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XXXIII CICLO DEL DOTTORATO DI RICERCA IN FISICA

GRAVITATIONAL DECOHERENCE

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Chapter 1

Introduction

The recent exciting first detections of gravitational waves [1, 2], which marked a new era in astrophysics and cosmology, have pushed the scientific community towards the construction of ever more sophisticated ground and space based detectors [3, 4, 5, 6, 7] to observe waves in a variety of ranges, possibly down to the cosmic background gravitational radiation. Detecting the latter would open the possibility to gain crucial information about the universe at its very primordial stage, at about 10^{-22} s after the Big Bang [8], where we expect our description of gravity to fail [9, 10], especially because of its unclear relation with quantum matter.

Most gravitational waves (which can be thought of as small perturbations of the metric propagating through spacetime at the speed of light [11, 12, 13, 14]) that arrive on the Earth are produced by different unresolved mechanisms and sources [8, 15], and thus result in a stochastic perturbation of the flat spacetime background. Within the framework of quantum theory, this altered background affects the dynamics of matter propagation [16, 17] and, when the quantum state is in a superposition, it leads to decoherence effects, as it's typical of any noisy environment.

In this scenario, the extreme sensitivity of matter waves [18, 19, 20, 21] to gravity gradients [22, 23, 24, 25, 26, 27, 28] makes matter-wave interferometers a perfect candidate for exploring the gravitational wave background [8, 29, 30] and, at the same time, for possibly answering some fundamental questions regarding the nature of gravity [31, 32, 33, 34, 35], and its coupling to quantum matter. Besides the technological challenge of building sensitive (therefore large) enough matter-wave interferometers, which realistically would have to operate in outer space, even from the theoretical point of view it is not clear how they would respond to a gravitational background produced by random sources, as no comprehensive dynamical description of the gravity induced decoherence process has been so far proposed.

The decoherence effect of a stochastic (or quantum) perturbation of the metric has in fact been studied by several authors [36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47],

each of whom has produced a different model for the evolution of off-diagonal elements of the density matrix of a quantum state or, more generally, the loss of interference in the system. However, that of giving a universal and meaningful description of the phenomenon is still an open problem, as the different models so far proposed refer to particular regimes of approximation and thus seem to lead to different and apparently incompatible conclusions.
The goal of our work is to formulate a more general description of gravity induced deocherence, in the form of a master equation, which is able to encompass the existing literature and explain the apparent discrepancies, as well as extend the so far know results. With a more general and unambiguous dynamics, we aim at assessing whether and to what extent matter-wave interferometers constitute a viable platform for probing of the cosmic gravitational background.

We will start our thesis with an overview of the basic concepts and the literature about gravitational decoherence in Chapter 2. Section 2.1 consists of a brief introduction to decoherence; we will make the reader familiar with the fundamental concepts of reduced density matrix, interference terms and decoherence rate.
In Section 2.1.1 we will show how a weak gravitational fluctuations affects the dynamics of a matter wave. In particular we will derive an equation for the induced phase accumulation in the limit of large wavelength of such a weak gravitational fluctuation, and show how this stochastic phase is responsible for a decoherence effect.
We will conclude the chapter with an essential review of the gravitational decoherence literature in Section 2.2. There, we will sketch the essential steps necessary to understand and reproduce some among the most relevant models present in the literature. We will also classify the major models according to the different approximations and assumptions employed in their derivation, as well as their final predictions: the eigenbasis and rate of decoherence. We will motivate the need of a novel model for the dynamics of the gravitational decoherence process in order to explain and overcome such differences.
In chapter 3 we will derive a novel model for the description of the dynamics of a scalar bosonic particle under the effects of a weak stochastic gravitational perturbation. The specific details of the derivation are reported in Sections 3.1-3. Our model predicts decoherence in the position, and momentum and energy eigenbasis as opposed to the result in the present literature. In Sections 3.4 and 3.5, we will show that the apparent contradictory results in the present literature can be described as different limiting case of our more general model.
In chapter 4 we will extend the model derived in chapter 2 to describe spin $1/2$ fermionic particles interacting with an external electromagnetic field. The details of the deriva-

tion are reported in Sections 4.1-3. As one might expect, the dynamics will predict deocherence in position, momentum and energy also in this case. In Section 4.2.1 we will discuss the differences with the bosonic model. We will conclude the chapter with a discussion of the limits under which also this master equation describes decoherence in only position or momentum and energy.
The results of Chapter 3 will be applied to matter-wave interferometry in Chapter 5. In section 5.1 we will provide the reader with very essential traits of Mach-Zehnder atomic interferometry. We will also introduce the interferometric visibility, which is the observable we will use to quantify the gravitational deocherence effect in such devices. The details of the theoretical analysis are illustrated in Sections 5.2 and 5.3, where we work out the dynamics of a wavepacket travelling through a symmetric Mach-Zehnder interferometer in presence of respectively a scalar and a tensorial stochastic gravitational perturbation.
In section 5.4 we will answer the question whether atom interferometers are a viable platform to detect the cosmic gravitational background by simulating the visibility of the interferometer for different values of the perturbation's parameters for a selected sample of actual and proposed interferometric experiments.
We will conclude the chapter with the analysis of environmental decoherence in space-based atom interferometers.
Finally we will give our conclusions in chapter 6.

Chapter 2

Gravitational decoherence: state of the art

In this chapter we revise the state of the art literature on gravitational decoherence. We start by providing the unfamiliar reader with a crash course on decoherence. For a more extensive take on the subject, which is beyond the purpose of this thesis, we address the reader to [48, 49]. Finally, we critically review the main models of gravitational decoherence in the present literature and discuss their features. In particular, we analyse their critical issues and open problems, thus providing the motivation for a more general model able to solve the literature's apparent contradictions.

2.1 Decoherence: a brief introduction

Before entering into the core of the thesis, it might be useful to recall some basic notions about decoherence for the unfamiliar reader.
In a nutshell, decoherence is the mechanism that explains the classicality of the world that we all experience in everyday life starting from a quantum microscopic dynamics. On a more technical level, decoherence is the reduction of the quantum coherences in a quantum system due to the interaction with the surrounding environment. In order to better understand how it works, let's consider for instance a very general quantum system in a superposition of normalized states $|\alpha\rangle$ and $|\beta\rangle$, interacting with a generic environment described by the normalised state $|\chi\rangle$, as shown in figure 2.1. We assume the system and the environment to be uncorrelated at initial time, i.e.:

$$|\psi_{t=0}\rangle = \frac{|\alpha_{t=0}\rangle + |\beta_{t=0}\rangle}{\sqrt{2}} \otimes |\chi_{t=0}\rangle \tag{2.1}$$

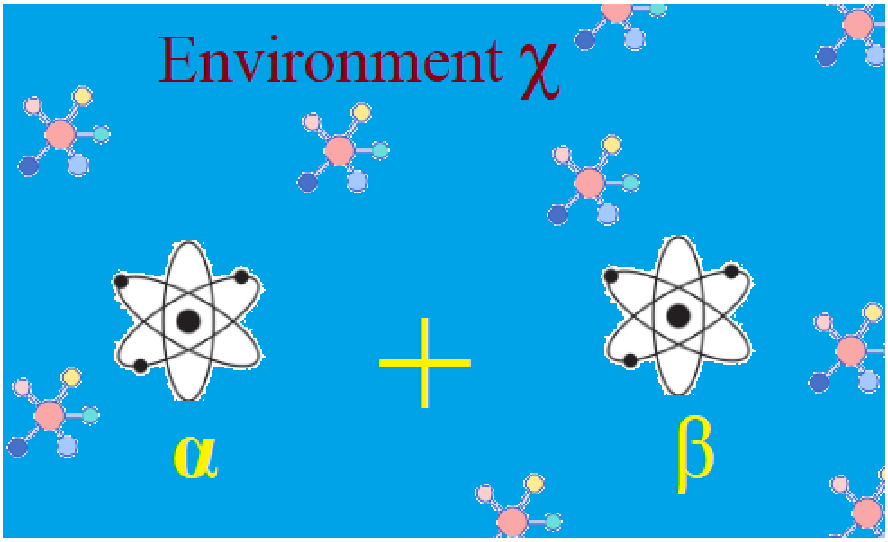

Figure 2.1: Interaction between a quantum system and a surrounding environment. Image realized with flaticon.com

or, alternatively, in the language of the density operator [48]:

$$
\begin{aligned}
\rho_{t=0} \equiv& |\psi_{t=0}\rangle\langle\psi_{t=0}| \\
=& \frac{1}{2}\Big(|\alpha_{t=0}\rangle\langle\alpha_{t=0}| + |\alpha_{t=0}\rangle\langle\beta_{t=0}| + |\beta_{t=0}\rangle\langle\alpha_{t=0}| + |\beta_{t=0}\rangle\langle\beta_{t=0}|\Big) \otimes |\chi_{t=0}\rangle\langle\chi_{t=0}| \\
=:& \rho^{(\mathrm{s})}_{t=0} \otimes \rho^{(\mathrm{E})}_{t=0}
\end{aligned}
\tag{2.2}
$$

where the superscripts (s) and (E) stand for system and environment respectively. Note that, as the system and the environment are uncorrelated at initial time $t = 0$, the initial state is factorized and the description of the system can be completely decoupled from that of the environment.
We then let the system evolve for a certain time τ. Due to the unavoidable interactions between system and environment, the total state at time $t = \tau$ will take the form [48]:

$$
|\psi_{t=\tau}\rangle = \frac{1}{\sqrt{2}}|\alpha^{\chi}_{t=\tau}\rangle \otimes |\chi^{\alpha}_{t=\tau}\rangle + |\beta^{\chi}_{t=\tau}\rangle \otimes |\chi^{\beta}_{t=\tau}\rangle \tag{2.3}
$$

where the superscripts α, β in the environmental states and χ in the system states denote the fact that the interaction with the quantum state alters the initial state of the environment and viceversa. In the language of the density operator, the above expression equivalently reads:

$$
\begin{aligned}
\rho_{t=\tau} = \frac{1}{2}\Big(&|\alpha^{\chi}_{t=\tau}\rangle\langle\alpha^{\chi}_{t=\tau}| \otimes |\chi^{\alpha}_{t=\tau}\rangle\langle\chi^{\alpha}_{t=\tau}| + |\alpha^{\chi}_{t=\tau}\rangle\langle\beta^{\chi}_{t=\tau}| \otimes |\chi^{\alpha}_{t=\tau}\rangle\langle\chi^{\beta}_{t=\tau}| \\
&+|\beta^{\chi}_{t=\tau}\rangle\langle\alpha^{\chi}_{t=\tau}| \otimes |\chi^{\beta}_{t=\tau}\rangle\langle\chi^{\alpha}_{t=\tau}| + |\beta^{\chi}_{t=\tau}\rangle\langle\beta^{\chi}_{t=0}| \otimes |\chi^{\beta}_{t=\tau}\rangle\langle\chi^{\beta}_{t=\tau}|\Big)
\end{aligned}
\tag{2.4}
$$

Note that because of the interaction the state is no longer factorized and, in order to obtain the effective state of the quantum system alone, one has then to integrate the density operator (2.4) over the environmental degrees of freedom (dof). In the language of density operator this is represented by taking the partial trace [48], whose result is the reduced density operator of the system only (ρ_s):

$$\rho_s = \frac{1}{2}\left(|\alpha^{\chi}_{t=\tau}\rangle\langle\alpha^{\chi}_{t=\tau}| + |\beta^{\chi}_{t=\tau}\rangle\langle\alpha^{\chi}_{t=\tau}|\langle\chi^{\alpha}_{t=\tau}|\chi^{\beta}_{t=\tau}\rangle + |\alpha^{\chi}_{t=\tau}\rangle\langle\beta^{\chi}_{t=\tau}|\langle\chi^{\beta}_{t=\tau}|\chi^{\alpha}_{t=\tau}\rangle + |\beta^{\chi}_{t=\tau}\rangle\langle\beta^{\chi}_{t=0}|\right) \tag{2.5}$$

The terms mixing $|\alpha\rangle$ and $|\beta\rangle$ encode the ability of the particle to self interfere, its quantum core. They are proportional to the overlap between the environment states that have interacted respectively with the α and β component of the wavefunction:

$$\langle\chi^{\alpha}_{t=\tau}|\chi^{\beta}_{t=\tau}\rangle = \int d\eta \chi^{*}_{\alpha}(\eta,\tau)\chi_{\beta}(\eta,\tau) \tag{2.6}$$

or its complex conjugate $\langle\chi^{\beta}_{t=\tau}|\chi^{\alpha}_{t=\tau}\rangle$, where η labels the environmental d.o.f. (e.g. position, spin, momentum,...) and $|\eta\rangle$ is a complete basis of the environment's Hilbert space. Note that, if the presence of the quantum state shifts the environment into states χ_α, χ_β that are almost orthogonal, the interference terms become almost zero, and the terms describing the quantum coherence between states $|\alpha_{t=\tau}\rangle$ and $|\beta_{t=\tau}\rangle$ are almost zero too. This means that the system has lost (nearly) all its quantum features, and the reduced density operator effectively describes a statistical mixture of classical states. This is the case, for example, of macroscopic objects, where the interaction with the environment involves a large number, typically an Avogadro number, of molecules, which may be scattered away by the object being in the state $\alpha(x)$ but not in state $\beta(x)$, and viceversa.

As an illustrative example, let us consider a table in a superposition of two different positions x_1 and x_2 in a room full of air, as depicted in Fig. (2.2). We model the quantum state of the center of mass of the table as a superposition of two (very narrow) Gaussian states $|\alpha\rangle$ and $|\beta\rangle$ localized in space around two macroscopically different positions x_1 and x_2 respectively[1]. In Fig. (2.3) we report an illustrative plot for the reduced density matrix at the initial time $t = 0$.
As for the environmental state, we assume it to be a product state of each of the air molecules normalized states present in the room:

$$|\chi_{t=0}\rangle = |\xi_{t=0}\rangle_1 \otimes |\xi_{t=0}\rangle_2 \otimes |\xi_{t=0}\rangle_3 \otimes ... \tag{2.7}$$

to keep the argument at a simple level. Furthermore, we assume that the table and the air in the room do not interact at the initial time $t = 0$. It follows that the initial state reads as in Eq. (2.2). As one might expect, as time passes and the air molecules move

[1]The state α represented in position reads $\alpha(x) = \langle x|\alpha\rangle = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-x_1)^2}{2\sigma^2}}$

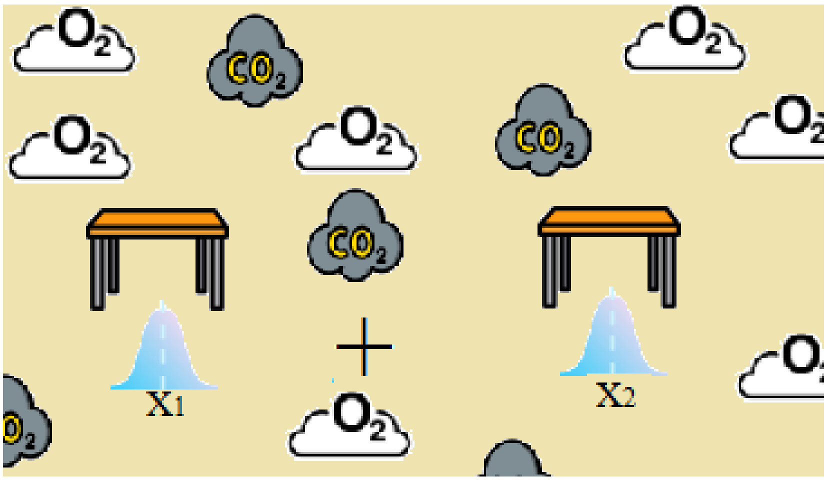

Figure 2.2: A table in a spatial superposition surrounded by air. Image realized with flaticon.com. Note that air is made of mostly N_2 and, in minor part, of other rare gasses, which are not represented here.

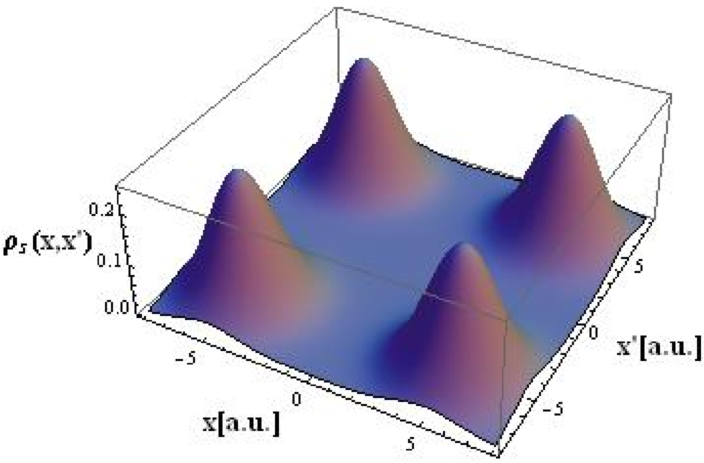

Figure 2.3: Reduced density matrix of a Gaussian state in a spatial superposition at time t=0.

around the room and scatter off the surface of the table, the description of the state of the system and the environment will be of the form of Eq. (2.3). The superscript α and β now denote the fact that the table being located around x_1 or x_2 might affect the position (if the table is or isn't in say x_1 there cannot or has to be air there) and the trajectory (air molecules moving towards say x_1 might or not bounce off the table) of the air molecules. Due to its way heavier mass, the state of table is practically not affected by the air, therefore we can safely drop the superscript χ.
The reduced density operator describing the state of the table alone then reads as in

Eq. (2.4), and the interference term reads:

$$\langle \chi^{\alpha}_{t=\tau} | \chi^{\beta}_{t=\tau} \rangle = \int dx \chi^{*}_{\alpha}(x,\tau)\chi_{\beta}(x,\tau) \tag{2.8}$$

Because of the presence (absence) of the table around x_1 and x_2, the position of the air molecules in the proximity of such locations differs in the $|\xi^{\alpha}_{t=\tau}\rangle$ and $|\xi^{\beta}_{\tau}\rangle$ states, and their overlap is:

$$\langle \xi^{\alpha} | \xi^{\beta} \rangle \sim \epsilon \lesssim 1 \tag{2.9}$$

as we assumed all the single molecule states $|\xi\rangle$ to be normalized. Given the macroscopic size of the table, it is clear that there will be an order of an Avogadro number of air molecules whose position is affected by the presence of the table. The interference terms in the reduced density matrix will therefore be proportional to

$$\langle \chi_{\alpha} | \chi_{\beta} \rangle = \langle \xi_{\alpha} | \xi_{\beta} \rangle_1 \langle \xi_{\alpha} | \xi_{\beta} \rangle_2 \langle \xi_{\alpha} | \xi_{\beta} \rangle_3 ... \sim (\epsilon)^{10^{23}} \sim 0 \tag{2.10}$$

The reduced density operator will therefore diagonalize in the basis of Gaussian spatially localized states, which are eigenstates of the position operator, so that the table will be in a statistical mixture of classical, localized in position states.
In Fig. (2.4) it is shown the time evolution of the reduced density matrix relative to a superposition of position Gaussian state whose off diagonal terms are exponentially suppressed, as it is typical of many collisional decoherence models.

Even if the above presented example might trick the reader into thinking the contrary, note that decoherence can be relevant even for single quantum particles since, as time passes, the free time evolution drives the environmental states ever further apart. Furthermore, the decoherence phenomenon does not occur in the position eigenbasis only, but can in fact happen in any basis of the system's Hilbert space. In the next chapters, we will in fact encounter decoherence in the momentum and energy eignebasis.

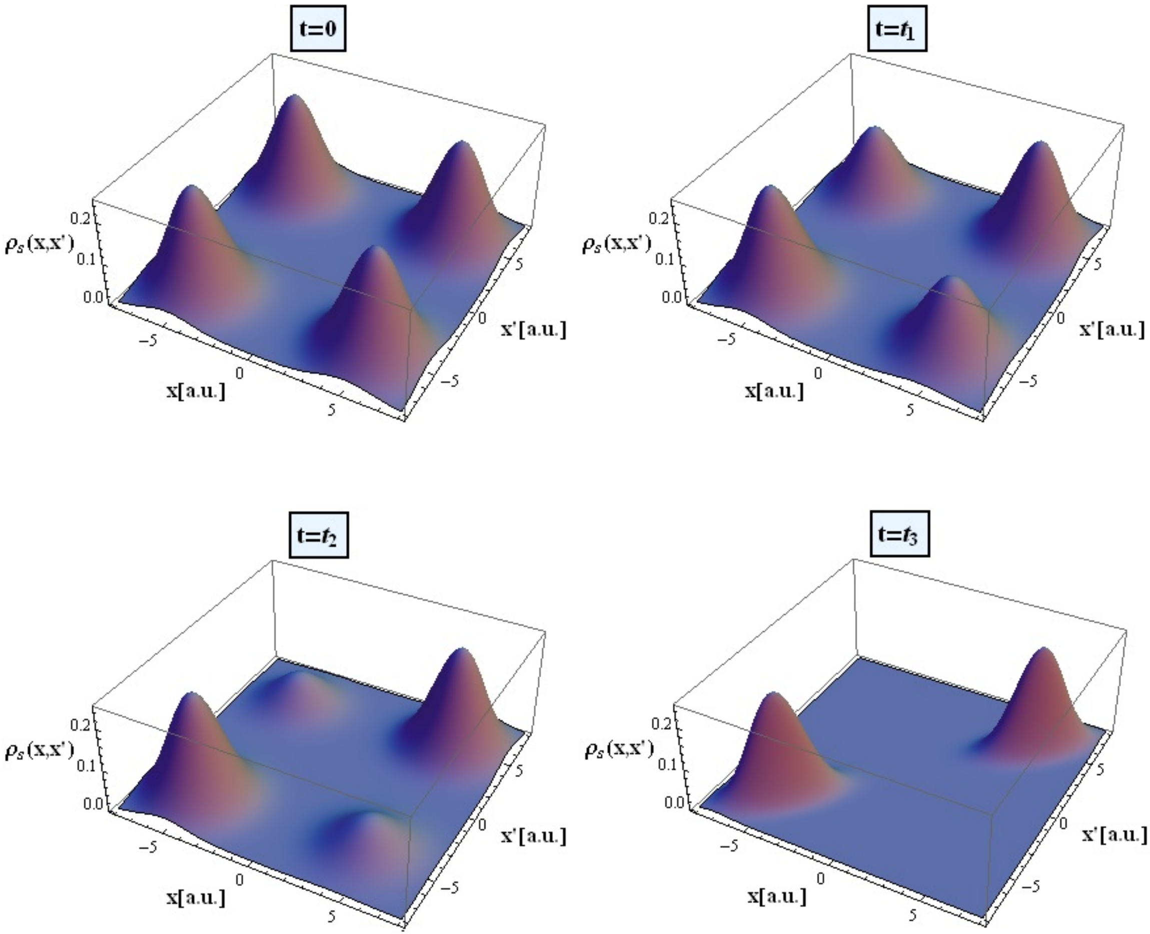

Figure 2.4: Time evolution of the reduced density matrix for a superposition of Gaussian position states with exponential off diagonal suppression. Note that $0 < t_1 < t_2 < t_3$, and that the full description of the time evolution of the reduce density matrix should also take into account the free evolution of the diagonal terms.

2.1.1 Gravity and decoherence

We complete the introduction by adding gravity to the game: we illustrate how a gravitational perturbation affects the dynamics of a matter field, eventually leading to a decoherence effect. In the following we will only consider a classical stochastic gravitational perturbation for the simplicity of the argument. The extension to a (quantum) graviton bath requires the introduction of quite some technical tools which lie beyond the purpose of this thesis, and is thus not as illustrative as the classical case. Nevertheless, we address the interested reader to [38, 42] and the references therein included for a complete treatment.

Let us begin by considering a matter wavepacket which we describe by means of a

first quantized scalar field $\psi(x)$. Such a wavepacket propagates in a region of spacetime in which the flat Minkowski background (with metric $\eta_{\mu\nu}$) is perturbed by a weak stochastic gravitational fluctuation $h_{\mu\nu}(x)$. The fluctuation is in general a function of spacetime (x), and by weak we mean that $|h_{\mu\nu}(x)| = \epsilon \ll 1$.
Due to the interaction between the matter field and the gravitational perturbation, the scalar field satisfies the Klein Gordon equation minimally coupled to gravity:

$$g^{\mu\nu}\nabla_\mu\partial_\nu\psi(x) - \frac{m^2c^2}{\hbar^2}\psi(x) = 0 \tag{2.11}$$

where $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ is the total metric tensor, whose spacetime dependence $(g_{\mu\nu}(x))$ was made implicit for convenience, and ∇_μ is the covariant derivative with respect to the Christoffel connection. Following the work of Linet and Tourrenc [17] we write the wave packet as $\psi(x) = e^{\frac{i\phi(x)}{\hbar}}$. We then reformulate the Klein Gordon equation as an equation for the wave's phase $\phi(x)$ with the help of the semi-classical WKB approximation [50] :

$$g^{\mu\nu}\partial_m\phi(x)\partial_\nu\phi(x) = \frac{m^2c^2}{\hbar^2} \tag{2.12}$$

Because of the weak field limit for the gravitational field $|h_{\mu\nu}| = \epsilon$, we are allowed to expand the phase of the matter field as $\phi = \phi^{(0)}(x) + \epsilon\phi^{(1)}(x)$. Thus we obtain the following system of coupled differential equations:

$$\frac{m^2c^2}{\hbar^2} = \eta^{\mu\nu}\partial_\mu\phi^{(0)}(x)\partial_\nu\phi^{(0)}(x) \tag{2.13}$$

$$0 = 2\eta^{\mu\nu}\partial_\mu\phi^{(0)}\partial_\nu\phi^{(1)}(x) + h^{\mu\nu}(x)\partial_\mu\phi^{(0)}(x)\partial_\nu\phi^{(0)}(x) \tag{2.14}$$

The first equation gives the expected plane wave solution in flat spacetime:

$$\phi^{(0)}(x) = \eta_{\mu\nu}\xi^\mu x^\nu + \phi_0^{(0)} \tag{2.15}$$

where ξ is a constant four-vector whose components are $\xi^0 = E/(\hbar c)$, $\xi^i = p^i/\hbar$, with E and $\mathbf{p}$ respectively the energy and the momentum of the wavepacket, and $\phi_0^{(0)}$is a constant. Upon plugging the above solution into Eq. (2.14), we get:

$$\xi^\nu\partial_\nu\phi^{(1)}(x) = \frac{1}{2}h_{\alpha\beta}(x)\xi^\alpha\xi^\beta. \tag{2.16}$$

The above equation can be conveniently simplified in the long wavelength limit, where the typical wavelength of the gravitational perturbation (λ_{gw}) is much larger than the typical length (λ_{wp}) of the wavepacket, i.e. $\lambda_{gw} \gg \lambda_{wp}$, so that the wavepacket can be assimilated to a point-like particle. In this limit the particle follows a well defined trajectory, which can be parametrized by a timelike parameter τ as: $x^\mu = \tau\xi^\mu + x_0^\mu$. Since the dynamics of the particle is influenced only by the fluctuations $h(x)$ that occurs on its path, we can also parametrize $h_{\mu\nu}(x) = h_{\mu\nu}(x(\tau))$ in Eq. (2.16) to obtain:

$$\frac{d\phi^{(1)}}{d\tau} = \frac{1}{2} h_{\mu\nu} \xi^{\mu} \xi^{\nu}, \tag{2.17}$$

Its solution is:

$$\phi^{(1)} = \frac{1}{2} \int_0^t d\tau \, h_{\mu\nu}(x(\tau)) \xi^{\mu} \xi^{\nu}, \tag{2.18}$$

whose physical interpretation is immediate: the presence of a gravitational perturbation induces a change into the phase of matter waves.
Let us now suppose that the wavepacket is initially prepared in a spatial superposition of Gaussian spatially localized quantum states $|\alpha\rangle$ and $|\beta\rangle$, as in the previous section:

$$|\psi(t=0)\rangle = \frac{|\alpha(t=0)\rangle + |\beta(t=0)\rangle}{\sqrt{2}} \tag{2.19}$$

It follows that, at a later time t, the state of the system plus environment will be:

$$|\psi(t)\rangle = \frac{(e^{i\phi_{\alpha}^{(1)}} |\alpha(t)\rangle + e^{i\phi_{\beta}^{(1)}} |\beta(t)\rangle)}{\sqrt{2}} \tag{2.20}$$

where the subscripts α and β label the trajectories of the two different bunches of the wavepacket. Note that the phase shifts $\phi_i^{(1)} = \int dt \, h_{\mu\nu}(x_{(i)}(\tau)) \xi^{\mu} \xi^{\nu}$, $i = \alpha, \beta$ are stochastic variables since they depend on the stochastic quantity $h_{\mu\nu}$. As a consequence, the expected value for every observable of the system is obtained upon taking the stochastic average over the random phase probability distribution $P(\phi^{(1)})$. This of course applies to the density operator describing the quantum system, which therefore reads:

$$\begin{aligned} \rho_s(t) = \frac{1}{2} \Big(& |\alpha(t)\rangle\langle\alpha(t)| + |\beta(t)\rangle\langle\alpha(t)| \mathbb{E}[e^{-i(\phi_{\alpha}^{(1)} - \phi_{\beta}^{(1)})}] \\ & + |\alpha(t)\rangle\langle\beta(t)| \mathbb{E}[e^{i(\phi_{\alpha}^{(1)} - \phi_{\beta}^{(1)})}] + |\beta(t)\rangle\langle\beta(t)| \Big) \end{aligned} \tag{2.21}$$

with $\mathbb{E}[e^{i\phi^{(1)}}] = \int d\phi^{(1)} P(\phi^{(1)}) e^{i\phi^{(1)}}$. The interference term then reads:

$$I = Re[|\beta(t)\rangle\langle\alpha(t)\rangle \mathbb{E}[e^{i(\phi_{\alpha}^{(1)} - \phi_{\beta}^{(1)})}]] \tag{2.22}$$

By comparing Eq. (2.22) with Eq. (2.6), we can conclude that the interaction with a stochastic gravitational perturbation will induce a decoherence phenomenon in the matter wave dyinamics. Equations (3.13-14) also tell us that the characterization of the decoherence effect depends crucially on the probability density for the phase shift. For the typical case of interest, when the phase is accumulated in a series of non correlated events, as it is reasonable to happen when the noise is produced by different unresolved astronomical or cosmological sources, the central limit theorem suggests that $P(\phi^{(1)})$ can be taken to be a normal distribution. It is then a standard result to show that:

$$\mathbb{E}[e^{i\phi^{(1)}}] = e^{i\mathbb{E}[\phi^{(1)}]-\frac{1}{2}\mathbb{E}[(\delta\phi^{(1)})^2]} \tag{2.23}$$

with $\delta\phi^{(1)} = \phi^{(1)} - \mathbb{E}[\phi^{(1)}]$, and $\phi^{(1)} = \phi^{(1)}_\alpha - \phi^{(1)}_\beta$ for short. The first term in Eq. (2.23) ($e^{i\mathbb{E}[\phi^{(1)}]}$) is just a global phase, while the second ($e^{-\frac{1}{2}\mathbb{E}[(\delta\phi^{(1)})^2]}$) is a damping that grows with the accumulated phase, acting on the coherences of the quantum systems.

We have thus shown that the presence of a gravitational perturbation affects the dynamics of a matter particle so that it induces a phase accumulation. In the case the perturbation is stochastic, such a phase shift is then responsible for the loss of coherence of the system.

The scheme implemented in this section for the description of gravity induced decoherence is rather immediate and illustrative, but it surely is not a rigorous, comprehensive and exhaustive treatment of the phenomenon. In the next section we will recap the results proposed in the literature of the most relevant models describing the effects of a (stochastic or quantum) gravitational perturbation on the dynamics of a particle or a matter field.

2.2 Literature and state of the art

The literature concerning gravitational decoherence is a young but rather rich one. Although the first isolated works [36, 37] trace back to the late '80s and early '90s, the subject has gained the interest of a wider part of the scientific community only since the turn of the century [38, 40, 41, 42, 43, 44, 45, 51, 46, 52, 47]. In what follows we will revise the main features of some of the most relevant models present in the literature. In particular, we highlight the main differences between the various models, explaining how they give rise to some open issues and questions that will be addressed throughout the rest of this thesis. For a complete classification of the literature, which lies beyond the purpose of this work, we refer the reader to the following review [46].

