathbf{v}_j(p)$ such that $\Omega(\mathbf{v}_i(p), \mathbf{v}_j(p)) \neq 0$, it follows that all the columns in (3.6) are linearly independent. We can therefore apply the implicit function theorem which yields the statement. □

It can be proved (see [17, Lemma 2.3]) that in a sufficiently small neighborhood $\mathcal{V}$ of p^1 in $\mathcal{P}$, for any $k \geqslant -n_1$, the projection

$$P_p : \ T^{\perp\Omega}_{\Phi_{p^1}}\mathcal{M} \cap \Sigma_k(\mathbb{R}^3, \mathbb{C}^2) \longrightarrow T^{\perp\Omega}_{\Phi_p}\mathcal{M} \cap \Sigma_k(\mathbb{R}^3, \mathbb{C}^2) \tag{3.7}$$

is an isomorphism. From Lemma 3.1 we have the parametrization

$$\mathcal{P} \times (\mathbb{R}^3 \times \mathbb{T}) \times D_{\mathbb{C}}(0, \varepsilon_0) \times (T^{\perp \Omega}_{\Phi_{p^1}} \mathcal{M} \cap H^1(\mathbb{R}^3, \mathbb{C}^2)) \to H^1(\mathbb{R}^3, \mathbb{C}^2) \tag{3.8}$$

with the *modulation coordinates*

$$(p, \tau, b, r) \mapsto u = e^{-\mathbb{i}\sigma_3 \tau \cdot \Diamond} \mathfrak{s}(b)(\Phi_p + P_p r). \tag{3.9}$$

We choose $p^0 \in \mathcal{P}$ so that

$$\Pi_j(u_0) = p^0_j \text{ for } j \in I = \{1, 2, 3, 4\} \tag{3.10}$$

(that is $p^0_j = 0$ for $j = 1, 2, 3$ and $\Pi_4(u_0) = p^0_4$, i.e. u_0 and Φ_{p^0} have same charge).

In terms of coordinates (3.9), system (1.1), which we have also written as $\dot{u} = X_E(u)$, see (2.11), can be expressed in terms of the Poisson brackets as follows (see [17, Lemma 2.6]):

$$\dot{p} = \{p, E\}, \quad \dot{\tau} = \{\tau, E\}, \quad \dot{b} = \{b, E\}, \quad \dot{r} = \{r, E\}. \tag{3.11}$$

By the intrinsic definition of partial derivative on manifolds (see [23, p. 25]) we have the following vector fields (recall $b_R = \mathrm{Re}(b)$ and $b_I = \mathrm{Im}(b)$):

$$\begin{aligned} \partial_{\tau_j} &= -\mathbb{i}\sigma_3 \Diamond_j u \;\text{ for } 1 \leqslant j \leqslant 4, \\ \partial_{p_k} &= e^{-\mathbb{i}\sigma_3 \tau \cdot \Diamond} \mathfrak{s}(b)(\partial_{p_k} \Phi_p + \partial_{p_k} P_p r) \;\text{ for } 1 \leqslant k \leqslant 4, \\ \partial_{b_A} &= e^{-\mathbb{i}\sigma_3 \tau \cdot \Diamond} \partial_{b_A} \mathfrak{s}(b)(\Phi_p + P_p r) \;\text{ for } A = R, I, \end{aligned} \tag{3.12}$$

which are obtained by differentiating by the various coordinates the r.h.s. of the equality in (3.9). By (3.12), we have an elementary and crucial fact that $X_{\Pi_j}(u) = \mathbb{i}\sigma_3 \nabla \Pi_j(u) = \mathbb{i}\sigma_3 \Diamond_j u$ for $1 \leqslant j \leqslant 7$ which corresponds to formulae (2.5)–(2.6) in [25]. In particular, we have

$$X_{\Pi_j}(u) = \partial_{\tau_j} \quad \text{for } 1 \leqslant j \leqslant 4,$$

which immediately implies

$$\{\Pi_j, \tau_k\} = -\delta_{jk}, \quad \{\Pi_j, b_A\} = 0, \quad \{\Pi_j, p_k\} = 0, \quad \{r, \Pi_j\} = 0 \;\text{ for } 1 \leqslant j \leqslant 4.$$

A natural step, which helps to reduce the number of equations in (3.11) and corresponds to an application of Noether's Theorem to Hamiltonian systems, see [28, Theorem 6.35, p. 402], is to substitute each function $p_j|_{j=1}^4$ in the coordinate system (p, τ, b, r) with the functions $\Pi_j|_{j=1}^4$ and move to coordinates $(\Pi_j|_{j=1}^4, \tau, b, r)$. Indeed, as in [17, formula (34)], we have, for $\varrho_j := \Pi_j(r)$ with $1 \leqslant j \leqslant 4$,

$$\Pi_j = p_j + \varrho_j + \Pi_j((P_p - P_{p^1})r) + \langle r, \Diamond_j (P_p - P_{p^1})r \rangle, \quad \varrho_j := \Pi_j(r). \tag{3.13}$$

This allows one to move from (p, τ, b, r) to $(\Pi_j|_{j=1}^4, \tau, b, r)$. Furthermore, $\partial_{\tau_k} \Pi_j(u) \equiv 0$ for $k \leqslant 4$ implies that the vector fields $\partial_{\tau_k}|_{k=1}^4$ are the same whether defined using the coordinates $(p, \tau_j|_{j=1}^4, b, r)$ or the coordinates $(\Pi_j|_{j=1}^4, \tau_j|_{j=1}^4, b, r)$. Hence, exploiting the invariance $E(e^{\mathbb{i}\sigma_3 \tau \cdot \Diamond} u) = E(u)$,

$$\{\Pi_j, E\} = -\{E, \Pi_j\} = -dE X_{\Pi_j} = -dE \partial_{\tau_j} = -\partial_{\tau_j} E = 0 \quad \text{for } 1 \leqslant j \leqslant 4.$$

By these identities, (1.1) in the new coordinates $(\Pi_j|_{j=1}^4, \tau, b, r)$ becomes

$$\begin{aligned} &\dot{\Pi}_j = 0 \;\text{ for } 1 \leqslant j \leqslant 4, \quad \dot{\tau} = \{\tau, E\}, \\ &\dot{b} = \{b, E\}, \quad \dot{r} = \{r, E\}. \end{aligned} \tag{3.14}$$

Notice that we have produced a Noetherian reduction of coordinates, because the equations of b and r are independent from the ones in the first line. We point out that by Lemma 2.2 we have also

$$\dot{\Pi}_j = \{\Pi_j, E\} = 0 \quad \text{for } 5 \leqslant j \leqslant 7. \tag{3.15}$$

4. **Expansion of the Hamiltonian.**

We introduce now the following new Hamiltonian,

$$K(u) := E(u) - E\left(\Phi_{p^0}\right) - \sum_{j=1,\dots,4} \lambda_j(p)\left(\Pi_j - p_j^0\right). \tag{4.1}$$

For solutions v of (1.1) with initial value v_0 satisfying $\Pi_j(v_0) = p_j^0$ for $1 \leqslant j \leqslant 4$, we have

$$\begin{aligned}
&\{\Pi_j, K\} = \{\Pi_j, E\} = 0 \qquad \text{for } 1 \leqslant j \leqslant 7,\\
&\{b, K\} = \{b, E\}, \qquad \{r, K\} = \{r, E\}, \qquad \{\tau_j, K\} = \{\tau_j, E\} - \lambda_j(p) \quad \text{for } 1 \leqslant j \leqslant 4.
\end{aligned}$$

Indeed, for example, since $\{\Pi_j, \Pi_k\} = 0$ for $j \leqslant 7$ and any $k \leqslant 4$ (which follows from $[\Diamond_j, \Diamond_k] = 0$ for $j \leqslant 7$ and any $k \leqslant 4$, cf. (2.7)–(2.9)), we have by Lemma 2.2:

$$\begin{aligned}
\{\Pi_j, K\}(v) &= \{\Pi_j, E\}(v) - \sum_{j=1,\dots,4} \big(\lambda_k\{\Pi_j, \Pi_k\}(v) + (\Pi_j(v) - p_j^0)\{\Pi_j, \lambda_k\}(v)\big)\\
&= \{\Pi_j, E\}(v),
\end{aligned}$$

where we used $\Pi_j(v) = p_j^0$. Other Poisson brackets are computed similarly.

By $\partial_{\tau_j} K \equiv 0$ for $1 \leqslant j \leqslant 4$, the evolution of the variables $(\Pi_j)|_{j=1}^{7}, b, r$ is unchanged if we consider the following new Hamiltonian system:

$$\dot{\Pi}_j = \{\Pi_j, K\} = 0 \quad \text{for } 1 \leqslant j \leqslant 4, \quad \dot{\tau} = \{\tau, K\}, \quad \dot{b} = \{b, K\}, \quad \dot{r} = \{r, K\}, \tag{4.2}$$

where $(\Pi_j)|_{j=1}^{4}$, τ, b, r is a system of independent coordinates, and where we consider also

$$\dot{\Pi}_j = \{\Pi_j, K\} = 0 \quad \text{for } 5 \leqslant j \leqslant 7. \tag{4.3}$$

Key in our discussion is the expansion of $K(u)$ in terms of the coordinates $((\Pi_j)|_{j=1}^{4}, r)$. We consider the expansion, with the canceled term equal to 0 by (2.18) and (2.16),

$$\begin{aligned}
K(u) &= K(\Phi_p + P_p r) = K(\Phi_p) + \cancel{\langle \nabla E(\Phi_p) - \sum_{j=1,\dots,4} \lambda_j(p) \nabla \Pi_j(\Phi_p), P_p r\rangle}\\
&+ \int_0^1 (1-t)\Big\langle \Big[\nabla^2 E(\Phi_p + tP_p r) - \sum_{j=1,\dots,4} \lambda_j(p) \nabla^2 \Pi_j(\Phi_p + tP_p r)\Big] P_p r, P_p r\Big\rangle\, dt.
\end{aligned}$$

The last line equals (cf. [17, (99)])

$$\begin{aligned}
&2^{-1}\langle(-\Delta + \sum_{j=1,\dots,4} \lambda_j(p)\Diamond_j) P_p r, P_p r\rangle + \int_0^1 (1-t)\langle \nabla^2 E_P(\Phi_p + tP_p r) P_p r, P_p r\rangle\, dt\\
&= 2^{-1}\langle(-\Delta + \sum_{j=1,\dots,4} \lambda_j(p)\Diamond_j) P_p r, P_p r\rangle
\end{aligned}$$

$$+ \int_{\mathbb{R}^3} dx \int_{[0,1]^2} \frac{t^2}{2} (\partial_t^2)\big|_{t=0} \partial_s [B(|s\Phi_p + tP_p r|^2)]\, dt\, ds$$
$$+ \sum_{j=2,3} \int_{\mathbb{R}^3} dx \int_{[0,1]^2} \frac{t^j}{j!} (\partial_t^{j+1})\big|_{t=0} \partial_s [B(|s\Phi_p + tP_p r|^2)]\, dt\, ds$$
$$+ \int_{\mathbb{R}^3} dx \int_{[0,1]^2} dt\, ds \int_0^t \partial_\tau^5 \partial_s [B(|s\Phi_p + \tau P_p r|^2)] \frac{(t-\tau)^3}{3!} d\tau + E_P(P_p r).$$

The second term in the second line is $2^{-1}\langle \nabla^2 E_P(\Phi_p) P_p r, P_p r\rangle$ and so in particular the second line is

$$2^{-1}\langle (-\Delta + \nabla^2 E_P(\Phi_p) - \sum_{j=1,\dots,4} \lambda_j(p) \Diamond_j) P_p r, P_p r\rangle = 2^{-1}\langle \mathrm{i}\sigma_3 \mathcal{H}_p P_p r, P_p r\rangle.$$

By (4.1), we have

$$K(\Phi_p) = d(p) - d(p^0) + (\lambda(p) - \lambda(p^0)) \cdot p^0, \tag{4.4}$$

where

$$d(p) := E(\Phi_p) - \lambda(p) \cdot p. \tag{4.5}$$

Since $\partial_{p_j} d(p) = -p \cdot \partial_{p_j} \lambda(p)$, we conclude $K(\Phi_p) = O((p-p^0)^2)$. Furthermore, from (3.13) we have

$$K(\Phi_p) = \mathfrak{G}\Big((\Pi_j - p_j^0)|_{j=1}^4, \Pi_j(r)|_{j=1}^4, \big(\Pi_j\big((P_p - P_{p^1})r\big) + \big\langle r, \Diamond_j (P_p - P_{p^1}) r\big\rangle\big)\big|_{j=1}^4 \Big), \tag{4.6}$$

with $\mathfrak{G}$ smooth and equal to zero at $(0,0,0)$ up to second order. Summing up, we have the following.

Lemma 4.1. *There is an expansion*

$$K(u) = K(\Phi_p) + 2^{-1}\Omega(\mathcal{H}_p P_p r, P_p r) + E_P(P_p r) \tag{4.7}$$
$$+ \sum_{d=3,4} \langle B_d(p), (P_p r)^d \rangle + \int_{\mathbb{R}^3} B_5(x, p, r(x)) (P_p r)^5(x)\, dx,$$ *where for any $k \in \mathbb{N}$:*

- $K(\Phi_p)$ *satisfies* (4.4)–(4.6);
- $(P_p r)^d(x)$ *represents d-products of components of* $P_p r$;
- $B_d \in C^\infty(\mathcal{P}, \Sigma_k(\mathbb{R}^3, B((\mathbb{R}^4)^{\otimes d}, \mathbb{R})))$ *for* $3 \leqslant d \leqslant 4$;
- *for* $\zeta \in \mathbb{R}^4$, B_5 *depends smoothly on its variables, so that* $\forall\ i \in \mathbb{N}$, *there is a constant* C_i *such that*

$$\|\nabla_{p,\zeta}^i B_5(\cdot, p, \zeta)\|_{\Sigma_k(\mathbb{R}^3, B((\mathbb{R}^4)^{\otimes 5}, \mathbb{R}))} \leqslant C_i. \tag{4.8}$$

□

We will perform a normal form argument on the expansion (4.7), eliminating some terms from the expansion by means of changes of variables. The first step in a normal forms argument is the diagonalization of the *homological equation*, see [1, p. 182], which is discussed in Section 10.

5. **Symbols $\mathcal{R}^{i,j}_{k,m}$, $\mathbf{S}^{i,j}_{k,m}$ and restrictions of K on submanifolds.** We begin with the following elementary lemma.

Lemma 5.1. *Set $u = \mathfrak{s}(b)\psi$. Then, for $b_R = \mathrm{Re}(b)$ and $b_I = \mathrm{Im}(b)$, we have:*

$$\begin{aligned}
&\Pi_5(u) = (1-2b_R^2)\Pi_5(\psi) - 2b_I b_R \Pi_6(\psi) - 2\sqrt{1-|b|^2}\, b_R \Pi_7(\psi);\\
&\Pi_6(u) = -2b_I b_R \Pi_5(\psi) + (1-2b_I^2)\Pi_6(\psi) - 2\sqrt{1-|b|^2}\, b_I \Pi_7(\psi);\\
&\Pi_7(u) = 2\sqrt{1-|b|^2}\, b_R \Pi_5(\psi) + 2\sqrt{1-|b|^2}\, b_I \Pi_6(\psi) + (1-2|b|^2)\Pi_7(\psi).
\end{aligned} \tag{5.1}$$

Proof. We have

$$\begin{aligned}
2\Pi_5(u) &= \langle \sigma_3\sigma_2 \boldsymbol{K} u, u\rangle = \langle \mathfrak{s}(-b)\sigma_3\sigma_2\boldsymbol{K}\mathfrak{s}(b)\psi,\psi\rangle = \langle \sigma_3\sigma_2\mathfrak{s}(-b)\mathfrak{s}\left(-\boldsymbol{K}b\right)\boldsymbol{K}\psi,\psi\rangle\\
&= \langle \sigma_3\sigma_2\left[(1-|b|^2 + b\sigma_2\boldsymbol{K}(\boldsymbol{K}b)\sigma_2\boldsymbol{K}) - \sqrt{1-|b|^2}(b+(\boldsymbol{K}b))\sigma_2\boldsymbol{K}\right]\boldsymbol{K}\psi,\psi\rangle\\
&= \langle \sigma_3\sigma_2\left[1-b_R^2-\cancel{b_I^2}-(b_R^2-\cancel{b_I^2}+2\mathbb{i}b_Rb_I)-2\sqrt{1-|b|^2}\,b_R\sigma_2\boldsymbol{K}\right]\boldsymbol{K}\psi,\psi\rangle\\
&= (1-2b_R^2)\langle\sigma_3\sigma_2\boldsymbol{K}\psi,\psi\rangle - 2b_Rb_I\langle\mathbb{i}\sigma_3\sigma_2\boldsymbol{K}\psi,\psi\rangle - 2\sqrt{1-|b|^2}\,b_R\langle\sigma_3\psi,\psi\rangle.
\end{aligned}$$

This yields the formula for $\Pi_5(u)$. By a similar computation

$$\begin{aligned}
2\Pi_6(u) &= \langle \mathbb{i}\sigma_3\sigma_2 \boldsymbol{K} u, u\rangle = \langle \mathfrak{s}(-b)\mathbb{i}\sigma_3\sigma_2\boldsymbol{K}\mathfrak{s}(b)\psi,\psi\rangle = \langle \mathbb{i}\sigma_3\sigma_2\mathfrak{s}(b)\mathfrak{s}(-\boldsymbol{K}b)\boldsymbol{K}\psi,\psi\rangle\\
&= \langle \mathbb{i}\sigma_3\sigma_2\left[(1-|b|^2 - b\sigma_2\boldsymbol{K}(\boldsymbol{K}b)\sigma_2\boldsymbol{K}) + \sqrt{1-|b|^2}(b-(\boldsymbol{K}b))\sigma_2\boldsymbol{K}\right]\boldsymbol{K}\psi,\psi\rangle\\
&= \langle \mathbb{i}\sigma_3\sigma_2\left[1-\cancel{b_R^2}-b_I^2+\cancel{b_R^2}-b_I^2+2\mathbb{i}b_Rb_I+2\mathbb{i}\sqrt{1-|b|^2}\,b_I\sigma_2\boldsymbol{K}\right]\boldsymbol{K}\psi,\psi\rangle\\
&= (1-2b_I^2)\langle\mathbb{i}\sigma_3\sigma_2\boldsymbol{K}\psi,\psi\rangle - 2b_Rb_I\langle\sigma_3\sigma_2\boldsymbol{K}\psi,\psi\rangle - 2\sqrt{1-|b|^2}\,b_I\langle\sigma_3\psi,\psi\rangle.
\end{aligned}$$

This yields the formula for $\Pi_6(u)$. Finally, the formula for $\Pi_7(u)$ is obtained from

$$\begin{aligned}
2\Pi_7(u) &= \langle \sigma_3 u, u\rangle = \langle \mathfrak{s}(-b)\sigma_3\mathfrak{s}(b)\psi,\psi\rangle = \langle \sigma_3\mathfrak{s}(b)\mathfrak{s}(b)\psi,\psi\rangle\\
&= \langle \sigma_3\left[(1-|b|^2+b\sigma_2\boldsymbol{K}b\sigma_2\boldsymbol{K}) + 2\sqrt{1-|b|^2}\,b\sigma_2\boldsymbol{K}\right]\psi,\psi\rangle\\
&= \langle \sigma_3\left[1-2|b|^2+2\sqrt{1-|b|^2}\,b_R\sigma_2\boldsymbol{K}+2\mathbb{i}\sqrt{1-|b|^2}\,b_I\sigma_2\boldsymbol{K}\right]\psi,\psi\rangle\\
&= (1-2|b|^2)\langle\sigma_3\psi,\psi\rangle + 2\sqrt{1-|b|^2}\,b_R\langle\sigma_3\sigma_2\boldsymbol{K}\psi,\psi\rangle + 2\sqrt{1-|b|^2}\,b_I\langle\mathbb{i}\sigma_3\sigma_2\boldsymbol{K}\psi,\psi\rangle.
\end{aligned}$$

□

We introduce the following spaces:

$$\Xi_k := \{(\Pi_4,\varrho,r)\in\mathbb{R}_+\times\mathbb{R}^7\times(T^{\perp\Omega}\mathcal{M}_{p^1}\cap\Sigma_k)\} \quad \text{for } k\in\mathbb{Z}, \tag{5.2}$$

where ϱ is an auxiliary variable which we will use to represent $\Pi(r)$. We now introduce two classes of symbols which will be important in the sequel.

Definition 5.2. For $A\subset\mathbb{R}^d$ an open set, $k\in\mathbb{N}_0$, $\mathcal{A}\subset\Xi_{-k}$ an open neighborhood of $(p_4^1,0,0)$, we say that $F\in C^m(A\times\mathcal{A},\mathbb{R})$ is $\mathcal{R}^{i,j}_{k,m}$ if there exists $C>0$ and an open neighborhood $\mathcal{A}'\subset\mathcal{A}$ of $(p_4^1,0,0)$ in Ξ_{-k} such that

$$|F(a,\Pi_4,\varrho,r)| \leqslant C\|r\|^j_{\Sigma_{-k}}(\|r\|_{\Sigma_{-k}}+|\varrho|+|\Pi_4-p_4^1|)^i \text{ in } I\times\mathcal{A}'. \tag{5.3}$$

We will write also $F=\mathcal{R}^{i,j}_{n,m}$ or $F=\mathcal{R}^{i,j}_{k,m}(a,\Pi_4,\varrho,r)$. We say $F=\mathcal{R}^{i,j}_{k,\infty}$ if $F=\mathcal{R}^{i,j}_{k,l}$ for all $l\geqslant m$. We say $F=\mathcal{R}^{i,j}_{\infty,m}$ if for all $l\geqslant k$ the above F is the restriction of an $F\in C^m(A\times\mathcal{A}_l,\mathbb{R})$ with $\mathcal{A}_l$ an open neighborhood of $(0,0)$ in $\mathbb{R}^7\times(T^{\perp\Omega}\mathcal{M}_{p^1}\cap\Sigma_{-l})$ and $F=\mathcal{R}^{i,j}_{l,m}$. If $F=\mathcal{R}^{i,j}_{\infty,m}$ for any m, we set $F=\mathcal{R}^{i,j}_{\infty,\infty}$.

Remark 5.3. Above, we can have $d = 0$ (that is, A is missing). We will also use the following cases: $d = 1$ with a time parameter; A an open neighborhood of the origin of $\mathbb{R} \times \mathfrak{su}(2)$. The last case is used only in Appendix A.

Definition 5.4. $T \in C^m(A \times \mathcal{A}, \Sigma_k(\mathbb{R}^3, \mathbb{C}^2))$, with $A \times \mathcal{A}$ like above, is $\mathbf{S}^{i,j}_{k,m}$, and we write as above $T = \mathbf{S}^{i,j}_{k,m}$ or $T = \mathbf{S}^{i,j}_{k,m}(a, \Pi_4, \varrho, r)$, if there exists $C > 0$ and a smaller open neighborhood $\mathcal{A}'$ of $(0,0)$ such that

$$\|T(a, \Pi_4, \varrho, r)\|_{\Sigma_k} \leqslant C\|r\|^j_{\Sigma_{-k}}(\|r\|_{\Sigma_{-k}} + |\varrho| + |\Pi_4 - p_4^1|)^i \text{ in } I \times \mathcal{A}'. \tag{5.4}$$

We use notation $T = \mathbf{S}^{i,j}_{k,\infty}$, $T = \mathbf{S}^{i,j}_{\infty,m}$ and $T = \mathbf{S}^{i,j}_{\infty,\infty}$ as above.

Lemma 5.5. *On the manifold $\Pi_j = p_j^0$ for $1 \leqslant j \leqslant 4$ there exist functions $\mathcal{R}^{1,2}_{\infty,\infty}$ such that*

$$p_j = p_j^0 - \Pi_j(r) + \mathcal{R}^{1,2}_{\infty,\infty}(p_4^0, \Pi_j(r)\ |_{j=1}^4, r). \tag{5.5}$$

Proof. The conclusion follows by the implicit function theorem applied to (3.13). $\square$

Inside the space parametrized by $(\Pi_j|_{j=1}^4, \tau, b, r)$, we consider

$$\mathcal{M}_1^6(p^0) \text{ defined by } \Pi_j|_{j=1}^6 = p_j^0|_{j=1}^6. \tag{5.6}$$

Notice that the intersection of $\mathcal{M}_1^6(p^0)$ with a small neighborhood of $\{e^{\mathrm{i}\vartheta}\Phi_{p^1} : \vartheta \in \mathbb{R}\}$ is a manifold. Indeed, on the soliton manifold $\mathcal{M}$ the differential forms $dp_j|_{j=1}^4$, db_R, db_I are linearly independent. At the points of $\mathcal{M}$ formula (3.13) implies $dp_j = d\Pi_j$ for $1 \leqslant j \leqslant 4$ while the first two lines of (5.1) imply $d\Pi_5 = -2p_4 db_R$ and $d\Pi_5 = -2p_4 db_I$. Hence, since $\Pi_j \in C^\infty(H^1(\mathbb{R}^3, \mathbb{C}^2), \mathbb{R})$, it follows that $d\Pi_j|_{j=1}^6$ are linearly independent in a neighborhood of $\{e^{\mathrm{i}\vartheta}\Phi_{p^1} : \vartheta \in \mathbb{R}\}$. Then since $\mathcal{M}_1^6(p^0)$ is defined by $\Pi_j = p_j^0$ for $j \leqslant 6$ we obtain our claim on $\mathcal{M}_1^6(p^0)$ for any p^0 sufficiently close to p^1.

