

## A SURVEY ON ASYMPTOTIC STABILITY OF GROUND STATES OF NONLINEAR SCHRÖDINGER EQUATIONS II

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ABSTRACT. We give short survey on the question of asymptotic stability of ground states of nonlinear Schrödinger equations, focusing primarily on the so called nonlinear Fermi Golden Rule.

**1. Introduction.** In 2004 one of us authored a survey [26] on the asymptotic stability of ground states of the nonlinear Schrödinger equation (NLS). Since then there has been considerable progress on this topic, so that it is worthwhile to write a review with some updates.

For  $d \geq 1$ , we consider the NLS

$$i\partial_t u = -\Delta u + \beta(|u|^2)u, \quad u|_{t=0} = u_0 \in H^1(\mathbb{R}^d, \mathbb{C}), \quad (1)$$

where  $\beta \in C^\infty(\mathbb{R}, \mathbb{R})$  satisfies, for  $d^* = \infty$  for  $d = 1, 2$  and  $d^* = \frac{d+2}{d-2}$  for  $d \geq 3$ ,

$$|\partial_t^n (\beta(t^2)t)| \leq C_n t^{p-n} \text{ for } t \geq 1, n \geq 0 \text{ and for a } p < d^*. \quad (2)$$

This guarantees that the Cauchy problem (1) is locally well posed, see Cazenave [20].

We are concerned with a spatially localized solution called soliton. In particular, we assume there exists an open interval  $\mathcal{O} \subset (0, \infty)$  such that

$$\Delta u - \omega u - \beta(|u|^2)u = 0 \quad \text{for } x \in \mathbb{R}^d, \quad (3)$$

admits a  $C^\infty$ -family of ground states  $\mathcal{O} \ni \omega \mapsto \phi_\omega \in H_{\text{rad}}^1(\mathbb{R}^d)$  with  $\phi_\omega(x) > 0$  everywhere. In fact, under these hypotheses, we have  $\phi_\omega \in C^\infty(\mathbb{R}^d)$  and

$$|\partial_x^\alpha \phi_\omega(x)| \leq C_{\alpha, \omega} (1 + |x|)^{-\frac{d-1}{2}} e^{-\sqrt{\omega}|x|} \text{ for all multiindexes } \alpha. \quad (4)$$

Then  $e^{i(\frac{1}{2}v \cdot x - \frac{1}{4}|v|^2 t + t\omega + \vartheta)} \phi_\omega(x - vt - D)$ , for any choice of  $(\omega, \vartheta, v, D) \in \mathcal{O} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$ , are solitonic solutions of the NLS. An important question is whether these ground state solutions are stable. A first notion of stability is the following.

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2020 *Mathematics Subject Classification.* Primary: 35Q55; secondary: 35B40.

*Key words and phrases.* Ground state, stability, dispersion, normal form, metastability.

S.C. was supported by grant FRA 2018 from the University of Trieste; M.M. was supported by the JSPS KAKENHI Grant Number 19K03579, JP17H02851 and JP17H02853.

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**Definition 1.1** (Orbital stability). A ground state  $\phi_\omega$  of (10) is *orbitally stable* if  $\forall \epsilon > 0, \exists \delta > 0$  s.t.  $\|\phi_\omega - u_0\|_{H^1} < \delta \Rightarrow \sup_{t>0} \inf_{(\vartheta, D) \in \mathbb{R} \times \mathbb{R}^d} \|e^{i\vartheta} \phi_\omega(\cdot + D) - u(t)\|_{H^1} < \epsilon$ , where  $u$  is the solution of the NLS with  $u(0) = u_0$ .

The literature on this is large, see the survey papers [47, 124] and the references therein.

**Theorem 1.2.** *If for  $\omega \in \mathcal{O}$  both the two conditions (H1), (H2) listed below are satisfied, then the corresponding ground state is orbitally stable:*

- (H1)  $\ker L_{+, \omega} \cap H_{\text{rad}}^1(\mathbb{R}^d) = \{0\}$  for the operator  $L_{+, \omega} := -\Delta + \omega + \beta(\phi_\omega^2) + 2\beta'(\phi_\omega^2)\phi_\omega^2$ ;  
 (H2) we have the Vakhitov–Kolokolov condition  $q'(\omega) > 0$ , where  $q(\omega) := Q(\phi_\omega)$ .

The study of equilibria and of solitons of NLS's or of more complex models and their orbital stability is not the topic of this paper. The notion of orbital stability applies also to other functions. For example, if  $B(t) \geq 0$  for  $t \geq 0$ , where  $B(t)$  is an antiderivative of  $\beta(t)$ , then energy and mass conservation and Gagliardo–Nirenberg inequalities imply in an elementary fashion the orbital stability of the 0 solution. Less elementary is the following fact. If 0 is stable and

$$d \geq 2 \text{ and for } d = 1 \text{ furthermore } \beta'(0) = 0, \quad (5)$$

then there is an  $\epsilon_0 > 0$  s.t.

$$\|u_0\|_{H^1(\mathbb{R}^d)} < \epsilon_0 \implies \exists u_+ \in H^1(\mathbb{R}^d) \text{ s.t. } \|u(t) - e^{it\Delta} u_+\|_{H^1} \xrightarrow{t \rightarrow +\infty} 0. \quad (6)$$

The theme of the present paper is an analogue of (6) in the case of solitons of the NLS. Specifically, we will give an outline of some of the most basic ideas behind the following analogous rough statement, which we call the asymptotic stability of solitons.

**Theorem 1.3.** *Let  $d \geq 3$ . Let  $\omega_1 \in \mathcal{O}$  satisfy the two conditions listed in Theorem 1.2. Then, under further hypotheses, which the authors of this review believe to hold generically, there exists  $\epsilon_1 > 0$  s.t. for any  $u_0 \in B_{H^1}(\phi_{\omega_1}, \epsilon_1) := \{v \in H^1 \mid \|v - \phi_{\omega_1}\| < \epsilon_1\}$  there exist  $\omega_+ \in \mathcal{O}$ ,  $v_+ \in \mathbb{R}^d$  and  $(\vartheta, D) \in C^0([0, +\infty), \mathbb{R} \times \mathbb{R}^d)$  s.t. the solution of the NLS with  $u(0) = u_0$  satisfies*

$$\|u(t) - e^{i\vartheta(t) + \frac{1}{2}v_+ \cdot x} \phi_{\omega_+}(\cdot - D(t)) - e^{it\Delta} h_+\|_{H^1(\mathbb{R}^d)} \xrightarrow{t \rightarrow +\infty} 0. \quad (7)$$

**Remark 1.4.** For dimensions 1 and 2 the same theorem is known to be true only under conditions that break translation, as when  $u_0$  is an even function or there is an additional translation breaking inhomogeneity in (10), like a linear potential. The proof in the case with translation is an open problem.

**Remark 1.5.** In dimension 1, well known is the case when  $\beta(|u|^2)u = -|u|^2u$ , where it is possible to apply methods from the theory of integrable systems [12, 117], which require  $u_0$  s.t.  $\langle x \rangle^s u_0 \in L^2(\mathbb{R})$  for  $s > 1/2$ , see [41].

**Remark 1.6.** The additional hypotheses required are (H3) (see Theorem 2.2), (H4)–(H7) in Sect. 4 and (H8) under (61). The most delicate condition in (H8) requires that the terms in (61) be non zero. This happens when the Fourier transform of certain functions has nonzero restriction on certain spheres of phase space. When  $\beta$  is real analytic, then the dependence of the coefficients on  $\omega$  is analytic.

The generic condition has not been proved rigorously, except in very special situations, see [86, 16, 1]. Even the question of checking numerically the generic condition seems to have attracted very little interest.

Theorem 1.3 has a long history. The theory was initiated by Soffer and Weinstein [120, 121] for small solitons bifurcating from linear potential, see also [115], followed by the important paper by Buslaev and Perelman [17] which proved the asymptotic stability for the case  $d = 1$ . Both of [120, 121] and [17] considers the case where the linearized operator have no non-zero eigenvalues (a recent paper involving this situation is [99]). The basic idea of these works is to divide the solution into a soliton part and a remainder part by modulation argument and then prove the decay of the remainder part by the dispersive properties of the linearized operator. The remainder is small and satisfies a complicated equation that looks like a NLS. The linear part of the equation of the remainder, has continuous spectrum and eigenvalues.

Very early the literature provided a theory of the dispersive properties of the continuous part of the linearized operator. In dimension  $d = 1$  this is in [17], which can be supplemented with Krieger and Schlag [90], see also [56]. The case  $d \geq 3$  is in [29], which has to be supplemented with [42], and  $d = 2$  in [44]. More effective use of dispersion, of Strichartz estimates and especially of the endpoint Strichartz estimate in  $d \geq 3$ , see Keel and Tao [76], is in Gustafson, Nakanishi and Tsai [61]. Smoothing estimates as a surrogate of the endpoint Strichartz estimate when  $d = 1, 2$  are in Mizumachi [103, 104]. A substantial simplification of Mizumachi's smoothing estimates is in [45]. Other early contributions are [79, 81]. Obviously, dispersion is a hard problem in the presence of strong nonlinearities, where one cannot hope to prove dispersive properties of the remainder just by Strichartz estimates, and here the literature is not as rich. Remarkable nonetheless are [87, 88] as well as the very recent [89].

While, to some extent, linear dispersion of the continuous mode was understood quite early, it took some time to understand how to treat the nonzero eigenvalues of the linearized operator. The starting point seems to be Sigal [119] which, for a different problem, showed the existence of a nonlinear damping mechanism by which the discrete modes lose energy which, by nonlinear interaction, spills in the continuous part of the equation and then scatters by essentially linear mechanisms. Sigal called this damping mechanism “nonlinear Fermi Golden Rule” (FGR). The first successful implementation of this idea in our context was obtained by Buslaev and Perelman [18]. For almost 15 years there was no major improvement on this part of the proof in [18]. Here we recall that [18] treats the case where there is just a single  $\mathbf{e}(\omega) \in (0, \omega)$  of multiplicity 1 of the linearization operator  $\mathcal{H}_\omega$ , with  $2\mathbf{e}(\omega) > \omega$ . Later Soffer and Weinstein [122] developed a similar idea in the context of the NLKG equation. See also [110]. Various papers were written in the early '00 [19, 128, 129, 130, 127, 30, 123, 57, 58] articulating the idea. A novelty was in Gang Zhou and Sigal [55], with still just one eigenvalue but with 2 replaced by  $N + 1$  for  $N \in \mathbb{N}$ , see also [40]. However, all these rather restrictive conditions on the spectrum of the linearized operator  $\mathcal{H}_\omega$ , were finally lifted only with [6, 27] around 2010. These papers introduced a more natural framework for a problem that, approached from a different viewpoint, could look impossibly complex, as can be seen, for example, by tracing the argument in [54]. It should be remarked, that quite independently from the theory we are discussing here, Perelman [114] and Merle and Raphael [100] exploited a form of FGR in their masterly analysis

of the  $\|\nabla u(t)\|_{L^2(\mathbb{R}^d)} \sim \frac{\sqrt{|\log|\log t|}}{t}$  blow up in the NLS with  $\beta(|u|^2)u = -|u|^{\frac{4}{d}}u$ . The connections between the two theories have not been explored yet, although [39] exploits ideas originating from the work of Merle and Raphael to simplify considerably the proof of the result in [32].

