A higher dimensional Poincaré–Birkhoff theorem for Hamiltonian flows

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Abstract

We propose an extension to higher dimensions of the Poincaré–Birkhoff Theorem which applies to Poincaré time-maps of Hamiltonian systems. Examples of applications to pendulum-type systems and weakly-coupled superlinear systems are also given.

Keywords: Poincaré–Birkhoff; Periodic solutions; Hamiltonian systems

1. Introduction

The classical Poincaré–Birkhoff fixed point theorem, also called Poincaré's last geometric theorem, affirms the existence of at least two fixed points for area-preserving homeomorphisms of the planar annulus keeping both boundary circles invariant and twisting them in opposite directions. Going to the universal cover it can be stated as follows (see e.g. [8]):

Theorem *(Poincaré–Birkhoff). Let $\mathcal{P} : \mathbb{R} \times [a, b] \to \mathbb{R} \times [a, b]$ be an area-preserving homeomorphism of the form*

$$\mathcal{P}(x, y) = (x + \vartheta(x, y), \rho(x, y)),$$

where the functions $\vartheta(x, y)$ and $\rho(x, y)$ are 2π-periodic in their first variable x, with $\rho(x, a) = a$ and $\rho(x, b) = b$, for every $x \in \mathbb{R}$. Assume the boundary twist condition:

$$\vartheta(x, a)\,\vartheta(x, b) < 0, \text{ for every } x \in \mathbb{R}.$$

Then, $\mathcal{P}$ has at least two fixed points in $[0, 2\pi[\times]a, b[$.

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This theorem was conjectured by Poincaré shortly before his death in 1912. The original manuscript [34] contained not only the proof of the theorem in some special cases but also two examples of applications in Dynamics, namely the search of closed geodesic lines on a convex surface, and the study of periodic solutions in the restricted three body problem. The full proof of the theorem is due to Birkhoff [2,4], who was also motivated by its applications to the search of periodic solutions of conservative dynamical systems [3,6].

Quoting Arnold, *attempts to generalize it* [the Poincaré–Birkhoff Theorem] *to higher dimensions are important for the study of periodic solutions of problems with many degrees of freedom* [1, page 416]. Birkhoff himself was aware of the importance of this problem, which he qualified as an *outstanding question* [4, page 299]. He proposed [5] a $2N$-dimensional version of the theorem in which the main assumption was the existence of a manifold, diffeomorphic to the N-torus, where the exact symplectic map preserves the first N coordinates. In this case one can use an idea which goes back to Poincaré [33, Chap. 28] and reduce the problem to that of the critical points of a function on the manifold. This argument would be used more recently by Moser and Zehnder [31, Theorem 2.21, p. 135] to deal with maps which are either close to the identity or have a monotone twist.

Using a different approach which combined Lusternik–Schnirelmann variational methods with the Conley index theory for flows, Conley and Zehnder [11, Theorem 3] proved another version of the Poincaré–Birkhoff Theorem in higher dimensions. Their result concerns the multiplicity of periodic solutions for time-dependent Hamiltonian vector fields provided that the C^2-smooth Hamiltonian function $H = H(t, x, y)$ is periodic in t and in the variables x_i, and quadratic on a neighborhood of infinity. Then, they obtained the existence of at least $N + 1$ periodic solutions. Remarkably, their result does not need the Poincaré time-map to be close to the identity, nor to have a monotone twist. The development of infinite-dimensional Lusternik–Schnirelmann methods would allow Szulkin [37, Theorem 4.2] to generalize the Conley–Zehnder theorem by making it applicable to a wider class of Hamiltonian systems. Further results along these lines can be found in [10,22,24].

Despite the efforts done for more than a century it seems that, for the time being, *there is no genuine generalization of the Poincaré–Birkhoff theorem to higher dimensions* [31, page 140]. The aim of this paper is to take a further step in this direction and propose a new higher-dimensional version which will apply to Poincaré time-maps of Hamiltonian systems.

In order to describe our results, let $J = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix}$ denote the standard $2N \times 2N$ symplectic matrix, and consider the (time dependent) Hamiltonian system

$$\dot{z} = J\nabla H(t, z), \qquad z = (x, y) \in \mathbb{R}^{2N}. \tag{HS}$$

Let the function $H : \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}$ be an *admissible Hamiltonian*, by which we mean that $H = H(t, z) = H(t, x, y)$ is continuous, T-periodic in t, 2π-periodic in each of the variables x_i with $i = 1, \dots, N$, and it has a continuously defined gradient with respect to z, denoted by ∇H.

Notice that, once a T-periodic solution $z(t) = (x(t), y(t))$ has been found, many others appear by just adding an integer multiple of 2π to some of the components $x_i(t)$; for this reason, we will call *geometrically distinct* two solutions of *(HS)* which can not be obtained from each other in this way.

Let $D \subseteq \mathbb{R}^N$ be a *convex body*, i.e., D is open, bounded and convex, and assume that every solution $z(t) = (x(t), y(t))$ of *(HS)* starting with $y(0) \in \overline{D}$ is defined for every $t \in [0, T]$. In the theorem below we also assume that D has a C^1-smooth boundary and denote by $\nu : \partial D \to \mathbb{R}^N$ the unit outward normal vector field. Let us recall that a square matrix $\mathbb{A}$ is *regular* if $\det \mathbb{A} \neq 0$, and *involutory* if $\mathbb{A}^2 = I_N$. The main result of these notes is the following:

Theorem 1.1. *Let one of the following conditions hold:*

(a) *There exists a symmetric regular $N \times N$ matrix $\mathbb{A}$ such that, for all solutions starting with $y(0) \in \partial D$, one has*

$$\langle x(T) - x(0), \mathbb{A}\nu(y(0)) \rangle > 0 \,.$$

(b) *There exist an involutory $N \times N$ matrix $\mathbb{A}$ and a point $d_0 \in D$ such that, for all solutions starting with $y(0) \in \partial D$, one has*

$$\langle x(T) - x(0), \mathbb{A}(y(0) - d_0) \rangle > 0 \,.$$

(c) *All solutions starting with* $y(0) \in \partial D$ *satisfy:*

$$x(T) - x(0) \notin \{\alpha \nu(y(0)) : \alpha \geq 0\}.$$

Then, (HS) has at least $N+1$ *geometrically distinct* T*-periodic solutions* $z^{(0)}(t), \dots, z^{(N)}(t)$ *starting with* $y^{(k)}(0) \in D$*, for every* $k = 0, \dots, N$*. Moreover, if the Hamiltonian function* H *is twice continuously differentiable with respect to* z *and the* T*-periodic solutions with initial condition on* $\mathbb{R}^N \times D$ *are nondegenerate, then there are at least* 2^N *of them.*

A few comments are in order. Condition *(a)* can be thought of as a generalization of the quadratic-near-infinity assumption of Conley and Zehnder in [11]: indeed, it suffices to take D as a sufficiently large ball centered at the origin. Condition *(b)* was introduced in the case $\mathbb{A} = I_N$ by Moser and Zehnder [31, Theorem 2.21]; for this reason, our theorem can also be seen as a generalization in the Hamiltonian case of this result. Notice also that we do not need the monotone twist condition required there. An approximation argument shows that the assumption of ∂D being C^1 can be dropped in this case. Condition *(c)* is a variant of the usual assumption made in the Poincaré–Bohl theorem. One can think of it as an *avoiding outer rays condition*; it is worth mentioning that Theorem 1.1 still keeps its validity when it is replaced with the *avoiding inner rays condition*, obtained by reversing the inequality $\alpha \geq 0$. Finally, we emphasize that in all three cases we do not assume the invariance of the domain $\mathbb{R}^N \times D$, nor we require any uniqueness for the solutions of initial value problems associated with *(HS)*.

We now consider an apparently different situation which, however, will be reduced to Theorem 1.1*(a)* by means of changes of variables and approximation arguments. Let the continuous Hamiltonian function $H : \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}$, $H = H(t, x, y)$ be T-periodic in time and continuously differentiable with respect to the state variables (x, y); however, in contrast with what was our framework until now, *we do not assume anymore the periodicity in the state variables* x_i. Instead, we assume that for each $i = 1, \dots, N$ we have selected two planar strictly star-shaped Jordan curves around the origin $\Gamma_1^i, \Gamma_2^i \subseteq \mathbb{R}^2$, such that

$$0 \in \mathcal{D}(\Gamma_1^i) \subseteq \overline{\mathcal{D}(\Gamma_1^i)} \subseteq \mathcal{D}(\Gamma_2^i).$$

Here we denote by $\mathcal{D}(\Gamma)$ the planar open bounded region delimited by the Jordan curve Γ. We consider the annular region

$$\mathcal{A} = \left[\overline{\mathcal{D}(\Gamma_2^1)} \setminus \mathcal{D}(\Gamma_1^1)\right] \times \dots \times \left[\overline{\mathcal{D}(\Gamma_2^N)} \setminus \mathcal{D}(\Gamma_1^N)\right] \subseteq \mathbb{R}^{2N}.$$

We write $z_i(t) = (x_i(t), y_i(t))$ and assume that every solution $z(t) = (z_1(t), \dots, z_N(t))$ of *(HS)* departing from $z(0) \in \mathcal{A}$ is defined for $t \in [0, T]$, and satisfies

$$z_i(t) \neq (0,0), \text{ for every } t \in [0, T] \text{ and } i = 1, \dots, N.$$

This condition allows us to consider continuous determinations $\arg z_i$ of the argument function along these curves, and we denote by $\mathrm{Rot}\,(z_i(t); [0, T]) = (\arg z_i(T) - \arg z_i(0))/2\pi$ their rotation numbers. Since the z_i are not necessarily closed, these rotation numbers can take any real value. We assume that there are integer numbers $\nu_1, \dots, \nu_N \in \mathbb{Z}$ such that, for each $i = 1, \dots, N$ either

$$\mathrm{Rot}\,(z_i(t); [0, T]) \begin{cases} < \nu_i, & \text{if } z_i(0) \in \Gamma_1^i, \\ > \nu_i, & \text{if } z_i(0) \in \Gamma_2^i, \end{cases} \tag{1}$$

or

$$\mathrm{Rot}\,(z_i(t); [0, T]) \begin{cases} > \nu_i, & \text{if } z_i(0) \in \Gamma_1^i, \\ < \nu_i, & \text{if } z_i(0) \in \Gamma_2^i. \end{cases} \tag{2}$$

Theorem 1.2. *Under the above assumptions, the Hamiltonian system (HS) has at least* $N+1$ *distinct* T*-periodic solutions* $z^{(0)}(t), \dots, z^{(N)}(t)$*, starting with* $z^{(k)}(0) \in \mathcal{A}$*, such that*

$$\mathrm{Rot}\,(z_i^{(k)}(t); [0, T]) = \nu_i, \quad \textit{for every } k = 0, \dots, N \textit{ and } i = 1, \dots, N.$$

Moreover, if H is twice continuously differentiable with respect to z and all T-periodic solutions with initial condition on $\mathcal{A}$ are nondegenerate, then there are at least 2^N of them.