$\mathcal{M}_1^6(p^0)$ is invariant by the system (4.2). The following shows that, when we factor $\mathcal{M}_1^6(p^0)$ by the action of $\mathbb{R}^3 \times \mathbb{T}$, the corresponding manifold is parametrized by $r \in T^{\perp\Omega}\mathcal{M}_{p^1} \cap H^1(\mathbb{R}^3, \mathbb{C}^2)$.

Lemma 5.6. *There exist functions $\mathcal{R}^{1,2}_{\infty,\infty}(p_4^0, \Pi(r), r)$ and functions $\mathcal{R}^{2,0}_{\infty,\infty}(p_4^0, \Pi(r))$ dependent only on $(p_4^0, \Pi(r))$ such that on $\mathcal{M}_1^6(p^0)$*

$$\begin{aligned} b_R &= (2p_4^0)^{-1}\Pi_5(r) + \mathcal{R}^{2,0}_{\infty,\infty}(p_4^0, \Pi(r)) + \mathcal{R}^{1,2}_{\infty,\infty}(p_4^0, \Pi(r), r)\,, \\ b_I &= (2p_4^0)^{-1}\Pi_6(r) + \mathcal{R}^{2,0}_{\infty,\infty}(p_4^0, \Pi(r)) + \mathcal{R}^{1,2}_{\infty,\infty}(p_4^0, \Pi(r), r). \end{aligned} \tag{5.7}$$

Proof (sketch). Since $\Pi_5 = \Pi_6 = 0$ by the first two equations in (5.1), by $\Pi_j(\Phi_p + P_p r) = \Pi_j(P_p r)$ for $j = 5, 6$ and by $\Pi_7(\Phi_p + P_p r) = p_4 + \Pi_7(P_p r)$ we have

$$\begin{aligned} 2\sqrt{1 - |b|^2} b_R(p_4 + \Pi_7(P_p r)) &= (1 - 2b_R^2)\Pi_5(P_p r) - 2b_I b_R \Pi_6(P_p r), \\ 2\sqrt{1 - |b|^2} b_I(p_4 + \Pi_7(P_p r)) &= -2b_I b_R \Pi_5(P_p r) + (1 - 2b_I^2)\Pi_6(P_p r). \end{aligned} \tag{5.8}$$

We consider the following change of coordinates, which defines x_R and x_I:

$$2p_4^0 b_R = \Pi_5(r) + x_R \text{ and } 2p_4^0 b_I = \Pi_6(r) + x_I. \tag{5.9}$$

Substitute in the l.h.s. of (5.8) both (5.9) and (5.5), and write $\Pi_j(P_p r) = \Pi_j(r) + \mathcal{R}^{1,2}_{\infty,\infty}(p_4^0, \Pi(r), r)$ everywhere in (5.8). Then from the first equation in (5.8) we get

$$\begin{aligned} &(1 + O(b^2))\left[1 - \Pi_4(r)/p_4^0 + \Pi_7(r)/p_4^0 + \mathcal{R}^{1,2}_{\infty,\infty}(p_4^0, \Pi(r), r)\right](\Pi_5(r) + x_R) \\ &= \Pi_5(r) + O(b^2\Pi(r)) + \mathcal{R}^{1,2}_{\infty,\infty}(p_4^0, \Pi(r), r)). \end{aligned}$$

So, after an obvious cancellation, we have

$$\begin{aligned}&(1+O(b^2))\left[1-\Pi_4(r)/p_4^0+\Pi_7(r)/p_4^0+\mathcal{R}^{1,2}_{\infty,\infty}(p_4^0,\Pi(r),r)\right]x_R\\&=\mathcal{R}^{2,0}_{\infty,\infty}(\Pi(r))+O\left(b^2\Pi(r)\right)+\mathcal{R}^{1,2}_{\infty,\infty}(p^0,\Pi(r),r)).\end{aligned}$$

which in turn implies, for $A=R$,

$$x_A=\mathcal{R}^{2,0}_{\infty,\infty}(p_4^0,\Pi(r))+O(b^2\Pi(r))+\mathcal{R}^{1,2}_{\infty,\infty}(p_4^0,\Pi(r),r))$$

where the big O is smooth. Since a similar equality holds also for $A=I$, substituting again b by means of (5.9) and applying the implicit function theorem, we obtain

$$x_A=\mathcal{R}^{2,0}_{\infty,\infty}(p_4^0,\Pi(r))+\mathcal{R}^{1,2}_{\infty,\infty}(p_4^0,\Pi(r),r))\ \text{ for } A=R,I.$$

□

Lemma 5.7. *In* $\mathcal{M}_1^6(p^0)$ *we have*

$$\Pi_7=p_4^0+\Pi_7(r)+\mathcal{R}^{2,0}_{\infty,\infty}(p^0,\Pi(r))+\mathcal{R}^{1,2}_{\infty,\infty}(p^0,\Pi(r),r). \tag{5.10}$$

Proof. By the third identity in (5.1) and by the definition of P_p, we have

$$\Pi_7=2\sqrt{1-|b|^2}b_R\Pi_5(P_pr)+2\sqrt{1-|b|^2}b_I\Pi_6(P_pr)+(1-2|b|^2)(p_4+\Pi_7(P_pr)).$$

Using Lemmata 5.5 and 5.6, we obtain (5.10). □

6. **Expressing Ω in coordinates.** Normal forms arguments are crucial in the proof of Theorem 1.1. It is important to settle on a coordinate system where the homological equations look manageable. While the symplectic form Ω has a very simple definition (2.6) in terms of the hermitian structure of $L^2(\mathbb{R}^3,\mathbb{C}^2)$, it has a rather complicated representation in terms of the coordinates $(\Pi_j|_{j=1}^4,\tau,b,r)$. Eventually we will settle on a coordinate system where the symplectic form is equal to the form Ω_0 to be introduced in Section 7. In this section we consider some preliminary material.

We consider $\widetilde{\Gamma}:=2^{-1}\langle \mathbb{i}\sigma_3 u,\cdot\rangle$. Using the definition of the exterior differentiation it is elementary to show that $d\widetilde{\Gamma}=\Omega$. We consider now the function

$$\psi(u):=2^{-1}\langle \mathbb{i}\sigma_3 e^{-\mathbb{i}\sigma_3\tau\cdot\Diamond}\mathfrak{s}(b)\Phi_p,u\rangle$$

and set $\Gamma:=\widetilde{\Gamma}-d\psi+d\sum_{j=1,\dots,4}\Pi_j\tau_j$. Obviously $d\Gamma=\Omega$. We have the following.

Lemma 6.1. *We have*

$$\Gamma=\sum_{j=1,\dots,4}\tau_jd\Pi_j+2^{-1}\Omega(P_pr,dr)+\sum_{j=1,\dots,4}2^{-1}\Omega(r,P_p\partial_{p_j}P_pr)dp_j+\varsigma, \tag{6.1}$$

$$\begin{aligned}\text{where }\ \varsigma:=&\left(\Pi_5\frac{b_Rb_I}{\sqrt{1-|b|^2}}-\Pi_6\frac{1-b_I^2}{\sqrt{1-|b|^2}}-\Pi_7b_I\right)db_R\\&+\left(\Pi_5\frac{1-b_R^2}{\sqrt{1-|b|^2}}-\Pi_6\frac{b_Rb_I}{\sqrt{1-|b|^2}}+\Pi_7b_R\right)db_I.\end{aligned}$$

Proof. The proof is elementary. The identity operator is du, which can be expanded as

$$\begin{aligned}du=&-\sum_{j=1,\dots,4}\mathbb{i}\sigma_3\Diamond_jud\tau_j+\sum_{j=1,\dots,4}e^{-\mathbb{i}\sigma_3\tau\cdot\Diamond}\mathfrak{s}(b)\partial_{p_j}(\Phi_p+P_pr)dp_j\\&+\sum_{A=,R,I}e^{-\mathbb{i}\sigma_3\tau\cdot\Diamond}\partial_{b_A}\mathfrak{s}(b)(\Phi_p+P_pr)db_A+e^{-\mathbb{i}\sigma_3\tau\cdot\Diamond}\mathfrak{s}(b)P_pdr.\end{aligned}$$

Then, inserting this into $\widetilde{\Gamma}$ and after some elementary simplification which uses also (3.4), we obtain

$$\begin{aligned}\widetilde{\Gamma} &= 2^{-1}\langle \mathbb{i}\sigma_3 u, du\rangle = -\sum_{j=1,\dots,4}\Pi_j d\tau_j \\ &+ \sum_{A=,R,I} 2^{-1}\langle \mathbb{i}\sigma_3\mathfrak{s}(b)(\Phi_p + P_p r), \partial_{b_A}\mathfrak{s}(b)(\Phi_p + P_p r)\rangle db_A \\ &+ \sum_{j=1,\dots,4} 2^{-1}\langle \mathbb{i}\sigma_3(\Phi_p + P_p r), \partial_{p_j}(\Phi_p + P_p r)\rangle dp_j + 2^{-1}\langle \mathbb{i}\sigma_3(\Phi_p + P_p r), P_p dr\rangle.\end{aligned} \tag{6.2}$$

We have:

$$\begin{aligned}&\text{second line of (6.2)} \\ &= \sum_{j=1,\dots,4} 2^{-1}\langle \mathbb{i}\sigma_3 P_p r, \partial_{p_j} P_p r\rangle dp_j + 2^{-1}\langle \mathbb{i}\sigma_3 P_p r, P_p dr\rangle + d2^{-1}\langle \mathbb{i}\sigma_3\Phi_p, P_p r\rangle,\end{aligned} \tag{6.3}$$

where we used what follows:

$\langle \mathbb{i}\sigma_3 P_p r, \partial_{p_j}\Phi_p\rangle = 0$ from the definition of P_p;

$\langle \mathbb{i}\sigma_3 \Phi_p, \partial_{p_j}\Phi_p\rangle = \langle \mathbb{i}e^{\frac{\mathbb{i}}{2}v\cdot x}\phi_\omega, \partial_{p_j} e^{\frac{\mathbb{i}}{2}v\cdot x}\phi_\omega\rangle = 0$ from formula (2.14).

Hence, by the definition of Γ and $\psi(u)$, we obtain:

$$\begin{aligned}\Gamma &= \sum_{j=1,\dots,4}\tau_j d\Pi_j + \sum_{j=1,\dots,4} 2^{-1}\langle \mathbb{i}\sigma_3 P_p r, \partial_{p_j} P_p r\rangle dp_j + 2^{-1}\langle \mathbb{i}\sigma_3 r, P_p dr\rangle \\ &- 2^{-1}\sum_{A=R,I}\langle \mathbb{i}\sigma_3\partial_{b_A}\mathfrak{s}(b)(\Phi_p + P_p r), \mathfrak{s}(b)(\Phi_p + P_p r)\rangle db_A.\end{aligned} \tag{6.4}$$

For $A = R$, by the definition of $\mathfrak{s}(b)$ the bracket in the last line equals

$$\begin{aligned}&\langle \mathbb{i}\sigma_3\Big(\frac{-b_R}{\sqrt{1-|b|^2}} + \sigma_2\boldsymbol{K}\Big)(\sqrt{1-|b|^2} - b\sigma_2\boldsymbol{K})u, u\rangle \\ &= \langle \mathbb{i}\sigma_3\left[-b_R + \bar{b} + \Big(\sqrt{1-|b|^2} + \frac{b_R b}{\sqrt{1-|b|^2}}\Big)\sigma_2\boldsymbol{K}\right]u, u\rangle \\ &= \langle \mathbb{i}\sigma_3\left[-\mathbb{i}b_I + \frac{1-b_I^2}{\sqrt{1-|b|^2}}\sigma_2\boldsymbol{K} + \frac{b_R b_I}{\sqrt{1-|b|^2}}\mathbb{i}\sigma_2\boldsymbol{K}\right]u, u\rangle \\ &= b_I\Pi_7 + \frac{1-b_I^2}{\sqrt{1-|b|^2}}\Pi_6 - \frac{b_R b_I}{\sqrt{1-|b|^2}}\Pi_5.\end{aligned}$$

For $A = I$, the bracket in the last line of (6.4) equals

$$\begin{aligned}&\langle \mathbb{i}\sigma_3\Big(\frac{-b_I}{\sqrt{1-|b|^2}} + \mathbb{i}\sigma_2\boldsymbol{K}\Big)(\sqrt{1-|b|^2} - b\sigma_2\boldsymbol{K})u, u\rangle \\ &= \langle \mathbb{i}\sigma_3\left[-b_I + \mathbb{i}\bar{b} + \Big(\mathbb{i}\sqrt{1-|b|^2} + \frac{b_I b}{\sqrt{1-|b|^2}}\Big)\sigma_2\boldsymbol{K}\right]u, u\rangle \\ &= \langle \mathbb{i}\sigma_3\left[\mathbb{i}b_R + \frac{1-b_R^2}{\sqrt{1-|b|^2}}\mathbb{i}\sigma_2\boldsymbol{K} + \frac{b_R b_I}{\sqrt{1-|b|^2}}\sigma_2\boldsymbol{K}\right]u, u\rangle \\ &= -b_R\Pi_7 - \frac{1-b_R^2}{\sqrt{1-|b|^2}}\Pi_5 + \frac{b_R b_I}{\sqrt{1-|b|^2}}\Pi_6.\end{aligned}$$

This completes the proof of Lemma 6.1. □

Lemma 6.2. *Consider the immersion $i : \mathcal{M}_1^6(p^0) \hookrightarrow H^1(\mathbb{R}^3, \mathbb{C}^2)$ and the pullback $i^*\Gamma$, which by an abuse of notation we will still denote by Γ. We have:*

$$\Gamma = i^*\Gamma = 2^{-1}\Omega(r, dr) + \langle \mathcal{R}^{0,2}_{\infty,\infty}(p_4^0, \Pi(r), r) \cdot \Diamond r + \mathbf{S}^{1,1}_{\infty,\infty}(p_4^0, \Pi(r), r), dr\rangle + \Pi_7 \varpi, \tag{6.5}$$

where

$$\begin{aligned}\varpi =& (b_R\, db_I - b_I\, db_R) = \frac{1}{4(p_4^0)^2}(\Pi_5(r) d\Pi_6(r) - \Pi_6(r) d\Pi_5(r)) \\ &+ \mathcal{R}^{2,0}_{\infty,\infty}(p_4^0, \Pi(r)) d\Pi(r) + \big\langle \mathbf{S}^{2,1}_{\infty,\infty}(p_4^0, \Pi(r), r), dr\big\rangle.\end{aligned} \tag{6.6}$$

Proof. The starting point is formula (6.1) for Γ. Obviously for the restrictions we have $d\Pi_k|_{\mathcal{M}_1^6(p^0)} = 0$ for $1 \leqslant k \leqslant 6$. So that the first summation in the r.h.s. of (6.1) contributes 0.

Next, notice that for $1 \leqslant j \leqslant 4$ from (5.5) we obtain

$$dp_j = -\big\langle \Diamond_j r + \mathbf{S}^{1,1}_{\infty,\infty}, dr\big\rangle + \sum_{k \leqslant 4} \mathcal{R}^{0,2}_{\infty,\infty} dp_k,$$

which, solved in terms of the dp_j's, gives

$$dp_j = -\sum_{k \leqslant 4} \Big\langle (\delta_{jk} + \mathcal{R}^{0,2}_{\infty,\infty}) \Diamond_k r + \mathbf{S}^{1,1}_{\infty,\infty}, dr\Big\rangle. \tag{6.7}$$

Substituting dp_j from (6.7) into (6.1) and using and $P_p r = r + \mathbf{S}^{1,1}_{\infty,\infty}(p^0, \Pi(r), r)$ on $\mathcal{M}_1^6(p^0)$, we obtain terms like the second in the r.h.s. of (6.5).

Finally, by $\Pi_5 = \Pi_6 = 0$, we obtain $\varsigma = \Pi_7 \varpi$. To get the r.h.s. in (6.6), we use the following formulae:

$$\begin{aligned} db_R &= (2p_4^0)^{-1}\langle \sigma_3 \sigma_2 \boldsymbol{K} r, dr\rangle + \mathcal{R}^{1,0}_{\infty,\infty}(p_4^0, \Pi(r)) d\Pi(r) + \langle \mathbf{S}^{1,1}_{\infty,\infty}, dr\rangle, \\ db_I &= (2p_4^0)^{-1}\langle \mathrm{i} \sigma_3 \sigma_2 \boldsymbol{K} r, dr\rangle + \mathcal{R}^{1,0}_{\infty,\infty}(p_4^0, \Pi(r)) d\Pi(r) + \langle \mathbf{S}^{1,1}_{\infty,\infty}, dr\rangle, \end{aligned} \tag{6.8}$$

where $\mathcal{R}^{1,0}_{\infty,\infty}(p_4^0, \Pi(r)) d\Pi(r)$ stands for $\sum_{j=1,\dots,7} \mathcal{R}^{1,0}_{\infty,\infty}(p_4^0, \Pi(r)) d\Pi_j(r)$ with different real-valued functions from the class $\mathcal{R}^{1,0}_{\infty,\infty}(p_4^0, \Pi(r))$. Formulae (6.8) are obtained by differentiating in (5.7). ☐

Substituting Π_7 by (5.10) in (6.5) and using (2.7)–(2.9), we obtain

$$\begin{aligned}\Gamma = & 2^{-1}\Omega(r, dr) + \langle \mathbf{S}^{1,1}_{\infty,\infty}(p_4^0, \Pi(r), r), dr\rangle \\ &+ (4p_4^0)^{-1}(\Pi_5(r) d\Pi_6(r) - \Pi_6(r) d\Pi_5(r)) \\ &+ \big(\mathcal{R}^{2,0}_{\infty,\infty}(p_4^0, \Pi(r)) + \mathcal{R}^{0,2}_{\infty,\infty}(p_4^0, \Pi(r), r)\big)\, d\Pi(r).\end{aligned} \tag{6.9}$$

7. **Spectral coordinates associated to $\mathcal{H}_{p^1}$.** By assumption, $p^1 = p(\omega^1, 0)$. Recall that the operator $\mathcal{H}_{p^1}$ defined in $L^2(\mathbb{R}^3, \mathbb{C}^2)$ is not $\mathbb{C}$-linear (because of $\mathfrak{L}^{(1)}_{\omega^1}$), but rather $\mathbb{R}$-linear. To make it $\mathbb{C}$-linear, we consider the complexification

$$L^2(\mathbb{R}^3, \mathbb{C}^2) \otimes_{\mathbb{R}} \mathbb{C}.$$

To avoid the confusion between $\mathbb{C}$ in the left factor and $\mathbb{C}$ on the right, we will use $\boldsymbol{\imath}$ to denote the imaginary unit in the latter space; that is, given $u \in L^2(\mathbb{R}^3, \mathbb{C}^2)$, we will have $u \otimes (a + \boldsymbol{\imath} b) \in L^2(\mathbb{R}^3, \mathbb{C}^2) \otimes_{\mathbb{R}} \mathbb{C}$. Notice that the domain of $\mathcal{H}_{p^1}$ in $L^2(\mathbb{R}^3, \mathbb{C}^2)$ is $H^2(\mathbb{R}^3, \mathbb{C}^2)$; we extend it to $L^2(\mathbb{R}^3, \mathbb{C}^2) \otimes_{\mathbb{R}} \mathbb{C}$ with the domain $H^2(\mathbb{R}^3, \mathbb{C}^2) \otimes_{\mathbb{R}} \mathbb{C}$ by setting $\mathcal{H}_{p^1}(v \otimes z) = (\mathcal{H}_{p^1} v) \otimes z$.

We extend the bilinear form $\langle\,,\,\rangle$ in (2.5) to a $\mathbb{C}$-bilinear form on $L^2(\mathbb{R}^3,\mathbb{C}^2)\otimes_{\mathbb{R}}\mathbb{C}$ by

$$\langle u\otimes z, v\otimes\zeta\rangle = z\zeta\langle u,v\rangle,\qquad u,\, v\in L^2(\mathbb{R}^3,\mathbb{C}^2),\quad z,\,\zeta\in\mathbb{C}.$$

We also extend Ω onto $L^2(\mathbb{R}^3,\mathbb{C}^2)\otimes_{\mathbb{R}}\mathbb{C}$, setting $\Omega(X,Y)=\langle \mathrm{i}\sigma_3 X, Y\rangle$. Then the decomposition (3.3) extends into

$$L^2(\mathbb{R}^3,\mathbb{C}^4)\otimes_{\mathbb{R}}\mathbb{C} = (T^{\perp\Omega}_{\Phi_{p^1}}\mathcal{M}\otimes_{\mathbb{R}}\mathbb{C})\oplus(T_{\Phi_{p^1}}\mathcal{M}\cap H^1(\mathbb{R}^3,\mathbb{C}^2))\otimes_{\mathbb{R}}\mathbb{C}. \tag{7.1}$$

Note that the extension of $\mathcal{H}_{p^1}$ onto $L^2(\mathbb{R}^3,\mathbb{C}^2)\otimes_{\mathbb{R}}\mathbb{C}$ is such that its action preserves the decomposition (7.1). The complex conjugation on $L^2(\mathbb{R}^3,\mathbb{C}^2)\otimes_{\mathbb{R}}\mathbb{C}$ is defined by $\overline{v\otimes z}:=v\otimes\overline{z}$.

Notice that if $\imath\mathcal{H}_{p^1}\xi_l=\mathbf{e}_l\xi_l$ with $\mathbf{e}_l>0$, then by complex conjugation we obtain $\imath\mathcal{H}_{p^1}\overline{\xi}_l=-\mathbf{e}_l\overline{\xi}_l$.

By Weyl's theorem, $\sigma_{\mathrm{ess}}(\imath\mathcal{H}_{p^1})=(-\infty,-\omega^1]\cup[\omega^1,\infty)$. We assume spectral stability, i.e. $\sigma_{\mathrm{ess}}(\imath\mathcal{H}_{p^1})\subset\mathbb{R}$. We assume that the set of eigenvalues satisfies $\sigma_{\mathrm{p}}(\imath\mathcal{H}_{p^1})\subset(-\omega^1,\omega^1)$, that $\pm\omega^1$ are not resonances, and the following:

(H6) For any $\mathfrak{e}\in\sigma_{\mathrm{p}}(\imath\mathcal{H}_{p^1})\backslash\{0\}$, algebraic and geometric multiplicities coincide and are finite.

(H7) There is a number $\mathfrak{N}\in\mathbb{N}$ and positive numbers $0<\mathbf{e}_1<\mathbf{e}_2<\ldots<\mathbf{e}_{\mathfrak{N}}<\omega^1$ such that $\sigma_{\mathrm{p}}(\mathcal{H}_{p^1})$ consists exactly of the numbers $\pm\imath\mathbf{e}_\ell$ and 0. Furthermore, the points $\pm\imath\omega^1$ are not resonances (that is, if $\mathcal{H}_{p^1}\Theta=\pm\imath\omega^1\Theta$ for one of the two signs, and if $\langle x\rangle\Theta\in L^\infty$, then $\Theta=0$).

Denote $d_\ell:=\dim\ker(\mathcal{H}_{p^1}-\imath\mathbf{e}_\ell)$ and let

$$\mathbf{n}:=\sum_{\ell=1,\ldots,\mathfrak{N}} d_\ell.$$

(H8) Define

$$\mathbf{N}:=\sup_\ell\inf\{n\in\mathbb{N}:\ n\mathbf{e}_\ell\in\sigma_{\mathrm{ess}}(\imath\mathcal{H}_{p^1})\}-1. \tag{7.2}$$

If $\mathbf{e}_{\ell_1}<\ldots<\mathbf{e}_{\ell_i}$ are distinct and $\mu\in\mathbb{Z}^i$ satisfies $|\mu|:=\sum_{j=1}^i\mu_j\leqslant 4\mathbf{N}+4$, we assume that

$$\mu_1\mathbf{e}_{\ell_1}+\cdots+\mu_k\mathbf{e}_{\ell_i}=0\qquad\Longleftrightarrow\qquad\mu=0.$$

It is easy to prove the symmetry of $\sigma_{\mathrm{p}}(\imath\mathcal{H}_{p^1})\subset\mathbb{R}$ around 0. We have

$$\ker(\imath\mathcal{H}_{p^1}\mp\mathbf{e}_l))\subset\mathcal{S}(\mathbb{R}^3,\mathbb{C}^2)\otimes_{\mathbb{R}}\mathbb{C}$$

and using Ω we consider the set $X_c\subset\mathcal{S}'(\mathbb{R}^3,\mathbb{C}^2)\otimes_{\mathbb{R}}\mathbb{C}$ defined by

$$X_c:=\left[\left(T\mathcal{M}_{\Phi_{p^1}}\otimes_{\mathbb{R}}\mathbb{C}\right)\oplus_\pm\oplus_{l=1}^N\left(\ker(\imath\mathcal{H}_{p^1}\mp\mathbf{e}_l)\right)\right]^{\perp\Omega}. \tag{7.3}$$

It is possible to prove the following decomposition:

$$\begin{aligned}&(T^{\perp\Omega}_{\Phi_{p^1}}\mathcal{M}\cap L^2(\mathbb{R}^3,\mathbb{C}^2))\otimes_{\mathbb{R}}\mathbb{C} \\ &=\left(\oplus_\pm\oplus_{l=1}^N\ker(\imath\mathcal{H}_{p^1}\mp\mathbf{e}_l)\right)\oplus\left(X_c\cap\left(L^2(\mathbb{R}^3,\mathbb{C}^2)\otimes_{\mathbb{R}}\mathbb{C}\right)\right).\end{aligned} \tag{7.4}$$

The decomposition in (7.4) is $\mathcal{H}_{p^1}$-invariant.

