DE GRUYTER Open Mathematics 2020; 18: 1185–1205

Research Article

Franco Obersnel and Pierpaolo Omari*

Revisiting the sub- and super-solution method for the classical radial solutions of the mean curvature equation

https://doi.org/10.1515/math-2020-0097
received July 31, 2020; accepted October 9, 2020

Abstract: This paper focuses on the existence and the multiplicity of classical radially symmetric solutions of the mean curvature problem:

$$\begin{cases} -\operatorname{div}\left(\dfrac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right) = f(x, v, \nabla v) & \text{in } \Omega, \\ a_0 v + a_1 \frac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases}$$

with Ω an open ball in $\mathbb{R}^N$, in the presence of one or more couples of sub- and super-solutions, satisfying or not satisfying the standard ordering condition. The novel assumptions introduced on the function f allow us to complement or improve several results in the literature.

Keywords: prescribed mean curvature equation, Dirichlet, Neumann, Robin boundary conditions, radial symmetry, classical solution, sub- and super-solutions

MSC 2020: 35J62, 35J93, 35J25, 34C25

1 Introduction and main results

This paper deals with the existence of classical solutions of the mean curvature problem:

$$\begin{cases} -\left(\dfrac{t^{N-1}u'}{\sqrt{1+u'^2}}\right)' = t^{N-1}f(t, u, u') & \text{in }]0, R[, \\ u'(0) = 0,\ a_0 u(R) + a_1 u'(R) = 0, \end{cases} \tag{1.1}$$

in the presence of couples of sub- and super-solutions, satisfying or not satisfying the standard ordering condition, that is, the sub-solution is smaller than the super-solution. It is assumed throughout this work that

(h_1) $R > 0$, $N \in \mathbb{N}^+$, $a_0, a_1 \in \mathbb{R}$ are given constants, with $a_0 \cdot a_1 \geq 0$, $a_0 + a_1 > 0$,
and
(h_2) $f : [0, R] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous.

*** Corresponding author: Pierpaolo Omari,** Dipartimento di Matematica e Geoscienze, Università degli Studi di Trieste, via A. Valerio 12/1, 34127 Trieste, Italy, e-mail: omari@units.it
Franco Obersnel: Dipartimento di Matematica e Geoscienze, Università degli Studi di Trieste, via A. Valerio 12/1, 34127 Trieste, Italy, e-mail: obersnel@units.it

Definition 1.1. (Notion of solution) By a classical solution of (1.1) we mean a function $u \in C^1([0, R]) \cap C^2(]0, R[)$ which satisfies the differential equation for all $t \in]0, R[$ and the boundary condition.

It is easy to verify, arguing like, e.g., in [1, Lemma 3.1], that a classical solution u of (1.1) actually belongs to $C^2([0, R])$ and satisfies the equation:

$$-\left(\frac{u'}{\sqrt{1+u'^2}}\right)' - \frac{N-1}{t}\frac{u'}{\sqrt{1+u'^2}} = f(t, u, u') \quad \text{for all} \quad t \in [0, R],$$

provided that the convention $\frac{u'(t)}{t} = u''(0)$ if $t = 0$ is set. Accordingly, solutions of (1.1) will always be assumed to belong to $C^2([0, R])$.

From [1, Remark 3.2] it also follows that if u is a solution of (1.1), then, setting $v(x) = u(|x|)$ for all $x \in \overline{\Omega}$, with Ω the open ball in $\mathbb{R}^N$ of center 0 and radius R, the function v belongs to $C^2(\overline{\Omega})$ and is a classical solution of

$$\begin{cases} -\operatorname{div}\left(\dfrac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right) = f\left(|x|, v, \nabla v \cdot \frac{x}{|x|}\right) & \text{in } \Omega, \\ a_0 v + a_1 \nabla v \cdot \frac{x}{|x|} = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

The existence of classical solutions of boundary value problems associated with the mean curvature equation

$$-\operatorname{div}\left(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}}\right) = f(x, v, \nabla v), \tag{1.3}$$

in a bounded domain Ω, is a central and delicate topic in the field of the quasilinear elliptic partial differential equations and has been largely discussed in the literature, also due to the applications of (1.3) in numerous geometrical and physical issues. Although breakthrough results were proven between the late sixties and the early seventies, starting with [2–8], this study has been carried out further in the subsequent decades by many authors; for a wide, but far from exhaustive, overview we refer to the monographs [9–12] and references therein.

However, in spite of such a large amount of work, a general sub- and super-solution method – a quite powerful technique for the study of semilinear and quasilinear elliptic problems [13–17] – has not been established yet for the classical solutions of boundary value problems associated with Eq. (1.3). Actually, some partial results have been formulated in [18, Theorems 2.1 and 2.2] and [19, Theorem 1.1], but unfortunately the given proofs do not seem complete, as all of them rely on the unproven Lemma 3.2 in [20]. This substantial lack of results within the existing literature might be attributed to the fact that mean curvature problems are fraught with a number of difficulties which do not arise in dealing with other quasilinear problems, such as the possible occurrence of gradient blow up phenomena and the corresponding development of singularities for the solutions, up to the formation of discontinuities. In order to overcome these drawbacks and to include in the analysis such kind of singular solutions, it appeared natural, starting with [21–26], to settle the problem in the space of bounded variation functions. In particular, this approach allowed to develop in [27–29] a sub- and super-solution method for the bounded variation solutions of various boundary value problems associated with Eq. (1.3), according to the notion of solution introduced in [30]. The likely most relevant feature of these results is that, assuming the sub- and super-solutions satisfy the standard ordering condition, no bound and no monotonicity are imposed on the function f, contrary to what was required in all preceding studies.

When limiting the discussion to the radial problem (1.2) with homogeneous Dirichlet boundary conditions, which has also been the subject of extensive study since the late eighties [31–36], the existence of classical solutions can be inferred from [28, Theorem 2.4] if the right-hand side f of the equation is gradient independent, locally Lipschitz and non-increasing with respect to the state variable, and further the sub- and the super-solutions are regular and satisfy the boundary conditions. A complementary result for the same problem was obtained by a different approach – monotone iteration versus variational techniques – in [19, Theorem 3.1], assuming that f is gradient independent, continuous, but non-decreasing with respect to the state variable, and again the sub- and the super-solutions are regular and fulfill the Dirichlet boundary conditions.

Since these results require that f satisfies some monotonicity property, it seems interesting to investigate the situation where such assumptions fail. Nevertheless, simple one-dimensional examples, given, e.g., in [37,38], show that the sole existence of a couple of sub- and super-solutions, although yielding the existence of a bounded variation solution, cannot guarantee the existence of a classical one. Therefore, as some supplementary assumption should be added, in this paper we adopt an alternative approach that allows us to remove any monotonicity requirement on f at the expense of imposing a local growth condition, which is independent of the size of the interval $[0, R]$ and, therefore, completely different from those considered in the literature, and in turn is almost necessary for guaranteeing the regularity of the solutions of the associated Cauchy problems. Namely, we introduce the following assumption:

(h_3) there exist an interval $I \subseteq \mathbb{R}$ and a continuous function $g : I \to \mathbb{R}$ such that, for all $t \in [0, R]$, $s \in I$ and $\xi \in \mathbb{R}$,

$$f(t, s, \xi)\cdot \operatorname{sgn}(\xi) \geq -g(s) \tag{1.4}$$

and

$$\int_I g(s)\,\mathrm{d}s < 1. \tag{1.5}$$

It is apparent that the condition (1.4) in (h_3) implies $g(s) \geq 0$ for all $s \in I$.

Remark 1.1. One sample case where (h_3) holds is the following. Suppose that the function f admits the decomposition

$$f(t, s, \xi) = f_1(t, s) + f_2(\xi)\cdot \xi,$$

with $f_2(\xi) \geq 0$ for all $\xi \in \mathbb{R}$. Then, f satisfies (h_3) provided that f_1 does. This means that no growth restriction with respect to the gradient variable is required here.

It is worth observing that the condition (h_3) guarantees that, for every $u_0 \in I$, any non-extendible solution u of the Cauchy problem:

$$\begin{cases} -\left(\dfrac{t^{N-1}u'}{\sqrt{1+u'^2}}\right)' = t^{N-1}f(t, u, u'), \\ u(0) = u_0,\ u'(0) = 0, \end{cases}$$

with range $u \subseteq I$, exists on the whole of $[0, R]$. This is a direct consequence of the inequality

$$\begin{aligned} 1 - \frac{1}{\sqrt{1+u'(t)^2}} &= \left[\frac{u'(\tau)^2}{\sqrt{1+u'(\tau)^2}}\right]_{t_0}^{t} - \int_{t_0}^{t} \frac{u'(\tau)u''(\tau)}{\sqrt{1+u'(\tau)^2}}\mathrm{d}\tau \\ &= -\int_{t_0}^{t} \frac{N-1}{\tau}\frac{u'(\tau)^2}{\sqrt{1+u'(\tau)^2}}\mathrm{d}\tau - \int_{t_0}^{t} f(\tau, u(\tau), u'(\tau))u'(\tau)\mathrm{d}\tau \\ &\leq \int_{t_0}^{t} g(u(\tau))|u'(\tau)|\mathrm{d}\tau \leq \int_{\operatorname{range}\, u} g(s)\mathrm{d}s, \quad \text{for all } t \in \operatorname{dom}\, u, \end{aligned}$$

where $t_0 \in [0, t]$ is the critical point of u closest to t.

