

Optimal stability in the identification of a rigid inclusion in an isotropic Kirchhoff-Love plate

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June 25, 2018

Abstract

In this paper we consider the inverse problem of determining a rigid inclusion inside a thin plate by applying a couple field at the boundary and by measuring the induced transversal displacement and its normal derivative at the boundary of the plate. The plate is made by non-homogeneous, linearly elastic and isotropic material. Under suitable a priori regularity assumptions on the boundary of the inclusion, we prove a constructive stability estimate of log type. Key mathematical tool is a recently proved optimal three spheres inequality at the boundary for solutions to the Kirchhoff-Love plate's equation.

Mathematics Subject Classification (2010): Primary 35B60. Secondary 35B30, 35Q74, 35R30.

Keywords: Inverse problems, elastic plates, stability estimates, unique continuation, rigid inclusion.

1 Introduction

In this paper we consider the inverse problem of the stable determination of a rigid inclusion embedded in a thin elastic plate by measuring the transverse displacement and its normal derivative at the boundary induced by a

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couple field applied at the boundary of the plate. We prove that the stability estimate of log-log type found in [M-Ro-Ve2] can be improved to a single logarithm in the case in which the plate is made of isotropic linear elastic material. From the point of view of applications, modern requirements of structural condition assessment demand the identification of defects using non-destructive methods, and, therefore, the present results can be useful in quality control of plates. We refer, among other contributions, to Bonnet and Constantinescu [Bo-Co] for a general overview of inverse problems arising in diagnostic analysis applied to linear elasticity and, in particular, to plate theory ([Bo-Co, Section 5.3]), and to [K] for the identification of a stiff inclusion in a composite thin plate based on wavelet analysis of the eigenfunctions.

In order to describe our stability result, let us introduce the Kirchhoff-Love model of thin, elastic isotropic plate under infinitesimal deformation; see, for example, [G]. Let the middle plane of the plate Ω be a bounded domain of \mathbb{R}^2 with regular boundary. The rigid inclusion D is modelled as a simply connected domain compactly contained in Ω . Under the assumptions of vanishing transversal forces in Ω and assigned couple field \widehat{M} acting on $\partial\Omega$, the transversal displacement $w \in H^2(\Omega)$ of the plate satisfies the following mixed boundary value problem

$$\begin{aligned}
 (1.1) \quad & \left\{ \begin{array}{ll} \operatorname{div}(\operatorname{div}(\mathbb{P}\nabla^2 w)) = 0, & \text{in } \Omega \setminus \overline{D}, \\ \end{array} \right. \\
 (1.2) \quad & \left\{ \begin{array}{ll} (\mathbb{P}\nabla^2 w)n \cdot n = -\widehat{M}_n, & \text{on } \partial\Omega, \\ \end{array} \right. \\
 (1.3) \quad & \left\{ \begin{array}{ll} \operatorname{div}(\mathbb{P}\nabla^2 w) \cdot n + ((\mathbb{P}\nabla^2 w)n \cdot \tau)_{,s} = (\widehat{M}_\tau)_{,s}, & \text{on } \partial\Omega, \\ \end{array} \right. \\
 (1.4) \quad & \left\{ \begin{array}{ll} w|_{\overline{D}} \in \mathcal{A}, & \text{in } \overline{D}, \\ \end{array} \right. \\
 (1.5) \quad & \left\{ \begin{array}{ll} \frac{\partial w^e}{\partial n} = \frac{\partial w^i}{\partial n}, & \text{on } \partial D, \\ \end{array} \right.
 \end{aligned}$$

coupled with the *equilibrium conditions* for the rigid inclusion D

$$(1.6) \quad \int_{\partial D} (\operatorname{div}(\mathbb{P}\nabla^2 w) \cdot n + ((\mathbb{P}\nabla^2 w)n \cdot \tau)_{,s}) g - ((\mathbb{P}\nabla^2 w)n \cdot n)g_{,n} = 0,$$

for every $g \in \mathcal{A}$,

where \mathcal{A} denotes the space of affine functions. We recall that, from the physical point of view, the boundary conditions (1.4)-(1.5) correspond to ideal connection between the boundary of the rigid inclusion and the surrounding elastic material, see, for example, [O-Ri, Section 10.10]. The unit vectors n and τ are the outer normal to $\Omega \setminus \overline{D}$ and the tangent vector to ∂D , respectively. Moreover, we have defined $w^e \equiv w|_{\Omega \setminus \overline{D}}$ and $w^i \equiv w|_{\overline{D}}$. The functions \widehat{M}_τ , \widehat{M}_n are the twisting and bending component of the assigned couple field

\widehat{M} , respectively. The plate tensor \mathbb{P} is given by $\mathbb{P} = \frac{h^3}{12}\mathbb{C}$, where h is the constant thickness of the plate and \mathbb{C} is the non-homogeneous Lamé elasticity tensor describing the response of the material.

The existence of a solution $w \in H^2(\Omega)$ of the problem (1.1)–(1.6) is ensured by general results, provided that $\widehat{M} \in H^{-\frac{1}{2}}(\partial\Omega, \mathbb{R}^2)$, with $\int_{\partial\Omega} \widehat{M}_i = 0$, for $i = 1, 2$, and \mathbb{P} is bounded and strongly convex. Let us notice that w is uniquely determined up to addition of an affine function.

Let us denote by w_i a solution to (1.1)–(1.6) for $D = D_i$, $i = 1, 2$. In order to deal with the stability issue, we found it convenient to replace each solution w_i with $v_i = w_i - g_i$, where g_i is the affine function which coincides with w_i on ∂D_i , $i = 1, 2$. By this approach, maintaining the same letter to denote the solution, the equilibrium problem (1.1)–(1.5) can be rephrased in terms of the following mixed boundary value problem with homogeneous Dirichlet conditions on the boundary of the rigid inclusion

$$\begin{aligned}
 (1.7) \quad & \left\{ \begin{array}{ll} \operatorname{div}(\operatorname{div}(\mathbb{P}\nabla^2 w)) = 0, & \text{in } \Omega \setminus \overline{D}, \\ (1.8) \quad & (\mathbb{P}\nabla^2 w)n \cdot n = -\widehat{M}_n, & \text{on } \partial\Omega, \\ (1.9) \quad & \operatorname{div}(\mathbb{P}\nabla^2 w) \cdot n + ((\mathbb{P}\nabla^2 w)n \cdot \tau)_{,s} = (\widehat{M}_\tau)_{,s}, & \text{on } \partial\Omega, \\ (1.10) \quad & w = 0, & \text{on } \partial D, \\ (1.11) \quad & \frac{\partial w}{\partial n} = 0, & \text{on } \partial D, \end{array} \right.
 \end{aligned}$$

for which there exists a unique solution $w \in H^2(\Omega \setminus \overline{D})$. The arbitrariness of this normalization, related to the fact that g_i is unknown, $i = 1, 2$, leads to the following formulation of the stability issue.