Consider now the coordinate $r\in T^{\perp\Omega}_{p^1}\mathcal{M}\cap L^2(\mathbb{R}^3,\mathbb{C}^2)$ from the coordinate system (3.8); it corresponds to the second summand in (7.1). Then, considered as an element from $L^2(\mathbb{R}^3,\mathbb{C}^2)\otimes_{\mathbb{R}}\mathbb{C}$, it can be decomposed into

$$r(x)=\sum_{l=1,\ldots,\mathbf{n}} z_l\xi_l(x)+\sum_{l=1,\ldots,\mathbf{n}}\overline{z}_l\overline{\xi}_l(x)+f(x),\quad f\in X_c\text{ with }f=\overline{f}\ , \tag{7.5}$$

with ξ_l eigenfunctions of $\mathcal{H}_{p^1}$ corresponding to $\imath \mathbf{e}_l$. We claim that it is possible to choose them so that

$$\begin{aligned} &\langle \mathrm{i}\sigma_3 \xi_i, \xi_l \rangle = \langle \mathrm{i}\sigma_3 \xi_i, f \rangle = 0 \quad \text{for all } i, l \text{ and for all } f \in X_c, \\ &\langle \mathrm{i}\sigma_3 \xi_i, \overline{\xi}_l \rangle = -\imath \delta_{il} \quad \text{for all } i, l. \end{aligned} \tag{7.6}$$

To see the second line, observe that on one hand for $\Theta \in (T^{\perp \Omega}_{\Phi_{p^1}} \mathcal{M} \otimes_{\mathbb{R}} \mathbb{C}) \backslash \{0\}$ we have $\langle \mathrm{i}\sigma_3 \mathcal{H}_{p^1} \Theta, \overline{\Theta} \rangle > 0$. Indeed, for $\Theta = (\Theta_1, \Theta_2)$ we have

$$\langle \mathrm{i}\sigma_3 \mathcal{H}_{p^1} \Theta, \overline{\Theta} \rangle = \langle \mathrm{i}\sigma_3 \mathcal{H}_{p^1} \Theta, \overline{\Theta} \rangle = \langle \mathfrak{L}^{(1)}_{\omega^1} \Theta_1, \Theta_1^* \rangle + \langle \mathfrak{L}^{(2)}_{\omega^1} \Theta_2, \overline{\Theta}_2 \rangle$$

with $\langle \Theta_2, \phi_{\omega^1} \rangle = 0$, which implies $\langle \mathfrak{L}^{(2)}_{\omega^1} \Theta_2, \overline{\Theta}_2 \rangle > c_0 \|\Theta_2\|^2_{L^2}$ and with $\langle \Theta_1, \partial_a \phi_{\omega^1} \rangle = \langle \Theta_1, x_a \phi_{\omega^1} \rangle = \langle \Theta_1, \mathrm{i}\phi_{\omega^1} \rangle = 0$ which implies $\langle \mathfrak{L}^{(1)}_{\omega^1} \Theta_1, \overline{\Theta}_1 \rangle > c_0 \|\Theta_1\|^2_{L^2}$, for a fixed $c_0 > 0$. On the other hand,

$$0 < \langle \mathrm{i}\sigma_3 \mathcal{H}_{p^1} \xi_i, \overline{\xi}_i \rangle = \imath \mathbf{e}_i \langle \mathrm{i}\sigma_3 \xi_i, \overline{\xi}_i \rangle.$$

It is then possible to choose ξ_i so that (7.6) is true. Notice that (7.6) means that the nonzero eigenvalues have positive *Krein signature*. This proves the second line of (7.6). The proof of the first line is elementary.

By (7.5) and (7.6), we have

$$2^{-1} \langle \mathrm{i}\sigma_3 \mathcal{H}_{p^1} r, r \rangle = \sum_{l=1,\dots,\mathbf{n}} \mathbf{e}_l |z_l|^2 + 2^{-1} \langle \mathrm{i}\sigma_3 \mathcal{H}_{p^1} f, f \rangle =: H_2. \tag{7.7}$$

In terms of (z, f), the Fréchet derivative dr can be expressed as

$$dr = \sum_{l=1,\dots,\mathbf{n}} (dz_l \, \xi_l + d\overline{z}_l \, \overline{\xi}_l) + df, \tag{7.8}$$

and by (7.6) we have

$$2^{-1} \langle \mathrm{i}\sigma_3 r, dr \rangle = 2^{-1} \imath \sum_{l=1,\dots,\mathbf{n}} (\overline{z}_l \, dz_l - z_l \, d\overline{z}_l) + 2^{-1} \langle \mathrm{i}\sigma_3 f, df \rangle. \tag{7.9}$$

Notice now that, in terms of (7.5) and (7.8),

$$\begin{aligned} d\Pi_j(r) &= \langle \Diamond_j (z\xi + \overline{z}\, \overline{\xi} + f), \; \xi\, dz + \overline{\xi}\, d\overline{z} + df \rangle \\ &= \sum_{l=1,\dots,\mathbf{n}} (\mathcal{R}^{0,1}_{\infty,\infty} \, dz_l + \mathcal{R}^{0,1}_{\infty,\infty} \, d\overline{z}_l) + \langle \Diamond_j f + \mathbf{S}^{0,1}_{\infty,\infty}, df \rangle. \end{aligned}$$

Hence, we obtain from (6.9):

$$\begin{aligned} &\Gamma = \Gamma_0 + \sum_{l=1,\dots,\mathbf{n}} (\mathcal{R}^{1,1}_{\infty,\infty} \, dz_l + \mathcal{R}^{1,1}_{\infty,\infty} \, d\overline{z}_l) + \langle \sum_{j \leqslant 7} \mathcal{R}^{0,2}_{\infty,\infty} \Diamond_j f + \mathbf{S}^{1,1}_{\infty,\infty}, df \rangle, \quad \text{where} \\ &\Gamma_0 := 2^{-1} \imath \sum_{l=1,\dots,\mathbf{n}} (\overline{z}_l dz_l - z_l d\overline{z}_l) + 2^{-1} \langle \mathrm{i}\sigma_3 f, df \rangle + \sum_{j \leqslant 7} \mathcal{R}^{1,0}_{\infty,\infty} (p^0, \Pi(f)) \langle \Diamond_j f, df \rangle. \end{aligned} \tag{7.10}$$

Then

$$\begin{aligned} \Omega_0 := d\Gamma_0 = &- \imath \sum_{l=1,\dots,\mathbf{n}} dz_l \wedge d\overline{z}_l + \langle \mathrm{i}\sigma_3 df, df \rangle \\ &+ \sum_{j,k} \mathcal{R}^{0,0}_{\infty,\infty} (p^0, \Pi(f)) \langle \Diamond_k f, df \rangle \wedge \langle \Diamond_j f, df \rangle, \end{aligned} \tag{7.11}$$

and, schematically, using $\partial_\rho \mathbf{S}^{1,1}_{\infty,\infty}\Big|_{(p_4^0,\rho,z,f)=(p_4^0,\Pi(f),z,f)} = \mathbf{S}^{0,1}_{\infty,\infty}$ and defining

$$\widehat{\nabla}_f F(\Pi(f), f) := \nabla_f F - \partial_{\Pi(f)} F \cdot \nabla_f \Pi(f), \tag{7.12}$$

we have

$$\begin{aligned} \Omega - \Omega_0 = & \mathcal{R}^{1,0}_{\infty,\infty}\, dz \wedge\, dz + \langle \widehat{\nabla}_f \mathbf{S}^{1,1}_{\infty,\infty} df, df\rangle \\ & + dz \wedge \langle \sum_{j\leqslant 7} \mathcal{R}^{0,1}_{\infty,\infty} \lozenge_j f + \mathbf{S}^{1,0}_{\infty,\infty}, df\rangle + d\Pi(f) \wedge \langle \mathbf{S}^{0,1}_{\infty,\infty}, df\rangle. \end{aligned} \tag{7.13}$$

We will transform Ω into Ω_0 by means of the Darboux Theorem, performed in a non-abstract way, to make sure that the coordinate transformation is as in Lemma 8.1.

8. **Flows.**

The following lemma is a consequence of Lemma A.1 in Appendix A:

Lemma 8.1. *For n, M, M_0, s, s', k, $l \in \mathbb{N}_0$ with $1 \leqslant l \leqslant M$, for Π_4 a parameter and for $\widetilde{\varepsilon}_0 > 0$, consider*

$$\begin{cases} \dot z(t) = \mathcal{R}^{0,M_0}_{n,M}(t, \Pi_4, \Pi(f), z, f) \\ \dot f(t) = \mathrm{i}\sigma_3 \sum_{j\leqslant 7} \mathcal{R}^{0,M_0+1}_{n,M}(t, \Pi_4, \Pi(f), z, f)\lozenge_j f + \mathbf{S}^{i,M_0}_{n,M}(t, \Pi_4, \Pi(f), z, f), \end{cases} \tag{8.1}$$

with the coefficients defined for $|t| < 5$, $|\Pi(f)| < \widetilde{\varepsilon}_0$, $|z| < \widetilde{\varepsilon}_0$, $\|r\|_{\Sigma_{-n}} < \widetilde{\varepsilon}$ and $|\Pi_4 - p_4^1| \leqslant \widetilde{\varepsilon}_0$.

Let $k \in \mathbb{Z} \cap [0, n-(l+1)]$ and set, for $s'' \geqslant 1$ and $\varepsilon > 0$,

$$\mathcal{U}^{s''}_{\varepsilon,k} := \{(z, f) \in \mathbb{C}^{\mathbf{n}} \times (X_c \cap \Sigma_{s''}) \ : \ |z| + \|f\|_{\Sigma_{-k}} + |\Pi(f)| \leqslant \varepsilon\}. \tag{8.2}$$

Let $a_0 \in A$. Then, for $\varepsilon > 0$ small enough, (8.1) defines a flow $(z^t, f^t) = \mathfrak{F}_t(z, f)$ with

$$\begin{aligned} z^t &= \mathcal{R}^{0,M_0}_{n-l-1,l}(*) \ , \text{ where } * = (t, \Pi_4, \Pi(f), z, f)\,, \\ f^t &= e^{\mathrm{i}\sigma_3 \sum_{j=1}^4 \mathcal{R}^{0,M_0+1}_{n-l-1,l}(*)\lozenge_j} T(e^{\sum_{i=1}^3 \mathcal{R}^{0,M_0+1}_{n-l-1,l}(*)\mathrm{i}\sigma_i}) \left(f + \mathbf{S}^{i,M_0}_{n-l-1,l}(*)\right), \end{aligned} \tag{8.3}$$

where for

$$n - l - 1 \geqslant s' \geqslant s + l \geqslant l \text{ and } k \in \mathbb{Z} \cap [0, n-l-1] \tag{8.4}$$

and for $\varepsilon_1 > \varepsilon_2 > 0$ sufficiently small we have

$$\mathfrak{F}_t \in C^l((-4,4) \times \mathcal{U}^{s'}_{\varepsilon_2,k}, \mathcal{U}^{s}_{\varepsilon_1,k}). \tag{8.5}$$

□

In (8.5) the C^l-regularity comes at the cost of a loss of l derivatives in the space $\Sigma_{s''}$, which is accounted for by $s' \geqslant s + l$.

In Proposition 10.3 we will need the following elementary technical lemmata.

Lemma 8.2. *Consider two systems for $\ell = 1, 2$:*

$$\begin{cases} \dot z(t) = \mathcal{B}^{(\ell)}(t, \Pi_4, \Pi(f), z, f) \\ \dot f(t) = \mathrm{i}\sigma_3 \sum_{j\leqslant 7} \mathcal{A}^{(\ell)}_j(t, \Pi_4, \Pi(f), z, f)\lozenge_j f + \mathcal{D}^{(\ell)}(t, \Pi_4, \Pi(f), z, f), \end{cases} \tag{8.6}$$

with the hypotheses of Lemma 8.1 satisfied, and suppose that

$$\begin{aligned} &\mathcal{B}^{(1)}(t, \Pi_4, \Pi(f), z, f) - \mathcal{B}^{(2)}(t, \Pi_4, \Pi(f), z, f) = \mathcal{R}^{0,M_0+1}_{n,M}(t, \Pi_4, \Pi(f), z, f) \\ &\mathcal{D}^{(1)}(t, \Pi_4, \Pi(f), z, f) - \mathcal{D}^{(2)}(t, \Pi_4, \Pi(f), z, f) = \mathbf{S}^{0,M_0+1}_{n,M}(t, \Pi_4, \Pi(f), z, f). \end{aligned} \tag{8.7}$$

Let $(z, f) \mapsto (z^t_{(\ell)}, f^t_{(\ell)})$ with $\ell = 1, 2$ be the two flows. Then for s, s' as in Lemma 8.1,

$$|z^1_{(1)} - z^1_{(2)}| + \|f^1_{(1)} - f^1_{(2)}\|_{\Sigma_{-s'}} \leqslant C\left(|z| + \|f\|_{\Sigma_{-s}}\right)^{M_0+1}. \tag{8.8}$$

For the proof, see Lemma A.2.

Lemma 8.3. *Under the hypotheses and notation of Lemma 8.2, we have:*

$$\Pi_j(f^1_{(1)}) - \Pi_j(f^1_{(2)}) = \mathcal{R}^{0,M_0+2}_{n-l-3,l}\big(\Pi_4, \Pi(f), z, f\big) \quad \textit{for } j = 1,\, 2,\, 3,\, 4. \tag{8.9}$$

Proof (sketch). For $\ell = 1, 2$ and $j = 1, 2, 3, 4$ we have

$$\Pi_j(f^1_{(\ell)}) = \Pi_j(f + \mathbf{S}^{(\ell)}) = \Pi_j(f) + \langle f, \Diamond_j \mathbf{S}^{(\ell)}\rangle + \Pi_j(\mathbf{S}^{(\ell)}), \tag{8.10}$$

where the r.h.s.'s are equal to the terms of (8.3) for $t = 1$ for each of the two flows,

$$\mathbf{S}^{(\ell)} = \mathbf{S}^{i,M_0}_{n-l-1,l}\big(\Pi_4, \Pi(f), z, f\big), \qquad \ell = 1,\, 2.$$

Hence $\Pi_j(\mathbf{S}^{(\ell)}) = \mathcal{R}^{i,2M_0}_{n-l-2,l}$, and this term can be absorbed into the r.h.s. of (8.9).

Next, observe that $\mathbf{S}^{(\ell)}$ is the integral $\int_0^1 \mathcal{D}^{(\ell)} dt$ of the terms $\mathcal{D}^{(\ell)}$ of Lemma 8.2. Formula (8.7) implies

$$\mathbf{S}^{(1)} - \mathbf{S}^{(2)} = \mathbf{S}^{0,M_0+1}_{n-l-2,l}\big(\Pi_4, \Pi(f), z, f\big),$$

as can be seen by elementary computations, and this in turn implies

$$\big\langle r, \Diamond_j\big(\mathbf{S}^{(1)} - \mathbf{S}^{(2)}\big)\big\rangle = \mathcal{R}^{0,M_0+2}_{n-l-3,l}\big(\Pi_4, \Pi(f), z, f\big). \qquad \square$$

We consider $f \in X_c \cap \Sigma_{\mathbf{N}_0}$ for $\mathbf{N}_0$ a large number. We can pick $\mathbf{N}_0 > 2\mathbf{N} + 2$ where $\mathbf{N}$ is defined in (7.2). Notice that (3.14) preserves this space. We have the following, which is proved as in [17], and which we discuss in Appendix B.

Lemma 8.4. *Consider $\mathfrak{F} = \mathfrak{F}^1 \circ \cdots \circ \mathfrak{F}^L$ with $\mathfrak{F}^j = \mathfrak{F}^j_{t=1}$ transformations as in Lemma 8.1 on the manifold $\mathcal{M}^6_1(p^0)$. Suppose that for any $\mathfrak{F}^j$ the M_0 in Lemma 8.1 equals m_j, where $1 = m_1 \leqslant \ldots \leqslant m_L$ with the constant i in Lemma 8.1 (ii) equal to 1 when $m_j = 1$. Fix M, k with $n_1 \gg k \geqslant \mathbf{N}_0$ (n_1 picked in Lemma 3.1). Then there is a $n = n(L, M, k)$ such that if the assumptions of Lemma 8.1 apply to each of operators $\mathfrak{F}^j$ for (M, n), there exist $\psi(p_4, \varrho) \in C^\infty$ with $\psi((p_4, \varrho) = O(|\varrho|^2)$ and a small $\varepsilon > 0$ such that in $\mathcal{U}^s_{\varepsilon,k}$ for $s \geqslant n - (M+1)$ we have the expansion*

$$K \circ \mathfrak{F} = \psi(p^0_4, \Pi(f)) + H_2 + \boldsymbol{R}, \tag{8.11}$$

and with what follows.

(1) *We have*

$$H_2 = \sum_{|\mu+\nu|=2\,,\ \mathbf{e}\cdot(\mu-\nu)=0} g_{\mu\nu}(p^0_4, \Pi(f)) z^\mu \overline{z}^\nu + 2^{-1}\langle \mathbb{i}\sigma_3 \mathcal{H}_{p^1} f, f\rangle. \tag{8.12}$$

(2) *Denote $\varrho = \Pi(f)$. There is the expansion $\boldsymbol{R} = \sum_{j=-1,\ldots,3} \boldsymbol{R}_j + \mathcal{R}^{1,2}_{k,m}(p^0_4, \varrho, f)$,*

$$\mathbf{R}_{-1} = \sum_{|\mu+\nu|=2\,,\ \mathbf{e}\cdot(\mu-\nu)\neq 0} g_{\mu\nu}(p^0_4, \varrho) z^\mu \overline{z}^\nu + \sum_{|\mu+\nu|=1} z^\mu \overline{z}^\nu \langle \mathbb{i}\sigma_3 G_{\mu\nu}(p^0_4, \varrho), f\rangle;$$

$$|\mathcal{R}^{1,2}_{k,m}(\Pi_4, \varrho, f)| \leqslant C\|f\|^2_{\Sigma_{-k}}(\|f\|_{\Sigma_{-k}} + |\varrho| + |\Pi_4 - p^1_4| + |z|);$$

for $\mathbf{N}$ *as in (H8),*

$$\boldsymbol{R}_0 = \sum_{|\mu+\nu|=3,\dots,2\mathbf{N}+2} z^\mu \overline{z}^\nu g_{\mu\nu}(p_4^0, \varrho);$$

$$\boldsymbol{R}_1 = \imath \sum_{|\mu+\nu|=2,\dots,2\mathbf{N}+1} z^\mu \overline{z}^\nu \langle \mathbb{i}\sigma_3 G_{\mu\nu}(p_4^0, \varrho), f\rangle;$$

$$\boldsymbol{R}_2 = \sum_{|\mu+\nu|=2\mathbf{N}+3} z^\mu \overline{z}^\nu g_{\mu\nu}(p_4^0, \varrho, z, f) - \sum_{|\mu+\nu|=2\mathbf{N}+2} z^\mu \overline{z}^\nu \langle \mathbb{i}\sigma_3 G_{\mu\nu}(p_4^0, \varrho, z, f), f\rangle;$$

$$\boldsymbol{R}_3 = \sum_{d=2,3,4} \langle B_d(p_4^0, \varrho, z, f), f^d\rangle + \int_{\mathbb{R}^3} B_5(x, p_4^0, \varrho, z, f, f(x)) f^5(x)\, dx + E_P(f),$$

with $B_2(p^1, 0, 0, 0) = 0.$ (8.13)

Above, $f^d(x)$ *schematically represents d-products of components of* f*.*

(3) *For* $\delta_j \in \mathbb{N}_0^m$ *the vectors defined in terms of the Kronecker symbols by* $\delta_j := (\delta_{1j}, \dots, \delta_{mj})$*,*

$$g_{\mu\nu} = \mathcal{R}^{1,0}_{k,m} \quad \text{for } |\mu+\nu| = 2 \text{ for } (\mu,\nu) \neq (\delta_j,\delta_j),\ 1 \leqslant j \leqslant m;$$
$$g_{\delta_j\delta_j} = \mathbf{e}_j + \mathcal{R}^{1,0}_{k,m},\ 1 \leqslant j \leqslant m; \quad G_{\mu\nu} = \mathbf{S}^{1,0}_{k,m} \text{ for } |\mu+\nu| = 1; \tag{8.14}$$

$g_{\mu\nu}$ *and* $G_{\mu\nu}$ *satisfy symmetries analogous to* (10.3)*.*

(4) *All the other* $g_{\mu\nu}$ *are* $\mathcal{R}^{0,0}_{k,m}$ *and all the other* $G_{\mu\nu}$ *are* $\mathbf{S}^{0,0}_{k,m}$*.*

(5) $B_d(p^0, \varrho, z, f) \in C^m(\mathcal{U}_{-k}, \Sigma_k(\mathbb{R}^3, B((\mathbb{R}^4)^{\otimes d}, \mathbb{R})))$ *for* $2 \leqslant d \leqslant 4$ *with* $\mathcal{U}_{-k} \subset \mathbb{R}^8 \times \mathbb{C}^{\mathbf{n}} \times (X_c \cap \Sigma_{-k})$ *an open neighborhood of* $(p_4^1, \varrho, z, f) = (0,0,0,0)$*.*

(6) *Let* $\zeta \in \mathbb{C}^2$*. Then for* $B_5(\cdot, \varrho, z, f, \zeta)$ *we have, for fixed constants* C_l *(the derivatives are not in the holomorphic sense),*

$$\text{for } |l| \leqslant m, \quad \|\nabla^l_{p^0,\varrho,z,f,\zeta} B_5(p_4^0, \varrho, z, f, \zeta)\|_{\Sigma_k(\mathbb{R}^3, B((\mathbb{C}^2)^{\otimes 5}, \mathbb{R})} \leqslant C_l. \tag{8.15}$$

For the proof, see Appendix B.

9. **Darboux theorem.**

Recall that we have introduced a model symplectic form Ω_0 in $\mathcal{M}_1^6(p^0)$ by formula (7.11). Now we transform Ω into Ω_0 by means of the Darboux Theorem, performed in a non-abstract way, to make sure that the coordinate transformation is as in Lemma 8.1.

Lemma 9.1. *For* n_1 *the constant in Lemma 3.1 and* $\varepsilon_2 > 0$ *consider the set*

$$\mathcal{U}_2 = \left\{(z, f) \in \mathbb{C}^{\mathbf{n}} \times (X_c \cap H^1) : \quad \|f\|_{\Sigma_{-n_1}} \leqslant \varepsilon_2, \quad |\Pi(f)| \leqslant \varepsilon_2, \quad |z| \leqslant \varepsilon_2\right\}.$$

Then for $\varepsilon_2 > 0$ *small enough there exists a unique vector field* $\mathcal{Y}^t$ *in* $\mathcal{U}_2$ *such that* $i_{\mathcal{Y}^t}(\Omega_0 + t(\Omega - \Omega_0)) = \Gamma_0 - \Gamma$ *for* $|t| < 5$ *with components, where* $\Pi_4 = p_4^0$*,*

$$(\mathcal{Y}^t)_{z_j} = \mathcal{R}^{1,1}_{n_1,\infty}(\Pi_4, \Pi(f), z, f)\,,$$
$$(\mathcal{Y}^t)_f = \mathbb{i}\sigma_3 \mathcal{R}^{0,2}_{n,\infty}(\Pi_4, \Pi(f), z, f) \cdot \Diamond f + \mathbf{S}^{1,1}_{n_1,\infty}(\Pi_4, \Pi(f), z, f).$$

Proof. The proof is essentially the same as that of [17, Lemma 3.4]. The first step is to consider a field Z such that $i_Z\Omega_0 = \Gamma_0 - \Gamma$. We claim that

$$(Z)_z = \mathcal{R}^{1,1}_{\infty,\infty}(\Pi_4, \Pi(f), z, f)\,,$$
$$(Z)_f = \mathbb{i}\sigma_3 \mathcal{R}^{0,2}_{\infty,\infty}(\Pi_4, \Pi(f), z, f) \cdot \Diamond f + \mathbf{S}^{1,1}_{\infty,\infty}(\Pi_4, \Pi(f), z, f).$$

Schematically, the equation for Z is of the form

$$\begin{aligned}&(Z)_z\, dz + \langle \big[\mathbb{i}\sigma_3 (Z)_f + \mathcal{R}^{0,0}_{\infty,\infty} \langle \Diamond f, (Z)_f \rangle \big] \Diamond f, df \rangle \\ &= \mathcal{R}^{1,1}_{\infty,\infty}\, dz + \langle \mathbb{i}\sigma_3 \mathcal{R}^{0,2}_{\infty,\infty} \cdot \Diamond f + \mathbf{S}^{1,1}_{\infty,\infty}, df \rangle.\end{aligned}$$

This immediately yields $(Z)_z = \mathcal{R}^{1,1}_{\infty,\infty}$. The equation for $(Z)_f$ is of the form

$$(Z)_f + \mathcal{R}^{0,0}_{\infty,\infty} \langle \Diamond f, (Z)_f \rangle \mathbb{i}\sigma_3 \Diamond f = \mathbb{i}\sigma_3 \mathcal{R}^{0,2}_{\infty,\infty} \cdot \Diamond f + \mathbf{S}^{1,1}_{\infty,\infty}, \tag{9.1}$$

with a solution in the form $(Z)_f = \sum_{i=0}^{\infty} (Z)^{(i)}_f$, with $(Z)^{(0)}_f = \mathbb{i}\sigma_3 \mathcal{R}^{0,2}_{\infty,\infty} \cdot \Diamond f + \mathbf{S}^{1,1}_{\infty,\infty}$ and

$$(Z)^{(i+1)}_f = \mathcal{R}^{0,0}_{\infty,\infty} \langle \Diamond f, (Z)^{(i)}_f \rangle \mathbb{i}\sigma_3 \Diamond f = (\mathcal{R}^{0,0}_{\infty,\infty})^{i+1} \langle \Diamond f, \mathbb{i}\sigma_3 \Diamond f \rangle^i \langle \Diamond f, (Z)^{(0)}_f \rangle \mathbb{i}\sigma_3 \Diamond f,$$

where by direct computation $\langle \Diamond_j f, \mathbb{i}\sigma_3 \Diamond_k f \rangle$ is a bounded bilinear form in $X_c \cap L^2(\mathbb{R}^3, \mathbb{C}^4)$ for all j, k. This implies that the series defining $(Z)_f$ is convergent and that $(Z)_f$ is as in (9.1).

