Lorenzo D'Ambrosio and Enzo Mitidieri*

Uniqueness and comparison principles for semilinear equations and inequalities in Carnot groups

Accepted July 30, 2017

Abstract: Variants of the Kato inequality are proved for distributional solutions of semilinear equations and inequalities on Carnot groups. Various applications to uniqueness, comparison of solutions and Liouville theorems are presented.

Keywords: Kato inequality, comparison principles, uniqueness property, semilinear inequalities, Liouville theorems, Carnot groups

MSC 2010: 35R03, 35R45, 35A05, 35B51

1 Introduction

It is well known that one of the fundamental tools for studying different questions related to coercive elliptic equations and inequalities on $\mathbb{R}^N$ is the so-called *Kato inequality* [14].

One of the earlier and main contributions in this direction has been proved by Brezis [3]. As a consequence of a *modified Kato inequality* he considered, among other things, distributional solutions of elliptic inequalities of the form

$$\Delta u \geq |u|^{q-1}u \quad \text{on } \mathbb{R}^N, \tag{1.1}$$

where $q > 1$. The main conclusion of Brezis is that if $u \in L^q_{\mathrm{loc}}(\mathbb{R}^N)$ solves (1.1), then

$$u(x) \leq 0 \quad \text{a.e. on } \mathbb{R}^N.$$

A number of important results can be deduced from this simple statement (see [3] for details).

Quasilinear versions of the Kato inequality have been studied recently in [8], where general a-priori estimates and Liouville theorems have been proved for weak solutions of coercive quasilinear elliptic equations and inequalities in divergence form; see also [1, 5, 6, 10, 11, 16] for related results.

The goal of this paper is to prove a *modified version of the Kato inequality* (see (3.1) below) for *distributional solutions* for a Laplacian operator on a Carnot group; see [2].

It should be noted that a similar Kato inequality has been proved in [8] for weak solutions, i.e., $W^{1,2}_{\mathrm{loc}}$ solutions. We point out, see Remark 3.5 below, that a Kato inequality for *distributional solutions* cannot be deduced from the corresponding inequality valid for *weak solutions* even in the standard Euclidean framework; see [8, Theorem 2.1].

This paper is organized as follows: Section 2 contains some preliminary material on Carnot groups. In Section 3, we prove one of the main results of this paper (see Theorem 3.2) and discuss its relation with the

Lorenzo D'Ambrosio: Dipartimento di Matematica, Università degli Studi di Bari, via E. Orabona, 4, 70125 Bari, Italy, e-mail: lorenzo.dambrosio@uniba.it. http://orcid.org/0000-0003-0677-056X
***Corresponding author: Enzo Mitidieri:** Dipartimento di Matematica e Geoscienze, Università degli Studi di Trieste, via A. Valerio, 12/1, 34127 Trieste, Italy, e-mail: mitidier@units.it

results proved in [8]. In Section 4, we prove some uniqueness results for a general semilinear second-order inequality and give some concrete applications. In Section 5, we shall briefly discuss the ideas pointed out in the preceding section to systems of semilinear inequalities; see [9] for other applications of Kato inequalities to semilinear elliptic systems. Finally, in Section 6 we prove a modified version of Kato complex inequalities in the setting of Carnot groups and present some applications to the so-called *reduction principles* and to uniqueness of solutions of complex problems; see [6].

2 Preliminaries on Carnot groups

In this section, we recall some preliminary facts concerning Carnot groups (for more information and proofs we refer the interested reader to [2, 12]).

A Carnot group is a connected, simply connected, nilpotent Lie group $\mathbb{G}$ of dimension $N \geq 2$ with graded Lie algebra $\mathcal{G} = V_1 \oplus \cdots \oplus V_r$ such that $[V_1, V_i] = V_{i+1}$ for $i = 1, \ldots, r-1$ and $[V_1, V_r] = 0$. A Carnot group $\mathbb{G}$ of dimension N can be identified, up to an isomorphism, with the structure of a *homogeneous Carnot group* $(\mathbb{R}^N, \circ, \delta_\lambda)$ defined as follows: We identify $\mathbb{G}$ with $\mathbb{R}^N$ endowed with a Lie group law $\circ$. We consider $\mathbb{R}^N$ split into r subspaces $\mathbb{R}^N = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_r}$ with $n_1 + n_2 + \cdots + n_r = N$ and $\xi = (\xi^{(1)}, \ldots, \xi^{(r)})$ with $\xi^{(i)} \in \mathbb{R}^{n_i}$. We shall assume that there exists a family of Lie group automorphisms, called *dilation*, δ_λ with $\lambda > 0$ of the form $\delta_\lambda(\xi) = (\lambda\xi^{(1)}, \lambda^2\xi^{(2)}, \ldots, \lambda^r\xi^{(r)})$. The Lie algebra of left-invariant vector fields on $(\mathbb{R}^N, \circ)$ is $\mathcal{G}$. For $i = 1, \ldots, n_1 = l$, let X_i be the unique vector field in $\mathcal{G}$ that coincides with $\partial/\partial\xi_i^{(1)}$ at the origin. We require that the Lie algebra generated by $X_1, \ldots, X_{n_1}$ is the whole $\mathcal{G}$.