The specific mathematical tools and techniques involved in the derivation of each of the above cited models differ from work to work. However, it is possible to highlight a common pattern (although not always followed by each model in the same order as presented below) for the formulation of the dynamical equations describing gravity induced decoherence.
The starting point is the promotion to a generally covariant setting of the flat scalar bosonic field (ϕ) action ($\mathcal{S}$), in order to account for curved spacetime effects [53]:

$$\mathcal{S} = \int d^4x\, c^2(\partial_\mu\phi^*\partial^\mu\phi - \frac{m^2c^2}{\hbar^2}|\phi|^2) \tag{2.24}$$

$$\begin{cases} \eta_{\mu\nu} \to g_{\mu\nu} \\ d^4x \to \sqrt{-g}\, d^4x \\ \partial_\mu \to \nabla_\mu \end{cases} \tag{2.25}$$

where $\eta_{\mu\nu}$ and $g_{\mu\nu}$ are respectively the flat and the curved spacetime metric, and ∇_μ denotes the covariant derivative with respect to the Christoffel connection. The metric is then expanded around a flat Minkowski background for small perturbations ($h_{\mu\nu}$) as:

$$\begin{cases} g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \\ |h_{\mu\nu}| \ll 1 \end{cases} \tag{2.26}$$

so that a new effective action for the scalar field can be derived at first order in the gravitational perturbation $h_{\mu\nu}$:

$$\mathcal{S} \sim \int d^4x \left(\mathcal{L}\Big|_{g=\eta} + \frac{\partial(\mathcal{L}\sqrt{-g})}{\partial g_{\mu\nu}}\Big|_{g=\eta} h_{\mu\nu} + O(h^2) \right) \tag{2.27}$$

Such an expression is then considerably simplified by fixing a particular gauge, and restricting to a specific type of gravitational perturbation. For example, Anastopoulos and Hu in [38] impose the Transverse Traceless (TT) gauge [11], Power and Percival and Sanchez-Gomez consider only conformal gravitational perturbations in [41] and [37] respectively, and Breuer et al. assume the typical wavelenght of the perturbation to be much smaller than the resolution scale of the quantum particle that described by the scalar bosonic field in [40].

Then, the equations of motions (EOM) for the scalar field are derived, either exploiting the Feynman-Vernon's influence functional [54] in the path integral formalism [37, 38, 42], or via a variational principle [40, 41], and subsequently projected to the one particle sector.

The EOM are further expanded and specialized to the non relativistic limit $|\mathbf{p}| \ll mc$, where $\mathbf{p}$ is the momentum of the particle. Finally, if not already in this form, the EOM are translated in the density matrix ($\hat{\rho}$) formalism, where the quantum [38, 42] or stochastic [41, 40, 37] average over a Gaussian gravitational noise can be taken, provided the specific form of the noise's two point correlation function (which again varies from model to model).

The resulting master equations can at this point be regrouped in two distinct classes: those that predict decoherence in the position eigenbasis, and those that predict decoherence in the energy ($E = \frac{\hat{\mathbf{p}}^2}{2m}$) eigenbasis.

In the first class fall the models of Sanchez-Gomez [37]:

$$\partial_t \rho(\mathbf{x}, \mathbf{x}'; t) = \frac{i\hbar}{2m}(\nabla^2_{\mathbf{x}} - \nabla^2_{\mathbf{x}'})\rho(\mathbf{x}, \mathbf{x}', t) - \frac{2m^2\alpha^2 Lc^3}{\hbar^2}(1 - e^{-(\mathbf{x}-\mathbf{x}')^2/L^2})\rho(\mathbf{x}, \mathbf{x}'; t) \tag{2.28}$$

of Power and Percival [41]:

$$\partial_t \rho(\mathbf{x}, \mathbf{x}'; t) = \frac{i\hbar}{2m}(\nabla^2_{\mathbf{x}} - \nabla^2_{\mathbf{x}'})\rho(\mathbf{x}, \mathbf{x}', t) - \frac{\sqrt{\pi}m^2\alpha^4 Lc^3}{\sqrt{2}\hbar^2}(1 - e^{-(\mathbf{x}-\mathbf{x}')^2/L^2})\rho(\mathbf{x}, \mathbf{x}'; t) \tag{2.29}$$

and of Blencowe [42]:

$$\partial_t \rho(\mathbf{x}, \mathbf{x}'; t) = \Big[\frac{i\hbar}{2M}(\nabla^2_{\mathbf{x}} - \nabla^2_{\mathbf{x}'}) - \frac{\mathcal{K}_B \Theta G}{\hbar c} \int d^3 r \Big(m(\mathbf{r} - \mathbf{x}) - m(\mathbf{r} - \mathbf{x}') \Big)^2 \Big] \rho(\mathbf{x}, \mathbf{x}'; t) \tag{2.30}$$

where $\mathcal{K}_B$ is the Boltzmann constant, G the gravitational constant L is the correlation length of the noise, α its amplitude, Θ its temperature, and τ_c its correlation time. In the second class fall instead the models of Breuer et al [40]:

$$\partial_t \hat{\rho}(t) = -\frac{i}{\hbar}\Big[\frac{\hat{\mathbf{p}}^2}{2m}, \hat{\rho}(t)\Big] - \frac{\tau_c}{2\hbar^2}\Big[\frac{\hat{\mathbf{p}}^2}{2m}, \Big[\frac{\hat{\mathbf{p}}^2}{2m}, \hat{\rho}(t)\Big]\Big] \tag{2.31}$$

and of Anastopoulos and Hu [38]:

$$\partial_t \hat{\rho}(t) = -\frac{i}{\hbar}\Big[\frac{\hat{\mathbf{p}}^2}{2m}, \hat{\rho}(t)\Big] - \frac{16\pi \mathcal{K}_B G \Theta}{9\hbar c}\Big[\frac{\hat{\mathbf{p}}^2}{2m}, \Big[\frac{\hat{\mathbf{p}}^2}{2m}, \hat{\rho}(t)\Big]\Big] \tag{2.32}$$

We summarize the features of each of the above analyzed models in Table I.

Table I: Gravitational decoherence models

	Nature of perturbation	**Shape and/or size**	**Gauge**	**correlation function**	**decoherence eigenbasis**
Sanchez-Gomez	classical	conformal	harmonic	Gaussian	position
Anastopoulos and Hu	quantum	generic	TT	thermal bath	energy
Blencowe	quantum	smaller than particle's size	harmonic	thermal bath	position
Breuer	classical	smaller than particle's resolution	harmonic	Gaussian	energy
Power and Percival	classical	conformal	harmonic	Gaussian	position

From the table is not possible to straigthforwardly assess whether and which particular assumption (for instance the gauge or the shape of the correlation function) is responsible for the determination of the eigenbase of decoherence. As an example in fact, both the Blencowe and Anastopoulos Hu models describe a quantum graviton bath with thermal correlation function, but nevertheless do not agree on the basis of

decoherence. Or, both the Breuer et al. and Power Percival ones choose the harmonic gauge, but the first predicts decoherence in energy while the second in position, and so on.
It follows that a general description of the underlying dynamics of gravitational decoherence able to encompass and connect the various results is needed.

In order to fill the gap in the literature and solve the decoherence eigenbasis puzzle, in the next chapter we will derive a novel model describing the effects of a stochastic gravitational perturbation on the dynamics of a scalar field in the non relativistic limit, which predicts deocherence in a variety of eigenbasis including position and energy in the appropriate limits. We will keep the treatment as general as possible and specialize the properties of the gravitational perturbation only as the last step. With the appropriate choices for such properties, we will be able to quantitatively recover the results of the literature.

Chapter 3

Gravitational decoherence: bosonic matter

In this chapter we develop a novel and general model for the decoherence induced by a stochastic gravitational perturbation on non relativistic scalar matter. The dynamics predicts decoherence in position, momentum and energy, depending on the properties of the metric perturbation. We show how our master equation reproduces the results present in the literature by taking appropriate limits, thus explaining the apparent contradiction in their dynamical description.

3.1 Hamiltonian equations of motion

We begin our analysis by considering the effects of a weak gravitational perturbation on the dynamics of scalar matter. We therefore derive the equations of motion (EOM) for a scalar bosonic field minimally coupled to linearized gravity. Let us consider the action for the charged Klein Gordon field in curved spacetime [53]:

$$S = \int d^4x \sqrt{-g}\mathcal{L} \tag{3.1}$$

with the Lagrangian density:

$$\mathcal{L} = (c^2 g^{\mu\nu} \nabla_\mu \psi^* \nabla_\nu \psi - \frac{m^2 c^4}{\hbar^2} \psi^* \psi) \tag{3.2}$$

where ∇_μ is the covariant derivative with respect to the Christoffel connection. We write the metric as the sum of a flat background $\eta_{\mu\nu} = \text{diag}(+---)$, and a perturbation $h_{\mu\nu}$:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{3.3}$$

We are interested in studying the dynamics of the Klein Gordon field in presence of a weak gravitational perturbation. Therefore we perform a Taylor expansion of the action around the flat background metric and truncate the series at the first perturbative order. Thus, we obtain the effective Lagrangian $\mathcal{L}_{eff}$ acting on flat spacetime:

$$
\begin{aligned}
S &= \int d^4x \Big[c^2(\eta^{\mu\nu}\partial_\mu\psi^*\partial_\nu\psi - \frac{m^2c^2}{\hbar^2}\psi^*\psi)(1+\frac{tr(h^{\mu\nu})}{2}) - c^2h^{\mu\nu}\partial_\mu\psi^*\partial_\nu\psi + O(h^2)\Big] \\
&=: \int d^4x(\mathcal{L}_{eff} + O(h^2))
\end{aligned}
\tag{3.4}
$$

Note that in doing so we are implicitely restricting the analysis to the class of reference frames in which the coordinates are described by rigid rulers, which are negligibly affected by the gravitational perturbation. This assumption though reasonable, as measuring devices are held together by intra and iter molecular forces, is arbitrary (it may be possible that a gravitational perturbation bends a measuring device).
The equations of motion for the matter field are obtained (at first order in the perturbation $h_{\mu\nu}$) from the Euler-Lagrange equations:

$$
\frac{\partial\mathcal{L}_{eff}}{\partial\psi^*} - \partial_\alpha\frac{\partial\mathcal{L}_{eff}}{\partial\partial_\alpha\psi^*} = 0 \tag{3.5}
$$

and in the harmonic gauge[1] they read:

$$
\Big[-\partial_t^2 + c^2(1+h^{00})\nabla^2 + 2ch^{0i}\partial_t\partial_i + c^2h^{ij}\partial_i\partial_j - \frac{m^2c^4}{\hbar^2}(1+h^{00}) + O(h^2)\Big]\psi = 0 \tag{3.6}
$$

We are interested in the description of the dynamics of a positive energy particle system in the non relativistic limit. In such a limit, the particle and antiparticle sectors are non interacting with one another, that is to say, the EOM (3.6) can be recast to a system of two uncoupled equations, one for each species sector. While this is evident and straightforward for the free case, for an interacting theory the decoupling is very complicated and achievable only perturbatively.
The first step is to explicitely express the field in a two component form. This can be done following the Feshbach-Villars formulation [55]. Accordingly we define a new field:

$$
\Psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix} \tag{3.7}
$$

such that:

$$
\begin{cases} \psi = \phi + \chi \\ i\hbar\Big(\partial_t - ch^{0i}\partial_i\Big)\psi = mc^2(\phi - \chi) \end{cases} \tag{3.8}
$$

[1]The harmonic Gauge implies translational invariance of the infinitesimal volume in the chosen coordinate system as e.g. in cartesian coordinates.

We note that such a formulation does not allow for a probabilistic interpretation of the field Ψ, as the conserved charged Q associated to the internal $U(1)$ symmetry ($\psi \to e^{ie}\psi$; $\psi^* \to e^{-ie}\psi^*$) via Noether's Theorem reads:

$$Q = 2e\, mc^2 \int d^3x \begin{pmatrix} \phi & \chi \end{pmatrix} \sigma_3 (1 + \frac{tr(h^{\mu\nu})}{2} - h^{00}) \begin{pmatrix} \phi \\ \chi \end{pmatrix} \tag{3.9}$$

instead of the required:

$$\rho = 2e\, mc^2 \int d^3x \begin{pmatrix} \phi & \chi \end{pmatrix} \sigma_3 \begin{pmatrix} \phi \\ \chi \end{pmatrix} \tag{3.10}$$

We therefore apply the transformation:

$$\begin{cases} T & = & (1 + \frac{tr(h)}{4} - \frac{h_{00}}{2}) \\ \Psi & \to & T\Psi \end{cases} \tag{3.11}$$

so that, in the new representation, the squared modulus of the field can be regarded as a probability density in the non relativistic limit.
With the help of Eq. (3.8) and after some algebra (See Appendix A) the EOM (3.6) read:

$$i\hbar\partial_t \Psi = [mc^2\sigma^3 + \mathfrak{E} + \mathcal{O}]\Psi \tag{3.12}$$

where:

$$\begin{aligned} \mathfrak{E} =& \frac{mc^2}{2} h^{00}\sigma_3 - \frac{\hbar^2}{2m}(1 + h^{00})\sigma_3\nabla^2 - \frac{\hbar^2}{2mc}\partial_t(h^{0i})\sigma_3\partial_i - \frac{\hbar^2}{2m}h^{ij}\sigma_3\partial_i\partial_j + i\hbar c h^{0i}\partial_i \\ & - \frac{i\hbar}{2}\partial_t(\frac{tr(h^{\mu\nu})}{2} - h^{00}) - \Big[\frac{\hbar^2}{4m}\nabla^2(h^{00}) - \frac{i\hbar^2}{8m}\nabla^2(tr(h^{\mu\nu}))\Big)\Big]\sigma_3 \end{aligned} \tag{3.13}$$

$$\begin{aligned} \mathcal{O} =& \frac{imc^2}{2} h^{00}\sigma_2 - \frac{i\hbar^2}{2m}(1 + h^{00})\sigma_2\nabla^2 - \frac{i\hbar^2}{2mc}\partial_t(h^{0i})\sigma_2\partial_i - \frac{i\hbar^2}{2m}h^{ij}\sigma_2\partial_i\partial_j \\ & - \Big[\frac{i\hbar^2}{4m}\nabla^2(h^{00}) - \frac{i\hbar^2}{8m}\nabla^2(tr(h^{\mu\nu}))\Big]\sigma_2 \end{aligned} \tag{3.14}$$

are respectively the diagonal and antidiagonal parts of the Hamiltonian $K = mc^2\sigma^3 + \mathfrak{E} + \mathcal{O}$, and σ_i, $i = 1, 2, 3$ are the Pauli matrices.

In the next section we will decouple the EOM to then take the non relativistic limit.

3.2 Non relativistic limit and canonical quantization

We aim to find a representation of the two component field Ψ in which the EOM (3.12) are diagonal. This representation can be found in the non relativistic limit following the Foldy-Wouthuysen Method [56], which allows one to write perturbatively (at any order in $\frac{v}{c}$) two decoupled equations, one for each component of the field. The method is operatively characterized by the application of an appropriate transformation U:

$$\Psi \to \Psi' = U\Psi \tag{3.15}$$

$$K \to K' = U(K - i\hbar\partial_t)U^{-1} = mc^2\sigma_3 + \mathfrak{E}' + \mathcal{O}' + O(h^2) \tag{3.16}$$

such that, in the new representation, the antidiagonal part $\mathcal{O}'$ is of higher order in $\frac{v}{c}$ than the diagonal $\mathfrak{E}'$. By neglecting $\mathcal{O}'$ one recovers two decoupled equations. By performing iteratively the transformation, one can always find a representation of the two component field for which the EOM are diagonal at any desired order in $\frac{v}{c}$.
In our case, we have that the task is easily achieved by applying the subsequent transformations:

$$\begin{cases} U &= e^{-i\sigma_3\mathcal{O}/(2mc^2)} \\ U' &= e^{-i\sigma_3\mathcal{O}'/(2mc^2)} \\ U'' &= e^{-i\sigma_3\mathcal{O}''/(2mc^2)} \end{cases} \tag{3.17}$$

after which, with some algebra (see Appendix B), the EOM read:

$$\begin{aligned} i\hbar\partial_t\Psi =& H\Psi \\ =& \Big[mc^2(1+\frac{h^{00}}{2})\sigma_3 - \frac{\hbar^2}{2m}(1+\frac{h^{00}}{2})\nabla^2\sigma_3 - \frac{\hbar^2}{2m}h^{ij}\partial_i\partial_j\sigma_3 + i\hbar c h^{0i}\partial_i + \frac{i\hbar}{2}\partial_t(h^{00}) \\ &- \frac{i\hbar}{4}\partial_t(tr(h^{\mu\nu})) + \frac{\hbar^2}{8m}\nabla^2(tr(h^{\mu\nu}))\sigma_3\Big]\Psi + O(c^{-4}) + O(h^2_{\mu\nu}) \end{aligned} \tag{3.18}$$

Note that as the transformations (3.17) are generalized unitary [57], they preserve the conserved charge in (3.9), i.e. the probability density in the non relativistic limit.
In the non relativistic limit the EOM (3.18) do not mix the two components ϕ and χ of the field (up to a very small correction). As we are interested in the dynamics of particles only, we restrict the analysis to the first field component ϕ.

Since the dynamics preserves the probability density, we are allowed to apply the canonical quantization prescription and impose the equal time commutation relations:

$$\begin{aligned} [\hat{\phi}(t,\mathbf{x}), \hat{\phi}(t,\mathbf{x}')] =& [\hat{\phi}^\dagger(t,\mathbf{x}), \hat{\phi}^\dagger(t,\mathbf{x}')] = 0 \\ [\hat{\phi}(t,\mathbf{x}), \hat{\phi}^\dagger(t,\mathbf{x}')] =& \delta^3(\mathbf{x}-\mathbf{x}') \end{aligned} \tag{3.19}$$

to obtain the EOM for the quantum field. The equation thus obtained does not allow for the creation or annihilation of particles. We can thus safely project it onto a single particle sector to obtain the single particle Schrödinger equation:

$$i\hbar\partial_t|\phi(t)\rangle = (\hat{H}_0 + \hat{H}_p + \hat{H}_d)|\phi(t)\rangle \tag{3.20}$$

with:

$$\begin{aligned}\hat{H}_0 =& mc^2 + \frac{\hat{\mathbf{p}}^2}{2m}\\ \hat{H}_p =& \frac{mc^2}{2}h^{00}(t,\hat{\mathbf{x}}) - \frac{\hbar^2}{8m}\{h^{00}(t,\hat{\mathbf{x}}),\hat{\mathbf{p}}^2\} + \frac{c}{2}\{h^{0i}(t,\hat{\mathbf{x}}),\hat{p}_i\} - \frac{1}{4m}\{h^{ij}(t,\hat{\mathbf{x}}),\hat{p}_i\hat{p}_j\}\\ \hat{H}_d =& \frac{\hbar^2}{8m}\nabla^2(tr[h^{\mu\nu}(t,\hat{\mathbf{x}})]) + \frac{i\hbar}{2}\partial_t(h^{00}(t,\hat{\mathbf{x}})) - \frac{i\hbar}{4}\partial_t(tr[h^{\mu\nu}(t,\hat{\mathbf{x}})])\end{aligned} \tag{3.21}$$

where $\hat{\mathbf{x}}, \hat{\mathbf{p}}$ are respectively the single particle position and the momentum operator. Note that the anticommutators between the gravitational perturbation (which is a function of the position operator) and the particle's momentum operator need to be included in the quantization prescription in order to guarantee the hermiticity of the Hamiltonian. The term H_0 is the standard free Hamiltonian plus an irrelevant global phase mc^2 that can be reabsorbed with the transformation:

$$|\phi(t)\rangle \to e^{imc^2t/\hbar}|\phi(t)\rangle \tag{3.22}$$

The terms $\hat{H}_p$ and $\hat{H}_d$ are a perturbation of $\hat{H}_0$, and encode the interaction between the scalar bosonic particle and a weak, otherwise generic, gravitational perturbation. We note that Eq. (3.20) correctly reduces to the usual Schrödinger equation for a particle in an external static Newtonian potential:

$$\begin{cases} i\hbar\partial_t|\phi(t)\rangle &= \left(\frac{\hat{\mathbf{p}}^2}{2m} - m\Phi\right)|\phi(t)\rangle \\ \Phi &= -\frac{c^2h^{00}}{2}\end{cases} \tag{3.23}$$

if we consider the external gravitational field to be of the same form of that of the Earth.

The generalization of Eq. (3.20) to an extended body is not an easy task, as one needs to take into account the degrees of freedom of all the elementary particles that constitute the body. However, it is rather simple to obtain the dynamics for just the center of mass if we assume that the internal degrees of freedom are frozen and cannot be excited by the gravitational perturbation as in the case of a rigid body. In such an approximation it is convenient to define the center of mass ($\hat{\mathbf{X}}$) and relative coordinate ($\hat{\mathbf{r}}_i$) operators:

$$\begin{cases} \hat{\mathbf{X}} = \int d^3r\, \mathbf{r}\frac{\hat{m}(\mathbf{r})}{M} \\ \hat{\mathbf{r}}_i = \hat{\mathbf{x}}_i - \hat{\mathbf{X}}\end{cases} \tag{3.24}$$

and their canonical conjugates, respectively $\hat{\mathbf{P}}$ and $\hat{\mathbf{k}}_i$, where $\hat{m}(\mathbf{r})$ is the mass density operator [2] and $M = \int d^3r\ \hat{m}(\mathbf{r})$ is the total mass. Upon tracing out the relative

[2] Under the assumption that the rigid body consists in an ensenble of a large number N of particles, the density operator can be defined as $\hat{m}(\mathbf{x}) = \sum_i \frac{m_i}{(2\pi\hbar)^3}\int d\mathbf{q}\, e^{-\frac{i}{\hbar}(\mathbf{x}-\hat{\mathbf{x}}_i)\cdot\mathbf{q}}$, where m_i and $\hat{x}_i$ are the mass and the position operator of the i-th particle.

degrees of freedom, the Hamiltonian for the center of mass of a rigid body reads:

$$\begin{aligned}\hat{H} =& Mc^2 + \frac{\hat{\mathbf{P}}^2}{2M} + \int d^3r\, h^{00}(\mathbf{r},t) m(\hat{\mathbf{X}}+\mathbf{r})c^2 - \int d^3r\, h^{00}(\mathbf{r},t)\frac{\{m(\mathbf{r}+\hat{\mathbf{X}}),\hat{\mathbf{P}}^2\}}{8M^2} \\ &+ c\int d^3r\, h^{0i}(\mathbf{r},t)\frac{\{m(\mathbf{r}+\hat{\mathbf{X}}),\hat{P}_i\}}{2M} - \int d^3r\, h^{ij}(\mathbf{r},t)\frac{\{m(\mathbf{r}+\hat{\mathbf{X}}),\hat{P}^i\hat{P}^j\}}{4M^2} \\ &+ \frac{i\hbar c^2}{2}\int d^3r\, \partial_t\Big(h^{00}(\mathbf{r},t) - \frac{1}{2}tr(h^{\mu\nu}(\mathbf{r},t))\Big)\frac{m(\hat{\mathbf{X}}+\mathbf{r})}{M} \\ &+ \frac{\hbar^2c^2}{8M}\int d^3r\, \nabla^2(tr[h^{\mu\nu}(\mathbf{r},t)])\frac{m(\hat{\mathbf{X}}+\mathbf{r})}{M}\end{aligned} \tag{3.25}$$

Eq. (3.25) was derived following the reference [58] where, however, the authors only consider the special case with $h^{0i} = h^{ij} = 0$.
In the next section we will specialize to the case of a (weak) stochastic gravitational background.

3.3 Stochastic gravitational perturbation: single particle master equation

The motivation to consider a stochastic weak gravitational perturbation is given by the interest towards Stochastic Semi-classical Gravity (an attempt to self-consistently describe the back-reaction of the quantum stress-energy fluctuations on the gravitational field, without having to invoke the quantization of the latter; see for example [59] and [60] for a review and further references), and by the interest in a stochastic gravitational background (see for instance [8, 15]), which we have already briefly introduced in Chapter 1.

If the metric is random, Eq. (3.20) becomes a stochastic differential equation. As a consequence the predictions are given by taking the stochastic average over the stochastic gravitational field. We then need to specify its stochastic properties.
We assume the noise to be Gaussian and with zero mean. The first assumption is justified by the law of large numbers, while the second by our choice of taking from the very beginning the Minkowski spacetime as the background spacetime around which the metric fluctuates. For the sake of simplicity, we also assume the different components of the metric fluctuation to be uncorrelated. This means that the noise is fully characterized by:

$$\begin{aligned}\mathbb{E}[h_{\mu\nu}(\mathbf{x},t)] =& 0 \\ \mathbb{E}[h_{\mu\nu}(\mathbf{x},t)h_{\mu\nu}(\mathbf{y},s)] =& \alpha^2 f_{\mu\nu}(\mathbf{x},\mathbf{y};t,s)\end{aligned} \tag{3.26}$$

where $\mathbb{E}[\cdot]$ denotes the stochastic average and α represents the strength of the gravitational fluctuations. The two point correlation function $f(\mathbf{x},\mathbf{y};t,s)$ is a real function

of order one, i.e. $0 \le |f_{\mu\nu}(\mathbf{x}, \mathbf{y}; t, s)| \le 1$.
We move to the density operator formalism [3]:

$$\hat{\Omega}(t) = |\phi(t)\rangle\langle\phi(t)| \tag{3.27}$$

As the only characterization of the noise is given by the stochastic average (Eq. (3.26)), we study the dynamics of the averaged operator:

$$\hat{\rho}(t) = \mathbb{E}[\hat{\Omega}(t)] \tag{3.28}$$

Let us consider the von Neumann equation for the averaged density matrix :

$$\begin{aligned} \partial_t \hat{\rho}(t) =& -\frac{i}{\hbar}\Big[\hat{H}_0(t), \hat{\rho}(t)\Big] - \frac{i}{\hbar}\mathbb{E}\Big[[\hat{H}_p(t) + \hat{H}_d(t), \hat{\Omega}(t)]\Big] \\ \equiv& \mathbb{E}\Big[\mathfrak{L}[\hat{\Omega(t)}]\Big] \end{aligned} \tag{3.29}$$

where $\mathfrak{L}[\,\cdot\,]$ denotes the Liouville superoperator. Equation (3.29) is in general difficult to tackle, because of the stochastic average, but it can be solved perturbatively by means of the cumulant expansion [61] (see Appendix F). With the further help of the Gaussianity, zero mean, uncorrelation of different components, we can rewrite

[3]We note that that of state vectors (and Schrödinger equation) is not the most appropriate formalism to adopt for the description of a quantum stochastic process in most experimental situation, as it does not allow one to describe statistical mixtures of quantum states.

Eq. (3.29) in Fourier space [4] as:

$$
\begin{aligned}
\partial_t\hat{\rho} = &-\frac{i}{\hbar}[\hat{H}_0,\hat{\rho}(t)] \\
&-\frac{\alpha^2}{\hbar^8}\int\frac{d^3q\,d^3q'}{(2\pi)^3}\int_0^t dt_1\,\tilde{f}^{00}(\mathbf{q},\mathbf{q}';t,t_1)\frac{m(\mathbf{q})m(\mathbf{q}')}{4M^2}\cdot \\
&\qquad\cdot\left[\left\{e^{i\mathbf{q}\cdot\hat{\mathbf{X}}/\hbar},(\frac{\hat{P}^2}{4M}+\frac{Mc^2}{2})\right\},\left[\left\{e^{i\mathbf{q}'\cdot\hat{\mathbf{X}}_{t_1}/\hbar},(\frac{\hat{P}^2}{4M}+\frac{Mc^2}{2})\right\},\hat{\rho}(t)\right]\right] \\
&-\frac{\alpha^2c^2}{\hbar^8}\int\frac{d^3q\,d^3q'}{(2\pi)^3}\int_0^t dt_1\,\tilde{f}^{0i}(\mathbf{q},\mathbf{q}';t,t_1)\frac{m(\mathbf{q})m(\mathbf{q}')}{4M^2}\cdot \\
&\qquad\cdot\left[\left\{e^{i\mathbf{q}\cdot\hat{\mathbf{X}}/\hbar},\hat{P}_i\right\},\left[\left\{e^{i\mathbf{q}'\cdot\hat{\mathbf{X}}_{t_1}/\hbar},\hat{P}_i\right\},\hat{\rho}(t)\right]\right] \\
&-\frac{\alpha^2}{\hbar^8}\int\frac{d^3q\,d^3q'}{(2\pi)^3}\int_0^t dt_1\,\tilde{f}^{ij}(\mathbf{q},\mathbf{q}';t,t_1)\frac{m(\mathbf{q})m(\mathbf{q}')}{4M^2}\cdot \\
&\qquad\cdot\left[\left\{e^{i\mathbf{q}\cdot\hat{\mathbf{X}}/\hbar},\frac{\hat{P}_i\hat{P}_j}{2M}\right\},\left[\left\{e^{i\mathbf{q}'\cdot\hat{\mathbf{X}}_{t_1}/\hbar}\frac{\hat{P}_i\hat{P}_j}{2M}\right\},\hat{\rho}(t)\right]\right] \\
&-\frac{\alpha^2}{\hbar^8}\int\frac{d^3q\,d^3q'}{(2\pi)^3}\int_0^t dt_1\,\tilde{f}^{\mu}_{\mu}(\mathbf{q},\mathbf{q}';t,t_1)\frac{\mathbf{q}^2\mathbf{q}'^2}{64M^2}\frac{m(\mathbf{q})m(\mathbf{q}')}{M^2}\left[e^{i\mathbf{q}\cdot\hat{\mathbf{X}}/\hbar},\left[e^{i\mathbf{q}'\cdot\hat{\mathbf{X}}_{t_1}/\hbar},\hat{\rho}(t)\right]\right] \\
&-\frac{\alpha^2}{16\hbar^4}\int\frac{d^3q\,d^3q'}{(2\pi)^3}\int_0^t dt_1\,\partial_t\partial_{t_1}\tilde{f}^{\mu}_{\mu}(\mathbf{q},\mathbf{q}';t,t_1)\frac{m(\mathbf{q})m(\mathbf{q}')}{M^2}\left[e^{i\mathbf{q}\cdot\hat{\mathbf{X}}/\hbar},\left[e^{i\mathbf{q}'\cdot\hat{\mathbf{X}}_{t_1}/\hbar},\hat{\rho}(t)\right]\right] \\
&+O(t\alpha^4\tau_c^3)
\end{aligned}
\tag{3.30}
$$

where $\hat{x}_{t_1}=e^{i\hat{H}_0t_1}\hat{x}e^{-i\hat{H}_0t_1}$. Note that the above equation becomes exact if $[\hat{H}_0,\hat{H}_p]=0$ (see Appendix G). The above equation describes the dynamics of the rigid body's center of mass is in the presence of a weak, stochastic and Gaussian gravitational field with zero mean, and whose different components are uncorrelated.
In the following we will not consider the effect on the dynamics due to the derivatives of the metric perturbation, as in typical experimental situations [1, 3, 4, 5, 6] they are negligible and in any case they would not add any further informative content to the analysis. This means that we neglect the last two lines of Eq. (3.30).