The next step is to define an operator $\mathcal{K}$ by $i_X(\Omega - \Omega_0) = i_{\mathcal{K}X}\Omega_0$. We claim that

$$\begin{aligned}(\mathcal{K}X)_z &= \mathcal{R}^{1,0}_{\infty,\infty} (X)_z + \langle \mathcal{R}^{0,2}_{\infty,\infty} \Diamond f + \mathbf{S}^{1,0}_{\infty,\infty}, (X)_f \rangle \\ (\mathcal{K}X)_f &= \mathbb{i}\sigma_3 \langle \mathbf{S}^{0,1}_{\infty,\infty}, (X)_f \rangle \Diamond f + \partial_f\, \mathbf{S}^{1,1}_{\infty,\infty}\big|_{(\rho,z,f)=(\Pi(f),z,f)} (X)_f \\ &\quad + (X)_z \mathcal{R}^{0,1}_{\infty,\infty} \Diamond f + (X)_z \mathbf{S}^{1,0}_{\infty,\infty} + \langle \Diamond f, (X)_f \rangle \mathbf{S}^{0,1}_{\infty,\infty}.\end{aligned} \tag{9.2}$$

From (7.11)–(7.13) we have schematically

$$\begin{aligned}&\boldsymbol{\imath}(\mathcal{K}X)_z dz + \Big\langle \big[\mathbb{i}\sigma_3 (\mathcal{K}X)_f + \mathcal{R}^{0,0}_{\infty,\infty} \langle \Diamond f, (\mathcal{K}X)_f \rangle \Diamond f \big], df \Big\rangle \\ &= \big(\mathcal{R}^{1,0}_{\infty,\infty} (X)_z + \langle \mathcal{R}^{0,1}_{\infty,\infty} \Diamond f + \mathbf{S}^{1,0}_{\infty,\infty}, (X)_f \rangle \big)\, dz + \Big\langle \Big[\partial_f\, \mathbf{S}^{1,1}_{\infty,\infty}\big|_{(\rho,z,f)=(\Pi(f),z,f)} (X)_f \\ &+ (X)_z (\mathcal{R}^{0,1}_{\infty,\infty} \Diamond f + \mathbf{S}^{1,0}_{\infty,\infty}) + \langle \mathbf{S}^{0,1}_{\infty,\infty}, (X)_f \rangle \Diamond f + \langle \Diamond f, (X)_f \rangle \mathbf{S}^{0,1}_{\infty,\infty} \Big], df \Big\rangle\end{aligned}$$

which yields immediately the first equation in (9.2). We have $(\mathcal{K}X)_f = \sum_{i=0}^{\infty} (\mathcal{K}^{(i)} X)_f$ with

$$\begin{aligned}\mathbb{i}\sigma_3 (\mathcal{K}^{(0)} X)_f = \partial_f\, \mathbf{S}^{1,1}_{\infty,\infty}\big|_{(\rho,z,f)=(\Pi(f),z,f)} (X)_f + (X)_z (\mathcal{R}^{0,1}_{\infty,\infty} \Diamond f + \mathbf{S}^{1,0}_{\infty,\infty}) \\ + \langle \mathbf{S}^{0,1}_{\infty,\infty}, (X)_f \rangle \Diamond f + \langle \Diamond f, (X)_f \rangle \mathbf{S}^{0,1}_{\infty,\infty}\end{aligned}$$

and

$$\begin{aligned}(\mathcal{K}^{(i+1)} X)_f &= \mathcal{R}^{0,0}_{\infty,\infty} \langle \Diamond f, (\mathcal{K}^{(i)} X)_f \rangle \mathbb{i}\sigma_3 \Diamond f \\ &= (\mathcal{R}^{0,0}_{\infty,\infty})^{i+1} \langle \Diamond f, \mathbb{i}\sigma_3 \Diamond f \rangle^i \langle \Diamond f, (\mathcal{K}^{(0)} X)_f \rangle \mathbb{i}\sigma_3 \Diamond f.\end{aligned}$$

Then the series defining $(\mathcal{K}X)_f$ converges and we get in particular the second equation in (9.2). Now the equation defining $\mathcal{Y}^t$ is equivalent to $(1 + t\mathcal{K})\mathcal{Y}^t = Z$. So we have

$$\begin{aligned}&(\mathcal{Y}^t)_z + t\mathcal{R}^{1,0}_{\infty,\infty} (\mathcal{Y}^t)_z + t \langle \mathcal{R}^{0,2}_{\infty,\infty} \Diamond f + \mathbf{S}^{1,0}_{\infty,\infty}, (\mathcal{Y}^t)_f \rangle = \mathcal{R}^{1,1}_{\infty,\infty} \\ &(\mathcal{Y}^t)_f + \mathbb{i}t\sigma_3 \langle \mathbf{S}^{0,1}_{\infty,\infty}, (\mathcal{Y}^t)_f \rangle \Diamond f + t\partial_f\, \mathbf{S}^{1,1}_{\infty,\infty}\big|_{(\rho,z,f)=(\Pi(f),z,f)} (\mathcal{Y}^t)_f \\ &+ t(\mathcal{Y}^t)_z (\mathcal{R}^{0,2}_{\infty,\infty} \Diamond f + \mathbf{S}^{1,0}_{\infty,\infty}) + t \langle \Diamond f, (\mathcal{Y}^t)_f \rangle \mathbf{S}^{0,1}_{\infty,\infty} = \mathbb{i}\sigma_3 \mathcal{R}^{0,2}_{\infty,\infty} \cdot \Diamond f + \mathbf{S}^{1,1}_{\infty,\infty}.\end{aligned}$$

Solving this we get the desired formulae for $(\mathcal{Y}^t)_{z_j}$ and $(\mathcal{Y}^t)_f$. $\square$

We can apply Lemma 8.1 to the flow $\mathfrak{F}_t:\ (z,f)\mapsto(z^t,f^t)$ generated by $\mathcal{Y}^t$. In terms of the decomposition (7.5) of r formula (8.3) becomes, for $n=n_1$,

$$\begin{aligned} &z^t=z+\mathcal{R}^{1,1}_{n_1-l-1,l}(t,\Pi_4,\Pi(f),z,f),\\ &f^t=e^{\mathrm{i}\sum_{j=1}^4\sigma_3\mathcal{R}^{0,2}_{n_1-l-1,l}(t,\Pi_4,\Pi(f),z,f)\Diamond_j}T\big(e^{\sum_{a=1}^3\mathcal{R}^{0,2}_{n_1-l-1,l}(t,\Pi_4,\Pi(f),z,f)\mathrm{i}\sigma_a}\big)\\ &\times\big(f+\mathbf{S}^{1,1}_{n_1-l-1,l}(t,\Pi_4,\Pi(f),z,f)\big).\end{aligned}\tag{9.3}$$

Classically the Darboux Theorem follows by $i_{\mathcal{Y}^t}\Omega_t=\Gamma_0-\Gamma$, where $\Omega_t:=\Omega_0+t(\Omega-\Omega_0)$, and by

$$\partial_t(\mathfrak{F}_t^*\Omega_t)=\mathfrak{F}_t^*(L_{\mathcal{Y}^t}\Omega_t+\partial_t\Omega_t)=\mathfrak{F}_t^*(di_{\mathcal{Y}^t}\Omega_t'+d(\Gamma-\Gamma_0))=0\tag{9.4}$$

with L_X the Lie derivative, whose definition is not needed here. Since this $\mathfrak{F}_t$ is not a differentiable flow on any given manifold, (9.4) is formal. Still, [17, Sect. 3.3 and Sect. 7] (i.e. a regularization and a limit argument for $\mathfrak{F}_t$) yield the following, which we state without proof:

Lemma 9.2. *Consider (8.1) defined by the field $\mathcal{X}^t$ and indexes and notation of Lemma 8.1 (in particular $M_0=1$ and $i=1$; n and M can be arbitrary as long as we fix n_1 large enough). Consider l, s', s, and k as in (8.4). Then for $\mathfrak{F}_1\in C^l(\mathcal{U}^{\mathbf{s}'}_{\varepsilon_2,k},\mathcal{U}^{\mathbf{s}}_{\varepsilon_1,k})$ derived from (9.3), we have $\mathfrak{F}_1^*\Omega=\Omega_0$.* □

We now turn to the analysis of the hamiltonian vector fields in the new coordinate system. For a function F let us decompose X_F according to the spectral decomposition (7.5): for $(X_F)_f\in X_c$,

$$X_F=\sum_{j=1,\dots,\mathbf{n}}(X_F)_{z_j}\xi_j(x)+\sum_{j=1,\dots,\mathbf{n}}(X_F)_{\overline{z}_j}\xi_j^*(x)+(X_F)_f.\tag{9.5}$$

By (7.11) and by $i_{X_F}\Omega_0=dF$ we have, schematically (recall also that here and below $\Pi_4=p_4^0$),

$$\begin{aligned}&-\imath(X_F)_{z_l}d\overline{z}_l+(X_F)_{\overline{z}_l}\,dz_l+\big\langle\big[\mathrm{i}\sigma_3(X_F)_f+\mathcal{R}^{0,0}_{\infty,\infty}(\Pi_4,\Pi(f))\langle\Diamond f,(X_F)_f\rangle\Diamond f\big],df\big\rangle\\&=\partial_{z_l}F\,dz_l+\partial_{\overline{z}_l}F\,d\overline{z}_l+\langle\nabla_fF,df\rangle.\end{aligned}$$

and so, schematically,

$$\begin{aligned}&(X_F)_{z_l}=\imath\partial_{\overline{z}_l}F\,,\quad (X_F)_{\overline{z}_l}=-\imath\partial_{z_l}F\\&(X_F)_f+\mathcal{R}^{0,0}_{\infty,\infty}(\Pi_4,\Pi(f))\langle\Diamond f,(X_F)_f\rangle P_c\mathrm{i}\sigma_3\Diamond f=-\mathrm{i}\sigma_3\nabla_fF.\end{aligned}$$

We set

$$X_F=X_F^{(0)}+X_F^{(1)}\ \text{with}\tag{9.6}$$

$$(X_F^{(0)})_{z_l}=\imath\partial_{\overline{z}_l}F\,,\qquad(X_F^{(0)})_{\overline{z}_l}=-\imath\partial_{z_l}F\,,\qquad(X_F^{(0)})_f=-\mathrm{i}\sigma_3\nabla_fF\,,\tag{9.7}$$

and where the remainder is of the form $(X_F^{(1)})_{z_l}=(X_F^{(1)})_{\overline{z}_l}=0$,

$$(X_F^{(1)})_f=\mathcal{R}^{0,0}_{\infty,\infty}(\Pi_4,\Pi(f))\langle\Diamond f,\mathrm{i}\sigma_3\nabla_fF\rangle P_c\mathrm{i}\sigma_3\Diamond f.\tag{9.8}$$

Indeed, $(X_F^{(1)})_f$ has to satisfy an equation of the form

$$\begin{aligned}&(X_F^{(1)})_f+\mathcal{R}^{0,0}_{\infty,\infty}(\Pi_4,\Pi(f))\langle\Diamond f,(X_F)_f^{(1)}\rangle P_c\mathrm{i}\sigma_3\Diamond f\\&=\mathcal{R}^{0,0}_{\infty,\infty}(\Pi_4,\Pi(f))\langle\Diamond f,\mathrm{i}\sigma_3\nabla_fF\rangle P_c\mathrm{i}\sigma_3\Diamond f.\end{aligned}$$

This can be solved like in the proof of Lemma 9.1 by writing $(X_F)_f^{(1)} = \sum_{i=0}^{\infty} X_i$ with

$$\begin{aligned} X_0 &= \mathcal{R}^{0,0}_{\infty,\infty}(\Pi_4,\Pi(f))\langle \Diamond f, \mathbb{i}\sigma_3 \nabla_f F\rangle P_c \mathbb{i}\sigma_3 \Diamond f \quad \text{and} \\ X_{i+1} &= \mathcal{R}^{0,0}_{\infty,\infty}(\Pi_4,\Pi(f))\langle \Diamond f, X_i\rangle P_c \mathbb{i}\sigma_3 \Diamond f \\ &= (\mathcal{R}^{0,0}_{\infty,\infty})^{i+1}\langle \Diamond f, \mathbb{i}\sigma_3 \Diamond f\rangle^{i}\langle \Diamond f, \mathbb{i}\sigma_3 \nabla_f F\rangle P_c \mathbb{i}\sigma_3 \Diamond f \end{aligned}$$

which yields (9.8). For two functions F and G we have the Poisson brackets

$$\begin{aligned} \{F,G\} &:= dF(X_G) = \partial_{z_l} F (X_G)_{z_l} + \partial_{\overline{z}_l} F (X_G)_{\overline{z}_l} + \langle \nabla_f F, (X_G)_f\rangle \\ &= \{F,G\}_{(0)} + \{F,G\}_{(1)}, \end{aligned} \tag{9.9}$$

where $\{F,G\}_{(i)} := dF(X_G^{(i)})$ and where

$$\{F,G\}_{(0)} = \imath(\partial_{z_l} F \partial_{z_l} G - \partial_{\overline{z}_l} F \partial_{\overline{z}_l} G) - \langle \nabla_f F, \mathbb{i}\sigma_3 \nabla_f G\rangle \tag{9.10}$$

and, schematically,

$$\{F,G\}_{(1)} = \mathcal{R}^{0,0}_{\infty,\infty}(\Pi_4,\Pi(f))\langle \nabla_f F, \Diamond f\rangle\langle \Diamond f, \mathbb{i}\sigma_3 \nabla_f G\rangle. \tag{9.11}$$

Compared to [17], where the Poisson bracket equals (9.10), here we have an additional term contributed by (9.11), which however is of higher order and harmless, as we will see later.

10. **Birkhoff normal forms.** We will reduce now to [17, Sect. 6]. We set, for the $\mathbf{e}_j$'s in (H6), see Section 7,

$$\mathbf{e} := (\mathbf{e}_1, ..., \mathbf{e}_{\mathbf{n}}).$$

In the sequel, $\Pi_4 = p_4^0$.

Definition 10.1. A function $Z(\varrho, z, f)$ is in normal form if $Z = Z_0 + Z_1$, where Z_0 and Z_1 are finite sums of the following type:

$$Z_1 = \sum_{\mathbf{e}\cdot(\nu-\mu)\in\sigma_{\mathrm{ess}}(\mathcal{H}_{p^1})} z^\mu \overline{z}^\nu \langle \mathbb{i}\sigma_3 G_{\mu\nu}(p_4^0,\varrho), f\rangle \tag{10.1}$$

with $G_{\mu\nu}(x,p_4,\varrho) \in C^m(U, \Sigma_k(\mathbb{R}^3,\mathbb{C}^4))$ for fixed $k,m \in \mathbb{N}$ and $U \subseteq \mathbb{R}^8$ an open neighborhood of $(p_4^0, 0)$,

$$Z_0 = \sum_{\mathbf{e}\cdot(\mu-\nu)=0} g_{\mu\nu}(p_4^0,\varrho) z^\mu \overline{z}^\nu, \tag{10.2}$$

with $g_{\mu\nu}(p_4,\varrho) \in C^m(U,\mathbb{C})$. We assume furthermore the symmetries $\overline{g}_{\mu\nu} = g_{\nu\mu}$ and $\overline{G}_{\mu\nu} = G_{\nu\mu}$.

Lemma 10.2. *For $i \in \{0,1\}$ fixed and n, $M \in \mathbb{N}$ sufficiently large and for $m \leqslant M-1$ let*

$$\chi = \sum_{|\mu+\nu|=M_0+1} c_{\mu\nu}(p_4^0,\Pi(f)) z^\mu \overline{z}^\nu + \imath \sum_{|\mu+\nu|=M_0} z^\mu \overline{z}^\nu \langle \mathbb{i}\sigma_3 C_{\mu\nu}(p_4^0,\Pi(f)), f\rangle,$$

with $c_{\mu\nu}(p^0,\varrho) = \mathcal{R}^{i,0}_{n,M}(p^0,\varrho)$ and $C_{\mu\nu}(p^0,\varrho) = \mathbf{S}^{i,0}_{n,M}(p^0,\varrho)$ and with

$$\overline{c}_{\mu\nu} = c_{\nu\mu}, \qquad \overline{C}_{\mu\nu} = -C_{\nu\mu} \tag{10.3}$$

(so that χ is real-valued for $f = \overline{f}$). Then we have what follows:

(1) *For ϕ^t the flow of X_χ, see Lemma 8.1, and $(z^t, f^t) = (z, f) \circ \phi^t$,*

$$\begin{aligned} z^t =& z + \mathcal{R}^{0,M_0}_{n-m-1,m-1}(t, \Pi_4, \Pi(f), z, f); \\ f^t =& e^{\mathrm{i}\sigma_3 \sum_{j=1}^4 \mathcal{R}^{0,M_0+1}_{n-m-1,m-1}(t,\Pi_4,\Pi(f),z,f)\Diamond_j} T\big(e^{\sum_{i=1}^3 \mathcal{R}^{0,M_0+1}_{n-m-1,m-1}(t,\Pi_4,\Pi(f),z,f)\mathrm{i}\sigma_i}\big) \\ & \times \big(f + \mathbf{S}^{0,M_0}_{n-m-1,m-1}(t, \Pi_4, \Pi(f), z, f)\big). \end{aligned} \tag{10.4}$$

(2) *For $n - m - 1 \geqslant s' \geqslant s + m - 1 \geqslant m - 1$ and $k \in \mathbb{Z} \cap [0, n - m - 1]$ and for $\varepsilon_1 > \varepsilon_2 > 0$ sufficiently small, $\phi := \phi^1 \in C^{m-1}(\mathcal{U}^{s'}_{\varepsilon_2,k}, \mathcal{U}^{s}_{\varepsilon_1,k})$ satisfies $\phi^*\Omega_0 = \Omega_0$.*

Proof. This result is a simple corollary of Lemma 8.1. For the proof that $\phi^*\Omega_0 = \Omega_0$, which is obvious in the standard setups, see the comments in [17, Lemma 5.3]. □

Then we have the following result on Birkhoff normal forms.

Proposition 10.3. *For any integer $2 \leqslant \ell \leqslant 2\mathbf{N} + 2$ there are transformations $\mathfrak{F}^{(\ell)} = \mathfrak{F}_1 \circ \phi_2 \circ ... \circ \phi_\ell$, with $\mathfrak{F}_1$ the transformation in (9.3) and with the ϕ_j's like in Lemma 10.2, such that the conclusions of Lemma 8.4 hold; that is, such that we have the following expansion, with $\Pi_4 = p_4^0$:*

$$H^{(\ell)} := K \circ \mathfrak{F}^{(\ell)} = \psi(p_4^0, \Pi(f)) + H_2 + \mathcal{R}^{1,2}_{k,m}(\Pi_4, \Pi(f), f) + \sum_{j=-1,\dots,3} \boldsymbol{R}^{(\ell)}_j,$$

with H_2' defined in (8.12) and with the following additional properties:

(i) *$\boldsymbol{R}^{(\ell)}_{-1} = 0$;*

(ii) *all the nonzero terms in $\boldsymbol{R}^{(\ell)}_0$ with $|\mu + \nu| \leqslant \ell$ are in normal form, that is $\mathbf{e} \cdot (\mu - \nu) = 0$;*

(iii) *all the nonzero terms in $\boldsymbol{R}^{(\ell)}_1$ with $|\mu + \nu| \leqslant \ell - 1$ are in normal form, that is $\mathbf{e} \cdot (\mu - \nu) \in \sigma_{\mathrm{ess}}(\mathcal{H}_{p^0})$.*

Proof. The proof of the analogue of Proposition 10.3 in [17] involves the simpler symplectic form

$$\Omega^{(0)}_0 := -\imath \sum_{l=1,\dots,\mathbf{n}} dz_l \wedge d\overline{z}_l + \langle \mathrm{i}\sigma_3 df, df\rangle.$$

In (8.11), we replace $\Pi(f)$ with ϱ; then $\mathbf{h} = H^{(\ell)}(p^0, \varrho, z, f)$ is $C^{2\mathbf{N}+2}$ near $(0, 0, 0)$ in $(\varrho, z, f) \in \mathbb{R}^7 \times \mathbb{C} \times (X_c \cap \Sigma_k)$ and the statement of Proposition 10.3 is about the fact that some of the following derivatives vanish:

$$g_{\mu\nu}(p^0, \varrho) = \frac{1}{\mu!\nu!} \partial_z^\mu \partial_{\overline{z}}^\nu \mathbf{h}\big|_{(\varrho,z,f)=(\varrho,0,0)}, \quad |\mu + \nu| \leqslant 2\mathbf{N} + 2, \tag{10.5}$$

$$\mathrm{i}\sigma_3 G_{\mu\nu}(p^0, \varrho) = \frac{1}{\mu!\nu!} \partial_z^\mu \partial_{\overline{z}}^\nu \nabla_f \mathbf{h}\big|_{(\varrho,z,f)=(\varrho,0,0)}, \quad |\mu + \nu| \leqslant 2\mathbf{N} + 1. \tag{10.6}$$

The proof is iterative and consists in assuming the statement correct for a given ℓ and proving it for $\ell + 1$, by picking an unknown χ as in (10.2) such that $H^{(\ell)} \circ \phi$ satisfies the conclusions for $\ell + 1$, where $\phi = \phi^1$, for ϕ^t the flow for the Hamiltonian vector field of χ.

Now, let us pick χ provided by [17, Theorem 6.4] when we use the symplectic form $\Omega^{(0)}_0$. We will show that this same χ works here.

Let $\phi^{(0)}$ be the $t=1$ flow generated by $X_\chi^{(0)}$. Notice that $\phi^{(0)}$ is a symplectomorphism for $\Omega_0^{(0)}$. Set

$$\widehat{H}^{(\ell)} = \psi(p^0, \Pi(f)) + H_2 + \sum_{j=-1,0,1} \mathbf{R}_j^{(\ell)}. \tag{10.7}$$

Noticing that here $\psi(p^0, \Pi(f))$ contributes 0 because it is $\psi(p^0, \varrho)$ with ϱ an auxiliary independent variable,

$$\begin{aligned}&\partial_z^\mu \partial_{\overline{z}}^\nu H^{(\ell)}\big|_{(\varrho,z,f)=(\varrho,0,0)} = \partial_z^\mu \partial_{\overline{z}}^\nu \widehat{H}^{(\ell)}\big|_{(\varrho,z,f)=(\varrho,0,0)}, \quad 2 \leqslant |\mu+\nu| \leqslant 2\mathbf{N}+2,\\ &\partial_z^\mu \partial_{\overline{z}}^\nu \nabla_f H^{(\ell)}\big|_{(\varrho,z,f)=(\varrho,0,0)} = \partial_z^\mu \partial_{\overline{z}}^\nu \nabla_f \widehat{H}^{(\ell)}\big|_{(\varrho,z,f)=(\varrho,0,0)}, \quad 1 \leqslant |\mu+\nu| \leqslant 2\mathbf{N}+1\end{aligned} \tag{10.8}$$

since all the other terms of $H^{(\ell)}$ not contained in $\widehat{H}^{(\ell)}$ are higher order in some of the variables, for example order 2 or higher in f. As we pointed out, $\psi(p^0, \Pi(f))$ contributes nothing to (10.8). The same is true of the term $\frac{1}{2}\langle \mathrm{i}\sigma_3 \mathcal{H}_{p^1} f, f\rangle$ inside H_2', see (8.12) (however, the pullbacks of these terms are significant in the formulae below). So the only contributors of (10.7) to (10.8) are very regular functions in (ϱ, z, f), where $\varrho = \Pi(f)$ is as before treated as auxiliary variable and $f \in (X_c \cap \Sigma_{-k})$. This yields the useful result that while the l.h.s.'s in (10.8) require f quite regular, for example $f \in \Sigma_k$ for a sufficiently large k, the r.h.s.'s are defined for $f \in \Sigma_{-k}$ for a large preassigned k. This is because the only term in $\widehat{H}^{(\ell)}(p^0, \varrho, z, f)$ that, to make sense, requires some regularity in f, that is the $\frac{1}{2}\langle \mathrm{i}\sigma_3 \mathcal{H}_{p^1} f, f\rangle$ hidden inside H_2' (see (8.12)) does not contribute to (10.8).