Given an open portion Σ of $\partial\Omega$, satisfying suitable regularity assumptions, and given two solutions w_i to (1.7)–(1.11) when $D = D_i$, $i = 1, 2$, satisfying, for some $\epsilon > 0$,

$$(1.12) \quad \min_{g \in \mathcal{A}} \left\{ \|w_1 - w_2 - g\|_{L^2(\Sigma)} + \left\| \frac{\partial}{\partial n}(w_1 - w_2 - g) \right\|_{L^2(\Sigma)} \right\} \leq \epsilon,$$

to evaluate the rate at which the Hausdorff distance $d_{\mathcal{H}}(\overline{D_1}, \overline{D_2})$ between D_1 and D_2 tends to zero as ϵ tends to zero.

In this paper we prove the following quantitative stability estimate of log type for inclusions D of $C^{6,\alpha}$ class:

$$(1.13) \quad d_{\mathcal{H}}(\overline{D_1}, \overline{D_2}) \leq C |\log \epsilon|^{-\eta},$$

where $C, \eta, C > 0$ and $\eta > 0$, are constants only depending on the a priori data, see Theorem 3.1 for a precise statement.

The above estimate is an improvement of the log-log type stability estimate found in [M-Ro-Ve2], although it must be said that the latter is not restricted to isotropic materials and also applies to less regular inclusions (e.g., D of $C^{3,1}$ class). It is worth to notice that a single logarithmic rate of convergence for the fourth order elliptic equation modelling the deflection of a Kirchhoff-Love plate is expected to be optimal, as it is in fact for the analogous inverse problem in the scalar elliptic case, which models the detection of perfectly conducting inclusions in an electric conductor in terms of measurements of potential and current taken on an accessible portion of the boundary of the body, as shown by the counterexamples due to Alessandrini ([Al]), Alessandrini and Rondi ([Al-R]), see also [Dc-R].

The methods used to prove (1.13) are inspired to the approach presented in the seminal paper [Al-Be-Ro-Ve] where, for the first time, it was shown how logarithmic stability estimates for the inverse problem of determining unknown boundaries can be derived by using quantitative estimates of Strong Unique Continuation at the Boundary (SUCB) which ensure a polynomial vanishing rate of the solutions satisfying homogeneous Dirichlet or Neumann conditions at the boundary. Precisely, in [Al-Be-Ro-Ve] the key tool was a Doubling Inequality at the boundary established by Adolfsson and Escauriza in [A-E].

Following the direction traced in [Al-Be-Ro-Ve], other kinds of quantitative estimates of the SUCB turned out to be crucial properties to prove optimal stability estimates for inverse boundary value problems with unknown boundaries in different frameworks, see for instance [S] where the case of Robin boundary condition is investigated. Let us recall, in the context of the case of thermic conductors involving parabolic equations, the three cylinders inequality and the one-sphere two-cylinders inequality at the boundary ([Ca-Ro-Ve1], [Ca-Ro-Ve2], [E-F-Ve], [E-Ve], [Ve1]), and similar estimate at the boundary for the case of wave equation with time independent coefficients [S-Ve], [Ve2], [Ve3].

In the present paper, the SUCB property used to improve the double logarithmic estimate found in [M-Ro-Ve2] takes the form of an optimal three spheres inequality at the boundary. This latter result was recently proved in [Al-Ro-Ve] for isotropic elastic plates under homogeneous Dirichlet boundary conditions, and leads to a Finite Vanishing Rate at the Boundary (Proposition 3.6).

Other main mathematical tools are quantitative estimates of Strong Unique Continuation at the Interior, essentially based on a three spheres inequality at the interior obtained in [M-Ro-Ve1] which allows to derive quantitative estimates of unique continuation from Cauchy data (Proposition 3.2), a Lipschitz estimate of Propagation of Smallness (Proposition 3.3) and the Finite

Vanishing Rate at the Interior (Proposition 3.5) for the solutions to the plate equation.

Let us observe that estimate (1.13) is the first stability estimate with optimal rate of convergence in the framework of linear elasticity. Indeed, up to now, the analogous estimate for the determination, within isotropic elastic bodies, of rigid inclusions ([M-Ro2]), cavities ([M-Ro1]) or pressurized cavities ([As-Be-Ro]) show a double logarithmic character, and the same convergence rate has been established by Lin, Nakamura and Wang for star-shaped cavities inside anisotropic elastic bodies ([L-N-W]).

The plan of the paper is as follows. Main notation and a priori information are presented in section 2. In section 3, we first state some auxiliary propositions regarding the estimate of continuation from Cauchy data (Proposition 3.2) and from the interior (Proposition 3.3), and the determination of the finite vanishing rate of the solutions to the plate equation at the interior (Proposition 3.5) and at the Dirichlet boundary (Proposition 3.6). Finally, in the second part of section 3 we give a proof of the main theorem (Theorem 3.1).

2 Notation

Let $P = (x_1(P), x_2(P))$ be a point of \mathbb{R}^2 . We shall denote by $B_r(P)$ the disk in \mathbb{R}^2 of radius r and center P and by $R_{a,b}(P)$ the rectangle of center P and sides parallel to the coordinate axes, of length $2a$ and $2b$, namely $R_{a,b}(P) = \{x = (x_1, x_2) \mid |x_1 - x_1(P)| < a, |x_2 - x_2(P)| < b\}$. To simplify the notation, we shall denote $B_r = B_r(O)$, $R_{a,b} = R_{a,b}(O)$.

Given a bounded domain Ω in \mathbb{R}^2 we shall denote

$$(2.1) \quad \Omega_\rho = \{x \in \Omega \mid \text{dist}(x, \partial\Omega) > \rho\}.$$

When representing locally a boundary as a graph, we use the following definition.