The paper [27] considers equations without translation. Translation was later and independently introduced in [28] and [5]. However, there are aspects of the proof, which is rather long and with many details, that have been finalized in later papers, such as [31]. See also [7] for some more on [5].

In this paper we will just focus on the FGR. The generic conditions in Theorem 1.3 pertain to the FGR. As we mentioned, there is very little numerical work on them. The fact that a certain quadratic form is non-negative, is explained later. Strict positivity is unproven, theoretically as well as numerically. Numerical simulations are certainly not simplified by the fact that the crucial quadratic form is obtained after a rather complex sequence of coordinate changes. The coordinate changes are not discussed in this paper.

The aim of this survey is to give some basic intuition of the main ideas of the proof of Theorem 1.3, skipping completely on the most technical parts of the proof.

**2. Theorem 1.3 in the absence of nonzero eigenvalues.** We embed  $\mathbb{C} \hookrightarrow \mathbb{C}^2$  using the natural identification

$$\mathbb{C} \ni u \mapsto \tilde{u} := \begin{pmatrix} u \\ \bar{u} \end{pmatrix} \in \tilde{\mathbb{C}} := \left\{ \begin{pmatrix} z \\ \bar{z} \end{pmatrix} \in \mathbb{C}^2 : z \in \mathbb{C} \right\} \subset \mathbb{C}^2. \tag{8}$$

Here we set  $\langle U, V \rangle_{\mathbb{C}^2} := 2^{-1}(u_1v_1 + u_2v_2)$  for  $U = {}^t(u_1 \ u_2)$  and  $V = {}^t(v_1 \ v_2)$  in  $\mathbb{C}^2$ . By this definition,  $\langle \tilde{u}, \sigma_1 \tilde{v} \rangle_{\mathbb{C}^2} = \operatorname{Re} u\bar{v}$ , and in particular  $\langle \tilde{u}, \sigma_1 \tilde{u} \rangle_{\mathbb{C}^2} = |u|^2$ , where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{9}$$

Armed with this, we can equivalently write the NLS as

$$i\sigma_3 \partial_t \tilde{u} = -\Delta \tilde{u} + \beta(\langle \tilde{u}, \sigma_1 \tilde{u} \rangle_{\mathbb{C}^2}) \tilde{u}, \quad \tilde{u}|_{t=0} = \tilde{u}_0 \in H^1(\mathbb{R}^d, \tilde{\mathbb{C}}). \tag{10}$$

With the above definition of  $\langle \cdot, \cdot \rangle_{\mathbb{C}^2}$ , we define

$$\langle U, V \rangle := \int_{\mathbb{R}^3} \langle U(x), V(x) \rangle_{\mathbb{C}^2} dx \text{ for } U, V \in L^2(\mathbb{R}^3, \mathbb{C}^2) \tag{11}$$

(we emphasize, that here there is no complex conjugation). In  $L^2(\mathbb{R}^d, \tilde{\mathbb{C}})$  we consider the symplectic form  $\Omega$ , defined by

$$\Omega(X, Y) = i\langle X, \sigma_3 \sigma_1 Y \rangle \text{ for all } X, Y \in L^2(\mathbb{R}^d, \tilde{\mathbb{C}}). \tag{12}$$

Given a function  $F \in C^1(U, \mathbb{R})$ , with  $U$  an open subset of  $H^1(\mathbb{R}^d, \tilde{\mathbb{C}})$ , we denote by  $dF(u)$  the Fréchet derivative of  $F$ , and by  $\nabla F(\tilde{u})$  its gradient, defined by  $dF(\tilde{u}) = \langle \sigma_1 \nabla F(\tilde{u}), \cdot \rangle$ . The Hamiltonian vector-field  $X_F$  of  $F$  associated to  $\Omega$  is defined by  $\Omega(X_F, \cdot) = dF$ , that is  $X_F = -i\sigma_3 \nabla F$ .

If we consider, for  $B(0) = 0$  the primitive  $B' = \beta$ , the energy

$$E(\tilde{u}) := \frac{1}{2} \langle (-\Delta) \tilde{u}, \sigma_1 \tilde{u} \rangle + \frac{1}{2} \int_{\mathbb{R}^3} B(\langle \tilde{u}, \sigma_1 \tilde{u} \rangle_{\mathbb{C}^2}) dx, \tag{13}$$

then  $\nabla E(\tilde{u}) = -\Delta \tilde{u} + \beta(\langle \tilde{u}, \sigma_1 \tilde{u} \rangle_{\mathbb{C}^2}) \tilde{u}$  and (10) can be interpreted as  $\partial_t \tilde{u} = X_E(\tilde{u})$ .

Notice that  $E \in C^2 \left( H^1(\mathbb{R}^d, \tilde{\mathbb{C}}), \mathbb{R} \right)$  with

$$\begin{aligned} \nabla^2 E(\tilde{u})\tilde{X} &:= \frac{d}{dt} \nabla E(\tilde{u} + t\tilde{X})|_{t=0} \\ &= -\Delta\tilde{X} + \beta(\langle \tilde{u}, \sigma_1 \tilde{u} \rangle_{\mathbb{C}^2})\tilde{X} + 2\beta'(\langle \tilde{u}, \sigma_1 \tilde{u} \rangle_{\mathbb{C}^2}) \left\langle \tilde{u}, \sigma_1 \tilde{X} \right\rangle_{\mathbb{C}^2} \tilde{u} \\ &= -\Delta\tilde{X} + \beta(\langle \tilde{u}, \sigma_1 \tilde{u} \rangle_{\mathbb{C}^2})\tilde{X} + \beta'(\langle \tilde{u}, \sigma_1 \tilde{u} \rangle_{\mathbb{C}^2})|u|^2\tilde{X} + \beta'(\langle \tilde{u}, \sigma_1 \tilde{u} \rangle_{\mathbb{C}^2}) \begin{pmatrix} 0 & u^2 \\ \bar{u}^2 & 0 \end{pmatrix} \tilde{X}. \end{aligned} \tag{14}$$

We define also quadratic forms  $P_j(u) := 2^{-1} \langle \diamond_j \tilde{u}, \sigma_1 \tilde{u} \rangle$  for  $j = 0, 1, \dots, d$ , which are invariant by gauge and translation symmetries, with

$$Q(u) = P_0(u) \text{ for } \diamond_0 := 1 \text{ the mass and } P_a(u) \text{ for } \diamond_a := -\sigma_3 i \partial_a, \ a = 1, \dots, d, \tag{15}$$

the linear momenta.

Here we extend the hypotheses in Theorem 1.2, and assume that  $q'(\omega) > 0$  for all  $\omega \in \mathcal{O}$ , which can be assumed, if necessary, restricting  $\mathcal{O}$ . Under such assumption, the map  $(\omega, v) \rightarrow p = \Pi(e^{\sigma_3 \frac{1}{2} v \cdot x} \phi_\omega)$  is a diffeomorphism into an open subset  $\mathcal{P}$  of  $\mathbb{R}^d$ . This uses also  $\Pi_a(e^{\sigma_3 \frac{1}{2} v \cdot x} u) = \Pi_a(u) + 2^{-1} v_a Q(u)$  for  $a = 1, \dots, d$ . For  $p = p(\omega, v) \in \mathcal{P}$  set  $\Phi_p = e^{\sigma_3 \frac{1}{2} v \cdot x} \tilde{\phi}_\omega$ . The  $\Phi_p$  are constrained critical points of  $E$  with associated Lagrange multipliers  $\lambda(p) \in \mathbb{R}^{d+1}$  so that

$$\nabla E(e^{i\sigma_3 \tau \cdot \diamond} \Phi_p) = \lambda(p) \cdot \diamond e^{i\sigma_3 \tau \cdot \diamond} \Phi_p, \tag{16}$$

where we have

$$\lambda_0(p) = -\omega(p) - 4^{-1} v^2(p), \quad \lambda_a(p) := v_a(p) \text{ for } a = 1, \dots, d. \tag{17}$$

We now introduce the linearization, for  $(\omega, v) = (\omega(p), v(p))$ ,

$$\mathcal{H}_p := \sigma_3(\nabla^2 E(\Phi_p) - \lambda(p) \cdot \diamond) = \sigma_3(-\Delta + \omega + 4^{-1} v^2 + i v \cdot \nabla) + V_p \tag{18}$$

$$\text{where } V_p := \sigma_3 \left[ \beta(\phi_{\omega(p)}^2) + \beta'(\phi_{\omega(p)}^2) \phi_{\omega(p)}^2 \right] + i \sigma_2 \beta'(\phi_{\omega(p)}^2) \phi_{\omega(p)}^2 e^{-\sigma_3 \frac{1}{2} v(p) \cdot x},$$

which can be computed from (14). By an abuse of notation, we set

$$\mathcal{H}_\omega := \mathcal{H}_p \text{ when } v(p) = 0 \text{ and } \omega(p) = \omega. \tag{19}$$

It is easy that  $\mathcal{H}_p = e^{\sigma_3 \frac{1}{2} v(p) \cdot x} \mathcal{H}_{\omega(p)} e^{-\sigma_3 \frac{1}{2} v(p) \cdot x}$ , so that the spectrum of  $\mathcal{H}_p$  depends only on  $\omega(p)$ .

Hypothesis (H2) of Theorem 1.2 guarantee that the map  $p \rightarrow \lambda(p)$  is a local diffeomorphism and, in particular, it is invertible. In [134] it is shown that Hypothesis (H1) of Theorem 1.2 implies the following:

$$\ker \mathcal{H}_p = \text{Span}\{\sigma_3 \diamond_j \Phi_p : j = 0, \dots, d\} \text{ and} \tag{20}$$

$$N_g(\mathcal{H}_p) = \text{Span}\{\sigma_3 \diamond_j \Phi_p, \partial_{\lambda_j} \Phi_p : j = 0, \dots, d\}, \tag{21}$$

where  $N_g(L) := \cup_{j=1}^\infty \ker(L^j)$ . Notice that the  $\supseteq$  in (20) follows immediately differentiating in  $\tau$  the identity (16) while the opposite inclusion is a much harder proposition, which rests on  $\ker L_+ \cap H_{\text{rad}}^1(\mathbb{R}^d) = \{0\}$ . Setting  $\tau = 0$  in (16) and differentiating in  $\lambda_j$ , we obtain the  $\supseteq$  in (21). The  $\subseteq$  in (21) follows from (20), the fact that the correspondence  $p \longleftrightarrow \lambda$  is a diffeomorphism (this, in turn a consequence of  $q'(\omega) > 0$  for all  $\omega \in \mathcal{O}$ ), from Fredholm alternative and from

$$\delta_{jk} = \partial_{p_k} p_j = 2^{-1} \partial_{p_k} \langle \diamond_j \Phi_p, \Phi_p \rangle = \langle \diamond_j \Phi_p, \partial_{p_k} \Phi_p \rangle. \tag{22}$$

We have the decomposition

$$L^2(\mathbb{R}^d, \mathbb{C}^2) = N_g(\mathcal{H}_p) \oplus N_g^\perp(\mathcal{H}_p^*), \tag{23}$$

$$N_g(\mathcal{H}_p^*) = \text{Span}\{\diamond_j \Phi_p, \sigma_3 \partial_{\lambda_j} \Phi_p : j = 0, \dots, d\}. \tag{24}$$

Set  $P_{N_g}(p) = P_{N_g(\mathcal{H}_p)}$  for the projection on  $N_g(\mathcal{H}_p)$  and  $P(p) := 1 - P_{N_g}(p)$ . Notice that

$$P_{N_g}(p)X = \sum_{j=0}^d (\sigma_3 \diamond_j \Phi_p \langle \sigma_1 X, \sigma_3 \partial_{p_j} \Phi_p \rangle + \partial_{p_j} \Phi_p \langle \sigma_1 X, \diamond_j \Phi_p \rangle). \tag{25}$$

Then we have the following Modulation Lemma, which originates with Soffer and Weinstein [120].