We now restrict our analysis to the Markovian case, i.e. we assume the noise to be delta correlated in time:

$$f^{\mu\nu}(\mathbf{x},\mathbf{y};t,s)=j^{\mu\nu}(\mathbf{x},\mathbf{y};t)\delta(t-s) \tag{3.31}$$

[4] Our choice for the Fourier transform is:

$$f(\mathbf{x})=\frac{1}{(\sqrt{2\pi}\hbar)^3}\int d^3q\,\tilde{f}(\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{x}/\hbar}$$

A further reasonable assumption, motivated by the homogeneity of spacetime itself, is that of translational invariance of the two point correlation function:

$$f^{\mu\nu}(\mathbf{x}, \mathbf{y}; t, s) = \lambda u^{\mu\nu}(\mathbf{x} - \mathbf{y})\delta(t - s) \tag{3.32}$$

where the factor λ is in principle a generic coefficient with the dimension of a time. Note that the white noise assumption makes physical sense only if the correlation time (τ_c) of the gravitational fluctuations is much smaller than the free dynamics' characteristic time (τ_{free}), or in the case where the contribution to the dynamics due to the gravitational perturbation is not affected by the free evolution dynamics, i.e. the operators describing the perturbation commute with the free dynamics operator $\hat{H}_0$ (See Appendix G). In such cases, as a first approximation, we can take λ to be:

$$\lambda = \min(\tau_c\, ,\, t) \tag{3.33}$$

Note that this choice does not affect the generality of the analysis as we leave $u^{\mu\nu}(\mathbf{x}-\mathbf{y})$ unspecified.

In such a regime Eq. (3.30) is exact and it is easy to show that it reduces to:

$$\begin{aligned}
\partial_t\hat{\rho} =& -\frac{i}{\hbar}[\hat{H}_0, \hat{\rho}(t)] \\
&- \frac{\alpha^2\lambda c^4}{4(2\pi)^{3/2}\hbar^5}\int d^3q\, \tilde{u}^{00}(\mathbf{q})m^2(\mathbf{q})\left[e^{i\mathbf{q}\cdot\hat{\mathbf{X}}/\hbar}, \left[e^{-i\mathbf{q}\cdot\hat{\mathbf{X}}/\hbar}, \hat{\rho}(t)\right]\right] \\
&- \frac{\alpha^2\lambda}{(2\pi)^{3/2}\hbar^5}\int d^3q\, \tilde{u}^{00}(\mathbf{q})\frac{m^2(\mathbf{q})}{M^2}\left[\left\{e^{i\mathbf{q}\cdot\hat{\mathbf{X}}/\hbar}, \frac{\hat{\mathbf{P}}^2}{4M}\right\}, \left[\left\{e^{-i\mathbf{q}\cdot\hat{\mathbf{X}}/\hbar}, \frac{\hat{\mathbf{P}}^2}{4M}\right\}, \hat{\rho}(t)\right]\right] \\
&- \frac{\alpha^2\lambda c^2}{2(2\pi)^{3/2}\hbar^5}\int d^3q\, \tilde{u}^{00}(\mathbf{q})\frac{m^2(\mathbf{q})}{M}\left[e^{i\mathbf{q}\cdot\hat{\mathbf{X}}/\hbar}, \left[\left\{e^{-i\mathbf{q}\cdot\hat{\mathbf{X}}/\hbar}, \frac{\hat{\mathbf{P}}^2}{4M}\right\}, \hat{\rho}(t)\right]\right] \\
&- \frac{\alpha^2\lambda c^2}{2(2\pi)^{3/2}\hbar^5}\int d^3q\, \tilde{u}^{00}(\mathbf{q})\frac{m^2(\mathbf{q})}{M}\left[\left\{e^{i\mathbf{q}\cdot\hat{\mathbf{X}}/\hbar}, \frac{\hat{\mathbf{P}}^2}{4M}\right\}, \left[e^{-i\mathbf{q}\cdot\hat{\mathbf{X}}/\hbar}, \hat{\rho}(t)\right]\right] \\
&- \frac{\alpha^2\lambda c^2}{(2\pi)^{3/2}\hbar^5}\int d^3q\, \tilde{u}^{0i}(\mathbf{q})\frac{m^2(\mathbf{q})}{4M^2}\left[\left\{e^{i\mathbf{q}\cdot\hat{\mathbf{X}}/\hbar}, \hat{P}_i\right\}, \left[\left\{e^{-i\mathbf{q}\cdot\hat{\mathbf{X}}/\hbar}, \hat{P}_i\right\}, \hat{\rho}(t)\right]\right] \\
&- \frac{\alpha^2\lambda}{(2\pi)^{3/2}\hbar^5}\int d^3q\, \tilde{u}^{ij}(\mathbf{q})\frac{m^2(\mathbf{q})}{M^2}\left[\left\{e^{i\mathbf{q}\cdot\hat{\mathbf{X}}/\hbar}, \frac{\hat{P}_i\hat{P}_j}{4M}\right\}, \left[\left\{e^{-i\mathbf{q}\cdot\hat{\mathbf{X}}/\hbar}, \frac{\hat{P}_i\hat{P}_j}{4M}\right\}, \hat{\rho}(t)\right]\right]
\end{aligned} \tag{3.34}$$

Eq. (3.34) describes decoherence both in position and in momentum, as it contains double commutators of functions of the position, momentum and free kinetic energy operators respectively with the averaged density matrix. In particular, we immediately recognize the term in the second line of Eq. (3.34) to give decoherence in position, that in the third line might give decoherence in energy (in the regime in which $\frac{\mathbf{q}\cdot\hat{\mathbf{X}}}{\hbar}$ is small), and that in the sixth line decoherence in momentum (in the same regime).

In the next section we will investigate under which conditions Eq. (3.34) reduces the different models of gravitational decoherence present in the literature.

3.4 Decoherence in the position eigenbasis

In this section we specialize Eq. (3.34) to the regime in which the dominant contribution to the decoherence effect is in the position eigenbasis. This can be done under the following assumptions:

$$\begin{cases} h^{00} \gtrsim h^{0i} \\ h^{00} \gtrsim h^{ij} \\ \Delta E \ll Mc^2 \left(1 - u^{00}(\Delta \mathbf{x})\right) \end{cases} \tag{3.35}$$

where $\Delta \mathbf{x}$ and ΔE are the quantum coherences of the system, respectively the position and energy ($E = \frac{\mathbf{P}^2}{2M}$). It is then easy to show that the leading contribution to Eq. (3.34) is:

$$\begin{aligned} \partial_t \hat{\rho} = &- \frac{i}{\hbar}[\hat{H}, \hat{\rho}(t)] - \frac{\alpha^2 \tau_c c^4}{(2\pi)^{3/2}\hbar^5} \int d^3q\, \tilde{u}^{00}(\mathbf{q}) m^2(\mathbf{q}) \left[e^{i\mathbf{q}\cdot\hat{\mathbf{X}}/\hbar}, \left[e^{-i\mathbf{q}\cdot\hat{\mathbf{X}}/\hbar}, \hat{\rho}(t)\right]\right] \\ &+ O(h^{\mu i}) + O(\Delta E) \end{aligned} \tag{3.36}$$

where we have safely replaced $\lambda = \tau_c$. The above equation describes decoherence in the position eigenbasis as the Lindblad operator is a funciton of the position operator. It is actually of the same form of the Gallis-Fleming master equation [62], which describes the decoherence induced on a particle by collisions with a surrounding thermal gas, allowing for a collisional interpretation of the result.
To compare with the previous literature on gravitational deocherence, we must further characterize the spatial correlation function of the noise and the mass density distribution. We start by considering the model proposed by Blencowe [42]. In order to recover an analogous master equation we must assume the noise to be delta correlated in space:

$$u^{00}(\mathbf{x} - \mathbf{x}') = l^3 \delta^3(\mathbf{x} - \mathbf{x}') \tag{3.37}$$

where l is a generic coefficient with the dimension of a length. Under this assumptions Eq. (3.34), represented in the position eigenbasis, in fact becomes:

$$\partial_t \rho(\mathbf{x}, \mathbf{x}'; t) = \left[\frac{i\hbar}{2M}(\nabla_x^2 - \nabla_{x'}^2) - \frac{\alpha^2 \tau_c c^4 l^3}{4\hbar^2} \int d^3r \left(m(\mathbf{r} - \mathbf{x}) - m(\mathbf{r} - \mathbf{x}')\right)^2\right] \rho(\mathbf{x}, \mathbf{x}'; t) \tag{3.38}$$

which has the same form of the master equation obtained in [42], and describes decoherence in position. The different rate is due to the different treatment of the gravitational noise: Blencowe considers a quantum bosonic thermic bath whose correlation functions can not be reproduced by our classical description of the noise. If we further take the mass density function to be a Gaussian:

$$m(\mathbf{r}) = \frac{m}{(\sqrt{2\pi}R)^3} e^{-\mathbf{r}^2/(2R^2)} \tag{3.39}$$

as it is done in the same work, Eq. (3.38) then reads:

$$\partial_t \rho(\mathbf{x},\mathbf{x}';t) = \frac{i\hbar}{2M}(\nabla_x^2 - \nabla_{x'}^2)\rho(\mathbf{x},\mathbf{x}';t) - \frac{\alpha^2 M^2 \tau_c c^4 l^3}{4(\sqrt{\pi})^3 \hbar^2 R^3}\Big(1 - e^{-\frac{(\mathbf{x}-\mathbf{x}')^2}{4R^2}}\Big)\rho(\mathbf{x},\mathbf{x}';t) \quad (3.40)$$

To recover the results obtained by Sanchez Gomez [37], we need to restrict to the point-like particle case:

$$m(\mathbf{r}) = M\delta^3(\mathbf{r}) \quad (3.41)$$

as in [37], and then to assume the spatial correlation function to be Gaussian:

$$\tilde{u}^{00}(\mathbf{q}-\mathbf{q}') = L^3\hbar^3\delta(\mathbf{q}-\mathbf{q}')e^{-\hbar^2\mathbf{q}^2L^2/2} \quad (3.42)$$

where L is the correlation length of the noise. With this choice for the spatial correlation functions it is natural to assume

$$\tau_c = \frac{L}{c} \quad (3.43)$$

as it is the only time scale of the system left, and Eq. (5.2) represented in the position basis reduces to:

$$\partial_t \rho(\mathbf{x},\mathbf{x}';t) = \frac{i\hbar}{2m}(\nabla_x^2 - \nabla_{x'}^2)\rho(\mathbf{x},\mathbf{x}';t) + \frac{2\alpha^2 m^2 c^3 L}{\hbar^2}\Big(e^{-\frac{(\mathbf{x}-\mathbf{x}')^2}{2L^2}} - 1\Big)\rho(\mathbf{x},\mathbf{x}';t) \quad (3.44)$$

and exactly recovers Sanchez Gomez's result.
A very similar equation was also obtained by Power and Percival [41]. Our model is able to qualitative recover the shape of the master equation, but not the specific rate which depends of the fourth power of the noise's strength, being the anlysis in [41] at higher order in the gravitational perturbation expansion.

In the next section we will describe under which assumptions our model is able to describe decoherence in the momentum and energy eigenbasis thus encompassing the results of Breuer et al. [40] and of Anastopoulos and Hu [38] that predict gravitational decoherence to occur in the energy eigenbasis.

3.5 Decoherence in the momentum eigenbasis

In this section we specialize Eq. (3.34) to the regime in which the dominant contribution to the decoherence effect is in the momentum or energy eigenbasis. This is the case when we can approximate:

$$e^{i\mathbf{q}\cdot\hat{\mathbf{X}}/\hbar} \sim \hat{\mathbb{1}} \quad (3.45)$$

i.e. in the regime of low momentum transfer. In this case Eq. (3.34) reduces to:

$$
\begin{aligned}
\partial_t \hat{\rho} = & -\frac{i}{\hbar}[\hat{H}, \hat{\rho}(t)] \\
& -\frac{\alpha^2\lambda}{(2\pi)^{3/2}\hbar^5}\int d^3q\, \tilde{u}^{00}(\mathbf{q})\frac{m^2(\mathbf{q})}{M^2}\left[\frac{\hat{\mathbf{P}}^2}{2M},\left[\frac{\hat{\mathbf{P}}^2}{2M},\hat{\rho}(t)\right]\right] \\
& -\frac{\alpha^2\lambda c^2}{(2\pi)^{3/2}\hbar^5}\int d^3q\, \tilde{u}^{0i}(\mathbf{q})\frac{m^2(\mathbf{q})}{M^2}\left[\hat{P}_i,\left[\hat{P}_i,\hat{\rho}(t)\right]\right] \\
& -\frac{\alpha^2\lambda}{(2\pi)^{3/2}\hbar^5}\int d^3q\, \tilde{u}^{ij}(\mathbf{q})\frac{m^2(\mathbf{q})}{M^2}\left[\frac{\hat{P}_i\hat{P}_j}{2M},\left[\frac{\hat{P}_i\hat{P}_j}{2M},\hat{\rho}(t)\right]\right]
\end{aligned}
\tag{3.46}
$$

In order to recover the results of Breuer et al. [40], the following hierarchy of the gravitational fluctuation must be verified:

$$
h^{ij} \gg h^{00}, h^{0i} \tag{3.47}
$$

and the spatial correlation functions chosen to be:

$$
\tilde{u}^{ij}(\mathbf{q}-\mathbf{q}') = \delta^{ij}L^3\hbar^3\delta(\mathbf{q}-\mathbf{q}')e^{-\hbar^2\mathbf{q}^2L^2/2} \tag{3.48}
$$

Also in this case it is natural to choose $\tau_c = L/c$. We also assume the mass density distribution to describe a point-like particle as in Eq. (3.41).
Under these assumptions Eq. (3.46) in fact reduces to:

$$
\partial_t\hat{\rho}(t) = -\frac{i}{\hbar}[\hat{H},\hat{\rho}(t)] - \frac{\alpha^2\lambda}{\hbar^2}\left[\frac{\hat{\mathbf{P}}^2}{2M},\left[\frac{\hat{\mathbf{P}}^2}{2M},\hat{\rho}(t)\right]\right] \tag{3.49}
$$

Eq. (3.49) is indeed the same as the one obtained by Breuer et al. with the identification:

$$
\alpha^2\lambda = \frac{T_c}{2} \tag{3.50}
$$

where T_c is the spatially averaged correlation time of the noise present in the same paper [5].
With the same assumptions we are also able to reproduce the shape of the master equation derived by Anastopoulos and Hu [38], but not the exact rate. As in the case of the Blencowe model, this is due to their quantum treatment of the gravitational noise.

Our model has so far proven to be able to describe more general scenarios than those present in the literature, as it is able to qualitative recover them as appropriate limiting cases, thus solving the decoherence basis puzzle. However, it might not be general enough to describe the outcome of a real experiment. The particles commonly employed in experiments (atoms, neutrons, electrons...) in fact have a charge, a spin and

[5] In the work of Breuer et al. the symbol used for the spatially averaged correlation time is τ_c. It was here changed to T_c in order to avoid any confusion with our own notation.

could be coupled to other external fields, like the Maxwell one for instance. For the above reasons, in the next chapter we will derive an analogous model, this time for spin 1/2 fermions interacting with both a gravitational perturbation and an external electromagnetic field.

Chapter 4

Gravitational decoherence: fermionic matter

In this chapter we extend the results of the previous chapter by deriving a model to describe the effects of a gravitational perturbation on a spin 1/2 fermionic matter field in the non relativistic regime and interacting with an external electromagnetic field.

4.1 Hamiltonian equations of motion

We first derive the equations of motion (EOM) for a spin 1/2 fermionic field minimally coupled to linearized gravity. We start from the action for the Dirac field in curved spacetime [53]:

$$S = \int d^4x \sqrt{-g}\mathcal{L} \tag{4.1}$$

with the Lagrangian density:

$$\mathcal{L} = \frac{i\hbar c}{2}[\bar{\psi}\gamma^\mu e^A{}_\mu \mathcal{D}_A\psi - e^A{}_\mu \mathcal{D}_A\bar{\psi}\gamma^\mu\psi] - mc^2\bar{\psi}\psi \tag{4.2}$$

where $e^A{}_\mu(x)$ is the so called vierbein field [63], an auxiliary field used in order to extend the definition of fermions as irreducible spin $1/2$ representations of the Poincaré group to curved spacetimes (see appendix C), and

$$\mathcal{D}_\mu\psi = \partial_\mu\psi + \frac{1}{8}[\gamma_a, \gamma_b]\omega_\mu{}^{ab}\psi + \frac{ie}{\hbar c}A_\mu\psi \tag{4.3}$$

is the covariant derivative with respect to both the spin ($\omega_\mu{}^{ab}$) and the electromagnetic (A_μ) connections. The pair ($e_A{}^\mu, \omega_A{}^{\mu\nu}$) allows for an equivalent geometrization of the gravitational interaction to the standard one given in terms of the metric and the affine connection $(g_{AB}, \Gamma^A{}_{BC})$(see Appendix C) ; the relation between the two frameworks is given by:

$$\begin{cases} e_A{}^\mu \eta_{\mu\nu} e_B{}^\nu = g_{AB} \\ \omega_A{}^{\mu\nu} = e_B{}^\mu \eta^{\nu\rho} \partial_A e^B{}_\rho + e_B{}^\mu \eta^{\nu\rho} e^C{}_\rho \Gamma^B{}_{AC} \end{cases} \tag{4.4}$$

Note that Eq. (4.4) holds only for a torsion free, metric compatible connection [63]. We write the metric as the sum of a flat background $\eta_{\mu\nu} = \mathrm{diag}(+ - --)$, and a perturbation $h_{\mu\nu}$:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{4.5}$$

We are interested in studying the dynamics of the Dirac field interacting with a weak gravitational perturbation. We therefore perform a Taylor expansion of the action around the flat background metric and truncate the series at the first perturbative order (See Appendix D for the explicit calculation). Thus, we obtain the effective Lagrangian $\mathcal{L}_{eff}$ acting on flat spacetime:

$$\begin{aligned} S = &\int d^4x \, \Big(\frac{i\hbar c}{2} [\bar{\psi}\gamma^\mu \nabla_\mu \psi - \nabla_\mu(\bar{\psi})\gamma^\mu \psi](1 + \frac{tr(h)}{2}) - (1 + \frac{tr(h)}{2}) mc^2 \bar{\psi}\psi \\ &- \frac{i\hbar c}{4} h_{\mu\nu} [\bar{\psi}\gamma^\mu \nabla^\nu \psi - \nabla^\nu(\bar{\psi})\gamma^\mu \psi] \Big) + O(h^2) \\ \equiv &\int d^4x \, \mathcal{L}_{eff} + O(h^2) \end{aligned} \tag{4.6}$$

where ∇_α is the flat covariant derivative with respect to the electromagnetic connection. The EOM for the matter field are obtained (at first order in the perturbation $h_{\mu\nu}$) from the Euler Lagrange equations:

$$\frac{\partial \mathcal{L}_{eff}}{\partial \bar{\psi}} - \nabla_\alpha \frac{\partial \mathcal{L}_{eff}}{\partial \nabla_\alpha \bar{\psi}} = 0 \tag{4.7}$$

and in the harmonic gauge they read:

$$\begin{aligned} i\hbar\partial_t\psi = &eA_0\psi + mc^2(1 + \frac{h_{00}}{2})\gamma^0\psi - \frac{mc^2}{2} h_{0j}\gamma^j\psi - i\hbar c(1 + \frac{h_{00}}{2})\gamma^0\gamma^i(\partial_i + \frac{ie}{\hbar c}A_i)\psi \\ &+ \frac{i\hbar c}{2} h_{0i}\gamma^i\gamma^j(\partial_j + \frac{ie}{\hbar c}A_j)\psi + \frac{i\hbar c}{2} h_{ij}\gamma^0\gamma^i(\partial^j + \frac{ie}{\hbar c}A^j)\psi + \frac{i\hbar c}{2} h_{0i}(\partial^i + \frac{ie}{\hbar c}A^i)\psi \\ &- \frac{i\hbar c}{8}\partial_\alpha(tr(h))\gamma^0\gamma^\alpha\psi + O(h^2)\psi \\ =: &\mathcal{H}\psi + O(h^2)\psi \end{aligned} \tag{4.8}$$

As for the scalar field discussed in the previous chapter, we note that we cannot give a probabilistic interpretation of the field ψ, as the conserved charged Q associated to the internal $U(1)$ symmetry ($\psi \to e^{ie}\psi$; $\bar{\psi} \to e^{-ie}\bar{\psi}$) via Noether's Theorem reads:

$$\begin{aligned} Q \equiv & -ie \int d^3x \left(\frac{\partial \mathcal{L}_{eff}}{\partial(\nabla_0 \psi)} \psi - \psi^\dagger \frac{\partial \mathcal{L}_{eff}}{\partial(\nabla_0 \psi^\dagger)} \right) \\ & = \hbar ec \int d^3x \left(\psi^\dagger (1 - tr(h) - \frac{h_{00}}{2}) \psi - \psi^\dagger \frac{h_{0i}}{2} \gamma^0 \gamma^i \psi \right) \end{aligned} \tag{4.9}$$

instead of the required:

$$\rho = \int d^3x\, \psi^\dagger \psi \tag{4.10}$$

We therefore apply the transformation:

$$\begin{cases} T & = & (1 - \frac{tr(h)}{2} - \frac{h_{00}}{4} - \frac{h_{0i}}{4} \gamma^0 \gamma^i) \\ \psi & \to & T\psi \\ \mathcal{H} & \to & \mathfrak{H} := T\mathcal{H}T^{-1} + i\hbar T \partial_t (T^{-1}) \end{cases} \tag{4.11}$$

so that, in the new representation, the conserved charge can be expressed by the standard for in Eq. (4.10), and the field admits a probabilistic interpretation in the non relativistic limit.
After some algebra the EOM (4.8) read:

$$i\hbar \partial_t \psi = [mc^2 \gamma^0 + \mathfrak{E} + \mathcal{O}]\psi \tag{4.12}$$

where

$$\begin{aligned} \mathfrak{E} = & eA_0 + \frac{mc^2}{2} h_{00} \gamma^0 + i\hbar c\, h_{0i} (\partial^i - \frac{ie}{\hbar c} A^i) + \frac{i\hbar c}{4} \partial_i (h^i_0) + \frac{\hbar c}{4} \epsilon^{ijk} \partial_i (h_{0j}) \Sigma_k \\ & - \frac{3i\hbar}{8} \partial_t (tr(h)) + \frac{i\hbar}{4} \partial_t (h_{00}) \end{aligned} \tag{4.13}$$

$$\begin{aligned} \mathcal{O} = & - i\hbar c (1 + \frac{h_{00}}{2})(\partial_j - \frac{ie}{\hbar c} A_j) \alpha^j + \frac{i\hbar}{4} \partial_t (h_{0i}) \alpha^i + \frac{i\hbar c}{2} h_{ij} (\partial^j - \frac{ie}{\hbar c} A^j) \alpha^i \\ & + \frac{i\hbar c}{4} \partial_i (\frac{tr(h)}{2} - h_{00}) \alpha^i \end{aligned} \tag{4.14}$$

are respectively the even (diagonal) and odd (off diagonal) parts of the Hamiltonian $\mathfrak{H}$, with $\alpha^\mu = \gamma^0 \gamma^\mu$ and $\Sigma^i = \text{diag}(\sigma^i, \sigma^i)$.
We are interested in the description of the dynamics of a positive energy particle system in the non relativistic limit. In such a limit, the particle and antiparticle sectors are non interacting with one another, that is to say, the EOM (4.8) can be recast to a system of two uncoupled equations respectively for the large (ψ_L) and small (ψ_s) component of the bispinor $\psi = \begin{pmatrix} \psi_L \\ \psi_s \end{pmatrix}$. While this is evident and straightforward for the free case [64], for an interacting theory the decoupling is very complicated and achievable only perturbatively.

In the next section we will provide a standard prescription for the diagonalization of the EOM in the non relativistic limit.

4.2 Non relativistic limit and canonical quantization

We aim to find a representation of the bispinor field ψ in which the EOM (4.12) are diagonal. This representation can be found in non relativistic limit following the Foldy-Wouthuysen Method [56], which allows one to write perturbatively (at any order in $\frac{v}{c}$) two decoupled equations, one for each component of the field. The method is operatively characterized by the application of an appropriate unitary transformation U:

$$\psi \to \psi' = U\psi \tag{4.15}$$

$$\begin{aligned} \mathfrak{H} \to \mathfrak{H}' &= U(\mathfrak{H} - i\hbar\partial_t)U^{-1} \\ &= mc^2\gamma^0 + \mathfrak{E}' + \mathcal{O}' + O(h^2) \end{aligned} \tag{4.16}$$

such that, in the new representation, the antidiagonal part $\mathcal{O}'$ is of higher order in $\frac{v}{c}$ than the diagonal $\mathfrak{E}'$. By neglecting $\mathcal{O}'$ one recovers two decoupled equations. By performing iteratively the transformation, one can always find a representation of the bispinor field for which the EOM are diagonal at any desired order in $\frac{v}{c}$.
In our case, the task is easily achieved by applying the subsequent transformations:

$$\begin{cases} U = & e^{-i\gamma^0\mathcal{O}/(2mc^2)} \\ U' = & e^{-i\gamma^0\mathcal{O}'/(2mc^2)} \\ U'' = & e^{-i\gamma^0\mathcal{O}''/(2mc^2)} \end{cases} \tag{4.17}$$

after which, with some algebra (see Appendix E) and by neglecting the terms containing the derivatives of the gravitational perturbation of order $\frac{v^3}{c^3}$ or higher[1], the Hamiltonian

[1]For the sake of compactness we relegate such terms to Appendix E. Note also that in most experimental situations such contributions are negligible in the case of gravity, and they wouldn't add any further informative content to the analysis in any case.

density to order $\frac{v^4}{c^4}$ reads:

$$\begin{aligned}
H =& eA_0 + \gamma^0 \Bigg[mc^2(1+\frac{h_{00}}{2}) - \frac{\hbar^2}{2m}(1+\frac{h_{00}}{2})(\nabla - \frac{ie}{\hbar c}\mathbf{A})^2 - \frac{\hbar e}{2mc}(1+\frac{h_{00}}{2})B^k\Sigma_k \\
& - \frac{\hbar^2}{2m}h_{ij}(\partial^i - \frac{ie}{\hbar c}A^i)(\partial^j - \frac{ie}{\hbar c}A^j) + \frac{\hbar e}{4mc}\epsilon^{ijl}h_{jk}{F_i}^k\Sigma_l \Bigg] \\
& + \frac{i\hbar^2 e}{4m^2c^2}(1+\frac{h_{00}}{2})\Big(\frac{\nabla}{2}\times\mathbf{E} - \mathbf{E}\times\nabla\Big)\cdot\mathbf{\Sigma} - (1+h_{00})\frac{\hbar^2 e}{8m^2c^2}\nabla\cdot\mathbf{E} \\
& - \frac{i\hbar^2 e}{16m^2c^2}\epsilon^{ikl}h_{ij}\partial^j(E_k)\Sigma_l - \frac{i\hbar^2 e}{8m^2c^2}\epsilon^{ikl}h_{ij}E_k(\partial^j - \frac{ie}{\hbar c}A^j)\Sigma_l \\
& + \frac{i\hbar^2 e}{4m^2c^2}\epsilon^{ijl}h_{0k}{F_j}^k(\partial_i - \frac{ie}{\hbar c}A_i)\Sigma_l - \frac{\hbar^2 e}{8m^2c^2}h_{0j}\partial_i(F^{ij}) + \frac{i\hbar^2 e}{8m^2c^2}\epsilon^{ijl}h_{0k}\partial_i({F_j}^k)\Sigma_l \\
& - \frac{\gamma^0}{8m^3c^6}\Bigg[\hbar^4c^4(1+2h_{00})(\nabla - \frac{ie}{\hbar c}\mathbf{A})^4 + \hbar^2ec^2(1+2h_{00})B^2 \\
& + 2\hbar^4c^4h_{ij}(\nabla - \frac{ie}{\hbar c}\mathbf{A})^2(\partial^i - \frac{ie}{\hbar c}A^i)(\partial^j - \frac{ie}{\hbar c}A^j) - \frac{\hbar^3ec^3}{2}\epsilon^{ijl}h_{jm}{F_i}^mB^k\{\Sigma_k,\Sigma_l\} \\
& + \frac{\hbar^3ec^3}{2}\epsilon^{ijl}\{(\nabla - \frac{ie}{\hbar c}\mathbf{A})^2, h_{jk}{F_i}^k\}\Sigma_l - \hbar^3ec^3(1+2h_{00})\{(\nabla - \frac{ie}{\hbar c}\mathbf{A})^2, B^k\}\Sigma_k\Bigg] \\
& + H_d + O(h^2) + O(\partial h) + O(\frac{v^5}{c^5})
\end{aligned} \tag{4.18}$$

where B and E are the magnetic and electric field, and in terms of the four-potential they read:

$$\begin{cases} \mathbf{E} = & -\nabla A_0 - \frac{1}{c}\dot{\mathbf{A}} \\ \mathbf{B} = & \nabla\times\mathbf{A} \\ B^k = & -\frac{1}{2}\epsilon^{ijk}F_{ij} \\ F_{ij} = & -\epsilon_{ijk}B^k \end{cases} \tag{4.19}$$

ϵ_{ijk} represent the Levi-Civita symbol, and

$$\begin{aligned}
H_d =& -\frac{\hbar^2}{8m}\partial_i(h_{00})(\partial^i - \frac{ie}{\hbar c}A^i)\gamma^0 - \frac{\hbar^2}{16m}\partial^i\partial_i(h_{00})\gamma^0 + \frac{i\hbar c}{4}\partial_i(h_0^i) \\
& + \frac{\hbar c}{4}\epsilon^{ijk}\partial_i(h_{0j})\Sigma_k - \frac{3i\hbar}{8}\partial_t(tr(h)) + \frac{i\hbar}{4}\partial_t(h_{00}) \\
& + \gamma^0\Bigg[\frac{\hbar^2}{2m}\partial^i(h_{00})\nabla_i - \frac{\hbar^2}{4m}\partial^i(h_{ij})\nabla^j - \frac{\hbar^2}{2m}\partial_i(\frac{tr(h)}{2} - h_{00})\nabla^i \\
& - \frac{i\hbar^2}{4m}\epsilon^{ijk}\Big(\partial_i(h_{00})\nabla_j - \partial_i(h_{jl})\nabla^l\Big)\Sigma_k - \frac{\hbar^2}{4m}\partial^i\partial_i(\frac{tr(h)}{2} - h_{00})\Bigg]
\end{aligned} \tag{4.20}$$