Definition 2.1. ($C^{k,\alpha}$ regularity) Let Ω be a bounded domain in \mathbb{R}^2 . Given k, α , with $k \in \mathbb{N}$, $0 < \alpha \leq 1$, we say that a portion S of $\partial\Omega$ is of class $C^{k,\alpha}$ with constants $r_0, M_0 > 0$, if, for any $P \in S$, there exists a rigid transformation of coordinates under which we have $P = 0$ and

$$\Omega \cap R_{r_0, 2M_0r_0} = \{x \in R_{r_0, 2M_0r_0} \mid x_2 > g(x_1)\},$$

where g is a $C^{k,\alpha}$ function on $[-r_0, r_0]$ satisfying

$$g(0) = g'(0) = 0,$$

$$\|g\|_{C^{k,\alpha}([-r_0,r_0])} \leq M_0 r_0,$$

where

$$\|g\|_{C^{k,\alpha}([-r_0,r_0])} = \sum_{i=0}^k r_0^i \sup_{[-r_0,r_0]} |g^{(i)}| + r_0^{k+\alpha} |g|_{k,\alpha},$$

$$|g|_{k,\alpha} = \sup_{\substack{t,s \in [-r_0,r_0] \\ t \neq s}} \left\{ \frac{|g^{(k)}(t) - g^{(k)}(s)|}{|t-s|^\alpha} \right\}.$$

Given a bounded domain Ω in \mathbb{R}^2 such that $\partial\Omega$ is of class $C^{k,\alpha}$, with $k \geq 1$, we consider as positive the orientation of the boundary induced by the outer unit normal n in the following sense. Given a point $P \in \partial\Omega$, let us denote by $\tau = \tau(P)$ the unit tangent at the boundary in P obtained by applying to n a counterclockwise rotation of angle $\frac{\pi}{2}$, that is

$$(2.2) \quad \tau = e_3 \times n,$$

where \times denotes the vector product in \mathbb{R}^3 and $\{e_1, e_2, e_3\}$ is the canonical basis in \mathbb{R}^3 .

In the sequel we shall denote by C constants which may change from line to line.

2.1 A priori information

i) A priori information on the domain.

Let us consider a thin plate $\Omega \times [-\frac{h}{2}, \frac{h}{2}]$ with middle surface represented by a bounded domain Ω in \mathbb{R}^2 and having uniform thickness h , $h \ll \text{diam}(\Omega)$.

We shall assume that, given $r_0, M_1 > 0$,

$$(2.3) \quad \text{diam}(\Omega) \leq M_1 r_0.$$

We shall also assume that Ω contains an open simply connected rigid inclusion D such that

$$(2.4) \quad \text{dist}(D, \partial\Omega) \geq r_0.$$

Moreover, we denote by Σ an open portion within $\partial\Omega$ representing the part of the boundary where measurements are taken.

Concerning the regularity of the boundaries, given $M_0 \geq \frac{1}{2}$ and $\alpha, 0 < \alpha \leq 1$, we assume that

$$(2.5) \quad \partial\Omega \text{ is of class } C^{2,1} \text{ with constants } r_0, M_0,$$

$$(2.6) \quad \Sigma \text{ is of class } C^{3,1} \text{ with constants } r_0, M_0.$$

$$(2.7) \quad \partial D \text{ is of class } C^{6,\alpha} \text{ with constants } r_0, M_0.$$

Let us notice that, without loss of generality, we have chosen $M_0 \geq \frac{1}{2}$ to ensure that $B_{r_0}(P) \subset R_{r_0, 2M_0r_0}(P)$ for every $P \in \partial\Omega$.

Moreover, we shall assume that for some $P_0 \in \Sigma$ and some $\delta_0, 0 < \delta_0 < 1$,

$$(2.8) \quad \partial\Omega \cap R_{r_0, 2M_0r_0}(P_0) \subset \Sigma,$$

and that

$$(2.9) \quad |\Sigma| \leq (1 - \delta_0)|\partial\Omega|.$$

ii) Assumptions about the boundary data.

On the Neumann data \widehat{M} we assume that

$$(2.10) \quad \widehat{M} \in L^2(\partial\Omega, \mathbb{R}^2), \quad (\widehat{M}_n, (\widehat{M}_\tau)_{,s}) \neq 0,$$

$$(2.11) \quad \text{supp}(\widehat{M}) \subset\subset \Sigma,$$

the (obvious) compatibility condition

$$(2.12) \quad \int_{\partial\Omega} \widehat{M}_i = 0, \quad i = 1, 2,$$

and that, for a given constant $F > 0$,

$$(2.13) \quad \frac{\|\widehat{M}\|_{L^2(\partial\Omega, \mathbb{R}^2)}}{\|\widehat{M}\|_{H^{-\frac{1}{2}}(\partial\Omega, \mathbb{R}^2)}} \leq F.$$

iii) Assumptions about the elasticity tensor.

Let us assume that the plate is made by elastic isotropic material, the plate tensor \mathbb{P} is defined by

$$(2.14) \quad \mathbb{P}A = B [(1 - \nu)A^{sym} + \nu(\text{tr}A)I_2],$$

for every 2×2 matrix A , where I_2 is the 2×2 identity matrix and $\text{tr}(A)$ denotes the trace of the matrix A . The *bending stiffness* (per unit length) of the plate is given by the function

$$(2.15) \quad B(x) = \frac{h^3}{12} \left(\frac{E(x)}{1 - \nu^2(x)} \right),$$

where the *Young's modulus* E and the *Poisson's coefficient* ν can be written in terms of the Lamé moduli as follows

$$(2.16) \quad E(x) = \frac{\mu(x)(2\mu(x) + 3\lambda(x))}{\mu(x) + \lambda(x)}, \quad \nu(x) = \frac{\lambda(x)}{2(\mu(x) + \lambda(x))}.$$

Hence, in this case, the displacement equation of equilibrium (1.1) is

$$(2.17) \quad \operatorname{div} (\operatorname{div} (B((1 - \nu)\nabla^2 w + \nu\Delta w I_2))) = 0, \quad \text{in } \Omega.$$

We make the following strong convexity assumptions on the Lamé moduli

$$(2.18) \quad \mu(x) \geq \alpha_0 > 0, \quad 2\mu(x) + 3\lambda(x) \geq \gamma_0 > 0, \quad \text{a.e. in } \Omega,$$

where α_0, γ_0 are positive constants.