**Lemma 2.1** (Modulation). *Fix  $p_1 \in \mathcal{P}$ . Then there exists a neighborhood  $\mathcal{U}$  of  $\Phi_{p_1}$  in  $H^1(\mathbb{R}^d, \mathbb{C})$  and functions  $p \in C^\infty(\mathcal{U}, \mathcal{P})$  and  $\tau \in C^\infty(\mathcal{U}, \mathbb{R}^{d+1})$  s.t.  $p(\Phi_{p_1}) = p_1$  and  $\tau(\Phi_{p_1}) = 0$  and s.t.  $\forall u \in \mathcal{U}$*

$$u = e^{i\sigma_3 \tau \cdot \diamond}(\Phi_p + R) \text{ and } R \in N_g^\perp(\mathcal{L}_p^*). \tag{26}$$

We write (10) as  $i\dot{u} = \nabla E(\tilde{u})$ . Using (26),  $\nabla E(\Phi_p) = \lambda(p) \cdot \diamond \Phi_p$ , the definition of  $\mathcal{H}_p$  and for  $O(R^2)$  is non-linear in  $R$ , we obtain

$$-(\dot{\tau} - \lambda(p)) \cdot \sigma_3 \diamond \Phi_p + i \dot{p} \cdot \partial_p \Phi_p - (\dot{\tau} - \lambda(p)) \cdot \sigma_3 \diamond R + i\dot{R} = \mathcal{H}_p R + O(R^2). \tag{27}$$

Applying  $P_{N_g}(p)$  to (27), summing on repeated indexes, we obtain the Modulation Equations

$$\begin{aligned} \dot{\tau}_k - \lambda_k + (\dot{\tau} - \lambda) \cdot \langle \diamond R, \partial_{p_k} \Phi_p \rangle - \dot{p} \cdot \langle i\sigma_1 R, \sigma_3 \partial_{p_k} \partial_p \Phi_p \rangle &= \langle O(R^2), \partial_{p_k} \Phi_p \rangle \\ \dot{p}_k - (\dot{\tau} - \lambda) \cdot \langle i\sigma_3 \sigma_1 \diamond R, \diamond_k \Phi_p \rangle - \dot{p} \cdot \langle R, \diamond_k \partial_p \Phi_p \rangle &= \langle O(R^2), \diamond_k \Phi_p \rangle, \end{aligned} \tag{28}$$

which need to be coupled with the following equation on  $R$ , obtained applying  $P(p)$  to (27),

$$i\dot{R} - \mathcal{H}_p R = (\dot{\tau} - \lambda(p)) \cdot P(p)\sigma_3 \diamond R + i\dot{p} P(p)\partial_p P(p)R + O(R^2). \tag{29}$$

Equation (29) resembles a vectorial-like NLS. Soffer and Weinstein in [120, 121], for a somewhat simpler system, had the idea to use the dispersive properties of the linearized equation  $i\dot{R} - \mathcal{H}_p R = 0$ . Instrumental where advances in the dispersion theory of Schrödinger due to Journé, Soffer and Sogge [74]. Buslaev and Perelman in [17] for dimension 1 extended these results to the operator  $\mathcal{H}_p$  (the analysis in [17] can be supplemented by material in [90]) introduced the idea of proving dispersion to 0 of  $R$  by exploiting the dispersive properties of the group  $e^{it\mathcal{H}_p}$ . Specifically, Buslaev and Perelman in [17] prove the following result.

**Theorem 2.2** (Main Theorem in [17]). *In the  $d = 1$  dimension, suppose, in addition to the hypotheses in Theorem 1.2 that for  $\omega \in \mathcal{O}$  both the two conditions listed below are satisfied:*

- (1)  $0$  is the only eigenvalue of  $\mathcal{H}_\omega$ ;
- (H3) the points  $\pm\omega$  are not resonances of for  $\mathcal{H}_\omega$ .

Then for any  $\omega_0 \in \mathcal{O}$  there exists  $\epsilon_0 > 0$  and  $C_0 > 0$  s.t. for

$$\| \langle x \rangle^2 (u_0 - \phi_{\omega_0}) \|_{L^2} + \| \partial_x (u_0 - \phi_{\omega_0}) \|_{L^2} < \epsilon_0$$

we have

$$\begin{aligned} \|R(t)\|_{L^\infty} &< C_0(1 + |t|)^{-1/2}\epsilon_0, \\ |\dot{\tau}(t) - \lambda(p(t))| + |\dot{p}(t)| &< C_0(1 + |t|)^{-3}\epsilon_0^2. \end{aligned}$$

In [29] there is a version of the above result for dimension  $d \geq 3$ , here quoted as is stated in [29].

**Theorem 2.3.** *For  $d \geq 3$ , assume the hypotheses of Theorem 2.2. Then for any  $u_0 \in \mathcal{O}$  there exist  $\epsilon_0 > 0$  and  $C_0 > 0$  s.t., for*

$$\|\langle x \rangle (u_0 - \phi_{\omega_0})\|_{H^{2d+2[d/2]+2}} < \epsilon_0 \tag{30}$$

$$\|u_0 - \phi_{\omega_0}\|_{H^{2d+2[d/2]+3} \cap W^{d+2[d/2]+2,1}} < \epsilon_0, \tag{31}$$

we have

$$\|R(t)\|_{W^{d+[d/2]+1,\infty}} < C_0(1 + |t|)^{-d/2}\epsilon_0, \tag{32}$$

$$|\dot{\tau}(t) - \lambda(p(t))| + |\dot{p}(t)| < C_0(1 + |t|)^{-d}\epsilon_0^2. \tag{33}$$

Like in [120, 121, 18], crucial to [29] is information on the dispersion of the associated linearized evolution  $e^{it\mathcal{H}_\omega}$ . In fact, [29] contains the following theorem for  $d \geq 3$ , based on work by Yajima [135, 136, 3], see also Weder, [132, 133], which was inspired by Journé, Soffer and Sogge [74]. The case  $d = 2$  is in [44].

**Theorem 2.4.** *For  $d \geq 2$ , for any  $\omega \in \mathcal{O}$  if, under the hypotheses (1) and (2) of Theorem 2.2, we set  $L^q(\omega) = L^q(\mathbb{R}^d, \mathbb{C}) \cap N_g^\perp(\mathcal{L}_\omega^*)$ , then strong limits*

$$W(\omega) := s - \lim_{t \rightarrow +\infty} e^{it\mathcal{H}_\omega} e^{it\sigma_3(\Delta - \omega)} \text{ and } Z(\omega) := s - \lim_{t \rightarrow +\infty} e^{it\sigma_3(-\Delta + \omega)} e^{-it\mathcal{H}_\omega} \tag{34}$$

define isomorphisms  $L^q(\mathbb{R}^d, \mathbb{C}) \xrightarrow{W(\omega)} L^q(\omega)$  and  $L^q(\omega) \xrightarrow{Z(\omega)} L^q(\mathbb{R}^d, \mathbb{C})$  for any  $q \in [1, \infty]$  and yield isomorphisms also between to Sobolev spaces  $W^{s,q}(\mathbb{R}^d, \mathbb{C})$  and  $W^{s,q}(\mathbb{R}^d, \mathbb{C}) \cap N_g^\perp(\mathcal{H}_\omega^*)$  for any  $s \in \mathbb{R}$ . Furthermore, the norms of the operators are upper semicontinuous in  $\omega$ .

**Remark 2.5.** Unfortunately, in [29] the proof of the case  $q = 2$ , specifically [29, Corollary 3.2], is wrong. However, the correct proof is a rather direct consequence of classical arguments by Kato [75], and is in [42].

The proof of Theorem 2.3 involves applying  $\mathcal{H}_p^j$  to (29) and then applying to it  $\langle \cdot, \sigma_1 \sigma_3 \mathcal{H}_p \mathcal{H}_p^j R \rangle$  to get

$$2^{-1} \partial_t \sum_{j=0}^m \langle \mathcal{H}_p^j R, \sigma_1 \sigma_3 \mathcal{H}_p \mathcal{H}_p^j R \rangle = \sum_{j=0}^m \langle \mathcal{H}_p^j \text{ r.h.s. of (29), } \sigma_1 \sigma_3 \mathcal{H}_p \mathcal{H}_p^j R \rangle,$$

for a sufficiently large  $m$ . The summation in the l.h.s. is equivalent to  $\|R\|_{H^{2m+1}}^2$ . This is proved by induction from  $\langle R, \sigma_1 \sigma_3 R \rangle \sim \|R\|_{H^1}^2$  for  $R \in N_g^\perp(\mathcal{L}_p^*)$ , which is true under the hypotheses of Theorem 2.3, see [134, 42]. By standard argument

$$\|R(t)\|_{H^{2m+1}}^2 \leq C \|R(0)\|_{H^{2m+1}}^2 e^{\int_0^t (|\dot{\tau}(t') - \lambda(p(t'))| + |\dot{p}(t')| + \|R(t')\|_{W^{m,\infty}}) dt'}. \tag{35}$$

This needs to be used in conjunction with estimates of the terms in the exponential. To this effect, we need to use Theorem 2.4. Quite problematic is the term  $(\dot{\tau} - \lambda(p)) \cdot P(p) \sigma_3 \diamond R$  in (29), as we will see also in Sect. 4.1. Here we sketch the discussion in [29], which comes from [17]. In an interval  $[0, T]$ , we consider  $\dot{\tau}_1 = \lambda(p(T))$  with

$\tau_1(0) = \tau(0)$  and define  $R_1$  by  $R_1 = e^{i\sigma_3(\tau-\tau_1)\cdot\diamond} R$ . Then, elementary computations yield

$$\begin{aligned}
 P(p(T))R_1(t) &= e^{it\mathcal{H}_{p(T)}} P(p(T))R_1(0) \\
 &- i \int_0^t P(p(T))e^{i(t-t')\mathcal{H}_{p(T)}} (\dot{\tau} - \lambda(p)) \left( e^{i\sigma_3(\tau-\tau_1)\cdot\diamond} P(p)e^{-i\sigma_3(\tau-\tau_1)\cdot\diamond} - 1 \right) \cdot \sigma_3 \diamond R dt' \\
 &- i \int_0^t P(p(T))e^{i(t-t')\mathcal{H}_{p(T)}} \left( e^{i\sigma_3(\tau-\tau_1)\cdot\diamond} V_p e^{-i\sigma_3(\tau-\tau_1)\cdot\diamond} - V_{p(T)} \right) R_1 dt' + \dots
 \end{aligned} \tag{36}$$