Furthermore, by Lemma 10.2, we have

$$\begin{aligned}&\partial_z^\mu \partial_{\overline{z}}^\nu H^{(\ell)}\circ\phi^{(0)}\big|_{(\varrho,z,f)=(\varrho,0,0)} = \partial_z^\mu \partial_{\overline{z}}^\nu \widehat{H}^{(\ell)}\circ\phi^{(0)}\big|_{(\varrho,z,f)=(\varrho,0,0)},\\ &\qquad\qquad 2 \leqslant |\mu+\nu| \leqslant 2\mathbf{N}+1,\\ &\partial_z^\mu \partial_{\overline{z}}^\nu \nabla_f H^{(\ell)}\circ\phi^{(0)}\big|_{(\varrho,z,f)=(\varrho,0,0)} = \partial_{\overline{z}}^\nu \nabla_f \widehat{H}^{(\ell)}\circ\phi^{(0)}\big|_{(\varrho,z,f)=(\varrho,0,0)},\\ &\qquad\qquad 1 \leqslant |\mu+\nu| \leqslant 2\mathbf{N}\end{aligned} \tag{10.9}$$

since the pull-backs of the terms of $H^{(\ell)}$ not contained in $\widehat{H}^{(\ell)}$ have zero derivatives because they are higher order either in z or in f, as can be seen considering that $\phi^{(0)}$ acts like (10.4) for $M_0 = \ell$. Since ϕ too has this structure, (10.9) is true also with $\phi^{(0)}$ replaced by ϕ. Set now

$$\widehat{H}^{(\ell)}\circ\phi = \psi(p^0,\varrho) + F \text{ with } F := \widehat{H}^{(\ell)}\circ\phi - \psi(p^0,\varrho). \tag{10.10}$$

We have $dF\big|_{(\varrho,z,f)=(\varrho,0,0)} = 0$, since by Lemma 8.4 we see that F is at least quadratic in (z,f). Lemma 8.2 is telling us that $\phi^{-1}\circ\phi^{(0)}$ is the identity up to a zero of order $\ell+1$ at $(z,f)=(0,0)$ in $\mathbb{C}^{\mathbf{n}}\times(X_c\cap\Sigma_{-k})$. Then by an elementary application of the chain rule

$$\begin{aligned}&\partial_z^\mu \partial_{\overline{z}}^\nu F\big|_{(\varrho,z,f)=(\varrho,0,0)} = \partial_z^\mu \partial_{\overline{z}}^\nu F\circ\phi^{-1}\circ\phi^{(0)}\big|_{(\varrho,z,f)=(\varrho,0,0)}, \qquad 2 \leqslant |\mu+\nu| \leqslant \ell+1,\\ &\partial_z^\mu \partial_{\overline{z}}^\nu \nabla_f F\big|_{(\varrho,z,f)=(\varrho,0,0)} = \partial_{\overline{z}}^\nu \nabla_f F\circ\phi^{-1}\circ\phi^{(0)}\big|_{(\varrho,z,f)=(\varrho,0,0)}, \quad 1 \leqslant |\mu+\nu| \leqslant \ell.\end{aligned}$$

On the other hand, by Lemma 8.3 we have that $\psi(p^0,\varrho)$ and $\psi(p^0,\varrho)\circ\phi^{-1}\circ\phi^{(0)}$ differ by a zero of order $\ell+2$ in $(\varrho,0,0)$. Summing up, we conclude:

$$\begin{aligned}&\partial_z^\mu \partial_{\overline{z}}^\nu \widehat{H}^{(\ell)}\circ\phi\big|_{(\varrho,z,f)=(\varrho,0,0)} = \partial_z^\mu \partial_{\overline{z}}^\nu \widehat{H}^{(\ell)}\circ\phi^{(0)}\big|_{(\varrho,z,f)=(\varrho,0,0)}, \quad 2 \leqslant |\mu+\nu| \leqslant \ell+1,\\ &\partial_z^\mu \partial_{\overline{z}}^\nu \nabla_f \widehat{H}^{(\ell)}\circ\phi\big|_{(\varrho,z,f)=(\varrho,0,0)} = \partial_{\overline{z}}^\nu \nabla_f \widehat{H}^{(\ell)}\circ\phi^{(0)}\big|_{(\varrho,z,f)=(\varrho,0,0)}, \quad 1 \leqslant |\mu+\nu| \leqslant \ell.\end{aligned}$$

Hence we have shown that [17, Theorem 6.4] implies Proposition 10.3. □

11. **Formulation of the system.** We consider the Hamiltonian $H := H^{(2\mathbf{N}+1)}$ and the reduced system

$$\dot{z} = \{z, H\}\,, \quad \dot{f} = \{f, H\}. \tag{11.1}$$

Recall that

$$H = \psi(p_4^0, \Pi(f)) + H_2 + Z_0 + Z_1 + \mathcal{R}, \tag{11.2}$$

H_2' like (8.12), Z_0 like (10.2), Z_1 like (10.1), $\mathcal{R} = \sum_{j=2,3} \mathbf{R}_j + \mathcal{R}^{1,2}_{k,m}(\Pi_4, \Pi(f), f)$.

We recall that, in the context of Strichartz estimates, a pair (p, q) is called *admissible* if

$$2/p + 3/q = 3/2, \qquad 2 \leqslant q \leqslant 6, \qquad p \geqslant 2. \tag{11.3}$$

Theorem 11.1. *For the constants $0 < \epsilon < \epsilon_0$ of Theorem 1.1, there is a fixed $C > 0$ such that*

$$\|f\|_{L^p_t(\mathbb{R}_+, W^{1,q}_x)} \leqslant C\epsilon \quad \textit{for all admissible pairs } (p, q), \tag{11.4}$$

$$\|z^\mu\|_{L^2_t(\mathbb{R}_+)} \leqslant C\epsilon \quad \textit{for all multi-indexes } \mu \textit{ with } \mathbf{e}\cdot\mu > \omega_1, \tag{11.5}$$

$$\|z\|_{W^{1,\infty}_t(\mathbb{R}_+)} \leqslant C\epsilon. \tag{11.6}$$

Furthermore, we have $\lim_{t\to+\infty} z(t) = 0$.

By standard arguments that we skip, such as a simpler version of [19, Sect. 7], Theorem 11.1 is a consequence of the following continuity argument.

Proposition 11.2. *For the constants $0 < \epsilon < \epsilon_0$ of Theorem 1.1, there exists a constant $\kappa > 0$ such that for any $C_0 > \kappa$ there is $\epsilon_0 > 0$ such that if the inequalities (11.4)–(11.6) hold for $I = [0, T]$ for some $T > 0$ and for $C = C_0$, then in fact the inequalities (11.4)–(11.6) hold for $I = [0, T]$ for $C = C_0/2$.*

We now discuss the proof of Proposition 11.2, which is similar to the proof for the scalar NLS; see for example [19] or [18]. We have, see (9.6), $\dot{f} = (X_H^{(0)})_f + (X_H^{(1)})_f$.

In [18], the equation was $\dot{f} = (X_H^{(0)})_f$. Given multi-indexes $\Theta', \Theta \in \mathbb{N}_0^m$ we write $\Theta' < \Theta$ if $\Theta' \neq \Theta$ and $\Theta'_l \leqslant \Theta_l$, $1 \leqslant l \leqslant m$. We now introduce

$$\mathbf{M}_0 = \left\{\mu \in \mathbb{N}_0^{\mathbf{n}} : \ |\mathbf{e}\cdot\mu| > \omega^1\,, \quad |\mu| \leqslant 2\mathbf{N}+2, \quad |\mathbf{e}\cdot\mu'| < \omega^1 \text{ if } \mu' < \mu\right\}, \tag{11.7}$$

$$\begin{aligned}\mathbf{M} = \big\{(\mu,\nu) \in \mathbb{N}_0^{2\mathbf{n}} : \ |\mathbf{e}\cdot(\mu-\nu)| > \omega^1\,, \quad |\mu+\nu| \leqslant 2\mathbf{N}+2 \ \text{ and} \\ |\mathbf{e}\cdot(\mu'-\nu')| < \omega^1 \text{ if } (\mu',\nu') < (\mu,\nu)\big\}.\end{aligned} \tag{11.8}$$

Notice that

$$\text{if } (\mu,\nu) \in \mathbf{M} \text{ we have either } \mu = 0 \text{ and } \nu \in \mathbf{M}_0, \text{ or } \nu = 0 \text{ and } \mu \in \mathbf{M}_0. \tag{11.9}$$

In [19, 18] it is shown that for $G^0_{\mu\nu} := G_{\mu\nu}(p^0, 0)$ we have

$$(X_H^{(0)})_f = \mathcal{H}_{p^1} f + \sum_{j=1,\dots,7} (\partial_{\Pi_j(f)} H) P_c \mathbb{i}\sigma_3 \Diamond_j f - \sum_{(\mu,\nu)\in\mathbf{M}} z^\mu \overline{z}^\nu G^0_{\mu\nu} + R_1 + R_2, \tag{11.10}$$

P_c the projection on X_c in (7.4), and there is a constant $C(C_0)$ independent of ϵ such that

$$\|R_1\|_{L^1_t([0,T],H^1)} + \|R_2\|_{L^2_t([0,T],W^{1,\frac{6}{5}})} \leqslant C(C_0)\epsilon^2. \tag{11.11}$$

We sketch briefly this point. With $\widehat{\nabla}_f$ defined in (7.12), we define

$$R_2 = \sum_{(\mu,\nu)\in \mathbf{M}} z^\mu \overline{z}^\nu \left(G^0_{\mu\nu} - G_{\mu\nu}\right) - \mathbb{i}\sigma_3 \widehat{\nabla}_f \mathbf{R}_2 - \mathbb{i}\sigma_3 B_2 f,$$

where the last term is defined schematically from $\widehat{\nabla}_f \langle B_2, f^2\rangle \sim \langle \widehat{\nabla}_f B_2, f^2\rangle + B_2 f$. Then the desired estimate on R_2 in (11.11) is elementary. For example,

$$\|B_2 f\|_{L^2([0,T],W^{1,\frac{6}{5}})} \leqslant \|B_2\|_{L^\infty([0,T],L^{3/2})} \|f\|_{L^2([0,T],W^{1,6})} \lesssim \epsilon \|f\|_{L^2([0,T],W^{1,6})} \lesssim \epsilon^2$$

by (8.13) and (11.4) in $[0,T]$. R_1 is formed by the other terms and it is standard to show that it satisfies the bound (11.11). For example, for $2 \leqslant d \leqslant 4$,

$$\begin{aligned}\|\langle \widehat{\nabla}_f B_d, f^d\rangle\|_{L^1_t H^1_x} &\leqslant \Big\| \sup_{\|g\|_{H^{-1}}=1} \langle \widehat{\nabla}_f B_d g, f^d\rangle \Big\|_{L^1_t} \\ &\leqslant \left\| \sup_{\|g\|_{H^{-1}}=1} \|\widehat{\nabla}_f B_d g\|_{\Sigma_k} \|f^d\|_{L^{6/d}_x} \right\|_{L^1_t} \lesssim \|f\|^2_{L^2_t L^6_x} \|f\|^{d-2}_{L^\infty_t H^1_x} \lesssim \epsilon^d\end{aligned}$$

and for $d = 3, 4$, for $(d-1, q_d)$ admissible,

$$\|B_d f^{d-1}\|_{L^1_t H^1_x} \lesssim \|f\|_{L^\infty_t H^1_x} \|f\|^{d-1}_{L^{d-1}_t L^{q_d}_x} \lesssim \epsilon^{d-1}. \tag{11.12}$$

The $d = 5$ term can be treated similarly, but has an additional part when the f derivative is applied to the ζ variable in (8.15). But the resulting term is like (11.12) for $d = 6$. Finally, $\|\nabla E_P(f)\|_{L^1_t H^1_x} \lesssim \epsilon^2$ by hypotheses (H1)–(H2). Having discussed (11.11), by (9.8) we get

$$\begin{aligned} X^{(1)}_H = \mathcal{R}^{0,0}_{\infty,\infty}(\Pi(f)) \big[&\langle \Diamond f, \mathcal{H}_{p^1} f\rangle + (\partial_{\Pi(f)} H)\langle \Diamond f, \mathbb{i}\sigma_3 \Diamond f\rangle + \langle \Diamond f, R_1 + R_2\rangle \\ &- \sum_{(\mu,\nu)\in\mathbf{M}} z^\mu \overline{z}^\nu \langle \Diamond f, G^0_{\mu\nu}\rangle \big] P_c \mathbb{i}\sigma_3 \Diamond f. \end{aligned} \tag{11.13}$$

Then, for $\mathbf{v}$ obtained summing contributions from (11.13) and the $\sum_{j=1,\dots,7}$ in (11.10), we obtain

$$\dot{f} - \left(\mathcal{H}_{p^1} f + P_c \mathbb{i}\sigma_3 \mathbf{v}\cdot\Diamond f\right) = -\sum_{(\mu,\nu)\in\mathbf{M}} z^\mu \overline{z}^\nu G^0_{\mu\nu} + R_1 + R_2. \tag{11.14}$$

It is easy to see from (11.4)–(11.6) and (11.11) that

$$\|\mathbf{v}\|_{L^1([0,T],\mathbb{R}^7)+L^\infty([0,T],\mathbb{R}^7)} \leqslant C(C_0)\epsilon. \tag{11.15}$$

Strichartz and smoothing estimates on f are a consequence of well-known estimates for the group $e^{t\mathcal{H}_{p^1}} P_c$ which resemble those valid for $e^{\mathbb{i}t\Delta}$; see [16] for references.

To deal with the term $P_c \mathbb{i}\sigma_3 \mathbf{v}\cdot\Diamond f$, where the operator $P_c \mathbb{i}\sigma_3 \mathbf{v}\cdot\Diamond$ does not commute with $\mathcal{H}_{p^1}$ we adopt an idea by Beceanu [4]. We consider the system $\dot{f} = \mathbb{i}\sigma_3 \mathbf{v}\cdot\Diamond f$, writing it in the form

$$\dot{f} = A(t)f + B(t)f\ ,\ A(t) := \sum_{j=1,\dots,4} \mathbb{i}\sigma_3 \mathbf{v}_j(t)\Diamond_j \text{ and } B(t) := \sum_{j=5,6,7} \mathbb{i}\sigma_3 \mathbf{v}_j(t)\Diamond_j. \tag{11.16}$$

Since $A(t)$ and $B(t)$ commute and the terms of the sum defining $A(t)$ commute, if we denote by $W(t,s)$ the fundamental solution of the system (11.16), that is,

$$\partial_t W(t,s) = (A(t) + B(t))W(t,s) \text{ with } W(s,s) = I, \tag{11.17}$$

and by $W_A(t,s) = e^{\int_s^t A(t')\,dt'}$ (resp. $W_B(t,s)$) the fundamental solution of $\dot{f} = A(t)f$ (resp. $\dot{f} = B(t)f$), then we have $W(t,s) = W_A(t,s)W_B(t,s)$.

Lemma 11.3. *Let $M > 5/2$ and $\alpha \in [0, 1/2)$. Then there exists a constant $C > 0$ dependent only on M such that for all $s < t$ in $[0, T]$*

$$\begin{aligned}\|\langle x\rangle^{-M}\,(W(t,s)-1)\,e^{\mathrm{i}\sigma_3(\Delta-\omega^1)(t-s)}\langle x\rangle^{-M}\|_{B(L^2,L^2)}\\ \leqslant C\psi_\alpha(t-s)\|\mathbf{v}\|^\alpha_{L^1([s,t])+L^\infty([s,t])}\end{aligned} \tag{11.18}$$

with $\psi_\alpha(t) = \langle t\rangle^{-\frac{3}{2}+\alpha}$ for $t \geqslant 1$ and $\psi_\alpha(t) = t^{-\alpha}$ for $t \in (0, 1)$.

Proof. We have

$$W(t,s)-1 = [(W_A(t,s)-1)W_B(t,s)] + [W_B(t,s)-1]\,. \tag{11.19}$$

In the first term in the r.h.s. $W_B(t,s)$ commutes with the other operators and is an isometry in L^2:

$$\begin{aligned}&\|\langle x\rangle^{-M}W_A(t,s)-1)W_B(t,s)e^{\mathrm{i}\sigma_3(\Delta-\omega^1)(t-s)}\langle x\rangle^{-M}\|_{B(L^2,L^2)}\\ &=\|\langle x\rangle^{-M}W_A(t,s)-1)e^{\mathrm{i}\sigma_3(\Delta-\omega^1)(t-s)}\langle x\rangle^{-M}\|_{B(L^2,L^2)}.\end{aligned}$$

Then the desired estimate of this is that of [19, Lemma 9.4]. We next consider the second term in the r.h.s. of (11.19). By the commutation properties of $W_B(t,s)$ we are reduced to bound

$$\|\langle x\rangle^{-M}e^{\mathrm{i}\sigma_3(\Delta-\omega^1)(t-s)}\langle x\rangle^{-M}\|_{B(L^2,L^2)}\left(\int_s^t\|B(t')W_B(t',s)dt'\|_{B(L^2,L^2)}\right)^\alpha.$$

The first factor is bounded by $c_0\langle t-s\rangle^{-\frac{3}{2}}$, the second by $|t-s|^\alpha\|B\|^\alpha_{L^\infty((s,t),B(L^2,L^2))}$, where the last factor is bounded by $\|\mathbf{v}\|^\alpha_{L^\infty((s,t),\mathbb{R}^7)}$. □

Proposition 11.4. *Let $F(t)$ satisfy $P_cF(t) = F(t)$ Consider the equation*

$$\dot{u} - \mathcal{H}_{p^1}u - P_c\mathrm{i}\sigma_3\mathbf{v}\cdot\Diamond u = F. \tag{11.20}$$

Then there exist fixed $\sigma > 3/2$, and an $\epsilon_0 > 0$ such that if $\epsilon \in (0, \epsilon_0)$ then we have

$$\|u\|_{L^p([0,T],W^{1,q})} \leqslant C(\|P_cu(0)\|_{H^1} + \|F\|_{L^2([0,T],H^{1,\sigma})+L^1([0,T],H^1)})\,, \tag{11.21}$$

for any admissible pair (p, q).

Before the proof, we observe that Proposition 11.4 implies the following.

Corollary 11.5. *Under the hypotheses of Theorem 11.1 there exist two constants c_0 and $\epsilon_0 > 0$ such that if $\epsilon \in (0, \epsilon_0)$ then*

$$\|f\|_{L^p_t([0,T],W^{1,q}_x)} \leqslant c_0\epsilon + c_0\sum_{(\mu,\nu)\in\mathbf{M}}\|z^{\mu+\nu}\|_{L^2(0,T)},\quad \text{for any admissible pair } (p,q). \tag{11.22}$$

For the elementary proof of this corollary, see for instance [19, Lemma 8.1].

Proof of Proposition 11.4. We follow [4]. Denote $u_0 = P_cu(0)$. We set $P_d := 1 - P_c$, fix $\delta > 0$ and consider

$$\dot{Z} - \mathcal{H}_{p^1}P_cZ - P_c\mathrm{i}\sigma_3\mathbf{v}\cdot\Diamond P_cZ = F - \delta P_dZ\,,\quad Z(0) = u_0. \tag{11.23}$$

Notice that, see (2.24),

$$\mathcal{H}_{p^1} = \mathrm{i}\sigma_3(-\Delta+\omega^1) + V \text{ with } V \in \mathcal{S}(\mathbb{R}^3, B(\mathbb{C}^2,\mathbb{C}^2)); \tag{11.24}$$

we then rewrite (11.23) as

$$\dot{Z} - \mathrm{i}\sigma_3(\Delta-\omega^1)Z - \mathrm{i}\sigma_3\mathbf{v}\cdot\Diamond Z = F + V_1V_2Z - \widetilde{P}_d(\mathbf{v})Z \ \text{ with } \ Z(0) = u_0,$$

$\widetilde{P}_d(\mathbf{v}) := P_d \mathrm{i}\sigma_3 \mathbf{v}\cdot\Diamond + \mathrm{i}\sigma_3\mathbf{v}\cdot\Diamond P_d$ and $V_1V_2 = V - \mathcal{H}_{p^1}P_d - \delta P_d$ with $V_2(x)$ a smooth exponentially decaying and invertible matrix, and with the multiplication operator $V_1 : H^{k,s'} \to H^{k,s}$ bounded for all k, s and s'. We have:

$$Z(t) = W(t,0)e^{\mathrm{i}\sigma_3(-\Delta+\omega^1)t}Z(0) \\ + \int_0^t e^{\mathrm{i}\sigma_3(-\Delta+\omega^1)(t-t')}W(t,t')\left[F(t') + V_1V_2Z(t') - \widetilde{P}_d(\mathbf{v}(t'))Z(t')\right] dt'. \tag{11.25}$$

For arbitrarily fixed pairs (K,S) and (K',S') there exists a constant C such that we have

$$\|\widetilde{P}_d(\mathbf{v})V_2^{-1}\|_{B(H^{-K',-S'},H^{K,S})} \leqslant C\epsilon.$$

By picking ϵ small enough, we can assume that the related operator norm is small. We have

$$\|Z\|_{L^p_tW^{q,1}\cap L^2_tH^{k,-\tau_0}} \leqslant C\|Z(0)\|_{H^1} + C\|F\|_{L^1_tH^1+L^2_tH^{1,\tau_0}} \\ + \|V_1 - \widetilde{P}_d(\mathbf{v}(t'))V_2^{-1}\|_{L^\infty_t(B(H^1,H^{1,\tau_0}))}\|V_2Z(t)\|_{L^2_tH^1}.$$

For $\widetilde{T}_0f(t) = V_2\int_0^t e^{\mathrm{i}\sigma_3(-\Delta+\omega^1)(t-t')}W(t,t')V_1f(t')dt'$, by (11.25), we obtain:

$$(I-\widetilde{T}_0)V_2Z(t) = V_2W(t,0)e^{\mathrm{i}\sigma_3(-\Delta+\omega^1)t}Z(0) \\ - V_2\int_0^t e^{\mathrm{i}\sigma_3(-\Delta+\omega^1)(t-t')}W(t,t')\left[F(t') - \widetilde{P}_d(\mathbf{v}(t'))Z(t')\right] dt'.$$

We then obtain the desired result if we can show that

$$\|(I-\widetilde{T}_0)^{-1}\|_{L^2([0,T),H^1(\mathbb{R}^3))\circlearrowleft} < C_1, \tag{11.26}$$

for ϵC_1 smaller than a fixed number. Thanks to Lemma 11.3 it is enough to prove (11.26) with $\widetilde{T}_0$ replaced by

$$T_0f(t) = V_2\int_0^t e^{\mathrm{i}\sigma_3(-\Delta+\omega^1)(t-t')}V_1f(t')dt'.$$

Set

$$T_1f(t) = V_2\int_0^t e^{(-\mathcal{H}_{p^1}P_c+\delta P_d)(t'-t)}V_1f(t')\,dt'.$$

By [15] we have $\|T_1\|_{L^2([0,T),H^1(\mathbb{R}^3))\circlearrowleft} < C_2$ for a fixed C_2. By elementary arguments, see [27],

$$(I-T_0)(I+T_1) = (I+T_1)(I-T_0) = I.$$

This yields (11.26) with $\widetilde{T}_0$ replaced by T_0 and with $C_1 = 1 + C_2$. □

Now we turn to the equations $\dot{z}_l = \imath\partial_{\overline{z}_l}H$. We will prove the following.

Proposition 11.6. *There exists a fixed $c_0 > 0$ and a constant $\epsilon_0 > 0$ which depends on C_0 such that*

$$\sum_l |z_l(t)|^2 + \sum_{(\mu,\nu)\in\mathbf{M}} \|z^{\mu+\nu}\|^2_{L^2(0,t)} \leqslant c_0(1+C_0)\epsilon^2, \quad \forall t\in[0,T], \quad \forall\epsilon\in(0,\epsilon_0). \tag{11.27}$$

Proposition 11.6 allows to conclude the proof of Proposition 11.2. The proof of Proposition 11.6 follows a series of standard steps, and is basically the same as the analogous proof in [16], or in [3].

The first step in the proof of Proposition 11.2 consists in splitting f as follows:

$$g = f + Y\,, \quad Y := -\imath\sum_{(\mu,\nu)\in\mathbf{M}} z^\mu\overline{z}^\nu R^+_{\imath\mathcal{H}_{p^1}}(\mathbf{e}\cdot(\nu-\mu))G^0_{\mu\nu}, \tag{11.28}$$

where $R^+_{\imath\mathcal{H}_{p^1}}$ is extension from above of the resolvent and makes sense because the theory of Jensen and Kato [26] holds also for these operators; see for example Perelman [30, Appendix 4].

The part of f that acts effectively on the variables z will be shown to be Y, while g is small, thanks to the following lemma.