Conversely, in the one-dimensional case, the condition (1.5) in (h_3) turns out to be necessary for the existence of solutions. Namely, if $g : \mathbb{R} \to \mathbb{R}$ is an arbitrary continuous function, any solution $u \in C^2([a, b])$ of the equation

$$-\left(\frac{u'}{\sqrt{1+u'^2}}\right)' = g(u)$$

satisfies the identity

$$\frac{1}{\sqrt{1+u'(t_1)^2}} - \frac{1}{\sqrt{1+u'(t_2)^2}} = \int_{u(t_2)}^{u(t_1)} g(s)\,\mathrm{d}s, \quad \text{for all } t_1, t_2 \in [a, b],$$

and hence, setting $I = \operatorname{range} u$, condition (1.5) must hold.

On the other hand, it is clear that assumptions (h_1), (h_2), (h_3) alone do not imply the solvability of the problem (1.1). A very simple example is given by

$$\begin{cases} -\left(\dfrac{t^{N-1}u'}{\sqrt{1+u'^2}}\right)' = Nt^{N-1} \quad \text{in }]0, 1[, \\ u'(0) = 0, \; a_0 u(1) + a_1 u'(1) = 0. \end{cases} \tag{1.6}$$

Indeed, integrating the equation on $[0, 1]$ shows that (1.6) has no classical solutions although the constant function $f(t, s, \xi) = N$ satisfies (h_3), for any interval I such that $|I| < \frac{1}{N}$ and for $g = N$.

The solvability of (1.1) can actually be achieved assuming the existence of couples of sub- and super-solutions, whose definition is given below taking into account the previously set convention: $\frac{u'(t)}{t} = u''(0)$ if $t = 0$, for all $u \in C^2([0, R])$ such that $u'(0) = 0$. We refer to [28,29,35,37,39–41] for the concrete construction of sub- and super-solutions in the frame of the mean curvature equation.

Definitions 1.1. (Notion of sub- and super-solution).

– A function $\alpha : [0, R] \to \mathbb{R}$ is a sub-solution of the problem (1.1) if there exist $\alpha_1, \ldots, \alpha_p \in C^2([0, R])$ such that $\alpha = \max\{\alpha_1, \ldots, \alpha_p\}$ and for each $i \in \{1, \ldots, p\}$

$$\begin{cases} -\left(\dfrac{t^{N-1}\alpha_i'(t)}{\sqrt{1+\alpha_i'(t)^2}}\right)' \le t^{N-1} f(t, \alpha_i(t), \alpha_i'(t)) \quad \text{for all } t \in]0, R[, \\ \alpha_i'(0) \ge 0, \; a_0\alpha_i(R) + a_1\alpha_i'(R) \le 0. \end{cases}$$

– A sub-solution α is strict if for each $i \in \{1, \ldots, p\}$ either

$$\begin{cases} -\left(\dfrac{t^{N-1}\alpha_i'(t)}{\sqrt{1+\alpha_i'(t)^2}}\right)' < t^{N-1} f(t, \alpha_i(t), \alpha_i'(t)) \quad \text{for all } t \in]0, R[, \\ \alpha_i'(0) > 0, \; a_0\alpha_i(R) + a_1\alpha_i'(R) < 0, \end{cases}$$

or

$$\begin{cases} -\left(\dfrac{\alpha_i'(t)}{\sqrt{1+\alpha_i'(t)^2}}\right)' - \dfrac{N-1}{t}\dfrac{\alpha_i'(t)}{\sqrt{1+\alpha_i'(t)^2}} < f(t, \alpha_i(t), \alpha_i'(t)) \quad \text{for all } t \in [0, R[, \\ \alpha_i'(0) = 0, \; a_0\alpha_i(R) + a_1\alpha_i'(R) < 0, \end{cases}$$

or

$$\begin{cases} -\left(\dfrac{t^{N-1}\alpha_i'(t)}{\sqrt{1+\alpha_i'(t)^2}}\right)' < t^{N-1}f(t,\alpha_i(t),\alpha_i'(t)) \quad \text{for all } t \in]0,R], \\ \alpha_i'(0) > 0,\ a_0\alpha_i(R) + a_1\alpha_i'(R) = 0, \end{cases}$$

or

$$\begin{cases} -\left(\dfrac{\alpha_i'(t)}{\sqrt{1+\alpha_i'(t)^2}}\right)' - \dfrac{N-1}{t}\dfrac{\alpha_i'(t)}{\sqrt{1+\alpha_i'(t)^2}} < f(t,\alpha_i(t),\alpha_i'(t)) \quad \text{for all } t \in [0,R], \\ \alpha_i'(0) = 0,\ a_0\alpha_i(R) + a_1\alpha_i'(R) = 0. \end{cases}$$

– A function $\beta : [0,R] \to \mathbb{R}$ is a super-solution of the problem (1.1) if there exist $\beta_1,\dots,\beta_q \in C^2([0,R])$ such that $\beta = \min\{\beta_1,\dots,\beta_q\}$ and for each $j \in \{1,\dots,q\}$

$$\begin{cases} -\left(\dfrac{t^{N-1}\beta_j'(t)}{\sqrt{1+\beta_j'(t)^2}}\right)' \geq t^{N-1}f(t,\beta_j(t),\beta_j'(t)) \quad \text{for all } t \in]0,R[, \\ \beta_i'(0) \leq 0,\ a_0\beta_j(R) + a_1\beta_j'(R) \geq 0. \end{cases}$$

– A super-solution β is strict if for each $j \in \{1,\dots,q\}$ either

$$\begin{cases} -\left(\dfrac{t^{N-1}\beta_j'(t)}{\sqrt{1+\beta_j'(t)^2}}\right)' > t^{N-1}f(t,\beta_j(t),\beta_j'(t)) \quad \text{for all } t \in]0,R[, \\ \beta_j'(0) < 0,\ a_0\beta_j(R) + a_1\beta_j'(R) > 0, \end{cases}$$

or

$$\begin{cases} -\left(\dfrac{\beta_j'(t)}{\sqrt{1+\beta_j'(t)^2}}\right)' - \dfrac{N-1}{t}\dfrac{\beta_j'(t)}{\sqrt{1+\beta_j'(t)^2}} > f(t,\beta_j(t),\beta_j'(t)) \quad \text{for all } t \in [0,R[, \\ \beta_j'(0) = 0,\ a_0\beta_j(R) + a_1\beta_j'(R) > 0, \end{cases}$$

or

$$\begin{cases} -\left(\dfrac{t^{N-1}\beta_j'(t)}{\sqrt{1+\beta_j'(t)^2}}\right)' > t^{N-1}f(t,\beta_j(t),\beta_j'(t)) \quad \text{for all } t \in]0,R], \\ \beta_j'(0) < 0,\ a_0\beta_j(R) + a_1\beta_j'(R) = 0, \end{cases}$$

or

$$\begin{cases} -\left(\dfrac{\beta_j'(t)}{\sqrt{1+\beta_j'(t)^2}}\right)' - \dfrac{N-1}{t}\dfrac{\beta_j'(t)}{\sqrt{1+\beta_j'(t)^2}} > f(t,\beta_j(t),\beta_j'(t)) \quad \text{for all } t \in [0,R], \\ \beta_j'(0) = 0,\ a_0\beta_j(R) + a_1\beta_j'(R) = 0. \end{cases}$$

Notation. For all $u, v \in C^1([0,R])$, we write $u \leq v$ if $u(t) \leq v(t)$ for all $t \in [0,R]$; $u < v$ if $u \leq v$ and $u \neq v$; $u \ll v$ if $u(t) < v(t)$ for all $t \in]0,R[$ and, in addition, either $u(0) < v(0)$ or else $u(0) = v(0)$ and $u'(0) < v'(0)$, as well as either $u(R) < v(R)$ or else $u(R) = v(R)$ and $u'(R) > v'(R)$.

The following existence and localization result holds in the presence of one couple of sub- and super-solutions satisfying the standard ordering condition.

Theorem 1.1. *Assume* (h_1), (h_2), (h_3) *and*
(h_4) *there exist a sub-solution* α *and a super-solution* β *of the problem* (1.1) *satisfying*

$$\alpha \le \beta \quad \textit{and} \quad \text{range } \alpha, \text{range } \beta \subseteq I.$$

Then, the problem (1.1) *has at least one classical solution* u, *with*

$$\alpha \le u \le \beta.$$

Furthermore, the multiplicity of solutions can be detected in the presence of two couples of sub- and super-solutions, if one of them does not satisfy the standard ordering condition. Indeed, the following version of the celebrated Amann three-solution theorem [13,16,42] holds true for the problem (1.1).

Theorem 1.2. *Assume* (h_1), (h_2), (h_3) *and* (h_4). *Suppose further that*
(h_5) *there exist a strict sub-solution* α_1 *and a strict super-solution* β_1 *of the problem* (1.1) *satisfying*

$$\alpha \le \alpha_1 \le \beta, \quad \alpha \le \beta_1 \le \beta \quad \textit{and} \quad \alpha_1 \nleq \beta_1.$$

Then, the problem (1.1) *has at least three classical solutions* u_1, u_2, u_3, *with*

$$u_1 < u_3 < u_2, \quad \alpha \le u_1 \ll \beta_1, \quad \alpha_1 \ll u_2 \le \beta, \quad u_3 \ngeq \alpha_1 \ \textit{and} \ u_3 \nleq \beta_1.$$

Remark 1.2. Although not explicitly stated here, the conclusions of Theorems 1.1 and 1.2 can be completed with the existence of extremal solutions. This easily follows from the adopted notion of sub- and super-solutions arguing like, e.g., in [43].

