Branched coverings of $\mathbb{C}\mathrm{P}^2$ and other basic 4-manifolds

Riccardo Piergallini and Daniele Zuddas

Abstract

We give necessary and sufficient conditions for a 4-manifold to be a branched covering of $\mathbb{C}\mathrm{P}^2$, $S^2 \times S^2$, $S^2 \tilde{\times} S^2$ or $S^3 \times S^1$, which are expressed in terms of the Betti numbers and the signature of the 4-manifold. Moreover, we extend these results to include branched coverings of connected sums of the above manifolds. This leads to some new examples of closed simply connected quasiregularly elliptic 4-manifolds.

1. *Introduction*

In [**24**] the first author proved that any closed orientable PL 4-manifold M is a simple 4-fold covering of S^4 branched over a closed locally flat PL surface self-transversally immersed in S^4. Subsequently, in [**15**] the self-intersections of the branch surface were shown to be removable once the covering has been stabilized to degree five, obtaining M as a 5-fold covering of S^4 branched over a closed locally flat PL surface embedded in S^4.

On the other hand, it is a classical fact of algebraic geometry that any smooth irreducible projective surface $S \subset \mathbb{C}\mathrm{P}^n$ is a holomorphic branched covering of $\mathbb{C}\mathrm{P}^2$ obtained by taking a general projection, where the branch set is an irreducible nodal cuspidal algebraic curve in $\mathbb{C}\mathrm{P}^2$. Even though this result is folklore, a proof appeared surprisingly only in a 2011 paper by Ciliberto and Flamini [**9**].

Furthermore, Auroux in [**3**] extended this result to all closed integral symplectic 4-manifolds M, proving that, roughly, they are realizable as "symplectic" coverings of $\mathbb{C}\mathrm{P}^2$ branched over a symplectic nodal cuspidal surface in $\mathbb{C}\mathrm{P}^2$. In fact, every closed integral symplectic 4-manifold (M, ω) admits a branched covering $M \to \mathbb{C}\mathrm{P}^2$, such that the pullback of the Fubini-Study form ω_{FS} can be suitably perturbed to a symplectic form which is ambient isotopic to $k\omega$ for some sufficiently large integer k. Moreover, any symplectic form on M is homotopic, through symplectic forms, to an integral one realizable as above. It is worth noting that there is a subtle difference between holomorphic and symplectic singular surfaces in $\mathbb{C}\mathrm{P}^2$: holomorphic nodal singularities are always positive, while the symplectic ones may also be negative.

Hence, it is interesting to study the topology of branched coverings of $\mathbb{C}\mathrm{P}^2$, and a natural question is the following: *which closed oriented 4-manifolds are realizable as branched coverings of* $\mathbb{C}\mathrm{P}^2$*?*

In this paper we give a complete answer to this question, by proving that a closed connected orientable PL 4-manifold M is a simple branched covering of $\mathbb{C}\mathrm{P}^2$ (branched over an embedded locally flat surface) if and only if the second Betti number $b_2(M)$ is positive. In addition, we also characterize the 4-manifolds M that are branched coverings of $S^2 \times S^2$, $S^2 \tilde{\times} S^2$ and $S^3 \times S^1$.

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Finally, we generalize these results to branched coverings of $\#_m \mathbb{C}\mathrm{P}^2 \#_n \overline{\mathbb{C}\mathrm{P}}^2$, $\#_n(S^2 \times S^2)$ and $\#_n(S^3 \times S^1)$.

The proofs of all these results follow the same idea: we split M into two pieces, based on certain submanifolds $N \subset M$, and represent them as branched coverings of standard bounded 4-manifolds by using [**25**], then we glue such branched coverings together. As a consequence of this argument, we also obtain a representation of the submanifolds $N \subset M$ as branched coverings of suitable standard submanifolds of the base spaces considered above, see Section 4.

For the sake of convenience, we work in the PL category. Nevertheless, our results can be easily translated into the smooth category as well, being PL = Diff in dimension four.

The following notations is used throughout the paper: $\mathbb{C}\mathrm{P}^2$ and $\overline{\mathbb{C}\mathrm{P}}^2$ for the complex projective space with the standard and the opposite orientation, respectively; $S^2 \mathbin{\tilde{\times}} S^2 \cong \mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}}^2$ for the twisted S^2-bundle over S^2; $b_i(M)$ for the i-th Betti number of M;

$$\beta_M : H_2(M)/\operatorname{Tor} H_2(M) \times H_2(M)/\operatorname{Tor} H_2(M) \to \mathbb{Z}$$

for the intersection form of M; $b_2^+(M)$ (resp. $b_2^-(M)$) for the maximal dimension of a vector subspace of $H_2(M;\mathbb{R})$ where β_M is positive (resp. negative) definite; and finally $\sigma(M)$ for the signature of M (see [**12**], [**16**] or [**20**]). Now we state our main theorems.

THEOREM 1.1. *Let M be a closed connected oriented PL 4-manifold. Then, there exists a branched covering $p\colon M \to N$ with:*
(a) $N = \mathbb{C}\mathrm{P}^2 \Leftrightarrow b_2^+(M) \geq 1$;
(b) $N = \overline{\mathbb{C}\mathrm{P}}^2 \Leftrightarrow b_2^-(M) \geq 1$;
(c) $N = S^2 \mathbin{\tilde{\times}} S^2 \Leftrightarrow b_2^+(M) \geq 1$ *and* $b_2^-(M) \geq 1$;
(d) $N = S^2 \times S^2 \Leftrightarrow b_2^+(M) \geq 1$ *and* $b_2^-(M) \geq 1$;
(e) $N = S^3 \times S^1 \Leftrightarrow b_1(M) \geq 1$.
In all cases, we can assume that p is a simple branched covering of degree $d \leq 4$, whose branch set B_p is a closed locally flat PL surface self-transversally immersed in N. Moreover, B_p can be desingularized to become embedded in N, with the following estimates for the degree d: $d \leq 5$ in cases (a) and (b) for $b_2(M) \geq 2$ and β_M odd, case (c) for β_M odd, case (d) for β_M even, and case (e); $d \leq 6$ in cases (a) and (b) for $b_2(M) \geq 2$ and β_M even, case (c) for β_M even, and case (d) for β_M odd; $d \leq 9$ in cases (a) and (b) for $b_2(M) = 1$.

REMARK 1. If β_M is indefinite, then M is a simple branched covering of all of $\mathbb{C}\mathrm{P}^2$, $\overline{\mathbb{C}\mathrm{P}}^2$, $S^2 \mathbin{\tilde{\times}} S^2$ and $S^2 \times S^2$. On the other hand, if β_M is positive (resp. negative) definite, then among these manifolds $\mathbb{C}\mathrm{P}^2$ (resp. $\overline{\mathbb{C}\mathrm{P}}^2$) is the only one of which M is a branched covering.

For the sake of completeness, we also state the following generalization of Theorem 1.1. The proof is based on the same methods of that of Theorem 1.1, and we will only sketch it.

THEOREM 1.2. *Let M be a closed connected oriented PL 4-manifold and let m and n be non-negative integers. Then, there exists a branched covering $p\colon M \to N$ with:*
(a) $N = \#_m \mathbb{C}\mathrm{P}^2 \#_n \overline{\mathbb{C}\mathrm{P}}^2 \Leftrightarrow b_2^+(M) \geq m$ *and* $b_2^-(M) \geq n$;
(b) $N = \#_n(S^2 \times S^2) \Leftrightarrow b_2^+(M) \geq n$ *and* $b_2^-(M) \geq n$;
(c) $N = \#_n(S^3 \times S^1) \Leftrightarrow \pi_1(M)$ *admits a free group of rank n as a quotient.*
In all cases, we can assume that p is a simple branched covering of degree $d \leq 4$, whose branch set B_p is a closed locally flat PL surface self-transversally immersed in N. Moreover, B_p can be desingularized to become embedded in N, with the following estimates for the degree d: $d \leq 5$ in case (a) for $b_2(M) \geq 2(m+n)$ and β_M odd, case (b) for β_M even, and case (c); $d \leq 6$ in

case (a) *for* $b_2(M) \geq 2(m+n)$ *and* β_M *even, and case* (b) *for* β_M *odd;* $d \leq 9$ *in case* (a) *for* $b_2(M) < 2(m+n)$.

We observe that Theorem 1.2 (a) includes Theorem 1.1 (a), (b) and (c), being $S^2 \mathbin{\widetilde{\times}} S^2 \cong \mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}}^2$. Similarly, it includes the case of $N = \#_m(S^2 \times S^2) \#_n(S^2 \mathbin{\widetilde{\times}} S^2)$ with $n \geq 1$, being $(S^2 \times S^2) \# \mathbb{C}\mathrm{P}^2 \cong (S^2 \mathbin{\widetilde{\times}} S^2) \# \mathbb{C}\mathrm{P}^2$.

REMARK 2. As a consequence of Theorem 1.2 (a) and (b), we obtain some simply connected 4-manifolds N admitting a simple branched covering $p\colon T^4 \to N$. Namely, they are $\#_m \mathbb{C}\mathrm{P}^2 \#_n \overline{\mathbb{C}\mathrm{P}}^2$ and $\#_n(S^2 \times S^2)$ for any $m \leq 3$ and $n \leq 3$. This extends the previous result by Rickman [**27**] concerning the case when N is $\#_2(S^2 \times S^2)$. All such manifolds N are *quasiregularly elliptic* (see Bonk and Heinonen [**8**] for the definition), since the composition of the universal covering of T^4 with p is a *quasiregular map* $\mathbb{R}^4 \to N$. The question of which closed simply connected manifolds are quasiregularly elliptic was posed by Gromov in [**13**, **14**]. According to Prywes [**26**], $b_2(M) \leq 6$ for any closed connected orientable quasiregularly elliptic 4-manifold M, in particular $\#_n(S^2 \times S^2)$ is not quasiregularly elliptic for $n \geq 4$. Hence our result implies a sharp answer to the Gromov question for such connected sums, while the cases of $\#_m \mathbb{C}\mathrm{P}^2 \#_n \overline{\mathbb{C}\mathrm{P}}^2$ with $m + n \leq 6$ and $\max(m, n) \geq 4$, as well as the exotic counterparts of all the above manifolds, remain still open.