Notice that, assuming (33),

$$|\tau - \tau_1| \leq \int_0^t |\lambda(p) - \lambda(p(T))| dt' \lesssim \int_0^t dt' \int_{t'}^T |\dot{p}| ds \lesssim \epsilon_0^2 \int_0^t dt' \int_{t'}^T \langle s \rangle^{-2d} ds \lesssim \epsilon_0^2.$$

This implies, by

$$P(p(T))R_1(t) - R_1 = \left[ P(p(T)), e^{i\sigma_3(\tau-\tau_1)\cdot\diamond} \right] R + e^{i\sigma_3(\tau-\tau_1)\cdot\diamond} (P(p(T)) - P(p)) R,$$

that, assuming (32), then the r.h.s. in the last equation is small, and so  $P(p(T))R_1(t) \sim R_1(t)$ . In turn, by (2.4), which implies

$$\|e^{it\mathcal{H}_{p(T)}} P(p(T))R_1(0)\|_{L^\infty} \lesssim \langle t \rangle^{-\frac{d}{2}} \|R_1(0)\|_{L^1 \cap L^2} \text{ where } \langle t \rangle := \sqrt{1 + |t|^2}$$

we get, also from  $\|R_1(0)\|_{L^1 \cap L^2} \lesssim \epsilon_0$  and from  $\langle t \rangle^{-\frac{d}{2}} * \langle t \rangle^{-\frac{d}{2}} \lesssim \langle t \rangle^{-\frac{d}{2}}$ ,

$$\|P(p(T))R_1(t)\|_{L^\infty} \lesssim \langle t \rangle^{-\frac{d}{2}} \epsilon_0 + \langle t \rangle^{-\frac{d}{2}} \epsilon_0^2.$$

This because the terms in the last two lines of (36), being nonlinear, are smaller than the 1st term in the r.h.s. of (36). Taking derivatives, one gets back (32). Inserting this in the modulation equations (28), one proves (33). This of course is just a caricature, but the rigorous argument is similar, assuming the (32)–(33) with some large constant  $C_0$  and then proving, by taking  $\epsilon_0$  sufficiently small, that the constant can be taken to be similar to

$$\sup_{\omega \in \mathcal{O}, t \geq 0} \|\langle t \rangle^{d/2} e^{it\mathcal{H}_\omega} P(p(\omega, 0))\|_{L_x^1 \cap L_x^2 \rightarrow L_x^\infty} < \infty.$$

Notice that (30) is unnecessary (and is due to a non optimal choice in [17, 29] of the modulation).  $\square$

Theorems 2.2 and 2.3 are based in a significant way on the fact that 0 is the only eigenvalue of  $\mathcal{H}_\omega$ .

**Lemma 2.6.** *If we drop the hypothesis (1) in Theorems 2.2 and 2.3, then the conclusions of these theorems are false.*

*Proof (sketch).* Buslaev and Perelman showed, in [18], see also [19], that the continuous spectrum component of  $R(t)$  decays slowly if  $\mathcal{H}_\omega$  has exactly one eigenvalue  $\mathbf{e}(\omega) \in (0, \omega)$ . We give a sketch of this, assuming for simplicity that  $u \in C^0(\mathbb{R}, H_{\text{rad}}^1(\mathbb{R}^d))$ , thus excluding translations. Let us suppose that  $N\lambda(\omega) < \omega < (N + 1)\mathbf{e}(\omega)$ , for an  $N \in \mathbb{N}$ , and let  $\ker(\mathcal{H}_\omega - \mathbf{e}(\omega))$  be generated by an appropriately normalized  $\xi_\omega$ . Then, using the symmetry of  $\sigma(\mathcal{H}_\omega)$  with respect to the coordinate axes and the fact that  $\sigma(\mathcal{H}_\omega) = \sigma(\mathcal{H}_\omega^*)$ ,

$$\begin{aligned}
 r(t) &= z(t)\xi_{\omega(t)} + \bar{z}(t)\sigma_1\xi_{\omega(t)} + f(t) \text{ with} \\
 f(t) &\in \left( N_g(\mathcal{H}_{\omega(t)}) \oplus \ker(\mathcal{H}_{\omega(t)}^* - \mathbf{e}(\omega(t))) \oplus \ker(\mathcal{H}_{\omega(t)}^* + \mathbf{e}(\omega(t))) \right)^\perp.
 \end{aligned} \tag{37}$$

Then, in [18] for the case  $N = 1$  and in [55] for generic  $N$ , it is shown that after a normal forms argument, for  $P(|z|^2)$  real valued we have

$$\begin{aligned} i\dot{z} - \mathbf{e}z &= P(|z|^2)z + \bar{z}^N \langle f, \sigma_1 G(\omega) \rangle_{L_x^2} + \dots \\ i\dot{f} - \mathcal{H}_\omega f &= z^{N+1}M(\omega) + \dots \end{aligned} \tag{38}$$

If, in the equation for  $z$ , we substitute  $f$  with  $-z^{N+1} \lim_{\varepsilon \rightarrow 0^+} R_{\mathcal{H}_\omega}((N+1)\mathbf{e}(\omega) + i\varepsilon)M$ , where the latter exists, see Proposition 3.12 later, and use formula

$$\lim_{\varepsilon \rightarrow 0^+} R_{\mathcal{H}_\omega}^+(\kappa + i\varepsilon) = P.V. \frac{1}{\mathcal{H}_\omega - \kappa} - i\pi\delta(\mathcal{H}_\omega - \kappa) \text{ for } \kappa \in \mathbb{R}, \tag{39}$$

which can be understood of the theory of distorted plane waves, but which we will not discuss in any detail, then the equation of  $z$  becomes

$$\begin{aligned} i\dot{z} - \mathbf{e}(\omega)z &= P(|z|^2)z - |z|^{2N} z \langle P.V. \frac{1}{\mathcal{H}_\omega - (N+1)\mathbf{e}(\omega)} M, \sigma_1 G \rangle_{L_x^2} \\ &\quad - i|z|^{2N} z \pi \langle \delta(\mathcal{H}_\omega - (N+1)\mathbf{e}(\omega))M, \sigma_1 G \rangle. \end{aligned}$$

Multiplying by  $\bar{z}$  and taking imaginary part, it can be shown that

$$\Rightarrow \frac{d}{dt}|z|^2 = -|z|^{2N+2}\Gamma(\omega) \text{ where } \Gamma(\omega) := 2\pi \langle \delta(\mathcal{H}_\omega - (N+1)\mathbf{e}(\omega))M, \sigma_1 G \rangle.$$

Notice that, using an appropriate distorted Fourier transform associated to  $\mathcal{H}_\omega$ , we have

$$\Gamma(\omega) \sim \int_{|\xi|=\sqrt{(N+1)\mathbf{e}(\omega)-\omega}} \langle \widehat{M}(\xi), \sigma_1 \widehat{G}(\xi) \rangle_{\mathbb{C}^2} dS. \tag{40}$$

In the case  $N = 1$ , Buslaev and Perelman [18] are able to show that the integral is nonnegative. Zhou and Sigal [55] develop rigorously the argument and assume that  $\Gamma(\omega) > 0$  to prove their own version of Theorem 1.3 for  $N > 1$  in the case of a single  $\mathbf{e}(\omega)$ . In [40] it is shown that  $\Gamma(\omega) < 0$  is incompatible with orbital stability. This means that, if there is a single  $\mathbf{e}(\omega)$ , if hypothesis (H1) and (H2) hold (they imply orbital stability), then, in the presumably generic case  $\Gamma(\omega) \neq 0$ , we need to have  $\Gamma(\omega) > 0$ . Notice that, if we assume  $\Gamma(\omega) = \Gamma$  constant, then

$$|z(t)| = |z(0)| (1 + N\Gamma|z(0)|^{2N}t)^{-\frac{1}{2N}}. \tag{41}$$

The above discussion is purely heuristic, but indicative of the arguments in [18, 122, 128, 129, 130, 19, 30, 123, 55, 40]. Notice that by (41), eventually  $|z(t)| \sim t^{-\frac{1}{2N}}$  as  $t \rightarrow +\infty$ . In fact, since  $|z(0)|$  is small,  $|z(t)|$  remains almost constant in the time interval  $[0, |z(0)|^{-2N}]$ . Because of the forcing term  $z^{N+1}M$  in (38), also  $f$  cannot be counted to disperse for a long time. These arguments show that the decay of  $R(t) = P(p(t)r(t))$  in Theorem 2.3, in general cannot be expected to be true.  $\square$

The discussion in Lemma 2.6 indicates the relevance of the eigenvalues of  $\mathcal{H}_\omega$  in the analysis of the problem. In principle, eigenvalues of  $\mathcal{H}_\omega$  could lead to invariant tori near the solitons, which would prevent the result in Theorem 1.3. In fact we will discuss the fact that there are no invariant tori, and this thanks to the a mechanism related to the fact that  $\Gamma(\omega) > 0$  in (40). The reader might wonder why we should have  $\Gamma(\omega) \geq 0$ . Heuristically this should be related to the fact that our NLS is Hamiltonian. If the coordinates  $(z, f)$  in (38) were normal, we could expect (38) to be of the form

$$i\dot{z} = \partial_{\bar{z}}E, \quad i\dot{f} = \nabla_{\bar{f}}E. \tag{42}$$

Then by the Schwartz lemma, at  $z = 0$  and  $f = 0$  we would get  $(N + 1)!M = \partial_z^{N+1} \nabla_{\bar{f}} E = \overline{\partial_{\bar{z}}^N \nabla_f \partial_{\bar{z}} E} = N! \sigma_1 \bar{G}$ . But  $M \sim \sigma_1 \bar{G}$  would imply  $\widehat{M} \sim \sigma_1 \widehat{\bar{G}}$ , yielding the  $\Gamma(\omega) \geq 0$ .

**3. The case when  $\mathcal{H}_\omega$  has positive eigenvalues.** We will assume that  $\Pi(\tilde{u}_0) = \Pi(\tilde{\phi}_{\omega^1}) = p^1$ . This can be obtained using appropriate boosts. We need some information on the spectrum of  $\mathcal{H}_{\omega^1}$ . The following is elementary.