Theorems 1.1 and 1.2 are the main results obtained in this work and the entire Section 2 is devoted to their proofs. Finally, in Section 3 we provide some additional results for the Neumann and the periodic problems, where some alternative boundedness conditions are imposed on the function f (see (h_6), or (h_6'), below) that unlike (h_3) involve the length of the interval $[0, R]$, or, respectively, the period T. It is clear that the complementary nature of the assumptions introduced in this paper allows us to deal with manifold situations, which can be treated by suitably combining our statements, but are not covered by other results available in the existing literature, such as those in [29,34,37,40,44–49].

2 Proofs

In this section, we deliver the proofs of Theorems 1.1 and 1.2. We first introduce a modified version of the problem (1.1), inspired from [16], and show that any solution of the modified problem is actually a solution of the original one. Next, we prove the existence of a solution of the modified problem, thus completing the proof of Theorem 1.1. We treat separately the Neumann case and the Dirichlet and the Robin cases since the proof, although basically the same, requires more care in the former situation. We also provide some information on the topological degree which will be crucial in detecting the existence of multiple solutions. The proof of Theorem 1.2 is then given in the final part of this section, relying on the previous degree calculations. For convenience, we henceforth set

$$\varphi(\xi) = \frac{\xi}{\sqrt{1+\xi^2}} \quad \text{for all } \xi \in \mathbb{R}.$$

2.1 A modified problem

In this section, we introduce a modified version of the problem (1.1) and prove that any solution of the modified problem is actually a solution of the original one. Assume (h_1), (h_2), (h_3) and (h_4). Let us set

$$K = \int_I g(s)\,\mathrm{d}s \quad \text{and} \quad L = \frac{\sqrt{K(2-K)}}{1-K}. \tag{2.1}$$

For any $M > 0$, we define

$$\varphi_M(\xi) = \begin{cases} \varphi(\xi) & \text{if } 0 \le \xi \le M, \\ \varphi(M) + \varphi'(M)(\xi - M) & \text{if } \xi > M, \\ -\varphi_M(-\xi) & \text{if } \xi < 0. \end{cases} \tag{2.2}$$

We also write

$$\Phi_M(\xi) = \int_0^\xi \varphi_M(\sigma)\,\mathrm{d}\sigma.$$

Setting

$$\psi(\xi) = \varphi_M(\xi)\xi - \Phi_M(\xi) - 1 + \frac{1}{\sqrt{1+\xi^2}}$$

and observing that ψ is even, $\psi(0) = 0$ and $\psi'(\xi) \ge 0$ for all $\xi \ge 0$, we see that

$$\varphi_M(\xi)\xi - \Phi_M(\xi) \ge 1 - \frac{1}{\sqrt{1+\xi^2}} \quad \text{for all } \xi \in \mathbb{R}. \tag{2.3}$$

Take $\tilde{L} \in \mathbb{R}$ such that

$$\tilde{L} > L + \max\{\|\alpha_i'\|_\infty, \|\beta_j'\|_\infty : i = 1,\dots,p,\ j = 1,\dots,q\}. \tag{2.4}$$

We introduce the following truncation of the function f:

$$h(t,s,\xi) = \begin{cases} f(t,s,-\tilde{L}) & \text{if } \xi < -\tilde{L}, \\ f(t,s,\xi) & \text{if } |\xi| \le \tilde{L}, \\ f(t,s,\tilde{L}) & \text{if } \xi > \tilde{L}. \end{cases} \tag{2.5}$$

We set, for each $i = 1,\dots,p$,

$$\gamma_i(t,s) = \begin{cases} \alpha_i(t) & \text{if } s < \alpha_i(t), \\ s & \text{if } s \ge \alpha_i(t) \end{cases}$$

and, for each $j = 1,\dots,q$,

$$\delta_j(t,s) = \begin{cases} \beta_j(t) & \text{if } s > \beta_j(t), \\ s & \text{if } s \le \beta_j(t). \end{cases}$$

Define, for each $i = 1,\dots,p$, the operator $\underline{\mathcal{H}_i} : C^1([0,R]) \to L^\infty(0,R)$ by

$$\underline{\mathcal{H}_i}(u)(t) = h\Big(t, \gamma_i(t,u(t)), \frac{\mathrm{d}}{\mathrm{d}t}\gamma_i(t,u(t))\Big) \quad \text{for a.e. } t \in [0,R]$$

and, for each $j = 1,\dots,q$, the operator $\overline{\mathcal{H}_j} : C^1([0,R]) \to L^\infty(0,R)$ by

$$\overline{\mathcal{H}_j}(u)(t) = h\Big(t, \delta_j(t,u(t)), \frac{\mathrm{d}}{\mathrm{d}t}\delta_j(t,u(t))\Big) \quad \text{for a.e. } t \in [0,R].$$

Finally, we introduce the operator $\mathcal{H} : C^1([0,R]) \to L^\infty(0,R)$ by setting, for all $u \in C^1([0,R])$ and a.e. $t \in [0,R]$,

$$\mathcal{H}(u)(t) = \begin{cases} \max\limits_{i=1,\dots,p} \underline{\mathcal{H}_i}(u)(t) + \arctan(\alpha(t) - u(t)) & \text{if } u(t) < \alpha(t), \\ h(t, u(t), u'(t)) & \text{if } \alpha(t) \le u(t) \le \beta(t), \\ \min\limits_{j=1,\dots,q} \overline{\mathcal{H}_j}(u)(t) - \arctan(u(t) - \beta(t)) & \text{if } u(t) > \beta(t). \end{cases} \tag{2.6}$$

Let us consider the modified problem

$$\begin{cases} -(\varphi_M(u')t^{N-1})' = t^{N-1}\mathcal{H}(u) & \text{in }]0,R[, \\ u'(0) = 0,\ a_0 u(R) + a_1 u'(R) = 0. \end{cases} \tag{2.7}$$

By a solution of (2.7), we mean a function $u \in W^{2,\infty}(0,R)$ which satisfies the equation a.e. in $]0,R[$ as well as the boundary conditions.

We prove that, taking $M > \tilde{L}$, any solution u of (2.7) is a solution of (1.1) too. We first verify that $u \ge \alpha$ by showing that $u \ge \alpha_i$ for each $i \in \{1,\dots,p\}$. Let us fix $i \in \{1,\dots,p\}$. By the monotonicity of φ_M we have that

$$0 \le \int\limits_{\{t:u(t)<\alpha_i(t)\}} t^{N-1}(\varphi_M(u') - \varphi_M(\alpha_i'))(u' - \alpha_i')\,\mathrm{d}t = -\int\limits_0^R t^{N-1}(\varphi_M(u') - \varphi_M(\alpha_i'))((u-\alpha_i)^-)'\,\mathrm{d}t.$$

Integrating by parts and observing that, from Definitions 1.1 and from (2.7),

$$[(t^{N-1}(\varphi_M(u'(t)) - \varphi_M(\alpha_i'(t))))(u(t) - \alpha_i(t))^-]_0^R \le 0,$$

we get

$$0 \le \int\limits_0^R (t^{N-1}(\varphi_M(u') - \varphi_M(\alpha_i')))'(u-\alpha_i)^-\,\mathrm{d}t. \tag{2.8}$$

Note that, as $M > \tilde{L}$, α_i satisfies the inequality

$$-(\varphi_M(\alpha_i')t^{N-1})' \le t^{N-1} f(t, \alpha_i, \alpha_i') \quad \text{in }]0,R[. \tag{2.9}$$

Hence we obtain, from (2.8), (2.7), (2.9) and from the definitions of $\mathcal{H}$ and h,

$$\begin{aligned} 0 &\le \int\limits_0^R t^{N-1}(-\mathcal{H}(u) + f(t,\alpha_i,\alpha_i'))(u-\alpha_i)^-\mathrm{d}t \\ &\le \int\limits_0^R t^{N-1}(-\underline{\mathcal{H}_i}(u) - \arctan(\alpha - u) + f(t,\alpha_i,\alpha_i'))(u-\alpha_i)^-\mathrm{d}t \\ &= \int\limits_{\{t:u(t)<\alpha_i(t)\}} t^{N-1}\Big(h\Big(t, \gamma_i(t,u), \tfrac{d}{\mathrm{d}t}\gamma_i(t,u)\Big) + \arctan(\alpha - u) - f(t,\alpha_i,\alpha_i')\Big)(u-\alpha_i)\,\mathrm{d}t \\ &= \int\limits_{\{t:u(t)<\alpha_i(t)\}} t^{N-1}\arctan(\alpha - u)(u-\alpha_i)\,\mathrm{d}t \\ &\le \int\limits_{\{t:u(t)<\alpha_i(t)\}} t^{N-1}\arctan(\alpha_i - u)(u-\alpha_i)\,\mathrm{d}t \le 0. \end{aligned}$$

Thus, we infer that $(u-\alpha_i)^- = 0$, that is, $u \ge \alpha_i$. Since this is valid for each $i \in \{1,\dots,p\}$, we conclude that $u \ge \alpha$. In a symmetric way, we verify that $u \le \beta$. Therefore, u is a solution of the problem:

$$\begin{cases} -(\varphi_M(u')t^{N-1})' = t^{N-1}h(t,u,u') & \text{in }]0,R[, \\ u'(0) = 0,\ a_0 u(R) + a_1 u'(R) = 0. \end{cases} \tag{2.10}$$

To prove that u is a solution of (1.1) it remains to show that $\|u'\|_\infty \le \tilde{L}$. Assume first that $\max_{[0,R]} u' = u'(t_1) > 0$ for some $t_1 \in]0, R]$ and take $t_0 = \max\{t \in [0, t_1] : u'(t) = 0\}$. We have $u'(t) > 0$ for all $t \in]t_0, t_1]$ and hence, using also (2.3),

$$
\begin{aligned}
1 - \frac{1}{\sqrt{1 + u'(t_1)^2}} &\le \varphi_M(u'(t_1))u'(t_1) - \Phi_M(u'(t_1)) \\
&= [\varphi_M(u'(t))u'(t)]_{t_0}^{t_1} - \int_{t_0}^{t_1} \varphi_M(u'(t))u''(t)\,\mathrm{d}t \\
&= (1 - N) \int_{t_0}^{t_1} \frac{u'(t)}{t}\varphi_M(u'(t))\,\mathrm{d}t - \int_{t_0}^{t_1} h(t, u(t), u'(t))u'(t)\,\mathrm{d}t \\
&\le \int_{t_0}^{t_1} g(u(t))u'(t)\,\mathrm{d}t = \int_{u(t_0)}^{u(t_1)} g(s)\,\mathrm{d}s \le K.
\end{aligned}
$$

Recalling (2.1) we find that $u'(t_1) \le L < \tilde{L}$. Similarly, we show that if $\min_{[0,R]} u' = u'(t_1) < 0$ for some $t_1 \in]0, R]$, then we have $|u'(t_1)| \le L < \tilde{L}$.