We assume that the Lamé moduli λ, μ satisfy the following regularity assumptions

$$(2.19) \quad \|\lambda\|_{C^4(\bar{\Omega})}, \quad \|\mu\|_{C^4(\bar{\Omega})} \leq \Lambda_0.$$

Under the above assumptions, by standard variational arguments (see, for example, [Ag]), problem (1.7)–(1.11) has a unique solution $w \in H^2(\Omega \setminus \bar{D})$ satisfying

$$(2.20) \quad \|w\|_{H^2(\Omega \setminus \bar{D})} \leq Cr_0^2 \|\widehat{M}\|_{H^{-\frac{1}{2}}(\partial\Omega, \mathbb{R}^2)},$$

where $C > 0$ only depends on α_0, γ_0, M_0 , and M_1 .

In the sequel, we shall refer to the set of constants $\alpha_0, \gamma_0, \Lambda_0, \alpha, M_0, M_1, \delta_0$ and F as to the *a priori* data.

3 Statement and proof of the main result

Here and in the sequel we shall denote by G the connected component of $\Omega \setminus \overline{(D_1 \cup D_2)}$ such that $\Sigma \subset \partial G$.

Theorem 3.1 (Stability result). *Let Ω be a bounded domain in \mathbb{R}^2 satisfying (2.3) and (2.5). Let $D_i, i = 1, 2$, be two simply connected open subsets of Ω satisfying (2.4) and (2.7). Moreover, let Σ be an open portion of $\partial\Omega$ satisfying (2.6), (2.8) and (2.9). Let $\widehat{M} \in L^2(\partial\Omega, \mathbb{R}^2)$ satisfy (2.10)–(2.13) and let the plate tensor \mathbb{P} given by (2.14) with Lamé moduli satisfying the regularity assumptions (2.19) and the strong convexity condition (2.18). Let*

$w_i \in H^2(\Omega \setminus \overline{D_i})$ be the solution to (1.7)–(1.11) when $D = D_i$, $i = 1, 2$. If, given $\epsilon > 0$, we have

$$(3.1) \quad \min_{g \in \mathcal{A}} \left\{ \|w_1 - w_2 - g\|_{L^2(\Sigma)} + r_0 \left\| \frac{\partial}{\partial n} (w_1 - w_2 - g) \right\|_{L^2(\Sigma)} \right\} \leq \epsilon,$$

then we have

$$(3.2) \quad d_{\mathcal{H}}(\overline{D_1}, \overline{D_2}) \leq r_0 \omega \left(\frac{\epsilon}{r_0^2 \|\widehat{M}\|_{H^{-\frac{1}{2}}(\partial\Omega, \mathbb{R}^2)}} \right),$$

where ω is an increasing continuous function on $[0, \infty)$ which satisfies

$$(3.3) \quad \omega(t) \leq C(|\log t|)^{-\eta}, \quad \text{for every } t, \quad 0 < t < 1,$$

and $C, \eta, C > 0, \eta > 0$, are constants only depending on the a priori data.

The proof of Theorem 3.1 is obtained from a sequence of propositions. The following proposition can be derived by merging Proposition 3.4 of [M-Ro-Ve2] and geometrical arguments contained in Proposition 3.6 of [Al-Be-Ro-Ve].

Proposition 3.2 (Stability Estimate of Continuation from Cauchy Data [M-Ro-Ve2, Proposition 3.4]). *Let the hypotheses of Theorem 3.1 be satisfied. We have*

$$(3.4) \quad \int_{(\Omega \setminus \overline{G}) \setminus \overline{D_1}} |\nabla^2 w_1|^2 \leq r_0^2 \|\widehat{M}\|_{H^{-\frac{1}{2}}(\partial\Omega, \mathbb{R}^2)}^2 \omega \left(\frac{\epsilon}{r_0^2 \|\widehat{M}\|_{H^{-\frac{1}{2}}(\partial\Omega, \mathbb{R}^2)}} \right),$$

$$(3.5) \quad \int_{(\Omega \setminus \overline{G}) \setminus \overline{D_2}} |\nabla^2 w_2|^2 \leq r_0^2 \|\widehat{M}\|_{H^{-\frac{1}{2}}(\partial\Omega, \mathbb{R}^2)}^2 \omega \left(\frac{\epsilon}{r_0^2 \|\widehat{M}\|_{H^{-\frac{1}{2}}(\partial\Omega, \mathbb{R}^2)}} \right),$$

where ω is an increasing continuous function on $[0, \infty)$ which satisfies

$$(3.6) \quad \omega(t) \leq C(\log |\log t|)^{-\frac{1}{2}}, \quad \text{for every } t < e^{-1},$$

with $C > 0$ only depending on $\alpha_0, \gamma_0, \Lambda_0, M_0, M_1$ and δ_0 .

Moreover, there exists $d_0 > 0$, with $\frac{d_0}{r_0}$ only depending on M_0 , such that if $d_{\mathcal{H}}(\overline{\Omega \setminus D_1}, \overline{\Omega \setminus D_2}) \leq d_0$ then (3.4)–(3.5) hold with ω given by

$$(3.7) \quad \omega(t) \leq C|\log t|^{-\sigma}, \quad \text{for every } t < 1,$$

where $\sigma > 0$ and $C > 0$ only depend on $\alpha_0, \gamma_0, \Lambda_0, M_0, M_1, \delta_0, L$ and $\frac{\tilde{r}_0}{r_0}$.

Next two propositions are quantitative versions of the SUCP property at the interior for solutions to the plate equilibrium problem. Precisely, Proposition 3.3 has global character and gives a lower bound of the strain energy density over any small disc compactly contained in $\Omega \setminus \overline{D}$ in terms of the Neumann boundary data. Proposition 3.5 establishes a polynomial order of vanishing for solutions to the plate problem at interior points of $\Omega \setminus \overline{D}$.

Proposition 3.3 (Lipschitz Propagation of Smallness [M-Ro-Ve2, Proposition 3.3]). *Let Ω be a bounded domain in \mathbb{R}^2 satisfying (2.3) and (2.5). Let D be an open simply connected subset of Ω satisfying (2.4), (2.7). Let $w \in H^2(\Omega \setminus \overline{D})$ be the solution to (1.7)–(1.11), coupled with the equilibrium condition (1.6), where the plate tensor \mathbb{P} is given by (2.14) with Lamé moduli satisfying the regularity assumptions (2.19) and the strong convexity condition (2.18) and with $\widehat{M} \in L^2(\partial\Omega, \mathbb{R}^2)$ satisfying (2.10)–(2.13).*

There exists $s > 1$, only depending on $\alpha_0, \gamma_0, \Lambda_0, M_0$ and δ_0 , such that for every $\rho > 0$ and every $\bar{x} \in (\Omega \setminus \overline{D})_{s\rho}$, we have

$$(3.8) \quad \int_{B_\rho(\bar{x})} |\nabla^2 w|^2 \geq \frac{Cr_0^2}{\exp \left[A \left(\frac{r_0}{\rho} \right)^B \right]} \|\widehat{M}\|_{H^{-\frac{1}{2}}(\partial\Omega, \mathbb{R}^2)}^2,$$

where $A > 0$, $B > 0$ and $C > 0$ only depend on $\alpha_0, \gamma_0, \Lambda_0, M_0, M_1, \delta_0$ and F .