Note that as the transformations (4.17) are unitary [57], they preserve the conserved charge in (4.9), i.e. the probability density in the non relativistic limit.
In the non relativistic limit the EOM (4.18) do not mix the two components ψ_L and ψ_s of the field (up to a very small correction). As we are interested in the dynamics of particles only, we restrict the analysis to the first field component ψ_L, that we rename as ψ in what follows.
Since the dynamics preserves the probability density, we are allowed to apply the canonical quantization prescription and impose the equal time commutation relations:

$$\begin{aligned}[\hat{\psi}(t,\mathbf{x}),\hat{\psi}(t,\mathbf{x}')] =&[\hat{\psi}^\dagger(t,\mathbf{x}),\hat{\psi}^\dagger(t,\mathbf{x}')] = 0\\ [\hat{\psi}(t,\mathbf{x}),\hat{\psi}^\dagger(t,\mathbf{x}')] =& \delta^3(\mathbf{x}-\mathbf{x}')\end{aligned} \tag{4.21}$$

to obtain the EOM for the quantum field. The equation thus obtained does not allow for the creation or annihilation of particles. We can thus safely project it onto a single particle sector to obtain the single particle Schrödinger like equation:

$$i\hbar\partial_t|\phi(t)\rangle = (\hat{H}_0 + \hat{H}_r + \hat{H}_p + \hat{H}_{rp} + \hat{H}_d)|\phi(t)\rangle \tag{4.22}$$

with:

$$\begin{aligned}\hat{H}_0 =& mc^2 + \frac{1}{2m}\Big(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}(\hat{\mathbf{x}})\Big)^2 + eA_0(\hat{\mathbf{x}}) - \frac{\hbar e}{2mc}\mathbf{B}(\hat{\mathbf{x}})\cdot\boldsymbol{\sigma}\\ \hat{H}_r =& \frac{\hbar e}{4m^2c^2}\Big(\frac{\hat{\mathbf{p}}}{2}\times\mathbf{E}(\hat{\mathbf{x}}) - \mathbf{E}(\hat{\mathbf{x}})\times\hat{\mathbf{p}}\Big)\cdot\boldsymbol{\sigma} - \frac{\hbar^2 e}{8m^2c^2}\nabla\cdot\mathbf{E}(\hat{\mathbf{x}})\\ &- \frac{\gamma^0}{8m^3c^6}\Bigg[c^4(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}(\hat{\mathbf{x}}))^4 + \hbar^2 ec^2B^2(\hat{\mathbf{x}}) - \hbar ec^3\{(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}(\hat{\mathbf{x}}))^2, B^k(\hat{\mathbf{x}})\}\sigma_k\Bigg]\\ \hat{H}_p =& \frac{mc^2}{2}h^{00}(t,\hat{\mathbf{x}}) - \frac{1}{8m}\{h^{00}(t,\hat{\mathbf{x}})\,,\,\Big(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}(\hat{\mathbf{x}})\Big)^2\} + \frac{c}{2}\{h_{0i}(t,\hat{\mathbf{x}})\,,\,\hat{p}^i\}\\ &- \frac{1}{4m}\{h^{ij}(t,\hat{\mathbf{x}})\,,\,\Big(\hat{p}_i - \frac{e}{c}A_i(t,\hat{\mathbf{x}})\Big)\Big(\hat{p}_j - \frac{e}{c}A_j(t,\hat{\mathbf{x}})\Big)\} - \frac{\hbar e}{4mc}\epsilon^{ikl}h_{ij}(\hat{\mathbf{x}},t)F^j{}_k(\hat{\mathbf{x}})\sigma_l\\ &- \frac{\hbar e}{4mc}h_{00}(\hat{\mathbf{x}},t)\mathbf{B}(\hat{\mathbf{x}})\cdot\boldsymbol{\sigma}\end{aligned} \tag{4.23}$$

and

$$
\begin{aligned}
\hat{H}_{rp} =& \frac{\hbar e}{16m^2c^2}\{h_{00}(\hat{\mathbf{x}},t),\left(\frac{\hat{\mathbf{p}}}{2}\times\mathbf{E}(\hat{\mathbf{x}})-\mathbf{E}(\hat{\mathbf{x}})\times\hat{\mathbf{p}}\right)\cdot\boldsymbol{\sigma}\} \\
&-\frac{i\hbar^2 e}{16m^2c^2}\epsilon^{ikl}h_{ij}(\hat{\mathbf{x}})\partial^j(E_k(\hat{\mathbf{x}}))\Sigma_l-\frac{\hbar^2 e}{8m^2c^2}h_{00}(\hat{\mathbf{x}})(\hat{\mathbf{x}},t)\nabla\cdot\mathbf{E}(\hat{\mathbf{x}}) \\
&-\frac{\hbar e}{16m^2c^2}\epsilon^{ikl}\{h_{ij}(\hat{\mathbf{x}},t),E_k(\hat{\mathbf{x}})(\hat{p}^j-\frac{e}{c}A^j(\hat{\mathbf{x}}))\}\Sigma_l+ \\
&+\frac{\hbar e}{8m^2c^2}\epsilon^{ijl}\{h_{0k}(\hat{\mathbf{x}}),{F_j}^k(\hat{p}_i-\frac{e}{c}A_i)\}\Sigma_l-\frac{\hbar^2 e}{8m^2c^2}h_{0j}(\hat{\mathbf{x}})\partial_i(F^{ij}(\hat{\mathbf{x}})) \\
&+\frac{i\hbar^2 e}{8m^2c^2}\epsilon^{ijl}h_{0k}(\hat{\mathbf{x}})\partial_i({F_j}^k(\hat{\mathbf{x}}))\Sigma_l \\
&-\frac{\gamma^0}{8m^3c^6}\Bigg[c^4\{h_{00}(\hat{\mathbf{x}}),(\nabla-\frac{e}{c}\mathbf{A}(\hat{\mathbf{x}}))^4+2\hbar^2ec^2h_{00}(\hat{\mathbf{x}})B^2(\hat{\mathbf{x}}) \\
&+c^4\{h_{ij}(\hat{\mathbf{x}}),(\hat{\mathbf{p}}-\frac{e}{c}\mathbf{A}(\hat{\mathbf{x}}))^2(\hat{p}^i-\frac{e}{c}A^i(\hat{\mathbf{x}}))(\hat{p}^j-\frac{e}{c}A^j(\hat{\mathbf{x}}))\} \\
&-\frac{\hbar^3ec^3}{2}\epsilon^{ijl}h_{jm}(\hat{\mathbf{x}}){F_i}^m(\hat{\mathbf{x}})B^k(\hat{\mathbf{x}})\{\Sigma_k,\Sigma_l\} \\
&+\frac{\hbar ec^3}{2}\epsilon^{ijl}\{(\hat{\mathbf{p}}-\frac{e}{c}\mathbf{A})^2,h_{jk}(\hat{\mathbf{x}}){F_i}^k(\hat{\mathbf{x}})\}\Sigma_l-\hbar ec^3\{h_{00}(\hat{\mathbf{x}})\{(\hat{\mathbf{p}}-\frac{e}{c}\mathbf{A})^2,B^k\}\}\Sigma_k\Bigg] \\
\hat{H}_d =& -\frac{\hbar}{16m}\{\partial_i(h_{00}(\hat{\mathbf{x}})),(\hat{\mathbf{p}}^i-\frac{e}{c}A^i(\hat{\mathbf{x}}))\}\gamma^0-\frac{\hbar^2}{16m}\partial^i\partial_i(h_{00}(\hat{\mathbf{x}}))\gamma^0+\frac{i\hbar c}{4}\partial_i(h^i_0(\hat{\mathbf{x}})) \\
&+\frac{\hbar c}{4}\epsilon^{ijk}\partial_i(h_{0j}(\hat{\mathbf{x}}))\sigma_k-\frac{3i\hbar}{8}\partial_t(tr(h(\hat{\mathbf{x}})))+\frac{i\hbar}{4}\partial_t(h_{00}(\hat{\mathbf{x}})) \\
&+\gamma^0\Bigg[\frac{\hbar^2}{4m}\{\partial^i(h_{00}(\hat{\mathbf{x}})),(\hat{\mathbf{p}}_i-\frac{e}{c}A_i(\hat{\mathbf{x}}))\}-\frac{\hbar^2}{8m}\{\partial^i(h_{ij}(\hat{\mathbf{x}})),(\hat{\mathbf{p}}^j-\frac{e}{c}A^j(\hat{\mathbf{x}}))\} \\
&-\frac{\hbar^2}{4m}\{\partial_i(\frac{tr(h(\hat{\mathbf{x}}))}{2}-h_{00}(\hat{\mathbf{x}})),(\hat{\mathbf{p}}^i-\frac{e}{c}A^i(\hat{\mathbf{x}}))\}-\frac{\hbar^2}{4m}\partial^i\partial_i(\frac{tr(h(\hat{\mathbf{x}}))}{2}-h_{00}(\hat{\mathbf{x}})) \\
&-\frac{i\hbar^2}{8m}\epsilon^{ijk}\Big(\{\partial_i(h_{00}(\hat{\mathbf{x}})),(\hat{\mathbf{p}}_j-\frac{e}{c}A_j(\hat{\mathbf{x}}))\}-\{\partial_i(h_{jl}(\hat{\mathbf{x}})),(\hat{\mathbf{p}}^l-\frac{e}{c}A^l(\hat{\mathbf{x}}))\}\Big)\sigma_k\Bigg]
\end{aligned}
\tag{4.24}
$$

where $\hat{\mathbf{x}},\hat{\mathbf{p}}$ are respectively the single particle position and the momentum operator. The term $\hat{H}_0$ is the usual Pauli Hamiltonian [64] plus an irrelevant global phase mc^2 that can be reabsorbed with the transformation:

$$
|\phi(t)\rangle\to e^{imc^2t/\hbar}|\phi(t)\rangle \tag{4.25}
$$

The term $\hat{H}_r$ encodes the standard relativistic corrections [57] due to the presence of the electromagnetic field up to order $\frac{v^4}{c^4}$. Finally, $\hat{H}_p$, $\hat{H}_d$ and $\hat{H}_{rp}$ account for the corrections due to the presence of the weak gravitational field respectively to $\hat{H}_0$ and $\hat{H}_r$.

4.2.1 Differences with the bosonic model

Equation (4.18) is rather instructive as it extends the non relativistic Hamiltonian obtained in the previous chapter for the scalar field, namely Eq. (3.21), which we recall below [2]:

$$\begin{aligned}
\hat{H}_0^{(B)} =& mc^2 + \frac{\hat{\mathbf{p}}^2}{2m} \\
\hat{H}_p^{(B)} =& \frac{mc^2}{2} h^{00}(t,\hat{\mathbf{x}}) - \frac{\hbar^2}{8m}\{h^{00}(t,\hat{\mathbf{x}}),\hat{\mathbf{p}}^2\} + \frac{c}{2}\{h^{0i}(t,\hat{\mathbf{x}}),\hat{p}_i\} - \frac{1}{4m}\{h^{ij}(t,\hat{\mathbf{x}}),\hat{p}_i\hat{p}_j\} \\
\hat{H}_d^{(B)} =& \frac{\hbar^2}{8m}\nabla^2(tr[h^{\mu\nu}(t,\hat{\mathbf{x}})]) + \frac{i\hbar}{2}\partial_t(h^{00}(t,\hat{\mathbf{x}})) - \frac{i\hbar}{4}\partial_t(tr[h^{\mu\nu}(t,\hat{\mathbf{x}})])
\end{aligned} \tag{4.26}$$

to describe the dynamics of a quantum particle subject to a gravitational perturbation and to an external electromagnetic field. It is however as instructive to consider the electromagnetic free case , i.e. the limit $A(t,\hat{\mathbf{x}}) \to 0$, in order to directly compare the fermionic and bosonic Hamiltonians. By taking the limit $A(t,\hat{\mathbf{x}}) \to 0$ of Eq. (4.18), we obtain:

$$\begin{aligned}
\hat{H}_0^{(F)} =& mc^2 + \frac{\hat{\mathbf{p}}^2}{2m} \\
\hat{H}_p^{(F)} =& \frac{mc^2}{2} h^{00}(t,\hat{\mathbf{x}}) - \frac{\hbar^2}{8m}\{h^{00}(t,\hat{\mathbf{x}}),\hat{\mathbf{p}}^2\} + \frac{c}{2}\{h^{0i}(t,\hat{\mathbf{x}}),\hat{p}_i\} - \frac{1}{4m}\{h^{ij}(t,\hat{\mathbf{x}}),\hat{p}_i\hat{p}_j\} \\
\hat{H}_d^{(F)} =& -\frac{\hbar}{16m}\{\partial_i(h_{00}(\hat{\mathbf{x}})),(\hat{\mathbf{p}}^i - \frac{e}{c}A^i(\hat{\mathbf{x}}))\}\gamma^0 - \frac{\hbar^2}{16m}\partial^i\partial_i(h_{00}(\hat{\mathbf{x}}))\gamma^0 + \frac{i\hbar c}{4}\partial_i(h_0^i(\hat{\mathbf{x}})) \\
&+ \frac{\hbar c}{4}\epsilon^{ijk}\partial_i(h_{0j}(\hat{\mathbf{x}}))\sigma_k - \frac{3i\hbar}{8}\partial_t(tr(h(\hat{\mathbf{x}}))) + \frac{i\hbar}{4}\partial_t(h_{00}(\hat{\mathbf{x}})) \\
&+ \gamma^0\Bigg[\frac{\hbar^2}{4m}\{\partial^i(h_{00}(\hat{\mathbf{x}})),(\hat{\mathbf{p}}_i - \frac{e}{c}A_i(\hat{\mathbf{x}}))\} - \frac{\hbar^2}{8m}\{\partial^i(h_{ij}(\hat{\mathbf{x}})),(\hat{\mathbf{p}}^j - \frac{e}{c}A^j(\hat{\mathbf{x}}))\} \\
&- \frac{\hbar^2}{4m}\{\partial_i(\frac{tr(h(\hat{\mathbf{x}}))}{2} - h_{00}(\hat{\mathbf{x}})),(\hat{\mathbf{p}}^i - \frac{e}{c}A^i(\hat{\mathbf{x}}))\} - \frac{\hbar^2}{4m}\partial^i\partial_i(\frac{tr(h(\hat{\mathbf{x}}))}{2} - h_{00}(\hat{\mathbf{x}})) \\
&- \frac{i\hbar^2}{8m}\epsilon^{ijk}\Big(\{\partial_i(h_{00}(\hat{\mathbf{x}})),(\hat{\mathbf{p}}_j - \frac{e}{c}A_j(\hat{\mathbf{x}}))\} - \{\partial_i(h_{jl}(\hat{\mathbf{x}})),(\hat{\mathbf{p}}^l - \frac{e}{c}A^l(\hat{\mathbf{x}}))\}\Big)\sigma_k\Bigg]
\end{aligned} \tag{4.27}$$

As expected, the bosonic and the fermionc description match for the gravity free case ($\hat{H}_0^{(B)} = \hat{H}_0^{(F)}$). They also match for the terms proportional to the gravitational perturbation $h_{\mu\nu}$. This is also to be expected: suppose in fact that there were actually a difference in the terms $h_{00}\hat{\mathbf{p}}^2/2m$ or $mc^2 h_{00}$. This would imply that e.g. a simple change form Cartesian to Rindler [12] coordinates would predict for a boson and a fermion to

[2] We have added the superscripts (B) and (F) for respectively bosonic and fermionic in order to avoid confusion

fall with the same acceleration in the first (Cartesian) but not the second (Rindler) reference frame, which would violate the weak equivalence principle. The same line of reasoning can be applied to the other terms containing $h_{ij}$and h_{0i}. It is interesting however to notice that some differences arise for the terms containing the derivatives of the gravitational perturbation $\partial h_{\mu\nu}$. Such differences originated when we required the matter field to allow for a probabilistic interpretation in order to canonically quantize the system.

4.3 Master equation with electromagnetic field

In this section we derive a master equation to describe the decoherence effect induced by a weak stochastic gravitational perturbation on a spin 1/2 fermionic particle, as done for the scalar case in the previous chapter. For the sake of simplicity and compactness of the result, we will restrict our analysis to the Pauli Hamiltonian $\hat{H}_0$ and its gravitational corrections $\hat{H}_p$, as the terms $\hat{H}_r$ and $\hat{H}_{rp}$ are of higher order in the non relativistic expansion [3], and the term $\hat{H}_d$ contains derivatives of the gravitational perturbations, as in typical experimental situations [1, 3, 4, 5, 6] they are negligible and would not add any further informative content to the analysis in any case.
This means that we approximate Eq. (4.22) to:

$$i\hbar\partial_t|\phi(t)\rangle = (\hat{H}_0 + \hat{H}_p)|\phi(t)\rangle \tag{4.28}$$

If the metric is random, Eq. (4.28) becomes a stochastic differential equation. As a consequence the predictions are given by taking the stochastic average over the random gravitational field. We then need to specify its stochastic properties.
As done for the bosonic particle case, we assume the noise to be Gaussian and with zero mean. For the sake of simplicity, we also assume the different components of the metric fluctuation to be uncorrelated. This means that the noise is fully characterized by:

$$\begin{aligned} \mathbb{E}[h_{\mu\nu}(\mathbf{x},t)] =& 0 \\ \mathbb{E}[h_{\mu\nu}(\mathbf{x},t)h_{\mu\nu}(\mathbf{y},s)] =& \alpha^2 f_{\mu\nu}(\mathbf{x},\mathbf{y};t,s) \end{aligned} \tag{4.29}$$

where we recall that $\mathbb{E}[\,\cdot\,]$ denotes the stochastic average, α represents the strength of the gravitational fluctuations, and $f(\mathbf{x},\mathbf{y};t,s)$ is the two point correlation function.
We move to the density operator formalism, and write the von Neumann equation for the averaged density matrix :

$$\begin{aligned} \partial_t\hat{\rho}(t) =& -\frac{i}{\hbar}\Big[\hat{H}_0(t),\hat{\rho}(t)\Big] - \frac{i}{\hbar}\mathbb{E}\Big[[\hat{H}_p(t),\hat{\Omega}(t)]\Big] \\ \equiv& \mathbb{E}\Big[\mathfrak{L}[\hat{\Omega(t)}]\Big] \end{aligned} \tag{4.30}$$

[3]Note however that one needs to be careful when applying the results of this section to a real experimental situations, as the term $\hat{H}_r$ might dominate over $\hat{H}_p$ (depending on the size of $\mathbf{E}$,$\mathbf{B}$, and $h_{\mu\nu}$) and should therefore be taken in consideration.

Also in this case we solve the above equation perturbatively exploiting the cumulant expansion [61]. With the further help of the Gaussianity, zero mean, uncorrelation of different components, we can rewrite Eq. (4.30) in Fourier space [4] as:

$$
\begin{aligned}
\partial_t\hat{\rho} = &-\frac{i}{\hbar}[\hat{H}_0,\hat{\rho}(t)]+\\
&-\frac{\alpha^2}{\hbar^8}\int\frac{d^3q\,d^3q'}{(2\pi)^3}\int_0^t dt_1\,\frac{\tilde{f}^{00}(\mathbf{q},\mathbf{q}';t,t_1)}{4}\cdot\\
&\qquad\cdot\left[\left\{e^{i\mathbf{q}\cdot\hat{\mathbf{x}}/\hbar},\Xi_{00}(\hat{\mathbf{x}},\hat{\mathbf{p}})\right\},\left[\left\{e^{i\mathbf{q}'\cdot\hat{\mathbf{x}}_{t_1}/\hbar},\Xi_{00}(\hat{\mathbf{x}}_{t_1},\hat{\mathbf{p}})\right\},\hat{\rho}(t)\right]\right]\\
&-\frac{\alpha^2c^2}{\hbar^8}\int\frac{d^3q\,d^3q'}{(2\pi)^3}\int_0^t dt_1\,\frac{\tilde{f}^{0i}(\mathbf{q},\mathbf{q}';t,t_1)}{4}\left[\left\{e^{i\mathbf{q}\cdot\hat{\mathbf{x}}/\hbar},\hat{p}_i\right\},\left[\left\{e^{i\mathbf{q}'\cdot\hat{\mathbf{x}}_{t_1}/\hbar},\hat{p}_i\right\},\hat{\rho}(t)\right]\right]\\
&-\frac{\alpha^2}{\hbar^8}\int\frac{d^3q\,d^3q'}{(2\pi)^3}\int_0^t dt_1\,\frac{\tilde{f}^{ij}(\mathbf{q},\mathbf{q}';t,t_1)}{4}\cdot\\
&\qquad\cdot\left[\left\{e^{i\mathbf{q}\cdot\hat{\mathbf{x}}/\hbar},\Xi_{ij}(\hat{\mathbf{x}},\hat{\mathbf{p}})\right\},\left[\left\{e^{i\mathbf{q}'\cdot\hat{\mathbf{x}}_{t_1}/\hbar},\Xi_{ij}(\hat{\mathbf{x}}_{t_1},\hat{\mathbf{p}})\right\},\hat{\rho}(t)\right]\right]\\
&+O(t\alpha^4\tau_c^3)
\end{aligned}
\tag{4.31}
$$

where we have introduced:

$$
\Xi_{00}(\hat{\mathbf{x}},\hat{\mathbf{p}}) = \frac{(\hat{\mathbf{p}}-\frac{e}{c}\mathbf{A}(t,\hat{\mathbf{x}}))^2}{4m}+\frac{mc^2}{2}-\frac{\hbar e}{2mc}\mathbf{B}(t,\hat{\mathbf{x}})\cdot\sigma \tag{4.32}
$$

$$
\Xi_{ij}(\hat{\mathbf{x}},\hat{\mathbf{p}}) = \frac{(\hat{p}_i-\frac{e}{c}A_i(t,\hat{\mathbf{x}}))(\hat{p}_j-\frac{e}{c}A_j(t,\hat{\mathbf{x}}))}{4m}+\frac{mc^2}{2}+\frac{\hbar e}{2mc}\epsilon_{kil}F^k{}_j(t,\hat{\mathbf{x}})\sigma^l \tag{4.33}
$$

for the sake of compactness.
The above equation describes the dynamics of a point-like spin 1/2 fermionic particle in presence of an external weak, stochastic gravitational field (with the further assumptions made in this section) and an external electromagnetic field.

Also in this case we specialize Eq. (4.31) to the Markovian limit, with the further assumptions of isotropy and homogeneity of the noise, so that its correlation function again reads:

$$
f^{\mu\nu}(\mathbf{x},\mathbf{y};t,s) = \tau u^{\mu\nu}(\mathbf{x}-\mathbf{y})\delta(t-s) \tag{4.34}
$$

[4] Recall that our choice for the Fourier transform is:

$$
f(\mathbf{x}) = \frac{1}{(\sqrt{2\pi}\hbar)^3}\int d^3q\,\tilde{f}(\mathbf{q})e^{i\mathbf{q}\cdot\mathbf{x}/\hbar}
$$

In such a regime Eq. (4.31) is exact and it is easy to show that it reduces to:

$$
\begin{aligned}
\partial_t\hat{\rho} = &-\frac{i}{\hbar}[\hat{H},\hat{\rho}(t)] \\
&-\frac{\alpha^2\lambda}{(2\pi)^{3/2}\hbar^5}\int d^3q\,\tilde{u}^{00}(\mathbf{q})\left[\left\{e^{i\mathbf{q}\cdot\hat{\mathbf{X}}/\hbar},\Xi_{00}(\hat{\mathbf{x}},\hat{\mathbf{p}})\right\},\left[\left\{e^{-i\mathbf{q}\cdot\hat{\mathbf{X}}/\hbar},\Xi_{00}(\hat{\mathbf{x}},\hat{\mathbf{p}})\right\},\hat{\rho}(t)\right]\right] \\
&-\frac{\alpha^2\lambda c^2}{(2\pi)^{3/2}\hbar^5}\int d^3q\,\tilde{u}^{0i}(\mathbf{q})\left[\left\{e^{i\mathbf{q}\cdot\hat{\mathbf{X}}/\hbar},\hat{P}_i\right\},\left[\left\{e^{-i\mathbf{q}\cdot\hat{\mathbf{X}}/\hbar},\hat{P}_i\right\},\hat{\rho}(t)\right]\right] \\
&-\frac{\alpha^2\lambda}{(2\pi)^{3/2}\hbar^5}\int d^3q\,\tilde{u}^{ij}(\mathbf{q})\left[\left\{e^{i\mathbf{q}\cdot\hat{\mathbf{X}}/\hbar},\Xi_{ij}(\hat{\mathbf{x}},\hat{\mathbf{p}})\right\},\left[\left\{e^{-i\mathbf{q}\cdot\hat{\mathbf{X}}/\hbar},\Xi_{ij}(\hat{\mathbf{x}},\hat{\mathbf{p}})\right\},\hat{\rho}(t)\right]\right]
\end{aligned}
\tag{4.35}
$$

Eq. (4.35) describes decoherence in an intricate combination of position momentum and energy bases, as it contains double commutators of functions of the position, momentum and free kinetic energy operators with the averaged density matrix.

In what follows we will specialize Eq. (4.35) to determine under which approximations it recovers decoherence in the position or momentum eigenbasis only.
As for the bosonic case, the conditions:

$$
\begin{cases}
h^{00} \gtrsim h^{0i} \\
h^{00} \gtrsim h^{ij} \\
\Delta E \ll Mc^2\left(1-u^{00}(\Delta\mathbf{x})\right)
\end{cases}
\tag{4.36}
$$

are sufficient for our master equation to describe decoherence in the position eigenbasis only, where in this case the energy coherence needs to be modified to take into account the presence of the electromagnetic field, as $E=\frac{(\mathbf{p}-\frac{e}{c}\mathbf{A})^2}{2m}$.
Under the above assumptions, Eq. (4.35) reads:

$$
\begin{aligned}
\partial_t\hat{\rho} = -\frac{i}{\hbar}[\hat{H},\hat{\rho}(t)] - \frac{\alpha^2\lambda}{(2\pi)^{3/2}\hbar^5}\int d^3q\,\tilde{u}^{00}(\mathbf{q})\cdot \\
\cdot\left[e^{i\mathbf{q}\cdot\hat{\mathbf{X}}/\hbar}\left(\frac{mc^2}{2}-\frac{\hbar e\sigma}{2mc}\cdot\mathbf{B}(t,\hat{\mathbf{x}})\right),\left[\left\{e^{-i\mathbf{q}\cdot\hat{\mathbf{X}}/\hbar}\left(\frac{mc^2}{2}-\frac{\hbar e\sigma}{2mc}\cdot\mathbf{B}(t,\hat{\mathbf{x}})\right),\hat{\rho}(t)\right]\right]
\end{aligned}
\tag{4.37}
$$

Contrary to the bosonic case, the condition of low momentum transfer

$$
e^{i\mathbf{q}\cdot\hat{\mathbf{X}}/\hbar}\sim\hat{\mathbb{1}}
\tag{4.38}
$$

is necessary but not sufficient to recover deocherence in the momentum or energy eigenbasis starting from Eq. (4.35). One in fact needs the further condition:

$$
\begin{cases}
|\mathbf{p}| \gg |\frac{e}{c}\mathbf{A}| \\
\frac{p^2}{2m} \gg |\frac{\hbar e\sigma}{2mc}\cdot\mathbf{B}|
\end{cases}
\tag{4.39}
$$

In this regime, Eq. (4.35) can be approximated as:

$$
\begin{aligned}
\partial_t \hat{\rho} = &- \frac{i}{\hbar}[\hat{H}, \hat{\rho}(t)] \\
&- \frac{\alpha^2 \lambda}{(2\pi)^{3/2}\hbar^5} \int d^3q \, \tilde{u}^{00}(\mathbf{q}) \left[\frac{\hat{\mathbf{p}}^2}{2m}, \left[\frac{\hat{\mathbf{p}}^2}{2m}, \hat{\rho}(t) \right] \right] \\
&- \frac{\alpha^2 \lambda c^2}{(2\pi)^{3/2}\hbar^5} \int d^3q \, \tilde{u}^{0i}(\mathbf{q}) \left[\hat{p}_i, \left[\hat{p}_i, \hat{\rho}(t) \right] \right] \\
&- \frac{\alpha^2 \lambda}{(2\pi)^{3/2}\hbar^5} \int d^3q \, \tilde{u}^{ij}(\mathbf{q}) \left[\frac{\hat{p}_i \hat{p}_j}{2m}, \left[\frac{\hat{p}_i \hat{p}_j}{2m}, \hat{\rho}(t) \right] \right]
\end{aligned}
\tag{4.40}
$$

where λ is defined as in Eq. (3.33).

In this and in the previous chapter, we have set firm theoretical grounds for an unambiguous and comprehensive description of the decoherence phenomenon caused on a quantum particle by a stochastic gravitational perturbation. In the next chapter we will apply our novel results to matter-wave interferometery in order to discuss their sensitivity to cosmic gravitational perturbations.

Chapter 5

Application: matter-wave interferometry and gravitational decoherence

In this chapter we analyse the effect of a stochastic gravitational perturbation on matter-wave Mach-Zehnder interferometers by applying our gravitational deocherence model to such setups. There is another study [43] in the literature concerning the effects of a stochastic gravitational perturbation on a Mach-Zehnder atom interferometer, the HYPER interferometer [65]. Such a study however deals only with a specific type of gravitational noise, the so called Binary Confusion Background. Our work is also intended to extend such an analysis. We therefore develop a scheme for quantifying the gravity induce loss of contrast in the interference fringes in particular regimes of interest. Finally, we apply our analysis to a selected sample of proposed and actual experiments in order to assess to which kind of cosmic perturbation the state of the art technology is sensitive to.

5.1 Mach-Zehnder interferometry

A Mach-Zehnder interferometer is a device used to determine a path dependent phase variation between two branches of a matter-wave that originate from the same source. We do not provide here a complete and detailed description of such a class of devices, as that lies beyond the purpose of this thesis. We will limit ourselves to sketch the essential traits of its functioning mechanism in order to keep up with our analysis. We refer the reader to the following work [66] and the references therein included for a comprehensive discussion.
Given the above premise, in a nutshell a Mach-Zehnder interferometer works as follows: a wavepacket is produced at the source, is collimated, and then goes through a beam splitter; the two partial waves travel the same distance before they are reflected by two mirrors to eventually collimate at a second beam splitter and be directed towards

a screen (or, more generally, one or more detectors) where the interference fringes are observed. A variation in the optical path (caused by a sample with different refractive index, the motion of the mirrors, gravity, ecc..) in one or both of the two arms determines a shift in the interference fringes.
In the case of decoherence in the position eigenbasis, it is well know [62, 67] that the effect on such a device is a loss of contrast in the interference pattern produced at the detector. To quantify this loss we use the interferometric visibility ν, which is defined in terms of the maximum ($P_{\max}$) and minimum ($P_{\min}$) intensity of the interference pattern:

$$\nu = \frac{P_{\max} - P_{\min}}{P_{\max} + P_{\min}} \tag{5.1}$$

We therefore implement a model for the evolution of the probability density to then determine the visibility. Motivated by experimental interest, we consider only the pure position ($\mathcal{P}$) and energy ($\mathcal{E}$) decoherence cases described by the simpler Eqs. (3.36, 3.46). Furthermore, we take the spatial correlation function of the noise to be a Gaussian, as in Eqs. (3.42, 3.48).
For the sake of simplicity, we will only consider in our analysis symmetric Mach-Zehnder interferometers, like the one schematically depicted in Fig. (5.1). Furthermore, we restrict the analysis to point-like particles, as Mach-Zehnder interferometry is currently bound to neutrons, atoms and Bose-Einstein condensates (BEC) due to technical limitations [68]. We also assume the matter-wave to be collimated and the interaction time with the mirrors to be negligible, thus we can rely on the longitudinal-eikonal approximation and reduce the study to a one dimensional problem along the transverse axis of propagation, i.e. the x-axis in Fig. (5.1).

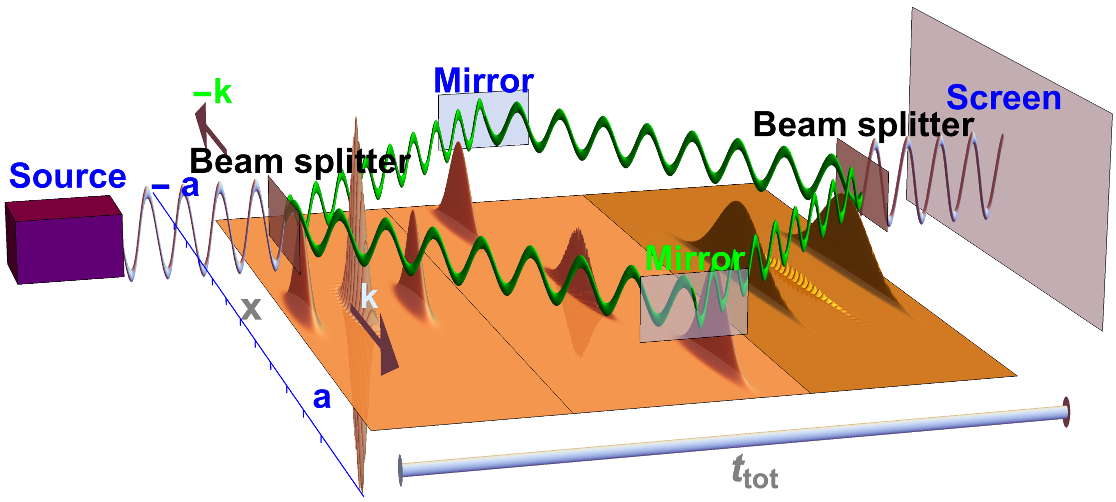

Figure 5.1: Symmetric Mach-Zehnder atom interferometer.