Theorem 1.2 generalizes some previous versions of the Poincaré–Birkhoff Theorem for planar annuli with star-shaped boundaries [12,25,36].

The paper is organized as follows. Section 2 is devoted to present and prove some facts on the extrinsic geometry of convex bodies which will be needed in the sequel. In Section 3 we introduce the notions of *basket functions* and their associated *differential cones*. They will provide a way to unify all three cases *(a)–(b)–(c)* appearing in Theorem 1.1. In Section 4 we consider an auxiliary class of admissible Hamiltonians, which we shall call *strongly admissible*; roughly speaking, they are somewhat more regular and can be used to approximate admissible Hamiltonians. The bulk of the proof of Theorem 1.1 is carried out in Section 5. In Section 6 we shall use an approximation argument to extend this result to domains D with corners, from which we will obtain Theorem 1.2. In Section 7 we illustrate some examples of applications, focusing our attention on two types of Hamiltonian systems, *pendulum-like systems* and *weakly-coupled superlinear systems*, and we extend some classical results. Finally, in Section 8 we briefly discuss some questions motivated by the paper, including the possibility to extend the main results to Hamiltonians which are discontinuous in the time variable.

2. Differential geometry of convex bodies

In this section, we recall some geometrical results on convex bodies (i.e., open, bounded and convex sets $D \subseteq \mathbb{R}^N$), which will be needed in the sequel. The convex body D will be called smooth if the boundary ∂D is C^∞-smooth. In this case, the unit normal outer vector field $\nu : \partial D \to \mathbb{R}^N$ is a C^∞-smooth map, and its differential $\nu'(p) : T_p\partial D \to T_p\partial D$ (the so-called *Weingarten map*) at a given point $p \in \partial D$ is a selfadjoint endomorphism. The associated quadratic form on $T_p\partial D$ is usually referred to as the *second fundamental form* of ∂D at p, and denoted by II_p.

We shall say that the convex body D is *strongly convex* provided that it is smooth and, for any $p \in \partial D$, the height function $h_p : \partial D \to \mathbb{R}$, $q \mapsto \langle q - p, -\nu(p)\rangle$ has a nondegenerate minimum at $q = p$. The corresponding Hessian quadratic form is II_p, meaning that D is strongly convex if and only if II_p is positive definite on $T_p\partial D$ for every $p \in \partial D$.

As an example, assume that the convex body D can be written as a sublevel set $D = \varphi^{-1}(]-\infty, c[)$, where the C^∞-smooth function $\varphi : \mathbb{R}^N \to \mathbb{R}$ is strongly convex in the sense that its hessian quadratic form is positive-definite everywhere. Then, D itself is strongly convex. Indeed, one checks that

$$II_p[u,u] = \frac{1}{|\nabla\varphi(p)|}\langle u, (\mathrm{Hess}_p\varphi)u\rangle\,,$$

which is positive definite on $T_p\partial D$.

On the other hand, if there exists a nontrivial segment $[q_1, q_2]$ contained in ∂D, then D is *not* strongly convex, since any point p in this segment must be a degenerate critical point of the height function h_p.

When the convex body D is not smooth, it may not make sense to consider the normal vector field, but at each point $q \in \partial D$ one has the *outer normal cone* $\mathcal{N}(q)$, defined by

$$\mathcal{N}(q) := \Big\{ w \in \mathbb{R}^N : \langle w, q - p\rangle \geq 0 \text{ for every } p \in D \Big\}\,.$$

Observe that this is indeed a nontrivial convex cone, for every $q \in \partial D$.

For instance, assume that $D =]-1, 1[^N$ is the standard cube. Then, for every $q = (q_1, q_2, \ldots, q_N) \in \partial D$,

$$\mathcal{N}(q) = I(q_1) \times I(q_2) \times \ldots \times I(q_N)\,, \tag{3}$$

where $I(q_i) = \begin{cases}]-\infty, 0]\,, & \text{if } q_i = -1\,, \\ [0, +\infty[\,, & \text{if } q_i = 1\,, \\ \{0\}\,, & \text{if } q_i \in]-1, 1[\,. \end{cases}$

On the other hand, if ∂D is C^1-smooth, then $\mathcal{N}(q) = \{\lambda\nu(q) : \lambda \geq 0\}$.

We now observe that convex bodies in $\mathbb{R}^N$ can always be approximated from inside by strongly convex ones. In the result below we denote by $\nu^*(p)$ the unit outward normal vector to ∂D^* at $p \in \partial D^*$ and by $\mathcal{N}(q)$ the outer normal cone to ∂D at $q \in \partial D$.

Lemma 2.1. *Let D be a convex body, let $K \subseteq D$ be a compact set, and let $\varepsilon > 0$ be given. Then, there exists a strongly convex body D^* such that*

$$K \subseteq D^* \subseteq \overline{D^*} \subseteq D\,,$$

and having the following property: for every $p \in \partial D^*$ there exists some $q \in \partial D$ with

$$|p - q| < \varepsilon\,, \qquad \operatorname{dist}(\nu^*(p), \mathcal{N}(q)) < \varepsilon\,.$$

Proof. Choose some continuous, convex function $\varphi : \mathbb{R}^N \to \mathbb{R}$ satisfying

$$\varphi(y) \begin{cases} < 1\,, & \text{if } y \in D\,,\\ = 1\,, & \text{if } y \in \partial D\,,\\ > 1\,, & \text{if } y \in \mathbb{R}^N \setminus \overline{D}\,. \end{cases}$$

For instance, one could take as φ the so-called *gauge* (or *Minkowski*) function associated with D, see, e.g. [23, Theorem 2, p. 21]. Using a standard convolution argument,[3] we may find a sequence of convex C^∞-smooth functions $\varphi_n : \mathbb{R}^N \to \mathbb{R}$ such that

$$|\varphi(y) - \varphi_n(y)| < \frac{1}{n}\,, \quad \text{for every } y \in \overline{D}\,.$$

After replacing $\varphi_n(y)$ by $\varphi_n(y) + \rho_n |y|^2$, for some small positive constant ρ_n, we may further assume that the Hessian matrix of φ_n is positive definite at every point. We set

$$D_n := \varphi^{-1}(\,]-\infty, 1 - 1/n[\,)\,,$$

which is a sequence of strongly convex bodies whose closure is contained in D. One easily checks that, if n is big enough then $D^* := D_n$ satisfies all the requirements, thus proving the result. □

To conclude this section, we study the projection map $\pi : \mathbb{R}^N \setminus \overline{D} \to \partial D$ associated with the smooth convex body D; it is defined by

$$y - \pi y = \operatorname{dist}(y, \partial D)\, \nu(\pi y)\,.$$

It is well-known that π is smooth and non-expansive, i.e.,

$$\|\pi' y\| \le 1\,, \quad \text{for every } y \in \mathbb{R}^N \setminus \overline{D}\,.$$

This inequality can be improved when D is strongly convex, as follows.

Lemma 2.2. *Let $D \subseteq \mathbb{R}^N$ be a strongly convex body. Then, $\|\pi' y\| < 1$ for every $y \in \mathbb{R}^N \setminus \overline{D}$. Moreover, there is a constant $C > 0$ such that*

$$|y|\, \|\pi' y\| \le C\,, \quad \text{for every } y \in \mathbb{R}^N \setminus \overline{D}\,.$$

Proof. We start from the equality

$$\pi(q + t\nu(q)) = q\,, \quad \text{for every } q \in \partial D \text{ and } t \ge 0\,.$$

Differentiating with respect to t, we get $(\pi' y)\nu(\pi y) = 0$, for every $y \in \mathbb{R}^N \setminus \overline{D}$, so that the norm of the linear map $\pi' y : \mathbb{R}^N \to T_{\pi y}\partial D$ coincides with that of its restriction to $T_{\pi y}\partial D$. On the other hand, differentiating with respect to q gives

$$(\pi' y) \circ [\mathrm{Id}_{T_{\pi y}\partial D} + \operatorname{dist}(y, \partial D)\nu'(\pi y)] = \mathrm{Id}_{T_{\pi y}\partial D}\,,$$

for every $y \in \mathbb{R}^N \setminus \overline{D}$. It means that $(\pi' y)\big|_{T_{\pi y}\partial D} : T_{\pi y}\partial D \to T_{\pi y}\partial D$ is an isomorphism and we have found its inverse:

[3] Indeed, the convolution of a convex function with a nonnegative one is still convex.

$$L_p := \Big((\pi' y)\big|_{T_{\pi y}\partial D}\Big)^{-1} = \mathrm{Id}_{T_{\pi y}\partial D} + \mathrm{dist}(y, \partial D)\nu'(\pi y)\,.$$

Since $\nu'(q) : T_q\partial D \to T_q\partial D$ is positive definite for every $q \in \partial D$, which is a compact set, there is a constant $\delta > 0$ (not depending on y) such that

$$\langle L_p u, u\rangle \geq \Big(1 + \delta\,\mathrm{dist}(y, \partial D)\Big)|u|^2\,, \quad \text{for every } u \in T_{\pi y}\partial D\,,$$

so that, using Schwarz inequality, $|L_p u| \geq \Big(1 + \delta\,\mathrm{dist}(y, \partial D)\Big)|u|$. Therefore,

$$\|\pi' y\| = \|L_p^{-1}\| \leq \frac{1}{1 + \delta\,\mathrm{dist}(y, \partial D)}\,,$$

for every $y \in \mathbb{R}^N \setminus \overline{D}$. The result follows. $\square$

3. Basket functions for convex bodies

Let $D \subseteq \mathbb{R}^N$ be a smooth convex body. A C^∞-smooth function $h : \mathbb{R}^N \to \mathbb{R}$ will be called a *basket function* for D provided that

(i) $h(y) = 0$ for any $y \in \overline{D}$.
(ii) $\nabla h(y) \neq 0$ if $y \in \mathbb{R}^N \setminus \overline{D}$
(iii) $\sup_{y\in\mathbb{R}^N} |\nabla h(y) - \mathbb{A}y| < +\infty$, for some regular symmetric matrix $\mathbb{A}$.

The *differential cone* $\mathcal{C}_p(h)$ associated with the basket function h at a given point $p \in \partial D$ is the closed convex cone generated by the set

$$\mathcal{K}_p := \bigcap_{r>0} \overline{\left\{\frac{\nabla h(y)}{|\nabla h(y)|} : y \in \mathbb{R}^N \setminus \overline{D},\ |y - p| < r\right\}}\,,$$

made of accumulation points of the map $\nabla h(y)/|\nabla h(y)|$, as $y \to p$.