With the above hypotheses, we call $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$ a *homogeneous Carnot group*. The *canonical sub-Laplacian* on $\mathbb{G}$ is the second-order differential operator $\mathcal{L} = \sum_{i=1}^{l} X_i^2$. Now, let $Y_1, \ldots, Y_l$ be a basis of $\operatorname{span}\{X_1, \ldots, X_l\}$; the second-order differential operator

$$\Delta_G = \sum_{i=1}^{l} Y_i^2$$

is called a *sub-Laplacian* on $\mathbb{G}$. We denote by $Q = \sum_{i=1}^{r} i n_i$ the *homogeneous dimension* of $\mathbb{G}$. In the sequel, we assume $Q \geq 3$.

A nonnegative continuous function $S : \mathbb{R}^N \to \mathbb{R}_+$ is called a *homogeneous norm* on $\mathbb{G}$ in the case that $S(\xi) = 0$ if and only if $\xi = 0$ and it is homogeneous of degree 1 with respect to δ_λ (i.e., $S(\delta_\lambda(\xi)) = \lambda S(\xi)$). We say that a homogeneous norm is symmetric if $S(\xi^{-1}) = S(\xi)$.

The Lebesgue measure is the bi-invariant Haar measure. For any measurable set $E \subset \mathbb{R}^N$, we have $|\delta_\lambda(E)| = \lambda^Q |E|$. Since $Y_1, \ldots, Y_l$ generate the whole $\mathcal{G}$, any sub-Laplacian Δ_G satisfies the Hörmander hypoellipticity condition. Moreover, the vector fields $Y_1, \ldots, Y_l$ are homogeneous of degree 1 with respect to δ_λ.

In what follows, we fix the vector fields $Y_1, \ldots, Y_l$. In this setting, we use the symbol ∇_0 to denote the vector field $(Y_1, \ldots, Y_l)$, and $-\operatorname{div}_0 := \nabla_0^*$, where ∇_0^* is the formal adjoint of ∇_0. Finally, we set

$$W^{1,2}_{\mathrm{loc}} := \{u \in L^2_{\mathrm{loc}} : |\nabla_0 u| \in L^2_{\mathrm{loc}}\}.$$

3 Kato's inequality for a sub-Laplacian operator on $\mathbb{G}$

In this section, we shall prove that a *modified version of the Kato inequality* for *distributional solutions* holds for a sub-Laplacian operator on a Carnot group $\mathbb{G}$.

Similar inequalities can be proved for more general classes of linear differential operators. For instance, one can handle second-order operators generated by a system of smooth vector fields in $\mathbb{R}^N$ satisfying the Hörmander condition, and left invariant differential operators on homogeneous groups; see [12]. However, we shall not discuss these kinds of generalizations here.

As usual, we denote by sign, sign^+ and u^+ the functions defined by

$$\text{sign}(t) := \begin{cases} 1 & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ -1 & \text{if } t < 0, \end{cases}$$

$$\text{sign}^+(t) := \begin{cases} 1 & \text{if } t > 0, \\ 0 & \text{if } t \le 0, \end{cases}$$

$$u^+ := \text{sign}^+(u)\, u.$$

Throughout this paper, $\Omega \subset \mathbb{R}^N$ denotes an open subset.

Definition 3.1. Let $f \in L^1_{\text{loc}}(\Omega)$. A distributional solution of the inequality

$$\Delta_G u \ge f \quad \text{in } \mathcal{D}'(\Omega)$$

is a function $u \in L^1_{\text{loc}}(\Omega)$ such that for any nonnegative $\phi \in \mathscr{C}^2_0(\Omega)$ we have that

$$\int_\Omega u \Delta_G \phi \ge \int_\Omega f\phi.$$

Theorem 3.2 (Kato inequality). *Let $u, f \in L^1_{\text{loc}}(\Omega)$ be such that*

$$\Delta_G u \ge f \quad \text{in } \mathcal{D}'(\Omega).$$

Then

$$\Delta_G u^+ \ge \text{sign}^+(u) f \quad \text{in } \mathcal{D}'(\Omega) \tag{3.1}$$

and

$$\Delta_G |u| \ge \text{sign}(u) f \quad \text{in } \mathcal{D}'(\Omega). \tag{3.2}$$

The proof is a consequence of the following lemma; see [1] for a related result.