2.2 Existence of a solution

In this section, we prove the existence of a solution of the modified problem (2.7) for any given $M > 0$, treating separately the Neumann case and the Dirichlet and Robin cases.

2.2.1 The Neumann problem

Assume $a_0 = 0$ in (2.7), i.e., consider the Neumann problem:

$$
\begin{cases}
-(\varphi_M(u')t^{N-1})' = t^{N-1}\mathcal{H}(u) & \text{in }]0, R[, \\
u'(0) = 0,\ u'(R) = 0,
\end{cases}
\tag{2.11}
$$

with $\mathcal{H}$ defined by (2.6).

Existence of solutions

Set

$$X = \{u \in C^1([0, R]) : u'(0) = 0,\ u'(R) = 0\}$$

and

$$\tilde{X} = \{w \in X : w(R) = 0\}.$$

We endow X with the norm

$$\|u\|_X = |u(R)| + \|u'\|_\infty. \tag{2.12}$$

It is clear that, with this norm, X is a Banach space and $\tilde{X}$ is a closed subspace of X. As any $u \in X$ admits a unique representation of the form $u = r + w$, with $r = u(R) \in \mathbb{R}$ and $w \in \tilde{X}$, it follows that $X = \mathbb{R} \oplus \tilde{X}$, algebraically and topologically. Define the operators:

$\mathcal{P} : L^\infty(0,R) \to \mathbb{R}$ by

$$\mathcal{P}v = NR^{-N}\int_0^R t^{N-1}v(t)\,\mathrm{d}t,$$

$\tilde{\mathcal{T}} : X \to \tilde{X}$ by

$$\tilde{\mathcal{T}}(u)(t) = \int_t^R \varphi_M^{-1}\left(\int_0^s \left(\frac{\sigma}{s}\right)^{N-1}(\mathcal{H}(u)(\sigma) - \mathcal{P}\mathcal{H}(u))\,\mathrm{d}\sigma\right)\mathrm{d}s \quad \text{for all } t \in [0,R],$$

and $\mathcal{T} : X \to X$ by

$$\mathcal{T}(r+w) = r + \mathcal{P}\mathcal{H}(r+w) + \tilde{\mathcal{T}}(r+w).$$

It is easy to verify that $u = r + w \in X$ is a fixed point of $\mathcal{T}$ if and only if u is a solution of (2.11). Compute, for a.e. $t \in [0,R]$,

$$\frac{d}{\mathrm{d}t}\tilde{\mathcal{T}}(u)(t) = -\varphi_M^{-1}\left(\int_0^t \left(\frac{\sigma}{t}\right)^{N-1}(\mathcal{H}(u)(\sigma) - \mathcal{P}\mathcal{H}(u))\,\mathrm{d}\sigma\right)$$

and

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\tilde{\mathcal{T}}(u)(t) &= -(\varphi_M^{-1})'\left(\int_0^t \left(\frac{\sigma}{t}\right)^{N-1}(\mathcal{H}(u)(\sigma) - \mathcal{P}\mathcal{H}(u))\,\mathrm{d}\sigma\right)\\ &\quad\cdot\left((1-N)t^{-N}\int_0^t \sigma^{N-1}(\mathcal{H}(u)(\sigma) - \mathcal{P}\mathcal{H}(u))\,\mathrm{d}\sigma + (\mathcal{H}(u)(t) - \mathcal{P}\mathcal{H}(u))\right).\end{aligned}$$

Since the operator $\mathcal{H} : C^1([0,R]) \to L^\infty(0,R)$ is continuous and has a bounded range, the operator $\mathcal{P} : L^\infty(0,R) \to \mathbb{R}$ is linear and continuous, and the function $\varphi_M^{-1} : \mathbb{R} \to \mathbb{R}$ is of class C^1 and has a bounded derivative, it follows that $\tilde{\mathcal{T}} : X \to \tilde{X}$ is continuous and there exists a constant $C > 0$ such that

$$\|\tilde{\mathcal{T}}(u)\|_{W^{2,\infty}} < C \quad \text{for all } u \in X.$$

For later use, it is convenient to assume further that $C > \tilde{L}$, where $\tilde{L}$ is the constant introduced in (2.4). By the Ascoli-Arzelà theorem, $\tilde{\mathcal{T}}$ has a relatively compact range in $\tilde{X}$, i.e., $\overline{\text{range } \tilde{\mathcal{T}}}$ is compact in $\tilde{X}$. Consequently, the operator $\mathcal{T} : X \to X$ is completely continuous, i.e., $\mathcal{T}$ is continuous and maps bounded subsets of X onto relatively compact subsets of X.

Choose a constant

$$D > \max_{i=1,\dots,p}\|\alpha_i\|_\infty + \max_{j=1,\dots,q}\|\beta_j\|_\infty + RC$$

and define in X the open bounded set

$$\mathcal{O} = \{u = r + w \in X : |r| < D \text{ and } \|w\|_X < C\}. \tag{2.13}$$

We prove that, for all $\lambda \in [0,1]$, any $u \in \bar{\mathcal{O}}$ such that

$$u = \lambda\mathcal{T}(u),$$

or equivalently

$$w = \lambda\tilde{\mathcal{T}}(u) \quad \text{and} \quad (1-\lambda)r = \lambda\mathcal{P}\mathcal{H}(u),$$

satisfies $u \in \mathcal{O}$. Since, for all $\lambda \in [0,1]$, we have that

$$\|w\|_X = \|\lambda\tilde{\mathcal{T}}(u)\|_X \le \|\tilde{\mathcal{T}}(u)\|_X < C,$$

it is enough to show that $|r| < D$. Assume, by contradiction, that there is $\lambda \in]0, 1]$ such that, e.g., $r \le -D$. We have, for all $t \in [0, R]$,

$$u(t) = r + w(t) \le -D + \|w\|_\infty < -\max_{i=1,\dots,p} \|\alpha_i\|_\infty - \max_{j=1,\dots,q} \|\beta_j\|_\infty - RC + RC \le \min_{i=1,\dots,p} \big(\min_{[0,R]} \alpha_i\big)$$

and hence

$$\frac{R^N}{N}\mathcal{PH}(u) = \int_0^R t^{N-1}\mathcal{H}(u)(t)\,\mathrm{d}t = \int_0^R t^{N-1}\big(\max_{i=1,\dots,p} \underline{\mathcal{H}_i}(u)(t) + \arctan(\alpha(t) - u(t))\big)\mathrm{d}t > \int_0^R t^{N-1} \max_{i=1,\dots,p} \underline{\mathcal{H}_i}(u)(t)\,\mathrm{d}t.$$

Thus we get, for each $i = 1,\dots, p$,

$$\begin{aligned}\frac{R^N}{N}\mathcal{PH}(u) &> \int_0^R t^{N-1}h\Big(t, \gamma_i(t, u(t)), \frac{\mathrm{d}}{\mathrm{d}t}\gamma_i(t, u(t))\Big)\mathrm{d}t = \int_0^R t^{N-1}f(t, \alpha_i(t), \alpha_i'(t))\,\mathrm{d}t \\ &\ge -\int_0^R (t^{N-1}\varphi_M(\alpha_i'(t)))'\,\mathrm{d}t = -R^{N-1}\varphi_M(\alpha_i'(R) \ge 0,\end{aligned}$$

as α_i satisfies (2.9) and $\alpha_i'(R) \le 0$, thus yielding the contradiction

$$0 \ge (1 - \lambda)r = \lambda\mathcal{PH}(u) > 0.$$

Similarly, we can verify that $r < D$. Consequently, the homotopy invariance of the degree implies that

$$\deg(\mathcal{I} - \mathcal{T}, \mathcal{O}) = 1.$$

In particular, $\mathcal{T}$ has a fixed point $u \in X$ which in turn is a solution of (2.11).