It is easy to check that any smooth convex body admits basket functions. For instance, one can take

$$h_1(y) = \begin{cases} 0\,, & \text{if } y \in \overline{D}\,, \\ \rho(|y - \pi y|)|y - \pi y|^2\,, & \text{if } y \in \mathbb{R}^N \setminus \overline{D}\,, \end{cases}$$

where $\rho : \mathbb{R} \to \mathbb{R}$ is a cutoff, C^∞-smooth function satisfying

$$\rho(s) = 0 \ \text{ if } s \leq 0\,, \qquad \rho(s) = \tfrac{1}{2} \ \text{ if } s \geq 1\,, \qquad \rho'(s) > 0 \ \text{ if } s \in]0, 1[\,.$$

Combining the facts that $y - \pi y = |y - \pi y|\nu(\pi y)$ and $(\pi' y)^*\nu(\pi y) = 0$ for any $y \in \mathbb{R}^N \setminus D$, one checks that

$$\nabla h_1(y) = \Big(|y - \pi y|^2\rho'(|y - \pi y|) + 2|y - \pi y|\rho(|y - \pi y|)\Big)\,\nu(\pi y)\,, \quad y \in \mathbb{R}^N \setminus D\,,$$

and it follows that h_1 is indeed a basket function for D (take $\mathbb{A} = \mathrm{Id}$ in *(iii)*); moreover,

$$\mathcal{C}_p(h_1) = \{\lambda\nu(p) : \lambda \geq 0\}\,, \qquad p \in \partial D\,. \tag{4}$$

Assuming that D is a *strongly* convex body, it is possible to construct other basket functions h whose associated differential cones are contained into some prefixed half spaces, which vary along ∂D according to some (possibly indefinite) matrix. We check this below:

Lemma 3.1. *Let $D \subseteq \mathbb{R}^N$ be a strongly convex body, let $d_0 \in D$ be given, and let the $N \times N$ regular matrices $\mathbb{A}_2, \mathbb{A}_3$ be symmetric and involutory, respectively. Then, D has basket functions h_2, h_3 satisfying*

$$\langle v_2, \mathbb{A}_2\nu(p)\rangle \geq 0\,,\ \langle v_3, \mathbb{A}_3(p - d_0)\rangle \geq 0\,, \qquad \text{if } v_2 \in \mathcal{C}_p(h_2) \text{ and } v_3 \in \mathcal{C}_p(h_3)\,,$$

for any $p \in \partial D$.

Proof. We first show the statement concerning h_2. Set

$$h_2(y) = \begin{cases} 0\,, & \text{if } y \in \overline{D}\,, \\ \rho(|y-\pi y|)\langle y-\pi y, \mathbb{A}_2(y-\pi y)\rangle\,, & \text{if } y \in \mathbb{R}^N \setminus \overline{D}\,, \end{cases}$$

the function ρ being defined as above. Then, for $|y|$ large enough, $h_2(y) = \frac{1}{2}\langle y-\pi y, \mathbb{A}_2(y-\pi y)\rangle$, and assumption *(iii)* (with $\mathbb{A} = \mathbb{A}_2$) follows from the last part of Lemma 2.2. Moreover, for any $y \in \mathbb{R}^N \setminus \overline{D}$ one has:

$$\langle \nabla h_2(y), \mathbb{A}_2\nu(\pi y)\rangle = |y-\pi y|^2\rho'(|y-\pi y|)\,\langle \nu(\pi y), \mathbb{A}_2\nu(\pi y)\rangle^2 + \\ + 2|y-\pi y|\,\rho(|y-\pi y|)\,\langle(\mathrm{Id}-\pi' y)^*\mathbb{A}_2\nu(\pi y), \mathbb{A}_2\nu(\pi y)\rangle\,.$$

The last part of Lemma 2.2 implies that $\mathrm{Id}-\pi' y$ is positive definite for any $y \in \mathbb{R}^N \setminus \overline{D}$. Therefore, the final term in the equality above is positive, while the middle one is nonnegative, and then, $\langle \nabla h_2(y), \mathbb{A}_2\nu(\pi y)\rangle > 0$ for any $y \in \mathbb{R}^N \setminus \overline{D}$. It implies assumption *(ii)*, as well as the statement on the differential cones associated with h_2.

Let us check the statement concerning h_3 now. There is no loss of generality in assuming that $d_0 = 0 \in D$. As a first step we also assume that $\mathbb{A}_3$ is orthogonal and symmetric, and define

$$h_3(y) = \begin{cases} 0\,, & \text{if } y \in \overline{D}\,, \\ \rho(|y-\pi y|)\langle y-\pi y, \mathbb{A}_3 y\rangle\,, & \text{if } y \in \mathbb{R}^N \setminus \overline{D}\,. \end{cases}$$

Assumption *(iii)* (with $\mathbb{A} = \mathbb{A}_3$) follows immediately from this definition and the last part of Lemma 2.2. On the other hand simple computations show that, if $y \in \mathbb{R}^N \setminus \overline{D}$,

$$\langle \nabla h_3(y), \mathbb{A}_3 y\rangle = |y-\pi y|\,\rho'(|y-\pi y|)\,\langle \nu(\pi y), \mathbb{A}_3 y\rangle^2 + \\ + \rho(|y-\pi y|)\,\langle \mathbb{A}_3(y-\pi y), \mathbb{A}_3 y\rangle + \rho(|y-\pi y|)\,\langle(\mathrm{Id}-\pi' y)^*\mathbb{A}_3 y, \mathbb{A}_3 y\rangle\,.$$

The first term on the right hand side is nonnegative, while the remaining ones are positive. Thus, $\langle \nabla h_3(y), \mathbb{A}_3 y\rangle > 0$ for any $y \in \mathbb{R}^N \setminus \overline{D}$, implying both *(ii)* and the statement about the differential cones of h_3. This concludes the proof when $\mathbb{A}_3$ is orthogonal and symmetric.

In the general case, when $\mathbb{A}_3$ is an arbitrary involutory matrix, there are a diagonal matrix $\mathbb{D}$ having only ± 1 in its diagonal and a regular matrix $\mathbb{P}$ such that $\mathbb{A}_3 = \mathbb{P}^{-1}\mathbb{D}\mathbb{P}$. By the previously considered case, the convex body $\mathbb{P}(D)$ admits a basket function ψ with $\mathcal{C}_{\mathbb{P}p}(\psi) \subseteq \{v \in \mathbb{R}^N : \langle v, \mathbb{D}\mathbb{P}p\rangle \geq 0\}$ for any $p \in \partial D$, and then $h_3(y) := \psi(\mathbb{P}y)$ satisfies the requirements for D. The proof is complete. □

We close this section by stating a general result concerning the existence and multiplicity of periodic solutions for Hamiltonian vector fields whose flow avoids the differential cones of a basket function for some convex body. In view of (4) and Lemma 3.1, it unifies all three cases *(a), (b), (c)* of Theorem 1.1, thus generalizing this result when D is strongly convex:

Theorem 3.2. *Let the Hamiltonian function H be admissible, and assume the existence of a convex body $D \subseteq \mathbb{R}^N$ and an associated basket function h such that any solution $z(t) = (x(t), y(t))$ of (HS) starting with $y(0) \in \overline{D}$ is defined for every $t \in [0, T]$ and, moreover, those with $y(0) \in \partial D$ satisfy $x(T) - x(0) \notin \mathcal{C}_{y(0)}(h)$. Then, the same conclusion of Theorem 1.1 holds.*

The proof of this result will be carried out in Section 5. We now introduce the class of *strongly admissible* Hamiltonians as a way to deal with the difficulties arising from the nonuniqueness of initial value problems associated with the Hamiltonian flow.

4. Strongly admissible Hamiltonians

We required in the Introduction that admissible Hamiltonians should be continuously defined on $\mathbb{R} \times \mathbb{R}^{2N}$. However, this assumption will be slightly relaxed through Sections 4–5, as we shall work with Hamiltonians which are defined on $[0, T] \times \mathbb{R}^{2N}$ and it may happen that $H(0, z) \neq H(T, z)$. Thus, from now on we shall say that the continuously defined Hamiltonian $H : [0, T] \times \mathbb{R}^{2N} \to \mathbb{R}$, $H = H(t, x, y)$ is *admissible* if it is 2π-periodic in each variable x_i

and continuously differentiable with respect to $z = (x, y)$. Allowing some abuse of terminology, we shall say that the solution $z : [0, T] \to \mathbb{R}^{2N}$ of (HS) is T-periodic if $z(0) = z(T)$.

So, let $H : [0, T] \times \mathbb{R}^{2N} \to \mathbb{R}$ be admissible and let $D \subseteq \mathbb{R}^N$ be a smooth convex body. H will be called *strongly admissible with respect to D* provided that:

[1.] There exists a relatively open set $\mathcal{V} \subseteq [0, T] \times \mathbb{R}^N$, containing $\{0\} \times (\mathbb{R}^N \setminus D)$, such that H is C^∞-smooth with respect to the state variables $z = (x, y)$ on the set $\mathcal{V}_\sharp := \{(t, x, y) : (t, y) \in \mathcal{V},\ x \in \mathbb{R}^N\}$.

[2.] There exists some $R > 0$ such that $H(t, x, y) = 0$, if $|y| \geq R$.

Observe that condition **[2.]** implies in particular that ∇H is bounded and the solutions of (HS) cannot explode in finite time. Thus, if H is strongly admissible with respect to some set D, then the solutions of (HS) are defined on the whole time interval $[0, T]$.

As usually, we denote by (HS) the Hamiltonian system associated with the admissible Hamiltonian $H : [0, T] \times \mathbb{R}^{2N} \to \mathbb{R}$; correspondingly, $(\widehat{HS})$ will stand for the Hamiltonian system associated with $\widehat{H}$. The main result of this section is the following:

Proposition 4.1. *Assume that all solutions $z(t) = (x(t), y(t))$ of (HS) starting with $y(0) \in \overline{D}$ are defined on the whole interval $[0, T]$, and no T-periodic solution $z(t) = (x(t), y(t))$ starts with $y(0) \in \partial D$. Then, for every $\varepsilon > 0$ there exists a Hamiltonian $\widehat{H} : [0, T] \times \mathbb{R}^{2N} \to \mathbb{R}$ which is strongly admissible with respect to D and satisfies:*

($\diamond$) *H and $\widehat{H}$ coincide on a relatively open subset of $[0, T] \times \mathbb{R}^{2N}$ which contains the graph of any T-periodic solution $\hat{z}(t) = (\hat{x}(t), \hat{y}(t))$ of $(\widehat{HS})$ starting with $\hat{y}(0) \in D$.*

($\diamond\diamond$) *For any solution $\hat{z}(t) = (\hat{x}(t), \hat{y}(t))$ of $(\widehat{HS})$ starting with $\hat{y}(0) \in \partial D$ there exists a solution $z(t) = (x(t), y(t))$ of (HS) starting with $y(0) \in \partial D$, such that*

$$|z(t) - \hat{z}(t)| < \varepsilon\,, \quad \text{for every } t \in [0, T]\,. \tag{5}$$

Before going to the proof of Proposition 4.1 we observe that, under the conditions there, uniqueness for initial value problems associated with (HS) is not guaranteed. Still, the possibly multivalued flow of our Hamiltonian system possesses some properties which evoke continuity. To start with, one has the following 'boundedness on compact sets' result:

Lemma 4.2. *Let H and D be under the conditions of Proposition 4.1. Then, the set of solutions $z(t) = (x(t), y(t))$ of (HS) departing with $y(0) \in \overline{D}$ is uniformly bounded in the y components on the time interval $[0, T]$, i.e.,*

$$|y(t)| \leq R_1\,, \qquad \text{for every } t \in [0, T]\,,$$

for some constant $R_1 > 0$ (not depending on the solution z).