Lemma 11.7. *For fixed $s > 1$ there exist a fixed c such that if ϵ_0 is sufficiently small we have $\|g\|_{L^2((0,T),H^{0,-s}(\mathbb{R}^3,\mathbb{C}^4)} \leqslant c\epsilon$.*

Proof. In the same way as the proof of Proposition 11.4 (which we wrote explicitly) is similar to analogous proofs valid for the scalar NLS (1.3), the proof of Lemma 11.7 is analogous to the proof of [19, Lemma 8.5] contained in [19, Sect. 10] and is skipped here. The only difference between [19] and the present situation is notational, in the sense that inside (11.20) one has $\mathrm{i}\sigma_3\mathbf{v}\cdot\Diamond u = \mathrm{i}\sigma_3\sum_{j\leqslant 7}\mathbf{v}_j\Diamond_j u$, as opposed to [19, (10.1)], where the corresponding terms are $\mathrm{i}\sigma_3\sum_{j=1}^4\mathbf{v}_j\Diamond_j u$. But this does not make any difference in the proof because what matters is simply that each $\Diamond_j$ commutes with $-\Delta+\omega^1$, which was used to get (11.25). □

Now we examine the equations on z. We have

$$-\imath\dot z_j = \partial_{\overline{z}_j}(H_2 + Z_0 + Z_1 + \mathcal{R}).$$

When we substitute (11.28) and we set $R^+_{\mu\nu} := R^+_{\imath\mathcal{H}_{p^1}}(\mathbf{e}\cdot(\nu-\mu))$ we obtain

$$\begin{aligned}-\imath\dot z_l - \partial_{\overline{z}_l}H_2 = \partial_{\overline{z}_l}Z_0 + \imath\sum_{(\alpha,\beta),(\mu,\nu)\in\mathbf{M}}\nu_l\frac{z^{\mu+\alpha}\overline{z}^{\nu+\beta}}{\overline{z}_l}\langle R^+_{\alpha\beta}G^0_{\alpha\beta}, \mathrm{i}\sigma_3 G_{\mu\nu}\rangle\\ + \sum_{(\mu,\nu)\in\mathbf{M}}\nu_l\frac{z^\mu\overline{z}^\nu}{\overline{z}_l}\langle g, \mathrm{i}\sigma_3 G_{\mu\nu}\rangle + \partial_{\overline{z}_l}\mathcal{R}.\end{aligned} \tag{11.29}$$

Using (11.9), we rewrite this as

$$-\imath\dot z_j - \partial_{\overline{z}_j}H_2 = \partial_{\overline{z}_j}Z_0 + \sum_{(\mu,\nu)\in\mathbf{M}}\nu_j\frac{z^\mu\overline{z}^\nu}{\overline{z}_j}\langle g, \mathrm{i}\sigma_3 G_{\mu\nu}\rangle + \mathcal{E}_j \tag{11.30}$$

$$+\imath\sum_{\beta,\nu\in\mathbf{M}_0}\nu_j\frac{\overline{z}^{\nu+\beta}}{\overline{z}_j}\langle R^+_{0\beta}G^0_{0\beta}, \mathrm{i}\sigma_3 G^0_{0\nu}\rangle \tag{11.31}$$

$$+\imath\sum_{\alpha,\nu\in\mathbf{M}_0}\nu_j\frac{z^\alpha\overline{z}^\nu}{\overline{z}_j}\langle R^+_{\alpha 0}G^0_{\alpha 0}, \mathrm{i}\sigma_3 G^0_{0\nu}\rangle. \tag{11.32}$$

Here the elements in (11.31) can be eliminated through a new change of variables that we will see momentarily and $\mathcal{E}_j$ is a remainder term defined by

$$\mathcal{E}_j := \sum_{(\mu,\nu)\in\mathbf{M}}\nu_j\frac{z^\mu\overline{z}^\nu}{\overline{z}_j}\langle g, \mathrm{i}\sigma_3 G_{\mu\nu}\rangle + \partial_{\overline{z}_j}\mathcal{R} - (11.31) - (11.32). \tag{11.33}$$

Set $\zeta_l = z_l + F_l(z,\overline{z})$ with

$$\begin{aligned}F_l(z,\overline{z}) = \sum_{\beta,\nu\in\mathbf{M}_0}\frac{\imath\nu_l\overline{z}^{\nu+\beta}}{\mathbf{e}\cdot(\beta+\nu)\overline{z}_l}\langle R^+_{0\beta}G^0_{0\beta}, \mathrm{i}\sigma_3 G^0_{0\nu}\rangle\\ - \sum_{\substack{\alpha,\nu\in\mathbf{M}_0\\ \mathbf{e}\cdot\alpha\neq\mathbf{e}\cdot\nu}}\frac{\imath\nu_l z^\alpha\overline{z}^\nu}{\mathbf{e}\cdot(\alpha-\nu)\overline{z}_l}\langle R^+_{\alpha 0}G^0_{\alpha 0}, \mathrm{i}\sigma_3 G^0_{0\nu}\rangle.\end{aligned}$$

This change of variables is such that, setting $F=(F_1,...,F_{\mathbf{n}})$, we get

$$\begin{aligned}\mathfrak{L}_j(z,f)|_{f=0}:&=\sum_{l=1,\dots,\mathbf{n}}(\partial_{\overline{z}_l}F_j(z,\overline{z})\partial_{z_l}H_2(z,0)-\partial_{z_l}F_j(z,\overline{z})\partial_{\overline{z}_l}H_2(z,0))\\&=\partial_{\overline{z}_l}H_2(F(z,\overline{z}),0)+(11.31)+(11.32).\end{aligned}$$

Furthermore, by $\nu\in\mathbf{M}_0$, which implies $\nu\cdot\mathbf{e}>\omega^1$, we have $|\nu|>1$. Then, by (11.5)–(11.6),

$$\|\zeta-z\|_{L^2(0,T)}\leqslant C\epsilon\sum_{\alpha\in\mathbf{M}_0}\|z^\alpha\|_{L^2(0,T)}\leqslant C(C_0)\epsilon^2,\quad \|\zeta-z\|_{L^\infty(0,T)}\leqslant C(C_0)\epsilon^3. \tag{11.34}$$

In the new ζ variables, (11.30) takes the form

$$-\imath\dot{\zeta}_j=\partial_{\overline{\zeta}_j}H_2(\zeta,f)+\partial_{\overline{\zeta}_j}Z_0(\zeta,f)+\mathcal{D}_j+\imath\sum_{\substack{\alpha,\nu\in\mathbf{M}_0\\ \mathbf{e}\cdot\alpha=\mathbf{e}\cdot\nu}}\nu_j\frac{\zeta^\alpha\overline{\zeta}^\nu}{\overline{\zeta}_j}\langle R^+_{\alpha 0}G^0_{\alpha 0},\mathrm{i}\sigma_3G^0_{0\nu}\rangle, \tag{11.35}$$

with for A_l =r.h.s. of (11.29),

$$\mathcal{D}_j=\mathcal{E}_j+\mathfrak{L}_j(z,0)-\mathfrak{L}_j(z,f)+\sum_{l=1,\dots,\mathbf{n}}\left(\partial_{z_l}F_j(z,\overline{z})A_l-\partial_{\overline{z}_l}F_j(z,\overline{z})\overline{A}_l\right). \tag{11.36}$$

From these equations by $\sum_l\mathbf{e}_l\big(\overline{\zeta}_l\partial_{\overline{\zeta}_l}(H_2+Z_0)-\zeta_j\partial_{\zeta_l}(H_2+Z_0)\big)=0$ we get

$$\begin{aligned}&\partial_t\sum_{l=1,\dots,\mathbf{n}}\mathbf{e}_l|\zeta_l|^2\\&=2\sum_{l=1,\dots,\mathbf{n}}\mathbf{e}_l\mathrm{Im}\left(\mathcal{D}_l\overline{\zeta}_l\right)+2\sum_{\substack{\alpha,\nu\in\mathbf{M}_0\\ \mathbf{e}\cdot\alpha=\mathbf{e}\cdot\nu}}\mathbf{e}\cdot\nu\,\mathrm{Re}\left(\zeta^\alpha\overline{\zeta}^\nu\langle R^+_{\alpha 0}G^0_{\alpha 0},\mathrm{i}\sigma_3G^0_{0\nu}\rangle\right).\end{aligned} \tag{11.37}$$

Lemma 11.8. *Assume inequalities* (11.4)–(11.6). *Then for a fixed constant c_0 we have*

$$\sum_{j=1,\dots,\mathbf{n}}\|\mathrm{Im}\left(\mathcal{D}_j\overline{\zeta}_j\right)\|_{L^1[0,T]}\leqslant(1+C_0)c_0\epsilon^2. \tag{11.38}$$

Proof (sketch). For a detailed proof we refer to [3, Appendix B]: here we give a sketch. First of all, we consider the contribution of $\mathcal{E}_j$. This, in turn, is a sum of various terms. For the terms originating from $\mathbf{R}_3$, cf. Lemma 8.4, we have

$$\left\|\left\langle\partial_{\overline{z}_j}B_d(p_4^0,\Pi(f),z,f),f^d\right\rangle\zeta_j\right\|_{L^1_t}\leqslant\|f\|^d_{L^d_tL^{p_d}_x}\|\zeta\|_{L^\infty_t}\lesssim\epsilon^{d+1},$$

with (d,p_d) admissible, and for $d=2,3,4,5$. For the following term, we claim

$$\|\partial_{\overline{z}_j}\mathbf{R}_2\zeta_j\|_{L^1_t}\lesssim\epsilon^3. \tag{11.39}$$

From Lemma 8.4 we know that $\mathbf{R}_2$ is basically a sum of degree $2\mathbf{N}+3$ monomials in $(z,\overline{z},f)$, which are at most degree 1 in f. Let us take a term which is degree 0 in f. Then its $\partial_{\overline{z}_j}$ derivative is in absolute value bounded above by a term $|z^{\mu+\nu}|$ with $|\mu|+|\nu|\geqslant 2\mathbf{N}+2$. So we can write it as $|z^{\alpha+\beta+\gamma}|$ with $|\alpha|\geqslant\mathbf{N}+1$, $|\beta|\geqslant\mathbf{N}+1$. But then $\alpha\cdot\mathbf{e}>\omega^1$ $\beta\cdot\mathbf{e}>\omega^1$. Then

$$\|z^{\alpha+\beta+\gamma}\zeta_j\|_{L^1_t}\leqslant\|z^\alpha\|_{L^2_t}\|z^\beta\|_{L^2_t}\|\zeta_j\|_{L^\infty_t}\lesssim\epsilon^3.$$

Terms of degree 1 in f can be treated similarly, yielding (11.39). We claim also

$$\left\|\nu_j\frac{z^{\mu+\alpha}\overline{z}^{\nu+\beta}}{\overline{z}_j}\zeta_j\right\|_{L^1_t}\lesssim\epsilon^3\quad\text{for }|(\mu-\nu)\cdot\mathbf{e}|>\omega^1\text{ and }(\mu,\nu)\notin\mathbf{M}. \tag{11.40}$$

In this case we can write $z^\mu \overline{z}^\nu = z^{\mu'} \overline{z}^{\nu'} z^\gamma \overline{z}^\delta$ with $(\mu', \nu') \in \mathbf{M}$ and $|\gamma| + |\delta| > 0$. Then we consider

$$\nu_j \frac{z^{\mu+\alpha} \overline{z}^{\nu+\beta}}{\overline{z}_j} \overline{\zeta}_j = \nu_j z^{\mu'+\alpha} \overline{z}^{\nu'+\beta} z^\gamma \overline{z}^\delta + \nu_j \frac{z^{\mu'+\alpha} \overline{z}^{\nu'+\beta}}{\overline{z}_j} (\overline{\zeta}_j - \overline{z}_j).$$

By (11.5) and (11.6),

$$\|z^{\mu'+\alpha} \overline{z}^{\nu'+\beta} z^\gamma \overline{z}^\delta\|_{L^1_t} \lesssim \|z^{\mu'} \overline{z}^{\nu'}\|_{L^2_t} \|z^\alpha \overline{z}^\beta\|_{L^2_t} \|z\|_{L^\infty_t}^{|\gamma|+|\delta|} \lesssim \epsilon^3,$$

and by (11.34)

$$\|\nu_j \frac{z^{\mu+\alpha} \overline{z}^{\nu+\beta}}{\overline{z}_j} (\overline{\zeta}_j - \overline{z}_j)\|_{L^1_t} \lesssim \|z^\alpha \overline{z}^\beta\|_{L^2_t} \|z - \zeta\|_{L^2_t} \lesssim \epsilon^3.$$

This yields (11.40). By similar arguments, one can prove

$$\|\nu_j \frac{z^\mu \overline{z}^\nu}{\overline{z}_j} \langle g, \mathrm{i}\sigma_3 G_{\mu\nu} \rangle \zeta_j\|_{L^1_t} \lesssim \epsilon^3 \quad \text{for } |(\mu - \nu) \cdot \mathbf{e}| > \omega^1 \text{ and } (\mu, \nu) \notin \mathbf{M}.$$

We next consider the following, see [3, Lemma B.1],

$$\|\overline{\partial}_j (Z_0(\zeta, f) - Z_0(z, f)) \zeta_j\|_{L^1_t} \lesssim \epsilon^3. \tag{11.41}$$

Is enough to consider $z^\alpha \frac{\overline{z}^\beta}{\overline{z}_j} \overline{\zeta}_j - \zeta^\alpha \frac{\overline{\zeta}^\beta}{\overline{\zeta}_j} \overline{\zeta}_j$ with $\mathbf{e} \cdot \alpha = \mathbf{e} \cdot \beta$ and $\beta_j > 0$. By the Taylor expansion these are

$$\sum_k \partial_k \left(\frac{z^\alpha \overline{z}^\beta}{\overline{z}_j} \right) (\zeta_k - z_k) \overline{\zeta}_j + \sum_k \overline{\partial}_k \left(\frac{z^\alpha \overline{z}^\beta}{\overline{z}_j} \right) (\overline{\zeta}_k - \overline{z}_k) \overline{\zeta}_j + \overline{\zeta}_j O(|z - \zeta|^2).$$

The remainder term is the easiest, the other two terms similar. Substituting the definition of ζ, a typical term in the first summation is $\frac{z^{\alpha+A} \overline{z}^{B+\beta}}{|z_k|^2}$, with $\alpha \cdot \mathbf{e} > \omega^1$, $\beta \cdot \mathbf{e} > \omega^1$, $A \cdot \mathbf{e} > \omega^1$ and $B \cdot \mathbf{e} > \omega^1$. and with $\alpha_k \neq 0 \neq B_k$. By (H8), $\mathbf{e} \cdot \alpha = \mathbf{e} \cdot \beta$ implies that there is at least one index $\beta_\ell \neq 0$ such that $\mathbf{e}_\ell = \mathbf{e}_k$. Then, by the fact that monomials $z^\alpha \overline{z}^\beta$ in Z_0 are such that $|\alpha| = |\beta| \geqslant 2$,

$$\left\| \frac{z^\alpha \overline{z}^\beta z^A \overline{z}^B}{|z_k|^2} \right\|_{L^1_t} \leqslant \|z^A\|_{L^2_t} \left\| \frac{z^B z_\ell}{z_k} \right\|_{L^2_t} \left\| \frac{z^\alpha \overline{z}^\beta}{z_\ell z_k} \right\|_{L^\infty_t} \lesssim C_2^2 \epsilon^{|\alpha|+|\beta|} \leqslant C_2^2 \epsilon^4. \tag{11.42}$$

Other contributions from (11) can be treated similarly, yielding (11.41).

The main contribution to the l.h.s. of (11.38) is originated from the following terms:

$$\left\| \nu_j \frac{z^\mu \overline{z}^\nu}{\overline{z}_j} \langle g, \mathrm{i}\sigma_3 G_{\mu\nu} \rangle \overline{\zeta}_j \right\|_{L^1_t} \leqslant c_1 C_0 \epsilon^2 \text{ for } (\mu, \nu) \in \mathbf{M} \tag{11.43}$$

with c_1 a fixed constant. Indeed the term to bound equals

$$\nu_j z^\mu \overline{z}^\nu \langle g, \mathrm{i}\sigma_3 G_{\mu\nu} \rangle + \nu_j \frac{z^\mu \overline{z}^\nu}{\overline{z}_j} \langle g, \mathrm{i}\sigma_3 G_{\mu\nu} \rangle (\overline{\zeta}_j - \overline{z}_j).$$

By Lemma 11.7, the first term has L^1_t norm bounded by

$$\|G_{\mu\nu}\|_{L^\infty_t H^{0,s}} \|z^\mu \overline{z}^\nu\|_{L^2_t} \|g\|_{L^2_t H^{0,-s}} \leqslant \|G_{\mu\nu}\|_{L^\infty_t H^{0,s}} C_0 \epsilon c \epsilon \leqslant c_1 C_0 \epsilon^2$$

for a fixed c_1. The second term has L^1_t norm bounded by the following, which yields (11.43),

$$\left\| \nu_j \frac{z^\mu \overline{z}^\nu}{\overline{z}_j} \right\|_{L^\infty_t} \|G_{\mu\nu}\|_{L^\infty_t H^{0,s}} \|g\|_{L^2_t H^{0,-s}} \|\zeta - z\|_{L^2_t} \lesssim \epsilon^4.$$

We estimated the contribution to the l.h.s. of (11.38) of $\mathcal{E}_j$. There are further terms in (11.36) to estimate. We claim

$$\|(\mathfrak{L}_j(z,0) - \mathfrak{L}_j(z,f))\overline{\zeta}_j\|_{L^\infty_t} \lesssim \epsilon^4. \tag{11.44}$$

A typical contribution to the l.h.s. is

$$(g(\Pi(f)) - g(\Pi(0))\,\frac{\nu_j \overline{z}^{\nu+\beta}}{\overline{z}_j}(\overline{z}_j + (\overline{\zeta}_j - \overline{z}_j)) \text{ with } \alpha,\nu \in \mathbf{M}_0,$$

with $g \in C^1(\mathbb{R}^7,\mathbb{C})$. We can bound its L^1_t norm using

$$\|f\|^2_{L^\infty H^1}\|z^\nu\|_{L^2}\|z^\beta\|_{L^2} \lesssim \epsilon^4$$

and using the argument that leads to (11.42). For the discussion of the bound for the contribution originating from the $\sum_{l=1,\dots,\mathbf{n}}$ term in (11.36), which is also higher order; see [3]. □

The second term in the r.h.s. of (11.37) equals, using $G^0_{\mu\nu} = \overline{G^0}_{\nu\mu}$,

$$\begin{aligned}&2\sum_{\kappa\in\mathfrak{K}} \kappa \operatorname{Re}\left\langle R^+_{\boldsymbol{\imath}\mathcal{H}_{p^1}}(-\kappa) \sum_{\alpha\in\mathbf{M}_0\,,\ \mathbf{e}\cdot\alpha=\kappa} \zeta^\alpha G^0_{\alpha 0}, \mathbb{i}\sigma_3 \sum_{\nu\in\mathbf{M}_0\,,\ \mathbf{e}\cdot\nu=\kappa} \overline{\zeta}^\nu G^0_{0\nu}\right\rangle \\ &= \pi^{-1}\sum_{\kappa\in\mathfrak{K}} \kappa \operatorname{Re}\left\langle R^+_{\boldsymbol{\imath}\mathcal{H}_{p^1}}(-\kappa)\mathbf{G}, \mathbb{i}\sigma_3\overline{\mathbf{G}}\right\rangle \quad \text{for } \mathbf{G} := \sqrt{2\pi}\sum_{\alpha\in\mathbf{M}_0\,,\ \mathbf{e}\cdot\alpha=\kappa} \zeta^\alpha G^0_{\alpha 0},\end{aligned} \tag{11.45}$$

where $\mathfrak{K} = \{k \in \mathbb{R} : \exists\ \nu \in \mathbf{M}_0 \text{ s.t. } \kappa = \mathbf{e}\cdot\nu\}$. Notice that $\kappa \in \mathfrak{K} \Rightarrow \kappa > \omega^1$.

As in [16, Lemma 10.5], there exist $L_{\alpha 0} \in W^{k,p}(\mathbb{R}^3,\mathbb{C}^4)$ for all $k \in \mathbb{R}$ and $p \geqslant 1$ such that the r.h.s. of (11.45) is equal to

$$\sum_{\kappa\in\mathfrak{K}} \kappa\Lambda(\kappa,\zeta) \quad \text{for } \Lambda(\kappa,\zeta) = \frac{1}{\pi}\operatorname{Re}\left\langle R^+_{\boldsymbol{\imath}\mathbb{i}\sigma_3(-\Delta+\omega^1)}(-\kappa)\mathbf{L}(\zeta), \mathbb{i}\sigma_3\overline{\mathbf{L}}\right\rangle$$

$$\text{and } \mathbf{L}(\zeta) := \sqrt{2\pi}\sum_{\substack{\alpha\in\mathbf{M}_0\\ \mathbf{e}\cdot\alpha=\kappa}} \zeta^\alpha L^0_{\alpha 0}.$$

We claim that each term in the above summation is non-negative. Observe that $\Lambda(\kappa,\zeta) = \Lambda_1(\kappa,\zeta) + \Lambda_2(\kappa,\zeta)$, $\mathbf{L}(\zeta) = {}^t(\mathbf{L}_1(\zeta),\mathbf{L}_2(\zeta))$, with

$$\Lambda_i(\kappa,\zeta) = \pi^{-1}(-1)^{i+1}\operatorname{Re}\left\langle R^+_{\boldsymbol{\imath}\mathbb{i}(-1)^{i+1}(-\Delta+\omega^1)}(-\kappa)\mathbf{L}_i, \mathbb{i}\overline{\mathbf{L}}_i\right\rangle.$$

Introduce now

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ \boldsymbol{\imath} & -\boldsymbol{\imath}\end{pmatrix} \text{ such that } U^{-1}\mathbb{i}U = -\boldsymbol{\imath}\sigma_3,$$

with σ_3 the Pauli matrix (1.2). Taking the complex conjugation, $\overline{U}^{-1}\mathbb{i}\overline{U} = \boldsymbol{\imath}\sigma_3$. Then, using ${}^t\overline{U} = U^{-1}$, we have, for $U^{-1}\mathbf{L}_i = {}^t(\mathbf{L}_{i1},\mathbf{L}_{i2})$:

$$\begin{aligned}\pi\Lambda_i(\kappa,\zeta) &= (-1)^{i+1}\operatorname{Re}\left\langle U^{-1}R^+_{\boldsymbol{\imath}\mathbb{i}(-1)^{i+1}(-\Delta+\omega^1)}(-\kappa)UU^{-1}\mathbf{L}_i, \overline{U}^{-1}\mathbb{i}\overline{U}\overline{U}^{-1}\overline{\mathbf{L}}_i\right\rangle \\ &= (-1)^{i+1}\operatorname{Re}\left\langle R^+_{(-1)^{i+1}\sigma_3(-\Delta+\omega^1)}(-\kappa)U^{-1}\mathbf{L}_i, \boldsymbol{\imath}\sigma_3\overline{U}^{-1}\overline{\mathbf{L}}_i\right\rangle \\ &= (-1)^{i+1}\operatorname{Re}\left\langle R^+_{(-1)^{i+1}(-\Delta+\omega^1)}(-\kappa)\mathbf{L}_{i1}, \boldsymbol{\imath}\overline{\mathbf{L}}_{i1}\right\rangle \\ &\quad - (-1)^{i}\operatorname{Re}\left\langle R^+_{(-1)^{i}(-\Delta+\omega^1)}(-\kappa)\mathbf{L}_{i2}, \boldsymbol{\imath}\overline{\mathbf{L}}_{i2}\right\rangle.\end{aligned}$$

Using the Sokhotski–Plemelj formula, we have:

$$\Lambda_1(\kappa,\zeta) = \left\langle \boldsymbol{\imath}\delta(\Delta-\omega^1+\kappa)\mathbf{L}_{12}, \boldsymbol{\imath}\overline{\mathbf{L}}_{12}\right\rangle = -\left\langle \delta(\Delta-\omega^1+\kappa)\mathbf{L}_{12}, \overline{\mathbf{L}}_{12}\right\rangle \leqslant 0;$$

$$\Lambda_2(\kappa,\zeta) = \left\langle \imath\delta(\Delta-\omega^1+\kappa)\mathbf{L}_{21}, \imath\overline{\mathbf{L}}_{21}\right\rangle = -\left\langle \delta(\Delta-\omega^1+\kappa)\mathbf{L}_{21}, \overline{\mathbf{L}}_{21}\right\rangle \leqslant 0.$$

The Fermi Golden Rule consists of two parts. The first part consists in showing that $\Lambda(\kappa,\zeta)$ are negative quadratic forms for the vector $(\zeta^\alpha)_{\alpha\in\mathbf{M}_0 \text{ s.t. } \alpha\cdot\omega^1=\kappa}$. This was proved here. The second part is that the $\Lambda(\kappa,\zeta)$ are strictly negative quadratic forms. This is expected to be generically true (as a similar statement was expected to be true in [12, 35]). We do not know how to prove this. For a proof on a different problem, see [2, Proposition 2.2]. For specific systems the strict negative condition ought to be checked numerically. Here we assume it as an hypothesis:

(H9) (*Fermi Golden Rule*) the l.h.s. of (11.46), proved above to be negative, is strictly negative, that is for some fixed constants and for any vector $\zeta\in\mathbb{C}^{\mathbf{n}}$ we have

$$\sum_{\in\mathfrak{K}}\kappa\Lambda(\kappa,\zeta)\approx-\sum_{\alpha\in\mathbf{M}_0}|\zeta^\alpha|^2. \tag{11.46}$$

By (H9) we have

$$2\sum_{l=1,\dots,\mathbf{n}}\mathbf{e}_l\operatorname{Im}\left(\mathcal{D}_l\overline{\zeta}_l\right)\gtrsim\partial_t\sum_{l=1,\dots,\mathbf{n}}\mathbf{e}_l|\zeta_l|^2+\sum_{\alpha\in\mathbf{M}_0}|\zeta^\alpha|^2. \tag{11.47}$$

Then, for $t\in[0,T]$ and assuming Lemma 11.8, we have

$$\sum_{l=1,\dots,\mathbf{n}}\mathbf{e}_l|\zeta_l(t)|^2+\sum_{\alpha\in\mathbf{M}_0}\|\zeta^\alpha\|^2_{L^2(0,t)}\lesssim\epsilon^2+C_0\epsilon^2.$$

By (11.34) this implies $|z|^2_{L^\infty(0,t)}+\sum_{\alpha\in\mathbf{M}_0}\|z^\alpha\|^2_{L^2(0,t)}\lesssim\epsilon^2+C_0\epsilon^2$ and yields Proposition 11.6. □

In the course of the proof we have shown that $\|z^\alpha\|^2_{L^2(0,t)}\lesssim C_0^2\epsilon^2$ and (1.8) together imply $\|z^\alpha\|^2_{L^2(0,t)}\lesssim C_0\epsilon^2$. This means that we can take $C_0\approx 1$. With Corollary 11.5 this completes the proof of Proposition 11.2. □

12. Proof of Theorem 1.1.

Lemma 12.1. *There is $f_+\in H^1(\mathbb{R}^3,\mathbb{C}^4)$ such that $f(t)$ from (11.4) satisfies*

$$\lim_{t\to+\infty}\|f(t)-W(t,0)e^{\mathrm{i}\sigma_3(-\Delta+\omega^1)t}f_+\|_{H^1}=0, \tag{12.1}$$

where $W(t,s)$ is the fundamental solution from (11.17).