5.2 Mach-Zehnder interferometry: decoherence in the position eigenbasis

The positional decoherence process induced by the scalar component of the gravitational perturbation is described by Eq. (3.36):

$$\partial_t\hat{\rho} = -\frac{i}{\hbar}\Big[\frac{\hat{\mathbf{P}}^2}{2M},\hat{\rho}(t)\Big] - \frac{\alpha^2\tau_c c^4}{(2\pi)^{3/2}\hbar^5}\int d^3q\,\tilde{u}^{00}(\mathbf{q})m^2(\mathbf{q})\Big[e^{i\mathbf{q}\cdot\hat{\mathbf{X}}/\hbar},\Big[e^{-i\mathbf{q}\cdot\hat{\mathbf{X}}/\hbar},\hat{\rho}(t)\Big]\Big] \tag{5.2}$$

We apply the above equation to a Mach-Zehnder set up, like the one described in Fig. (5.1), in order to quantify the effect of decoherence on the visibility. We therefore specialize Eq. (3.36) by means of the longitudinal-eikonal approximation and of the point-like particle assumption $m(r) = m\delta(r)$. Furthermore, we assume the spatial correlation function to be a Gaussian $\tilde{u}^{00}(\mathbf{q}) = L^3\hbar^3 e^{-\mathbf{q}^2L^2/(2\hbar^2)}$. It is then only natural to take $\tau_c = L/c$. When represented in the position eigenbasis, the equation then reads:

$$\partial_t\rho_t(x,x') = -\frac{i\hbar}{2m}({\nabla_x}^2 - {\nabla_{x'}}^2)\rho_t(x,x') - \frac{2\alpha^2m^2\tau_c c^4}{\hbar^2}\left(1 - e^{\frac{(x-x')^2}{2L^2}}\right)\rho_t(x,x') \tag{5.3}$$

We solve the above equation with the help of the characteristic function [69, 70], which is defined in terms of the statistical operator $\hat{\rho}_t$ as:

$$\begin{aligned}\chi_t(s,q) =& \mathrm{Tr}[e^{i(\hat{x}q-\hat{p}s)/\hbar}\hat{\rho}_t]\\ =&\frac{1}{h}\int dxdpdy\; e^{ip(y-s)/\hbar}e^{iqx/\hbar}\rho_t(x-\frac{y}{2},x+\frac{y}{2})\\ =&\int dx e^{iqx/\hbar}\rho_t(x-\frac{s}{2},x+\frac{s}{2})\end{aligned} \tag{5.4}$$

and is connected to the probability density, and thus the interference fringes, through the relation:

$$P_t(x) = \frac{1}{h}\int \chi_t(0,q)e^{-iqx/\hbar}dq \tag{5.5}$$

In this formalism the Eq. (5.3) reads:

$$\partial_t\chi_t(s,q) = \left(-\frac{q}{m}\partial_s - \frac{2\alpha^2m^2Lc^3}{\hbar^2}\left(1 - e^{\frac{s^2}{2L^2}}\right)\right)\chi_t(s,q) \tag{5.6}$$

The formal solution to the above equation is:

$$\chi_t(s,q) = R'_{t,\mathcal{P}}(s)\chi_0(s-\frac{q}{m}t,q) \tag{5.7}$$

with $R'_{t,\mathcal{P}}(s,q) = e^{\int_0^t\Gamma_{\tau-t,\mathcal{P}}(s,q)d\tau}$, and $\Gamma_{t,\mathcal{P}}(s,q)$ is the transformed decoherence rate function:

$$\Gamma_{t,\mathcal{P}}(s,q) = \left(1 - e^{-(qt/m+s)^2/(2L^2)}\right) \tag{5.8}$$

Note that this result holds true independently of the choice of the initial state, and for any decoherence rate funtion such that $\Gamma_{t,\mathcal{P}}(x,x') = \Gamma_{t,\mathcal{P}}(x-x')$.

It follows that in a Mach-Zehnder interferometer the evolution of the system from the first beam splitter to the mirrors is easily described by Eq. (5.7). The state of the system right before the reflection at the mirrors therefore looks like:

$$\chi_{t_{ref}}(s,q) = R'_{t_{ref},\mathcal{P}}(s)\chi_0(s - \frac{q}{m}t_{ref}, q) \tag{5.9}$$

The reflection at the mirrors is instead modelled, following the principles of the "image charge" and exploiting the symmetries of the apparatus, as the sudden transformation:

$$\chi_{t_{\text{ref}}}(s,q) \to \chi_{mir}(s,q) = \chi_{t_{\text{ref}}}(-4a-s,-q) + \chi_{t_{\text{ref}}}(4a-s,-q) + 2\cos\left(\frac{aq}{\hbar}\right)\chi_{t_{\text{ref}}}(-s,-q) \tag{5.10}$$

which corresponds to

$$\begin{cases} \psi_{t_{\text{ref}}}(x) \to \psi_{t_{\text{ref}}}(2a-x) + \psi_{t_{\text{ref}}}(-2a-x) \\ p \to -p \end{cases} \tag{5.11}$$

at the level of the wavefunction, where $2a$ is the distance between the two mirrors. Finally, the evolution from the mirrors, just after the reflection, to the second beam splitter is again governed by Eq. (5.7), and the state therefore reads:

$$\chi_{t_{tot}}(s,q) = R'_{t_{tot},\mathcal{P}}(s)\chi_{mir}(s - \frac{q}{m}(t_{tot} - t_{ref}), q) \tag{5.12}$$

For the purpose of detecting the effects of decoherence, it is convenient to perform the measurement immediately after or in place of the second beam splitter, when the two beams are still overlapped and an interference pattern can be detected. For this reason we decide not to include the transformation induced by the second beam splitter[1], nor the subsequent evolution to the screen in the analysis.
This results into the following interference pattern at the screen:

$$\begin{aligned} P^{(\mathcal{P})}_{\text{scr}}(x) =& \frac{1}{h}\int dq\, e^{\frac{i}{\hbar}qx} e^{\int_0^{t_{\text{ref}}}(\Gamma_{\mathcal{P}}(\frac{q}{m}\tau))d\tau}\Big[e^{\int_0^{t_{\text{ref}}}\Gamma_{\mathcal{P}}(\frac{q}{m}\tau+4a)d\tau}\chi_0\big(-4a-\frac{2aq}{k},-q\big)+ \\ &+ e^{\int_0^{t_{\text{ref}}}\Gamma_{\mathcal{P}}(\frac{q}{m}\tau-4a)d\tau}\chi_0\big(4a-\frac{2aq}{k},-q\big) + 2\cos\left(\frac{aq}{\hbar}\right)\chi_0\big(-\frac{2aq}{k},-q\big)\Big] \end{aligned} \tag{5.13}$$

that can be used to estimate the visibility as in Eq. (5.1), given the explicit form of the state at the first beam splitter $\chi_0(s,q)$. We choose it to be a Gaussian wavepacket

[1]The transformation is $\hat{\rho} \to \cos(\hat{x}k/\hbar)\hat{\rho}\cos(\hat{x}k/\hbar)$. Note that it has no impact on the effects of decoherence on the probability density $\rho(x,x)$.

of spread σ in a superposition of momenta $\pm k$ (because of the action of the first beam splitter of Fig. (5.1)):

$$\chi_0(s,q) = \frac{e^{-\frac{q^2\sigma^2}{4\hbar^2}-\frac{s^2}{4\sigma^2}}\left(e^{-\frac{k^2\sigma^2}{\hbar^2}}\cosh\left(\frac{kq\sigma^2}{\hbar^2}\right)+\cos\left(\frac{ks}{\hbar}\right)\right)}{e^{-\frac{k^2\sigma^2}{\hbar^2}}+1} \tag{5.14}$$

which corresponds to

$$\psi_0(x) = \sqrt{\frac{1}{2\sqrt{\pi}\sigma[1+\exp(\frac{-k^2\sigma^2}{\hbar^2})]}}\exp(-\frac{x^2}{2\sigma^2})\cos\left(\frac{kx}{\hbar}\right) \tag{5.15}$$

in terms of the wavefunction. With this initial state, the time at which the reflection occurs trivially reads $t_{\text{ref}} = a/v$ where $v = k/m$. Below, we report a series of plots for the probability density obtained with such a choice for the initial state.

In the two series, the experimental parameters are set according to the figures' cap-

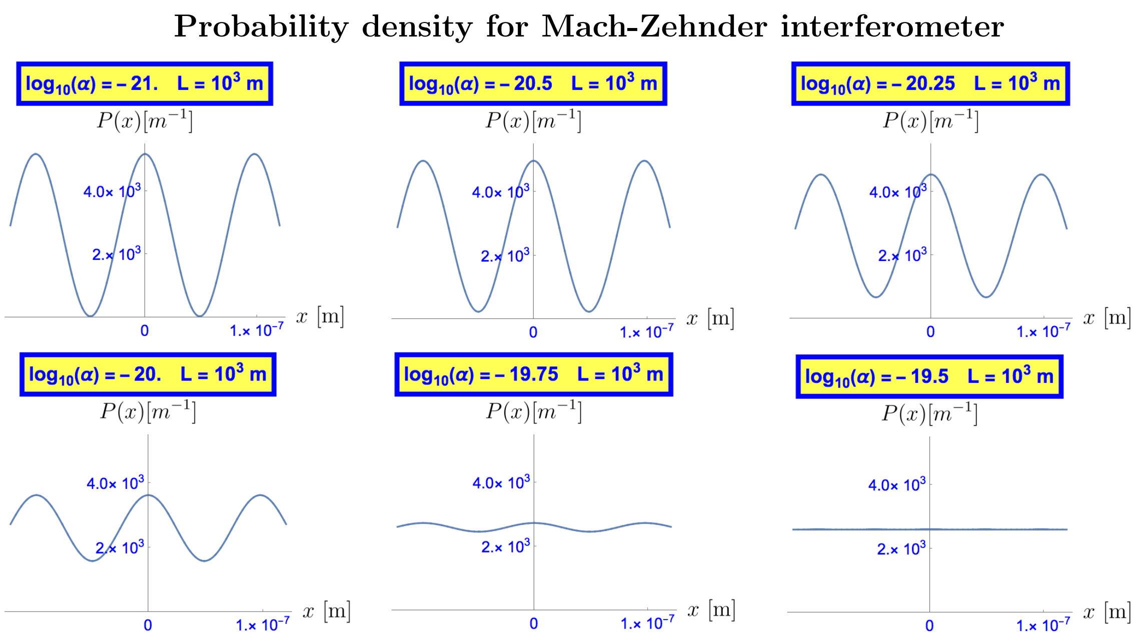

Figure 5.2: Plots of the probability density with running α and fixed L. The parameters are relative to a proposed Mach-Zehnder interferometer, the STE-QUEST interferometer, and are: $\sigma = 3 * 10^{-5}$ m; $m = 1.6 * 10^{-25}$ kg; $k = 3.4 * 10^{-27}\frac{\text{kg m}}{s}$; $t_{tot} = 10.0$ s; $a = 10.4$ cm.

tion [71] and varying respectively α with fixed L, and L with fixed α. Such plots highlight how the sole effect on the interference fringes due to a scalar gravitational perturbation is a reduction of the peaks.

The resulting formula for the visibility is very complicated. However, in the longitudinal-eikonal approximation the spread of the wavepacket (σ) is much smaller than the superposition distance, i.e. $\sigma \ll a$, and the formula can be simplified. Indeed, the initial

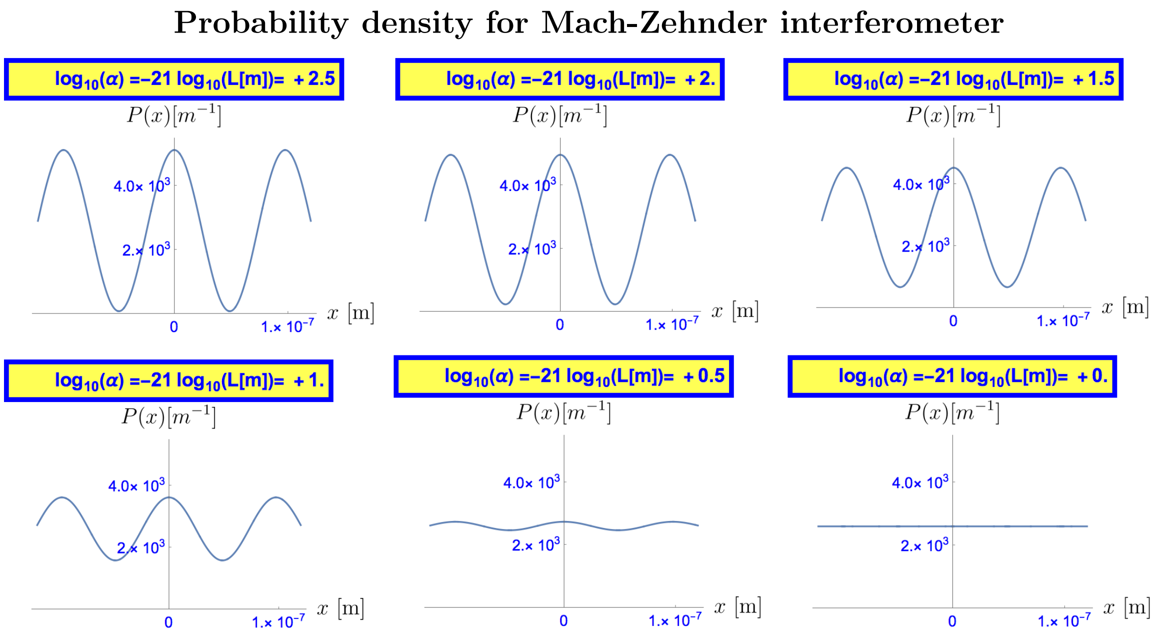

Figure 5.3: Plots of the probability density with running L and fixed α.The parameters are relative to a proposed Mach-Zehnder interferometer, the STE-QUEST interferometer, and are: $\sigma = 3 * 10^{-5}$ m; $m = 1.6 * 10^{-25}$ kg; $k = 3.4 * 10^{-27} \frac{\text{kg m}}{s}$; $t_{tot} = 10.0$ s; $a = 10.4$ cm.

state can be approximated in Eq. (5.13) as $\chi_0(s, q) \simeq \sqrt{\pi\sigma}\delta(s)$, in which case we are able to write the visibility as:

$$\nu^{(\mathcal{P})} \simeq e^{\int_0^{a/v} d\tau \Gamma_{\mathcal{P}}(2v\tau)} \tag{5.16}$$

with

$$\int_0^{a/v} d\tau\, \Gamma_{\mathcal{P}}(2v\tau) = \frac{\alpha^2 c^3 L m^2 \left(\sqrt{2\pi} L \operatorname{erf}\left(\frac{\sqrt{2}a}{L}\right) - 4a\right)}{2v\hbar^2} \tag{5.17}$$

This formula shows a reduction of the visibility proportional to square of the mass of the particle, meaning that a small increase in the latter will give an important gain in the sensitivity to spacetime fluctuations. For the sake of completeness we also analyse the interferometer sensitivity in the regime in which the spacial correlation of the stochastic perturbation is much bigger than the size of the superposition, i.e. $a \ll L$. In this regime Eq. (5.16) simplifies to:

$$\nu^{(\mathcal{P})} \simeq \exp\left(-\frac{4a^3\alpha^2 c^3 m^3}{3kL\hbar^2}\right) \tag{5.18}$$

It is interesting to notice that the effect scales with the cube power of the superposition distance a, or alternatively with the cube power of the of the experimental time (indeed

$a = t_{\text{tot}} k/2m$), while in gravimetry with atom interferometers it depends on the square of time [72]. In the opposite regime instead, where the size of the superposition is much larger than the spatial correlation of the noise, i.e. $a \gg L$, the visibility simplifies to:

$$\nu^{(\mathcal{P})} \simeq \exp\left(-\frac{2a\alpha^2 c^3 L m^3}{k\hbar^2}\right) \tag{5.19}$$

and, as expected, an increase of the size of the superposition will not give any significant improvement to the sensitivity.

5.3 Mach-Zehnder interferometry: decoherence in the energy eigenbasis

The energy decoherence process induced by the tensor component of the gravitational perturbation is described by Eq. (3.46) of the main text with the further assumption $h^{ij} \gg h^{0i}, h^{00}$:

$$\partial_t \hat{\rho} = -\frac{i}{\hbar}\Big[\frac{\hat{\mathbf{P}}^2}{2m}, \hat{\rho}(t)\Big] - \frac{\alpha^2\lambda}{(2\pi)^{3/2}\hbar^5}\int d^3q\, \tilde{u}^{ij}(\mathbf{q})\frac{m^2(\mathbf{q})}{m^2}\left[\frac{\hat{P}_i\hat{P}_j}{2m}, \Big[\frac{\hat{P}_i\hat{P}_j}{2m}, \hat{\rho}(t)\Big]\right] \tag{5.20}$$

where we recall[2] $\lambda = \min(\tau_c, t)$. As done for the position decoherence case, we apply the above equation to a Mach-Zehnder set up, in order to quantify the effect of decoherence on the visibility. We therefore specialize Eq. (3.46) by means of the longitudinal-eikonal approximation and of the point-like particle assumption $m(r) = m\delta(r)$. Furthermore, we assume the spatial correlation function to be a Gaussian: $\tilde{u}^{ij}(\mathbf{q}) = L^3\hbar^3 e^{-\mathbf{q}^2L^2/(2\hbar^2)}\delta^{ij}$ as a first approximation. Also in this case it is natural to take $\tau_c = L/c$. When represented in the momentum eigenbasis, the equation then reads:

$$\partial_t \rho_t(p,p') = -\frac{i}{2m}(p^2 - p'^2)\rho_t(p,p') - \frac{2\alpha^2\lambda(p^2-p'^2)^2}{4m^2\hbar^2}\rho_t(p,p') \tag{5.21}$$

We again solve the above equation with the help of the characteristic function defined in Eq. (5.4). In this formalism, Eq. (5.21) reads:

$$\partial_t \chi_t(s,q) = \left(\frac{q}{m}\partial_s + \frac{2\alpha^2\lambda q^2}{m^2}\partial_s^2\right)\chi_t(s,q) \tag{5.22}$$

The solution to the above equation is:

$$\chi_t(s,q) = \frac{1}{h}\int dp ds' e^{ip(s'-s+\frac{q}{m}t)/\hbar} R_{t,\mathcal{E}}(-2pq)\chi_0(s',q) \tag{5.23}$$

with $R_{t,\mathcal{E}}(p,p') = e^{-\Gamma_{\mathcal{E}}(p,p')t}$, where $\Gamma_{\mathcal{E}}(p,p')$ is the decoherence rate function:

$$\Gamma_{\mathcal{E}}(p,p') = \frac{\alpha^2\lambda(p^2-p'^2)^2}{2m^2\hbar^2} \tag{5.24}$$

[2] see Eq. (3.33) in Sec. 3.3

and we have used the property $R_{\mathcal{E}}(p,p') = R_{\mathcal{E}}(p^2 - p'^2)$ that can be easily checked to be valid for the decoherence kernel in Eq. (5.24). Note that, also in this case, the above Eq. (5.23) is valid independently of the choice of the initial state.
The application to Mach-Zehnder interferometry follows as for the position decoherence case: the evolution of the system from the first beam splitter to the mirrors and from the mirrors to the second beam splitter is easily described by Eq. (5.23), while the reflection by Eq (5.10):

$$\begin{cases} \chi_{t_{ref}}(s,q) = \frac{1}{\hbar}\int dpds' e^{ip(s'-s+\frac{q}{m}t_{ref})/\hbar} R_{t_{ref},\mathcal{E}}(-2pq)\chi_0(s',q) \\ \chi_{mir}(s,q) = \chi_{t_{\text{ref}}}(-4a-s,-q) + \chi_{t_{\text{ref}}}(4a-s,-q) + 2\cos\left(\frac{aq}{\hbar}\right)\chi_{t_{\text{ref}}}(-s,-q) \\ \chi_{t_{tot}}(s,q) = \frac{1}{\hbar}\int dpds' e^{ip(s'-s+\frac{q}{m}(t_{tot}-t_{ref}))/\hbar} R_{t_{tot},\mathcal{E}}(-2pq)\chi_{mir}(s',q) \end{cases} \tag{5.25}$$

Also in this case, we neglect the action of the second beam splitter and the subsequent evolution to the screen for the reason stated in the previous section. The resulting interference pattern at the screen is:

$$P^{(\mathcal{E})}_{\text{scr}}(x) = \int \frac{dqdpds'}{(2\pi\hbar)^2} e^{\frac{i}{\hbar}qx} e^{-2\Gamma_{\mathcal{E}}(p-\frac{q}{2},p+\frac{q}{2})t_{\text{ref}}} e^{\frac{ip}{\hbar}(\frac{2aq}{k}-s')} 2\left[\cos\left(\frac{4ap}{\hbar}\right) + \cos\left(\frac{aq}{\hbar}\right)\right]\chi_0(s',-q) \tag{5.26}$$

that can be used to estimate the visibility as in Eq. (5.1), given the explicit form of the state at the first beam splitter $\chi_0(s,q)$. We choose it to be a Gaussian wavepacket, Eq (5.14), as for the case of position decoherence. Upon plugging Eq. (5.14) into Eq. (5.26) and performing the integration in the s' and p variables, the probability density reads:

$$\begin{aligned} P^{(\mathcal{E})}_{\text{scr}}(x) = \int dq \Bigg(& \frac{\sigma e^{\left(-\frac{k^2\left(32a^2\hbar^2+q^2\sigma^4\right)+12a^2q^2\hbar^2+8akq\hbar^2\left(2a+\Gamma m q^3\sigma^2\right)}{4k\hbar^2(8a\Gamma m q^2\hbar^2+k\sigma^2)}\right)}}{\left(e^{\frac{k^2\sigma^2}{\hbar^2}}+1\right)\hbar}\sqrt{\frac{k\hbar^2}{8a\Gamma m q^2\hbar^2+k\sigma^2}}e^{iqx/\hbar}\cdot \\ & \cdot\Bigg[e^{\left(\frac{4a^2k^2\hbar^2+2a^2q^2\hbar^2+k^4\sigma^4}{8a\Gamma k m q^2\hbar^4+k^2\sigma^2\hbar^2}\right)}\Bigg[e^{\frac{8a^2q}{8a\Gamma m q^2\hbar^2+k\sigma^2}}\cos\left(\frac{2ak\sigma^2(2k-q)}{8a\Gamma m q^2\hbar^3+k\sigma^2\hbar}\right)+ \\ & + 2\cos\left(\frac{aq}{\hbar}\right)e^{\frac{4a^2(k+q)}{8a\Gamma m q^2\hbar^2+k\sigma^2}}\cos\left(\frac{akq\sigma^2}{8a\Gamma m q^2\hbar^3+k\sigma^2\hbar}\right)+\cos\left(\frac{2ak\sigma^2(2k+q)}{8a\Gamma m q^2\hbar^3+k\sigma^2\hbar}\right)\Bigg] \\ & + \cosh\left(\frac{kq\sigma^2}{\hbar^2}\right)\left(e^{\frac{2a^2\left(2k^2+q^2\right)}{k(8a\Gamma m q^2\hbar^2+k\sigma^2)}}+e^{\frac{2a^2\left(2k^2+4kq+q^2\right)}{k(8a\Gamma m q^2\hbar^2+k\sigma^2)}}+2\cos\left(\frac{aq}{\hbar}\right)e^{\frac{2a^2\left(4k^2+2kq+q^2\right)}{k(8a\Gamma m q^2\hbar^2+k\sigma^2)}}\right)\Bigg]\Bigg) \end{aligned} \tag{5.27}$$

where we have rewritten $\Gamma_{\mathcal{E}}(p-\frac{q}{2},p+\frac{q}{2}) = 4\Gamma p^2 q^2$ for simplicity. The above formula is very complicated, and we were not able to obtain a simpler analytical expression for it nor for the visibility, even in the long wavelength and longitudinal-eikonal regimes of approximation. However, Eq. (5.27) can be used to determine the visibility via a numerical analysis. In Fig. (5.4) we report a series of plots for the probability

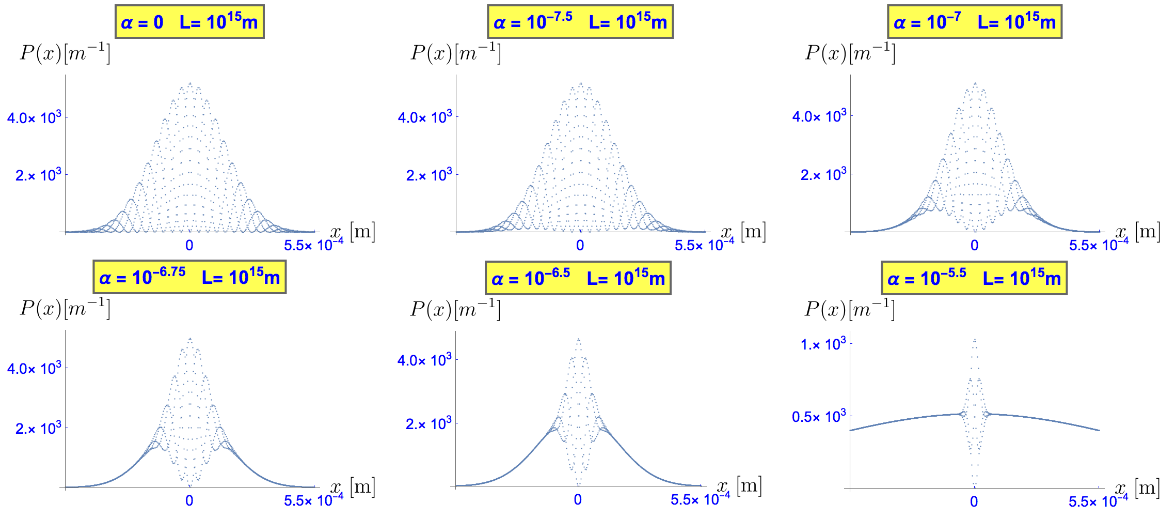

Figure 5.4: Numerical plots of the probability density with running α and fixed $L = 10^{15}$ m. The parameters are relative to a proposed Mach-Zehner interferometer, the STE-QUEST interferometer, and are $\sigma = 3 * 10^{-5}$ m; $m = 1.6 * 10^{-25}$ kg; $k = 3.4 * 10^{-27} \frac{\text{kg m}}{s}$; $t_{tot} = 10.0$ s; $a = 10.4$ cm.

density in order to illustrate the qualitative effect of a tensorial stochastic gravitational perturbation on the interference pattern.

Fig. (5.4) shows how the effect of decoherence is not a simple reduction of the peaks, as in the case of pure positional decoherence. Furhtermore, the tails of the probability density seem to be more affected than the center.

In the next section we will apply the results of this and the previous section to Mach-Zehnder interferometers.

5.4 Case studies

To make the study more concrete, we now apply the above results to test the sensitivity of a sample set of operative and proposed atom interferometry experiments, i.e. we study for which pairs of parameters (α, L) a gravitational perturbation induces a reduction of the visibility bigger than 10% ($\nu = 90\%$). If such a reduction were actually to be measured in a real experiment, and an accurate analysis of the noise (thermal, electrical, vibrational) in the experimental apparatus performed, it would allow one to use Eqs. (5.26) and (5.18) to set possibly stringent bounds on the pair of the perturbation's parameters $(\alpha,\, L)$.

We accordingly set the parameters of our simulated experiment for each of the selected experiments as reported in Table II:

Table II: Interferometers' parameters				
	m [kg]	**k** [$\frac{\text{kg m}}{\text{s}}$]	$\mathbf{t}_{\text{tot}}$ [s]	$\boldsymbol{\sigma}$ [m]
HYPER [65]	$2.5 * 10^{-25}$	$8.8 * 10^{-28}$	1.5	$6.4 * 10^{-7}$
STE-QUEST [71]	$1.6 * 10^{-25}$	$3.4 * 10^{-27}$	10.0	$3.0 * 10^{-5}$
Xu [73]	$2.5 * 10^{-25}$	$1.5 * 10^{-27}$	20.0	$3.9 * 10^{-6}$
Muntiga [74]	$1.6 * 10^{-25}$	$1.9 * 10^{-27}$	0.6	$5.0 * 10^{-5}$
Kovachy [75]	$1.6 * 10^{-25}$	$8.5 * 10^{-28}$	2.1	$5.6 * 10^{-5}$

In order to determine the reduction of the visibility, we use Eqs. (5.16,5.17) in the case of the pure position decoherence induced by the scalar component of the gravitational perturbation. In the case of pure energy decoherence instead, we numerically integrate Eq. (5.27) to get an expression for the probability density and from that we estimate its minimum and maximum to plug in Eq. (5.1). Since the effect of decoherence is more prominent on the tails of the interference figure (see Sec. 5.3), we take the relative minimum and maximum of just the tails, which we define as the areas in which the probability density is less or equal than a quarter of its absolute maximum value (which instead always corresponds to $x = 0$ m) in absence of decoherence.
The main results are summarized by Figs. (5.5, 5.6) and Figs.(5.7,5.8) respectively.
Our analysis shows a reduction of visibility of more than 10%, which can be detected in a real interferometric experiment [65, 74, 73, 76, 75], in presence of scalar fluctuations with correlation length and strength respectively down to $L \simeq 10^{-1}$ m, and $\alpha \sim 10^{-22}$ for state of the art matter-wave interferometers. In the case of the tensorial perturbation, we had to resort to a numerical analysis, as we were not able to find an analytic expression for the probability density and therefore the visibility. Such an analysis shows a much lower sensitivity to tensorial gravitational perturbation, as expected from [43]. A clear sign of decoherence can be observed only for perturbations whose strength α is of the order of 10^{-5}, which is too large to be produced by any expected source of tensorial fluctuations. As a final remark, note that the sensitivity curves in Fig. 5.8 are straight horizontal lines, as for all experiments the characteristic time of the fluctuations $\tau = \frac{L}{C}$ is greater than the time of the experiment t_{tot}, thus $\lambda = t$ and ν becomes L independent.

5.4.1 Other sources of decoherence

The study so far was carried out assuming no other source of decoherence except gravitational fluctuations, while in real experiments different sources of decoherence are always present [49]. We show that the most relevant source of decoherence, i.e. thermal gas collisions, gives a negligible effect in a spaced-based setup. We will not consider other sources of decoherence because they strongly depend on the specific setup.
The decoherence function $\Gamma_{\text{coll}}(x)$ describing gas collision can be quite complex [49], however in an interferometric experiment usually the superposition distance is much bigger than the typical thermal De Broglie wavelength of the gas allowing one to rely on the simplified expression [77]:

$$\Gamma_{\text{coll}} = \frac{4\pi\Gamma(9/10)}{5\sin(\pi/5)}\left(\frac{9\pi\beta_c\beta_g I_g I}{64\hbar\epsilon_0(I+I_g)}\right)\frac{p_g v_g^{3/5}}{K_b T_g} \tag{5.28}$$

where T_g,p_g, m_g are the temperature, the pressure and the mass of the gas, I, I_g are the ionization energies, β_c and β_g the static polarizabilities of the matter-wave and gas particle and $v_g = \sqrt{2K_bT_g/m_g}$ is the thermal velocity of the gas particle.
Upon plugging in the values of the parameter relative to a space based experiment, which are summarized in Table III, we get $\Gamma_{\text{coll}} \simeq 7.6 * 10^{-30} s^{-1}$, our analysis shows that the decoherence induced by thermal gas collisions is practically absent in such a setup.