This lemma follows directly from [14, Theorem 5, page 9]. We shall also need a 'continuous dependence' result for our possibly multivalued flow:

Lemma 4.3. *Let H and D be under the conditions of Proposition 4.1 and choose some $\varepsilon > 0$. Then, there exists some $\delta > 0$ such that, whenever $\widehat{H} : [0, T] \times \mathbb{R}^{2N} \to \mathbb{R}$ is an admissible Hamiltonian with*

$$|\nabla \widehat{H}(t, z) - \nabla H(t, z)| \leq \delta\,, \text{ for every } (t, z) \in [0, T] \times \mathbb{R}^{2N}\,, \tag{6}$$

then every solution $\hat{z}(t) = (\hat{x}(t), \hat{y}(t))$ of $(\widehat{HS})$ with $\operatorname{dist}(\hat{y}(t_0), \partial D) \leq \delta$ for some $t_0 \in [0, \delta]$ can be extended to the whole time interval $[0, T]$ and satisfies (5) for some solution $z = (x, y) : [0, T] \to \mathbb{R}^{2N}$ of (HS) starting with $y(0) \in \partial D$.

The proof this lemma follows standard arguments based on the combination of the Ascoli–Arzelà Theorem and Lemma 4.2, and will be omitted.

Proof of Proposition 4.1. In view of Lemma 4.2, after multiplying H by some cutoff function $a = a(y)$, we may assume that **[2.]** holds. On the other hand, a compactness argument based on Ascoli–Arzelà Theorem implies the existence of some number $\varepsilon_* > 0$ such that $|z(T) - z(0)| \geq \varepsilon_*$ whenever $z(t) = (x(t), y(t))$ is a solution of (HS) departing with $y(0) \in \partial D$.

Fix now $\varepsilon \in]0, \varepsilon_*/2[$, and let $\delta \in]0, T[$ be as given by Lemma 4.3 for this positive number. Choose some C^∞-smooth cutoff function $m : [0, 1] \times \mathbb{R}^N \to [0, 1]$ satisfying

$$m(t, y) = \begin{cases} 1, & \text{if } 0 \leq t \leq \delta/2 \text{ and } \operatorname{dist}(y, \mathbb{R}^N \setminus D) \leq \delta/2\,, \\ 0, & \text{if } \delta \leq t \leq 1 \text{ or } \operatorname{dist}(y, \mathbb{R}^N \setminus D) \geq \delta\,, \end{cases}$$

and let the sequence $H_n : [0, T] \times \mathbb{R}^{2N} \to \mathbb{R}$ of admissible C^∞-smooth Hamiltonians converge to H in the C^1-sense. We may assume, with no loss of generality, that all of them satisfy **[2.]**. Then, $\widehat{H}_n(t, x, y) := (1 - m(t, y))H(t, x, y) + m(t, y)H_n(t, x, y)$ is a sequence of strongly admissible Hamiltonians which converges to H in the C^1-sense; in particular, $\widehat{H} := \widehat{H}_n$ satisfies (6), for n large enough, and $(\diamond\diamond)$ follows directly from Lemma 4.3 (with $t_0 = 0$).

In order to check $(\diamond)$, we observe that $\widehat{H}$ and H coincide on the relatively open set

$$\mathcal{O} := \Big([0, \delta] \times \mathbb{R}^N \times \{y \in \mathbb{R}^N : \operatorname{dist}(y, \mathbb{R}^N \setminus D) > \delta\}\Big) \cup \Big(]\delta, T] \times \mathbb{R}^{2N}\Big).$$

We claim that the graph of any T-periodic solution $\hat{z}(t) = (\hat{x}(t), \hat{y}(t))$ of $(\widehat{HS})$ starting with $\hat{y}(0) \in D$ is contained in $\mathcal{O}$. If not, it has to cross the set

$$[0, \delta] \times \mathbb{R}^N \times \{y \in \mathbb{R}^N : \operatorname{dist}(y, \partial D) \leq \delta\}.$$

By Lemma 4.3, there is a solution $z(t) = (x(t), y(t))$ of (HS) such that $y(0) \in \partial D$ and $|z(t) - \hat{z}(t)| < \varepsilon$, for every $t \in [0, T]$. Being $\hat{z}(0) = \hat{z}(T)$, we conclude that $|z(T) - z(0)| \leq 2\varepsilon < \varepsilon_*$, a contradiction. The proof is complete. □

5. The proof

In this section we prove Theorem 1.1. The combination of Lemma 4.3 (for $\widehat{H} = H$ and $t_0 = 0$) and Lemma 2.1 implies that it suffices to show the result when D is a strongly convex body. Consequently, the results of Section 3 mean that we only have to prove Theorem 3.2. And, in view of Proposition 4.1, we may (and shall) assume that $\widehat{H}$ is strongly admissible with respect to D.

An important ingredient of our proof is the following theorem due to Szulkin (cf. [37, Theorem 4.2] and [38, Theorem 8.1]), which was proved by variational methods (see [10,17,24] for related results).

Theorem 5.1 *(Szulkin). Let $\mathbb{A}$ be a regular symmetric $N \times N$ matrix, and let $H : [0, T] \times \mathbb{R}^{2N} \to \mathbb{R}$ be an admissible Hamiltonian, with*

$$H(t, x, y) = \tfrac{1}{2}\langle \mathbb{A}y, y\rangle + G(t, x, y)\,, \tag{7}$$

where the gradient of G with respect to (x, y) is bounded. Then, the Hamiltonian system (HS) has at least $N + 1$ geometrically distinct T-periodic solutions. Moreover, if G is twice continuously differentiable with respect to (x, y) and the T-periodic solutions are known to be nondegenerate, then there are at least 2^N of them.

We point out that in [37,38] it is assumed that the Hamiltonian H is continuously defined (and T-periodic in time) on $\mathbb{R} \times \mathbb{R}^{2N}$. This is equivalent to adding the assumption $H(0, z) = H(T, z)$ in Theorem 5.1. Even though this was a natural assumption in Szulkin's work, his arguments, which were based on the study of the associated (strongly indefinite, $(\mathbb{R}/2\pi\mathbb{Z})^N$-invariant) action functional, can still be carried on without changes in this slightly more general situation.

Theorem 5.1 is not directly applicable to our situation; however, we will construct a modified Hamiltonian $\widetilde{H} : [0, T] \times \mathbb{R}^{2N} \to \mathbb{R}$, satisfying its assumptions, and being equal to H on a relatively open set of $[0, T] \times \mathbb{R}^{2N}$ which contains the graph of every T-periodic solution of the modified Hamiltonian system $(\widetilde{HS})$. In this way, Theorem 3.2 will follow from Szulkin's Theorem 5.1.

Following the usual practice, we shall say that a function defined on a non-open subset of $\mathbb{R}^N$ is smooth provided that it can be smoothly extended to an open set. Similarly, a map between non-open subsets of $\mathbb{R}^N$ will be called a diffeomorphism provided that it is bijective and can be extended to a diffeomorphism between open sets.

From now on we operate under the premises of Theorem 3.2, to which we add the assumption that H is strongly admissible with respect to D. We fix $\mathcal{V}$, $\mathcal{V}_\sharp$, R as given by **[1.]–[2.]**, and consider the sets

$$\begin{cases} \mathcal{F}_0 := \{\eta \in \mathbb{R}^N \setminus D : |\eta| \le R\}, \\ \mathcal{F}_1 := \{\eta \in \mathbb{R}^N : \mathrm{dist}(\eta, \mathbb{R}^N \setminus D) \le \varrho,\ |\eta| \le R\}, \\ \mathcal{G} := \{\eta \in \mathbb{R}^N : |\eta| > R\}, \\ \Delta := \big([0,\tau] \times \mathcal{F}_0\big) \cup \big([0,T] \times \mathcal{G}\big), \\ \mathcal{U}_\sharp := \mathcal{V}_\sharp \cup ([0,T] \times \mathbb{R}^N \times \mathcal{G}). \end{cases}$$

By combining the boundedness of ∇H and Lemma 4.3 we see that one can choose the small constants $\varrho > 0$ and $0 < \tau < T$ so that

$$[0,\tau] \times \mathcal{F}_1 \subseteq \mathcal{V}, \tag{8}$$

and the following implications hold for solutions $z(t) = (x(t), y(t))$ of (HS):

$$t \in [0,\tau] \quad \Rightarrow \quad |y(t) - y(0)| < \varrho, \tag{9}$$

and

$$\mathrm{dist}(y(0), \partial D) \le \varrho \quad \Rightarrow \quad x(T) - x(0) \notin \{\lambda \nabla h(y(0)) : \lambda \ge 0\}. \tag{10}$$

We denote by $\mathcal{Z} = (\mathcal{X}, \mathcal{Y})$ the flow map associated with (HS), i.e., $\mathcal{Z}(t;\zeta) = \mathcal{Z}(t;\xi,\eta) = (\mathcal{X}(t;\xi,\eta), \mathcal{Y}(t;\xi,\eta))$ is the value at time t of the solution starting from $\zeta = (\xi,\eta)$ at zero time. It follows from (8)–(9) that $\mathcal{Z}$ is well-defined and smooth on $\Delta_\sharp := \{(t,\xi,\eta) \in [0,T] \times \mathbb{R}^{2N} : (t,\eta) \in \Delta\}$; moreover, $\mathfrak{Z}(t;\zeta) = (t; \mathcal{Z}(t;\zeta))$ defines a diffeomorphism between $\Delta_\sharp$ and its image $\mathfrak{Z}(\Delta_\sharp)$. Notice also that

$$\mathfrak{Z}(\Delta_\sharp) \subseteq \big([0,\tau] \times \mathbb{R}^N \times \mathcal{F}_1\big) \cup \big([0,T] \times \mathbb{R}^N \times \mathcal{G}\big) \subseteq \mathcal{U}_\sharp. \tag{11}$$