Proof. Starting from (11.10) and using (11.24), we obtain the following analogue of (11.25):

$$f(t)=W(t,0)e^{\mathrm{i}\sigma_3(-\Delta+\omega^1)t}f(0)+$$
$$\int_0^t e^{\mathrm{i}\sigma_3(-\Delta+\omega^1)(t-t')}W(t,t')\Big[Vf(t')-\sum_{(\mu,\nu)\in\mathbf{M}}z^\mu(t')\overline{z}^\nu(t')G^0_{\mu\nu}+R_1(t')+R_2(t')\Big]\,dt'.$$

This implies $W(0,t)e^{\mathrm{i}\sigma_3(\Delta-\omega^0)t}f(t)\underset{t\to+\infty}{\longrightarrow}f_+$ in $H^1(\mathbb{R}^3,\mathbb{C}^4)$, by standard arguments (cf. [19, Sect. 11]). □

Completion of the proof of Theorem 10.3. Recall that expressing u in terms of the coordinates in (3.8) we have

$$u(t)=e^{-\mathrm{i}\sigma_3\sum_{j=1}^4\tau_j'(t)\Diamond_j}\left(\sqrt{1-|b'(t)|^2}+b'(t)\sigma_2\boldsymbol{K}\right)\left(\Phi_{p'(t)}+P_{p'(t)}r'(t)\right), \tag{12.2}$$

where we denote by (p', τ', b', r') the initial coordinates. Using the invariance $\Pi(u(t)) = \Pi(u_0)$ we can express (p', b') in terms of r' obtaining the following:

$$\begin{aligned} p'_j(t) &= \Pi_j(u_0) - \Pi_j(r'(t)) + \mathcal{R}^{1,2}_{\infty,\infty}\big(p^0_4, \Pi(r'(t)), r'(t)\big) \quad \text{for } j = 1,2,3,4; \\ b'_R(t) &= (2p^0_4)^{-1}\Pi_5(r'(t)) + \mathcal{R}^{2,0}_{\infty,\infty}\big(p^0_4, \Pi(r'(t))\big) + \mathcal{R}^{1,2}_{\infty,\infty}\big(p^0_4, \Pi(r'(t)), r'(t)\big); \\ b'_I(t) &= (2p^0_4)^{-1}\Pi_6(r'(t)) + \mathcal{R}^{2,0}_{\infty,\infty}\big(p^0_4, \Pi(r'(t))\big) + \mathcal{R}^{1,2}_{\infty,\infty}\big(p^0_4, \Pi(r'(t)), r'(t)\big). \end{aligned} \tag{12.3}$$

Furthermore, we can express r' in terms of the (z, f) of the last coordinate system for $\ell = 2\mathbf{N} + 1$ in Proposition 10.3:

$$\begin{aligned} r'(t) = e^{\mathbb{i}\sum_{j=1}^4 \sigma_3 \mathcal{R}^{0,2}_{k,m}(p^0_4, \Pi(f(t)), z(t), f(t))\diamondsuit_j} T\big(e^{\sum_{i=1}^3 \mathcal{R}^{0,2}_{k,m}(p^0_4, \Pi(f), z, f)\mathbb{i}\sigma_i}\big) \\ \times \Big(f(t) + \mathbf{S}^{0,1}_{k,m}\big(p^0_4, \Pi(f(t)), z(t), f(t)\big)\Big). \end{aligned} \tag{12.4}$$

While the changes of coordinates in Lemma 9.2 and in the normal forms in Section 10 involve loss of regularity of f, in order to be differentiable so that the pullback of the symplectic forms makes sense, nonetheless these maps are also continuous changes of coordinates inside in $H^1(\mathbb{R}^3, \mathbb{C}^2)$; see Lemma 8.1 for $l = 0$. Notice that (1.1) leaves $\Sigma_k(\mathbb{R}^3, \mathbb{C}^2)$ invariant for any $k \in \mathbb{N}$ and that, similarly, the system leaves $\mathbb{C}^{\mathbf{n}} \times (X_c \cap \Sigma_k(\mathbb{R}^3, \mathbb{C}^2))$ invariant.

By the well-posedness of (1.1) in $H^1(\mathbb{R}^3, \mathbb{C}^2)$ and of (11.1) in $\mathbb{C}^{\mathbf{n}} \times X_c$, a continuous change of coordinates (12.2)–(12.4) maps solutions of (11.1) in $\mathbb{C}^{\mathbf{n}} \times X_c$ into solutions in $H^1(\mathbb{R}^3, \mathbb{C}^2)$ of (1.1), capturing the solutions of (1.1) in the statement of Theorem 10.3. See also [20, Sect. 8].

By Lemma 12.1 it is easy to conclude that $\mathcal{R}^{0,2}_{k,m} \underset{t\to+\infty}{\longrightarrow} 0$ in $\mathbb{R}^7$ and $\mathbf{S}^{0,1}_{k,m} \underset{t\to+\infty}{\longrightarrow} 0$ in $\Sigma_k(\mathbb{R}^3, \mathbb{C}^4)$ for the terms in (12.4), and that $\mathcal{R}^{1,2}_{k,m} \underset{t\to+\infty}{\longrightarrow} 0$ for the terms in (12.3). Then for $1 \leqslant j \leqslant 4$ we have

$$\lim_{t\to+\infty} \Pi_j(r'(t)) = \lim_{t\to+\infty} \Pi_j(f(t)) = \lim_{t\to+\infty} \Pi_j\big(W(t,0)e^{\mathbb{i}\sigma_3(-\Delta+\omega^1)t} f_+\big) = \Pi_j(f_+)$$

since $\Pi_j\big(W(t,0)e^{\mathbb{i}\sigma_3(-\Delta+\omega^1)t} f_+\big) = \Pi_j(f_+)$. Hence, since p is characterized by the first four variables (cf. (2.12)), this defines p_+ in (1.9).

We consider a function $g \in C^1(\mathbb{R}_+, \mathbf{G})$ such that

$$e^{-\mathbb{i}\sigma_3 \sum_{j=1}^4 \tau'_j(t)\diamondsuit_j}\big(\sqrt{1 - |b'(t)|^2} + b'(t)\sigma_2 \boldsymbol{K}\big) = T(g(t)).$$

By (12.4) we have

$$T(g(t))P_{p'(t)}r'(t) = T(g(t))e^{\mathbb{i}\sigma_3 \sum_{j=1}^4 \mathcal{R}^{0,2}_{k,m}\diamondsuit_j} T(e^{\sum_{i=1}^3 \mathcal{R}^{0,2}_{k,m}\mathbb{i}\sigma_i})f + o_{\Sigma_k}(1), \tag{12.5}$$

where $o_{\Sigma_k}(1) \underset{t\to+\infty}{\to} 0$ in $\Sigma_k(\mathbb{R}^3, \mathbb{C}^2)$. We claim the following, with the proof in Appendix A.

Claim 12.2.

$$T(g(t))e^{\mathbb{i}\sigma_3 \sum_{j=1}^4 \mathcal{R}^{0,2}_{k,m}\diamondsuit_j} T(e^{\sum_{i=1}^3 \mathcal{R}^{0,2}_{k,m}\mathbb{i}\sigma_i}) = \widetilde{W}(0,t) \tag{12.6}$$

with $\widetilde{W}(t,s)$ the fundamental solution, in the sense of (11.17), of a system of the form

$$\dot{u} = \mathbb{i}\sigma_3 \widetilde{\mathbf{v}} \cdot \diamondsuit u, \qquad \text{where} \quad \widetilde{\mathbf{v}} \cdot \diamondsuit = \sum_{j=1,\dots,7} \mathbb{i}\sigma_3 \widetilde{\mathbf{v}}_j(t)\diamondsuit_j. \tag{12.7}$$

Substituting (3.9) and (12.4) into (1.1), for a $G_1 \in C^0(H^1(\mathbb{R}^3,\mathbb{C}^2), L^1(\mathbb{R}^3,\mathbb{C}^4))$ we get

$$\dot f = -\mathbb{i}\sigma_3\Delta f + \mathbb{i}\sigma_3\widetilde{\mathbf{v}}\cdot\Diamond f + G_1(u), \tag{12.8}$$

while from (11.14) we have for a $G_2 \in C^0(H^1(\mathbb{R}^3,\mathbb{C}^2), L^1(\mathbb{R}^3,\mathbb{C}^4))$

$$\dot f = -\mathbb{i}\sigma_3\Delta f + \mathbb{i}\sigma_3\omega^1 f + \mathbb{i}\sigma_3\mathbf{v}\cdot\Diamond f + G_2(u). \tag{12.9}$$

The fact that G_1, $G_2 \in C^0(H^1(\mathbb{R}^3,\mathbb{C}^2), L^1(\mathbb{R}^3,\mathbb{C}^4))$ is rather simple. For example, $G_2(u)$ is given by the sum of the r.h.s. of (11.14) with a linear term $V_{\omega^1} f$ where $V_{\omega^1} \in S(\mathbb{R}^3, M(\mathbb{C}^4))$ is the matrix-valued function in (11.24). It is elementary to show that $u \mapsto f$ is in $C^0(H^1(\mathbb{R}^3,\mathbb{C}^2), L^2(\mathbb{R}^3,\mathbb{C}^4))$.

The rest of $G_2(u)$ comes from the r.h.s. of (11.14), obtained applying $\widehat{\nabla}_f$ to the terms $\mathbf{R}|_{j=1}^3$ in the expansion (8.11). It is elementary that this, too, is in $C^0(H^1(\mathbb{R}^3,\mathbb{C}^2), L^1(\mathbb{R}^3,\mathbb{C}^4))$.

By comparing the equation for f with G_1 and the equation for f with G_2, it follows that we necessarily have $\widetilde{\mathbf{v}}\cdot\Diamond = \omega^1 + \mathbf{v}\cdot\Diamond$; see [18, Lemma 13.8]. Hence, returning to (12.5), we have

$$T(g(t))P_{p'(t)}r'(t) = \widetilde{W}(0,t)W(t,0)e^{\mathbb{i}\sigma_3\omega^1 t}e^{-\mathbb{i}\sigma_3\Delta t}f_+ + o_{H^1}(1),$$

for $W(t,0)$ defined by (11.17) and where

$$\partial_t(\widetilde{W}(0,t)W(t,0)e^{\mathbb{i}\sigma_3\omega^1 t}) = \widetilde{W}(0,t)\mathbb{i}\sigma_3\left((\mathbf{v}-\widetilde{\mathbf{v}})\cdot\Diamond + \omega^1\right)W(t,0) = 0.$$

We conclude that there exists $g_0 \in \mathbf{G}$ such that for $h_+ = T(g_0)f_+$ one has

$$T(g(t))P_{p'(t)}r'(t) = e^{-\mathbb{i}\sigma_3\Delta t}h_+ + o_{H^1}(1).$$

This completes the proof of (1.9).

Finally, we emphasize that the proof is predicated on the values $\Pi_j(u_0) = p_j^0$ for $j \leqslant 6$, with the coordinate changes and the manifold $\mathcal{M}_1^6(p^0)$ dependent on p^0. However, since the symbols $\mathcal{R}^{i,j}_{k,m}$ and $\mathbf{S}^{i,j}_{k,m}$ appearing in the coordinate changes depend continuously on p^0, the estimates are uniform in p^0, as long as this is close enough to p^1. This completes the proof of Theorem 1.1. □

Appendix A. Proofs of Lemma 8.1, Lemma 8.2 and Claim 12.2. Lemma 8.1 is obtained expressing r in terms of (z,f) from the following lemma, where we omit the dependence on the constant parameter Π_4.

Lemma A.1. *For $n, M, M_0, s, s', k, l \in \mathbb{N}_0$ with $1 \leqslant l \leqslant M$ such that (8.4) is satisfied, for $a \in A$ a parameter, with A an open subset in $\mathbb{R}^d$, and for $\widetilde{\varepsilon}_0 > 0$, consider*

$$\dot r(t) = \mathbb{i}\sigma_3 \sum_{j\leqslant 7} \mathcal{R}^{0,M_0+1}_{n,M}(t,a,\Pi(r),r)\Diamond_j r + \mathbf{S}^{i,M_0}_{n,M}(t,a,\Pi(r),r). \tag{A.1}$$

Let $k \in \mathbb{Z}\cap[0, n-(l+1)]$ and set for $s'' \geqslant 1$ and $\varepsilon > 0$

$$\mathcal{U}^{s''}_{\varepsilon,k} := \{r \in T^{\perp\Omega}_{\Phi_{p^1}}\mathcal{M} \cap \Sigma_{s''} \ : \ \|r\|_{\Sigma_{-k}} + |\Pi(r)| \leqslant \varepsilon\}. \tag{A.2}$$

Let $a_0 \in A$. Then, for $\varepsilon > 0$ small enough, (A.1) defines a flow $\mathfrak{F}_t$

$$\begin{aligned}\mathfrak{F}_t(r) = {}& e^{\mathbb{i}\sigma_3\sum_{j=1}^4 \mathcal{R}^{0,M_0+1}_{n-l-1,l}(t,a,\Pi(r),r)\Diamond_j} \times \\ & T(e^{\sum_{i=1}^3 \mathcal{R}^{0,M_0+1}_{n-l-1,l}(t,a,\Pi(r),r)\mathbb{i}\sigma_i})\left(r + \mathbf{S}^{i,M_0}_{n-l-1,l}(t,a,\Pi(r),r)\right),\end{aligned} \tag{A.3}$$

where for and for $\varepsilon_1 > \varepsilon_2 > 0$ sufficiently small we have

$$\mathfrak{F}_t \in C^l((-4,4) \times D_{\mathbb{R}^d}(a_0, \varepsilon_2) \times \mathcal{U}^{s'}_{\varepsilon_2,k}, \mathcal{U}^{s}_{\varepsilon_1,k}). \tag{A.4}$$

Proof (sketch). While the statement is the same of [17, Lemma 3.8] and [2, Lemma 3], we have to deal with operators $\diamondsuit_j$ for $j = 5,6,7$ which do not commute.

For $\xi \in \mathfrak{su}(2)$ and $q \in \mathbb{R}^4$ we consider $S := e^{-\mathbb{i}\sigma_3 \sum_{j=1}^4 q_j \diamondsuit_j} T(e^{-\xi}) r$, for T the representation in (2.19). It is elementary that for some $F_j \in C^\infty$ we have

$$\begin{aligned} &\Pi_j(r) = \Pi_j(S) \quad \text{for } j = 1,2,3,4, \\ &\Pi_j(r) = \Pi_j(S) + F_j(\xi, \Pi_k(S)|_{k=5}^7) \quad \text{for } j = 5,6,7, \end{aligned} \tag{A.5}$$

where $F_j(0,*) \equiv 0 \equiv F_j(*,0)$ for any $*$ and where for $j = 5,6,7$ the above equality is obtained proceeding like in Lemma 5.1. Then expressing the coefficients of (A.3) in terms of the new variables, we have new coefficients

$$\begin{aligned} &\mathfrak{D}(t,a,\xi,\varrho,S) := e^{-\mathbb{i}\sigma_3 \sum_{j=1}^4 q_j \diamondsuit_j} T(e^{-\xi}) \mathbf{S}^{i,M_0}_{n,M}(*), \text{ where} \\ &* := \left(t, a, \varrho_l|_{l=1}^4, \varrho_l|_{l=5}^7 + F_l(\xi, \varrho_k|_{k=5}^7)|_{l=5}^7, e^{\mathbb{i}\sigma_3 \sum_{j=1}^4 q_j \diamondsuit_j} T(e^{\xi}) S\right), \\ &\mathfrak{A}_j(t,a,\xi,\varrho,S) := \mathcal{R}^{0,M_0+1}_{n,M}(*). \end{aligned}$$

Notice that for $0 \leqslant \ell \leqslant M$ we have

$$\mathfrak{D}(t,a,\xi,\varrho,S) = \mathbf{S}^{i,M_0}_{n-\ell,\ell}(t,a,\xi,\varrho,S) \text{ and } \mathfrak{A}_j(t,a,\xi,\varrho,S) = \mathcal{R}^{0,M_0+1}_{n-\ell,\ell}(t,a,\xi,\varrho,S).$$

Then consider the following system which we explain below:

$$\begin{aligned} &\dot S = \mathfrak{D}(t,a,\xi,\varrho,S); \\ &\dot q_j = \mathfrak{A}_j(t,a,\xi,\varrho,S) \text{ for } j = 1,2,3,4, \text{ with } q_j(0) = 0; \\ &\sum_{k=1}^{\infty} \frac{1}{k!} (\mathrm{ad}(\xi))^{k-1} \dot\xi = \sum_{i=1}^{3} \mathfrak{A}_j(t,a,\xi,\varrho,S) \mathbb{i}\sigma_i \text{ with } \xi(0) = 0; \\ &\dot\varrho_j = \langle S, \diamondsuit_j \mathfrak{D}(t,a,\xi,\varrho,S)\rangle + A_j, \\ &\qquad A_j = \begin{cases} 0, & j = 1,2,3,4; \\ -\partial_\xi F_j(\xi, \varrho_k|_{k=5}^7)\dot\xi - \sum_{l=5}^7 \partial_{\varrho_l} F_j(\xi, \varrho_k|_{k=5,6,7})\dot\varrho_l, & j = 5,6,7. \end{cases} \end{aligned} \tag{A.6}$$

We explain now the above equations. The second and third line are defined in order to simplify the equation for S. Indeed, when we substitute S in the equation of r we get

$$\begin{aligned} &\partial_t(e^{\mathbb{i}\sigma_3 \sum_{j=1}^4 q_j \diamondsuit_j} T(e^{\xi}) S) = \\ &= e^{\mathbb{i}\sigma_3 \sum_{j=1}^4 q_j \diamondsuit_j} T(e^{\xi}) \Big(\mathbb{i}\sigma_3 \sum_{j=1}^4 \dot q_j \diamondsuit_j S + T(e^{-\xi}) \partial_t (T(e^{\xi})) S + \dot S \Big) \\ &= e^{\mathbb{i}\sigma_3 \sum_{j=1}^4 q_j \diamondsuit_j} \Big(T(e^{\xi}) \mathbb{i}\sigma_3 \sum_{j=1}^4 \mathcal{A}_j \diamondsuit_j S + \mathbb{i}\sigma_3 \sum_{j=5}^7 \mathcal{A}_j \diamondsuit_j T(e^{\xi}) S \Big) + \mathcal{D}. \end{aligned}$$

By the choice made in the second line of (A.6) the summations over $j = 1,2,3,4$ cancel out. We will show that the summations over $j = 5,6,7$ also cancel out. By the Baker–Campbell–Hausdorff formula, see [33, p. 15], we have

$$\partial_t e^{\xi} = \left(\sum_{k=1}^{\infty} \frac{1}{k!} \left(\mathrm{ad}(\xi)\right)^{k-1} \dot{\xi} \right) e^{\xi}, \quad \text{where } \mathrm{ad}(\xi) : \mathfrak{su}(2) \to \mathfrak{su}(2), \quad \vartheta \mapsto [\xi, \vartheta]. \tag{A.7}$$

So, for $1\!\!1_{\mathbb{C}^2}$ the unit element in $\mathbf{SU}(2)$, we have

$$\partial_t(T(e^{\xi})) = dT(1\!\!1_{\mathbb{C}^2}) \left(\sum_{k=1}^{\infty} \frac{1}{k!} \left(\mathrm{ad}(\xi)\right)^{k-1} \dot{\xi} \right) T(e^{\xi}). \tag{A.8}$$

On the other hand, by (2.9) and (2.20) we have

$$\sum_{j=5,6,7} \mathcal{A}_j \mathrm{i}\sigma_3 \Diamond_j T(e^{\xi}) = \sum_{i=1,2,3} \mathcal{A}_{i+4} dT(1\!\!1_{\mathbb{C}^2})(\mathrm{i}\sigma_i) T(e^{\xi}).$$

So the third equation in (A.6) yields the cancellation of these terms. Hence we conclude that the first equation in (A.6) is true.

We also derive equations for ϱ_j by differentiating $\partial_t \Pi_j(S)$ and by substituting $\Pi_j(S)$ with ϱ_j.

Solving the last equation in (A.6) in terms of $\dot{\varrho}_j|_{j=5}^{7}$ and replacing in the last equation $\dot{\xi}$ by means of the third equation, we obtain, for $1 \leqslant \ell \leqslant M$,

$$\begin{aligned}
\dot{S} &= \mathbf{S}^{i,M_0}_{n-\ell,\ell}(t, a, \xi, \varrho, S);\\
\dot{q}_j &= \mathcal{R}^{0,M_0+1}_{n-\ell,\ell}(t, a, \xi, \varrho, S) \ \text{ for } j = 1,2,3,4, \text{ with } q_j(0) = 0;\\
\dot{\xi} &= \mathcal{R}^{0,M_0+1}_{n-\ell,\ell}(t, a, \xi, \varrho, S) \ \text{ with } \xi(0) = 0;\\
\dot{\varrho}_j &= \mathcal{R}^{0,M_0+1}_{n-\ell-1,\ell}(t, a, \xi, \varrho, S) \ \text{ for } j = 1, ..., 7.
\end{aligned} \tag{A.9}$$

Taking as initial conditions $(r, 0, 0, \Pi(r))$, by elementary arguments, see [17, Lemma 3.8], we get from (A.9) a flow

$$\begin{aligned}
S(t) &= r + \int_0^t \mathbf{S}^{i,M_0}_{n-\ell-1,\ell}(t', a, \Pi(r), r) dt' = r + \mathbf{S}^{i,M_0}_{n-\ell-1,\ell}(t, a, \Pi(r), r);\\
q_j(t) &= \int_0^t \mathcal{R}^{0,M_0+1}_{n-\ell-1,\ell}(t', a, \Pi(r), r) dt' = \mathcal{R}^{0,M_0+1}_{n-\ell-1,\ell}(t, a, \Pi(r), r) \ \text{ for } j = 1,2,3,4;\\
\xi(t) &= \sum_{i=1}^{3} \int_0^t \mathcal{R}^{0,M_0+1}_{n-\ell-1,\ell}(t', a, \Pi(r), r) dt' \mathrm{i}\sigma_i = \sum_{i=1}^{3} \mathcal{R}^{0,M_0+1}_{n-\ell-1,\ell}(t, a, \Pi(r), r) \mathrm{i}\sigma_i;\\
\Pi_j(S(t)) &= \Pi_j(r) + \int_0^t \mathcal{R}^{0,M_0+1}_{n-\ell-1,\ell}(t', a, \Pi(r), r) dt'\\
&= \Pi_j(r) + \mathcal{R}^{0,M_0+1}_{n-\ell-1,\ell}(t, a, \Pi(r), r) \ \text{ for } j = 1, ..., 7.
\end{aligned} \tag{A.10}$$

In view of (A.5), we get also

$$\Pi_j(r(t)) = \Pi_j(r) + \mathcal{R}^{0,M_0+1}_{n-\ell-1,\ell}(t, a, \Pi(r), r) \text{ for } j = 1, ..., 7. \tag{A.11}$$

This ends the proof of the parts of Lemma 8.1 which differ from [17, Lemma 3.8]. □

The proof of Lemma 8.2 follows from the following result.