It is known that there are smooth 4-manifolds $X_{m,n}$ homeomorphic but not diffeomorphic to $\#_m \mathbb{C}\mathrm{P}^2 \#_n \overline{\mathbb{C}\mathrm{P}}^2$ for certain $m, n \geq 1$, see for example Donaldson [**10**], Akhmedov and Park [**1**, **2**] and Park, Stipsicz and Szabó [**23**]. As an immediate consequence of Theorem 1.2, we get the following corollary.

COROLLARY 1.3. *For every smooth 4-manifold* $X_{m,n}$ *homeomorphic to* $\#_m \mathbb{C}\mathrm{P}^2 \#_n \overline{\mathbb{C}\mathrm{P}}^2$, *there exists a smooth simple covering* $p\colon X_{m,n} \to \#_m \mathbb{C}\mathrm{P}^2 \#_n \overline{\mathbb{C}\mathrm{P}}^2$ *of degree* ≤ 4 (*resp.* ≤ 9) *branched over a smooth self-transversally immersed* (*resp. embedded*) *surface.*

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2. *Preliminaries*

We briefly recall the notion of branched covering, in order to introduce some terminology (see [**6**] or [**12**] for more details).

A map $p\colon M \to N$ between compact *oriented* PL manifolds having the same dimension n is called a *branched covering* if it is a non-degenerate *orientation preserving* PL map with the following properties: 1) there is an $(n-2)$-dimensional polyhedral subspace $B_p \subset N$, the *branch set* of p, such that the restriction $p_| : M - p^{-1}(B_p) \to N - B_p$ is an ordinary covering of finite *degree* $d(p)$ (we assume B_p to be minimal with respect to this property); 2) in the bounded case, $p^{-1}(\partial N) = \partial M$ and p preserves the product structure of a collar of the boundaries (which implies that the restriction to the boundary $p_| : \partial M \to \partial N$ is a branched covering of the same degree of p).

Moreover, p is called *simple* if the monodromy of the above mentioned ordinary covering sends every meridian around B_p to a transposition. In this case, also the restriction to the boundary $p_| : \partial M \to \partial N$ is simple.

Definition 1. Let M and N be compact oriented connected n-manifolds, and let $M_1, \dots, M_k \subset M$ and $N_1, \dots, N_k \subset N$ be compact oriented locally flat PL submanifolds embedded in M and N, respectively. By a *d-fold branched covering* $p\colon (M; M_1, \dots, M_k) \to (N; N_1, \dots, N_k)$ we mean a d-fold branched covering $p\colon M \to N$ whose branch set is transversal to all the submanifolds N_i and such that $p(M_i) = N_i$ and $p_i = p_{|M_i} : M_i \to N_i$ preserves the orientation for every $i = 1, \dots, k$.

We note that, if p is a (simple) d-fold branched covering as in the definition, then each restriction $p_i\colon M_i \to N_i$ is a (simple) d_i-fold branched covering for some $d_i \leq d$.

Given two closed oriented locally flat PL surfaces $F_1, F_2 \subset M$ in the closed oriented PL 4-manifold M, we will denote by $F_1 \cdot F_2$ their algebraic intersection, that is the number $\beta_M([F_1], [F_2]) \in \mathbb{Z}$.

We also need the following technical definition. First, we remind that a properly embedded locally flat PL surface $S \subset B^4$ is said to be *ribbon* if the distance function from the origin restricted to S, has no local maxima in $\operatorname{Int} S$. In particular, a push in of a PL surface embedded in $S^3 \subset B^4$ is ribbon.

Definition 2. A simple branched covering $p\colon M \to S^3$ is said to be *ribbon fillable* if it can be extended to a simple branched covering $q\colon W \to B^4$ whose branch set $B_q \subset B^4$ is a ribbon surface (which immediately implies that $M = \partial W$, $B_p = \partial B_q \subset S^3$ is a link, and $d(p) = d(q)$). For the sake of convenience, we also call ribbon fillable any simple branched cover $p\colon M \to S^3_1 \cup \dots \cup S^3_k$ that is a disjoint union of ribbon fillable coverings.

This definition is relevant in light of the following theorem, which summarises a classical result for 4-dimensional branched coverings due to Montesinos [**21**] (see also [**6**, **7**] for an explicit direct construction, starting from a Kirby diagram), and an application of it to 3-manifolds obtained by taking into account the Lickorish-Wallace theorem [**18**, **29**].

Theorem 2.1. *Any compact connected oriented 4-dimensional 2-handlebody W is a simple 3-fold covering of B^4 branched over a ribbon surface in B^4. Hence, every closed connected oriented 3-manifold is a ribbon fillable 3-fold branched covering of S^3.*

The degree of a branched covering of a sphere or a ball can be arbitrarily increased by iterating the operation of stabilisation, according to the following definition.

Definition 3. For any d-fold branched covering $p\colon M \to N$, where $N \cong S^n$ or $N \cong B^n$, the *covering stabilisation* of p is the $(d+1)$-fold branched covering $M \to N$ obtained from p by adding to the branch set B_p a separate trivial $(n-2)$-sphere in S^n or proper $(n-2)$-ball in B^n, respectively, with monodromy $(d\; d+1)$ for a meridian of it. By k subsequent applications of this operation, we get a $(d+k)$-fold branched covering $p' : M \to N$, which we call the *k-fold stabilisation of p*. By construction, p' turns out to be a simple branched covering if p is simple. Moreover, if $N \cong S^3$ and p is ribbon fillable, then also p' is ribbon fillable.

The proofs of our results depend on the following theorem, which was established in [**25**].

Theorem 2.2. *Let M be a compact connected oriented PL 4-manifold whose boundary has k connected components, and let $B^4_1, \dots, B^4_k \subset S^4$ be a collection of pairwise disjoint*

PL 4-balls bounded by the 3-spheres $S^3_1, \dots, S^3_k \subset S^4$, respectively. Any d-fold ribbon fillable simple branched covering $p\colon \partial M \to S^3_1 \cup \dots \cup S^3_k$ of degree $d \geq 4$, extends to a simple d-fold covering $q\colon M \to S^4 - \mathrm{Int}(B^4_1 \cup \dots \cup B^4_k)$ whose branch set B_q is a locally flat self-transversal PL surface properly immersed (embedded for $d \geq 5$) in $S^4 - \mathrm{Int}(B^4_1 \cup \dots \cup B^4_k)$.

3. *Branched coverings of disc bundles and their plumbings*

Given a closed connected oriented surface F, we denote by $\xi_{F,e} : D_{F,e} \to F$ the oriented disc bundle over F with Euler number $e \in \mathbb{Z}$. By abusing notation, we also write $F \subset \mathrm{Int}\, D_{F,e}$ to indicate (the properly embedded oriented surface image of) a PL section $F \to \mathrm{Int}\, D_{F,e}$ of $\xi_{F,e}$.

PROPOSITION 3.1. *If $p\colon F \to G$ is a (simple) branched covering of degree $d \geq 1$ between closed connected oriented surfaces, then the pullback $p^*(\xi_{G,e})$ is bundle isomorphic to $\xi_{F,de}$ for every $e \in \mathbb{Z}$. Moreover, for any PL sections $F \subset \mathrm{Int}\, D_{F,de}$ and $G \subset \mathrm{Int}\, D_{G,e}$, p lifts to a fiber-preserving (simple) branched covering $\widetilde{p}\colon (D_{F,de}; F) \to (D_{G,e}; G)$ having the same degree d and branch set the disjoint union of fiber discs $B_{\widetilde{p}} = \xi^{-1}_{G,e}(B_p)$.*

Proof. To prove the bundle isomorphism $p^*(\xi_{G,e}) \cong \xi_{F,de}$, it is enough to consider two sections $G', G'' \subset \mathrm{Int}\, D_{G,e}$ of $\xi_{G,e}$ that intersect each other transversally away from $\xi^{-1}_{G,e}(B_p)$, and observe that the pullback sections F', F'' of $p^*(\xi_{G,e})$ satisfy $F' \cdot F'' = d(G' \cdot G'') = de$.

Up to the above isomorphism, we obtain a lifting $\widetilde{p}\colon D_{F,de} \to D_{G,e}$ associated to the pullback, which is a (simple) branched covering with branch set $\xi^{-1}_{G,e}(B_p)$, due to the local product structure of the bundles. Moreover, given any two sections F and G as in second part of the statement, we can attain $\widetilde{p}(F) = G$ by composing $\widetilde{p}$ with an arbitrary bundle automorphism of $\xi_{F,de}$ that sends F to the pullback of G. □

PROPOSITION 3.2. *For any connected simple branched covering $p\colon F \to S^2$ of degree $d \geq 1$, the simple branched covering $\widetilde{p}\colon (D_{F,\pm d}; F) \to (D_{S^2,\pm 1}; S^2)$ given by the previous proposition, restricts to a ribbon fillable branched covering $\widetilde{p}_{|\partial} : \partial D_{F,\pm d} \to \partial D_{S^2,\pm 1} \cong S^3$.*

Proof. By the Lüroth-Clebsch theorem (see Berstein and Edmonds [**5**], or Bauer and Catanese [**4**] for a different approach), simple branched coverings from a closed connected oriented genus g surface to S^2 are classified by the degree. Therefore, up to covering equivalence we can assume that p is the $(d-2)$-fold stabilisation of the hyperelliptic 2-fold covering $F \to S^2$.

Let $n = g(F) + d - 1$. Then B_p consists of $2n$ points $a_1, a'_1, \dots, a_n, a'_n$ having monodromies $(1\,2), (1\,2), \dots, (1\,2), (1\,2), (2\,3), (2\,3), \dots, (d{-}1\,d), (d{-}1\,d)$, with respect to a suitable Hurwitz system. Thus, the branch set $B_{\widetilde{p}}$ consists of $2n$ discs with those monodromies.