Table III: Collisional parameters

$\mathbf{I_g}$ [eV]	$\beta_{\mathbf{g}}$ [m^3]	$\mathbf{T_g}$ [K]	$\mathbf{p_g}$ [Pa]	$\mathbf{I_c}$ [eV]	$\beta_{\mathbf{c}}$ [m^3]
13.6	$7.42 * 10^{-41}$	20	10^{-11}	3.89	$59.42*10^{-30}$

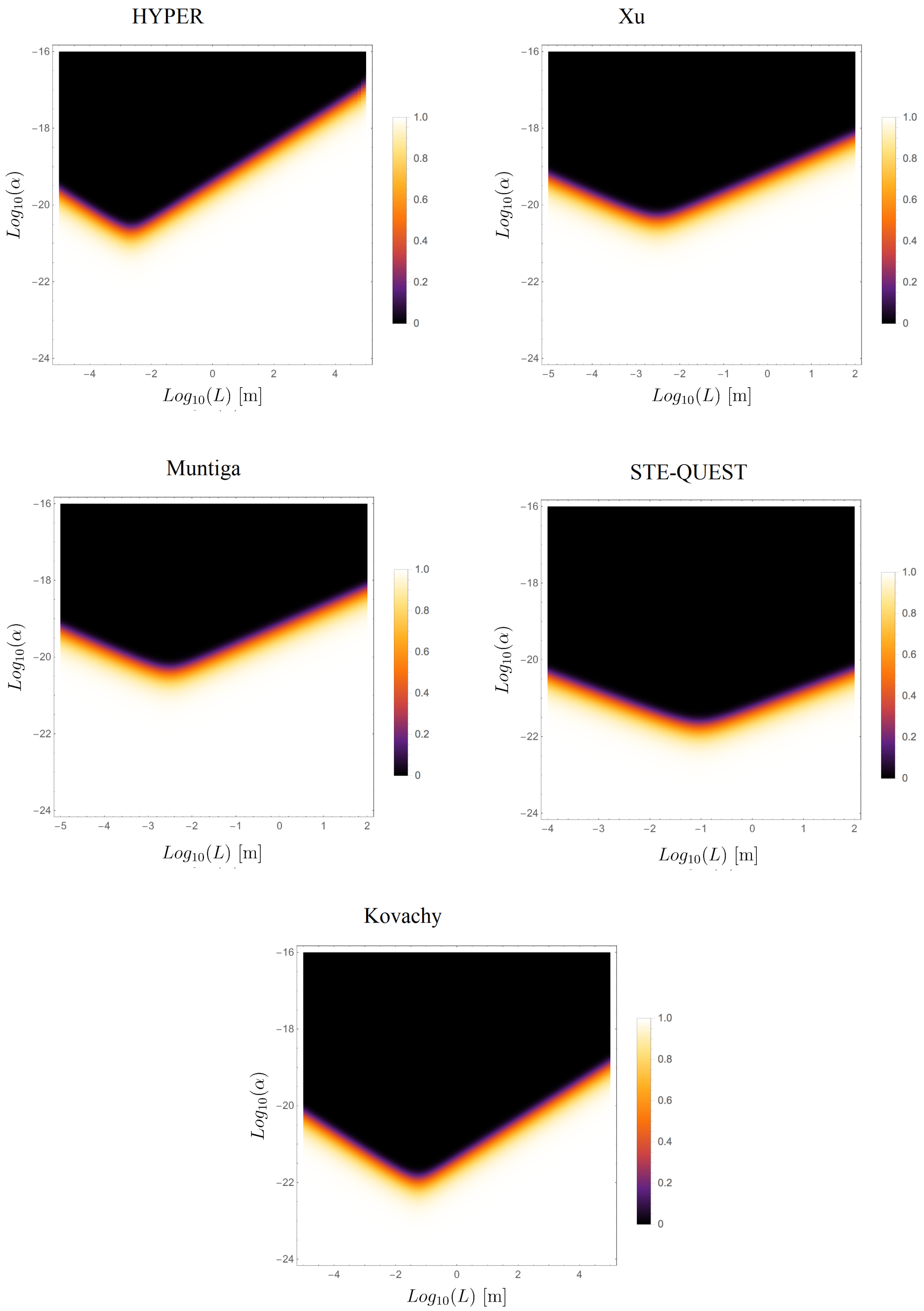

Figure 5.5: Colored plots showing the visibility as a function of the scalar perturbation's strength (α) and correlation length L. One plot for each interferometer of Table II

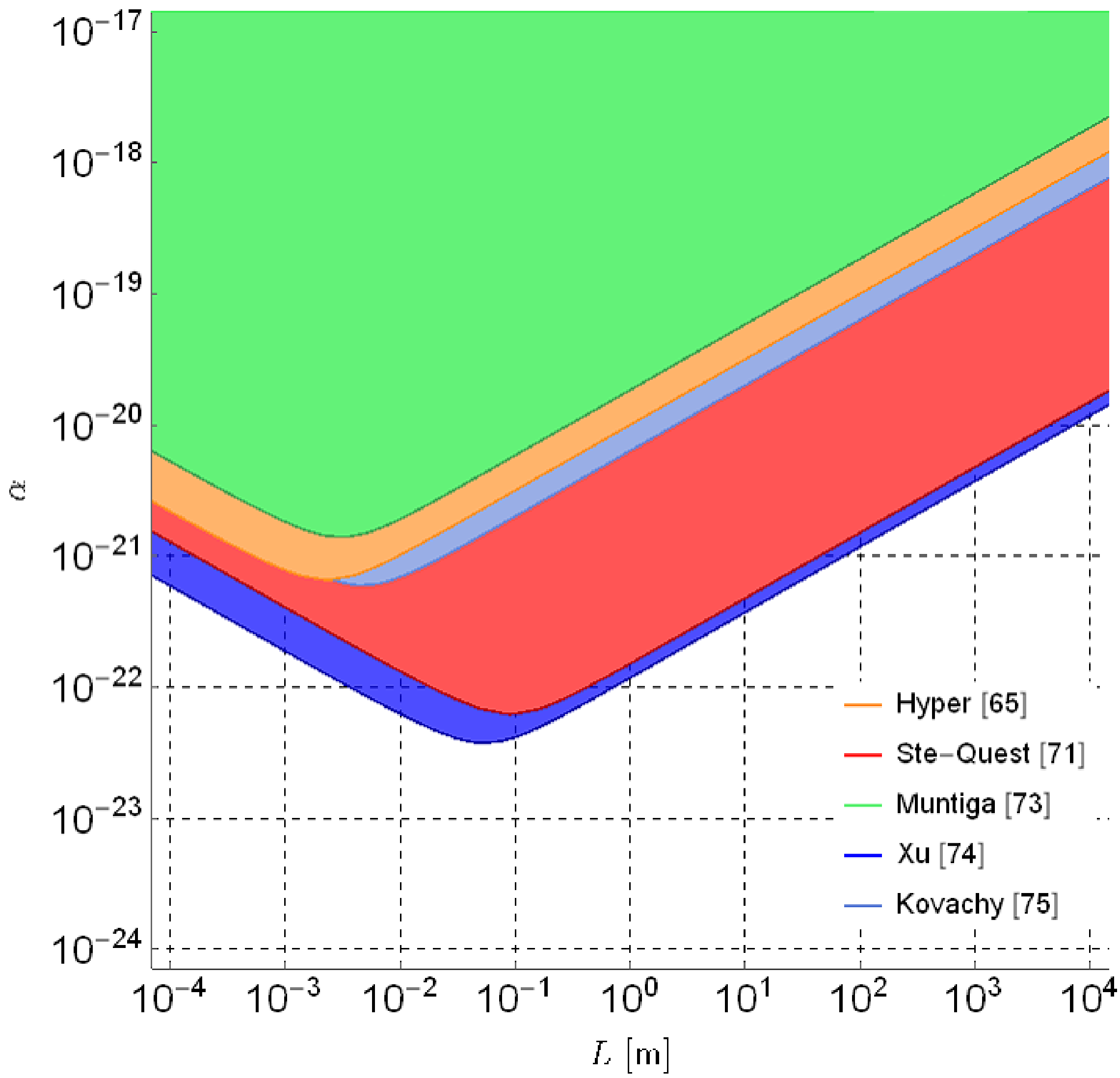

Figure 5.6: The different shaded area represent the region of parameters α and L in which a scalar gravitational perturbation induces a reduction of more that 10% in visibility for different experimental scenarios (see the legend).

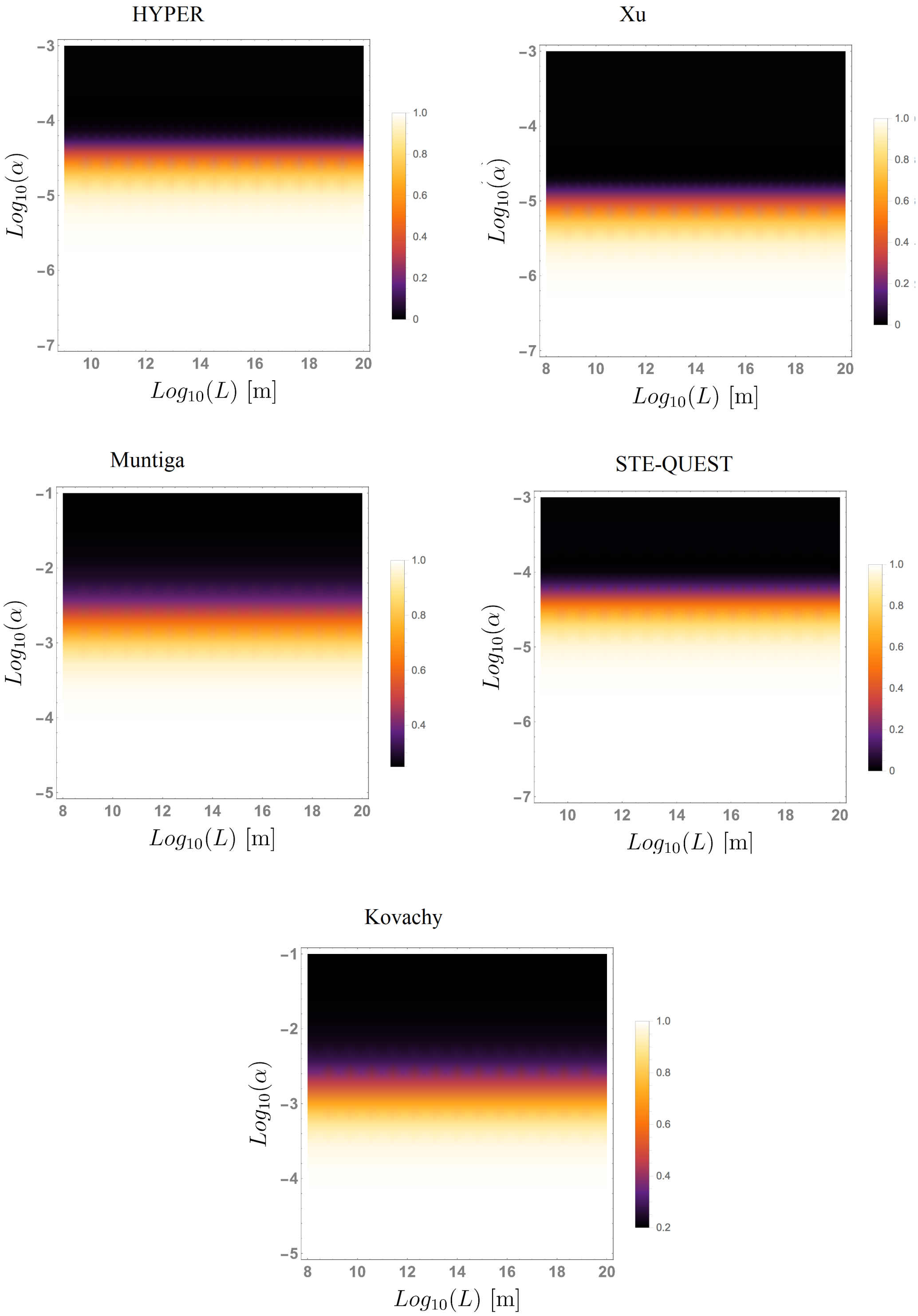

Figure 5.7: Colored plots showing the visibility as a function of the tensor perturbation's strength (α) and correlation length L. One plot for each interferometer of Table II

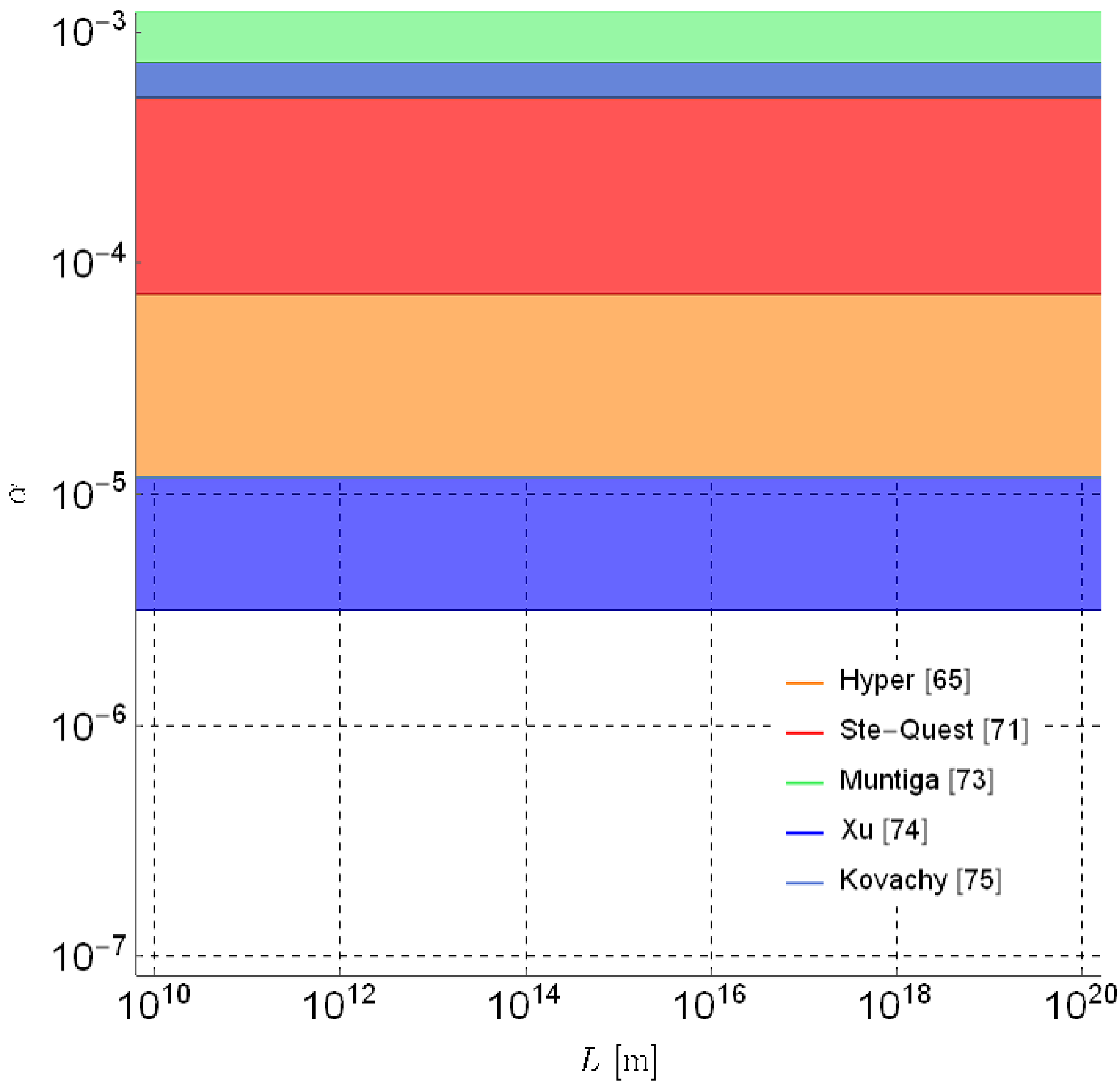

Figure 5.8: The different shaded areas represents the region of parameters alpha and L in which a tensorial gravitational perturbation induces a reduction of more that 10% of the interference fringes for different experimental scenarios (see the legend).

Chapter 6

Conclusions

In this thesis we have analyzed the decoherence effect induced on non relativistic quantum matter by a stochastic gravitational perturbation.
We have started by giving in chapter 2 an overview of the literature of gravitational decoherence models. Such models can be regrouped in two distinct classes: those that predict the gravitational decoherence phenomenon to occur in the position eignebasis, and those that predict decoherence in the momentum or energy eigenbasis. The different models also disagree on the rate of decoherence, i.e. the typical time scale over which a quantum system interacting with a gravitational perturbation becomes classical.
In order to solve such apparent contradictions and to determine the general underlying dynamics of gravitational decoherence, we have derived in chapter 3 a general model of decoherence for a non relativistic quantum particle interacting with a weak stochastic gravitational perturbation. We have specialized such an equation to the Markovian limit under some further reasonable assumptions on the stochastic properties of the gravitational noise motivated by simplicity arguments, cosmological models and observations.
We have extended our model to the description of the center of mass of a rigid extended body, which is a more realistic and experimentally interesting scenario.
Our Markovian master equation predicts decoherence in position, momentum and energy as it contains, among other terms, double commutators of functions of the position, momentum and free kinetic energy operators with the averaged density matrix. With such a novel model, we were able to successfully recover other results present in the literature as appropriate limiting cases of our general master equation (thus resolving the decoherence rate and eigenbasis puzzles), and to determine the regimes of validity for the gravitational decoherence phenomenon to occur in the position or momentum (energy) eigenbasis.
In chapter 4, we have extended the model derived in chapter 3 in order to describe the gravitational decoherence effect on a spin 1/2 fermionic particle under the influence of an external electromagnetic field. The resulting Hamiltonian and master equation correctly reproduce the results of the bosonic model up to very small corrections. Such

corrections account for relativistic effects (different spin of the two kind of particles) and the different quantization scheme employed.
Motivated by the interest to assess if and to what extent matter-wave interferometers can be used to probe the cosmic gravitational background, in chapter 5 we have applied the gravitational decoherence model developed in chapter 3 to atom interferometry. We have modelled the behaviour of the interferometric visibility as a function of the gravitational perturbation's parameters in order to quantify the decoherence effect. Among other results, with such an analysis we have provided a practical formula to estimate the sensitivity of such a class of experiments to stochastic scalar fluctuations of the metric.
We have applied our analysis to a selected sample of actual and proposed matter-wave interferometers, showing that state of the art technology is sensitive to scalar perturbation with strength and correlation length down to $\alpha \sim 10^{-23}$, $L \sim 1$ m, while it is practically unaffected by tensorial perturbations.
We have analysed the most relevant competing decoherence effect, namely thermal gas collisional decoherence, and shown that it is negligible with respect to gravitational decoherence.
Although based on strongly simplifying assumptions, this study shows that matter-wave interferometry is a promising avenue for testing the interface of quantum mechanics and gravity, and for the detection of a scalar gravitational stochastic background.

Chapter 7

Appendices

Appendix A: Feshbach Villars formalism

Here we provide explicit calculation for the derivation of Eq. (3.12).
Let us first rewrite Eq. (3.6) as:

$$(i\hbar\partial_t - i\hbar c h^{0i}\partial_i)^2\psi = \Big[\hbar^2 c\partial_t(h^{0i})\partial_i - \hbar^2 c^2(1+h^{00})\nabla^2 - \hbar^2 c^2 h^{ij}\partial_i\partial_j + m^2c^4(1+h^{00})\Big]\psi + O(h^2) \tag{A.1}$$

and the system of Eq. (3.8) as

$$\begin{cases} i\hbar(\partial_t - ch^{0i}\partial_i)\psi + mc^2\psi = 2mc^2\phi \\ i\hbar(\partial_t - ch^{0i}\partial_i)\psi - mc^2\psi = -2mc^2\chi \end{cases} \tag{A.2}$$

Casting Eq. (A.1) in the above system we get :

$$\begin{aligned} i\hbar(\partial_t - ch^{0i}\partial_i)\phi =& \frac{mc^2}{2}(\phi-\chi)\frac{m^2c^4}{2mc^2}(1+h^{00})(\phi+\chi) - \frac{\hbar^2}{2m}(1+h^{00})\nabla^2(\phi+\chi) \\ &- \frac{\hbar^2}{2m}h^{ij}\partial_i\partial_j(\phi+\chi) + \frac{\hbar^2}{2mc}\partial_t(h^{0i})\partial_i(\phi+\chi) \end{aligned} \tag{A.3}$$

$$\begin{aligned} i\hbar(\partial_t - ch^{0i}\partial_i)\chi =& -\frac{mc^2}{2}(\phi-\chi) - \frac{m^2c^4}{2mc^2}(1+h^{00})(\phi+\chi) + \frac{\hbar^2}{2m}(1+h^{00})\nabla^2(\phi+\chi) \\ &+ \frac{\hbar^2}{2m}h^{ij}\partial_i\partial_j(\phi+\chi) - \frac{\hbar^2}{2mc}\partial_t(h^{0i})\partial_i(\phi+\chi) \end{aligned} \tag{A.4}$$

Recalling now that $\Psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}$ and exploiting the Pauli matrices, the system reduces to:

$$\begin{aligned} i\hbar\partial_t\Psi =& \Big[mc^2\sigma_3 + \frac{mc^2}{2}h^{00}[\sigma_3 + i\sigma_2] + i\hbar c h^{0i}\partial_i - \frac{\hbar^2}{2m}(1+h^{00})[\sigma_3 + i\sigma_2]\nabla^2 + \\ & -\frac{\hbar^2}{2mc}\partial_t(h^{0i})[\sigma_3 + i\sigma_2]\partial_i - \frac{\hbar^2}{2m}h^{ij}[\sigma_3 + i\sigma_2]\partial_i\partial_j\Big]\Psi \\ =& : \mathfrak{H}\Psi \end{aligned} \tag{A.5}$$

Upon applying the transformation (3.11), the EOM transform as:

$$\mathfrak{H} \to K := T\mathfrak{H}T^{-1} + i\hbar T\partial_t(T^{-1}) \tag{A.6}$$

and read exactly as Eq. (3.12) of the main text.

Appendix B: Foldy Wouthuysen method - bosonic model

Here we illustrate the Fouldy Wouthuysen method applied to Eq. (3.12). Let us consider the transformations:

$$K \to K' = U(K - i\hbar\partial_t)U^{-1} \tag{B.1}$$

and specialize U to Eq. (3.17), i.e.

$$U = e^{-i\sigma_3\mathcal{O}/(2mc^2)} =: e^{iS} \tag{B.2}$$

With the help of the BCH identity:

$$\begin{aligned} K' =& e^{iS}(K - i\hbar\partial_t)e^{-iS} = K + i[S,K] + \frac{i^2}{2!}[S[S,K]] + \\ & + \frac{i^3}{3!}[S[S[S,K]]] + ... \\ & + \hbar(-\dot{S} - \frac{i}{2}[S,\dot{S}] + \frac{1}{6}[S,[S,\dot{S}]] + ...) \end{aligned} \tag{B.3}$$

Recalling that:

$$K = mc^2\sigma_3 + \mathfrak{E} + \mathcal{O} \tag{B.4}$$

and noticing that:

$$[\sigma_3, \mathfrak{E}] = 0 \tag{B.5}$$

$$\{\sigma_3, \mathcal{O}\} = 0 \tag{B.6}$$

$$[\sigma_3\mathcal{O}, \sigma_3] = -2\mathcal{O} \tag{B.7}$$

$$[\sigma_3\mathcal{O}, \mathfrak{E}] = \sigma_3[\mathcal{O}, \mathfrak{E}] \tag{B.8}$$

$$[\sigma_3\mathcal{O}, \mathcal{O}] = 2\sigma_3\mathcal{O}^2 \tag{B.9}$$

it is not difficult to check that:

$$K' = mc^2\sigma_3 + \mathfrak{E}' + \mathcal{O}' \tag{B.10}$$

where:

$$\mathfrak{E}' = \mathfrak{E} + \sigma_3(\frac{\mathcal{O}^2}{2mc^2} - \frac{\mathcal{O}^4}{8m^3c^6}) - \frac{i}{8m^2c^4}[\mathcal{O}, \dot{\mathcal{O}}] - \frac{1}{8m^2c^4}[\mathcal{O}, [\mathcal{O}, \mathfrak{E}]] + ... \tag{B.11}$$

$$\mathcal{O}' = \frac{1}{2mc^2}\sigma_3[\mathcal{O}, \mathfrak{E}] - \frac{\mathcal{O}^3}{3m^2c^4} + \frac{i}{2mc^2}\sigma_3\dot{\mathcal{O}} + ... \tag{B.12}$$

We note that $\mathcal{O}'$ is of order c^{-1}, meaning that we need to perform a further transformation if we want non trivial diagonal EOM. The transformation that we perform is:

$$U' = e^{-i\sigma_3\mathcal{O}'/(2mc^2)} \tag{B.13}$$

after which the Hamiltonian reads:

$$K'' = mc^2\sigma_3 + \mathfrak{E}' + \mathcal{O}'' + ... \tag{B.14}$$

with:

$$\mathcal{O}'' = \frac{\sigma_3}{2mc^2}[\mathcal{O}', \mathfrak{E}'] + \frac{i}{2mc^2}\sigma_3\dot{\mathcal{O}}' + ... \tag{B.15}$$

As $\mathcal{O}'' \sim O(\frac{v^3}{c^3})$ we need to perform a final transformation:

$$U'' = e^{-i\sigma_3\mathcal{O}''/(2mc^2)} \tag{B.16}$$

Finally the Hamiltonian reads:

$$H := K''' = mc^2\sigma_3 + \mathfrak{E}' + O(c^{-4}) \tag{B.17}$$

It is easy to note that the only (other than $\mathfrak{E}$) contribution to $\mathfrak{E}'$ at the desired order is:

$$\begin{aligned} \frac{\sigma_3}{2mc^2}\mathcal{O}^2 =& \frac{\sigma_3}{2mc^2}\{\frac{imc^2}{2}h^{00}\sigma_2\, ,\, -\frac{i\hbar^2}{2m}\nabla^2\sigma_2\} + O(h^2) + O(c^{-4}) \\ =& \frac{\hbar^2}{4m}(h^{00}\nabla^2 + \nabla^2(h^{00}))\sigma_3 + O(h^2_{\mu\nu}) + O(c^{-4}) \end{aligned} \tag{B.18}$$

so that the Hamiltonian becomes:

$$\begin{aligned} H =& mc^2(1 + \frac{h^{00}}{2})\sigma_3 - \frac{\hbar^2}{2m}(1 + \frac{h^{00}}{2})\nabla^2\sigma_3 - \frac{\hbar^2}{2m}h^{ij}\partial_i\partial_j\sigma_3 + i\hbar ch^{0i}\partial_i + \frac{i\hbar}{2}\partial_t(h^{00}) \\ &- \frac{i\hbar}{4}\partial_t(tr(h^{\mu\nu})) + \frac{\hbar^2}{8m}\nabla^2(tr(h^{\mu\nu}))\sigma_3 + O(c^{-4}) + O(h^2_{\mu\nu}) \end{aligned} \tag{B.19}$$

as in Eq. (3.18) of the main text.

Appendix C: Vierbein (or tetrad) formulation of gravity

We illustrate the basic ingredients of the tetrad formalism of the General Relativity theory. For a more complete treatment we address the reader to [63, 78].

The standard geometrical interpretation of the gravitational interaction is based on the notion of the Riemannian metric (g) and the Christoffel connection (Γ). The spacetime curvature, its dynamical evolution and the interaction with matter sources are described through differential equations involving g and Γ.
It is possible though to equivalently describe the geometry of a Riemannian maniflod (M) using the notion of vierbein and local connection. Such a formalism is particularly convenient when one wants to formulate a theory of gravity as a gauge theory, and wants to accomodate the notion of particles as irreducible representations of the Poincaré group in curved spacetimes [79, 80, 81].
We know that locally the laws of special relativity are valid. This translates into the consideration that we can attach at each and every point p of the Riemannian manifold M a flat tangent manifold equipped with the flat Minkowski metric.
There is a natural choice for the basis of such a tangent space (T_pM), the coordinate (or differential) basis:

$$\hat{e}_{(\mu)} = \partial_{(\mu)} \tag{C.1}$$

given by the partial derivatives of the coordinates. It follows that a given 4-vector $A \in T_pM$ has components:

$$A = A^\mu \hat{e}_{(\mu)} = A^\mu \partial_{(\mu)} \tag{C.2}$$

The dual basis:

$$\hat{e}^{(\mu)} = dx^{(\mu)} \tag{C.3}$$

spans the cotangent space, and it is given by the differential of the coordinates. A dual vector $B \in T_pM$ then has components:

$$B = B_\mu \hat{e}^{(\mu)} = B_\mu dx^{(\mu)} \tag{C.4}$$

As T_pM is a vector space, we are in principle free to choose any orthonormal basis to span it, as long as T_pM preserves the appropriate signature of the manifold. We therefore introduce a set of basis vectors $\hat{e}_a$, which we choose as non coordinate unit vectors, and we denote this choice by using capital Latin letters for indices of the non coordinate frame. Such a non coordinate basis is called a tetrad basis. The condition for preserving the signature of the metric therefore reads:

$$g(\hat{e}_A, \hat{e}_B) = \eta_{AB} = \mathrm{diag}(+, - - -) \tag{C.5}$$

With this choice, we can clearly find a fixed orthonormal basis that is independent of position. Then, form a local prospective, any vector can be expressed as a linear combination of the fixed tetrad basis vectors at the point in the following way:

$$\hat{e}_\mu(x) = e_\mu{}^A(x)\hat{e}_{(A)} \tag{C.6}$$

$$V^A = e_\mu{}^A V^\mu \tag{C.7}$$

The 4x4 invertible matrix $e_\mu{}^A(x)$ is called a vierbein field (or tetrad), and it is the transformation matrix that maps the tangent space $T_x M$ into Minkowski space preserving the inner product.
The inverse vierbein field (or tetrad) has components $e^\mu{}_A(x)$, and satisfies the orthonormality condition:

$$\begin{aligned} e^\mu{}_A e_\nu{}^A &= \delta^\mu_\nu \\ e_\mu{}^A e^\mu{}_B &= \delta^A_B \end{aligned} \tag{C.8}$$

which come from the preservation of the inner product.
The vierbein fields are mixed indices objects, in the sense that they carry one Minkowski spacetime index (A), and one Riemannian index (μ). Accordingly, they transform under coordinate and Lorentz transformations respectively as:

$$e_\mu{}^A \xrightarrow{\text{coord}} e'^{\,A}_\mu = \frac{\partial x^\nu}{\partial x'^\mu} e_\nu{}^A \tag{C.9}$$

$$e_\mu{}^A(x) \xrightarrow{\text{Lorentz}} e'^{\,A}_\mu(x) = \Lambda^A{}_B e_\mu{}^B \tag{C.10}$$

We now consider the covariant derivative ∇X of a vector (X) in the Minkowski frame. It will be given by the standard derivative (∂X) plus a correction given by the affine connection of the Minkowski frame:

$$(\nabla_\mu X^A) dx^\mu \otimes \hat{e}_{(A)} = (\partial_\mu X^A + \omega_\mu{}^A{}_B X^B) dx^\mu \otimes \hat{e}_{(A)} \tag{C.11}$$

The expression for the covariant derivative in the coordinate basis instead reads:

$$\begin{aligned} \nabla X =& (\nabla_\mu x^\nu) dx^\mu \otimes \partial_\nu \\ =& (\partial_\mu X^\nu + \Gamma^\nu_{\mu\alpha} X^\alpha) dx^\mu \otimes \partial_\nu \\ =& (\partial_\mu X^\nu + \Gamma^\nu_{\mu\alpha} X^\alpha) dx^\mu \otimes e_\nu{}^A(x)\hat{e}_{(A)} \\ =& e_\nu{}^A(x)\Big(\partial_\mu (e^\nu{}_B(x) X^B) + \Gamma^\nu_{\mu\alpha} e^\alpha{}_B(x) X^B\Big) dx^\mu \otimes \hat{e}_{(A)} \end{aligned} \tag{C.12}$$

Upon comparing Eq. (C.11) with Eq. (C.12), we can express the Minkowski frame or local affine connection in terms of the tetrads and the usual affine connection as:

$$\omega_\mu{}^A{}_B X^B = e_\nu{}^A(x)\partial_\mu e^\nu{}_B(x) X^B + e_\nu{}^A(x) e^\alpha{}_B(x) \Gamma^\nu_{\mu\alpha} X^B \tag{C.13}$$

Note that the above relation implies the metric compatibility condition:

$$\begin{aligned} \nabla_\mu e^\nu{}_B(x) &= 0 \\ \nabla_\mu g_{\alpha\beta} &= 0 \end{aligned} \tag{C.14}$$

Observing that $\nabla_\mu X^A$ must transform under a Lorentz boost as X^A, it follows:

$$\begin{aligned}\nabla_\mu(\Lambda^A{}_B) =&0\\ =&\partial_\mu(\Lambda^A{}_B)+\omega_\mu{}^A{}_C\Lambda^C{}_B-\omega_\mu{}^C{}_B\Lambda^A{}_C\end{aligned} \tag{C.15}$$

Upon multiplying the last line of Eq. (C.15) by $\Lambda^B{}_D$ on the left, we obtain the following relation:

$$\omega_\mu{}^A{}_D=\Lambda^B{}_D\Lambda^A{}_C\omega_\mu{}^C{}_B-\Lambda^B{}_D\partial_\mu(\Lambda^A{}_B) \tag{C.16}$$

which tells us that the affine connection transforms inhomogeneously under Lorentz transformations.