**Lemma 3.1.** *For any  $\omega \in \mathcal{O}$  following facts hold.*

- (1) *The spectrum  $\sigma(\mathcal{H}_\omega)$  is symmetric with respect to the coordinates axes. We have  $\sigma(\mathcal{H}_\omega) = \sigma(\mathcal{H}_\omega^*)$ .*
- (2) *Since  $\phi_\omega$  is a ground state, all the eigenvalues of  $\mathcal{H}_\omega$ , except possibly for a pair  $\pm i\mathbf{e}$  with  $\mathbf{e} > 0$ , are in  $\mathbb{R}$ .*
- (3) *If  $i\mathbf{e} \in \sigma_p(\mathcal{H}_\omega)$  with  $\mathbf{e} > 0$  then  $N_g(\mathcal{H}_\omega - i\mathbf{e})$ , the corresponding generalized eigenspace, has dimension 1.*
- (4) *If  $z \neq 0$  is an eigenvalue, then we have  $N_g(\mathcal{H}_\omega - z) = \ker(\mathcal{H}_\omega - z)$ .*

We assume that  $\mathcal{H}_{\omega^1}$  has no embedded solitons inside the essential spectrum.

(H4) There are no eigenvalues in  $\mathcal{H}_{\omega^1}$  in  $\mathbb{R} \setminus (-\omega^1, \omega^1)$ .

**Remark 3.2.** It is expected, but unproved yet, that, since  $\phi_\omega$  is a ground state, always there are no eigenvalues in  $\mathcal{H}_\omega$  in  $\sigma_e(\mathcal{H}_\omega) = \mathbb{R} \setminus [-\omega, \omega]$ , that is, no eigenvalues embedded in the “interior” of the continuous spectrum. Obviously, to our knowledge no embedded eigenvalues have been detected numerically in the case of ground states. For more general solitary waves which are not ground states, we expect that embedded eigenvalues could exist, but that they cannot have positive *Krein signature*. The signature of the eigenvalues of  $\mathcal{H}_\omega$  in  $\mathbb{R} \setminus \{0\}$  is always positive, in the case of ground states.

**Remark 3.3.** Even in the case they exist, the embedded eigenvalues are unstable, in the sense that, perturbing the equation, the  $\mathcal{H}_\omega$  of the new equation will in general not have these eigenvalues. Results of this type go back to Grillakis [60], are also in Tsai and Yau [130] and, as explained [42], can better be viewed in the classical framework of Howland [69, 70]

We allow  $\mathcal{H}_{\omega^1}$  to have a certain number of eigenvalues in the gap  $(-\omega_1, \omega_1)$ .

(H5) There is an  $m$  s.t.  $\mathcal{H}_{\omega^1}$  has  $m$  positive eigenvalues  $\mathbf{e}_1 \leq \mathbf{e}_2 \leq \dots \leq \mathbf{e}_m$ , where we repeat an eigenvalue a number of times equal to its multiplicity. We assume there are fixed integers  $m_0 = 0 < m_1 < \dots < m_{l_0} = m$  such that  $\mathbf{e}_j = \mathbf{e}_i$  exactly for  $i$  and  $j$  both in  $(m_l, m_{l+1}]$  for some  $l \leq l_0$ . In this case  $\dim \ker(\mathcal{H}_{\omega^1} - \mathbf{e}_j) = m_{l+1} - m_l$ . We assume there exist  $N_j \in \mathbb{N}$  such that  $0 < N_j \mathbf{e}_j < \omega_1 < (N_j + 1) \mathbf{e}_j$  with  $N_j \geq 1$ . We set  $N = N_1$ .

**Remark 3.4.** The literature considered for more than a decade only the case when  $m = 1$ , except for [127], where however  $N_j = 1$  for all  $j$ . These are very restrictive conditions. Only [6, 27] started to consider fairly general situations.

**Remark 3.5.** Hence we allow the eigenvalues to have finite multiplicity. The number  $(N_j + 1) \in \mathbb{N}$  is the smallest such that the corresponding multiple of  $\mathbf{e}_j$  is in  $\sigma_c(\mathcal{H}_{\omega^1})$ .

**Remark 3.6.** We give here a partial list of papers which have explored the spectrum of operators such as  $\mathcal{H}_{\omega^1}$ . Chang, Gustafson, Nakanish and Tsai [21] explore in great

detail and mostly numerically the spectrum of  $\mathcal{H}_{\omega^1}$  in the case of  $\beta(|u|^2) = -|u|^{p-1}$ . Their computations in dimensions  $d = 1, 2, 3$  show the presence of many eigenvalues for  $p \rightarrow 1^+$  and of just two real nonzero eigenvalues for  $p \rightarrow (1 + 4/d)^-$  which reach 0 at  $p = 1 + 4/d$  and bifurcate into two imaginary eigenvalues for  $p > 1 + 4/d$ .

**Remark 3.7.** Buslaev and Grikurov [15] and Marzuola, Raynor, and Simpson [94] study numerically situations when  $q(\omega)$ , the function in (H2) Theorem 1.2, has a minimum  $\omega_*$ . Then  $\mathcal{H}_\omega$  has two imaginary eigenvalues for  $\omega < \omega_*$  which converge to 0 as  $\omega \rightarrow \omega_*^-$  and bifurcate into two positive eigenvalues for  $\omega > \omega_*$ . This is explained analytically in Comech and Pelinovsky [24]. Interesting oscillating patterns are described numerically in [15, 94], with interesting conjectures, which are discussed analytically, but inconclusively, in [36], where the problem is shown to be similar to that of a soliton constrained in a potential.

**Remark 3.8.** The spectrum of  $\mathcal{H}_{\omega^1}$  for the equations with  $\beta(|u|^2) = -|u|^{p-1}$  and  $p > 1 + 4/d$ , in particular the case  $d = 3$  and  $p = 3$  have been studied in considerable detail. The case  $\beta(|u|^2) = -|u|^2$ ,  $d = 3$ , is considered in Schlag [118], where it is shown that  $\mathcal{H}_{\omega^1}$  has no eigenvalues other than 0 in  $[-\omega^1, \omega^1]$  if the operator  $L_{+\omega^1}$  in Theorem 1.2 and the operator  $L_{-\omega^1} := -\Delta - \omega^1 + \beta(\phi_{\omega^1}^2)$  don't have eigenvalues in  $(0, \omega^1]$ . This information on  $L_{\pm\omega^1}$  is verified numerically for the cubic NLS with  $d = 3$  in Demanet and Schlag [49] and proved rigorously in Costin, Huang and Schlag [25]. In Marzuola and Simpson [95], for the cubic NLS with  $d = 3$  it is proved numerically absence of nonzero real eigenvalues. Further cases of computer assisted proofs of absence of nonzero real eigenvalues for mass supercritical NLS with  $\beta(|u|^2) = -|u|^{p-1}$  are considered in Asad and Simpson [4].

We assume that the eigenvalues in (H5) satisfy the following non resonance condition.

(H6) If  $\mathbf{e}_{j_1} < \dots < \mathbf{e}_{j_k}$  are  $k$  distinct  $\mathbf{e}$ 's, and  $\mu \in \mathbb{Z}^k$  satisfies  $|\mu| := |\mu_1| + \dots + |\mu_m| \leq 2N_1 + 3$ , then we have

$$\mu_1 \mathbf{e}_{j_1} + \dots + \mu_k \mathbf{e}_{j_k} = 0 \iff \mu = 0 .$$

**Remark 3.9.** A more restrictive formulation would be to say that the eigenvalues are linearly independent in  $\mathbb{Z}$ . That would be a more stringent condition than necessary.

Another hypothesis it the following.

(H7) There is no multi index  $\mu \in \mathbb{Z}^m$  with  $|\mu| \leq 2N_1 + 3$  such that  $\mu \cdot \vec{\mathbf{e}} = \omega^1$  (where  $\vec{\mathbf{e}} := (\mathbf{e}_1, \dots, \mathbf{e}_m)$ ).

**Remark 3.10.** Notice that in [27] and in some of the subsequent papers, the hypotheses are more restrictive, because it is assumed that the multiplicities of the eigenvalues are constant, and the hypotheses (H4)–(H7) are assumed for all  $\omega$ . The hypotheses stated here, which require (H4)–(H7) just for  $\omega^1$ , come from [23].

We need to record the following version of Theorem 2.4, which has essentially the same proof, see [42] on how to deal with the eigenvalues.

**Theorem 3.11.** *Let*

$$X_c(\omega^1) := \{N_g(\mathcal{H}_{\omega^1}^*) \oplus (\oplus_{\mathbf{e} \in \sigma_p \setminus \{0\}} \ker(\mathcal{H}_{\omega^1}^* - \mathbf{e}))\}^\perp, \tag{43}$$

where we can take  $X_c(\omega^1) \subset \mathcal{S}'(\mathbb{R}^d, \mathbb{C}^2)$ , in the space of tempered distributions. Then the statement of Theorem 2.4 continues to be true for  $L^q(\omega) := L^q(\mathbb{R}^d, \mathbb{C}^2)$   $\cap$

$X_c(\omega^1)$ . In particular the wave operators yield isomorphisms also between to Sobolev spaces  $W^{s,q}(\mathbb{R}^d, \mathbb{C}^2)$  and  $W^{s,q}(\mathbb{R}^d, \mathbb{C}^2) \cap X_c(\omega^1)$  for any  $s \in \mathbb{R}$ .

Notice that

$$N_g^\perp(\mathcal{H}_{\omega^1}^*) = (\oplus_{\mathbf{e} \in \sigma_p \setminus \{0\}} \ker(\mathcal{H}_{\omega^1} - \mathbf{e})) \oplus (L^2(\mathbb{R}^d, \mathbb{C}^2) \cap X_c(\omega^1)) \tag{44}$$

and so the spectral decomposition

$$L^2(\mathbb{R}^d, \mathbb{C}^2) = N_g(\mathcal{H}_{\omega^1}) \oplus (\oplus_{\mathbf{e} \in \sigma_p \setminus \{0\}} \ker(\mathcal{H}_{\omega^1} - \mathbf{e})) \oplus (L^2(\mathbb{R}^d, \mathbb{C}^2) \cap X_c(\omega^1)). \tag{45}$$

We have the following useful result which, among other things, insures that  $\mathcal{H}_{\omega^1}$  satisfies the limiting absorption principle.

**Proposition 3.12.** *There exists  $\tau_d > 0$  s.t. for  $\tau \geq \tau_d$  the following hold.*

(1) *There exists  $C = C(\tau, \omega)$ , upper semicontinuous in  $\omega$  such that for any  $\varepsilon \neq 0$ ,*

$$\|R_{\mathcal{H}_\omega}(\lambda + i\varepsilon)P_c(\mathcal{H}_\omega)u\|_{L_x^2 L_x^{2,-\tau}} \leq C\|u\|_{L^2}.$$

where

$$L_x^{2,\tau} = L^{2,-\tau}(\mathbb{R}^d, \mathbb{C}^2) = \{f \in \mathcal{S}'(\mathbb{R}^d, \mathbb{C}^2) : \|(1 + |x|^2)^{-\frac{\tau}{2}} f\|_{L^2} < \infty\}.$$

(2) *For any  $u \in L_x^{2,\tau}$  the following limits exist:*

$$\lim_{\varepsilon \searrow 0} R_{\mathcal{H}_\omega}(\lambda \pm i\varepsilon)u = R_{\mathcal{H}_\omega}^\pm(\lambda)u \text{ in } C^0(\sigma_e(\mathcal{H}_\omega), L_x^{2,-\tau}).$$

(3) *There exists  $C = C(\tau, \omega)$ , upper semicontinuous in  $\omega$  such that*

$$\|R_{\mathcal{H}_\omega}^\pm(\lambda)P_c(\mathcal{H}_\omega)\|_{B(L_x^{2,\tau}, L_x^{2,-\tau})} < C\langle \lambda \rangle^{-\frac{1}{2}}.$$

(4) *Given any  $u \in L_x^{2,\tau}$  for the projection on the  $L^2(\mathbb{R}^d, \mathbb{C}^2) \cap X_c(\omega^1)$  term in (45) we have*

$$P_c(\mathcal{H}_\omega)u = \frac{1}{2\pi i} \int_{\sigma_e(\mathcal{H}_\omega)} (R_{\mathcal{H}_\omega}^+(\lambda) - R_{\mathcal{H}_\omega}^-(\lambda))u \, d\lambda.$$

**4. Idea of the proof of Theorem 1.3.** In this section, we will proceed to show heuristically how to prove Theorem 1.3.