Degree calculations

Assume now that the sub-solution α and the super-solution β are strict. We first show that if u is a solution of (1.1) with $\alpha \le u \le \beta$, then u satisfies $\alpha \ll u \ll \beta$. We already know that $\alpha \le u \le \beta$ and, in particular, we have $\varphi_M(u') = \varphi(u')$ as $M > \tilde{L} > \|u'\|_\infty$. We shall limit ourselves to prove that u satisfies $u \gg \alpha_i$ for each $i \in \{1,\dots, p\}$, as showing that $u \ll \beta_j$ for each $j \in \{1,\dots, q\}$ is totally similar. Fix $i \in \{1,\dots, p\}$ and set $v = u - \alpha_i$. Let us suppose, by contradiction, that $v \not\gg 0$. Hence, there exists $t_0 \in [0, R]$ such that $v(t_0) = \min v = 0$. By definition of sub-solution, we immediately realize that $v'(t_0) = 0$ and thus $\alpha_i(t_0) = u(t_0)$ and $\alpha_i'(t_0) = u'(t_0)$. In particular, if $t_0 = 0$ or $t_0 = R$, the definition of solution yields $\alpha_i'(t_0) = 0$. Recalling now the definition of strict sub-solution we observe that, if $t_0 = R$, only the third or the fourth alternative in Definition 1.1 may occur. Hence, if $0 < t_0 \le R$, we get

$$\begin{aligned}-\varphi_M'(u'(t_0))v''(t_0) &= -\varphi_M'(u'(t_0))u''(t_0) - \frac{N-1}{t_0}\varphi_M(u'(t_0)) + \varphi_M'(\alpha_i'(t_0))\alpha_i''(t_0) + \frac{N-1}{t_0}\varphi_M(\alpha_i'(t_0)) \\ &> f(t_0, u(t_0), u'(t_0)) - f(t_0, \alpha_i(t_0), \alpha_i'(t_0)) = 0.\end{aligned}$$

On the other hand, if $t_0 = 0$, only the second or the fourth alternative in Definition 1.1 may occur and we find

$$-Nv''(0) = -N\varphi_M'(0)v''(0) > f(0, u(0), 0) - f(0, \alpha_i(0), 0) = 0.$$

In all cases we infer $v''(t_0) < 0$, but this is impossible at a minimum point which is simultaneously a critical point.

Next we define in X the open bounded set

$$\mathcal{U} = \{u \in X : \alpha \ll u \ll \beta,\ \|u'\|_\infty < C\}.$$

It is clear that $\mathcal{U} \subseteq O$, where O is defined in (2.13). Since any fixed point u of $\mathcal{T}$ is a solution of (2.11), it satisfies

$$\alpha \ll u \ll \beta, \quad \|u'\|_\infty < C,$$

and hence $O \setminus \bar{\mathcal{U}}$ is fixed point free. The excision property of the degree thus yields

$$\deg(\mathcal{I} - \mathcal{T}, \mathcal{U}) = \deg(\mathcal{I} - \mathcal{T}, O) = 1.$$

2.2.2 The Dirichlet and the Robin problems

Assume now $a_0 > 0$ in (2.7), i.e., consider either the Dirichlet problem

$$\begin{cases} -(\varphi_M(u')t^{N-1})' = t^{N-1}\mathcal{H}(u) & \text{in }]0, R[, \\ u'(0) = 0, u(R) = 0, \end{cases}$$

in case $a_1 = 0$, or the Robin problem

$$\begin{cases} -(\varphi_M(u')t^{N-1})' = t^{N-1}\mathcal{H}(u) & \text{in }]0, R[, \\ u'(0) = 0, a_0 u(R) + a_1 u'(R) = 0, \end{cases}$$

in case $a_1 > 0$.

Existence of solutions

Much like in the Neumann case, we set

$$X = \{u \in C^1([0, R]) : u'(0) = 0\}$$

and we endow X with the norm defined by (2.12). We also introduce the operator $\mathcal{T} : X \to X$ by setting

$$\mathcal{T}(u)(t) = \frac{a_1}{a_0}\varphi_M^{-1}\left(\int_0^R \left(\frac{\sigma}{R}\right)^{N-1} \mathcal{H}(u)(\sigma)\,\mathrm{d}\sigma\right) + \int_t^R \varphi_M^{-1}\left(\int_0^s \left(\frac{\sigma}{s}\right)^{N-1} \mathcal{H}(u)(\sigma)\,\mathrm{d}\sigma\right)\mathrm{d}s \quad \text{for all } t \in [0, R]. \tag{77}$$

The fixed points $u \in X$ of $\mathcal{T}$ are precisely the solutions of (2.7). It is easy to see that $\mathcal{T}$ is completely continuous and there exists a constant $C > \tilde{L}$ such that

$$\|\mathcal{T}(u)\|_{W^{2,\infty}} < C, \quad \text{for all } u \in X.$$

Define in X the open ball

$$O = \{u \in X : \|u\|_X < C\}.$$

Since $\mathcal{T}(\bar{O}) \subseteq O$, the Schauder fixed point theorem implies the existence of a fixed point $u \in O$, which in turn is a solution of (2.7). Moreover, a simple homotopy argument yields

$$\deg(I - \mathcal{T}, O) = 1.$$

Degree calculations

We first show that if u is a solution of (1.1) with $\alpha \le u \le \beta$, then u satisfies $\alpha \ll u \ll \beta$, keeping the notation of Section 2.2.1. From the definition of strict sub-solution, we infer that $v(t) > 0$ for all $t \in [0, R[$. Assume that $v(R) = 0$. Since $v(R) = \min v$, we get either $v'(R) = 0$ or $v'(R) < 0$. In the former situation, a contradiction follows arguing as we did in the Neumann case. In the latter situation, we automatically have that $v \gg 0$. The rest of the proof then proceeds exactly in the same way as for the Neumann problem.

2.3 Multiplicity of solutions

2.3.1 Degree calculations

In this section, we consider the Neumann case and the Dirichlet and Robin cases simultaneously. In what follows, the symbols X, C, $\mathcal{T}$ refer to the symbols introduced in Section 2.2.1, in the case of the Neumann problem, or in Section 2.2.2, in the case of the Dirichlet and the Robin problems. Let us consider the modified problem (2.7). Since all solutions u of (2.7) satisfy $\alpha \le u \le \beta$, it is clear that $\alpha - 1 \ll u \ll \beta + 1$. Let us introduce the following open bounded subsets of X

$$\begin{aligned}\mathcal{U}_0 &= \{u \in X : \alpha - 1 \ll u \ll \beta + 1 \text{ and } \|u'\|_\infty < C\},\\ \mathcal{U}_1 &= \{u \in X : \alpha - 1 \ll u \ll \beta_1 \text{ and } \|u'\|_\infty < C\},\\ \mathcal{U}_2 &= \{u \in X : \alpha_1 \ll u \ll \beta + 1 \text{ and } \|u'\|_\infty < C\}.\end{aligned}$$

Note that $\mathcal{U}_1 \subset \mathcal{U}_0$, $\mathcal{U}_2 \subset \mathcal{U}_0$, and, as $\alpha_1 \nleq \beta_1$, $\mathcal{U}_1 \cap \mathcal{U}_2 = \varnothing$. Let $u \in X$ be a fixed point of $\mathcal{T}$. Then it satisfies $\alpha - 1 \ll u \ll \beta + 1$. Suppose further that $u \le \beta_1$. Since β_1 is a strict super-solution of the problem (1.1), we actually have $u \ll \beta_1$, as shown in Sections 2.2.1 and 2.2.2 when computing the degree. Similarly, if $u \ge \alpha_1$, since α_1 is a strict sub-solution of the problem (1.1), we actually have $u \gg \alpha_1$. Therefore, we conclude that

$$0 \notin (\mathcal{I} - \mathcal{T})(\partial\mathcal{U}_0 \cup \partial\mathcal{U}_1 \cup \partial\mathcal{U}_2). \tag{2.14}$$

Define now the open bounded subset of X

$$\mathcal{U}_3 = \mathcal{U}_0 \backslash \overline{\mathcal{U}_1 \cup \mathcal{U}_2}.$$

By (2.14), using the excision property of the degree, we get

$$\deg(\mathcal{I} - \mathcal{T}, \mathcal{U}_0) = \deg(\mathcal{I} - \mathcal{T}, \mathcal{U}_0 \backslash (\partial\mathcal{U}_1 \cup \partial\mathcal{U}_2))$$

and, hence, the additivity property of the degree implies that

$$\deg(\mathcal{I} - \mathcal{T}, \mathcal{U}_0) = \deg(\mathcal{I} - \mathcal{T}, \mathcal{U}_1) + \deg(\mathcal{I} - \mathcal{T}, \mathcal{U}_2) + \deg(\mathcal{I} - \mathcal{T}, \mathcal{U}_3).$$

Since we have that

$$\deg(\mathcal{I} - \mathcal{T}, \mathcal{U}_0) = \deg(\mathcal{I} - \mathcal{T}, \mathcal{U}_1) = \deg(\mathcal{I} - \mathcal{T}, \mathcal{U}_2) = 1,$$

we finally get

$$\deg(\mathcal{I} - \mathcal{T}, \mathcal{U}_3) = -1.$$

2.3.2 Existence of solutions

Since $\mathcal{U}_1$, $\mathcal{U}_2$, $\mathcal{U}_3$ are mutually disjoint, the previous degree calculations imply that there are three distinct fixed points u_1, u_2, u_3 of the operator $\mathcal{T}$, with

$$u_1 \in \mathcal{U}_1, \quad u_2 \in \mathcal{U}_2, \quad u_3 \in \mathcal{U}_3.$$

Since any fixed point u of $\mathcal{T}$ satisfies $\alpha \le u \le \beta$, u_1, u_2, u_3 are all solutions of (1.1). Moreover, we have that

$$\alpha \le u_1 \ll \beta_1, \quad \alpha_1 \ll u_2 \le \beta, \quad u_3 \ngeq \alpha_1, \quad u_3 \nleq \beta_1.$$

Since $\min\{u_3, \beta_1\}$ is a super-solution of (1.1), there exists a solution $\tilde{u}_1$ of (1.1) satisfying

$$\alpha \le \tilde{u}_1 \le \min\{u_3, \beta_1\} < u_3.$$

Similarly, as $\max\{u_3, \alpha_1\}$ is a sub-solution of (1.1), there exists a solution $\tilde{u}_2$ of (1.1) satisfying

$$u_3 < \max\{u_3, \alpha_1\} \le \tilde{u}_2 \le \beta.$$

Then, replacing u_1 with $\tilde{u}_1$ and u_2 with $\tilde{u}_2$, we can conclude that (1.1) has three solutions such that

$$u_1 < u_3 < u_2.$$

3 Complementary results

In this section, we provide some additional results, complementing Theorems 1.1 and 1.2, where alternative bounds are imposed on the function f, involving, unlike assumption (h_3), the length of the interval $[0, R]$. We shall mainly focus on the Neumann problem as, in the case of Dirichlet or Robin boundary conditions, a suitable boundedness assumption on f is enough to guarantee alone the existence of solutions. In the last part, we briefly discuss also the periodic problem, which is closely related to the one-dimensional Neumann problem.