Lemma 3.3. *Let $\gamma \in \mathscr{C}^2(\mathbb{R})$ be a convex function with bounded first derivative. Let $u, f \in L^1_{\text{loc}}(\Omega)$ be such that*

$$\Delta_G u \ge f \quad \text{in } \mathcal{D}'(\Omega).$$

Then $\gamma(u) \in L^1_{\text{loc}}(\Omega)$ and

$$\Delta_G \gamma(u) \ge \gamma'(u) f \quad \text{in } \mathcal{D}'(\Omega).$$

Proof. We need to prove that for any nonnegative $\phi \in \mathscr{C}^2_0(\Omega)$ we have

$$\gamma(u)\phi \in L^1(\Omega),$$

and that the following inequality holds:

$$\int_\Omega \gamma(u) \Delta_G \phi \ge \int_\Omega \gamma'(u) f\phi.$$

Fix $\phi \in \mathscr{C}^2_0(\Omega)$. Let $(m_\eta)_\eta$ be a family of symmetric mollifiers associated to a fixed homogeneous norm S. Set $u_\eta := u \star_{\mathbb{G}} m_\eta$ in $\Omega_\eta := \{x \in \Omega : \text{dist}(x, \partial\Omega) > \eta\}$, that is,

$$u_\eta(x) := \int_\Omega u(y) m_\eta(x \circ y^{-1})\, dy = \int_\Omega u(y^{-1} \circ x) m_\eta(y)\, dy, \quad x \in \Omega_\eta.$$

For η small enough, it follows that $\text{supp}(\phi) \subset \Omega_\eta$.

Let $x \in \Omega_\eta$. Since $m_\eta(x \circ \cdot^{-1})$ is a nonnegative test function in Ω, by using the Fubini–Tonelli theorem we obtain

$$\begin{aligned}
\int_\Omega u_\eta(x)\Delta_G\phi(x)\,dx &= \int_\Omega \Delta_G\phi(x)\Big(\int_\Omega u(y^{-1}\circ x)m_\eta(y)\,dy\Big)\,dx \\
&= \int_\Omega m_\eta(y)\Big(\int_\Omega u(y^{-1}\circ x)\Delta_G\phi(x)\,dx\Big)\,dy \\
&= \int_\Omega m_\eta(y)\Big(\int_\Omega u(z)(\Delta_G\phi)(y\circ z)\,dz\Big)\,dy \\
&= \int_\Omega m_\eta(y)\Big(\int_\Omega u(z)\Delta_G(z\to\phi(y\circ z))\,dz\Big)\,dy \\
&\geq \int_\Omega m_\eta(y)\Big(\int_\Omega f(z)\phi(y\circ z)\,dz\Big)\,dy \\
&= \int_\Omega m_\eta(y)\Big(\int_\Omega f(y^{-1}\circ x)\phi(x)\,dx\Big)\,dy \\
&= \int_\Omega \phi(x)\Big(\int_\Omega f(y^{-1}\circ x)m_\eta(y)\,dy\Big)\,dx \\
&= \int_\Omega \phi(x)f_\eta(x)\,dx,
\end{aligned}$$

that is,[1]

$$\Delta_G u_\eta(x) \geq f_\eta(x) \quad \text{on } \Omega_\eta.$$

On the other hand, by the convexity of γ it follows that

$$\Delta_G\gamma(u_\eta) = \gamma'(u_\eta)\Delta_G u_\eta + \gamma''(u_\eta)|\nabla_L u|^2 \geq \gamma'(u_\eta)\Delta_G u_\eta,$$

which implies

$$\int_\Omega \gamma(u_\eta)\Delta_G\phi \geq \int_\Omega \gamma'(u_\eta)\Delta_G u_\eta\phi \geq \int_\Omega \gamma'(u_\eta)f_\eta\phi.$$