Remark 3.4. The exponential character of the dependence on ρ in (3.8) comes from the fact that this global estimate follows from the trace-type inequality

$$\|\widehat{M}\|_{H^{-\frac{1}{2}}(\partial\Omega, \mathbb{R}^2)} \leq C \|\nabla^2 w\|_{L^2(\Omega \setminus \overline{D})},$$

with C only depending on M_0, M_1, δ_0 and Λ_0 (see [M-Ro-Ve2, Lemma 4.6]) and from a lower bound of the strain energy density over the disc $B_\rho(\bar{x})$ in terms of the strain energy density over all the domain $\Omega \setminus \overline{D}$. The latter estimate requires a geometrical construction involving a number of iterated applications of the three spheres inequality depending in the radius ρ which leads to an exponential dependence. The local polynomial vanishing rate at the interior is given in the following proposition.

Proposition 3.5 (Finite Vanishing Rate at the Interior). *Under the hypotheses of Proposition 3.3, there exist $\bar{c}_0 < \frac{1}{2}$ and $C > 1$, only depending on $\alpha_0, \gamma_0, \Lambda_0$, such that, for every $\bar{r} > 0$ and for every $x \in \Omega \setminus \overline{D}$ such that $B_{\bar{r}}(x) \subset \Omega \setminus \overline{D}$, and for every $r_1 < r_2 = \bar{c}_0 \bar{r}$, we have*

$$(3.9) \quad \int_{B_{r_1}(x)} |\nabla^2 w|^2 \geq C \left(\frac{r_1}{\bar{r}} \right)^{\tau_0} \int_{B_{r_2}(x)} |\nabla^2 w|^2,$$

where $\tau_0 \geq 1$ only depends on $\alpha_0, \gamma_0, \Lambda_0, \delta_0$ and F .

Proof. The above estimate is based on the following three spheres inequality at the interior, which was obtained in [M-Ro-Ve1, Theorem 6.6]: there exist $c_0, 0 < c_0 < 1$, and $C > 1$ only depending on $\alpha_0, \gamma_0, \Lambda_0$, such that for every $\bar{r} > 0$, for every $x \in (\Omega \setminus \overline{D})$ such that $B_{\bar{r}}(x) \subset \Omega \setminus \overline{D}$, and for every $r_1 < r_2 < r_3 < c_0 \bar{r}$, we have

$$(3.10) \quad \int_{B_{r_2}(x)} |\nabla^2 w|^2 \leq C \left(\frac{r_3}{r_1} \right)^C \left(\int_{B_{r_1}(x)} |\nabla^2 w|^2 \right)^{\vartheta_0} \left(\int_{B_{r_3}(x)} |\nabla^2 w|^2 \right)^{1-\vartheta_0},$$

where

$$(3.11) \quad \vartheta_0 = \frac{\log \left(\frac{c_0 r_3}{r_2} \right)}{2 \log \left(\frac{r_3}{r_1} \right)}.$$

Inequality (3.9) can be derived by exploiting the optimality of the exponent ϑ_0 and by reassembling the terms in (3.10). \square

As noticed in the introduction, our key SUCB property is stated in the following proposition, which is the counterpart at the boundary ∂D of Proposition 3.5.

Proposition 3.6 (Finite Vanishing Rate at the Boundary). *Under the hypotheses of Proposition 3.3, there exist $\bar{c} < \frac{1}{2}$ and $C > 1$, only depending on $\alpha_0, \gamma_0, \Lambda_0, M_0, \alpha$, such that, for every $x \in \partial D$ and for every $r_1 < r_2 = \bar{c} r_0$,*

$$(3.12) \quad \int_{B_{r_1}(x) \cap (\Omega \setminus \overline{D})} w^2 \geq C \left(\frac{r_1}{r_0} \right)^\tau \int_{B_{r_0}(x) \cap (\Omega \setminus \overline{D})} w^2.$$

where $\tau \geq 1$ only depends on $\alpha_0, \gamma_0, \Lambda_0, M_0, \alpha, M_1, \delta_0$ and F .

Proof. By Corollary 2.3 in [Al-Ro-Ve], there exist $c < 1$, only depending on M_0, α , and $C > 1$ only depending on $\alpha_0, \gamma_0, \Lambda_0, M_0, \alpha$, such that, for every $x \in \partial D$ and for every $r_1 < r_2 < c r_0$,

$$(3.13) \quad \int_{B_{r_1}(x) \cap (\Omega \setminus \overline{D})} w^2 \geq C \left(\frac{r_1}{r_0} \right)^{\frac{\log B}{\log \frac{c r_0}{r_2}}} \int_{B_{r_0}(x) \cap (\Omega \setminus \overline{D})} w^2,$$

where $B > 1$ is given by

$$(3.14) \quad B = C \left(\frac{r_0}{r_2} \right)^C \frac{\int_{B_{r_0}(x) \cap (\Omega \setminus \overline{D})} w^2}{\int_{B_{r_2}(x) \cap (\Omega \setminus \overline{D})} w^2},$$

Let us choose in the above inequalities $r_2 = \bar{c}r_0$, with $\bar{c} = \frac{c}{2}$.

By (2.20) we have

$$(3.15) \quad \int_{B_{r_0}(x) \cap (\Omega \setminus \bar{D})} w^2 \leq Cr_0^6 \|\widehat{M}\|_{H^{-\frac{1}{2}}(\partial\Omega, \mathbb{R}^2)}^2,$$

with C depending on $\alpha_0, \gamma_0, M_0, \alpha, M_1$. By interpolation estimates for solutions to elliptic equations (see, for instance, [Al-Ro-Ve, Lemma 4.7], stated for the case of hemidisks, but which holds also in the present context), we have that

$$\int_{B_{r_2}(x) \cap (\Omega \setminus \bar{D})} w^2 \geq Cr_2^4 \int_{B_{\frac{r_2}{2}}(x) \cap (\Omega \setminus \bar{D})} |\nabla^2 w|^2,$$

with C depending on α_0, γ_0 and Λ_0 . By Proposition 3.3 and recalling the definition of r_2 , we derive

$$(3.16) \quad \int_{B_{r_2}(x) \cap (\Omega \setminus \bar{D})} w^2 \geq Cr_0^6 \|\widehat{M}\|_{H^{-\frac{1}{2}}(\partial\Omega, \mathbb{R}^2)}^2,$$

with C depending on $\alpha_0, \gamma_0, \Lambda_0, M_0, \alpha, M_1, \delta_0$ and F .