Since the restriction to the boundary of the bundle $\xi_{S^2,\pm 1}$ is a Hopf fibration, the branch set of $\widetilde{p}_{|\partial}$, which is the boundary of $B_{\widetilde{p}}$, consists of $2n$ Hopf fibers $C_1, C'_1, \dots, C_n, C'_n$, such that C_i and C'_i have the same monodromy.

Now, there exist n pairwise disjoint properly embedded ribbon annuli $R_1, \dots, R_n \subset B^4$, such that $\partial R_i = C_i \cup C'_i$. Indeed, these can be obtained as the push in of the preimages by $\xi_{S^2,\pm 1}$ of n pairwise disjoint arcs $A_1, \dots, A_n \subset S^2$, such that each arc A_i joins a_i and a'_i. By choosing these arcs so that they meet the Hurwitz system only at their end points, the monodromy of $\widetilde{p}_{|\partial}$ can be extended over $B^4 - (R_1 \cup \dots \cup R_n)$, yielding a ribbon filling of that covering. □

For any disc bundles $\xi_{F_1,e_1} : D_{F_1,e_1} \to F_1$ and $\xi_{F_2,e_2} : D_{F_2,e_2} \to F_2$ over closed connected oriented surfaces F_1 and F_2, we can form the *positive n-fold plumbing* $X_n(\xi_{F_1,e_1}, \xi_{F_2,e_2})$ of

D_{F_1,e_1} and D_{F_2,e_2} as follows. We choose two families of pairwise disjoint discs $U_1,\dots,U_n \subset F_1$ and $V_1,\dots,V_n \subset F_2$, together with local trivializations $\xi^{-1}_{F_1,e_1}(U_i) \cong B^2 \times B^2$ and $\xi^{-1}_{F_2,e_2}(V_i) \cong B^2 \times B^2$ of the bundles, for $i = 1,\dots,n$. Then, we define the oriented PL 4-manifold

$$X_n(\xi_{F_1,e_1}, \xi_{F_2,e_2}) = D_{F_1,e_1} \cup_{\phi_1\cup\dots\cup\phi_n} D_{F_2,e_2}, \tag{1}$$

where the gluing homeomorphisms $\phi_i \colon \xi^{-1}_{F_1,e_1}(U_i) \to \xi^{-1}_{F_2,e_2}(V_i)$ are assumed to interchange the base and the fiber up to those local trivializations. We can consider D_{F_1,e_1} and D_{F_2,e_2} as subspaces of $X_n(\xi_{F_1,e_1}, \xi_{F_2,e_2})$, and we call each connected component $\xi^{-1}_{F_1,e_1}(U_i) \cong_{\phi_i} \xi^{-1}_{F_2,e_2}(V_i)$ of $D_{F_1,e_1} \cap D_{F_2,e_2}$ a *plumbing region*.

Given two PL sections $F_1 \subset \operatorname{Int} D_{F_1,e_1}$ and $F_2 \subset \operatorname{Int} D_{F_2,e_2}$ of ξ_{F_1,e_1} and ξ_{F_2,e_2}, respectively, we can choose the above trivializations in such a way that all the intersections $F_1 \cap \xi^{-1}_{F_1,e_1}(U_i)$ and $F_2 \cap \xi^{-1}_{F_2,e_2}(V_i)$ correspond to $B^2 \times \{0\} \subset B^2 \times B^2$. In this way, we can consider

$$F_1, F_2 \subset \operatorname{Int} X_n(\xi_{F_1,e_1}, \xi_{F_2,e_2}) \tag{2}$$

as properly embedded oriented PL surfaces which intersect transversally and positively at n points, in such way that $X_n(\xi_{F_1,e_1}, \xi_{F_2,e_2})$ can be thought as a regular neighborhood of $F_1 \cup F_2$.

REMARK 3. The triple $(X_n(\xi_{F_1,e_1}, \xi_{F_2,e_2}); F_1, F_2)$ does not depend, up to PL homeomorphisms, on the choices involved in the construction.

PROPOSITION 3.3. *For any (simple) d-fold branched coverings $p_1 \colon F_1 \to G_1$ and $p_2 \colon F_2 \to G_2$ between closed connected oriented surfaces, any disc bundles ξ_{F_i,de_i} and ξ_{G_i,e_i}, and any PL sections $F_i \subset \xi_{F_i,de_i}$ and $G_i \subset \xi_{G_i,e_i}$, for $i = 1, 2$, there exists a (simple) d-fold branched covering*

$$\widetilde{p} \colon (X_d(\xi_{F_1,de_1}, \xi_{F_2,de_2}); F_1, F_2) \to (X_1(\xi_{G_1,e_1}, \xi_{G_2,e_2}); G_1, G_2).$$

In addition, $\widetilde{p}$ is fiber-preserving away from the plumbing regions and sends each plumbing region upstairs homeomorphically to the plumbing region downstairs, and the branch set $B_{\widetilde{p}}$ is a disjoint union of fiber discs, coinciding with $B_{\widetilde{p}_1} \cup B_{\widetilde{p}_2}$.

Proof. Proposition 3.1 yields d-fold fiber-preserving branched coverings $\widetilde{p}_1 \colon D_{F_1,de_1} \to D_{G_1,e_1}$ and $\widetilde{p}_2 \colon D_{F_2,de_2} \to D_{G_2,e_2}$.

Let us consider the two discs $U \subset G_1$ and $V \subset G_2$ that determine the plumbing region of $X_1(\xi_{G_1,e_1}, \xi_{G_2,e_2})$ as $\xi^{-1}_{G_1,e_1}(U) \cong_\phi \xi^{-1}_{G_2,e_2}(V)$, where ϕ is the gluing homeomorphism. By Remark 3, we can assume that $U \cap B_{p_1} = \emptyset$ and $V \cap B_{p_2} = \emptyset$.

It follows that $p_1^{-1}(U)$ is a disjoint union of d discs $U_1,\dots,U_d \subset F_1$, and similarly $p_2^{-1}(V)$ is a disjoint union of d discs $V_1,\dots,V_d \subset F_2$. Taking into account that $\widetilde{p}_1$ and $\widetilde{p}_2$ are fiber-preserving, by Remark 3 again, we can assume that the plumbing regions of $X_d(\xi_{F_1,de_1}, \xi_{F_2,de_2})$ are $\xi^{-1}_{F_1,de_1}(U_i) \cong_{\phi_i} \xi^{-1}_{F_2,de_2}(V_i)$, and that the gluing homeomorphisms $\phi_i \colon \xi^{-1}_{F_1,de_1}(U_i) \to \xi^{-1}_{F_2,de_2}(V_i)$ are determined by the equations

$$\widetilde{p}_2 \circ \phi_i = \phi \circ \widetilde{p}_1,$$

for $i = 1,\dots,d$. Therefore, the maps $\widetilde{p}_1$ and $\widetilde{p}_2$ can be glued together to give a map

$$\widetilde{p} \colon X_d(\xi_{F_1,de_1}, \xi_{F_2,de_2}) \to X_1(\xi_{G_1,e_1}, \xi_{G_2,e_2}),$$

which in turn is a branched covering since the gluing is by homeomorphisms, and it is fiber-preserving away from the plumbing regions because so are $\widetilde{p}_1$ and $\widetilde{p}_2$. Thus, $B_{\widetilde{p}} = B_{\widetilde{p}_1} \cup B_{\widetilde{p}_2}$ is a disjoint union of fiber discs. Finally, the equalities $\widetilde{p}(F_1) = \widetilde{p}_1(F_1) = G_1$ and $\widetilde{p}(F_2) = \widetilde{p}_2(F_2) = G_2$, and the fact that $\widetilde{p}$ sends each plumbing region upstairs homeomorphically to the plumbing region downstairs, are obvious by the construction. $\square$

PROPOSITION 3.4. *Given any connected simple branched coverings $p_1 : F_1 \to S^2$ and $p_2 : F_2 \to S^2$ of degree $d \geq 1$ and any integer $e \in \mathbb{Z}$, we have that $\partial X_1(\xi_{S^2,e}, \xi_{S^2,0}) \cong S^3$ and the simple branched covering*

$$\widetilde{p}\colon X_d(\xi_{F_1,de}, \xi_{F_2,0}) \to X_1(\xi_{S^2,e}, \xi_{S^2,0}),$$

of the previous proposition, restricts to a ribbon fillable branched covering

$$\widetilde{p}_{|\partial} : \partial X_d(\xi_{F_1,de}, \xi_{F_2,0}) \to \partial X_1(\xi_{S^2,e}, \xi_{S^2,0}) \cong S^3.$$

Proof. The manifold $X_1(\xi_{S^2,e}, \xi_{S^2,0})$ admits a handlebody decomposition with two 2-handles attached to B^4 along the components of the Hopf link, one with framing e to give $D_{S^2,e}$ and the other with framing 0 to give $D_{S^2,0}$. In the corresponding Kirby diagram of the boundary, the 0-framed component of the framed link can be cancelled with the e-framed one to give a PL homeomorphism $\partial X_1(\xi_{S^2,e}, \xi_{S^2,0}) \cong S^3$.

The branch set $B_{\widetilde{p}}$ coincides with $B_{\widetilde{p}_1} \cup B_{\widetilde{p}_2}$ by the previous proposition, and in the above handlebody decomposition is given by $2(g(F_1) + d - 1)$ discs parallel to the co-core of the 2-handle with framing e and $2(g(F_2) + d - 1)$ discs parallel to the co-core of the other 2-handle. The discs of each family come in pairs with equal monodromies, as in the proof of Proposition 3.2.

By looking at the boundary, we get the left side of Figure 1, which depicts $B_{\widetilde{p}_1|\partial}$ and $B_{\widetilde{p}_2|\partial}$ as two families of circles linked with the corresponding framed unknots. Up to the PL homeomorphism $\partial X_1(\xi_{S^2,e}, \xi_{S^2,0}) \cong S^3$, we get the boundary link in the right side of Figure 1. To see this, we first slide the circles corresponding to $B_{\widetilde{p}_2|\partial}$, over the unknot with framing e, making them unlinked with the one with framing 0. Subsequently, we slide all the branch circles over the 0-framed unknot to separate them from the framed link, which can be now cancelled, realising the surgery that yields the PL homeomorphism with S^3.