On the other hand we recall that, by *(i)*, the basket function h vanishes on D. This fact will allow us to show the following:

Lemma 5.2. *There is a C^∞-smooth function $r : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ satisfying*

- $(\star)$ $r(t,\eta) = 0$, *if* $(t,\eta) \notin \Delta$,
- $(\star\star)$ $\frac{1}{T}\int_0^T r(t,\eta)\,dt = h(\eta)$, *for every* $\eta \in \mathbb{R}^N$,
- $(\star\star\star)$ $r(t,\eta) = h(\eta)$, *if* $|\eta|$ *is sufficiently large.*

Proof. There is no loss of generality in assuming that $0 \in D$. We choose some C^∞-smooth function $u : \mathbb{R} \to \mathbb{R}$ with

$$u(s) = 0 \ \text{ if } s \le 0, \quad u(s) = 1 \ \text{ if } s \ge 1, \quad u(s) \in \,]0,1[\ \text{ if } s \in \,]0,1[,$$

and define $r : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ by

$$r(t,\eta) = \begin{cases} 0, & \text{if } \eta = 0, \\ \dfrac{u(|\eta| - Rt/\tau)}{\frac{1}{T}\int_0^T u(|\eta| - Rs/\tau)\,ds}\, h(\eta), & \text{if } \eta \ne 0. \end{cases}$$

It is easy to check that r satisfies all the required properties. It proves the lemma. □

We continue now with the proof of Theorem 3.2. We consider the functions $r_\sharp, \mathcal{R} : [0,T] \times \mathbb{R}^{2N} \to \mathbb{R}$ defined by

$$r_\sharp(t,\xi,\eta) = r(t,\eta), \qquad \mathcal{R}(t,z) = \begin{cases} r_\sharp(\mathfrak{Z}^{-1}(t,z)), & \text{if } (t,z) \in \mathfrak{Z}(\Delta_\sharp), \\ 0, & \text{otherwise}. \end{cases}$$

Both of them are C^∞-smooth. Let

$$\widetilde{H}(t,z) = H(t,z) - \lambda \mathcal{R}(t,z)\,, \tag{12}$$

where $\lambda > 0$ is a constant, to be determined later. We observe that, for $|y|$ large enough,

$$\widetilde{H}(t,x,y) = -\lambda \mathcal{R}(t,x,y) = -\lambda r(t,y) = -\lambda h(y)\,,$$

and, in view of assumption *(iii)* for basket functions, $\nabla \widetilde{H}(t,x,y) + \lambda \mathbb{A} y$ is bounded on $[0,T] \times \mathbb{R}^{2N}$. Thus, we may apply Szulkin's Theorem 5.1 to the perturbed Hamiltonian system ($\widetilde{HS}$) associated with $\widetilde{H}$, to get at least $N+1$ geometrically distinct T-periodic solutions of ($\widetilde{HS}$), or 2^N of them if the Hamiltonian is twice continuously differentiable with respect to (x,y) and the periodic solutions are nondegenerate. In what follows, we will prove that, if λ is chosen large enough, these are actually T-periodic solutions of *(HS)*.

The following observation will be important: if $\tilde{z} = (\tilde{x}, \tilde{y}) : [0,T] \to \mathbb{R}^{2N}$ is a solution of ($\widetilde{HS}$) with $\tilde{y}(0) \in D$, then $(t, \tilde{z}(t)) \notin \mathfrak{Z}(\Delta_\sharp)$ for every $t \in [0,T]$; in particular, $\tilde{z}$ is a solution of *(HS)*. To check this we recall that the Hamiltonian H is C^∞-smooth on $\mathcal{U}_\sharp$, which is a relatively open subset of $[0,T] \times \mathbb{R}^{2N}$. Moreover, $\mathcal{U}_\sharp \supseteq \mathfrak{Z}(\Delta_\sharp)$, as noticed in (11). Our solution $\tilde{z}$ starts being a solution of the original system *(HS)*, because in some small region around the initial condition both systems are the same. Should it arrive at $\mathfrak{Z}(\Delta_\sharp)$ it would first cross $\mathfrak{Z}([0,\tau] \times \mathbb{R}^N \times \partial D) \cup \mathfrak{Z}([\tau, T] \times \mathbb{R}^N \times \partial \mathcal{G})$, which is made by the graph of solutions of the original system, and this cannot happen, by uniqueness.

By the previous discussion, one only has to show that no solution $\tilde{z}(t) = (\tilde{x}(t), \tilde{y}(t))$ of ($\widetilde{HS}$) departing with $\tilde{y}(0) \notin D$ is T-periodic. This assertion is easier to check when $\tilde{y}(0) \in \mathcal{G}$. Indeed, on $[0,T] \times \mathbb{R}^N \times \mathcal{G}$ one has

$$H(t,x,y) = 0, \qquad \mathcal{Z}(t;\xi,\eta) = (\xi,\eta), \qquad \widetilde{H}(t,x,y) = -\lambda r(t,y)\,,$$

and our modified Hamiltonian system ($\widetilde{HS}$) becomes (using the variables $\xi = \tilde{x}$, $\eta = \tilde{y}$):

$$\dot{\xi} = -\lambda \frac{\partial r}{\partial \eta}(t,\eta)\,, \qquad \dot{\eta} = 0\,. \tag{13}$$

In particular, $\tilde{y}(t) \equiv \tilde{y}(0)$, and $\tilde{x}(T) - \tilde{x}(0) = -\lambda T \nabla h(\tilde{y}(0)) \neq 0$.

It remains to consider the case $\tilde{y}(0) \in \mathcal{F}_0$. Under this assumption, let $\zeta = (\xi, \eta) : [0,\tau] \to \mathbb{R}^{2N}$ be the solution of (13) satisfying $\zeta(0) = \tilde{z}(0)$. Observe that $\eta(t)$ must be constant on $[0,T]$, so $\eta(t) \equiv \eta(0) \in \mathcal{F}_0$, and $\xi(t)$ must be constant on $[\tau, T]$; moreover, by ($\star\star$) in Lemma 5.2,

$$\zeta(\tau) - \zeta(0) = \zeta(T) - \zeta(0) = -\lambda T (\nabla h(\tilde{y}(0)), 0)\,. \tag{14}$$

The following lemma establishes a functional relation between $\tilde{z}$ and ζ.

Lemma 5.3. *The equality $\tilde{z}(t) = \mathcal{Z}(t; \zeta(t))$ holds for every $t \in [0,\tau]$.*

Proof. Recalling the definition of $\widetilde{H}$ in (12) and the fact that H is C^∞-smooth on $\mathcal{U}_\sharp$, we see that $\widetilde{H}$ is C^∞-smooth on $\mathcal{U}_\sharp$. In view of (11) and the fact that $(t, \zeta(t)) \in \Delta_\sharp$ for every $t \in [0,\tau]$, this relatively open subset of $[0,T] \times \mathbb{R}^{2N}$ contains the graph of $z_1(t) := \mathcal{Z}(t;\zeta(t))$ for $t \in [0,\tau]$; moreover, $\tilde{z}(0) = z_1(0)$. Thus, by uniqueness, it suffices to show that z_1 is a solution of ($\widetilde{HS}$).

Straightforward computations give:

$$\dot{z}_1 = J \left(\nabla H(t, z_1) + \lambda J \frac{\partial \mathcal{Z}}{\partial \zeta}(t;\zeta) J \nabla r_\sharp(t;\zeta) \right). \tag{15}$$

Being $\mathcal{Z}$ the flow map associated with a Hamiltonian system, $\mathcal{Z}(t;\cdot)$ is canonical, i.e.,

$$\left(\frac{\partial \mathcal{Z}}{\partial \zeta}(t, \zeta(t)) \right)^* J \, \frac{\partial \mathcal{Z}}{\partial \zeta}(t, \zeta(t)) = J\,,$$

and, consequently (15) becomes

$$\dot{z}_1 = J \left(\nabla H(t, z_1) - \lambda \left(\left(\frac{\partial \mathcal{Z}}{\partial \zeta}(t;\zeta) \right)^{-1} \right)^* \nabla r_\sharp(t,\zeta) \right) = J \nabla \widetilde{H}(t, z_1)\,,$$

proving the result. □

We arrive at the following:

Lemma 5.4. *There exists a solution $z : [0, T] \to \mathbb{R}^{2N}$ of (HS) for which*

$$z(0) = \tilde{z}(0) - \lambda T (\nabla h(\tilde{y}(0)), 0)\,, \qquad z(T) = \tilde{z}(T)\,.$$

Proof. We set

$$z(t) := \begin{cases} \mathcal{Z}(t; \zeta(\tau)) & \text{if } 0 \le t < \tau \\ \tilde{z}(t) & \text{if } \tau \le t \le T \end{cases}\,.$$

Then, $z(0) = \zeta(\tau)$, while $z(T) = \tilde{z}(T)$. Since we also have that $\tilde{z}(0) = \zeta(0)$, the result follows from (14). □

Remembering now that H is strongly admissible (and, in particular, assumption **[2.]**), and combining items *(ii)–(iii)* from the definition of basket functions we deduce the existence of some constant $\gamma > 0$ such that

$$\left| \frac{\partial H}{\partial y}(t, x, y) \right| \le \frac{1}{\gamma} \text{ on } [0, T] \times \mathbb{R}^{2N}\,, \qquad |\nabla h(q)| \ge \gamma \text{ if } \operatorname{dist}(q, D) \ge \varrho\,. \tag{16}$$

Observe that the constant γ does not depend on the solution $\tilde{z}$. The next lemma leads to the conclusion of the proof.