By (3.15)–(3.16), we can estimate B from above, obtaining the thesis. \square

Proof of Theorem 3.1. In order to estimate the Hausdorff distance between the inclusions,

$$(3.17) \quad \delta = d_{\mathcal{H}}(\bar{D}_1, \bar{D}_2),$$

it is convenient to introduce the following auxiliary distances:

$$(3.18) \quad d = d_{\mathcal{H}}(\overline{\Omega \setminus D_1}, \overline{\Omega \setminus D_2}),$$

$$(3.19) \quad d_m = \max \left\{ \max_{x \in \partial D_1} \text{dist}(x, \overline{\Omega \setminus D_2}), \max_{x \in \partial D_2} \text{dist}(x, \overline{\Omega \setminus D_1}) \right\}.$$

Let $\eta > 0$ such that

$$(3.20) \quad \max_{i=1,2} \int_{(\Omega \setminus \bar{G}) \setminus \bar{D}_i} |\nabla^2 w_i|^2 \leq \eta.$$

Following the arguments presented in [Al-Be-Ro-Ve], we first control d_m in terms of η , and then we use this estimate to control d in terms of η . Finally, by improving the results in [Ca-Ro-Ve2, proof of Theorem 1.1, step 2], we estimate δ in terms of d .

Let us start by proving the inequality

$$(3.21) \quad d_m \leq Cr_0 \left(\frac{\eta}{r_0^2 \|\widehat{M}\|_{H^{-1/2}(\partial\Omega)}^2} \right)^{\frac{1}{\tau}},$$

where τ has been introduced in Proposition 3.6 and C is a positive constant only depending on the a priori data.

Let us assume, without loss of generality, that there exists $x_0 \in \partial D_1$ such that

$$(3.22) \quad \text{dist}(x_0, \overline{\Omega \setminus D_2}) = d_m > 0.$$

Since $B_{d_m}(x_0) \subset D_2 \subset \Omega \setminus \overline{G}$, we have

$$(3.23) \quad B_{d_m}(x_0) \cap (\Omega \setminus \overline{D_1}) \subset (\Omega \setminus \overline{G}) \setminus \overline{D_1}$$

and then, by (3.20),

$$(3.24) \quad \int_{B_{d_m}(x_0) \cap (\Omega \setminus \overline{D_1})} |\nabla^2 w_1|^2 \leq \eta.$$

Let us assume that

$$(3.25) \quad d_m < \bar{c}r_0,$$

where \bar{c} is the positive constant appearing in Proposition 3.6. Since $w_1 = 0$, $\nabla w_1 = 0$ on ∂D_1 , by Poincaré inequality (see, for instance, [Al-M-Ro, Example 4.4]) and noticing that $d_m \leq \text{diam}(\Omega) \leq M_1 r_0$, we have

$$(3.26) \quad \eta \geq \frac{C}{r_0^4} \int_{B_{d_m}(x_0) \cap (\Omega \setminus \overline{D_1})} w_1^2,$$

where $C > 0$ is a positive constant only depending on α , M_0 , M_1 .

By Proposition 3.6, we have

$$(3.27) \quad \eta \geq \frac{C}{r_0^4} \left(\frac{d_m}{r_0} \right)^\tau \int_{B_{r_0}(x_0) \cap (\Omega \setminus \overline{D_1})} w_1^2,$$

where $C > 0$ is a positive constant only depending on α_0 , γ_0 , Λ_0, α , M_0 , M_1 and F .

By Lemma 4.7 in [Al-Ro-Ve], we have

$$(3.28) \quad \eta \geq C \left(\frac{d_m}{r_0} \right)^\tau \int_{B_{\frac{r_0}{2}}(x_0) \cap (\Omega \setminus \overline{D_1})} |\nabla^2 w_1|^2,$$

where $C > 0$ is a positive constant only depending on $\alpha_0, \gamma_0, \Lambda_0, \alpha, M_0, M_1$.
By Proposition 3.3, we have

$$(3.29) \quad \eta \geq C \left(\frac{d_m}{r_0} \right)^\tau r_0^2 \|\widehat{M}\|_{H^{-1/2}(\partial\Omega)}^2,$$

where $C > 0$ is a positive constant only depending on $\alpha_0, \gamma_0, \Lambda_0, \alpha, M_0, M_1, \delta_0, F$, from which we can estimate d_m

$$(3.30) \quad d_m \leq Cr_0 \left(\frac{\eta}{r_0^2 \|\widehat{M}\|_{H^{-1/2}(\partial\Omega)}^2} \right)^{\frac{1}{\tau}},$$

where $C > 0$ is a positive constant only depending on $\alpha_0, \gamma_0, \Lambda_0, \alpha, M_0, M_1, \delta_0, F$.

Now, let us assume that

$$(3.31) \quad d_m \geq \bar{c}r_0.$$

By starting again from (3.24), and applying Proposition 3.3 and recalling $d_m \leq M_1 r_0$, we easily have

$$(3.32) \quad d_m \leq Cr_0 \left(\frac{\eta}{r_0^2 \|\widehat{M}\|_{H^{-1/2}(\partial\Omega)}^2} \right),$$

where $C > 0$ is a positive constant only depending on $\alpha_0, \gamma_0, \Lambda_0, M_0, M_1, \delta_0, F$. Assuming $\eta \leq r_0^2 \|\widehat{M}\|_{H^{-1/2}(\partial\Omega)}^2$, we obtain (3.21).

Without loss of generality, let $y_0 \in \overline{\Omega \setminus D_1}$ such that

$$(3.33) \quad \text{dist}(y_0, \overline{\Omega \setminus D_2}) = d.$$

It is significant to assume $d > 0$, so that $y_0 \in D_2 \setminus D_1$. Let us define

$$(3.34) \quad h = \text{dist}(y_0, \partial D_1),$$

possibly $h = 0$.