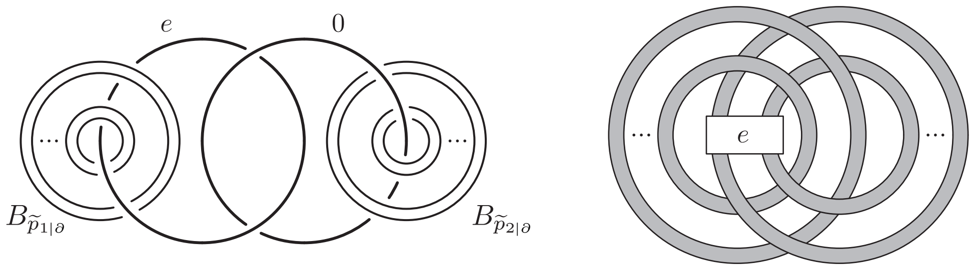

FIGURE 1. *Ribbon fillability of $\widetilde{p}_{|\partial}$.*

At this point, the ribbon fillability follows as in the last part of the proof of Proposition 3.2, by extending the monodromy over the complement in B^4 of a family of $g(F_1) + g(F_2) + 2d - 2$ bands, which are the push in of the bands showed in the right side of Figure 1. $\square$

4. *Branched coverings constructions for submanifolds*

We are ready to state and prove our results for branched coverings relative to certain submanifolds, as we mentioned in the Introduction.

THEOREM 4.1. *Let M be a closed connected oriented PL 4-manifold and $F \subset M$ be a closed connected oriented locally flat PL surface. If $d = |F \cdot F| \geq 4$, then there exists a simple d-fold branched covering:*
(*a*) $p\colon (M; F) \to (\mathbb{C}\mathrm{P}^2; \mathbb{C}\mathrm{P}^1)$ *if $F \cdot F$ is positive;*
(*b*) $p\colon (M; F) \to (\overline{\mathbb{C}\mathrm{P}}^2; \mathbb{C}\mathrm{P}^1)$ *if $F \cdot F$ is negative.*
In both cases, $F = p^{-1}(\mathbb{C}\mathrm{P}^1)$, and B_p is a closed locally flat PL surface self-transversally immersed (embedded for $d \geq 5$) in $\mathbb{C}\mathrm{P}^2$ or $\overline{\mathbb{C}\mathrm{P}}^2$.

Proof. Case (b) immediately follows from case (a) by reversing the orientation of M. So, it suffices to prove case (a), supposing $d = F \cdot F \geq 4$.

Let $T_F \subset M$ be a tubular neighborhood of F in M, and $T_{\mathbb{C}\mathrm{P}^1} \subset \mathbb{C}\mathrm{P}^2$ be a tubular neighborhood of $\mathbb{C}\mathrm{P}^1$ in $\mathbb{C}\mathrm{P}^2$. Then, given any simple d-fold branched covering $f\colon F \to S^2$ and taking into account the PL homeomorphisms $T_F \cong D_{F,d}$ and $T_{\mathbb{C}\mathrm{P}^1} \cong D_{\mathbb{C}\mathrm{P}^1,1} \cong D_{S^2,1}$, we can apply Proposition 3.1 to obtain a simple d-fold branched covering $t\colon (T_F, F) \to (T_{\mathbb{C}\mathrm{P}^1}, \mathbb{C}\mathrm{P}^1)$. Moreover, the restriction $t_{|\partial}\colon \partial T_F \to \partial T_{\mathbb{C}\mathrm{P}^1}$ is ribbon fillable by Proposition 3.2.

We set $W = \mathrm{Cl}(M - T_F)$ and $Y = \mathrm{Cl}(\mathbb{C}\mathrm{P}^2 - T_{CP^1}) \cong B^4$. Then, Theorem 2.2 allows us to extend $t_{|\partial}$ to a simple d-fold covering $q\colon W \to Y$ branched over a self-transversally immersed (embedded for $d \geq 5$) surface.

Finally, we can define the desired covering p as the union of the coverings t and q, which share the same restriction to the boundary. Namely, $p = t \cup_\partial q\colon M = T_F \cup_\partial W \to \mathbb{C}\mathrm{P}^2 = T_{\mathbb{C}\mathrm{P}^1} \cup_\partial Y$. □

THEOREM 4.2. *Let M be a closed connected oriented PL 4-manifold and $F_1, F_2 \subset M$ be two closed connected oriented locally flat PL surfaces transversal to each other, whose all intersection points are positive. If $F_1 \cdot F_1 = n\,d$, $F_1 \cdot F_2 = d$ and $F_2 \cdot F_2 = 0$ for some integers n and $d \geq 4$, then there exists a simple d-fold branched covering:*
(*a*) $p\colon (M; F_1, F_2) \to (S^2 \times S^2; S^2_1, S^2_2)$, *with S^2_1 and S^2_2 respectively a section with self-intersection n and a fiber of the trivial bundle $S^2 \times S^2 \to S^2$, if n is even;*
(*b*) $p\colon (M; F_1, F_2) \to (S^2 \mathbin{\tilde\times} S^2; S^2_1, S^2_2)$, *with S^2_1 and S^2_2 respectively a section with self-intersection n and a fiber of the twisted bundle $S^2 \mathbin{\tilde\times} S^2 \to S^2$, if n is odd.*
In both cases, $F_i = p^{-1}(S^2_i)$, and B_p is a closed locally flat PL surface self-transversally immersed (embedded for $d \geq 5$) in $S^2 \times S^2$ or $S^2 \mathbin{\tilde\times} S^2$.

We observe that a section as specified in the above statement exists for every integer n. In fact, given two copies of the trivial bundle $B^2 \times S^2 \to B^2$, we can glue them along the boundary by the map $(\alpha, x) \mapsto (\alpha, \rho_{n\alpha}(x))$, with $\rho_\alpha\colon \mathbb{R}^3 \to \mathbb{R}^3$ the rotation of α radians around the third axis. In this way, we get the trivial bundle $S^2 \times S^2 \to S^2$ or the twisted bundle $S^2 \mathbin{\tilde\times} S^2 \to S^2$, depending on the parity of n, with two natural sections deriving from the two copies of $B^2 \times \{(0, 0, \pm 1)\}$, both having self-intersection n.

Proof. For the sake of convenience, we denote by $\xi\colon E \to S^2$ the trivial bundle $S^2 \times S^2 \to S^2$ or the twisted bundle $S^2 \mathbin{\tilde\times} S^2 \to S^2$, depending on whether n is even or odd.

We can choose tubular neighborhoods T_{F_1} of F_1 and T_{F_2} of F_2, in such a way that their union $T_{F_1} \cup T_{F_2}$ is a regular neighborhood of $F_1 \cup F_2$ in M. It follows that there is a PL homeomorphism

$$(T_{F_1} \cup T_{F_2}; F_1, F_2) \cong (X_d(\xi_{F_1,dn}, \xi_{F_2,0}); F_1, F_2),$$

where $X_d(\xi_{F_1,dn}, \xi_{F_2,0})$ is the d-fold plumbing defined in Section 3.

Similarly, we can choose tubular neighborhoods $T_{S_1^2}$ of S_1^2 and $T_{S_2^2}$ of S_2^2, in such a way that their union $T_{S_1^2} \cup T_{S_2^2}$ is a regular neighborhood of $S_1^2 \cup S_2^2$ in E. As above, there is a PL homeomorphism

$$(T_{S_1^2} \cup T_{S_2^2}; S_1^2, S_2^2) \cong (X_1(\xi_{S_1^2,n}, \xi_{S_2^2,0}); S_1^2, S_2^2).$$

Now, let $f_1 : F_1 \to S_1^2$ and $f_2 : F_2 \to S_2^2$ be simple d-fold branched coverings. By Proposition 3.3, we get a simple d-fold branched covering

$$t\colon (T_{F_1} \cup T_{F_2}; F_1, F_2) \to (T_{S_1^2} \cup T_{S_2^2}; S_1^2, S_2^2),$$

whose restriction $t_{|\partial} : \partial(T_{F_1} \cup T_{F_2}) \to \partial(T_{S_1^2} \cup T_{S_2^2})$ is ribbon fillable by Proposition 3.4.

Looking at the complement of those tubular neighborhoods, we put $W = \mathrm{Cl}(M - (T_{F_1} \cup T_{F_2}))$ and $Y = \mathrm{Cl}(E - (T_{S_1^2} \cup T_{S_2^2})) \cong B^4$. Then, we can use Theorem 2.2 for extending $t_{|\partial}$ to a simple d-fold covering $q\colon W \to Y$ branched over a self-transversally immersed (embedded for $d \geq 5$) PL surface, and conclude the proof by putting $p = t \cup_\partial q\colon M = (T_{F_1} \cup T_{F_2}) \cup_\partial W \to E = (T_{S_1^2} \cup T_{S_2^2}) \cup_\partial Y$. □

THEOREM 4.3. *Let M be a closed connected oriented PL 4-manifold and $N \subset M$ be a closed connected oriented (locally flat) PL 3-manifold. For any $d \geq 4$ there exists a simple d-fold branched covering:*
(a) $p\colon (M; N) \to (S^4; S^3)$ if N disconnects M;
(b) $p\colon (M; N) \to (S^3 \times S^1; S^3 = S^3 \times \{*\})$ if N does not disconnect M.
In both cases, $N = p^{-1}(S^3)$, and B_p is a closed locally flat PL surface self-transversally immersed (embedded for $d \geq 5$) in S^4 or $S^3 \times S^1$.

Proof. According to Theorem 2.1, and up to covering stabilization, there exists a ribbon fillable d-fold branched covering $c\colon N \to S^3$.

If N disconnects M, let $M_1, M_2 \subset M$ be the closures of the two connected components of $M - N$. Then, M_1 and M_2 are two PL compact oriented 4-manifolds with $\partial M_1 = \partial M_2 = N$, such that $M = M_1 \cup M_2$. By Theorem 2.2, the branched covering c extends to two simple d-fold branched coverings $p_1 : M_1 \to S^4_-$ and $p_2 : M_2 \to S^4_+$, both branched over a locally flat PL surface self-transversally immersed (embedded if $d \geq 5$) in the base space, where $S^n_\pm \subset S^n$ are the two hemispheres bounded by $S^{n-1} \subset S^n$. Therefore, we can put $p = p_1 \cup p_2 : M \to S^4$.