Lemma 5.5. *If $\lambda > 1/\gamma^2$, then $\tilde{x}(0) \ne \tilde{x}(T)$. In particular, $\tilde{z}$ is not T-periodic.*

Proof. According to Lemma 5.4, one has

$$\tilde{x}(T) - \tilde{x}(0) = x(T) - x(0) - \lambda T \nabla h(\tilde{y}(0))\,. \tag{17}$$

In case $\operatorname{dist}(\tilde{y}(0), \partial D) < \varrho$, the result follows from (10) (recall that $\tilde{y}(0) = y(0)$). If, on the contrary, $\operatorname{dist}(\tilde{y}(0), \partial D) \ge \varrho$, the combination of (16)–(17) gives:

$$|\tilde{x}(T) - \tilde{x}(0)| \ge \lambda T |\nabla h(\tilde{y}(0))| - |x(T) - x(0)| \ge \lambda T \gamma - T/\gamma > 0\,.$$

The proof is thus complete. □

6. Nonsmooth sets, tubes, and products of annuli

In this section we are going to deduce Theorem 1.2 from Theorem 1.1*(a)*. With this goal we shall first extend this latter result in two directions. Firstly, we would like to have a version of the theorem for general convex bodies, not necessarily C^1-smooth. With this aim, we will have to replace the outer normal vector field $\nu = \nu(y)$ by the outer normal cone $\mathcal{N}(y)$ at each point $y \in \partial D$. Secondly, we would like to generalize the notion of admissible Hamiltonians to allow them to be periodic in the x variable with respect to some basis of $\mathbb{R}^N$ which is not necessarily the usual one. More specifically, we shall say that the Hamiltonian function[4] $H : \mathbb{R} \times \mathbb{R}^{2N} \to \mathbb{R}$, $H = H(t, x, y)$ is *admissible with respect to the basis* $\mathcal{B} = \{b_1, \dots, b_N\}$ of $\mathbb{R}^N$ provided that, besides the usual regularity assumptions, it satisfies

$$H(t, x + b_i, y) = H(t, x, y)\,, \quad \text{for every } (t, x, y) \in [0, T] \times \mathbb{R}^{2N}\,.$$

Theorem 6.1. *Let the Hamiltonian $H = H(t, x, y)$ be admissible with respect to some basis of $\mathbb{R}^N$. Let the $N \times N$ matrix $\mathbb{A}$ be regular and symmetric, let $D \subseteq \mathbb{R}^N$ be a convex body, and assume that every solution $z(t) = (x(t), y(t))$ of (HS) departing with $y(0) \in \partial D$ is defined for every $t \in [0, T]$ and satisfies*

$$\langle x(T) - x(0), \mathbb{A} w \rangle > 0\,, \quad \text{for every } w \in \mathcal{N}(y(0)) \setminus \{0\}\,. \tag{18}$$

Then, the same conclusion of Theorem 1.1 holds.

[4] At this moment we go back to Hamiltonians which are continuously defined on $\mathbb{R} \times \mathbb{R}^{2N}$ and T-periodic in time.

Proof. An approximation argument which combines Lemma 2.1 and Lemma 4.3 (for $\widehat{H} = H$ and $t_0 = 0$) shows that it suffices to prove the result when D is smooth. In this case, (18) becomes the usual condition $\langle x(T) - x(0), \mathbb{A}\nu(y(0))\rangle > 0$. Let $\mathbb{P}$ be the (nonsingular) matrix whose columns are the elements of $\mathcal{B}$ and consider the (canonical) change of variables

$$x_1 = \mathbb{P}^{-1}x, \qquad y_1 = \mathbb{P}^* y. \tag{19}$$

The transformed Hamiltonian $H_1(t, x_1, y_1) = H(t, x, y)$ is admissible with respect to the usual basis and together with the convex body $D_1 := \mathbb{P}^*(D)$ and the matrix $\mathbb{A}_1 = \mathbb{P}^*\mathbb{A}\mathbb{P}$, it lies under the assumptions of Theorem 1.1 *(a)*. The result follows. □

We now consider a situation where the set D may vary with the point x. Let us introduce the following definition. By a *tube* we mean a set of the form

$$\mathcal{T} = \{(x, y) \in \mathbb{R}^{2N} : a_i(x_i) < y_i < b_i(x_i),\ i = 1, \dots, N\},$$

where $a_i, b_i : \mathbb{R} \to \mathbb{R}$ are given 2π-periodic continuous functions, for $i = 1, \dots, N$, with $a_i(s) < b_i(s)$ for every $s \in \mathbb{R}$. For any $j \in \{1, \dots, N\}$, the j-th (closed) top face of $\mathcal{T}$ is the set

$$\mathcal{T}_j^+ = \{(x, y) \in \mathbb{R}^{2N} :\ y_j = b_j(x_j) \text{ and } a_i(x_i) \le y_i \le b_i(x_i),\ \text{if } i \ne j\},$$

while the j-th (closed) bottom face of $\mathcal{T}$ is given by

$$\mathcal{T}_j^- = \{(x, y) \in \mathbb{R}^{2N} :\ y_j = a_j(x_j) \text{ and } a_i(x_i) \le y_i \le b_i(x_i),\ \text{if } i \ne j\}.$$

Notice that $\partial\mathcal{T}$ is the union of all the top and bottom faces of $\mathcal{T}$.

Theorem 6.2. *Let the Hamiltonian function $H : [0, T] \times \mathbb{R}^{2N} \to \mathbb{R}$ be admissible, and let $\mathcal{T}$ be a tube such that every solution $z(t) = (x(t), y(t))$ of (HS) with $z(0) \in \overline{\mathcal{T}}$ is defined for every $t \in [0, T]$. Moreover, assume that for any $j = 1, \dots, N$, either*

$$x_j(T) - x_j(0) \begin{cases} > 0, & \text{if } z(0) \in \mathcal{T}_j^+, \\ < 0, & \text{if } z(0) \in \mathcal{T}_j^-, \end{cases}$$

or

$$x_j(T) - x_j(0) \begin{cases} < 0, & \text{if } z(0) \in \mathcal{T}_j^+, \\ > 0, & \text{if } z(0) \in \mathcal{T}_j^-. \end{cases}$$

Then, the Hamiltonian system (HS) has at least $N + 1$ geometrically distinct T-periodic solutions $z^{(0)}(t), \dots, z^{(N)}(t)$ with $z^{(k)}(0) \in \mathcal{T}$, for every $k = 0, \dots, N$. Moreover, if the Hamiltonian function H is twice continuously differentiable with respect to z and the T-periodic solutions with initial condition on $\mathcal{T}$ are nondegenerate, then there are at least 2^N of them.

Proof. We first assume that $\mathcal{T} = \mathbb{R}^N \times\,]-1, 1[^N$, i.e., $a_i \equiv -1$ and $b_i \equiv 1$, for every $i = 1, \dots, N$. Then, recalling (3), if $(x(t), y(t))$ is a solution of *(HS)* starting with $y(0)$ on the boundary of $]-1, 1[^N$, we see that

$$\langle x(T) - x(0), \mathbb{A}v\rangle > 0, \quad \text{for every } v \in \mathcal{N}(y(0)) \setminus \{0\},$$

where $\mathbb{A}$ is a diagonal matrix, whose elements on the diagonal are $+1$ or -1. The result then follows from Theorem 6.1. (Indeed, we have thus proved a slightly more general result than the one claimed, namely that it holds for Hamiltonians H which are admissible with respect to some basis $\mathcal{B}$.)

We now treat the general case. After an approximation argument based on the Féjer Theorem and Lemma 4.3 (for $t_0 = 0$, $\widehat{H} = H$), there is no loss of generality in assuming that the functions a_i, b_i are C^∞-smooth. We define the functions $c_i, l_i : \mathbb{R} \to \mathbb{R}$ by

$$c_i(s) = \frac{a_i(s) + b_i(s)}{2}, \qquad l_i(s) = \frac{b_i(s) - a_i(s)}{2},$$

for $i=1,\dots,N$, and let $\Psi:\mathbb{R}^{2N}\to\mathbb{R}^{2N}$ be defined by

$$\Psi(x,y)=\left(\int_0^{x_1} l_1(s)\,ds\,,\dots,\int_0^{x_N} l_N(s)\,ds\,,\ \frac{y_1-c_1(x_1)}{l_1(x_1)},\dots,\frac{y_N-c_N(x_N)}{l_N(x_N)}\right).$$

It can be verified that Ψ is a symplectic diffeomorphism. Hence, the change of variables $(\hat{x},\hat{y})=\Psi(x,y)$ (which, for $N=1$ was proposed in [31, Exercise 1, p. 132]), transforms our Hamiltonian system *(HS)* into a new one, with Hamiltonian function

$$\widehat{H}(t,\hat{z})=H(t,\Psi^{-1}(\hat{z}))\,.$$

The new Hamiltonian is still periodic in the variables $x_1,\dots,x_N$, but the corresponding periods have changed and are now $T_1=\int_0^{p_1} l_1(s)\,ds,\dots,T_N=\int_0^{p_N} l_N(s)\,ds$, respectively. In other words, $\widehat{H}$ is now admissible with respect to the basis $\mathcal{B}=\{T_1b_1,\dots,T_Nb_n\}$, where $\{b_1,\dots,b_N\}$ is the canonical basis of $\mathbb{R}^N$. Moreover, the change of variables transforms the tube $\mathcal{T}$ into $\mathbb{R}^N\times\,]-1,1[^N$. We are thus reduced to the first step, and the result follows. □

Theorem 1.2 is an easy consequence of Theorem 6.2 by going to polar coordinates. We check the details below.

Proof of Theorem 1.2. Since the solutions $z(t)$ departing from $z(0)\in\mathcal{A}$ are defined on $[0,T]$ and none of their components attain the origin, we can find a constant $\delta_0>0$ such that $|z_i(t)|>2\delta_0$, for every $t\in[0,T]$ and $i=1,\dots,N$, for each of those solutions. We now modify the Hamiltonian function near the origin, as follows. Let $\omega:\mathbb{R}\to\mathbb{R}$ be a C^∞-smooth function such that

$$\omega(r)=\begin{cases}0\,, & \text{if } r\le\delta_0\,,\\ 1\,, & \text{if } r\ge 2\delta_0\,.\end{cases}$$

Then, we consider the new Hamiltonian system

$$\dot{z}=J\nabla\bar{H}(t,z)\,,$$

with

$$\bar{H}(t,z)=\omega\big(\min\{|z_i|:i=1,\dots,N\}\big)H(t,z)\,,$$

so that $\bar{H}(t,z)=0$ when one of the components of z is too near the origin. This will not affect the solutions starting from $\mathcal{A}$, as long as $t\in[0,T]$. We now consider the (time-dependent) change of variables

$$x_i=\sqrt{2\rho_i}\,\cos\big(\theta_i-(2\pi/T)\nu_it\big)\,,\quad y_i=-\sqrt{2\rho_i}\,\sin\big(\theta_i-(2\pi/T)\nu_it\big)\,, \tag{20}$$

so to get the Hamiltonian system

$$\dot{\theta}_i=\frac{\partial\mathcal{H}}{\partial\rho_i}(t,\rho,\theta)\,,\quad \dot{\rho}_i=-\frac{\partial\mathcal{H}}{\partial\theta_i}(t,\rho,\theta)\,,$$

defined for $\theta=(\theta_1,\dots,\theta_N)\in(\mathbb{R}/2\pi\mathbb{Z})^N$ and $\rho=(\rho_1,\dots,\rho_N)\in\mathbb{R}^N$ with $\rho_i>0$ for every i. Here,

$$\mathcal{H}(t,\theta,\rho):=\bar{H}(t,x,y)+\sum_{i=1}^N(2\pi/T)\nu_i\rho_i\,,$$

the variables x,y in the argument of $\bar{H}$ being related to θ,ρ by (20). Notice that the change of variables is justified if $z(0)\in\mathcal{A}$, since then $z_i(t)\neq(0,0)$ for every $t\in[0,T]$ and $i=1,\dots,N$. This system can now be extended also when $\rho_i\le 0$ for some i, by simply setting $\mathcal{H}(t,\theta,\rho):=\sum_{i=1}^N(2\pi/T)\nu_i\rho_i$ there. Now, Theorem 6.2 applies. Indeed, the star-shaped curves Γ_1^i,Γ_2^i are transformed into the continuous and 2π-periodic functions a_i,b_i, and the twist condition follows from (1) and (2). Going back to the original variables, the proof is easily concluded. □

7. Applications

In this section, we illustrate how our theorems may be applied to two types of situations, which we call *pendulum-like systems*, and *weakly-coupled superlinear systems*. For briefness, we only concentrate on the search of T-periodic solutions, but the experienced reader will recognize the possibility of proving the existence of periodic solutions of the second kind, for the pendulum-like systems, and of subharmonic solutions, for the superlinear systems. The stated results provide existence of $N+1$ solutions. Needless to say, the number of solutions we find will be 2^N in the nondegenerate situation.