3.1 The Neumann problem

In this section, we consider the Neumann problem:

$$\begin{cases} -\left(\dfrac{t^{N-1}u'}{\sqrt{1+u'^2}}\right)' = t^{N-1}f(t,u,u') \quad \text{in }]0,R[, \\ u'(0) = 0,\ u'(R) = 0. \end{cases} \tag{3.1}$$

It has already been observed in [46, Lemma 2.5] that a global bound on f yields an estimate on the derivative of any solution of (3.1) and eventually, in [46, Theorem 2.6], the solvability of (3.1) if an asymptotic sign condition holds. The next result provides instead the existence of a solution of (3.1) assuming a local bound on f with a sharper constant, together with the existence of a couple of sub- and super-solutions satisfying the standard ordering condition. In the light of the non-existence results in [50] the bound on the function f is optimal in dimension $N = 1$.

Proposition 3.1. *Assume (h_1), (h_2)*
(h_6) *there exist an interval $I \subseteq \mathbb{R}$ and a constant K, with*

$$0 < K < \frac{N\sqrt[N]{2}}{R},$$

such that, for all $t \in [0, R]$, $s \in I$ and $\xi \in \mathbb{R}$,

$$|f(t,s,\xi)| \le K \tag{3.2}$$

and
(h_7) *there exist a sub-solution α and a super-solution β of the problem* (3.1) *satisfying*

$$\alpha \le \beta \text{ and } \operatorname{range}\alpha,\ \operatorname{range}\beta \subseteq I.$$

Then, the problem (3.1) *has at least one classical solution u, with*

$$\alpha \le u \le \beta.$$

Proof. Suppose u is a solution of the problem (3.1) with range $u \subseteq I$. If $t \in]0, \frac{R}{\sqrt[N]{2}}]$, integrating the equation in (3.1) between 0 and t, we find

$$|\varphi(u'(t))| \le t^{1-N}\int_0^t s^{N-1}|f(s,u(s),u'(s))|\,ds \le K\frac{t}{N} \le K\frac{R}{N\sqrt[N]{2}} < 1,$$

whereas, if $t \in \,]\frac{R}{\sqrt[N]{2}}, R[$, integrating between t and R, we get

$$|\varphi(u'(t))| \leq t^{1-N} \int_t^R s^{N-1}|f(s, u(s), u'(s))|\mathrm{d}s \leq K\frac{R^N - t^N}{Nt^{N-1}} \leq K\frac{R}{N\sqrt[N]{2}} < 1.$$

Set

$$L = \varphi^{-1}\Big(K\frac{R}{N\sqrt[N]{2}}\Big). \tag{3.3}$$

Then any solution u of the problem (3.1), with range $u \subseteq I$, satisfies the estimate

$$\|u'\|_\infty \leq L. \tag{3.4}$$

Next we proceed as in the proof of Theorem 1.1, by introducing, for any $M > 0$, the function φ_M as in (2.2), the constant $\tilde{L}$ as in (2.4), the truncated function h as in (2.5), the operator $\mathcal{H}$ as in (2.6) and the modified problems (2.10) and (2.7) with $a_0 = 0$. As in that proof we show that a solution u of the modified problem (2.7) exists and is a solution of the problem (2.10) too. To conclude we observe that the function h satisfies assumption (h_6) and therefore, arguing as in the first part of this proof, the estimate (3.4) holds. Hence, u is a solution of (3.1), provided that $M > \tilde{L}$. □

In the following result, we consider the case where α and β do not satisfy the standard ordering condition. In this situation, we need to ask the bound in (3.2) to be global, in the sense that $I = \mathbb{R}$.

Proposition 3.2. *Assume* (h_1), (h_2), (h_6) *with* $I = \mathbb{R}$, *and*
(h_8) *there exist a sub-solution* α *and a super-solution* β *of the problem* (3.1) *satisfying*

$$\alpha \nleq \beta.$$

Then, the problem (3.1) *has at least one classical solution* u, *with*

$$u(t_1) \leq \alpha(t_1) \quad \text{and} \quad u(t_2) \geq \beta(t_2) \quad \text{for some} \;\; t_1, t_2 \in [0, R].$$

Proof. We define L as in (3.3) and

$$D = \max\{\|\alpha_i\|_{C^1}, \|\beta_j\|_{C^1} : i = 1,\ldots, p, j = 1,\ldots, q\} + L(1 + R).$$

As noted in the proof of Proposition 3.1, condition (h_6) implies that any solution u of the problem (3.1) satisfies (3.4). Therefore, all solutions u of (3.1) such that $u(t_1) \leq \alpha(t_1)$ and $u(t_2) \geq \beta(t_2)$, for some $t_1, t_2 \in [0, R]$, satisfy

$$\|u\|_{C^1} \leq \|\alpha\|_\infty + \|\beta\|_\infty + (1 + R)\|u'\|_\infty \leq D. \tag{3.5}$$

Choose $\tilde{L} > D$ and define a continuous function $\eta : \mathbb{R}^2 \to [0, 1]$ satisfying

$$\eta(s, \xi) = \begin{cases} 1 & \text{if } (s, \xi) \in [-D, D]^2, \\ 0 & \text{if } (s, \xi) \in \mathbb{R}^2 \backslash [-\tilde{L}, \tilde{L}]^2. \end{cases}$$

Choose any ε, with $0 < \varepsilon < \frac{2}{\pi}K$, and define the function $h : [0, R] \times \mathbb{R}^2 \to \mathbb{R}$ by

$$h(t, s, \xi) = \eta(s, \xi)f(t, s, \xi) - \varepsilon \arctan^+(s - \tilde{L}) + \varepsilon \arctan^-(s + \tilde{L}).$$

Consider the Neumann problem

$$\begin{cases} -\left(\dfrac{t^{N-1}u'}{\sqrt{1 + u'^2}}\right)' = t^{N-1}h(t, u, u') \;\; \text{in } \;]0, R[, \\ u'(0) = 0, u'(R) = 0. \end{cases} \tag{3.6}$$

Since for all $t \in [0, R]$

$$h(t, \alpha(t), \alpha'(t)) = f(t, \alpha(t), \alpha'(t)) \quad \text{and} \quad h(t, \beta(t), \beta'(t)) = f(t, \beta(t), \beta'(t)),$$

α is a sub-solution and β is a super-solution of the problem (3.6). Furthermore, as for all $t \in [0, R]$:

$$h(t, -\tilde{L} - 1, 0) = \varepsilon\frac{\pi}{4} \quad \text{and} \quad h(t, \tilde{L} + 1, 0) = -\varepsilon\frac{\pi}{4},$$

the constant function $\alpha_0 = -\tilde{L} - 1$ is a strict sub-solution and the constant function $\beta_0 = \tilde{L} + 1$ is a strict super-solution of the problem (3.6). We have that

$$\alpha_0 \le \alpha \le \beta_0, \quad \alpha_0 \le \beta \le \beta_0 \quad \text{and} \quad \alpha \nleq \beta.$$

Let us distinguish three possible cases.

We first suppose that there exists a solution u of (3.6) such that $u \ge \alpha$ but $u \not\gg \alpha$. In this case, there exists $t_1 \in [0, R]$ with $u(t_1) = \alpha(t_1)$. As $\alpha \nleq \beta$ there exists also $t_2 \in [0, R]$ with $u(t_2) \ge \beta(t_2)$.

Next, we suppose that there exists a solution u of (3.6) such that $u \le \beta$ but $u \not\ll \beta$. In this case, there exists $t_2 \in [0, R]$ with $u(t_2) = \beta(t_2)$. As $\alpha \nleq \beta$ there exists also $t_1 \in [0, R]$ with $u(t_1) \le \alpha(t_1)$.

Finally, we suppose that, for any solution u of (3.6), if $u \ge \alpha$, then $u \gg \alpha$, and if $u \le \beta$, then $u \ll \beta$. In this case, degree calculations similar to those performed in Sections 2.2 and 2.3 yield the existence of a solution u of (3.6) such that $u \ngeq \alpha$ and $u \nleq \beta$, that is, $u(t_1) < \alpha(t_1)$ and $u(t_2) > \beta(t_2)$ for some $t_1, t_2 \in [0, R]$.

To conclude we observe that the function h satisfies assumption (h_6) with $I = \mathbb{R}$ and, therefore, arguing as in the first part of this proof, the estimate (3.5) holds and, hence, u is a solution of (3.1) as well. □

3.2 The Dirichlet and the Robin problems

In this section, we consider the Dirichlet and the Robin problems:

$$\begin{cases} -\left(\dfrac{t^{N-1}u'}{\sqrt{1+u'^2}}\right)' = t^{N-1}f(t, u, u') & \text{in }]0, R[, \\ u'(0) = 0,\ a_0 u(R) + a_1 u'(R) = 0, \end{cases} \tag{3.7}$$

with $a_0 > 0$. In this case, if f satisfies a bound similar to (3.2), which yields an estimate on the derivative, the solvability of (3.7) is always guaranteed, without requiring the existence of sub- and super-solutions.