In the case where N does not disconnect M, we consider the decomposition $M = M_1 \cup M_2$, with M_1 a collar of N in M, and $M_2 = \mathrm{Cl}(M - M_1)$. The simple d-fold covering $p_1 = c \times \mathrm{id}_{S^1_-} : M_1 \cong N \times S^1_- \to S^3 \times S^1_-$ is branched over the locally flat PL surface $B_c \times S^1_-$, which is properly embedded in $S^3 \times S^1_-$. The restriction of p_1 to the boundary is a ribbon fillable d-fold branched covering $\partial M_1 = \partial M_2 \to \partial(S^3 \times S^1_-) = \partial(S^3 \times S^1_+)$, which by Theorem 2.2 admits a simple d-fold extension $p_2 : M_2 \to S^3 \times S^1_+$ branched over a locally flat PL surface self-transversally immersed (embedded if $d \geq 5$) in $S^3 \times S^1_+$. So, also in this case we can conclude by putting $p = p_1 \cup p_2 : M \to S^3 \times S^1$. □

Our last result of this section is not related to the main theorem. Still, we include it for the sake of completeness, since it provides a representation of surfaces in 4-manifolds as branched covering of trivial 2-spheres in S^4 (cf. [**22**] for links in 3-manifolds).

THEOREM 4.4. *Let M be a closed connected oriented PL 4-manifold and $F \subset M$ be a closed oriented locally flat PL surface with k connected components $F_1, \dots, F_k$, such that $F_i \cdot F_i = 0$ for every $i = 1, \dots, k$ (that is, the normal bundle ν_F is trivial). Then, for any $d \geq 4$ there is a simple d-fold branched covering $p\colon (M; F) \to (S^4; T_k)$, with $T_k \subset S^4$ the trivial 2-link with k*

spherical components and $B_p \subset S^4$ a closed locally flat PL surface self-transversally immersed (embedded for $d \geq 5$) in S^4, which is transversal to T_k. Moreover, p can be chosen in such a way that each restriction $p_{|F_i} : F_i \to p(F_i) \cong S^2$ is equivalent to any given simple branched covering of degree $d_i \leq d-2$. In particular, if F is consists of 2-spheres, we can assume $B_p \cap T_k = \emptyset$, hence p is the trivial d-fold covering over T_k.

We note that any closed oriented locally flat PL surface $F \subset S^4$ admits a branched covering representation as in the theorem.

For the proof of Theorem 4.4 we need two lemmas.

LEMMA 4.5. *Let $C \subset B^3$ be a properly embedded (not necessarily connected) compact curve. Then, the surface $F = C \times B^1 \subset B^3 \times B^1 \cong B^4$ is ribbon.*

Proof. Up to ambient isotopy, we can assume that the origin $0 \in B^3$ does not belong to C and that the image $D = \pi_0(C) \subset S^2$ of C under the radial projection $\pi_0 : B^3 - \{0\} \to S^2$ from 0 forms only transversal double points (it gives a diagram of C). Let $\pi_{(0,0)} : (B^3 \times B^1) - \{(0,0)\} \to \partial(B^3 \times B^1) = (S^2 \times B^1) \cup (B^3 \times S^0) \cong S^3$ the radial projection from the origin $(0,0) \in B^3 \times B^1$. Then, for each $x \in C$ the image under $\pi_{(0,0)}$ of the segment $\{x\} \times B^1$ is given by $\pi_{(0,0)}(\{x\} \times B^1) = (\pi_0(\{x\}) \times B^1) \cup ([x, \pi_0(x)] \times S^0) \subset \partial(B^3 \times B^1)$, where $[x, \pi_0(x)] \subset B^3$ denotes the segment spanned by x and $\pi_0(x)$. It follows that the image $\pi_{(0,0)}(F) \subset \partial(B^3 \times B^1)$ forms only ribbon intersections, consisting of a single double arc for each double point of D. Hence, F is a ribbon surface. □

REMARK 4. In the smooth category, one could argue that the surface F can be realized in B^4 as a ruled surface, not passing through the origin. Then, the distance from the origin restricts to a function on F without local maxima in $\operatorname{Int} F$, which implies that F is ribbon.

LEMMA 4.6. *Let $N_1, \dots, N_k \subset M$ be pairwise disjoint compact oriented (locally flat) PL 3-manifolds with non-empty boundary, and let $B^3_1, \dots, B^3_k \subset S^4$ be pairwise disjoint PL 3-balls. For every $i = 1, \dots, k$, let $c_i : N_i \to B^3_i$ be a simple d-fold branched covering, with $B_{c_i} \subset B^3_i$ a properly embedded compact curve and $d \geq 4$. Then, $c = c_1 \cup \dots \cup c_k$ extends to a simple d-fold branched covering $p : M \to S^4$ with B_p a locally flat PL surface self-transversally immersed (embedded for $d \geq 5$) in S^4.*

Proof. We consider pairwise disjoint collars $C_i = C(N_i) \subset M$ of the 3-manifolds N_i in M and pairwise disjoint collars $D_i = C(B^3_i) \subset S^4$ of the 3-balls B^3_i in S^4. Then, we have $C_i \cong N_i \times B^1$ with N_i canonically identified to $N_i \times \{0\}$, and $D_i \cong B^3_i \times B^1$ with B^3_i canonically identified to $B^3_i \times \{0\}$. Up to these identifications and assuming all the collars positively oriented, the branched coverings c_i extend to simple d-fold coverings $c'_i = c_i \times \mathrm{id}_{B^1} : C_i \to D_i$. By Lemma 4.5, each branch set $B_{c'_i}$ is a ribbon surface in $D_i \cong B^4$. Now, we consider the simple d-fold branched covering $p_1 = \cup_i c'_i : \cup_i C_i \to \cup_i D_i$, and put $X = \mathrm{Cl}(M - \cup_i C_i)$ and $Y = \mathrm{Cl}(S^4 - \cup_i D_i)$. The restriction to the boundary of p_1 gives a simple d-fold branched covering $p_{1|\partial} : \partial X \to \partial Y$, which is ribbon fillable by construction. Therefore, Theorem 2.2 allows us to extend $p_{1|\partial}$ to a simple d-fold branched covering $p_2 : X \to Y$ with B_{p_2} a locally flat PL surface self-transversally immersed (embedded for $d \geq 5$) in Y. Thus, we can conclude the proof by putting $p = p_1 \cup_\partial p_2$. □

Proof Theorem 4.4. Since the normal bundle ν_F is trivial, for every $i = 1, \dots, k$ we can find a 3-dimensional locally flat PL ribbon $N_i \cong F_i \times [0,1]$ in M such that $\partial N_i = F_i \cup F_i'$, with $F_i' \subset M$ a "parallel" copy of F_i oriented in the opposite way. We assume the N_i's to be pairwise disjoint. Let $N_i' \subset M$ be the 3-manifold obtained by removing the interiors of $d - d_i - 2$ disjoint PL 3-balls from $\operatorname{Int} N_i$.

Each surface $\partial N_i'$ admits a d-fold simple branched covering $f_i : \partial N_i' \to S^2$, where F_i consists of the sheets 1 to d_i, F_i' consists of the sheets $d_i + 1$ and $d_i + 2$, while the boundaries of the removed 3-balls consists of the remaining $d - d_i - 2 \geq 0$ sheets trivially covering S^2. By Corollary 6.3 in [**5**], this can be extended to a d-fold simple branched covering $c_i : N_i' \to B^3$. After having identified the base spaces of such coverings with a family of disjoint PL 3-balls $B^3_1, \dots, B^3_k \subset S^4$, we can apply Lemma 4.6 to get a simple covering $p\colon (M; N_1', \dots, N_k') \to (S^4; B^3_1, \dots, B^3_k)$ of degree d, branched over a locally flat PL surface self-transversally immersed (embedded for $d \geq 5$) in S^4. Then, p is the desired branched covering, since $p(F) = \partial(\cup_i B^3_i)$ is a trivial link of k spheres. Moreover, by the Lüroth-Clebsch classification of simple branched coverings of S^2 (see [**5**] or [**4**]), the restrictions $p_{|F_i}$ can be arbitrarily chosen, up to isotopy, with the given degrees d_i.

If $F_i \cong S^2$ for every i, we set $d_i = 1$ and at the beginning of the proof we remove the interiors of $d - 2$ balls from N_i (instead of $d - 3$) so that N_i' has d boundary components, all homeomorphic to a sphere. Then, by following the same argument, we obtain the desired simple branched covering $p\colon M \to S^4$ such that $T_k \cap B_p = \emptyset$. $\square$

5. *The proofs of the main theorems*

In this section we prove the Theorems 1.1 and 1.2 stated in the Introduction. For that we need some algebraic properties of the intersection forms of PL 4-manifolds, which are stated in the next lemmas.

Lemma 5.1. *Let M be a closed connected oriented PL 4-manifold. If $b_2^+(M) \geq 1$, there exists a class $\phi \in H_2(M)/\operatorname{Tor} H_2(M)$ such that $\beta_M(\phi, \phi) = k$ for each of the followings*

$$k = \begin{cases} 4 & \text{in any case} \\ 6 & \text{if } \beta_M \text{ is even} \\ 9 & \text{if } \beta_M \text{ is odd} \\ 5 & \text{if } \beta_M \text{ is odd and } b_2(M) \geq 2. \end{cases}$$

If in addition $b_2^-(M) \geq 1$, there exist two classes $\phi_1, \phi_2 \in H_2(M)/\operatorname{Tor} H_2(M)$ whose intersection matrix $\Phi = (\beta_M(\phi_i, \phi_j))$ is

$$\Phi = \begin{pmatrix} kn & k \\ k & 0 \end{pmatrix},$$

for every $n = 0, 1$ and each of the followings

$$k = \begin{cases} 4 & \text{in any case} \\ 5 + n & \text{if } \beta_M \text{ is even} \\ 6 - n & \text{if } \beta_M \text{ is odd.} \end{cases}$$

Proof. We start by proving the first part, where $b_2^+(M) \geq 1$. If β_M is odd, then it is diagonalizable. This follows by a theorem of Donaldson for definite intersection forms of closed oriented PL 4-manifolds [**11**], and by the Serre classification theorem of indefinite unimodular integral forms [**28**, **20**]. Hence, there exists $\delta_1 \in H_2(M)/\operatorname{Tor} H_2(M)$ such that $\beta_M(\delta_1, \delta_1) = 1$, and for $b_2(M) \geq 2$ there exists also $\delta_2 \in H_2(M)/\operatorname{Tor} H_2(M)$ such that $\beta_M(\delta_1, \delta_2) = 0$ and $\beta_M(\delta_2, \delta_2) = \pm 1$.