The convergence of $\gamma(u_\eta)\to\gamma(u)$ in $L^1_{\mathrm{loc}}(\Omega)$ is assured by the convergence of $u_\eta\to u$ in $L^1_{\mathrm{loc}}(\Omega)$ and the fact that γ is a Lipschitz function (since γ' is bounded). By observing that

$$\begin{aligned}
\int_\Omega \gamma'(u_\eta)(x)f_\eta(x)\phi(x)\,dx &= \int_\Omega\int_\Omega \gamma'(u_\eta)(x)\phi(x)m_\eta(x\circ y^{-1})f(y)\,dy\,dx \\
&= \int_\Omega m_\eta \star_{\mathbb{G}} (\gamma'(u_\eta)\phi)(y)f(y)\,dy,
\end{aligned}$$

it suffices to prove that

$$\int_\Omega m_\eta \star_{\mathbb{G}} (\gamma'(u_\eta)\phi)(y)f(y) \to \int_\Omega \gamma'(u)\phi f.$$

To this end, we first claim that

$$m_\eta \star_{\mathbb{G}} (\gamma'(u_\eta)\phi) \to \gamma'(u)\phi \quad \text{in } L^1(\Omega). \tag{3.3}$$

1 The above argument can be applied to a general second-order linear operator which is translation left invariant in a nilpotent Lie group.

Indeed, since γ' is continuous and $u_\eta \to u$ a.e. in Ω (if necessary by passing to a subsequence), it follows that $\gamma'(u_\eta)\phi \to \gamma'(u)\phi$ a.e. in Ω. Now, by γ' being bounded, an application of the Lebesgue dominated convergence gives $\gamma'(u_\eta)\phi \to \gamma'(u)\phi$ in $L^1(\Omega)$. Moreover,

$$\begin{aligned}\Big|\int_\Omega m_\eta \star_{\mathbb{G}} (\gamma'(u_\eta)\phi)(y) - \gamma'(u)\phi\Big| &= \Big|\int_\Omega m_\eta \star_{\mathbb{G}} (\gamma'(u_\eta)\phi)(y) - m_\eta \star_{\mathbb{G}} \gamma'(u)\phi + m_\eta \star_{\mathbb{G}} \gamma'(u)\phi - \gamma'(u)\phi\Big| \\ &\le \Big|\int_\Omega m_\eta \star_{\mathbb{G}} (\gamma'(u_\eta)\phi)(y) - m_\eta \star_{\mathbb{G}} \gamma'(u)\phi\Big| + \Big|\int_\Omega m_\eta \star_{\mathbb{G}} \gamma'(u)\phi - \gamma'(u)\phi\Big| \\ &\le \Big(\int_\Omega m_\eta\Big)\Big|\int_\Omega (\gamma'(u_\eta)\phi) - \gamma'(u)\phi\Big| + \Big|\int_\Omega m_\eta \star_{\mathbb{G}} \gamma'(u)\phi - \gamma'(u)\phi\Big| \to 0.\end{aligned}$$

Next, if necessary by passing to a subsequence, we may suppose that the convergence in (3.3) is a.e. on Ω.

Now, since $\gamma'(u_\eta)\phi$ is uniformly bounded by $M := |\gamma'|_\infty |\phi|_\infty$, we deduce that

$$|m_\eta \star_{\mathbb{G}} (\gamma'(u_\eta)\phi)| \le \int_\Omega m_\eta M \le M.$$

Noticing that $m_\eta \star_{\mathbb{G}} (\gamma'(u_\eta)\phi)$ has compact support contained in $\operatorname{supt}\phi + B_\eta \subset \operatorname{supt}\phi + B_1 =: K$, it follows that

$$|m_\eta \star_{\mathbb{G}} (\gamma'(u_\eta)\phi)|f \le Mf\chi_K \in L^1_{\mathrm{loc}}(\Omega).$$

Finally, by the Lebesgue theorem we have

$$\int_\Omega \gamma'(u_\eta) f_\eta \phi = \int_\Omega m_\eta \star_{\mathbb{G}} (\gamma'(u_\eta)\phi)(y) f(y)\,dy \to \int_\Omega \gamma'(u) f\phi\,dy.$$

This completes the proof. □

Proof of Theorem 3.2. The idea is first to approximate the function sign^+ with a family of convex functions γ_ϵ having bounded derivatives, and then apply Lemma 3.3 above.