7.1. Pendulum-like systems

One year after the publication of the 1983 paper by Conley and Zehnder [11], Mawhin and Willem [28] studied some pendulum-like scalar second order differential equations, by the use of a variational method. They proved that, if the T-periodic forcing term has zero mean value, then there are at least two T-periodic solutions. The papers by Conley–Zehnder and Mawhin–Willem attracted a lot of attention. They were further extended in [24,27,35,37], always using variational methods. As observed by Rabinowitz [35] in 1988, the Mawhin–Willem result could have been obtained (in the smooth case) from the Conley–Zehnder theorem, after a suitable modification of the nonlinearity. Alternatively, as noticed in [19,20], it could also have been obtained directly from some generalized version of the Poincaré–Birkhoff theorem.

In this subsection, we exploit this idea and study Hamiltonian systems whose behavior reminds that of pendulum-like equations. Our main result will be the following

Theorem 7.1. *Let the Hamiltonian $H:[0,T]\times\mathbb{R}^{2N}\to\mathbb{R}$ be admissible, and assume that*

$$\lim_{|y|\to\infty}\frac{\nabla_x H(t,x,y)}{|y|}=0\,,\quad\text{uniformly in } (t,x)\in[0,T]\times\mathbb{R}^N\,. \tag{21}$$

If, moreover, there are two positive constants r, ρ and a regular $N\times N$ matrix $\mathbb{A}$, having only real eigenvalues, such that

$$|y|\ge r\quad\Rightarrow\quad\big\langle\nabla_y H(t,x,y),\mathbb{A}y\big\rangle>\rho\,\big|\nabla_y H(t,x,y)\big|\,|y|\,, \tag{22}$$

then the Hamiltonian system (HS) has at least $N+1$ distinct T-periodic solutions.

Proof. Let $z(t)=(x(t),y(t))$ be a solution of (*HS*), with $z(0)=z_0=(x_0,y_0)$. Even if no uniqueness is assumed, we will denote any such solution by $\mathcal{Z}(t;z_0)=(\mathcal{X}(t;z_0),\mathcal{Y}(t;z_0))$. By (21) and the periodicity of H in the x_i variables, such a solution has to be globally defined on $[0,T]$. Another consequence of (21), which can be obtained with the help of Gronwall's Lemma, is that

$$\lim_{|y_0|\to\infty}\frac{\mathcal{Y}(t;x_0,y_0)-y_0}{|y_0|}=0\,,\qquad\text{uniformly in } (t,x_0)\in[0,T]\times\mathbb{R}^N\,,$$

which, in its turn, implies that

$$\lim_{|y_0|\to\infty}\left(\frac{\mathcal{Y}(t;x_0,y_0)}{|\mathcal{Y}(t;x_0,y_0)|}-\frac{y_0}{|y_0|}\right)=0\,,\quad\text{uniformly in } (t,x_0)\in[0,T]\times\mathbb{R}^N. \tag{23}$$

Assume for a moment the matrix $\mathbb{A}$ to be symmetric. Since the solutions of the initial value problems are globally defined, we can find an $R>r$ such that, if $(x(t),y(t))$ is a solution of (*HS*) with $(x(0),y(0))=(x_0,y_0)$ and $|y_0|\ge R$, then $|y(t)|\ge r$, for every $t\in[0,T]$. Then, by (22) and (23), if R is sufficiently large,

$$\begin{aligned}\left\langle\frac{\nabla_y H(t,x(t),y(t))}{|\nabla_y H(t,x(t),y(t))|}\,,\ \mathbb{A}\frac{y_0}{|y_0|}\right\rangle&=\left\langle\frac{\nabla_y H(t,x(t),y(t))}{|\nabla_y H(t,x(t),y(t))|}\,,\mathbb{A}\frac{y(t)}{|y(t)|}\right\rangle+\\&\quad+\left\langle\frac{\nabla_y H(t,x(t),y(t))}{|\nabla_y H(t,x(t),y(t))|}\,,\mathbb{A}\left(\frac{y(t)}{|y(t)|}-\frac{y_0}{|y_0|}\right)\right\rangle>0\,.\end{aligned}$$

Consequently,

$$\langle x(T) - x(0), \mathbb{A} y(0)\rangle = \int_0^T \langle \nabla_y H(t, x(t), y(t)), \mathbb{A} y(0)\rangle \, dt$$

$$= \int_0^T \left\langle \frac{\nabla_y H(t, x(t), y(t))}{|\nabla_y H(t, x(t), y(t))|}, \mathbb{A} \frac{y_0}{|y_0|} \right\rangle |y_0| \, |\nabla_y H(t, x(t), y(t))| \, dt > 0 .$$

The conclusion follows from Theorem 1.1*(a)*, taking $D = B(0, R)$.

Let us now treat the case when $\mathbb{A}$ is diagonalizable. Let $\mathbb{P}$ be a regular matrix for which $\mathbb{D} = \mathbb{P}^* \mathbb{A} (\mathbb{P}^*)^{-1}$ is diagonal. With the canonical change of variables (19) we get a new Hamiltonian system, where the Hamiltonian function H_1 still satisfies

$$\lim_{|y_1| \to \infty} \frac{\nabla_{x_1} H_1(t, x_1, y_1)}{|y_1|} = 0 , \quad \text{uniformly in } (t, x_1) \in [0, T] \times \mathbb{R}^N . \tag{24}$$

On the other hand, (22) implies the existence of some positive constants r_1, ρ_1 such that

$$|y_1| \geq r_1 \quad \Rightarrow \quad \langle \nabla_{y_1} H_1(t, x_1, y_1), \mathbb{D} y_1 \rangle > \rho_1 \, |y_1| \, \big| \nabla_{y_1} H_1(t, x_1, y_1) \big| ,$$

so that we are reduced to the case of a diagonal (hence symmetric) matrix.

Finally, let $\mathbb{A}$ be any regular matrix having only real eigenvalues. Then, $\mathbb{A}$ can be approximated by diagonalizable matrices: there is a sequence $(\mathbb{A}_n)_n$ of matrices, all of which are regular and have distinct real eigenvalues, which converges to $\mathbb{A}$ in the usual operator norm topology. This can be easily seen using the Jordan canonical form. Then, if $|y| \geq r$, taking n large enough, we have

$$\begin{aligned} \left\langle \nabla_y H(t, x, y), \mathbb{A}_n \frac{y}{|y|} \right\rangle = & \\ &= \left\langle \nabla_y H(t, x, y), \mathbb{A} \frac{y}{|y|} \right\rangle + \left\langle \nabla_y H(t, x, y), (\mathbb{A}_n - \mathbb{A}) \frac{y}{|y|} \right\rangle \\ &> \frac{\rho}{2} \, |\nabla_y H(t, x, y)| , \end{aligned}$$

so that we are back to the previous case. The proof is thus complete. □

As a possible example of application, we can deal with second order systems of the type

$$\ddot{x} + \nabla F(t, x) = e(t) ,$$

where $F(t, x_1, \ldots, x_N)$ is 2π-periodic with respect to each variable $x_1, \ldots, x_N$ (so ∇F is bounded), and $e : \mathbb{R} \to \mathbb{R}^N$ is a T-periodic forcing with zero mean value, i.e.,

$$\int_0^T e(t) \, dt = 0 . \tag{25}$$

(As an example, if $N = 1$, we have in mind the pendulum equation.) Writing the equivalent Hamiltonian system

$$\dot{x} = y + E(t) , \qquad \dot{y} = -\nabla F(t, x) ,$$

with $E(t) = \int_0^t e(s) \, ds$, we see that Theorem 7.1 directly applies, taking as $\mathbb{A}$ the identity matrix. Similar results have been obtained in [26,35].

Another example is given by equations of the type

$$\frac{d}{dt} (\nabla \Phi \circ \dot{x}) + \nabla F(t, x) = e(t) , \tag{26}$$

where Φ is a real valued, strictly convex C^1-smooth function defined on a ball $B(0, a) \subseteq \mathbb{R}^N$, with $\nabla \Phi : B(0, a) \to \mathbb{R}^N$ being a homeomorphism, and $\nabla \Phi(0) = 0$. Denoting by Φ^* the Legendre–Fenchel transform of Φ, we can write the equivalent Hamiltonian system

$$\dot{x} = \nabla \Phi^*(y + E(t)) , \qquad \dot{y} = -\nabla F(t, x) . \tag{27}$$

Recall that $\nabla\Phi^* = (\nabla\Phi)^{-1} : \mathbb{R}^N \to B(0,a)$ and, since Φ^* is strictly convex and coercive, it satisfies

$$\liminf_{|y|\to\infty} \frac{\langle \nabla\Phi^*(y), y\rangle}{|y|} > 0\,.$$

So, assuming (25), Theorem 7.1 easily applies, again with $\mathbb{A} = I_N$. We thus obtain as a corollary a result by Mawhin [27]. As a particular case, one can take $\Phi(y) = 1 - \sqrt{1-|y|^2}$ (leading to the so-called 'relativistic operator').

A rather similar situation is encountered in a result by Golé [21, Theorem 42.2], where the Hamiltonian function is assumed to be *uniformly optical*. Under his assumptions, the gradient of the Hamiltonian with respect to the first state variable turns out to be bounded, while the gradient with respect to the second one satisfies our condition (22), for some positive definite matrix $\mathbb{A}$. Hence, his result can also be obtained from our theorem.