Otherwise, if β_M is even, then, again by Donaldson's theorem [**11**], it is indefinite, and so it contains a hyperbolic direct summand (see [**20**], [**12**] or [**16**]). This is a sublattice having a basis $\eta_1, \eta_2 \in H_2(M)/\operatorname{Tor} H_2(M)$, such that $\beta_M(\eta_1, \eta_1) = \beta_M(\eta_2, \eta_2) = 0$ and $\beta_M(\eta_1, \eta_2) = 1$.

In both cases, odd and even, there exists $\phi \in H_2(M)/\operatorname{Tor} H_2(M)$ such that $\beta_M(\phi, \phi) = 4$, with $\phi = 2\delta_1$ for β_M odd, and $\phi = \eta_1 + 2\eta_2$ for β_M even. For the remaining cases, we take: $\phi = \eta_1 + 3\eta_2$, giving $k = 6$, if β_M is even; $\phi = 3\delta_1$, giving $k = 9$, if β_M is odd; $\phi = (2 - \beta_M(\delta_2, \delta_2))\,\delta_1 + 2\delta_2$, giving $k = 5$, if β_M is odd and $b_2(M) \geq 2$.

Next, we prove the second part, where $b_2^+(M) \geq 1$ and $b_2^-(M) \geq 1$. If β_M is odd, then it is diagonalizable and so there exist $\delta_1, \delta_2 \in H_2(M)/\operatorname{Tor} H_2(M)$ such that $\beta_M(\delta_1, \delta_1) = 1$, $\beta_M(\delta_1, \delta_2) = 0$, and $\beta_M(\delta_2, \delta_2) = -1$. Then, we get: $k = 4$ and $n = 0$, for $\phi_1 = \delta_1 + \delta_2$ and $\phi_2 = 2(\delta_1 - \delta_2)$; $k = 4$ and $n = 1$, for $\phi_1 = 2\delta_1$ and $\phi_2 = 2(\delta_1 - \delta_2)$; $k = 6$ and $n = 0$, for $\phi_1 = \delta_1 + \delta_2$ and $\phi_2 = 3(\delta_1 - \delta_2)$; $k = 5$ and $n = 1$, for $\phi_1 = 3\delta_1 + 2\delta_2$ and $\phi_2 = \delta_1 - \delta_2$.

If instead β_M is even, there exists a hyperbolic pair $\eta_1, \eta_2 \in H_2(M)/\operatorname{Tor} H_2(M)$, as in the analogous case of the previous part of the proof. Then, we get: $k = 4$ and $n = 0$, for $\phi_1 = \eta_1$ and $\phi_2 = 4\eta_2$; $k = 4$ and $n = 1$, for $\phi_1 = \eta_1 + 2\eta_2$ and $\phi_2 = 4\eta_2$; $k = 5$ and $n = 0$, for $\phi_1 = \eta_1$ and $\phi_2 = 5\eta_2$; $k = 6$ and $n = 1$, for $\phi_1 = \eta_1 + 3\eta_2$ and $\phi_2 = 6\eta_2$. □

LEMMA 5.2. *Let M be a closed connected oriented PL 4-manifold with $b_2(M) \geq 1$. Then, for every non-negative integers $m \leq b_2^+(M)$ and $n \leq b_2^-(M)$ there exists a sublattice $\Lambda_{m,n}(k) \subset (H_2(M)/\operatorname{Tor} H_2(M), \beta_M)$ such that*

$$\Lambda_{m,n}(k) \cong \oplus_m \langle k \rangle \oplus_n \langle -k \rangle,$$

for each of the followings

$$k = \begin{cases} 4 & \text{in any case} \\ 6 & \text{if } \beta_M \text{ is even} \\ 9 & \text{if } \beta_M \text{ is odd} \\ 5 & \text{if } \beta_M \text{ is odd and } b_2(M) \geq 2(m+n) \end{cases}$$

where $\langle k \rangle$ is the integral rank 1 lattice of determinant k.

Proof. If β_M is odd, arguing as in the proof of Lemma 5.1, we have that the lattice $(H_2(M)/\operatorname{Tor} H_2(M), \beta_M)$ is isomorphic to

$$\oplus_{b_2^+(M)} \langle 1 \rangle \oplus_{b_2^-(M)} \langle -1 \rangle.$$

Thus, $\Lambda_{m,n}(4)$ and $\Lambda_{m,n}(9)$ can be obtained by taking the doubles of some generators in the former case, and the triples in the latter. Moreover, we can obtain $\Lambda_{m,n}(5)$ if $b_2(M) \geq 2(m+n)$ by the same argument as in the proof of Lemma 5.1 applied to pairs of generators.

If β_M is even, then the lattice $(H_2(M)/\operatorname{Tor} H_2(M), \beta_M)$ is isomorphic to $\oplus_a (\pm E_8) \oplus_b H$ for $a = |\sigma(M)|/8$ and $b = b_2^{\mp}(M) \geq 1$, where E_8 is the symmetric rank 8 positive definite indecomposable unimodular lattice and H is the unimodular hyperbolic rank 2 integral lattice.

With respect to a suitable basis, E_8 can be represented by the matrix

$$A_8 = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix}.$$

In this basis, the sublattice of E_8 spanned by the columns $g_1, \dots, g_8$ of the matrix

$$G = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & -2 & -2 & -4 \\ 0 & 1 & 0 & 0 & 1 & 2 & 3 & 5 \\ 0 & 0 & 0 & 0 & -2 & -2 & -4 & -6 \\ 0 & 0 & 1 & 0 & 1 & 1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 1 & 1 & 1 & 2 & 3 \end{pmatrix}$$

is isomorphic to $\oplus_8\langle 2\rangle$. We then obtain $\oplus_8\langle 4\rangle \subset E_8$ as the sublattice spanned by all vectors of the form $g_{2i-1} \pm g_{2i}$ for $i \in \{1, 2, 3, 4\}$.

Moreover, we obtain $\oplus_8\langle 6\rangle \subset E_8$ as the sublattice spanned by all vectors of the form

$$\begin{aligned} &g_{i+1} + g_{i+2} - g_{i+3}, \qquad && g_{i+1} - g_{i+2} + g_{i+4}, \\ &g_{i+1} + g_{i+3} - g_{i+4}, \qquad && g_{i+2} + g_{i+3} + g_{i+4}, \end{aligned}$$

for $i \in \{0, 4\}$.

On the other hand, we can find sublattices $\langle k\rangle \oplus \langle -k\rangle \subset H$, for $k = 4, 6$. Therefore, the lattice $(H_2(M)/\operatorname{Tor} H_2(M), \beta_M)$ with β_M even, contains a sublattice isomorphic to

$$\oplus_{b_2^+(M)}\langle k\rangle \oplus_{b_2^-(M)}\langle -k\rangle$$

for $k = 4, 6$, from which we get a sublattice $\Lambda_{m,n}(k)$ for $k = 4, 6$. $\square$

Lemma 5.3. *Let M be a closed connected oriented PL 4-manifold. Let n be an integer such that $1 \le n \le \min(b_2^+(M), b_2^-(M))$. Then, the lattice $(H_2(M)/\operatorname{Tor} H_2(M), \beta_M)$ contains a sublattice isomorphic to $\oplus_n kH$ if β_M is even, and sublattice isomorphic to $\oplus_n 2kH$ if β_M is odd, for every integer $k \ge 1$, where H is the unimodular hyperbolic rank 2 integral lattice. If in addition $n < \max(b_2^+(M), b_2^-(M))$, there is a sublattice isomorphic to $\oplus_n kH$ for every integer $k \ge 1$ also when β_M is odd.*

Proof. If β_M is even, then there is a sublattice of $(H_2(M)/\operatorname{Tor} H_2(M), \beta_M)$ which is isomorphic to $\oplus_n H$. Then, chosen a basis of this sublattice formed by hyperbolic pairs $\eta_1, \eta_1', \dots, \eta_n, \eta_n'$ such that $\beta_M(\eta_i, \eta_i) = \beta_M(\eta_i', \eta_i') = 0$ and $\beta_M(\eta_i, \eta_i') = 1$ for every $i = 1, \dots, n$, we can take the sublattice spanned by all vectors of the form $\eta_i, k\eta_i'$, for $i = 1, \dots, n$.

If instead β_M is odd, then the intersection form is diagonalisable, hence it contains a sublattice isomorphic to $\oplus_n(\langle 1\rangle \oplus \langle -1\rangle)$. Let $\{\phi_1, \phi_1', \dots, \phi_n, \phi_n'\}$ be an orthogonal basis of this sublattice such that $\beta_M(\phi_i, \phi_i) = -\beta_M(\phi_i', \phi_i') = 1$, for every $i = 1, \dots, n$. Then, it is enough to take the sublattice spanned by all vectors of the form $\phi_i + \phi_i'$, $k(\phi_i - \phi_i')$, for $i = 1, \dots, n$.

For the last part of the statement, suppose β_M odd and $n < \max(b_2^+(M), b_2^-(M))$. Let $a = b_2^+(M) - n$ and $b = b_2^-(M) - n$. Then, by Serre's classification [**28**, **20**], the intersection lattice of M is isomorphic to $\oplus_n H \oplus_a \langle 1\rangle \oplus_b \langle -1\rangle$, since this last form is indefinite, has the same rank and signature of M, and it is odd because a or b is non-zero. Hence, we get a sublattice isomorphic to $\oplus_n kH$ for every integer $k \geq 1$. $\square$

We are now ready to prove Theorems 1.1 and 1.2, which we state again here below for the reader convenience.