There are three cases to consider:

- i) $h \leq \frac{d}{2}$;
- ii) $h > \frac{d}{2}, h \leq \frac{d_0}{2}$;
- iii) $h > \frac{d}{2}, h > \frac{d_0}{2}$.

Here the number $d_0, 0 < d_0 < r_0$, is such that $\frac{d_0}{r_0}$ only depends on M_0 , and it is the same constant appearing in Proposition 3.4. In particular, Proposition 3.6 in [Al-Be-Ro-Ve] shows that there exists an absolute constant $C > 0$ such that if $d \leq d_0$, then $d \leq Cd_m$.

Case i).

By definition, there exists $z_0 \in \partial D_1$ such that $|z_0 - y_0| = h$. By applying the triangle inequality, we get $\text{dist}\left(z_0, \overline{\Omega \setminus D_2}\right) \geq \frac{d}{2}$. Since, by definition, $\text{dist}\left(z_0, \overline{\Omega \setminus D_2}\right) \leq d_m$, we obtain $d \leq 2d_m$.

Case ii).

It turns out that $d < d_0$ and then, by the above recalled property, again we have that $d \leq Cd_m$, for an absolute constant C .

Case iii).

Let $\tilde{h} = \min\{h, r_0\}$. We obviously have that $B_{\tilde{h}}(y_0) \subset \Omega \setminus \overline{D_1}$ and $B_d(y_0) \subset D_2$. Let us set

$$d_1 = \min\left\{\frac{d}{2}, \frac{\bar{c}_0 d_0}{4}\right\}.$$

Since $d_1 < d$ and $d_1 < \tilde{h}$, we have that $B_{d_1}(y_0) \subset D_2 \setminus \overline{D_1}$ and therefore $\eta \geq \int_{B_{d_1}(y_0)} |\nabla^2 w_1|^2$.

Since $\frac{d_0}{2} < \tilde{h}$, $B_{\frac{d_0}{2}}(y_0) \subset \Omega \setminus \overline{D_1}$ so that we can apply Proposition 3.5 with $r_1 = d_1, \bar{r} = \frac{d_0}{2}, r_2 = \bar{c}_0 \bar{r}$, obtaining $\eta \geq C \left(\frac{2d_1}{d_0}\right)^{\tau_0} \int_{B_{\frac{d_0}{2}}(y_0)} |\nabla^2 w_1|^2$, with $C > 0$ only depending on $\alpha_0, \gamma_0, \Lambda_0, \delta_0$ and F . Next, by Proposition 3.3, recalling that $\frac{d_0}{r_0}$ only depends on M_0 , we derive that

$$d_1 \leq Cr_0 \left(\frac{\eta}{r_0^2 \|\widehat{M}\|_{H^{-1/2}(\partial\Omega)}^2} \right)^{\frac{1}{\tau_0}},$$

where $C > 0$ only depends on $\alpha_0, \gamma_0, \Lambda_0, M_0, M_1, \delta_0$ and F . For η small enough, $d_1 < \frac{\bar{c}_0 d_0}{4}$, so that $d_1 = \frac{d}{2}$ and

$$d \leq Cr_0 \left(\frac{\eta}{r_0^2 \|\widehat{M}\|_{H^{-1/2}(\partial\Omega)}^2} \right)^{\frac{1}{\tau_0}}.$$

Collecting the three cases, we have

$$(3.35) \quad d \leq Cr_0 \left(\frac{\eta}{r_0^2 \|\widehat{M}\|_{H^{-1/2}(\partial\Omega)}^2} \right)^{\frac{1}{\tau_1}},$$

with $\tau_1 = \max\{\tau, \tau_0\}$ and $C > 0$ only depends on $\alpha_0, \gamma_0, \Lambda_0, \alpha, M_0, M_1, \delta_0$ and F .

Finally, let us estimate δ in terms of d . By (3.35), for η small enough, we have that

$$d < \frac{r_0}{4\sqrt{1+M_0^2}}.$$

Let us notice that if a point y belongs to $\overline{D_1} \setminus D_2$, then $\text{dist}(y, \partial D_1) \leq d$.

Without loss of generality let $\bar{x} \in \overline{D_1}$ such that $\text{dist}(\bar{x}, \overline{D_2}) = \delta > 0$. Then $\bar{x} \in \overline{D_1} \setminus D_2$ and therefore $\text{dist}(\bar{x}, \partial D_1) \leq d$.

Let $w \in \partial D_1$ such that $|w - \bar{x}| = \text{dist}(\bar{x}, \partial D_1) \leq d$.

Letting n the outer unit normal to D_1 at w , we may write $\bar{x} = w - |w - \bar{x}|n$. By our regularity assumptions on D_1 , the truncated cone $C(\bar{x}, -n, 2(\frac{\pi}{2} - \arctan M_0)) \cap B_{\frac{r_0}{4}}(\bar{x})$ having vertex \bar{x} , axis $-n$ and width $2(\frac{\pi}{2} - \arctan M_0)$, is contained in $\overline{D_1}$.

On the other hand, by definition of δ , $B_\delta(\bar{x}) \subset \Omega \setminus \overline{D_2}$, so that the truncated cone $C(\bar{x}, -n, 2(\frac{\pi}{2} - \arctan M_0)) \cap B_{\min\{\delta, r_0/4\}}(\bar{x})$ is contained in $D_1 \setminus \overline{D_2}$.

Let us see that $\delta < \frac{r_0}{4}$. In fact if, by contradiction, $\delta \geq \frac{r_0}{4}$, we can consider the point $z = \bar{x} - \frac{r_0}{4}n$. Since $z \in D_1 \setminus D_2$, as noticed above, $\text{dist}(z, \partial D_1) \leq d$. On the other hand, by using the fact that $|z - w| \leq \frac{r_0}{2}$ and by the regularity of D_1 , it is easy to compute that $\text{dist}(z, \partial D_1) \geq \frac{r_0}{4\sqrt{1+M_0^2}}$, obtaining a contradiction.

Hence $\min\{\delta, \frac{r_0}{4}\} = \delta$ and, by defining $\bar{z} = \bar{x} - \delta n$ and by analogous calculations, we can conclude that $\delta \leq (\sqrt{1+M_0^2})d$, which is the desired estimate of δ in terms of d .