First of all, by some small boosts we reduce to the case when  $\Pi_a(u_0) = 0$  for all  $a = 1, \dots, d$ . We notice that the we can write  $R = P(p)r$  with  $r \in N_g^\perp(\mathcal{H}_{\omega^1}^*)$ . Now we have

$$\dot{p} = \{p, E\}, \quad \dot{\tau} = \{\tau, E\}, \quad \dot{r} = \{r, E\}. \tag{46}$$

We can substitute the coordinates  $p$  with the coordinates  $\Pi$ . In the new coordinates, the system becomes

$$\begin{aligned} \dot{\Pi}_j &= 0, & \dot{\tau} &= \{\tau, E\}, \\ \dot{r} &= \{r, E\}. \end{aligned} \tag{47}$$

Notice that in the coordinates  $(\Pi, \tau, r)$ , as well as in the system of coordinates  $(p, \tau, r)$ , we have  $\partial_\tau E = 0$ . Then we have a reduction of the system to  $\dot{r} = \{r, E\}$ .

We also choose  $p^0 \in \mathcal{P}$  so that

$$\Pi(u_0) = p^0. \tag{48}$$

Notice that if we consider the equations  $\Pi = p^0$ , they define a submanifold in  $H^1(\mathbb{R}^d, \mathbb{C})$  in a neighborhood of  $\{e^{i\tau \cdot \diamond} \Phi_{p^1} : \tau \in \mathbb{R}^{d+1}\}$  This set is parametrized by  $(\tau, r)$ . Taking the quotient by the group  $e^{i\tau \cdot \diamond}$ , we obtain a manifold, parametrized

by  $r$ . This manifold inherits in a natural way a symplectic structure, which is inherited from the  $\Omega$  defined in (12).

Now we split according to (44)

$$r(x) = \sum_{l=1, \dots, n} z_l \xi_l(x) + \sum_{l=1, \dots, n} \bar{z}_l \sigma_1 \xi_l(x) + f(x), \quad f \in X_c \text{ with } f = \bar{f}, \quad (49)$$

We will assume that it is possible to change the  $(z, f)$  coordinate so that the symplectic form is given by

$$\Omega = i \sum_{l=1, \dots, n} dz_l \wedge d\bar{z}_l + i \langle \sigma_3 df, \sigma_1 df \rangle. \quad (50)$$

The correct version is just slightly more complex, see [23, formula (7.11)], and contains some additional higher order terms that we skip in the following heuristic discussion.

It is critical, to consider an expansion of the energy in terms of these coordinates. It is important to expand

$$\begin{aligned} E &= E(\Phi_p) + \langle \nabla E(\Phi_p), \sigma_1 P_p r \rangle + 2^{-1} \langle \nabla^2 E(\Phi_p) P_p r, P_p r \rangle + \dots \\ &= E(\Phi_p) + 2^{-1} \langle \nabla^2 E(\Phi_p) P_p r, P_p r \rangle + \dots \end{aligned}$$

where we exploit  $\langle \nabla E(\Phi_p), \sigma_1 P_p r \rangle = \lambda \cdot \langle \diamond \Phi_p, \sigma_1 P_p r \rangle = 0$ . Adding and subtracting  $\lambda(p) \cdot \Pi(P_p r)$  in the r.h.s., we obtain

$$E = E(\Phi_p) + \lambda(p) \cdot \Pi(P_p r) + 2^{-1} \langle (\nabla^2 E(\Phi_p) - \lambda \cdot \diamond) P_p r, P_p r \rangle + \dots$$

Substituting  $\Pi = p + \Pi(P_p r)$ , subtracting on both sides  $E(\phi_{\omega^0})$ , we get

$$\begin{aligned} E - E(\phi_{\omega^0}) &= E(\Phi_p) - \lambda \cdot p - (E(\phi_{\omega^0}) - \lambda_0 \cdot p^0) + (\lambda - \lambda_0) \cdot p^0 \\ &\quad + 2^{-1} \langle (\nabla^2 E(\Phi_p) - \lambda \cdot \diamond) P_p r, P_p r \rangle + \dots \\ &= d(\omega) - d(\omega^0) + (\omega - \omega^0)q(\omega^0) + 2^{-1}v^2q(\omega^0) + 2^{-1} \langle \sigma_3 \mathcal{H}_{\omega^1} r, r \rangle + \dots \end{aligned}$$

Let us now substitute  $r$  with the expansion in (49). Then, for  $d(\omega) := E(\phi_\omega) - \omega q(\omega)$ ,

$$\begin{aligned} E - E(\phi_{\omega^0}) &= E(\Phi_p) - \lambda \cdot p - (E(\phi_{\omega^0}) - \lambda_0 \cdot p^0) + (\lambda - \lambda_0) \cdot p^0 \\ &\quad + 2^{-1} \langle (\nabla^2 E(\Phi_p) - \lambda \cdot \diamond) P_p r, P_p r \rangle + \dots \\ &= d(\omega) - d(\omega^0) + (\omega - \omega^0)q(\omega^0) + 2^{-1}v^2q(\omega^0) + 2^{-1} \langle \sigma_3 \mathcal{H}_{\omega^1} r, r \rangle + \dots \end{aligned}$$

Then we obtain an expression of the form

$$\begin{aligned} E &= \psi(\Pi(f)) + E_{\text{discr}} + 2^{-1} \langle \sigma_3 \mathcal{H}_{\omega^1} f, \sigma_1 f \rangle \\ &\quad + \sum z^\mu \bar{z}^\nu a_{\mu\nu}(\Pi(f)) + \sum z^\mu \bar{z}^\nu \langle \sigma_3 A_{\mu\nu}(\Pi(f)), \sigma_1 f \rangle + \dots \end{aligned}$$

where  $E_{\text{discr}} := \sum_j \mathbf{e}_j |z_j|^2$  and where we sum over finitely many multi-indexes. We remark that  $\bar{a}_{\mu\nu} = a_{\mu\nu}$  and  $\bar{A}_{\mu\nu} = -\sigma_1 A_{\nu\mu}$ , by the fact that  $E$  is real valued.

Non resonant terms of the form  $z^\mu \bar{z}^\nu a_{\mu\nu}$  for  $(\mu - \nu) \cdot \vec{\mathbf{e}} \neq 0$  can be eliminated by considering appropriate canonical transformations given by  $\phi^t|_{t=1}$ , using the flow of the Hamiltonian vector-field associated to functions of the form  $\chi = z^\mu \bar{z}^\nu b_{\mu\nu}$ , with the coefficient unknown. Indeed, concisely,

$$\begin{aligned} E \circ \phi^t|_{t=1} &= E + \{E_{\text{discr}} + 2^{-1} \langle \sigma_3 \mathcal{H}_{\omega^1} f, \sigma_1 f \rangle, z^\mu \bar{z}^\nu\} b_{\mu\nu} + \dots \\ &= E + \{E_{\text{discr}}\} b_{\mu\nu} + \dots = E + \vec{\mathbf{e}} \cdot (\mu - \nu) z^\mu \bar{z}^\nu b_{\mu\nu} + \dots \end{aligned}$$

can be used to eliminate the non resonant  $z^\mu \bar{z}^\nu a_{\mu\nu}$  term, just by solving  $a_{\mu\nu} + \vec{e} \cdot (\mu - \nu) b_{\mu\nu} = 0$ .

Similarly, terms of the form  $z^\mu \bar{z}^\nu \langle \sigma_3 A_{\mu\nu}, \sigma_1 f \rangle$  with  $|\vec{e} \cdot (\mu - \nu)| < \omega^1$  are *non resonant*, and can be eliminated similarly using  $\chi = z^\mu \bar{z}^\nu \langle \sigma_3 B_{\mu\nu}, \sigma_1 f \rangle$ . Indeed, concisely,

$$\begin{aligned} & E \circ \phi^t|_{t=1} \\ &= E + \{E_{\text{discr}}, z^\mu \bar{z}^\nu\} \langle \sigma_3 B_{\mu\nu}, \sigma_1 f \rangle + z^\mu \bar{z}^\nu \{2^{-1} \langle \sigma_3 \mathcal{H}_{\omega^1} f, \sigma_1 f \rangle, \langle \sigma_3 B_{\mu\nu}, \sigma_1 f \rangle\} + \dots \\ &= E + z^\mu \bar{z}^\nu \langle \sigma_3 (\vec{e} \cdot (\mu - \nu) + \mathcal{H}_{\omega^1}) B_{\mu\nu}, \sigma_1 f \rangle + \dots \end{aligned}$$

so that the non resonant term can be canceled solving  $(\vec{e} \cdot (\mu - \nu) + \mathcal{H}_{\omega^1}) B_{\mu\nu} = A_{\mu\nu}$ . Here, the fact that the coefficients  $a_{\mu\nu}$  and  $A_{\mu\nu}$ , and so also  $b_{\mu\nu}$  and  $B_{\mu\nu}$ , depend on  $\Pi(f)$  and are not constant, is not an obstacle for a rigorous implementation of the above ideas, because  $\Pi(f)$  remain constant, up to an error which is higher order and does not affect the computations.

Eventually we find a a system of coordinates, where the significant terms are

$$\begin{aligned} E &= \psi(\Pi(f)) + E_{\text{discr}} + 2^{-1} \langle \sigma_3 \mathcal{H}_{\omega^1} f, \sigma_1 f \rangle + Z_0 + Z_1 + \dots, \text{ where} \\ Z_0 &= \sum_{(\mu-\nu) \cdot \vec{e}=0} z^\mu \bar{z}^\nu a_{\mu\nu}(\Pi(f)) \\ Z_1 &:= \sum_{\vec{e} \cdot \mu > \omega^1} z^\mu \langle \sigma_3 A_{\mu 0}, \sigma_1 f \rangle + \sum_{\vec{e} \cdot \nu > \omega^1} \bar{z}^\nu \langle \sigma_3 A_{0\nu}, \sigma_1 f \rangle. \end{aligned}$$

Here  $E$  real valued, we have  $\bar{f} = \sigma_1 f$ , and as a consequence

$$\bar{A}_{\mu\nu} = -\sigma_1 A_{\nu\mu}.$$

The system reads

$$i\dot{f} = \mathcal{H}_{\omega^1} f + \sigma_3 \nabla_{\Pi(f)} E \cdot \diamond f + \sum_{\mathbf{e} \cdot \alpha > \omega^1} z^\alpha A_{\alpha 0} + \sum_{\mathbf{e} \cdot \beta > \omega^1} \bar{z}^\beta A_{0\beta} + \dots \tag{51}$$

$$i\dot{z}_j = \partial_{\bar{z}_j} E = \mathbf{e}_j z_j + \partial_{\bar{z}_j} Z_0 + \sum_{\vec{e} \cdot \nu > \omega^1} \nu_j \frac{\bar{z}^\nu}{z_j} \langle A_{0\nu}, \sigma_3 \sigma_1 f \rangle + \dots \tag{52}$$

We remark that the very recent notion of Refined Profiles, introduced in [39], should allow to avoid completely the above normal forms arguments: in [39] this is proved at small energies. Continuing with the argument, the crux of the proof consists in proving the following.