Proposition 3.3. *Assume* (h_1), *with* $a_0 > 0$, (h_2) *and*
(h_9) *there exists a constant* K, *with*

$$0 < K < \frac{N}{R}$$

such that, for all $t \in [0, R]$, $s \in \mathbb{R}$ *and* $\xi \in \mathbb{R}$,

$$|f(t, s, \xi)| \le K.$$

Then, the problem (3.7) *has at least one classical solution* u.

Proof. Let us consider, depending on $\lambda \in [0, 1]$, the problem

$$\begin{cases} -\left(\dfrac{t^{N-1}u'}{\sqrt{1+u'^2}}\right)' = t^{N-1}\lambda f(t, u, u') & \text{in }]0, R[, \\ u'(0) = 0,\ a_0 u(R) + a_1 u'(R) = 0, \end{cases} \tag{3.8}$$

with $a_0 > 0$. Suppose u is a solution of (3.8) for some $\lambda \in [0, 1]$. As in the proof of Proposition 3.1, integrating the equation in (3.8) between 0 and any $t \in]0, R]$, we find

$$|\varphi(u'(t))| \leq K\frac{R}{N} < 1.$$

Using the boundary condition we then infer the existence of a constant $C > 0$ such that

$$\|u\|_{C^1} < C,$$

for any such u and λ. A simple degree argument then yields the existence of a solution of the problem (3.7). □

3.3 The periodic problem

We discuss in this section the existence of classical periodic solutions of the equation:

$$-\left(\frac{u'}{\sqrt{1+u'^2}}\right)' = f(t, u, u'). \tag{3.9}$$

For the sake of simplicity and to point out the symmetry with the previous sections, we shall denote the assumptions considered in this part with the symbols (h_i'), $i = 1, 2, 3 \ldots$. It is assumed throughout this section that
(h_1') $T > 0$ *is a given period*
and
(h_2') $f : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is continuous and T-periodic with respect to the first variable.*

Definition 3.1. (Notion of solution) A classical T-periodic solution of (3.9) is a function $u \in C^2(\mathbb{R})$, which is T-periodic and satisfies the equation for all $t \in \mathbb{R}$.

We note that, if $u \in C^1(\mathbb{R})$ is any T-periodic function, then there exists $t_0 \in [0, T[$ such that $u'(t_0) = 0$. Therefore, if u is any T-periodic solution of (3.9), then, for some t_0, u satisfies the Neumann boundary conditions $u'(t_0) = u'(t_0 + T) = 0$. Consequently, all gradient estimates established in the previous sections for the one-dimensional Neumann problem extend to the periodic case. Taking this fact into account, we omit the proofs of the following propositions, as they parallel those of Theorems 1.1, 1.2 and Propositions 3.1, 3.2. Before stating our results we need to introduce the notion of sub-solution and super-solution for the periodic problem.

Definitions 3.1. (Notion of sub- and super-solution).

- A function $\alpha : [0, T] \to \mathbb{R}$ is a sub-solution of the T-periodic problem for (3.9) if there exist $\alpha_1, \ldots, \alpha_p \in C^2([0, T])$ such that $\alpha = \max\{\alpha_1, \ldots, \alpha_p\}$ and for each $i \in \{1, \ldots, p\}$

$$\begin{cases} -\left(\dfrac{\alpha_i'(t)}{\sqrt{1+\alpha_i'(t)^2}}\right)' \leq f(t, \alpha_i(t), \alpha_i'(t)) \quad \text{for all } t \in]0, T[, \\ \alpha_i(0) = \alpha_i(T),\ \alpha_i'(0) \geq \alpha_i'(T). \end{cases}$$

- A sub-solution α is strict if for each $i \in \{1, \ldots, p\}$ either

$$\begin{cases} -\left(\dfrac{\alpha_i'(t)}{\sqrt{1+\alpha_i'(t)^2}}\right)' < f(t, \alpha_i, (t)\alpha_i'(t)) \quad \text{for all } t \in]0, T[, \\ \alpha_i'(0) > \alpha_i'(T) \end{cases}$$

or

$$-\left(\frac{\alpha_i'(t)}{\sqrt{1+\alpha_i'(t)^2}}\right)' < f(t,\alpha_i(t),\alpha_i'(t)) \quad \text{for all } t \in [0,T].$$

– A function $\beta : [0,T] \to \mathbb{R}$ is a super-solution of the T-periodic problem for (3.9) if there exist $\beta_1,\dots,\beta_q \in C^2([0,T])$ such that $\beta = \min\{\beta_1,\dots,\beta_q\}$ and for each $j \in \{1,\dots,q\}$

$$\begin{cases} -\left(\frac{\beta_j'(t)}{\sqrt{1+\beta_j'(t)^2}}\right)' \geq f(t,\beta_j(t),\beta_j'(t)) \quad \text{for all } t \in]0,T[, \\ \beta_i(0)=\beta_i(T),\, \beta_i'(0) \leq \beta_i'(T). \end{cases}$$

– A super-solution β is strict if for each $j \in \{1,\dots,q\}$ either

$$\begin{cases} -\left(\frac{\beta_j'(t)}{\sqrt{1+\beta_j'(t)^2}}\right)' > f(t,\beta_j(t),\beta_j'(t)) \quad \text{for all } t \in]0,T[, \\ \beta_i'(0) < \beta_i'(T) \end{cases}$$

or

$$-\left(\frac{\beta_j'(t)}{\sqrt{1+\beta_j'(t)^2}}\right)' > f(t,\beta_j,(t)\beta_j'(t)) \quad \text{for all } t \in [0,T].$$

Notation. For all $u, v \in C^1([0,T])$ satisfying $u(0)=u(T)$ and $v(0)=v(T)$, we write $u \leq v$ if $u(t) \leq v(t)$ for all $t \in [0,T]$, $u < v$ if $u \leq v$ and $u \neq v$, $u \ll v$ if $u(t) < v(t)$ for all $t \in [0,T]$.

Proposition 3.4. *Assume* (h_1'), (h_2') *and*
(h_3') *there exist an interval* $I \subseteq \mathbb{R}$ *and a continuous function* $g : I \to \mathbb{R}$ *such that, for all* $t \in \mathbb{R}$, $s \in I$ *and* $\xi \in \mathbb{R}$,

$$f(t,s,\xi)\cdot \operatorname{sgn}(\xi) \geq -g(s) \quad \textit{and} \quad \int_I g(s)\,\mathrm{d}s < 1$$

and
(h_4') *there exist a sub-solution* α *and a super-solution* β *of the* T*-periodic problem for* (3.9) *satisfying*

$$\alpha \leq \beta \quad \textit{and} \quad \text{range}\,\alpha, \text{range}\,\beta \subseteq I.$$

Then, Eq. (3.9) *has at least one classical T-periodic solution u, with*

$$\alpha \leq u \leq \beta.$$

Proposition 3.5. *Assume* (h_1'), (h_2'), (h_3'), (h_4') *and*
(h_5') *there exist a strict sub-solution* α_1 *and a strict super-solution* β_1 *of the T- periodic problem for* (3.9) *satisfying*

$$\alpha \leq \alpha_1, \quad \beta_1 \leq \beta \quad \textit{and} \quad \alpha_1 \nleq \beta_1.$$

Then, Eq. (3.9) *has at least three classical T-periodic solutions* u_1, u_2, u_3, *with*

$$u_1 < u_3 < u_2, \quad \alpha \leq u_1 \ll \beta_1, \quad \alpha_1 \ll u_2 \leq \beta_1, \quad u_3 \ngeq \alpha_1 \quad \textit{and} \quad u_3 \nleq \beta_1.$$

Proposition 3.6. *Assume* (h_1'), (h_2'),

(h_6') *there exist an interval* $I \subseteq \mathbb{R}$ *and a constant* K, *with*

$$0 < K < \frac{2}{T},$$

such that, for all $t \in [0, R]$, $s \in I$ *and* $\xi \in \mathbb{R}$,

$$|f(t, s, \xi)| \leq K$$

and

(h_7') *there exist a sub-solution* α *and a super-solution* β *of the* T*-periodic problem for* (3.9) *satisfying*

$$\alpha \leq \beta \quad \textit{and} \quad \text{range}\,\alpha, \text{range}\,\beta \subseteq I.$$

Then, Eq. (3.9) *has at least one classical* T*-periodic solution* u, *with*

$$\alpha \leq u \leq \beta.$$

Proposition 3.7. *Assume* (h_1'), (h_2'), (h_6'), *with* $I = \mathbb{R}$, *and*

(h_8') *there exist a sub-solution* α *and a super-solution* β *of the* T*-periodic problem for* (3.9) *satisfying*

$$\alpha \nleq \beta.$$

Then, Eq. (3.9) *has at least one classical* T*-periodic solution* u, *with*

$$u(t_1) \leq \alpha(t_1) \quad \textit{and} \quad u(t_2) \geq \beta(t_2) \quad \textit{for some} \;\; t_1, t_2 \in [0, R].$$

Propositions 3.4–3.7, also in combination with Remark 1.1, complete and extend several results previously obtained in [29,37,40,44,45,48,49].

Acknowledgments: This research has been performed under the auspices of "Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni – Istituto Nazionale di Alta Matematica." The authors have been supported by "Università degli Studi di Trieste – Finanziamento di Ateneo per Progetti di Ricerca Scientifica – FRA 2018, FRA 2020."