One can construct the usual geometric objects from $(e,\,\omega)$, as it is typically done from $(g,\,\Gamma)$, such as the Curvature Tensor:

$$R^{AB}{}_{\mu\nu}=\partial_\mu\omega_\nu{}^A{}_B-\partial_\nu\omega_\mu{}^A{}_B+\omega_\mu{}^A{}_C\omega_\nu{}^C{}_B-\omega_\nu{}^A{}_C\omega_\mu{}^C{}_B \tag{C.17}$$

and the Torsion:

$$\mathfrak{T}^A_{\mu\nu}=\partial_\mu e_\nu{}^A-\partial_\nu e_\mu{}^A+\omega_\mu{}^A{}_B e_\nu{}^B-\omega_\nu{}^A{}_B e_\mu{}^B \tag{C.18}$$

The field equations for the vierbein field can be derived from a variational principle in the same fashion it is typically done for the metric. In order to show it, Let us recall the inner product-signature preservation condition Eq. (C.5), which can be equivalently recast into:

$$g_{\mu\nu}=e_\mu{}^A\eta_{AB}e_\nu{}^B \tag{C.19}$$

It then follows that the variation of the metric can be expressed in terms of the variation of the vierbein filed as:

$$\delta g_\mu=e_{\nu A}\delta e_\mu{}^A+e_{\mu A}\delta e_\nu{}^A=-(g_{\mu\lambda}e_\nu{}^A+g_{\nu\lambda}e_\mu{}^A)\delta e_\lambda{}^A \tag{C.20}$$

The variation of the Einstein-Hilbert action ($S=\frac{1}{8\pi G}\int d^4x\sqrt{-g}R$ [12]) then reads:

$$\begin{aligned}\partial_g S=&\frac{1}{8\pi G}\int d^4x\,\sqrt{-g}(R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R)\delta g_{\mu\nu}\\ =&\frac{1}{8\pi G}\int d^4x\,e(R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R)\frac{\partial g_{\mu\nu}}{\partial e^\lambda{}_A}\delta e^\lambda{}_A\\ =&\frac{1}{8\pi G}\int d^4x\,e\Big(R_\lambda{}^\nu e_\nu{}^A-\frac{1}{2}R\delta^\nu_\lambda e_\nu{}^A+R^\mu{}_\lambda e_\mu{}^A-\frac{1}{2}R\delta^\mu_\lambda e_\mu{}^A\Big)\delta e^\lambda{}_A\\ =&\frac{1}{8\pi G}\int d^4x\,e\Big(R^\mu{}_\nu-\frac{1}{2}\delta^\mu_\lambda R\Big)e_\mu{}^A\delta e^\lambda{}_A\end{aligned} \tag{C.21}$$

Recalling the expression for the Einstein tensor ($G^{\mu\nu}=R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R$), the above equation yields:

$$G^{\mu\nu}e_\mu{}^A=0 \tag{C.22}$$

which must be interpreted as the Einstein's equations for the vierbein field e. Note that in order to switch back to the usual metric formulation it is sufficient to multiply the above equation by $e^\lambda{}_A$. We have thus shown that the vierbein formulation of General relativity is equivalent to the standard metric one.

Appendix D: EOM for the spin 1/2 fermionic field

We present the explicit steps for the derivation of the effective action of a spin 1/2 fermionic matter field coupled to a weak gravitational perturbation.

Consider the action for the Dirac field in curved spacetime:

$$S = \int d^4x \sqrt{-g}\mathcal{L}_D \tag{D.1}$$

with the Lagrangian density:

$$\mathcal{L}_D = \frac{i\hbar c}{2}[\bar{\psi}\gamma^\mu e^A{}_\mu \mathcal{D}_A\psi - e^A{}_\mu \mathcal{D}_A\bar{\psi}\gamma^\mu\psi] - mc^2\bar{\psi}\psi \tag{D.2}$$

where $e^A{}_\mu(x)$ is the so called vierbein field, the field that maps the tangent space to the manifold M at point x T_xM (coordinate basis ∂_A) into Minkowski space (non coordinate basis $\mathbf{e}_\mu$), and

$$\mathcal{D}_\mu\psi = \partial_\mu\psi + \frac{1}{8}[\gamma_a, \gamma_b]\omega_\mu{}^{ab}\psi + \frac{ie}{\hbar c}A_\mu\psi \tag{D.3}$$

is the covariant derivative with respect to both the spin and the electromagnetic connections. The pair $(e_A{}^\mu, \omega_A{}^{\mu\nu})$ allows for an equivalent geometrization of the gravitational interaction to the standard one given in terms of the metric and the affine connection $(g_{AB}, \Gamma^A{}_{BC})$; the relation between the two frameworks is given by:

$$\begin{cases} e_A{}^\mu \eta_{\mu\nu} e_B{}^\nu = g_{AB} \\ \omega_A{}^{\mu\nu} = e_B{}^\mu \eta^{\nu\rho}\partial_A e^B{}_\rho + e_B{}^\mu \eta^{\nu\rho} e^C{}_\rho \Gamma^B{}_{AC} \end{cases} \tag{D.4}$$

Note that (D.4) holds only for a torsion free, metric compatible connection.
We write the metric as the sum of a flat background $\eta_{\mu\nu} = \mathrm{diag}(+ - --)$, and a perturbation $h_{\mu\nu}$:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{D.5}$$

We are interested in studying the dynamics of the Dirac field in presence of a weak gravitational perturbation. We therefore perform a Taylor expansion of the action around the flat background metric and truncate the series at the first perturbative order:

$$S \approx \int d^4x \, (\sqrt{-g}\mathcal{L})\Big|_{g=\eta} - h^{\mu\nu}\Big(\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial g^{\mu\nu}}\Big)\Big|_{g=\eta} + O(h^2) \tag{D.6}$$

In order to work out the explicit expression for $\frac{\partial(\sqrt{-g}\mathcal{L})}{\partial g^{\mu\nu}}$, we look at the variation of the action with respect to the metric tensor:

$$\begin{aligned} \delta_g S &= -\frac{1}{2}\int d^4x \, \sqrt{-g}T^{AB}\delta g_{AB} \\ &= \int d^4x \, \frac{\partial(\sqrt{-g}\mathcal{L})}{\partial g_{AB}}\delta g_{AB} \end{aligned} \tag{D.7}$$

Notice that the above expression can be equivalently rewritten for a torsion free, metric compatible connection as:

$$\begin{aligned}\delta_g S &= \int d^4x\, \frac{\partial(\sqrt{-g}\mathcal{L})}{\partial {e^C}_\alpha}\delta {e^C}_\alpha + \frac{\partial(\sqrt{-g}\mathcal{L})}{\partial \omega_{A\mu\nu}}\delta\omega_{A\mu\nu} \\ &= \int d^4x\, \sqrt{-g}\frac{\partial\mathcal{L}}{\partial {e^C}_\alpha}\delta {e^C}_\alpha + \sqrt{-g}\frac{\partial\mathcal{L}}{\partial\omega_{A\mu\nu}}\delta\omega_{A\mu\nu} + \mathcal{L}\Big(\frac{\partial\sqrt{-g}}{\partial {e^C}_\alpha}\delta {e^C}_\alpha + \frac{\partial\sqrt{-g}}{\partial\omega_{A\mu\nu}}\delta\omega_{A\mu\nu}\Big)\end{aligned} \tag{D.8}$$

By noticing that $\frac{\partial\sqrt{-g}}{\partial\omega_{A\mu\nu}} = 0$, and defining $\frac{\partial\mathcal{L}}{\partial {e^C}_\alpha} =: {\mathcal{T}_C}^\alpha$ and $\frac{\partial\mathcal{L}}{\partial\omega_{A\mu\nu}} =: \mathcal{S}^{A\mu\nu}$, we rewrite the above equation as:

$$\begin{aligned}\delta S &= \int d^4x\sqrt{-g}\Big[{\mathcal{T}_C}^\alpha\delta {e^C}_\alpha + \mathcal{S}^{A\mu\nu}\delta\omega_{A\mu\nu} + 2{e_C}^\alpha\mathcal{L}_D\delta {e^C}_\alpha\Big] \\ &= \int d^4x\sqrt{-g}\Big[\Big({\mathcal{T}_C}^\alpha - \mathcal{D}_A[{{\mathcal{S}^A}_C}^\alpha - {{\mathcal{S}^{A\alpha}}_C} + {\mathcal{S}_C}^{A\alpha} + {\mathcal{S}^{\alpha A}}_C - {\mathcal{S}_C}^{\alpha A} - {{\mathcal{S}^\alpha}_C}^A]+ \\ &\qquad + 2{e_C}^\alpha\mathcal{L}_D\Big)\delta {e^C}_\alpha\Big] \\ &= \int d^4x\sqrt{-g}({B_C}^\alpha + 2{e_C}^\alpha\mathcal{L}_D)\delta {e^C}_\alpha \\ &=: \int d^4x\sqrt{-g}{\Theta_C}^\alpha\delta {e^C}_\alpha\end{aligned} \tag{D.9}$$

where ${B_C}^\alpha$ is the Belinfante stress energy tensor [82]. In the case of a fermionic field it reads [83]:

$$\begin{aligned}{B_C}^\alpha &= \frac{i\hbar c}{4}[\bar{\psi}\gamma^\alpha\mathcal{D}_C\psi - \mathcal{D}_C\bar{\psi}\gamma^\alpha\psi + \bar{\psi}\gamma_C\mathcal{D}^\alpha\psi - \mathcal{D}^\alpha\bar{\psi}\gamma_C\psi] \\ &= \frac{1}{2}({\mathcal{T}_C}^\alpha + {\mathcal{T}^\alpha}_C)\end{aligned} \tag{D.10}$$

Comparing Eq. (D.7) and Eq. (D.9), we notice:

$$\begin{aligned}{\Theta_C}^\alpha\delta {e^C}_\alpha &= -\frac{1}{2}T^{AB}\delta g_{AB} \\ &= \frac{1}{2}T^{AB}(g_{AC}{e_B}^\alpha + g_{BC}{e_A}^\alpha)\delta {e^C}_\alpha \\ &= {T_C}^\alpha\delta {e^C}_\alpha\end{aligned} \tag{D.11}$$

Thus we can write Eq. (D.6) as:

$$\begin{aligned} S &\approx \int d^4x \Big[(\sqrt{-g}\mathcal{L}_D)\Big|_{g=\eta} + \frac{\partial(\mathcal{L}_D\sqrt{-g})}{\partial g_{AB}}\Big|_{g=\eta} h_{AB} + O(h^2) \Big] \\ &= \int d^4x \Big[(\sqrt{-g}\mathcal{L}_D)\Big|_{g=\eta} - \frac{1}{2}(\Theta^{A\alpha} e^B{}_\alpha)\Big|_{g=\eta} h_{AB} + O(h^2) \Big] \\ &= \int d^4x \, \Big(\frac{i\hbar c}{2}[\bar{\psi}\gamma^\mu \nabla_\mu \psi - \nabla_\mu(\bar{\psi})\gamma^\mu\psi](1+\frac{tr(h)}{2}) - (1+\frac{tr(h)}{2}) mc^2\bar{\psi}\psi \\ &\quad - \frac{i\hbar c}{4} h_{\mu\nu}[\bar{\psi}\gamma^\mu\nabla^\nu\psi - \nabla^\nu(\bar{\psi})\gamma^\mu\psi] \Big) + O(h^2) \\ &=: \int d^4x \, \mathcal{L}_{eff} + O(h^2) \end{aligned} \tag{D.12}$$

and recover Eq. (4.6) of the main text.

Appendix E: Foldy Wouthuysen method - fermionic model

Here we illustrate the Fouldy Wouthuysen method applied to Eq. (4.12). Let us consider the transformations:

$$\mathfrak{H} \to \mathfrak{H}' = U(\mathfrak{H} - i\hbar\partial_t)U^{-1} \tag{E.1}$$

and specialize U to Eq. (4.17), i.e.

$$U = e^{-i\gamma^0\mathcal{O}/(2mc^2)} =: e^{iS} \tag{E.2}$$

With the help of the BCH identity:

$$\begin{aligned} \mathfrak{H}' =& e^{iS}(\mathfrak{H} - i\hbar\partial_t)e^{-iS} = \mathfrak{H} + i[S,\mathfrak{H}] + \frac{i^2}{2!}[S[S,\mathfrak{H}]] + \\ &+ \frac{i^3}{3!}[S[S[S,\mathfrak{H}]]] + \ldots \\ &+ \hbar(-\dot{S} - \frac{i}{2}[S,\dot{S}] + \frac{1}{6}[S,[S,\dot{S}]] + \ldots) \end{aligned} \tag{E.3}$$

Recalling that:

$$\mathfrak{H} = mc^2\gamma^0 + \mathfrak{E} + \mathcal{O} \tag{E.4}$$

and noticing that:

$$\begin{aligned} [\gamma^0, \mathfrak{E}] &= 0 && \text{(E.5)} \\ \{\gamma^0, \mathcal{O}\} &= 0 && \text{(E.6)} \\ [\gamma^0\mathcal{O}, \gamma^0] &= -2\mathcal{O} && \text{(E.7)} \\ [\gamma^0\mathcal{O}, \mathfrak{E}] &= \gamma^0[\mathcal{O}, \mathfrak{E}] && \text{(E.8)} \\ [\gamma^0\mathcal{O}, \mathcal{O}] &= 2\gamma^0\mathcal{O}^2 && \text{(E.9)} \end{aligned}$$

we get:

$$\mathfrak{H}' = mc^2\gamma^0 + \mathfrak{E}' + \mathcal{O}' \tag{E.10}$$

where:

$$\mathfrak{E}' = \mathfrak{E} + \gamma^0\left(\frac{\mathcal{O}^2}{2mc^2} - \frac{\mathcal{O}^4}{8m^3c^6}\right) - \frac{1}{8m^2c^4}[\mathcal{O}, [\mathcal{O}, \mathfrak{E}] + i\hbar\dot{\mathcal{O}}] + ... \tag{E.11}$$

$$\mathcal{O}' = \frac{1}{2mc^2}\gamma^0[\mathcal{O}, \mathfrak{E}] - \frac{\mathcal{O}^3}{3m^2c^4} + \frac{i}{2mc^2}\gamma^0\dot{\mathcal{O}} + ... \tag{E.12}$$

We note that $\mathcal{O}'$ is of order c^{-1}, meaning that we need to perform a further transformation if we want non trivial diagonal EOM. The transformation that we perform is:

$$U' = e^{-i\gamma^0\mathcal{O}'/(2mc^2)} \tag{E.13}$$

after which the Hamiltonian reads:

$$\mathfrak{H}'' = mc^2\gamma^0 + \mathfrak{E}' + \mathcal{O}'' + ... \tag{E.14}$$

with:

$$\mathcal{O}'' = \frac{\gamma^0}{2mc^2}[\mathcal{O}', \mathfrak{E}'] + \frac{i}{2mc^2}\gamma^0\dot{\mathcal{O}}' + ... \tag{E.15}$$

As $\mathcal{O}'' \sim O(\frac{v^3}{c^3})$ we need to perform a final transformation:

$$U'' = e^{-i\gamma^0\mathcal{O}''/(2mc^2)} \tag{E.16}$$

Finally the Hamiltonian reads:

$$H := \mathfrak{H}''' = mc^2\gamma^0 + \mathfrak{E}' + O(\frac{v^5}{c^5}) \tag{E.17}$$

In order to calculate the explicit expression of the Hamiltonian in Eq. (E.17), we pick the Pauli representation for the Dirac gamma matrices:

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & \mathbb{0} \\ \mathbb{0} & -\mathbb{1} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} \mathbb{0} & \sigma^i \\ -\sigma^i & \mathbb{0} \end{pmatrix}, \quad \alpha^i \equiv \gamma^0\gamma^i = \begin{pmatrix} \mathbb{0} & \sigma^i \\ \sigma^i & \mathbb{0} \end{pmatrix}, \quad \Sigma_i = \begin{pmatrix} \sigma_i & \mathbb{0} \\ \mathbb{0} & \sigma_i \end{pmatrix} \tag{E.18}$$

By exploiting the identities:

$$\begin{aligned} \left[(\partial^j - \frac{ie}{\hbar c}A^j), (\partial_k - \frac{ie}{\hbar c}A_k)\right] &= -\frac{ie}{\hbar c}F^j{}_k \\ \alpha^i\alpha^j &= -\eta^{ij} + \epsilon^{ijk}\Sigma_k \\ \{\alpha^i, \alpha^j\} &= -2\eta^{ij} \\ \eta^{ij} &= -\delta^{ij} \end{aligned} \tag{E.19}$$

it only takes a bit of algebra to show that:

$$
\begin{aligned}
\frac{\gamma^0}{2mc^2}\mathcal{O}^2 =& \frac{\gamma^0}{2mc^2}\Big(- i\hbar c(1+\frac{h_{00}}{2})(\partial_j - \frac{ie}{\hbar c}A_j)\gamma^0\gamma^j + \frac{i\hbar c}{2}h_{ij}(\partial^j - \frac{ie}{\hbar c}A^j)\gamma^0\gamma^i \\
& - \frac{i\hbar c}{4}\partial_i(\frac{tr(h)}{2} - h_{00})\gamma^0\gamma^i + \frac{i\hbar}{4}\partial_t(h_{0i})\gamma^0\gamma^i\Big)^2 \\
=& \gamma^0\Bigg[- \frac{\hbar^2}{2m}(1+h_{00})(\nabla - \frac{ie}{\hbar c}\mathbf{A})^2 - \frac{\hbar e}{2mc}(1+h_{00})B^k\Sigma_k \\
& - \frac{\hbar^2}{2m}h_{ij}(\partial^i - \frac{ie}{\hbar c}A^i)(\partial^j - \frac{ie}{\hbar c}A^j) + \frac{\hbar e}{4mc}\epsilon^{ijl}h_{jk}F_i{}^k\Sigma_l\Bigg] + \\
& + \gamma^0\Bigg[\frac{\hbar^2}{2m}\partial^i(h_{00})\nabla_i - \frac{\hbar^2}{4m}\partial^i(h_{ij})\nabla^j - \frac{\hbar^2}{2m}\partial_i(\frac{tr(h)}{2} - h_{00})\nabla^i \\
& - \frac{i\hbar^2}{4m}\epsilon^{ijk}\Big(\partial_i(h_{00})\nabla_j - \partial_i(h_{jl})\nabla^l\Big)\Sigma_k - \frac{\hbar^2}{4m}\partial^i\partial_i(\frac{tr(h)}{2} - h_{00})\Bigg] + O(h^2)
\end{aligned}
\tag{E.20}
$$

As the above term is of order $\gamma^0\frac{\mathcal{O}^2}{2mc^2} \sim O(\frac{v^2}{c^2})$, it follows that the next term in Eq. (E.11) is of order $\gamma^0\frac{\mathcal{O}^4}{8m^3c^6} \sim O(\frac{v^4}{c^4})$. After some algebra it reads:

$$
\begin{aligned}
\mathcal{O}^4 =&\Bigg(\hbar^2c^2(1+h_{00})(\nabla-\frac{ie}{\hbar c}\mathbf{A})^2-\hbar ec(1+h_{00})B^k\Sigma_k-\frac{i\hbar^2c^2}{2}\epsilon^{ijk}\partial_i(h_{00})(\partial_j-\frac{ie}{\hbar c}A_j)\Sigma_k\\
&+\frac{\hbar^2c^2}{2}\partial^i(h_{00})(\partial_i-\frac{ie}{\hbar c}A_i)-\hbar^2c^2h_{ij}(\partial^i-\frac{ie}{\hbar c}A^i)(\partial^j-\frac{ie}{\hbar c}A^j)+\frac{\hbar ec}{2}\epsilon^{ijl}h_{jl}{F_i}^k\Sigma_l\\
&-\frac{\partial^i}{(}h_{ij})(\partial^j-\frac{ie}{\hbar c}A^j)+\frac{i\hbar^2c^2}{2}\epsilon^{ijl}\partial_i(h_{jk})(\partial^k-\frac{ie}{\hbar c}A^k)\Sigma_l-\hbar^2c^2\partial_i(\frac{tr(h)}{2}-h_{00})\partial^i\\
&-\frac{\hbar^2c^2}{2}\partial^i\partial_i(\frac{tr(h)}{2}-h_{00})\Bigg)^2\\
=&\hbar^4c^4(1+2h_{00})(\nabla-\frac{ie}{\hbar c}\mathbf{A})^4+\hbar^2ec^2(1+2h_{00})B^2\\
&-\hbar^3ec^3(1+2h_{00})\{(\nabla-\frac{ie}{\hbar c}\mathbf{A})^2,B^k\}\Sigma_k+\frac{\hbar^3ec^3}{2}\epsilon^{ijl}\{(\nabla-\frac{ie}{\hbar c}\mathbf{A})^2,h_{jk}{F_i}^k\}\Sigma_l\\
&+2\hbar^4c^4h_{ij}(\nabla-\frac{ie}{\hbar c}\mathbf{A})^2(\partial^i-\frac{ie}{\hbar c}A^i)(\partial^j-\frac{ie}{\hbar c}A^j)\\
&-\frac{\hbar^3ec^3}{2}\epsilon^{ijl}h_{jm}{F_i}^mB^k\{\Sigma_k,\Sigma_l\}+\frac{i\hbar^4c^4}{2}\epsilon^{ijk}\{(\nabla-\frac{ie}{\hbar c}\mathbf{A})^2,\partial_i(h_{00})(\partial_j-\frac{ie}{\hbar c}A_j)\}\Sigma_k\\
&-\frac{i\hbar^4c^4}{2}\{(\nabla-\frac{ie}{\hbar c}\mathbf{A})^2,\partial^i(h_{00})(\partial_i-\frac{ie}{\hbar c}A_i)\}\\
&+\frac{\hbar^4c^4}{2}\{(\nabla-\frac{ie}{\hbar c}\mathbf{A})^2,\partial_i(h_{ij})(\partial^j-\frac{ie}{\hbar c}A^j)\}\\
&-\frac{\hbar^4c^4}{2}\epsilon^{ijl}\{(\nabla-\frac{ie}{\hbar c}\mathbf{A})^2,\partial_i(h_{jk})(\partial^k-\frac{ie}{\hbar c}A^k)\}\Sigma_l\\
&+\hbar^4c^4\{(\nabla-\frac{ie}{\hbar c}\mathbf{A})^2,\partial_i(\frac{tr(h)}{2}-h_{00})\partial_i\}\\
&+\frac{\hbar^4c^4}{2}\{(\nabla-\frac{ie}{\hbar c}\mathbf{A})^2,\partial^i\partial_i(\frac{tr(h)}{2}-h_{00})\}+\frac{\hbar^3ec^3}{2}\{B^k,\partial^i(h_{ij})(\partial^j-\frac{ie}{\hbar c}A^j)\}\Sigma_k\\
&+\frac{i\hbar^3ec^3}{e}\epsilon^{ijl}\{B^k\Sigma_k,\partial_i(h_{00})(\partial_j-\frac{ie}{\hbar c}A_j)\Sigma_l\}-\frac{i\hbar^3ec^3}{e}\{B^k,\partial^i(h_{00})(\partial_i-\frac{ie}{\hbar c}A_i)\}\Sigma_k\\
&+\frac{i\hbar^3ec^3}{e}\epsilon^{ijl}\{B^k\Sigma_k,\partial_i(h_{jm})(\partial^m-\frac{ie}{\hbar c}A^m)\Sigma_l\}+\hbar^3ec^3\{B^k,\partial_i(\frac{tr(h)}{2}-h_{00})\partial^i\}\Sigma_k\\
&+\frac{\hbar^3ec^3}{2}\{B^k,\partial_i\partial^i(\frac{tr(h)}{2}-h_{00})\}\Sigma_k
\end{aligned}
\tag{E.21}
$$

The last term in Eq. (E.11) requires lengthy intermediate calculations in order to get to the final result. We start by considering the expressions $[\mathcal{O},\mathfrak{E}]$ and $\dot{\mathcal{O}}$ separately.

With the help of Eqs. (E.19) and some algebra:

$$\begin{aligned}\Big[\mathcal{O},\mathfrak{E}\Big] =&\Big[-i\hbar c(1+\frac{h_{00}}{2})(\partial_j-\frac{ie}{\hbar c}A_j)\gamma^0\gamma^j+\frac{i\hbar c}{2}h_{ij}(\partial^j-\frac{ie}{\hbar c}A^j)\gamma^0\gamma^i+\frac{i\hbar c}{4}\partial_t(h_{0i})\gamma^0\gamma^i\\ &+\frac{i\hbar c}{4}\partial_i(\frac{tr(h)}{2}-h_{00})\gamma^0\gamma^i\,,eA_0+\frac{mc^2}{2}h_{00}\gamma^0+i\hbar c\,h_{0i}(\partial^i-\frac{ie}{\hbar c}A^i)+\frac{i\hbar c}{4}\partial_i(h^i_0)\\ &+\frac{\hbar c}{4}\epsilon^{ijk}\partial_i(h_{0j})\Sigma_k-\frac{3i\hbar}{8}\partial_t(tr(h))+\frac{i\hbar}{4}\partial_t(h_{00})\Big]\\ =&\,i\hbar mc^3h_{00}\nabla_i\gamma^i+\frac{i\hbar mc^3}{2}\partial_i(h_{00})\gamma^i-i\hbar ec\,h_{0j}F_i^j\alpha^i+\hbar^2c^2\partial_i(h_{0j})\nabla^j\alpha^i\\ &-i\hbar ec(1+\frac{h_{00}}{2})\partial_i(A_0)\alpha^i+\frac{i\hbar ec}{2}h_{ij}\partial^j(A_0)\alpha^i\\ &+\frac{\hbar^2c^2}{2}\Big(\partial_j(h_{0i})\nabla^i\alpha^j-\partial_j(h_{0i}\nabla^j\alpha^i)\Big)\\ &-\frac{i\hbar^2c^2}{4}\epsilon^{jkl}\partial_i\partial_j(h_{0k})\alpha^i\Sigma_l-\frac{3\hbar^2c}{8}\partial_i\partial_t(tr(h))\alpha^i+\frac{\hbar^2c}{4}\partial_i\partial_t(h_{00})\alpha^i\end{aligned} \tag{E.22}$$

$$\begin{aligned}\dot{\mathcal{O}}=&-\frac{i\hbar c}{2}\partial_t(h_{00})\nabla_i\alpha^i-\frac{e}{2}h_{00}\partial_t(A_i)\alpha^i+\frac{i\hbar c}{2}\partial_t(h_{ij})\nabla^j\alpha^i+\frac{e}{2}h_{ij}\partial_t(A^j)\alpha^i\\ &+\frac{i\hbar c}{4}\partial_t\partial_i(\frac{tr(h)}{2}-h_{00})\alpha^i\end{aligned} \tag{E.23}$$

Upon plugging the Eq. (E.22,E.23) into the last term in Eq: (E.11), exploiting again the identities in Eqs. (E.19), and with a lot of algebra, we arrive at the final expression:

$$
\begin{aligned}
-\frac{[\mathcal{O},[\mathcal{O},\mathfrak{E}]+i\hbar\dot{\mathcal{O}}]}{8m^2c^4} &= +\frac{\hbar^2}{4m}h_{00}(\nabla-\frac{ie}{\hbar c}\mathbf{A})^2\gamma^0+\frac{e\hbar}{4mc}h_{00}\mathbf{B}\cdot\gamma^0\mathbf{\Sigma} \\
&+\frac{i\hbar^2 e}{4m^2c^2}(1+\frac{h_{00}}{2})\Big(\frac{\nabla}{2}\times\mathbf{E}-\mathbf{E}\times\nabla\Big)\cdot\mathbf{\Sigma}-(1+h_{00})\frac{\hbar^2 e}{8m^2c^2}\nabla\cdot\mathbf{E} \\
&-\frac{i\hbar^2 e}{16m^2c^2}\epsilon^{ikl}h_{ij}\partial^j(E_k)\Sigma_l-\frac{i\hbar^2 e}{8m^2c^2}\epsilon^{ikl}h_{ij}E_k(\partial^j-\frac{ie}{\hbar c}A^j)\Sigma_l \\
&+\frac{i\hbar^2 e}{4m^2c^2}\epsilon^{ijl}h_{0k}F_j{}^k(\partial_i-\frac{ie}{\hbar c}A_i)\Sigma_l-\frac{\hbar^2 e}{8m^2c^2}h_{0j}\partial_i(F^{ij})+\frac{i\hbar^2 e}{8m^2c^2}\epsilon^{ijl}h_{0k}\partial_i(F_j{}^k)\Sigma_l \\
&-\frac{\hbar^2 e}{16m^2c^2}\partial_i(h_{00})E^i-\frac{i\hbar^2 e}{16m^2c^2}\epsilon^{ijk}\partial_i(h_{00})E_j\Sigma_k-\frac{\hbar^2 e}{16m^2c^2}\partial^i(h_{ij})E^j \\
&+\frac{i\hbar^2 e}{16m^2c^2}\epsilon^{ijl}\partial_i(h_{jk})E^k\Sigma_l+\frac{\hbar^2}{8m}\partial_i(h_{00})(\partial^i-\frac{ie}{\hbar c}A^i)-\frac{i\hbar^2}{8m}\epsilon^{ijk}\partial_i(h_{00})(\partial_j-\frac{ie}{\hbar c}A_j)\Sigma_k \\
&-\frac{\hbar^2 e}{8m^2c^2}\partial^i(h_{0j})F_i{}^j+\frac{i\hbar^2 e}{8m^2c^2}\epsilon^{ijl}\partial_i(h_{0k})F_j{}^k\Sigma_l-\frac{i\hbar^2 e}{16m^2c^2}\epsilon^{ijk}\partial_i\left(\frac{tr(h)}{2}-h_{00}\right)E_j\Sigma_k \\
&-\frac{i\hbar^3}{16m^2c}\epsilon^{jkl}\epsilon^{ima}\partial_m(h_{0j})\{(\partial_i-\frac{ie}{\hbar c}A_i),(\partial_k-\frac{ie}{\hbar c}A_k)\}\Sigma_a\Sigma_l \\
&-\frac{i\hbar^3}{8m^2c}\epsilon^{jkl}\epsilon_l{}^{im}\partial_m(h_{0j})(\partial_k-\frac{ie}{\hbar c}A_k)(\partial_k-\frac{ie}{\hbar c}A_i)\Sigma_a\Sigma^a+\frac{i\hbar^2 e}{8m^2c^2}\epsilon^{jkl}\partial^i(h_{0j})F_{ki}\Sigma_l \\
&-\frac{\hbar^3}{8m^2c}\epsilon^{jkl}\partial^i(h_{0j})(\partial_k-\frac{ie}{\hbar c}A_k)(\partial_i-\frac{ie}{\hbar c}A_i)\Sigma_l \\
&+\frac{\hbar^3}{16m^2c}\epsilon^{jkl}\partial^i\partial_i(h_{0j})(\partial_k-\frac{ie}{\hbar c}A_k)\Sigma_l-\frac{i\hbar^2 e}{16m^2c^2}\epsilon^{ikl}\partial_i(h_{0j})F^j{}_k\Sigma_l \\
&-\frac{i\hbar^3}{8m^2c}\partial^i\partial_i(h_{0j})(\partial^j-\frac{ie}{\hbar c}A^j)+\frac{\hbar^3}{8m^2c}\epsilon^{jki}\partial_i(h_{0j})(\partial_k-\frac{ie}{\hbar c}A_k)(\partial^l-\frac{ie}{\hbar c}A^l)\Sigma_l \\
&-\frac{\hbar^3}{16m^2c}\epsilon^{kjl}\partial_k\partial_i(h_{0j})(\partial^i-\frac{ie}{\hbar c}A^i)\Sigma_l+\frac{h^3}{16m^2c}\epsilon^{jkl}\partial^i\partial_j(h_{0k})(\partial_i-\frac{ie}{\hbar c}A_i)\Sigma_l \\
&+\frac{i\hbar^3}{16m^2c}\epsilon^{jkl}\epsilon^{ima}\partial_m\partial_j(h_{0k})(\partial_i-\frac{ie}{\hbar c}A_i)\Sigma_l\Sigma_a-\frac{\hbar^3}{16m^2c}\epsilon^{jkl}\partial^i\partial_i\partial_j(h_{0k})\Sigma_l \\
&-\frac{\hbar^2}{8m}\partial_i(h_{00})(\partial^i-\frac{ie}{\hbar c}A^i)\gamma^0-\frac{\hbar^2}{16m}\partial^i\partial_i(h_{00})\gamma^0-\frac{i\hbar^3}{16m^2c^2}\partial^i\partial_t(h_{00})(\partial^i-\frac{ie}{\hbar c}A^i) \\
&-\frac{\hbar^3}{16m^2c^2}\epsilon^{ijk}\partial_j\partial_t(h_{00})(\partial_i-\frac{ie}{\hbar c}A_i)\Sigma_k-\frac{i\hbar^2 e}{8m^2c^3}\epsilon^{ijl}\partial_t(h_{jk})F_i{}^k\Sigma_l \\
&+\frac{i\hbar^3}{16m^2c^2}\epsilon^{ijk}\partial_t\partial_j(tr(h)-h_{00})(\partial_i-\frac{ie}{\hbar c}A_i)\Sigma_k+\frac{i\hbar^3}{32m^2c^2}\partial_t\partial^i(tr(h)-h_{00})(\partial_i-\frac{ie}{\hbar c}A_i) \\
&+\frac{i\hbar^3}{32m^2c^2}\partial^i\partial_i\partial_t(tr(h)-h_{00})+\frac{i\hbar^3}{32m^2c^2}\partial^i\partial_t(h_{ij})(\partial^j-\frac{ie}{\hbar c}A^j) \\
&+\frac{i\hbar^3}{32m^2c^2}\epsilon^{ijl}\partial_i\partial_t(h_{jk})(\partial^k-\frac{ie}{\hbar c}A^k)\Sigma_l
\end{aligned}
\tag{E.24}
$$