**Proposition 4.1.** *There is a fixed  $C_0 > 0$  such that for  $\varepsilon_0 > 0$  sufficiently small, for  $\varepsilon \in (0, \varepsilon_0)$  and for  $|z(0)| + \|f(0)\|_{H^1} < \varepsilon$ , then the following inequalities, for some  $T > 0$*

$$\|f\|_{L_t^r([0, T], W_x^{1,p})} \leq 2C_0 \varepsilon \text{ for all admissible pairs } (r, p) \tag{53}$$

$$\|z^\mu\|_{L_t^2(0, T)} \leq 2C_0 \varepsilon \text{ for all multi indexes } \mu \text{ with } \vec{e} \cdot \mu > \omega_0 \tag{54}$$

$$\|z_j\|_{W_t^{1,\infty}(0, T)} \leq 2C_0 \varepsilon \text{ for all } j \in \{1, \dots, m\} \tag{55}$$

imply improved inequalities obtained replacing  $2C_0$  with  $C_0$ .

4.1. **Analysis of the equation of  $f$ .** If we had  $\sigma_3 \nabla_{\Pi(f)} E \cdot \diamond f = 0$ , then we would have

$$\|f\|_{L^2_t([0,T], W_x^{1,p})} \leq c_0 \|f(0)\|_{H^1} + \sum_{\vec{e} \cdot \mu > \omega^1} \|z^\mu\|_{L^2_t(0,T)} + C(C_0)\epsilon^2, \tag{56}$$

implying that the key estimates are those on  $\|z^\mu\|_{L^2_t(0,T)}$  for  $\vec{e} \cdot \mu > \omega^1$ .

This in fact is true, but nonetheless

$$\sigma_3 \nabla_{\Pi(f)} E \cdot \diamond f = \varpi_0 \sigma_3 f + i\sigma_3 \vec{\omega} \cdot \nabla f \tag{57}$$

is nonzero. It helps that the hypotheses of Proposition 4.1 imply  $\|(\varpi_0, \vec{\omega})\|_{L^\infty(0,T)} \leq C(C_0)\epsilon$ . However the terms in 57 are not in  $L^2([0, T], W^{1, \frac{2d}{d-2}}) + L^1([0, T], H^1)$ , so that they cannot be incorporated with the terms on their right in (52). Nor they can be eliminated easily by some integrating factor, this because  $\sigma_3$  and  $i\sigma_3 \partial_a$  for  $a = 1, \dots, d$  do not commute with  $\mathcal{H}_{\omega^1}$ .

Nonetheless, a form of integrating factor has been proved by Beceanu [9], but only in dimensions  $d \geq 3$ . An alternative argument, attributed to Perelman, is presented in Bambusi [5, Appendix B], but that too, based on Proposition 1.1 [113], depends on dimensions  $d \geq 3$ . A different argument, due to Buslaev and Perelman [18] is known in dimensions  $d = 1, 2$ , but only when  $\vec{\omega} = 0$  in (57), that is, when there is no translation in the problem. This accounts for the fact that Theorem 1.3 has not been proved in dimensions 1 and 2, as we already mentioned, except under hypotheses (like extra symmetries, or in the presence of a potential) that break the translation invariance. Notice that the gauge change argument sketched near (36) depends on the hypothesis (1) stated in Theorem 2.2 (absence of nonzero eigenvalues).

**Remark 4.2.** What is crucial, for the integrating factor argument, is that

$$\| \langle x \rangle^{-M(d)} e^{it\Delta} \langle x \rangle^{-M(d)} \|_{L^2 \rightarrow L^2} \leq C_d \langle t \rangle^{-\frac{d}{2}}$$

for a sufficiently large  $M(d)$ , with  $\frac{d}{2} > 1$  for  $d \geq 3$ . In the cases  $d = 1, 2$  the lack of integrability is an obstruction for the argument. The case  $d = 1$  in particular, can be phrased by stating that  $-\Delta$  is in  $\mathbb{R}$  a non-generic Schrödinger operator, because of the fact that the point 0 is a resonance.

It is probably not coincidental that the proof of asymptotic stability of kinks for the  $\phi^4$  model by Kowalczyk, Martel and Muñoz [87] is valid only in the case of odd solutions, that is, by imposing a symmetry which allows to exclude translation, and that the main difficulty at removing the symmetry is the resonance at the threshold of the continuous spectrum of the linearization, see [87, Remark 1.2].

4.2. **Analysis of the equation of  $z$ .** Recall that we defined  $E_{\text{discr}} := \sum_j \mathbf{e}_j |z_j|^2$ . The idea of the proof consists, schematically, in showing that

$$\dot{E}_{\text{discr}} = \{E_{\text{discr}}, E\} \sim \{E_{\text{discr}}, Z_1\} \lesssim - \sum_{\vec{e} \cdot \mu > \omega^1} |z^\mu|^2 \tag{58}$$

This will imply

$$\sum_j \mathbf{e}_j |z_j(t)|^2 + \sum_{\mathbf{e} \cdot \mu > \omega^1} \int_0^t |z^\mu|^2 \lesssim \sum_j \mathbf{e}_j |z_j(0)|^2,$$

yielding the crucial bound that, in turn and thanks to (56), yields Proposition 4.1. We sketch a heuristic argument for (58) The starting point consists in considering

$$f = - \sum z^\alpha R_{\mathcal{H}_{\omega^1}}^+(\vec{e} \cdot \alpha) A_{\alpha 0} - \sum \bar{z}^\beta R_{\mathcal{H}_{\omega^1}}^+(-\vec{e} \cdot \beta) A_{0\beta} + g. \tag{59}$$

The effect of this change of variables is to show that  $g$  satisfies an equation where the terms  $z^\alpha A_{\alpha 0}$  and  $\bar{z}^\alpha A_{0\alpha}$  have been canceled out. While  $g(t) \notin H^1$ , nonetheless, using (53)–(55) it is possible to prove

$$\| \langle x \rangle^{-N(d)} g \|_{L_t^2([0,T], H_x^1)} \lesssim \| f(0) \|_{H^1} + |z(0)| + \epsilon^2 \leq C_0 \epsilon$$

for an appropriate  $N(d) > 0$ . Here  $c_0 \sim 1$  is  $c_0 \ll C_0$  and consequently in the sequel we ignore the term  $g$ .

Substituting (58) in (52), ignoring the terms in  $g$ , which are smaller, by elementary arguments we get to

$$i\dot{z}_j = \mathbf{e}_j z_j - \sum_{\substack{\vec{e} \cdot \nu > \omega^1 \\ \vec{e} \cdot \alpha > \omega^1}} \nu_j \frac{z^\alpha \bar{z}^\nu}{\bar{z}_j} \langle A_{0\nu}, \sigma_3 \sigma_1 R_{\mathcal{H}_{\omega^1}}^+(\vec{e} \cdot \alpha) A_{\alpha 0} \rangle + \dots$$

Generically, when all the eigenvalues of  $\mathcal{H}_{\omega^1}$  in  $(0, \omega_1)$  have multiplicity 1, and recalling  $A_{0\alpha} = -\sigma_1 \bar{A}_{\alpha 0}$ , this simplifies further

$$i\dot{z}_j = \mathbf{e}_j z_j - \sum_{\vec{e} \cdot \alpha > \omega^1} \alpha_j \frac{|z^\alpha|^2}{\bar{z}_j} \langle \bar{A}_{\alpha 0}, \sigma_3 R_{\mathcal{H}_{\omega^1}}^+(\vec{e} \cdot \alpha) A_{\alpha 0} \rangle + \dots \tag{60}$$

Recalling now Theorem 2.4, for  $A_{\alpha 0} = W(\omega^1) A_{\alpha 0}^{(1)}$ , we have the following steps, already discussed in the old survey [26],

$$\begin{aligned} \langle \bar{A}_{\alpha 0}, \sigma_3 R_{\mathcal{H}_{\omega^1}}^+(\vec{e} \cdot \alpha) A_{\alpha 0} \rangle &= \lim_{\epsilon \rightarrow 0^+} \langle \bar{A}_{\alpha 0}, \sigma_3 R_{\mathcal{H}_{\omega^1}}(\vec{e} \cdot \alpha + i\epsilon) A_{\alpha 0} \rangle \\ &= \lim_{\epsilon \rightarrow 0^+} \langle \overline{W(\omega^1) A_{\alpha 0}^{(1)}}, \sigma_3 W(\omega^1) R_{\sigma_3(-\Delta + \omega^1)}(\vec{e} \cdot \alpha + i\epsilon) A_{\alpha 0}^{(1)} \rangle \\ &= \lim_{\epsilon \rightarrow 0^+} \langle \overline{A_{\alpha 0}^{(1)}}, W(\omega^1)^* \sigma_3 W(\omega^1) R_{\sigma_3(-\Delta + \omega^1)}(\vec{e} \cdot \alpha + i\epsilon) A_{\alpha 0}^{(1)} \rangle. \end{aligned}$$

Using the identity  $W(\omega^1)^* \sigma_3 W(\omega^1) = \sigma_3 Z(\omega^1) W(\omega^1) = \sigma_3$ , we conclude that

$$\begin{aligned} \langle \bar{A}_{\alpha 0}, \sigma_3 R_{\mathcal{H}_{\omega^1}}^+(\vec{e} \cdot \alpha) A_{\alpha 0} \rangle &= \langle \overline{A_{\alpha 0}^{(1)}}, \sigma_3 R_{\sigma_3(-\Delta + \omega^1)}^+(\vec{e} \cdot \alpha) A_{\alpha 0}^{(1)} \rangle \\ &= \langle \overline{A_{\alpha 0}^{(1)}}, \sigma_3 P.V. \frac{1}{\sigma_3(-\Delta + \omega^1) - \vec{e} \cdot \alpha} A_{\alpha 0}^{(1)} \rangle \\ &\quad + \pi i \langle \overline{A_{\alpha 0}^{(1)}}, \sigma_3 \delta(\sigma_3(-\Delta + \omega^1) - \vec{e} \cdot \alpha) A_{\alpha 0}^{(1)} \rangle. \end{aligned}$$