A variant of the above concerns the case when Φ is a strictly convex C^1-smooth function defined on the whole $\mathbb{R}^N$, with $\nabla\Phi : \mathbb{R}^N \to B(0,a)$ being a homeomorphism, and $\nabla\Phi(0) = 0$. As a particular case, one can take $\Phi(y) = 1 - \sqrt{1+|y|^2}$ (leading to the so-called 'mean curvature operator'). Let $h : [0,T] \to \mathbb{R}$ be such that

$$|\nabla F(t,x)| \le h(t)\,, \quad \text{for every } t \in [0,T] \text{ and } x \in \mathbb{R}^N.$$

Writing equation (26) as the equivalent system (27), if we want to apply Theorem 1.1, with $D = B(0, \frac{1}{2}a)$, we must be careful that the solutions $z(t) = (x(t), y(t))$ starting with $y(0) \in \overline{D}$ remain with $y(t)$ in a compact set contained in $B(0,a)$, for $t \in [0,T]$, so that the Hamiltonian function can be modified outside this set and extended to the whole space. This will be guaranteed if

$$2(\|h\|_1 + \|E\|_\infty) < a\,.$$

We thus generalize a result obtained in [19] for the scalar equation (see also [32], where bounded variation solutions are obtained).

To conclude this subsection, we observe that Theorem 1.1 can also be applied to recover a result by Josellis [22] where, besides (21), it was assumed that

$$\lim_{|y|\to\infty} \frac{\nabla_y H(t,x,y) - A(t)y}{|y|} = 0\,, \quad \text{uniformly in } (t,x) \in [0,T]\times\mathbb{R}^N\,,$$

and that the matrix $\mathbb{A} := \int_0^T A(t)\,dt$ is regular and symmetric. We omit the details, for briefness.

7.2. Generalized annuli and weakly-coupled superlinear systems

We close this paper by considering systems of the form:

$$\begin{cases} \ddot{x}_1 + g_1(x_1) = \dfrac{\partial\mathcal{U}}{\partial x_1}(t, x_1, \dots, x_N)\,, \\ \dots \\ \ddot{x}_N + g_N(x_N) = \dfrac{\partial\mathcal{U}}{\partial x_N}(t, x_1, \dots, x_N)\,. \end{cases} \tag{28}$$

Here, all functions $g_i : \mathbb{R} \to \mathbb{R}$ are continuous, and $\mathcal{U} : [0,T]\times\mathbb{R}^N \to \mathbb{R}$ is continuous and continuously differentiable in $x_1, \dots, x_N$. The result presented below generalizes the first part of [7, Theorem 3.1]. It can also be seen as a version for systems of the main theorem of [13]. Possibly, it can be adapted to situations where the retractive forces $g_1, \dots, g_N$ can have one or two singularities (cf. [16]).

Theorem 7.2. *Assume that*

$$\lim_{|s|\to\infty} \frac{g_i(s)}{s} = +\infty\,,$$

and that there is a constant $K > 0$ such that

$$\left| \frac{\partial\mathcal{U}}{\partial x_i}(t, x_1, \dots, x_N) \right| \le K\,,$$

for every $i = 1, \dots, N$ and $(t, x_1, \dots, x_N) \in [0, T] \times \mathbb{R}^N$. Then, there is a positive integer $\bar{\nu}$ with the following property: for any fixed integers $\nu_1, \dots, \nu_N \geq \bar{\nu}$, system (28) has at least $N + 1$ distinct T-periodic solutions

$$x^{(k)}(t) = (x_1^{(k)}(t), \dots, x_N^{(k)}(t)), \qquad k = 0, \dots, N,$$

such that each $x_i^{(k)}(t)$ has exactly $2\nu_i$ simple zeros in $[0, T[$.

Proof. We consider the equivalent Hamiltonian system

$$\dot{x}_i = y_i, \quad \dot{y}_i = -g_i(x_i) - \frac{\partial \mathcal{U}}{\partial x_i}(t, x_1, \dots, x_N), \qquad i = 1, \dots, N, \tag{29}$$

corresponding to the Hamiltonian $H(t, x, y) := \frac{1}{2}|y|^2 + \sum_{i=1}^N \int_0^{x_i} g_i(u)du + \mathcal{U}(t, x, y)$. Using the arguments from [9, Lemma 1], one checks that the solutions of our system are globally defined. Moreover, following the lines in [9, Lemma 2] one checks that, for every $r > 0$ there is some $R(r) > r$ such that, if a solution $z(t) = (x(t), y(t))$ of (29) satisfies $x_i(0)^2 + y_i(0)^2 \geq R(r)$ for some $i = 1, \dots, N$, then $x_i(t)^2 + y_i(t)^2 \geq r$ for every $t \in [0, T]$. In particular, $z_i(t) \neq (0, 0)$, for every $t \in [0, T]$ (i.e., the zeroes of x_i are simple), and we can therefore compute the rotation number $\mathrm{Rot}(z_i(t); [0, T])$. It is standard to show (see [9, Lemma 3]) that the superlinear growth of g_i implies that the negative angular speed of $z_i(t)$ grows to infinity as the amplitude $|z_i(t)|$ increases. Thus, arguing as in [9, Lemma 4], taking e.g. $\bar{R} = R(1)$, there is an integer $\bar{\nu} \geq 1$ such that

$$(i) \text{ if } |z_i(0)| = \bar{R}, \text{ then } -\mathrm{Rot}(z_i(t); [0, T]) < \bar{\nu}, \qquad i = 1, \dots, N.$$

Choose numbers $\nu_1, \dots, \nu_N \geq \bar{\nu}$; there is a constant $\widehat{R} > \bar{R}$ such that

$$(ii) \text{ if } |z_i(0)| = \widehat{R}, \text{ then } -\mathrm{Rot}(z_i(t); [0, T]) > \nu_i, \qquad i = 1, \dots, N.$$

Applying Theorem 1.2, we find $N + 1$ distinct T-periodic solutions whose i-th component satisfies $\mathrm{Rot}(z_i(t); [0, T]) = -\nu_i$, for every $i = 1, \dots, N$. It implies that x_i has $2\nu_i$ simple zeroes on $[0, T[$, thus concluding the proof. □

8. Final remarks

The Carathéodory case. In many applications it can be important to consider Hamiltonian systems which are discontinuous in time. Thus, let us say that the Hamiltonian $H : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is (C^p, L^r)-Carathéodory provided that:

i) $H(t, z)$ is measurable in t for fixed z;
ii) $H(t, z)$ is p times continuously differentiable in z for a.e. fixed t;
iii) $|H(t, z)| + |\nabla H(t, z)| + \ldots + |D^p H(t, z)|$ is dominated on compact subsets of $\mathbb{R} \times \mathbb{R}^{2N}$ by L^r-functions of t.

For the sake of briefness, we shall simply call C^p-Carathéodory the Hamiltonians H which are (C^p, L^r)-Carathéodory for some $r > 1$.

It turns out that Theorems 1.1 and 1.2 (and in fact, also Theorems 3.2, 6.1 and 6.2) keep their validity for C^1-Carathéodory Hamiltonians (C^2-Carathéodory Hamiltonians in the nondegenerate case). The underlying reason here is that a version of Szulkin's Theorem 5.1 also holds in the Carathéodory setting. Indeed, if one assumes that the C^1-Carathéodory Hamiltonian H has the form (7), and one replaces Szulkin's boundedness condition by the assumption that $|\nabla G|$ is (globally) dominated by a $L^r(0, T)$-function of t for some $r > 1$, then the associated Hamiltonian system has at least $N + 1$ geometrically different T-periodic solutions (2^N of them if H is C^2-Carathéodory and the solutions are nondegenerate). This is so because the associated action functional is still well-defined (and continuously differentiable) on $H^{1/2}(\mathbb{R}/\mathbb{Z})$, and has a (strongly indefinite) saddle geometry; the abstract theorems designed by Szulkin still apply.

The proof of Theorem 3.2 (which gives rise to all other results of this paper), can be carried out with small changes to the Carathéodory case; maybe the greatest difference now is that one has to allow strongly admissible Hamiltonians to be discontinuous in t. Precisely, **[1.]** must be re-understood in the sense that H is C^∞-Carathéodory on $\mathcal{V}_\sharp$. An important step of our construction consists in approximating admissible Hamiltonians by strongly admissible ones so that the corresponding flows converge uniformly on compact sets; it can be easily done by means of a convolution argument in the z variables. Of course, the approximation will only be uniform in the z variables and for a.e. t, but this is sufficient for the convergence of the flows.

Exact symplectic vs. Hamiltonian maps. Our results do not require uniqueness for initial value problems. However, if there is uniqueness and we denote by

$$\mathcal{P} : \mathbb{R}^N \times \overline{D} \to \mathbb{R}^N \times \mathbb{R}^N, \qquad \mathcal{P}(x, y) = (x + \vartheta(x, y), \rho(x, y)),$$

the associated Poincaré time-map, then periodic solutions of (*HS*) correspond to fixed points of $\mathcal{P}$. Thus, Theorem 1.1 states the existence of fixed points for maps belonging to a certain class, and a natural question here concerns to having a way to know when a given map lies there. It is well known that, assuming some smoothness for the Hamiltonian, $\mathcal{P}$ must be a diffeomorphism into its image, differing from the identity on a periodic map in the x_i variables. Besides, it has to be *exact symplectic*, i.e. $\mathcal{P}^*\lambda - \lambda = dF$, for some smooth function $F : \mathbb{T}^N \times \overline{D} \to \mathbb{R}$, where $\lambda = \sum_{i=1}^N y_i dx_i$ is the canonical 1-form. On the other hand, a well known result (cf. [29, Proposition 9.19] or [21, Theorem 58.9]) states that $\mathcal{P}$ is the Poincaré map of a Hamiltonian system of the type we are dealing with if and only if it can be joined to the identity via a smooth isotopy of exact symplectic maps. However, this criterion could not be easy to check in practical situations. More explicit conditions are available when $\mathcal{P}$ is an exact symplectic *monotone twist* map. Indeed, as Moser has shown [30], in the two dimensional case all such maps are indeed Poincaré time maps of a Hamiltonian system. The higher dimensional case has been treated by Golé [21, Theorem 41.6], assuming that the map is globally defined and the twist is, in some sense, controlled at infinity.

Connection to the Brouwer degree. Concerning also the uniqueness setting, one easily checks that all three conditions *(a), (b), (c)* in Theorem 1.1 imply that the Brouwer degree $\deg\big(\vartheta(x, \cdot), D, 0\big)$ must be ± 1 for every $x \in \mathbb{R}^N$. We do not know whether our assumptions can be generalized to some condition on these degrees.

Addendum: This paper is a re-organization of a selection of the main results which were originally structured in the following two manuscripts:

> “On the higher dimensional Poincaré–Birkhoff theorem for Hamiltonian flows. 1. The indefinite twist; 2. The avoiding rays condition”.

They have been available online since 2013 and 2014, respectively, and have subsequently been used for further applications in [15,18]. More recently, their ideas, methods and results have been used by the first author and Paolo Gidoni to further investigate on the *avoiding cones condition*.

Conflict of interest statement

There is no conflict of interest.

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