So that the total Hamiltonian reads:

$$
\begin{aligned}
H =& eA_0 + \gamma^0 \Bigg[mc^2(1+\frac{h_{00}}{2}) - \frac{\hbar^2}{2m}(1+\frac{h_{00}}{2})(\nabla - \frac{ie}{\hbar c}\mathbf{A})^2 - \frac{\hbar e}{2mc}(1+\frac{h_{00}}{2})B^k\Sigma_k \\
& - \frac{\hbar^2}{2m}h_{ij}(\partial^i - \frac{ie}{\hbar c}A^i)(\partial^j - \frac{ie}{\hbar c}A^j) + \frac{\hbar e}{4mc}\epsilon^{ijl}h_{jk}F_i{}^k\Sigma_l \Bigg] \\
& + \frac{i\hbar^2 e}{4m^2c^2}(1+\frac{h_{00}}{2})\Big(\frac{\nabla}{2}\times\mathbf{E} - \mathbf{E}\times\nabla\Big)\cdot\mathbf{\Sigma} - (1+h_{00})\frac{\hbar^2 e}{8m^2c^2}\nabla\cdot\mathbf{E} \\
& - \frac{i\hbar^2 e}{16m^2c^2}\epsilon^{ikl}h_{ij}\partial^j(E_k)\Sigma_l - \frac{i\hbar^2 e}{8m^2c^2}\epsilon^{ikl}h_{ij}E_k(\partial^j - \frac{ie}{\hbar c}A^j)\Sigma_l \\
& + \frac{i\hbar^2 e}{4m^2c^2}\epsilon^{ijl}h_{0k}F_j{}^k(\partial_i - \frac{ie}{\hbar c}A_i)\Sigma_l - \frac{\hbar^2 e}{8m^2c^2}h_{0j}\partial_i(F^{ij}) + \frac{i\hbar^2 e}{8m^2c^2}\epsilon^{ijl}h_{0k}\partial_i(F_j{}^k)\Sigma_l \\
& - \frac{\gamma^0}{8m^3c^6}\Bigg[\hbar^4c^4(1+2h_{00})(\nabla - \frac{ie}{\hbar c}\mathbf{A})^4 + \hbar^2ec^2(1+2h_{00})B^2 \\
& + 2\hbar^4c^4h_{ij}(\nabla - \frac{ie}{\hbar c}\mathbf{A})^2(\partial^i - \frac{ie}{\hbar c}A^i)(\partial^j - \frac{ie}{\hbar c}A^j) + \frac{\hbar^3ec^3}{2}\epsilon^{ijl}\{(\nabla - \frac{ie}{\hbar c}\mathbf{A})^2, h_{jk}F_i{}^k\}\Sigma_l \\
& - \frac{\hbar^3ec^3}{2}\epsilon^{ijl}h_{jm}F_i{}^mB^k\{\Sigma_k,\Sigma_l\} - \hbar^3ec^3(1+2h_{00})\{(\nabla - \frac{ie}{\hbar c}\mathbf{A})^2, B^k\}\Sigma_k \Bigg] \\
& - \frac{\hbar^2}{8m}\partial_i(h_{00})(\partial^i - \frac{ie}{\hbar c}A^i)\gamma^0 - \frac{\hbar^2}{16m}\partial^i\partial_i(h_{00})\gamma^0 + \frac{i\hbar c}{4}\partial_i(h_0^i) \\
& + \frac{\hbar c}{4}\epsilon^{ijk}\partial_i(h_{0j})\Sigma_k - \frac{3i\hbar}{8}\partial_t(tr(h)) + \frac{i\hbar}{4}\partial_t(h_{00}) \\
& + \gamma^0\Bigg[\frac{\hbar^2}{2m}\partial^i(h_{00})\nabla_i - \frac{\hbar^2}{4m}\partial^i(h_{ij})\nabla^j - \frac{\hbar^2}{2m}\partial_i(\frac{tr(h)}{2} - h_{00})\nabla^i \\
& \quad - \frac{i\hbar^2}{4m}\epsilon^{ijk}\Big(\partial_i(h_{00})\nabla_j - \partial_i(h_{jl})\nabla^l\Big)\Sigma_k - \frac{\hbar^2}{4m}\partial^i\partial_i(\frac{tr(h)}{2} - h_{00})\Bigg] \\
& + H_{dd} + O(h^2) + O(\frac{v^5}{c^5})
\end{aligned}
\tag{E.25}
$$

with

$$
\begin{aligned}
H_{dd} =& -\frac{\hbar^2 e}{16m^2c^2}\partial_i(h_{00})E^i - \frac{i\hbar^2 e}{16m^2c^2}\epsilon^{ijk}\partial_i(h_{00})E_j\Sigma_k - \frac{\hbar^2 e}{16m^2c^2}\partial^i(h_{ij})E^j \\
& + \frac{i\hbar^2 e}{16m^2c^2}\epsilon^{ijl}\partial_i(h_{jk})E^k\Sigma_l + \frac{\hbar^2}{8m}\partial_i(h_{00})(\partial^i - \frac{ie}{\hbar c}A^i) - \frac{i\hbar^2}{8m}\epsilon^{ijk}\partial_i(h_{00})(\partial_j - \frac{ie}{\hbar c}A_j)\Sigma_k \\
& - \frac{\hbar^2 e}{8m^2c^2}\partial^i(h_{0j})F_i{}^j + \frac{i\hbar^2 e}{8m^2c^2}\epsilon^{ijl}\partial_i(h_{0k})F_j{}^k\Sigma_l - \frac{i\hbar^2 e}{16m^2c^2}\epsilon^{ijk}\partial_i\left(\frac{tr(h)}{2} - h_{00}\right)E_j\Sigma_k \\
& - \frac{i\hbar^3}{16m^2c}\epsilon^{jkl}\epsilon^{ima}\partial_m(h_{0j})\{(\partial_i - \frac{ie}{\hbar c}A_i),(\partial_k - \frac{ie}{\hbar c}A_k)\}\Sigma_a\Sigma_l
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{i\hbar^3}{8m^2c}\epsilon^{jkl}\epsilon_l{}^{im}\partial_m(h_{0j})(\partial_k - \frac{ie}{\hbar c}A_k)(\partial_k - \frac{ie}{\hbar c}A_i)\Sigma_a\Sigma^a + \frac{i\hbar^2 e}{8m^2c^2}\epsilon^{jkl}\partial^i(h_{0j})F_{ki}\Sigma_l \\
&- \frac{\hbar^3}{8m^2c}\epsilon^{jkl}\partial^i(h_{0j})(\partial_k - \frac{ie}{\hbar c}A_k)(\partial_i - \frac{ie}{\hbar c}A_i)\Sigma_l \\
&+ \frac{\hbar^3}{16m^2c}\epsilon^{jkl}\partial^i\partial_i(h_{0j})(\partial_k - \frac{ie}{\hbar c}A_k)\Sigma_l - \frac{i\hbar^2 e}{16m^2c^2}\epsilon^{ikl}\partial_i(h_{0j})F^j{}_k\Sigma_l \\
&- \frac{i\hbar^3}{8m^2c}\partial^i\partial_i(h_{0j})(\partial^j - \frac{ie}{\hbar c}A^j) + \frac{\hbar^3}{8m^2c}\epsilon^{jki}\partial_i(h_{0j})(\partial_k - \frac{ie}{\hbar c}A_k)(\partial^l - \frac{ie}{\hbar c}A^l)\Sigma_l \\
&- \frac{\hbar^3}{16m^2c}\epsilon^{kjl}\partial_k\partial_i(h_{0j})(\partial^i - \frac{ie}{\hbar c}A^i)\Sigma_l + \frac{h^3}{16m^2c}\epsilon^{jkl}\partial^i\partial_j(h_{0k})(\partial_i - \frac{ie}{\hbar c}A_i)\Sigma_l \\
&+ \frac{i\hbar^3}{16m^2c}\epsilon^{jkl}\epsilon^{ima}\partial_m\partial_j(h_{0k})(\partial_i - \frac{ie}{\hbar c}A_i)\Sigma_l\Sigma_a - \frac{\hbar^3}{16m^2c}\epsilon^{jkl}\partial^i\partial_i\partial_j(h_{0k})\Sigma_l \\
&- \frac{\hbar^2}{8m}\partial_i(h_{00})(\partial^i - \frac{ie}{\hbar c}A^i)\gamma^0 - \frac{\hbar^2}{16m}\partial^i\partial_i(h_{00})\gamma^0 - \frac{i\hbar^3}{16m^2c^2}\partial^i\partial_t(h_{00})(\partial^i - \frac{ie}{\hbar c}A^i) \\
&- \frac{\hbar^3}{16m^2c^2}\epsilon^{ijk}\partial_j\partial_t(h_{00})(\partial_i - \frac{ie}{\hbar c}A_i)\Sigma_k - \frac{i\hbar^2 e}{8m^2c^3}\epsilon^{ijl}\partial_t(h_{jk})F_i{}^k\Sigma_l \\
&+ \frac{i\hbar^3}{16m^2c^2}\epsilon^{ijk}\partial_t\partial_j(tr(h) - h_{00})(\partial_i - \frac{ie}{\hbar c}A_i)\Sigma_k \\
&+ \frac{i\hbar^3}{32m^2c^2}\partial^i\partial_i\partial_t(tr(h) - h_{00}) + \frac{i\hbar^3}{32m^2c^2}\partial^i\partial_t(h_{ij})(\partial^j - \frac{ie}{\hbar c}A^j) \\
&+ \frac{i\hbar^3}{32m^2c^2}\epsilon^{ijl}\partial_i\partial_t(h_{jk})(\partial^k - \frac{ie}{\hbar c}A^k)\Sigma_l + \frac{i\hbar^3}{32m^2c^2}\partial_t\partial^i(tr(h) - h_{00})(\partial_i - \frac{ie}{\hbar c}A_i) \\
&- \frac{\gamma^0}{8m^3c^6}\Bigg[\frac{i\hbar^4c^4}{2}\epsilon^{ijk}\{(\nabla - \frac{ie}{\hbar c}\mathbf{A})^2, \partial_i(h_{00})(\partial_j - \frac{ie}{\hbar c}A_j)\}\Sigma_k \\
&- \frac{i\hbar^4c^4}{2}\{(\nabla - \frac{ie}{\hbar c}\mathbf{A})^2, \partial^i(h_{00})(\partial_i - \frac{ie}{\hbar c}A_i)\} \\
&+ \frac{\hbar^4c^4}{2}\{(\nabla - \frac{ie}{\hbar c}\mathbf{A})^2, \partial_i(h_{ij})(\partial^j - \frac{ie}{\hbar c}A^j)\} \\
&- \frac{\hbar^4c^4}{2}\epsilon^{ijl}\{(\nabla - \frac{ie}{\hbar c}\mathbf{A})^2, \partial_i(h_{jk})(\partial^k - \frac{ie}{\hbar c}A^k)\}\Sigma_l \\
&+ \hbar^4c^4\{(\nabla - \frac{ie}{\hbar c}\mathbf{A})^2, \partial_i(\frac{tr(h)}{2} - h_{00})\partial_i\} \\
&+ \frac{\hbar^4c^4}{2}\{(\nabla - \frac{ie}{\hbar c}\mathbf{A})^2, \partial^i\partial_i(\frac{tr(h)}{2} - h_{00})\} + \frac{\hbar^3ec^3}{2}\{B^k, \partial^i(h_{ij})(\partial^j - \frac{ie}{\hbar c}A^j)\}\Sigma_k \\
&+ i\hbar^3ec^3\epsilon^{ijl}\{B^k\Sigma_k, \partial_i(h_{00})(\partial_j - \frac{ie}{\hbar c}A_j)\Sigma_l\} - i\hbar^3ec^3\{B^k, \partial^i(h_{00})(\partial_i - \frac{ie}{\hbar c}A_i)\}\Sigma_k \\
&+ i\hbar^3ec^3\epsilon^{ijl}\{B^k\Sigma_k, \partial_i(h_{ja})(\partial^a - \frac{ie}{\hbar c}A^a)\Sigma_l\} + \hbar^3ec^3\{B^k, \partial_i(\frac{tr(h)}{2} - h_{00})\partial^i\}\Sigma_k \\
&+ \frac{\hbar^3ec^3}{2}\{B^k, \partial_i\partial^i(\frac{tr(h)}{2} - h_{00})\}\Sigma_k\Bigg] \qquad \text{(E.26)}
\end{aligned}
$$

By neglecting the terms containing derivatives of the gravitational field of order $\frac{v^3}{c^3}$ or higher (namely the term H_{dd}), we recover Eq. (4.18) of the main text.

Appendix F: Cumulant Expansion Method

In this section we derive Eqs. (3.30, 4.31) and Eqs. (3.34,4.35) with the help of the Cumulant Expasion method [61]. We start by giving a brief presentation of the method, which is generally speaking a very useful tool for the solution of Stochastic Differential Equations (SDEs), to eventually apply it to our specific cases of interest.

Let us consider the rather generic multiplicative SDE:

$$\partial_t\hat{\Omega}(t) = [\mathcal{A} + \alpha\mathcal{B}(t)]\hat{\Omega}(t) \tag{F.1}$$

with $\hat{\Omega}$ being generic density operator, $\mathcal{A}$ a constant superoperator, $\mathcal{B}(t)$ a random superoperator with finite correlation time τ_c, and α the parameter measuring the magnitude of the fluctuations. Of this form are Eqs. (3.29) and (4.30) of the main text. Our goal will be to solve such an equation.
In the interaction picture, Eq. (F.1) reads:

$$\begin{aligned} \hat{\Omega}(t) &= e^{t\mathcal{A}}\hat{\tilde{\Omega}}(t) && \text{(F.2)} \\ \partial_t\hat{\tilde{\Omega}}(t) &= \alpha e^{-t\mathcal{A}}\mathcal{B}(t)e^{t\mathcal{A}}\hat{\tilde{\Omega}} \equiv \alpha\tilde{\mathcal{B}}(t)\hat{\tilde{\Omega}}(t) && \text{(F.3)} \end{aligned}$$

Its formal solution is:

$$\hat{\tilde{\Omega}}(t) = T[e^{\alpha\int_0^t\tilde{\mathcal{B}}(s)ds}]\hat{\tilde{\Omega}}(0) \tag{F.4}$$

Note that Eq. (F.4) represents the solution only for a given realization of the random process, while in experiments one is typically interested into averaged effects. We therefore consider the averaged differential equation:

$$\partial_t\hat{\tilde{\rho}}(t) = \mathbb{E}[\alpha\tilde{\mathcal{B}}(t)\hat{\Omega}(t)] \tag{F.5}$$

where we recall $\hat{\rho} = \mathbb{E}[\hat{\Omega}]$. Its formal solution reads:

$$\hat{\tilde{\rho}}(t) = \mathbb{E}\left[T\left[e^{\alpha\int_0^t\tilde{\mathcal{B}}(s)ds}\right]\right]\hat{\tilde{\Omega}}(0) \tag{F.6}$$

which is in most cases though of any practical use. In order to alternatively solve the averaged dynamics we note that, as $\mathcal{B}(t)$ is indeed a random variable, by definition it follows that $\mathbb{E}\left[e^{\alpha\int_0^t\tilde{\mathcal{B}}(s)ds}\right]$ is a moment generating function. We can then apply the standard cumulant expansion method (for all practical purposes a series expansion of the exponential, for more details see chapter III.4 of [61]). With such a method, we intend to find the generator of the averaged dynamics governing the statistical operator $\hat{\rho}(t)$, i.e. the non stochastic superoperator $\mathcal{G}$ such that:

$$\partial_t \hat{\hat{\rho}}(t) = \mathcal{G}(t)\hat{\hat{\rho}}(t) \tag{F.7}$$

Upon applying the cumulant expansion to Eq. (F.6), we obtain:

$$\begin{aligned}\hat{\hat{\rho}}(t) = T\Big[\exp\Big\{\alpha\int_0^t dt_1\langle\langle\tilde{\mathcal{B}}(t_1)\rangle\rangle + \frac{\alpha^2}{2}\int_0^t dt_1 dt_2\langle\langle\tilde{\mathcal{B}}(t_1)\tilde{\mathcal{B}}(t_2)\rangle\rangle + ...\\ + \frac{\alpha^m}{m!}\int_0^t dt_1...dt_m\langle\langle\tilde{\mathcal{B}}(t_1)...\tilde{\mathcal{B}}(t_m)\rangle\rangle + ...\Big\}\Big]\tilde{\Omega}(0)\end{aligned} \tag{F.8}$$

where $\langle\langle\ \tilde{\mathcal{B}}(t_1)...\tilde{\mathcal{B}}(t_m)\ \rangle\rangle$ denotes the mth cumulant. Note that each term in the cumulant expansion is of order $O(\alpha^m\tau_c^{m-1}t)$. In the case of a Gaussian and white noise however, all terms with m greater than 2 vanish [84]. Furthermore, In most physically interesting cases (like for Eq. (3.29), where the the stochastic noise has zero mean), the dominant contribution to Eq. (F.8) is given by the second order term. Eq. (F.8) therefore reads:

$$\hat{\hat{\rho}}(t) = T\left[e^{\frac{\alpha^2}{2}\int_0^t\int_0^{t_1}dt_1 dt_2\mathbb{E}\left[\tilde{\mathcal{B}}(t_1)\tilde{\mathcal{B}}(t_2)\right]}\right]\hat{\tilde{\Omega}}(0) \tag{F.9}$$

Eq. (F.9) is simpler than Eq. (F.8), but we are still not able to straightforwardly extract the generator of the averaged dynamics $\mathcal{G}$ from it. In order to do so, we make use of the Disentangling Theorem [85] as it is presented in [86]. We therefore define a generic non stochastic time dependent superoperator $\mathcal{K}(t)$ and the relative evolution superoperator:

$$\mathcal{V}(t,t_1) = T\left[e^{\int_{t_1}^t dt'\,\mathcal{K}(t')}\right] \tag{F.10}$$

With the help of $\mathcal{V}(t,t_1)$ we can define a new representation for $\hat{\tilde{\Omega}}(t)$ and $\tilde{\mathcal{B}}(t)$ as:

$$\hat{\tilde{\Omega}}(t) = \mathcal{V}(t,0)\hat{\tilde{\Omega}}^k(t) \tag{F.11}$$

$$\tilde{\mathcal{B}}^k(t) = \mathcal{V}(t,0)^{-1}\tilde{\mathcal{B}}(t)\mathcal{V}(t,0) \tag{F.12}$$

so that Eq. (F.9) reads:

$$\hat{\hat{\rho}}(t) = T\left[e^{\int_0^t\mathcal{K}(t_1)dt_1}\right]T\left[e^{\frac{\alpha^2}{2}\int_0^t\int_0^{t_1}dt_1 dt_2\mathbb{E}\left[\tilde{\mathcal{B}}^k(t_1)\tilde{\mathcal{B}}^k(t_2)\right]-\int_0^t\mathcal{K}^k(t_1)dt_1}\right]\hat{\tilde{\Omega}}(0) \tag{F.13}$$

We then conveniently choose $\mathcal{K}(t)$ such that:

$$\mathcal{K}^k(t_1) = \frac{\alpha^2}{2}\int_0^{t_1} dt_2\mathbb{E}[\tilde{\mathcal{B}}^k(t_1)\tilde{\mathcal{B}}^k(t_2)] \tag{F.14}$$

and we are able to cancel the terms of order $\alpha^2\tau_c$ in the second factor of Eq. (F.13) [86]. Note that the superoperator $\mathcal{K}(t)$ is to be intended as a time local superoperator, i.e. even if defined through the integral expression in eq. (F.14), the time ordering in eq. (F.13) will order the whole operator $K(t)$ only according to the time t. Furthermore, note that the expression for $\mathcal{K}$ is implicit:

$$\mathcal{K}(t_1) = \frac{\alpha^2}{2}\int_0^{t_1} dt_2 \mathbb{E}[\tilde{\mathcal{B}}(t_1)\mathcal{V}(t_1,t_2)\tilde{\mathcal{B}}(t_2)\mathcal{V}(t_1,t_2)^{-1}] \tag{F.15}$$

as on the r.h.s. $\mathcal{V}(t_1,t_2)$ depends on $\mathcal{K}$ itself. Noticing that $\mathcal{K}$ is of $O(\alpha^2\tau_c)$, we perform a perturbative expansion in $\alpha\tau_c$ ($\mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2 + \mathcal{K}_3 + ...$) in order to obtain its explicit expression. The first term ($\mathcal{K}_1$) is obtained by neglecting the action of $\mathcal{V}(t_1,t_2)$ on $\tilde{\mathcal{B}}(t_2)$ in Eq. (F.14) so that:

$$\mathcal{K}_1(t) = \frac{\alpha^2}{2}\int_0^{t_1} dt_2 \mathbb{E}[\tilde{\mathcal{B}}(t_1)\tilde{\mathcal{B}}(t_2)] \tag{F.16}$$

The next term ($\mathcal{K}_2$) is of order $O(\alpha^4\tau_c^2)$, and is obtained upon plugging the above expression in Eq. (F.10):

$$\mathcal{K}_2(t_1) = \int_0^{t_1} dt_2 \mathbb{E}\left[\tilde{\mathcal{B}}(t_1)T\left[e^{\int_{t_2}^{t_1} dt'\mathcal{K}_1(t')}\right]\tilde{\mathcal{B}}(t_2)T\left[e^{-\int_{t_2}^{t_1} dt'\mathcal{K}_1(t')}\right]\right] \tag{F.17}$$

Higher order terms can be obtained in a similar fashion. This procedure can be repeated for the other terms of the cumulant expansion, so to obtain a disentangled expression at the desired order in α and τ_c, see [86] for the explicit construction in a more general case. It follows that at $O(\alpha^2\tau_c)$ Eq. (F.9) reads:

$$\hat{\tilde{\rho}}(t) = \left(T\left[e^{\int_0^t \mathcal{K}(t_1)dt_1 + O(\alpha^4\tau_c^2)}\right]T\left[e^{\frac{\alpha^2}{2}\int_0^t\int_0^{t_1} dt_1 dt_2 \mathbb{E}\left[\tilde{\mathcal{B}}^k(t_1)\tilde{\mathcal{B}}^k(t_2)\right] - \int_0^t \mathcal{K}^k(t_1)dt_1}\right]\right)\hat{\tilde{\Omega}}(0) \tag{F.18}$$

Eq. (F.18) is the formal solution of the differential equation:

$$\partial_t\hat{\tilde{\rho}}(t) = \frac{\alpha^2}{2}\int_0^t dt' \mathbb{E}[\tilde{\mathcal{B}}(t)\tilde{\mathcal{B}}(t')]\hat{\tilde{\rho}}(t) + O(\alpha^4\tau_c^3 t) \tag{F.19}$$

which in the original representation reads:

$$\partial_t\hat{\rho}(t) = \left(\mathcal{A} + \frac{\alpha^2}{2}\int_0^t dt' \mathbb{E}[\mathcal{B}(t)e^{\mathcal{A}(t-t')}\mathcal{B}(0)e^{-A(t-t')}]\right)\hat{\rho}(t) + O(\alpha^4\tau_c^3 t) \tag{F.20}$$

In order to apply this result to Eqs. (3.29, 4.30), the mapping from Eq. (F.20) is given by:

$$\begin{cases} \mathcal{A} \to -\frac{i}{\hbar}\left(\hat{H}_{0,L}^{(B)} - \hat{H}_{0,R}^{(B)}\right) \\ \alpha\mathcal{B} \to -\frac{i}{\hbar}\left(\hat{H}_{p,L}^{(B)} + \hat{H}_{d,L}^{(B)} - \hat{H}_{p,R}^{(B)} - \hat{H}_{d,R}^{(B)}\right) \end{cases} \tag{F.21}$$

and:

$$\begin{cases} \mathcal{A} \to -\frac{i}{\hbar}\left(\hat{H}^{(F)}_{0,L} - \hat{H}^{(F)}_{0,R}\right) \\ \alpha\mathcal{B} \to -\frac{i}{\hbar}\left(\hat{H}^{(F)}_{p,L} - \hat{H}^{(F)}_{p,R}\right) \end{cases} \tag{F.22}$$

in the two respective case, where the subscripts L and R denote the fact that the operator is acting respectively on the left and on the right of the density operator $\hat{\Omega}$ (i.e. $\mathcal{A}_L\mathcal{A}_R\hat{\Omega} = \mathcal{A}\hat{\Omega}\mathcal{A}$), while the superscripts (B) stands for bosons and (F) for fermions. The final result (at order $\alpha^2\tau_c$) is:

$$\begin{aligned} \partial_t\hat{\rho} = &-\frac{i}{\hbar}[\hat{H}^{(B)}_0, \hat{\rho}(t)] - \frac{1}{\hbar^2}\int_0^t dt_1 \cdot \\ &\cdot \mathbb{E}\left[\left[\left(\hat{H}^{(B)}_p(t) + \hat{H}^{(B)}_d(t)\right), \left[e^{i\hat{H}^{(B)}_0(t_1-t)}\left(\hat{H}^{(B)}_p(t_1) + \hat{H}^{(B)}_d(t_1)\right)e^{-i\hat{H}^{(B)}_0(t_1-t)}, \hat{\rho}(t)\right]\right]\right] \end{aligned} \tag{F.23}$$

$$\partial_t\hat{\rho} = -\frac{i}{\hbar}[\hat{H}^{(F)}_0, \hat{\rho}(t)] - \frac{1}{\hbar^2}\int_0^t dt_1 \mathbb{E}\left[\left[\hat{H}^{(F)}_p(t), \left[e^{i\hat{H}^{(F)}_0(t_1-t)}\hat{H}^{(F)}_p(t_1)e^{-i\hat{H}^{(F)}_0(t_1-t)}, \hat{\rho}(t)\right]\right]\right] \tag{F.24}$$

precisely as in Eqs. (3.30, 4.31) of the main text.

Appendix G: Recovering Markovian master equation

In this section we specialize Eq. (F.20) to interesting limiting cases. We also recover the Markovian master equations (3.34, 4.35) of the main text.

We start by considering the special case in which the stochastic superoperator $\mathcal{B}$ can be factorized as:

$$\mathcal{B}(t) = \mathfrak{h}_i(t)\mathcal{F}^i(t) \tag{G.1}$$

where $\mathfrak{h}_i(t)$ is a (collection of) stochastic process(es) and $\mathcal{F}^i(t)$ a non stochastic superoperator. Of this form are in fact the stochastic superoperators defined in Eqs. (F.21, F.22) through the explicit expressions of Eqs. (3.25, 4.23) of the main text.
We then notice that Eq. (F.20) becomes exact if $[\mathcal{A}, \mathcal{B}] = 0$. In this case in fact $[\tilde{\mathcal{B}}(t), \tilde{\mathcal{B}}(t_1)] = 0 = [\tilde{\mathcal{B}}^k(t)\tilde{\mathcal{B}}^k(t_1)]$, so that $\mathcal{K} = \mathcal{K}_1$, and the factor inside the second time ordering in Eq. (F.20) vanishes. It follows that Eq. (F.20) can be further simplified as:

$$\partial_t\hat{\rho}(t) = \left(\mathcal{A} + \frac{\alpha^2}{2}\int_0^t dt' D_{ij}(t,t')\mathcal{F}^i(0)\mathcal{F}^j(0)\right)\hat{\rho}(t) \tag{G.2}$$

where $D_{ij}(t,t') = \mathbb{E}[\mathfrak{h}_i(t)\mathfrak{h}_j(t-t')]$ is the time correlation function of the noise. As a very rough approximation we take the time correlation function to be an Heaviside theta function [1]:

$$D_{ij}(t) = \sigma_{ij}\Theta(t-\tau_c) \tag{G.3}$$

where τ_c is the correlation time of the noise, and σ_{ij} depends on the explicit form of $\mathcal{B}$. In this case Eq. (G.2) reads:

$$\partial_t\hat{\rho}(t) = \Big(\mathcal{A} + \frac{\alpha^2\lambda\sigma_{ij}}{2}\mathcal{F}^i(0)\mathcal{F}^j(0)\Big)\hat{\rho}(t) \tag{G.4}$$

where $\lambda = \min(t,\tau_c)$.

A different interesting scenario to consider is when the Markovian limit of Eq. (F.20) can be taken, i.e. when the correlation time (τ_c) of the noise is much smaller than the characteristic time (τ_{free}) of the free dynamics, and the limit $\tau_c/\tau_{\text{free}} \to 0$ can be taken. I this limit the action of $e^{\mathcal{A}(t_1-t)}$ on $\mathcal{B}$ (and more generally of any of the evolution superoperators employed in the derivation of Eq. (F.20)) will vanish to zero and the equation Eq. (F.20) reads[2]:

$$\partial_t\hat{\rho}(t) = \Big(\mathcal{A} + \frac{\alpha^2}{2}\int_0^\infty dt' D_{ij}(t,t')\mathcal{F}^i(0)\mathcal{F}^j(0)\Big)\hat{\rho}(t) \tag{G.5}$$

This equation can be further simplified noticing that in the limit $\tau_c/\tau_{\text{free}} \to 0$ the time correlation function is naturally replaced by a Dirac delta function:

$$D_{ij}(t) = \sigma_{ij}\delta(t-\tau_c) \tag{G.6}$$

and Eq. (G.5) consequently reads:

$$\partial_t\hat{\rho}(t) = \Big(\mathcal{A} + \frac{\alpha^2\tau_c\sigma_{ij}}{2}\mathcal{F}^i(0)\mathcal{F}^j(0)\Big)\hat{\rho}(t) \tag{G.7}$$

As a final remark, note that the factor τ_c in the above equation can be safely replaced with λ, as the error made lies within the boundaries of the validity of the Markovian approximation.
Upon subsituting the explicit expression for $\mathfrak{h}_i(t)$ and $\mathcal{F}^i(t)$ according to Eqs. (G.1, F.21, F.22, 3.25, 4.23) and given the stochastic properties of the noise (Eqs. (3.26, 3.32)), we recover Eqs. (3.34,4.35) of the main text.

[1] A more physically meaningful result can be obtained by taking the correlation function as an exponential decay

[2] Note that we have safely replaced the upper limit of integration t with ∞, as the integrand vanishes anyway [61].

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