In the last line, the first term is real valued and the last is imaginary. Hence, when we multiply (60) by  $\mathbf{e}_j \bar{z}_j$ , sum up on  $j$  and take the imaginary part, we obtain

$$\begin{aligned} 2^{-1} \partial_t \sum_j \mathbf{e}_j |z_j|^2 &= -\pi \sum_{\vec{e} \cdot \alpha > \omega^1} |z^\alpha|^2 \langle \overline{A_{\alpha 0}^{(1)}}, \sigma_3 \delta(\sigma_3(-\Delta + \omega^1) - \vec{e} \cdot \alpha) A_{\alpha 0}^{(1)} \rangle \\ &= -\pi \sum_{\vec{e} \cdot \alpha > \omega^1} |z^\alpha|^2 \left\langle \begin{pmatrix} \overline{(A_{\alpha 0}^{(1)})_1} \\ \overline{(A_{\alpha 0}^{(1)})_2} \end{pmatrix}, \begin{pmatrix} \delta(-\Delta + \omega^1 - \vec{e} \cdot \alpha) & 0 \\ 0 & -\underbrace{\delta(\Delta - \omega^1 - \vec{e} \cdot \alpha)}_0 \end{pmatrix} \begin{pmatrix} (A_{\alpha 0}^{(1)})_1 \\ (A_{\alpha 0}^{(1)})_2 \end{pmatrix} \right\rangle \\ &= -\pi \sum_{\vec{e} \cdot \alpha > \omega^1} |z^\alpha|^2 \left\langle \overline{(A_{\alpha 0}^{(1)})_1}, \delta(-\Delta + \omega^1 - \vec{e} \cdot \alpha) (A_{\alpha 0}^{(1)})_1 \right\rangle, \end{aligned}$$

where  ${}^t A_{\alpha 0}^{(1)} = \left( (A_{\alpha 0}^{(1)})_1, (A_{\alpha 0}^{(1)})_2 \right)$  and where

$$\begin{aligned} & \left\langle \overline{(A_{\alpha 0}^{(1)})_1}, \delta(-\Delta + \omega^1 - \vec{e} \cdot \alpha)(A_{\alpha 0}^{(1)})_1 \right\rangle \\ &= \frac{1}{2(\vec{e} \cdot \alpha - \omega^1)} \int_{|\xi| = \vec{e} \cdot \alpha - \omega^1} \left| \widehat{(A_{\alpha 0}^{(1)})_1}(\xi) \right|^2 dS \geq 0. \end{aligned} \tag{61}$$

The following is an hypothesis.

(H8) We assume the last inequality to be strict, for appropriate choice of multi-indexes  $\alpha$ .

Then the argument closes up.

**Remark 4.3.** The coefficients in (61) are obtained after the NLS undergoes a significant number of coordinate changes. As a consequence, it is not easy to write concretely and check numerically (H8). Notice though that in [39] the argument is much simplified, there is no normal forms argument and the coefficients of the FGR are much simpler.

**5. Further remarks and references.** We add some further remarks.

**Remark 5.1.** The problem of the eventual behavior of a soliton of (10) in a confining well obtained adding to (10) a potential, is mostly open. For a non complete list of references see [7, 11, 52, 53, 67, 68, 46, 71], and see therein for further references. These papers treat long time behavior, but not asymptotic behavior. This problem is very similar to the oscillations discussed in Remark 3.7.

**Remark 5.2.** The effect of a potential on an escaping soliton is easier to track, because, while it is deviated, the soliton is preserved. There are various papers on the asymptotic behavior of escaping potentials like [31, 33, 50, 109]. A very suggestive analysis of a soliton of the cubic integrable NLS in dimension 1 hitting a defocusing delta potential is in Holmer, Marzuola and Zworski [65, 66]. But the discussion, which uses also the integrable structure, involves finite times only: it is not clear how to show that certain terms, that in [66] are remainder, do not develop in significant ones over larger intervals of time. Very little, beyond Deift and Zhou [48], is known about the use of the inverse scattering transform and the nonlinear steepest descent method in the context of non-integrable systems, with the problem in [66, 98] looking like natural for such a theory. For example, it would be natural to use the nonlinear steepest descent method to show that all solutions of a defocusing NLS in dimension 1 with a repulsive Dirac potential and with initial datum in  $H^1(\mathbb{R}) \cap \langle x \rangle^{-1} L^2(\mathbb{R})$  decay like  $t^{-1/2}$  and have the asymptotic profile that in Masaki, Murphi and Segata [98] is proved only for small initial data.

An asymptotic analysis over all times for a problem similar to [66] is in Perelman [112], which however discusses a very flat soliton. In [111] there is a finite time analysis of interaction of two solitons.

Substantial modifications of a moving soliton in the presence of a NLS with a slowly varying coefficient in front of the nonlinearity are in [105, 106].

**Remark 5.3.** There are deep connections between the Fermi Golden Rule discussed here and the problem of the  $\|\nabla u(t)\|_{L^2(\mathbb{R}^d)} \sim \sqrt{\frac{\log|\log t|}{t}}$  blow up in the NLS with  $\beta(|u|^2)u = -|u|^{\frac{4}{d}}u$ . The key in the proof is the extension in [114, 100] of the solitons

in a larger class of functions, which are only approximately solutions of the NLS and resemble the part of the solution here given by

$$u_{\text{approx}} = e^{i\sigma_3\tau\cdot\diamond} \left( \Phi_p + P(p) \left( \sum_{l=1,\dots,n} z_l \xi_l + \sum_{l=1,\dots,n} \bar{z}_l \sigma_1 \xi_l \right) \right),$$

obtained omitting the contribution of the continuous coordinate  $f$ . In [100] there is the discussion of a Lyapunov function that takes the role of  $E_{\text{discr}} := \sum_j \mathbf{e}_j |z_j|^2$ . In [100] the discussion is rather delicate because the coupling responsible for the Fermi Golden Rule is exponentially small, rather than polynomially small like in Sect. 4.2. In [114, 100] the choice of *profiles* (that is a generic solution is represented as a sum of a profile plus a remainder, where the profile is similar to a ground state) does not require an algorithmic procedure (normal forms) and is related to the framework introduced by previous authors and discussed by Sulem and Sulem [125, Chapter 7]. Obviously, the step by step normal forms argument glimpsed in Sect. 4, which eliminates resonant terms one at a time from the Taylor expansion of the energy  $E$ , would never yield the exponentially small resonant terms in [100]. It is interesting that in the first paper of their series [102], Merle and Raphael in Sect. 4.3 perform an argument similar to normal forms which yields the non sharp upper bound on blow up of [102]. Presumably further changes of variables would yield algebraic improvements, which nonetheless are not sharp. In any case, from [101] (the second paper in the series) on, they settle in the optimal coordinates obtaining the sharp upper bound on blow up. Notice that in [39], in analogy to [114, 100], there is a choice of profile that allows to avoid a normal forms argument.

Very delicate, especially because it is very difficult to estimate in a sharp way various remainders, is the proof of the sharp lower bound on blow up in [100], where a Lyapunov function is defined starting from the *local virial identity* (the latter is stated in Proposition 2 [100], see also the earlier [102, 101]) and then by various adjustments. The discussion is different from the one in the present survey, where the  $E_{\text{discr}}$  is defined using the discrete coordinates, which lose energy leaking in the background. Work needs to be done to compare the Lyapunov function in [100] with the  $E_{\text{discr}}$  of the present paper. It would be interesting to compare and unify the methods, considering problems mixing the frameworks in [114, 100] and in here. One such problem might be the one discussed in [36] and, by analogy, probably also problems involving solitons trapped in wells, and in general, problems where the linearizations have eigenvalues close to 0. Other similar problems are the ones involving the complicated patterns in [96, 59] near the bifurcating standing waves of [77, 78, 80].

**Remark 5.4.** Another topic which is not well studied is the relation between Fermi Golden Rule and the small “wings” of nanopteron/micropteron [14]. Here, a nanopteron/micropteron are infinite energy solutions which look like solitons locally but have a small nondecaying (or slow decaying) tail near spatial infinity. For mathematical results on the existence of such solutions for various equations see [8, 64, 73, 91, 92, 126]. When the tail is exponentially small w.r.t. a small parameter it is called nanopteron and, if it is polynomially small, it is called micropteron. Since the asymptotic stability result reviewed in this paper claims that there are no finite energy quasi-periodic solutions near solitons even though the linearized equation possesses quasi-periodic solutions due to the internal modes, it is natural to ask if there exist infinite energy quasi-periodic solutions near solitons, which should be

micropteron if they exist. Moreover, it is natural to guess that the “wing” of such micropteron are related to the Fermi Golden Rule, in particular the first two terms of the r.h.s. of (59). In connection of Merle-Raphael’s result on the blow up of critical NLS [101, 100], it was shown by Johnson and Pan [72] that there exist infinite energy solutions which blow up without the log log correction. From the above point of view, it is natural to ask the relation between Merle and Raphael’s optimal choice of the coordinate and Johnson-Pan’s solution. Moreover the exponentially small Fermi Golden Rule and the asymptotic behavior of Johnson-Pan’s solution at spatial infinity are of interest. However, these topics are completely open as far as the authors know.

**Remark 5.5.** Many of the papers on asymptotic stability of standing waves, focus on small standing waves which bifurcate from eigenvalues of a Schrödinger operator, see [120, 121, 115], [128]–[131], [127, 61, 55, 54, 57, 58, 108, 97, 37]. In these papers the spectrum of the Schrödinger operator is rather simple. More general situations are considered in [32], whose proof is much simplified in [39], which however treats only the case when the eigenvalues have multiplicity 1. Higher multiplicities, but under restrictive conditions on the spectrum, are considered in [62]. Analogues of [32] are for the NLKG in [38] and for Dirac in [43]. Notice that in [122, 6, 2], which treat NLKG, there are no standing waves because only real valued solutions are considered. In [38], since complex valued solutions of the NLKG are considered, the dynamics of small energy solutions of the NLKG are more complicated than in [122, 6].

**Remark 5.6.** The radiation damping also plays a role in the instability of excited states which are linearly stable. This mechanism was called “radiation induced instability” in [63] following the name “dissipation induced instability” [10]. See also, [34].

**Remark 5.7.** Global asymptotic results have been proved for equations where the nonlinearity is concentrated in a point, or in finitely many points, that is  $\beta(|u|^2)u$  is replaced by  $\delta(x - x_0)\beta(|u|^2)u$ , or by a linear combination of such terms. See [82]–[85], [22] and therein.

Another model with very remarkable results is the energy critical focusing wave equation in 3 D, especially in the radial case, see [51], where the proof is based on the *channel of energy* inequality, which is specific to wave equations, and on *nonlinear profile* decompositions. In the context of the NLS, the nonlinear profile decompositions are rather complicated, see [107], and the presence, in the terminology of [110], of *internal modes* of the solitons might render difficult proving the soliton decoupling, see also [35].

**Remark 5.8.** Little seems to be known about the nonlinear Klein Gordon Equations (NLKG). We do not know of any result analogous to Theorems 1.3, 2.2 or 2.3 for solitary waves of the NLKG. Notice that an analogue of Theorem 1.3 is known for solutions with appropriate symmetries of nonlinear Dirac Equations, see [13].

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Received December 2019; revised August 2020.

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