By Proposition 3.2,

$$(3.36) \quad d \leq Cr_0 \left(\log \left| \log \left(\frac{\epsilon}{r_0^2 \|\widehat{M}\|_{H^{-1/2}(\partial\Omega)}^2} \right) \right| \right)^{-\frac{1}{2\tau_1}},$$

with $\tau_1 \geq 1$ and $C > 0$ only depends on $\alpha_0, \gamma_0, \Lambda_0, \alpha, M_0, M_1, \delta_0$ and F . By this first rough estimate, there exists $\epsilon_0 > 0$, only depending on α_0 ,

$\gamma_0, \Lambda_0, \alpha, M_0, M_1, \delta_0$ and F , such that, if $\epsilon \leq \epsilon_0$, then $d \leq d_0$. Therefore the second part of Proposition 3.2 applies and the thesis follows. \square

Acknowledgements

Antonino Morassi is supported by PRIN 2015TTJN95 “Identificazione e diagnostica di sistemi strutturali complessi”. Edi Rosset and Sergio Vessella are supported by Progetto GNAMPA 2017 “Analisi di problemi inversi: stabilità e ricostruzione”, Istituto Nazionale di Alta Matematica (INdAM). Edi Rosset is supported by FRA2016 “Problemi inversi, dalla stabilità alla ricostruzione”, Università degli Studi di Trieste.

References

- [A-E] V. Adolfsson, L. Escauriaza, $C^{1,\alpha}$ domains and unique continuation at the boundary, *Comm. Pure Appl. Math.* L (1997) 935–969.
- [Ag] S. Agmon, *Lectures on Elliptic Boundary Value Problems*, Van Nostrand, New York, 1965.
- [Al] G. Alessandrini, Examples of instability in inverse boundary-value problems, *Inverse Problems* 13 (1997) 887–897.
- [Al-Be-Ro-Ve] G. Alessandrini, E. Beretta, E. Rosset, S. Vessella, Optimal stability for inverse elliptic boundary value problems with unknown boundaries, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* 29(4) (2000) 755–806.
- [Al-M-Ro] G. Alessandrini, A. Morassi, E. Rosset, The linear constraints in Poincaré and Korn type inequalities, *Forum Math.* 20 (2008) 557–569.
- [Al-R] G. Alessandrini, L. Rondi, Optimal stability for the inverse problem of multiple cavities, *J. Differential Equations* 176 (2001) 356–386.
- [Al-Ro-Ve] G. Alessandrini, E. Rosset, S. Vessella, Optimal three spheres inequality at the boundary for the Kirchhoff-Love plate’s equations with Dirichlet conditions, submitted (2018), arXiv:1802.08631.
- [As-Be-Ro] A. Aspri, E. Beretta, E. Rosset, On an elastic model arising from volcanology: an analysis of the direct and inverse problem, (submitted) (2018), arXiv:1705.11099.

- [Bo-Co] M. Bonnet, A. Constantinescu, Inverse problems in elasticity, *Inverse Problems* 21(2) (2005) 1–50.
- [Ca-Ro-Ve1] B. Canuto, E. Rosset, S. Vessella, Quantitative estimates of unique continuation for parabolic equations and inverse initial-boundary value problems with unknown boundaries, *Transactions Amer. Math. Soc.* 354 (2002) 491–535.
- [Ca-Ro-Ve2] B. Canuto, E. Rosset, S. Vessella, A stability result in the localization of cavities in a thermic conducting medium, *ESAIM: Control, Optimisation and Calculus of Variations* 7 (2002) 521–565.
- [Dc-R] M. Di Cristo, L. Rondi, Examples of exponential instability for inverse inclusion and scattering problems *Inverse Problems* 19 (2003) 685–701.
- [E-F-Ve] L. Escauriaza, F. J. Fernandez, and S. Vessella, Doubling properties of caloric functions, *Appl. Anal.* 85 (2006) 205–223.
- [E-Ve] L. Escauriaza, S. Vessella, Optimal three cylinder inequalities for solutions to parabolic equations with Lipschitz leading coefficients, in *Inverse Problems: Theory and Applications*, G. Alessandrini and G. Uhlmann, eds., *Contemp. Math.* 333, AMS, Providence, RI, 2003, pp. 79–87.
- [G] M. E. Gurtin, *The Linear Theory of Elasticity*, *Handbuch der Physik*, vol. VI, Springer-Verlag, Berlin, Heidelberg, New York, 1972.
- [K] A. Katunin, Identification of stiff inclusion in circular composite plate based on quaternion wavelet analysis of modal shapes, *J. of Vibroengineering* 16(5) (2014) 2545-2551.
- [L-N-W] C-L. Lin, G. Nakamura, J-N. Wang, Three spheres inequalities for a two-dimensional elliptic system and its application, *J. Differential Equations* 232 (2007) 329–351.
- [M-Ro1] A. Morassi, E. Rosset, Stable determination of cavities in elastic bodies, *Inverse Problems* 20 (2004) 453–480.
- [M-Ro2] A. Morassi, E. Rosset, Uniqueness and stability in determining a rigid inclusion in an elastic body, *Memoirs of the American Mathematical Society*, volume 200, issue 938 (2009).
- [M-Ro-Ve1] A. Morassi, E. Rosset, S. Vessella, Size estimates for inclusions in an elastic plate by boundary measurements, *Indiana Univ. Math. J.* 56(5) (2007) 2535–2384.

- [M-Ro-Ve2] A. Morassi, E. Rosset, S. Vessella, Stable determination of a rigid inclusion in an anisotropic elastic plate, *SIAM J. Math. Anal.* 44(3) (2012) 2204–2235.
- [O-Ri] J.T. Oden, E.A. Ripperger, *Mechanics of Elastic Structures*, Hemisphere Publishing Co., Washington, New York, London 1981.
- [S] E. Sincich, Stability for the determination of unknown boundary and impedance with a Robin boundary condition, *SIAM J. Math. Anal.* 42 (2010) 2922–2943.
- [S-Ve] E. Sincich, S. Vessella, Wave equation with Robin condition, quantitative estimates of strong unique continuation at the boundary. *Rend. Istit. Mat. Univ. Trieste* 48 (2016) 221–243.
- [Ve1] S. Vessella, Quantitative estimates of unique continuation for parabolic equations, determination of unknown time-varying boundaries and optimal stability estimates, *Inverse Problems*, 24 (2008) 1–81.
- [Ve2] S. Vessella, Quantitative estimates of strong unique continuation for wave equations, *Math. Ann.* 367(1-2) (2017) 135–164.
- [Ve3] S. Vessella, Stability estimates for an inverse hyperbolic initial boundary value problem with unknown boundaries, *SIAM J. Math. Anal.* 47 (2015) 1419–1457.