References

[1] C. Corsato, C. De Coster, N. Flora, and P. Omari, *Radial solutions of the Dirichlet problem for a class of quasilinear elliptic equations arising in optometry*, Nonlinear Anal. **181** (2019), 9–23.
[2] R. Finn, *Remarks relevant to minimal surfaces, and to surfaces of prescribed mean curvature*, J. Analyse Math. **14** (1965), 139–160.
[3] H. Jenkins and J. Serrin, *The Dirichlet problem for the minimal surface equation in higher dimensions*, J. Reine Angew. Math. **229** (1968), 170–187.
[4] J. Serrin, *The problem of Dirichlet for quasilinear elliptic differential equations with many independent variables*, Philos. Trans. Roy. Soc. London Ser. A **264** (1969), 413–496.
[5] O. A. Ladyzhenskaya and N. N. Ural'tseva, *Local estimates for gradients of solutions of non-uniformly elliptic and parabolic equations*, Comm. Pure Appl. Math. **23** (1970), 677–703.
[6] E. Bombieri, E. De Giorgi, and E. Giusti, *Minimal cones and the Bernstein problem*, Invent. Math. **7** (1969), 243–268.
[7] E. Bombieri, E. De Giorgi, and M. Miranda, *Una maggiorazione a priori relativa alle ipersuperfici minimali non parametriche*, Arch. Rational Mech. Anal. **32** (1969), 255–267.
[8] R. Temam, *Solutions généralisées de certaines équations du type hypersurfaces minima*, Arch. Rational Mech. Anal. **44** (1971), no. 72, 121–156.
[9] E. Giusti, *Minimal Surfaces and Functions of Bounded Variation*, Birkhäuser Verlag, Basel, 1984.
[10] U. Massari and M. Miranda, *Minimal Surfaces of Codimension One*, North-Holland Publishing Co., Amsterdam, 1984.
[11] R. Finn, *Equilibrium Capillary Surfaces*, Springer-Verlag, New York, 1986.
[12] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin, 2001.

[13] H. Amann, *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM Rev. **18** (1976), no. 4, 620–709.
[14] V. K. Le and K. Schmitt, *Sub-supersolution theorems for quasilinear elliptic problems: a variational approach*, Electron. J. Differ. Equ. **118** (2004), 7.
[15] V. K. Le and K. Schmitt, *Some general concepts of sub- and supersolutions for nonlinear elliptic problems*, Topol. Methods Nonlinear Anal. **28** (2006), no. 1, 87–103.
[16] C. De Coster and P. Habets, *Two-Point Boundary Value Problems: Lower and Upper Solutions*, Elsevier B.V., Amsterdam, 2006.
[17] K. Schmitt, *Revisiting the method of sub- and supersolutions for nonlinear elliptic problems*, Proceedings of the Sixth Mississippi State–UBA Conference on Differential Equations and Computational Simulations, vol. 15 of Electron. J. Differ. Equ. Conf., Southwest Texas State Univ., San Marcos, TX, 2007, pp. 377–385.
[18] E. S. Noussair, C. A. Swanson, and J. F. Yang, *A barrier method for mean curvature problems*, Nonlinear Anal. **21** (1993), no. 8, 631–641.
[19] H. Pan and R. Xing, *Sub- and supersolution methods for prescribed mean curvature equations with Dirichlet boundary conditions*, J. Differ. Equ. **254** (2013), no. 3, 1464–1499.
[20] E. S. Noussair and C. A. Swanson, *Positive solutions of the prescribed mean curvature equation in* $\boldsymbol{R}^N$, Indiana Univ. Math. J. **42** (1993), no. 2, 559–570.
[21] E. Giusti, *Superfici cartesiane di area minima*, Rend. Sem. Mat. Fis. Milano **40** (1970), 135–153.
[22] M. Emmer, *Esistenza, unicità e regolarità nelle superfici di equilibrio nei capillari*, Ann. Univ. Ferrara Sez. VII (N.S.) **18** (1973), 79–94.
[23] M. Giaquinta, *On the Dirichlet problem for surfaces of prescribed mean curvature*, Manuscripta Math. **12** (1974), 73–86.
[24] M. Miranda, *Dirichlet problem with* L^1 *data for the non-homogeneous minimal surface equation*, Indiana Univ. Math. J. **24** (1974), no. 75, 227–241.
[25] E. Giusti, *Boundary value problems for non-parametric surfaces of prescribed mean curvature*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **4** (1976), no. 3, 501–548.
[26] C. Gerhardt, *Boundary value problems for surfaces of prescribed mean curvature*, J. Math. Pures Appl. **58** (1979), no. 1, 75–109.
[27] V. K. Le, *On a sub-supersolution method for the prescribed mean curvature problem*, Czechoslovak Math. J. **58** (2008), no. 2, 541–560.
[28] F. Obersnel and P. Omari, *Existence and multiplicity results for the prescribed mean curvature equation via lower and upper solutions*, Differ. Integral Equ. **22** (2009), no. 9–10, 853–880.
[29] F. Obersnel, P. Omari, and S. Rivetti, *Existence, regularity and stability properties of periodic solutions of a capillarity equation in the presence of lower and upper solutions*, Nonlinear Anal. Real World Appl. **13** (2012), no. 6, 2830–2852.
[30] G. Anzellotti, *The Euler equation for functionals with linear growth*, Trans. Amer. Math. Soc. **290** (1985), no. 2, 483–501.
[31] W.-M. Ni and J. Serrin, *Nonexistence theorems for quasilinear partial differential equations,* Proceedings of the Conference commemorating the 1st Centennial of the Circolo Matematico di Palermo (Palermo, 1984), Rend. Circ. Mat. Palermo **8** (1985), no. 2, 171–185.
[32] J. Serrin, *Positive solutions of a prescribed mean curvature problem*, Proceedings of the Conference on Calculus of Variations and Partial Differential Equations (Trento, 1986), vol. 1340 of Lecture Notes in Math., Springer-Verlag, Berlin, 1988, pp. 248–255.
[33] P. Clément, R. Manásevich, and E. Mitidieri, *On a modified capillary equation*, J. Differ. Equ. **124** (1996), no. 2, 343–358.
[34] C. Bereanu, P. Jebelean, and J. Mawhin, *Radial solutions for some nonlinear problems involving mean curvature operators in Euclidean and Minkowski spaces*, Proc. Amer. Math. Soc. **137** (2009), no. 1, 161–169.
[35] C. Corsato, C. De Coster, F. Obersnel, P. Omari, and A. Soranzo, *A prescribed anisotropic mean curvature equation modeling the corneal shape: a paradigm of nonlinear analysis*, Discrete Contin. Dyn. Syst. Ser. S **11** (2018), no. 2, 213–256.
[36] C. Corsato, C. De Coster, F. Obersnel, and P. Omari, *Qualitative analysis of a curvature equation modelling MEMS with vertical loads*, Nonlinear Anal. Real World Appl. **55** (2020), 103123.
[37] F. Obersnel and P. Omari, *Multiple bounded variation solutions of a periodically perturbed sine-curvature equation*, Commun. Contemp. Math. **13** (2011), no. 5, 863–883.
[38] C. Corsato, P. Omari, and F. Zanolin, *Subharmonic solutions of the prescribed curvature equation*, Commun. Contemp. Math. **18** (2016), no. 3, 1550042.
[39] F. Obersnel and P. Omari, *Positive solutions of the Dirichlet problem for the prescribed mean curvature equation*, J. Differ. Equ. **249** (2010), no. 7, 1674–1725.
[40] F. Obersnel and P. Omari, *Multiple bounded variation solutions of a capillarity problem*, Discrete Contin. Dyn. Syst. (Dynamical systems, differential equations and applications. 8th AIMS Conference. Suppl. Vol. II) (2011), 1129–1137.
[41] P. Omari and E. Sovrano, *Positive solutions of indefinite logistic growth models with flux-saturated diffusion*, Nonlinear Anal. **201** (2020), 111949.
[42] H. Amann, *Existence and multiplicity theorems for semi-linear elliptic boundary value problems*, Math. Z. **150** (1976), no. 3, 281–295.

[43] C. De Coster, F. Obersnel, and P. Omari, *A qualitative analysis, via lower and upper solutions, of first order periodic evolutionary equations with lack of uniqueness*, in: Handbook of Differential Equations: Ordinary Differential Equations. Vol. III, Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam, 2006, pp. 203–339.

[44] C. Bereanu and J. Mawhin, *Boundary-value problems with non-surjective ϕ-Laplacian and one-sided bounded nonlinearity*, Adv. Differ. Equ. **11** (2006), no. 1, 35–60.

[45] C. Bereanu and J. Mawhin, *Multiple periodic solutions of ordinary differential equations with bounded nonlinearities and ϕ-Laplacian*, NoDEA Nonlinear Differ. Equ. Appl. **15** (2008), no. 1–2, 159–168.

[46] C. Bereanu, P. Jebelean, and J. Mawhin, *Radial solutions for Neumann problems involving mean curvature operators in Euclidean and Minkowski spaces*, Math. Nachr. **283** (2010), no. 3, 379–391.

[47] C. Bereanu, P. Jebelean, and J. Mawhin, *Radial solutions for Neumann problems with ϕ-Laplacians and pendulum-like nonlinearities*, Discrete Contin. Dyn. Syst. **28** (2010), no. 2, 637–648.

[48] J. A. Cid, *On the existence of periodic oscillations for pendulum-type equations*, Adv. Nonlinear Anal. **10** (2021), no. 1, 121–130.

[49] A. Fonda and R. Toader, *A dynamical approach to lower and upper solutions for planar systems*, preprint (2020).

[50] F. Obersnel, P. Omari, and S. Rivetti, *Asymmetric Poincaré inequalities and solvability of capillarity problems*, J. Funct. Anal. **267** (2014), no. 3, 842–900.