Lemma A.2. *Consider two systems for $\ell = 1, 2$:*

$$\dot{r}(t) = \mathrm{i}\sigma_3 \sum_{j=1,\dots,7} \mathcal{A}^{(\ell)}_j(t, \Pi(r), r) \Diamond_j r + \mathcal{D}^{(\ell)}(t, \Pi(r), r),$$

with the hypotheses of Lemma A.1 satisfied, and suppose that

$$\mathcal{D}^{(1)}(t,\Pi(r),r)-\mathcal{D}^{(2)}(t,\Pi(r),r)=\mathbf{S}^{0,M_0+1}_{n,M}(t,\Pi(r),r).$$

Let $r\mapsto r^t_{(\ell)}$ with $\ell=1,2$ be the flow for each of the two systems. Then, for s, s' as in Lemma A.1,

$$\|r^1_{(1)}-r^1_{(2)}\|_{\Sigma_{-s'}}\leqslant C\|r\|^{M_0+1}_{\Sigma_{-s}}.$$

Proof. The proof is elementary. We consider

$$\begin{aligned}\sum_{\ell=1,2}(-)^\ell\frac{d}{dt}r^t_{(\ell)}&=\sum_{\ell=1,2}(-)^\ell\mathbb{i}\sigma_3\mathcal{R}^{0,M_0+1}_{n,M}(t,\Pi(r^t_{(\ell)}),r^t_{(\ell)})\cdot\Diamond r^t_{(\ell)}\\&+\underbrace{\sum_{\ell=1,2}(-)^\ell\mathcal{D}^{(\ell)}(t,\Pi(r^t_{(2)}),r^t_{(2)})}_{S^{0,M_0+1}_{n,M}(t,\Pi(r^t_{(2)}),r^t_{(2)})}+\sum_{\ell=1,2}(-)^\ell\mathcal{D}^{(1)}(t,\Pi(r^t_{(\ell)}),r^t_{(\ell)}).\end{aligned}$$

Then for $\mathbf{x}^t_\ell:=(\Pi(r^t_{(\ell)}),r^t_{(\ell)})$

$$\begin{aligned}\|r^t_{(2)}-r^t_{(1)}\|_{\Sigma_{-\mathbf{s}'}}&\leqslant\sum_\ell\int_0^t\|r^{t'}_{(\ell)}\|^{M_0+2}_{\Sigma_{-s}}\,dt'+\int_0^t\|r^{t'}_{(2)}\|^{M_0+1}_{\Sigma_{-s}}\,dt'\\&+\int_0^t\int_0^1\|\partial_{\Pi(r)}\mathcal{D}^{(1)}(t',\mathbf{x}^{t'}_1+\tau(\mathbf{x}^{t'}_2-\mathbf{x}^{t'}_1))\|_{\Sigma_{-s}}|\Pi(r^{t'}_{(2)})-\Pi(r^{t'}_{(1)})|\,dt'\\&+\int_0^t\int_0^1\|\partial_r\mathcal{D}^{(1)}(t',\mathbf{x}^{t'}_1+\tau(\mathbf{x}^{t'}_2-\mathbf{x}^{t'}_1))\|_{\Sigma_{-s}\to\Sigma_{-s}}\|r^{t'}_{(2)}-r^{t'}_{(1)}\|_{\Sigma_{-s}}\,dt'.\end{aligned}$$

Since there is a fixed $C>0$ such that

$$\begin{aligned}&\|r_{(\ell)}(t')\|_{\Sigma_{-s}}\leqslant C\|r\|_{\Sigma_{-s}}\text{ from (8.3)},\\&|\Pi(r^{t'}_{(2)})-\Pi(r^{t'}_{(1)})|\leqslant C\|r\|^{M_0+1}_{\Sigma_{-s}}\text{ from the previous one and (8.3)},\\&\|\partial_r\mathcal{D}^{(1)}(t,\Pi,\varrho,r)\|_{\Sigma_{-\mathbf{s}}\to\Sigma_{-s}}\leqslant C\|r\|^{M_0-1}_{\Sigma_{-s}},\\&\|\partial_\varrho\mathcal{D}^{(1)}(t,\Pi,\varrho,r)\|_{\Sigma_{-s}}\leqslant C\|r\|^{M_0}_{\Sigma_{-s}},\end{aligned}$$

where the last inequalities follow from (5.4), for some fixed constant $C>0$ we obtain

$$\|r^t_{(2)}-r^t_{(1)}\|_{\Sigma_{-s'}}\leqslant C\left(t\|r\|^{M_0+1}_{\Sigma_{-s}}+\|r\|^{M_0-1}_{\Sigma_{-s}}\int_0^t\|r^{t'}_{(2)}-r^{t'}_{(1)}\|_{\Sigma_{-s}}\,dt\right),$$

for $t\in[0,1]$, which by Gronwall's inequality yields (8.8). □

Proof of Claim 12.2. Let $\mathfrak{g}=\mathbb{R}^4\times\mathfrak{su}(2)$ be the Lie algebra of $\mathbf{G}$. We can assume that the inverse of the l.h.s. of (12.6) is equal to $e^{\mathbb{i}\sigma_3\sum_{j=1}^4X_j(t)\Diamond_j}T(e^{\xi(t)})$ with $X\in C^1(\mathbb{R}_+,\mathbb{R}^4)$ and $\xi\in C^1(\mathbb{R}_+,\mathfrak{su}(2))$. Then, for $u(t):=e^{\mathbb{i}\sigma_3\sum_{j=1}^4X_j(t)\Diamond_j}T(e^{\xi(t)})u_0$, by (A.8) we have

$$\dot u(t)=\mathbb{i}\sigma_3\sum_{j=1}^4\dot X_j(t)\Diamond_ju(t)+dT(\mathbb{1}_{\mathbb{C}^2})\left(\sum_{k=1}^\infty\frac{1}{k!}\left(\mathrm{ad}(\xi(t))\right)^{k-1}\dot\xi(t)\right)u(t).$$

We set $\tilde{\mathbf{v}}_j(t)=\dot X_j(t)$ for $j\leqslant4$ and, exploiting that $\mathbb{i}\sigma_i|_{i=1}^3$ is a basis of $\mathfrak{su}(2)$, we define $\tilde{\mathbf{v}}_j(t)|_{j=5}^7$ by

$$\sum_{l=1}^3\tilde{\mathbf{v}}_{l+3}(t)\mathbb{i}\sigma_l=\sum_{k=1}^\infty\frac{1}{k!}\left(\mathrm{ad}(\xi(t))\right)^{k-1}\dot\xi(t).$$

Then we conclude that (12.7) it true for this choice of $u(t)$ and of $\tilde{\mathbf{v}}_j(t)|_{j=1}^7$. Then $u(t) = \widetilde{W}(t,0)u_0$ and $\widetilde{W}(0,t) = \widetilde{W}^{-1}(t,0)$ is such that equality (12.6) is true. This yields Claim 12.2. □

Appendix B. **Proof of Lemma 8.4.** The proof can be obtained from the following lemma, expressing r in terms of (z,f) and omitting again the dependence of the symbols on Π_4, which has constant value.

Lemma B.1. *Consider $\mathfrak{F} = \mathfrak{F}^1 \circ \cdots \circ \mathfrak{F}^L$ with $\mathfrak{F}^j = \mathfrak{F}^j_{t=1}$ transformations as in Lemma A.1 on the manifold $\mathcal{M}^6_1(p^0)$. Suppose that for any $\mathfrak{F}^j$ the M_0 in Lemma A.1 equals m_j, where $1 = m_1 \leqslant \ldots \leqslant m_L$ with the constant i in Lemma 8.1 (ii) equal to 1 when $m_j = 1$. Fix M, k with $n_1 \gg k \geqslant \mathbf{N}_0$ (n_1 picked in Lemma 3.1). Then there is a $n = n(L,M,k)$ such that if the assumptions of Lemma 8.1 apply to each of operators $\mathfrak{F}^j$ for (M,n), there exist $\underline{\psi}(\varrho) \in C^\infty$ with $\underline{\psi}(\varrho) = O(|\varrho|^2)$ and a small $\varepsilon > 0$ such that in $\mathcal{U}^s_{\varepsilon,k}$ for $s \geqslant n-(M+1)$ we have the expansion*

$$K \circ \mathfrak{F} = \underline{\psi}(\Pi(r)) + 2^{-1}\Omega(\mathcal{H}_p P_p r, P_p r) + \mathcal{R}^{1,2}_{k,M} + E_P(P_p r) + \boldsymbol{R}'', \tag{B.1}$$

$$\boldsymbol{R}'' := \sum_{d=2,3,4} \langle B_d(\Pi(r), r), (P_p r)^d\rangle + \int_{\mathbb{R}^3} B_5(x, \Pi(r), r, r(x))(P_p r)^5(x)\,dx,$$

with:

- *$B_2(0,0) = 0$;*
- *$B_d(\varrho, r) \in C^M(\mathcal{U}_{-k}, \Sigma_k(\mathbb{R}^3, B((\mathbb{R}^4)^{\otimes d}, \mathbb{R})))$, $2 \leqslant d \leqslant 4$, with $\mathcal{U}_{-k} \subset \mathbb{R}^7 \times (T^{\perp\Omega}_{\Phi_{p^1}}\mathcal{M} \cap \Sigma_{-k})$ an open neighborhood of $(0,0)$;*
- *for $\zeta \in \mathbb{R}^4$ $(\varrho, r) \in \mathcal{U}_{-k}$,*

$$\|\nabla^i_{r,\varrho,\zeta} B_5(\varrho, r, \zeta)\|_{\Sigma_k(\mathbb{R}^3, B((\mathbb{R}^4)^{\otimes 5}, \mathbb{R})} \leqslant C_i, \quad i \leqslant M.$$

Proof. The proof is in [17], but we sketch it. First of all, by (A.4) we have, for $k \leqslant n - L(M+1)$,

$$\mathcal{U}^{n-(M+1)}_{\varepsilon_{L+1},k} \xrightarrow{\mathfrak{F}^L} \mathcal{U}^{n-2(M+1)}_{\varepsilon_L,k} \ldots \xrightarrow{\mathfrak{F}^2} \mathcal{U}^{n-L(M+1)}_{\varepsilon_2,k} \xrightarrow{\mathfrak{F}^1} \mathcal{U}^{n-(L+1)(M+1)}_{\varepsilon_1,k} \subset \mathcal{U}^{k+3}_{\varepsilon_1,k} \subset \mathcal{U}^{\mathbf{N}_0}_{\varepsilon_1,k}, \tag{B.2}$$

where each map is C^M if we pick $n_1 \geqslant n = n(L,M,k) := k+3+(L+1)(M+1)$ and then we get $\mathfrak{F} \in C^M(\mathcal{U}^{n-(M+1)}_{\varepsilon_{L+1},k}, \mathcal{U}^{k+3}_{\varepsilon_1,k})$.

By (A.3), the r-th component of $\mathfrak{F}$ is of the form

$$\mathfrak{F}(\varrho, r) = e^{\mathrm{i}\sigma_3 \sum_{j=1}^4 \mathcal{R}^{1,1}_{k+3,M}(\varrho,r)\Diamond_j} T(e^{\sum_{i=1}^3 \mathcal{R}^{1,1}_{k+3,M}(\varrho,r)\mathrm{i}\sigma_i})(r + \mathbf{S}^{1,1}_{k+3,M}(\varrho, r)). \tag{B.3}$$

Then by $[\Diamond_j, \Diamond_k] = 0$ for all k if $j \leqslant 4$ we have

$$\Pi_j(r)|_{j=1}^4 \circ \mathfrak{F} = \Pi_j(r + \mathbf{S}^{1,1}_{k+3,M}\Pi(r), r))|_{j=1}^4 = \Pi(r)|_{j=1}^4 + \mathcal{R}^{1,2}_{k+2,M}(\Pi(r), r).$$

From (3.13) we have

$$p \circ \mathfrak{F} = p + \mathcal{R}^{1,2}_{k+2,M} \text{ and so } \Phi_p \circ \mathfrak{F} = \Phi_p + \mathbf{S}^{1,2}_{k+2,M}.$$

Then we have

$$\begin{aligned} E(u \circ \mathfrak{F}) &= E\Big(e^{-\mathrm{i}\sigma_3 \sum_{j=1}^4 \tau_j \Diamond_j}(\sqrt{1-|b|^2} + b\sigma_2 \boldsymbol{K})(\Phi_p + P_p r) \circ \mathfrak{F}\Big) \\ &= E((\Phi_p + P_p r) \circ \mathfrak{F}) = E(\Phi_p + \mathbf{S}^{1,2}_{k+2,M} + P_p(e^{\mathrm{i}\sigma_3 \mathcal{R}^{1,1}_{k+2,M}\cdot\Diamond}(r + \mathbf{S}^{1,1}_{k+2,M})) \\ &= E(\Phi_p + P_p r + \mathbf{S}^{1,2}_{k+2,M} + P_p \mathbf{S}^{1,1}_{k+2,M}), \end{aligned}$$

where we use the commutation (for the proof, see [17, Lemma 4.1])

$$[P_p, e^{\mathrm{i}\sigma_3 \sum_{j=1}^4 \mathcal{R}^{1,1}_{k+3,M}\diamondsuit_j} T(e^{\sum_{i=1}^3 \mathcal{R}^{1,1}_{k+3,M})\mathrm{i}\sigma_i}]r$$
$$= [e^{\mathrm{i}\sigma_3 \sum_{j=1}^4 \mathcal{R}^{1,1}_{k+3,M}\diamondsuit_j} T(e^{\sum_{i=1}^3 \mathcal{R}^{1,1}_{k+3,M})\mathrm{i}\sigma_i}, \widehat{P}_p]r = \mathbf{S}^{1,2}_{k+2,M}.$$

We get similarly for $1 \leqslant j \leqslant 4$

$$\Pi_j(u\circ\mathfrak{F})|_{j=1}^4 = \Pi_j(\Phi_p + P_p r + \mathbf{S}^{1,2}_{k+2,M} + P_p\mathbf{S}^{1,1}_{k+2,M})|_{j=1}^4$$
$$= \Pi_j(\Phi_p + P_p r)|_{j=1}^4 + \mathcal{R}^{1,2}_{k,m}.$$

Then

$$K(\mathfrak{F}(u)) = E(\Phi_p + P_p r + \mathbf{S}^{1,2}_{k+2,M} + P_p\mathbf{S}^{1,1}_{k+2,M}) - E\left(\Phi_{p^0}\right)$$
$$- \sum_{j\leqslant 4}(\lambda_j(p) + \mathcal{R}^{1,2}_{k+2,m})\left(\Pi_j(\Phi_p + P_p r) + \mathcal{R}^{1,2}_{k,m} - \Pi_j\left(\Phi_{p^0}\right)\right). \tag{B.4}$$

Like in [17, Lemma 4.3], we set

$$\Psi = \Phi_p + \mathbf{S}^{1,2}_{k+2,M} + P_p\mathbf{S}^{1,1}_{k+2,M};$$

we need to analyze $E(\Psi + P_p r)$ which we break into (cf. (2.10))

$$E(\Psi + P_p r) = E_P(\Psi + P_p r) + E_K(\Psi + P_p r).$$

It is also shown in [17, Lemma 4.3] that

$$E_P(\Psi + P_p r) = E_P(\Psi) + E_P(P_p r)$$
$$+ \text{terms that can be incorporated into } \mathbf{R}''$$
$$+ \sum_{j=0,1}\int_{\mathbb{R}^3} dx \int_{[0,1]^2} \frac{t^j}{j!}(\partial_t^{j+1})\big|_{t=0}\partial_s[B(|s\Psi + tP_p r|^2)]\,dt\,ds. \tag{B.5}$$

The second line of (B.5) equals

$$\int_{\mathbb{R}^3} dx \int_{[0,1]^2} dt\,ds \sum_{j=0,1}\frac{t^j}{j!}(\partial_t^{j+1})\big|_{t=0}\partial_s\ \Big\{\ B(|s\Phi_p + tP_p r|^2)+$$
$$+\int_0^1 d\tau\,\partial_\tau[B(|s(\Phi_p + \tau(\mathbf{S}^{1,2}_{k+2,M} + P_p\mathbf{S}^{1,1}_{k+2,M}) + tP_p r|^2)]\ \Big\}. \tag{B.6}$$

The contribution from the last line of (B.6) can be incorporated into $\mathbf{R}'' + \mathcal{R}^{1,2}_{k,m}$. Notice that from the $j=0$ term in the first line of (B.6) we get

$$2\int_{\mathbb{R}^3} dx \int_0^1 ds\partial_s[B'(|s\Phi_p|^2)s\Phi_p\cdot P_p r] \quad = \quad 2\int_{\mathbb{R}^3} dx B'(|\Phi_p|^2)\Phi_p\cdot P_p r$$
$$= \quad \langle\nabla E_P(\Phi_p), P_p r\rangle. \tag{B.7}$$

The $j=1$ term in the first line of (B.6) is $2^{-1}\langle\nabla^2 E_P(\Phi_p)P_p r, P_p r\rangle$; thus,

$$E_P(\Psi + P_p r) = E_P(\Psi) + E_P(P_p r)$$
$$+ \langle\nabla E_P(\Phi_p), P_p r\rangle + 2^{-1}\langle\nabla^2 E_P(\Phi_p)P_p r, P_p r\rangle + \mathbf{R}'' + \mathcal{R}^{1,2}_{k,m}. \tag{B.8}$$

Then,

$$E_K(\Psi + P_p r) \tag{B.9}$$
$$= E_K(\Psi) - \langle\Delta\Phi_p, P_p r\rangle + \underbrace{\Big\langle -\Delta\big(\mathbf{S}^{1,2}_{k+2,M} + P_p\mathbf{S}^{1,1}_{k+2,M}\big), P_p r\Big\rangle}_{\mathcal{R}^{1,2}_{k,m}} + E_K(P_p r).$$

Using (2.10), (2.6), (2.18) and the fact that $\mathrm{i}\sigma_3\lambda(p)\cdot\Diamond\Phi_p \in T_{\Phi_p}\mathcal{M}$, see (2.21), we have

$$\begin{aligned}\langle -\Delta\Phi_p + \nabla E_P(\Phi_p), P_p r\rangle &= \langle \nabla E(\Phi_p), P_p r\rangle = -\Omega(\mathrm{i}\sigma_3 \nabla E(\Phi_p), P_p r)\\ &= -\Omega(\mathrm{i}\sigma_3\lambda(p)\cdot\Diamond\Phi_p, P_p r) = 0.\end{aligned}$$

Adding (B.8) and (B.9) and using the cancellation of the sum of the second term in the right-hand side of (B.9) with the term (B.7) which follows from the above relation, we arrive at

$$E(\Psi + P_p r) = E(\Psi) + E(P_p r) + 2^{-1}\langle \nabla^2 E_P(\Phi_p) P_p r, P_p r\rangle + \mathbf{R}'' + \mathcal{R}^{1,2}_{k,m}, \quad \text{(B.10)}$$

where we used (2.10). From (2.18),

$$\begin{aligned}E(\Psi) &= E(\Phi_p) + \overbrace{\langle \nabla E(\Phi_p), P_p \mathbf{S}^{1,1}_{k+2,M}\rangle}^{0} + \overbrace{\langle \nabla E(\Phi_p), \mathbf{S}^{1,2}_{k+2,M}\rangle}^{\mathcal{R}^{1,2}_{k+2,M}} + \mathcal{R}^{1,2}_{k,M} \\ &= E(\Phi_p) + \mathcal{R}^{1,2}_{k,M},\end{aligned} \quad \text{(B.11)}$$

where the $\mathcal{R}^{1,2}_{k,M}$ in the right-hand side is absorbed into $\mathcal{R}^{1,2}_{k,M}$ in (B.1).

We have

$$\begin{aligned}-\lambda(p)\cdot\Pi(\Phi_p + P_p r) &= -\lambda(p)\cdot\Pi(\Phi_p) - \lambda(p)\cdot\Pi(P_p r) - \langle \lambda(p)\cdot\Diamond\Phi_p, P_p r\rangle\\ &= -\lambda(p)\cdot\Pi(\Phi_p) - \lambda(p)\cdot\Pi(P_p r),\end{aligned} \quad \text{(B.12)}$$

where we used $\langle \lambda(p)\cdot\Diamond\Phi_p, P_p r\rangle = \Omega(-\mathrm{i}\sigma_3\lambda(p)\cdot\Diamond\Phi_p, P_p r) = 0$.

Substituting (B.10) (where we apply (B.11)) and (B.12) into (B.4), we have:

$$\begin{aligned}K(\mathfrak{F}(u)) &= E(\Phi_p) + E(P_p r) + 2^{-1}\langle \nabla^2 E_P(\Phi_p) P_p r, P_p r\rangle - E(\Phi_{p^0})\\ &\quad - \lambda(p)\cdot\Pi(\Phi_p) - \lambda(p)\cdot\Pi(P_p r) + \lambda(p)\cdot\Pi(\Phi_{p^0}) + \mathbf{R}'' + \mathcal{R}^{1,2}_{k,m}.\end{aligned}$$

By (4.5), $d(p) = E(\Phi_p) - \lambda(p)\cdot\Pi(\Phi_p)$. Then we have

$$\begin{aligned}E(\Phi_p) - E(\Phi_{p^0}) - \lambda(p)\cdot(\Pi(\Phi_p) - \Pi(\Phi_{p^0})) &= d(p) - d(p^0) - (\lambda(p^0) - \lambda(p))\cdot p^0\\ = K(\Phi_p) = O((\Pi_j(r)|_{j=1}^4)^2) + \mathcal{R}^{2,2}_{\infty,\infty}, \end{aligned} \quad \text{(B.13)}$$

where $O((\Pi_j(r)|_{j=1}^4)^2)$ is $\underline{\psi}(\Pi_j(r))$ in (B.1) and $\mathcal{R}^{2,2}_{\infty,\infty}$ is absorbed inside $\mathcal{R}^{1,2}_{k,M}$. Thus,

$$K(\mathfrak{F}(u)) = \underline{\psi}(\Pi(r)) + E(P_p r) + 2^{-1}\langle \nabla^2 E_P(\Phi_p) P_p r, P_p r\rangle - \lambda(p)\cdot\Pi(P_p r) + \mathbf{R}'' + \mathcal{R}^{1,2}_{k,m}.$$

Breaking $E(P_p r) = E_P(P_p r) + E_K(P_p r)$ and using the relation

$$\begin{aligned}&2^{-1}\langle \nabla^2 E_P(\Phi_p) P_p r, P_p r\rangle + E_K(P_p r) - \lambda(p)\cdot\Pi(P_p r)\\ &\quad = 2^{-1}\langle (\nabla^2 E(\Phi_p) - \lambda(p)\cdot\Diamond) P_p r, P_p r\rangle = 2^{-1}\Omega(\mathcal{H}_p P_p r, P_p r),\end{aligned}$$

we arrive at the conclusion of the lemma. □

The following is an elementary consequence of Lemma B.1 and is proved in [17, Lemma 4.4].

Lemma B.2. *Under the hypotheses and notation of Lemma 8.4, for $\boldsymbol{R}'$ like $\boldsymbol{R}''$, for $\psi \in C^\infty(\mathbb{R}^4, \mathbb{R})$ with $\psi(\varrho) = O(|\varrho|^2)$, we have*

$$\begin{aligned}K\circ\mathfrak{F} &= \psi(\Pi_j(r)|_{j=1,\ldots,4}) + 2^{-1}\Omega(\mathcal{H}_{p^1} r, r) + \mathcal{R}^{1,2}_{k,m} + E_P(r) + \boldsymbol{R}', \quad \text{(B.14)}\\ \boldsymbol{R}' &:= \sum_{d=2,3,4} \langle B_d(\Pi(r), r), r^d\rangle + \int_{\mathbb{R}^3} B_5(x, \Pi(r), r, r(x)) r)^5(x)\, dx,\end{aligned}$$

the B_d for $2 \leqslant d \leqslant 5$ with similar properties of the functions in Lemma 4.1.

Proof. The proof, for whose details we refer to [17], is obtained by writing

$$P_p r = r + (P_p - P_{p^1})r = r + \mathbf{S}^{1,1}_{\infty,\infty}$$

and substituting $P_p r = r + \mathbf{S}^{1,1}_{\infty,\infty}$ inside (B.1). That from $E_P(P_p r) + \mathbf{R}''$ in (B.1) we obtain a term which is contained in $\mathcal{R}^{1,2}_{k,m} + E_P(r) + \mathbf{R}'$ in (B.14) is elementary and is discussed in [17]. We have

$$\frac{1}{2}\Omega(\mathcal{H}_p P_p r, P_p r) = \frac{1}{2}\langle -\Delta P_p r, P_p r\rangle - \lambda(p)\cdot \Pi(P_p r) + \frac{1}{2}\langle \nabla^2 E_P(\Phi_p) P_p r, P_p r\rangle. \tag{B.15}$$

Then

$$\langle -\Delta P_p r, P_p r\rangle = \langle -\Delta r, r\rangle + \mathcal{R}^{1,2}_{k,m}, \quad \Pi(P_p r) = \Pi(r) + \mathcal{R}^{1,2}_{k,m},$$

$$\langle \nabla^2 E_P(\Phi_p) P_p r, P_p r\rangle = \langle \nabla^2 E_P(\Phi_{p^1}) r, r\rangle + \mathcal{R}^{1,2}_{k,m} + \big\langle (\nabla^2 E_P(\Phi_p) - \nabla^2 E_P(\Phi_{p^1})) r, r\big\rangle,$$

$$\lambda(p) = \lambda(p^1) + \mathcal{R}^{1,0}_{\infty,\infty}(\Pi_j(r)|_{j=1}^4) + \mathcal{R}^{1,2}_{k,m},$$

where for the last line we considered (3.13) which implies

$$p = \Pi - \Pi(r) + \mathcal{R}^{1,2}_{\infty,\infty}$$

and where $\mathcal{R}^{1,0}_{\infty,\infty}(\Pi(r))$ is smooth in the argument and is $O(|\Pi(r)|)$.

Then we conclude that the right hand side of (B.15) is

$$\begin{aligned}&\overbrace{2^{-1}\big\langle(-\Delta - \lambda(p^1)\cdot \Diamond + \nabla^2 E_P(\Phi_{p^1}))r, r\big\rangle}^{2^{-1}\Omega(\mathcal{H}_{p^1} r, r)} + \mathcal{R}^{2,0}_{\infty,\infty}(\Pi_j(r)|_{j=1}^4) + \mathcal{R}^{1,2}_{k,m} \\ &+ 2^{-1}\big\langle (\nabla^2 E_P(\Phi_p) - \nabla^2 E_P(\Phi_{p^1}))r, r\big\rangle,\end{aligned} \tag{B.16}$$

where the last term can be absorbed in the $d = 2$ term of $\mathbf{R}'$ by (3.13). Setting $\psi(\varrho) = \underline{\psi}(\varrho) + \mathcal{R}^{2,0}_{\infty,\infty}(\varrho)$ with the $\mathcal{R}^{2,0}_{\infty,\infty}$ in (B.16), we get the desired result. ☐

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