THEOREM 1.1. *Let M be a closed connected oriented PL 4-manifold. Then, there exists a branched covering $p\colon M \to N$ with:*
(a) $N = \mathbb{C}\mathrm{P}^2 \Leftrightarrow b_2^+(M) \geq 1$;
(b) $N = \overline{\mathbb{C}\mathrm{P}}{}^2 \Leftrightarrow b_2^-(M) \geq 1$;
(c) $N = S^2 \mathbin{\tilde{\times}} S^2 \Leftrightarrow b_2^+(M) \geq 1$ and $b_2^-(M) \geq 1$;
(d) $N = S^2 \times S^2 \Leftrightarrow b_2^+(M) \geq 1$ and $b_2^-(M) \geq 1$;
(e) $N = S^3 \times S^1 \Leftrightarrow b_1(M) \geq 1$.
In all cases, we can assume that p is a simple branched covering of degree $d \leq 4$, whose branch set B_p is a closed locally flat PL surface self-transversally immersed in N. Moreover, B_p can be desingularized to become embedded in N, with the following estimates for the degree d: $d \leq 5$ in cases (a) and (b) for $b_2(M) \geq 2$ and β_M odd, case (c) for β_M odd, case (d) for β_M even, and case (e); $d \leq 6$ in cases (a) and (b) for $b_2(M) \geq 2$ and β_M even, case (c) for β_M even, and case (d) for β_M odd; $d \leq 9$ in cases (a) and (b) for $b_2(M) = 1$.

Proof. First of all, we recall the well known fact that in a closed connected oriented PL 4-manifold M any homology class $\alpha \in H_2(M)$ can be represented by a closed oriented locally flat PL surface $F \subset M$ (see [**12**] or [**16**]). Moreover, F can be easily made connected by embedded surgery. Similarly, any homology class $\alpha \in H_3(M)$ can be represented by a closed oriented locally flat PL 3-manifold $N \subset M$, but in this case N can be made connected only if α is primitive (see [**19**]).

(*a*). Given any d-fold branched covering $p\colon M \to \mathbb{C}\mathrm{P}^2$, we can assume up to PL isotopy that $B_p \subset \mathbb{C}\mathrm{P}^2$ meets $\mathbb{C}\mathrm{P}^1$ transversally. Then, $F = p^{-1}(\mathbb{C}\mathrm{P}^1) \subset M$ is a closed oriented locally flat PL surface, which represents a non-zero element $\phi \in H_2(M)/\operatorname{Tor} H_2(M)$ such that $\beta_M(\phi,\phi) = d > 0$. Hence, $b_2^+(M) \geq 1$.

For the converse, assume that $b_2^+(M) \geq 1$. By the first part of Lemma 5.1, there exists a class $\phi \in H_2(M)/\operatorname{Tor} H_2(M)$ such that $\beta_M(\phi,\phi) = 4$. Then, the desired 4-fold branched covering $p\colon M \to \mathbb{C}\mathrm{P}^2$ can be obtained by applying Theorem 4.1 (*a*) in the case $d = 4$ to any closed connected oriented locally flat PL surface $F \subset M$ representing the homology class ϕ. In this way, the branch set B_p turns out to be a closed locally flat PL surface self-transversally immersed in $\mathbb{C}\mathrm{P}^2$.

To obtain a non-singular branch surface according to the cases stated in the theorem, we apply Theorem 4.1 (*a*) with $d \geq 5$ to any closed connected oriented locally flat PL surface $F \subset M$ representing the homology class ϕ provided by the corresponding cases of the first part of Lemma 5.1 with $k = d$, taking into account that β_M is necessarily odd if $b_2(M) = 1$.

(*b*). This case immediately follows from case (*a*), by reversing the orientations.

(*c*) and (*d*). As in the proof of Theorem 4.2, denote by $\xi : E \to S^2$ the bundle $S^2 \times S^2 \to S^2$ or $S^2 \mathbin{\tilde{\times}} S^2 \to S^2$, depending on the case, and let S^2_1, $S^2_2 \subset E$ be any PL section and fiber of ξ, respectively.

Given a branched d-fold covering $p\colon M \to E$, we can assume up to PL isotopy that $B_p \subset E$ meets both the surfaces S^2_1 and S^2_2 transversally. Then, $F_1 = p^{-1}(S^2_1) \subset M$ and $F_2 = p^{-1}(S^2_2) \subset M$ are closed oriented locally flat PL surfaces such that $F_1 \cdot F_2 = d > 0$ and $F_2 \cdot F_2 = 0$. It follows that the homology class $\phi \in H_2(M)/\operatorname{Tor} H_2(M)$ represented by F_2 is non-zero and $\beta_M(\phi,\phi) = 0$. Therefore, β_M is indefinite, hence $b_2^+(M) \geq 1$ and $b_2^-(M) \geq 1$.

Conversely, assuming $b_2^+(M) \geq 1$ and $b_2^-(M) \geq 1$, let $\phi_1, \phi_2 \in H_2(M)/\operatorname{Tor} H_2(M)$ be the homology classes given by the second part of Lemma 5.1 with $n = 0$ for $E = S^2 \times S^2$ or $n = 1$ for $E = S^2 \widetilde{\times} S^2$, and $k = d$ depending on the case of the statement that we want to realize. We represent ϕ_1 and ϕ_2 by closed connected oriented locally flat PL surfaces $F_1, F_2 \subset M$, respectively, which can be assumed to be transversal to each other. Then, we can perform an embedded surgery, without changing the homology classes of the surfaces but increasing the genus of one of them, to eliminate each pair of opposite intersection points (if any) between F_1 and F_2. This determines new surfaces F_1 and F_2 with d transversal positive intersection points. At this point, the wanted branched covering $p\colon M \to E$ can be obtained by applying Theorem 4.2 to $(M; F_1, F_2)$.

(e). Given any d-fold branched covering $p\colon M \to S^3 \times S^1$, we can assume up to PL isotopy that $B_p \subset S^3 \times S^1$ meets $S^3 \times \{*\}$ transversally and is disjoint from $\{*\} \times S^1$. Then, $N = p^{-1}(S^3 \times \{*\}) \subset M$ and $C = p^{-1}(\{*\} \times S^1) \subset M$ are closed oriented locally flat PL submanifolds of dimensions 3 and 1, respectively, such that $N \cdot C = d > 0$. Then, C represents a non-trivial homology class in $H_1(M)/\operatorname{Tor} H_1(M)$, and so $b_1(M) \geq 1$.

Conversely, if $b_1(M) \geq 1$, and hence $b_3(M) \geq 1$, let $N \subset M$ be a closed connected oriented locally flat 3-manifold representing a primitive non-trivial element of $H_3(M)$. Then N does not disconnect M and we can apply Theorem 4.3 (b) to get the desired branched covering $p\colon M \to S^3 \times S^1$. □

Theorem 1.2. *Let M be a closed connected oriented PL 4-manifold and let m and n be non-negative integers. Then, there exists a branched covering $p\colon M \to N$ with:*

(a) *$N = \#_m \mathbb{C}\mathrm{P}^2 \#_n \overline{\mathbb{C}\mathrm{P}}^2 \Leftrightarrow b_2^+(M) \geq m$ and $b_2^-(M) \geq n$;*

(b) *$N = \#_n(S^2 \times S^2) \Leftrightarrow b_2^+(M) \geq n$ and $b_2^-(M) \geq n$;*

(c) *$N = \#_n(S^3 \times S^1) \Leftrightarrow \pi_1(M)$ admits a free group of rank n as a quotient.*

In all cases, we can assume that p is a simple branched covering of degree $d \leq 4$, whose branch set B_p is a closed locally flat PL surface self-transversally immersed in N. Moreover, B_p can be desingularized to become embedded in N, with the following estimates for the degree d: $d \leq 5$ in case (a) for $b_2(M) \geq 2(m+n)$ and β_M odd, case (b) for β_M even, and case (c); $d \leq 6$ in case (a) for $b_2(M) \geq 2(m+n)$ and β_M even, and case (b) for β_M odd; $d \leq 9$ in case (a) for $b_2(M) < 2(m+n)$.

Proof. We only sketch the proof, because it follows the same ideas of the proof of Theorem 1.1. For items (a) and (b) the implications to the right are straightforward, so we only discuss the implications to the left.

(a). We consider the proper sublattice $\Lambda_{m,n}(d) \subset H_2(M)/\operatorname{Tor} H_2(M)$ given by Lemma 5.2, according to the particular case of item (a) that we want to prove, and represent the base of $\Lambda_{m,n}(d)$ by disjoint embedded oriented connected locally flat PL surfaces $F_1, \ldots, F_{m+n} \subset M$. We also consider $\mathbb{C}\mathrm{P}^1_1, \ldots, \mathbb{C}\mathrm{P}^1_{m+n} \subset N$, where $\mathbb{C}\mathrm{P}^1_i$ is a projective line in the i-th connected summand of $N = \#_m \mathbb{C}\mathrm{P}^2 \#_n \overline{\mathbb{C}\mathrm{P}}^2$.

Next, we construct d-fold simple branched coverings $t_i\colon (T_{F_i}; F_i) \to (T_{\mathbb{C}\mathrm{P}^1_i}; \mathbb{C}\mathrm{P}^1_i)$ between tubular neighborhoods, based on Proposition 3.1 as in the proof of Theorem 4.1, whose restrictions on the boundary are ribbon fillable by Proposition 3.2.

Now, we put

$$t = \cup_i t_i : \cup_i (T_{F_i}; F_i) \to \cup_i (T_{\mathbb{C}\mathrm{P}^1_i}; \mathbb{C}\mathrm{P}^1_i),$$

$$W = \mathrm{Cl}(M - \cup_i T_{F_i}),$$

$$Y = \mathrm{Cl}(N - \cup_i T_{\mathbb{C}\mathrm{P}^1_i}) \cong \mathop{\#}_{m+n} B^4 \cong S^4 - \mathrm{Int}(B^4_1 \cup \cdots \cup B^4_{m+n}).$$

Then, we extend the ribbon fillable branched covering $t_{|\partial}: \partial W \to \partial Y \cong \cup_{m+n} S^3$ to a simple branched covering $q: W \to Y$ by means of Theorem 2.2, and finally we obtain the wanted branched covering by putting $p = q \cup t: M \to N$.

(b). By Lemma 5.3, we can find a sublattice of $(H_2(M)/\operatorname{Tor} H_2(M), \beta_M)$ which is isomorphic to $\oplus_n dH$, where d can be chosen according to the specific case that we want to obtain. For each direct summand dH, we choose a basis η_i, η_i' of it such that $\beta_M(\eta_i, \eta_i) = \beta_M(\eta_i', \eta_i') = 0$ and $\beta_M(\eta_i, \eta_i') = d$, for $i = 1, \ldots, n$.

Such homology classes can be represented by pairwise transversal closed connected oriented locally flat PL surfaces $F_i, F_i' \subset M$ such that their geometric intersections equal the algebraic ones, for $i = 1, \ldots, n$.

Then, we can find a simple d-fold branched covering as desired by repeating the argument used in case (a) (see also the proof of case (d) of Theorem 1.1), with the following setting

$$t = \cup_i t_i : \cup_i (T_{F_i} \cup T_{F_i'}; F_i, F_i') \to \cup_i \big(T_{S^2_{1i}} \cup T_{S^2_{2i}}; S^2_{1i}, S^2_{2i}\big),$$

$$W = \operatorname{Cl}\big(M - \cup_i (T_{F_i} \cup T_{F_i'})\big),$$

$$Y = \operatorname{Cl}\big(N - \cup_i (T_{S^2_{1i}} \cup T_{S^2_{2i}})\big) \cong \mathop{\#}_n B^4 \cong S^4 - \operatorname{Int}(B^4_1 \cup \cdots \cup B^4_n).$$

(c). Suppose that there is a d-fold branched covering $p: M \to N = \#_n (S^3 \times S^1)$ for some $d \geq 1$. Let $\gamma_1, \ldots, \gamma_n \in \pi_1(N) \cong \mathbb{F}_n$ be the free generators, where $\mathbb{F}_n$ is the free group of rank n. By lifting loops, we can find elements $\tilde\gamma_1, \ldots, \tilde\gamma_n \in \pi_1(M)$ such that $p_*(\tilde\gamma_i) = \gamma_i^{a_i}$ for certain $a_i \in \{1, \ldots, d\}$ and for all $i = 1, \ldots, n$, where $p_*: \pi_1(M) \to \pi_1(N)$ is the homomorphism induced by p. Then, $p_*(\pi_1(M))$ contains the subgroup $\langle \gamma_1^{a_1}, \ldots, \gamma_n^{a_n} \rangle$ of $\mathbb{F}_n$. It follows that $p_*(\pi_1(M))$ is free of rank at least n, implying that it admits $\mathbb{F}_n$ as a quotient.

For the converse, we observe that for every epimorphism $\phi: \pi_1(M) \to \mathbb{F}_n$ there exists a PL embedding $h: \vee_n S^1 \to M$ such that $h_*: \pi_1(\vee_n S^1) \to \pi_1(M)$ is a right inverse of ϕ. We want to define a PL map $g: M \to \vee_n S^1$, which is a left inverse of h, and such that $g_* = \phi: \pi_1(M) \to \pi_1(\vee_n S^1) \cong \mathbb{F}_n$. To define g, we consider a handlebody decomposition of M with only one 0-handle H^0 centered at $h(*)$, where $*$ is the join point of $\vee_n S^1$, and such that $H^0 \cup H^1_1 \cup \cdots \cup H^1_n$ is a regular neighborhood of $h(\vee_n S^1)$, for some 1-handles $H^1_1, \ldots, H^1_n$. At this point, the construction of g is as follows: over $H^0 \cup H^1_1 \cup \cdots \cup H^1_n$, the map g is a PL collapsing retraction over $h(\vee_n S^1)$ composed with $h^{-1}: h(\vee_n S^1) \to \vee_n S^1$; over the remaining 1-handles it is defined according to ϕ; then g can be extended over the 2-handles, thanks to the compatibility with ϕ over the generators of $\pi_1(M)$; finally, there is no obstruction to further extend g over the higher index handles.

For every $i = 1 \ldots, n$, let y_i be a point in the i-th component of $\vee_n S^1 - \{*\}$, over which g is transversal (y_i is a regular value), and let Y_i be the connected component of $g^{-1}(y_i)$ that contains $h(y_i)$. Then, Y_i is a connected orientable locally flat PL 3-manifold in M.

Let M' be M cut open along $Y_1, \ldots, Y_n$. By construction, M' is a connected 4-manifold with $2n$ boundary components $Y_1, \overline{Y}_1, \ldots, Y_n, \overline{Y}_n$ and there are identifications $Y_i \cong \overline{Y}_i$ coming from the cuts. By Theorem 2.2 there exists a simple d-fold branched covering $q: M' \to S^4 - \cup_{i=1}^n \operatorname{Int}(B^4_i \cup \overline{B}^4_i)$ such that the coverings $q_{|Y_i}: Y_i \to \partial B_i$ and $q_{|\overline{Y}_i}: \overline{Y}_i \to \partial \overline{B}^4_i$ match with respect to the above identifications, where B^4_i and $\overline{B}^4_i$ are disjoint 4-balls in S^4, for $i = 1, \ldots, n$. We can assume that B_q is a locally flat self-transversally immersed compact PL surface if $d \geq 4$, and that it is embedded if $d \geq 5$. Then, we can glue back Y_i with $\overline{Y}_i$, as well as ∂B^4_i with $\partial \overline{B}^4_i$, by means of the identifications needed to reconstruct M and $\#_n(S^3 \times S^1)$ respectively. Then we get a simple branched covering $p: M \to \#_n(S^3 \times S^1)$ as desired. $\square$

6. *Final remarks*

In Theorem 1.1 (a), the simple branched covering $p\colon M \to \mathbb{C}\mathrm{P}^2$ can be constructed such that $p^*(w_2(\mathbb{C}\mathrm{P}^2)) = w_2(M)$ if $w_2(M)^2 \neq 0$ in $H^4(M;\mathbb{Z}_2)$. Indeed, in the proof it is enough to take as $\phi \in H_2(M)/\operatorname{Tor} H_2(M)$ the Poincaré dual (modulo $\operatorname{Tor} H_2(M)$) of any integral lift of $w_2(M)$ with positive (odd) square. An analogous fact holds for Theorem 1.1 (b).

In Theorem 1.2 (b), for β_M odd and $n < \max(b_2^+(M), b_2^-(M))$, we can also obtain a 5-fold simple covering $p\colon M \to \#_n(S^2 \times S^2)$ branched over a non-singular PL surface, by using the last part of Lemma 5.3 and taking $d = 5$ in the proof.

In Theorem 4.3 we can take p such that its restriction $p_{|N}$ coincides with any given ribbon fillable d-fold branched covering $c\colon N \to S^3$. Indeed, in the proof the choice of c as such a covering is arbitrary.

The following Corollary to Theorem 4.3 is immediate but possibly interesting for the PL or smooth Schoenflies Conjecture in S^4.

Corollary 6.1. *Let $\Sigma^3 \subset S^4$ be a PL embedded 3-sphere and let $d \geq 4$. Then, there exists a d-fold simple covering $p\colon (S^4;\Sigma^3) \to (S^4;S^3)$ branched over a locally flat PL self-transversally immersed surface, which can be taken embedded for $d \geq 5$. Moreover, the restriction $p_{|\Sigma^3}\colon \Sigma^3 \to S^3$ can be arbitrarily chosen among d-fold ribbon fillable branched coverings.*

Moreover, for a PL 3-manifold $N \subset M$, one can prove that there is a simple branched covering $p\colon (M;N) \to (S^4;S^3)$ even though N does not disconnect M. In this case, we obtain an arbitrary degree $d \geq 6$ and a locally flat PL embedded branch surface. The proof goes as follows: following the proof of Theorem 4.3 (b), we begin with a ribbon fillable branched covering $c\colon \partial M_1 \to S^3$ of degree $d \geq 6$, with M_1 a collar of N in M. This is possible because ∂M_1 has two connected components homeomorphic to N. Then, by Theorem 2.2, there are two extensions of c as simple d-fold branched coverings $p_1 : M_1 \to S^4_-$ and $p_2 : M_2 \to S^4_+$, both branched over a locally flat properly embedded PL surface. Their union provides the desired branched covering $p\colon (M;N) \to (S^4;S^3)$.

In Theorem 4.4, for $k \geq 2$, we can take $S^2 \subset S^4$ instead of T_k, with $d = 4k$. Thus, there exists a simple branched covering $p\colon (M;F) \to (S^4;S^2)$ even though F is not connected. The proof is essentially the same, the only difference consisting in the identification of the base of $c_i : N_i \to B^3$ with a single copy of $B^3 \subset S^4$ instead of k copies of it.

The singularities of the branch surfaces of all the 4-dimensional simple branched coverings we have constructed, namely the transversal self-intersections, originate from the application of Theorem 2.2, which was proved in [**25**]. In the construction therein, such singularities appear in pairs, so one can investigate to what extent they can be eliminated, without increasing the covering degree. Then, we conclude by asking the following question (cf. Problem 4.113 (A) in Kirby's list [**17**]).

Question 1. Can the simple branched covering $p\colon M \to N$ in Theorem 1.1 be always chosen with a locally flat PL embedded branch surface even for $d = 4$?

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Riccardo Piergallini
Scuola di Scienze e Tecnologie
Università di Camerino
Via Madonna delle Carceri 9
62032 Camerino (MC)
Italy

riccardo.piergallini@unicam.it

Daniele Zuddas
Dipartimento di Matematica e Geoscienze
Università di Trieste
Via Alfonso Valerio 12/1
34127 Trieste
Italy

dzuddas@units.it