Let $m \in \mathscr{C}(\mathbb{R})$ be nonnegative with $\operatorname{supt}(m) \subset [-1, 1]$ and $\int m = 1$. For $\epsilon > 0$, set $m_\epsilon := \frac{1}{\epsilon} m(\frac{t-\epsilon}{\epsilon})$ and consider γ_ϵ as the solution of the problem

$$\gamma_\epsilon'' = m_\epsilon \quad \text{with} \quad \gamma_\epsilon'(0) = \gamma_\epsilon(0) = 0.$$

Clearly, we have $\gamma_\epsilon(t) = \gamma_\epsilon'(t) = 0$ for $t \le 0$. In addition, $\gamma_\epsilon'(t) = 1$ for $t > 2\epsilon$ and $0 \le \gamma_\epsilon(t) \le t^+$, $0 \le \gamma_\epsilon'(t) \le 1$. This implies the pointwise convergence, as $\epsilon \to 0$, of $\gamma_\epsilon(t) \to t^+$ and $\gamma_\epsilon'(t) \to \operatorname{sign}^+ t$. Finally, by Lemma 3.3 we have

$$\int_\Omega \gamma_\epsilon(u)\Delta_G\phi \ge \int_\Omega \gamma_\epsilon'(u) f\phi,$$

and by the Lebesgue theorem we obtain

$$\int_\Omega u^+\Delta_G\phi \ge \int_\Omega \operatorname{sign}^+ u f\phi.$$

The proof of (3.2) follows from a similar argument as above, so we shall omit it. □

Remark 3.4. Theorem 3.2 holds if we replace the functions sign^+ and u^+ respectively with

$$\operatorname{sign}_h^+(t) := \begin{cases} 1 & \text{if } t > h, \\ 0 & \text{if } t \le h, \end{cases}$$

and $u_h^+ := (u - h)^+$, where $h \in \mathbb{R}$. To this end, we can argue as in the proof of Theorem 3.2, replacing $\gamma_\epsilon(t)$ by $\gamma_\epsilon(t - h)$.

Remark 3.5. Theorem 3.2 deals with $L^1_{\mathrm{loc}}(\Omega)$ solutions of the inequality

$$\Delta_G u \geq f \quad \text{in } \Omega,$$

while [8, Theorem 2.1] allows to consider $W^{1,2}_{\mathrm{loc}}(\Omega)$ solutions.

One may try to prove (3.1) by mollifying the solution and then applying [8, Theorem 2.1]. In this case, one would obtain

$$\Delta_G u_\eta^+ \geq \operatorname{sign}^+(u_\eta) f_\eta \quad \text{in } \Omega.$$

Clearly, in order to prove (3.1) we need to know that

$$\operatorname{sign}^+(u_\eta) \to \operatorname{sign}^+(u)$$

at least a.e. This is not always possible. Indeed, we can construct a function u (even continuous) such that each mollification u_η has $\operatorname{sign}^+(u_\eta) \equiv 1$, while $\operatorname{sign}^+(u) \not\equiv 1$. We shall prove this when $\Omega =]0, 1[$.

Let $\{q_n\}_{n\geq 1}$ be the set of rational numbers contained in $]0, 1[$. Fix $1 > \epsilon > 0$ and set

$$I_n :=]q_n - \epsilon 2^{-n}, q_n + \epsilon 2^{-n}[\cap]0, 1[, \quad I := \bigcup_{n\geq 1} I_n, \quad S := [0, 1] \setminus I.$$

The set I is open and dense in $[0, 1]$. Moreover, $0 < |I| \leq \epsilon < 1$, thus $|S| > 0$.

Next, for each $n \geq 1$ let $\phi_n : [0, 1] \to \mathbb{R}$ be a continuous nonnegative function such that $\phi_n(x) > 0$ if and only if $x \in I_n$ and $\|\phi_n\|_\infty \leq 1$. Set

$$u := \sum_{n\geq 1} \phi_n 2^{-n}.$$

Since the above series is uniformly convergent, the function u is continuous. Moreover, $u(x) > 0$ if and only if there exists $n \geq 1$ such that $\phi_n(x) > 0$. This is obviously equivalent to the fact that $x \in I_n$. In other words, u vanishes on S and it is positive on I.

Let $\eta > 0$ and let u_η be a mollification of u, that is, $u \star m_\eta$, where $(m_\eta)_\eta$ is a standard family of mollifiers. We claim that $u_\eta(x_0) > 0$ for any $x_0 \in]0, 1[$. Indeed, let $x_0 \in]0, 1[$. By our choice of $\{q_n\}_n$ there exists $n \geq 1$ such that $|q_n - x_0| < \eta$. Hence $I_n \cap]x_0 - \eta, x_0 + \eta[\neq \emptyset$ and

$$u_\eta(x_0) = \int u(y) m_\eta(x_0 - y)\, dy \geq \int\limits_{I_n \cap |y - x_0| < \eta} \phi_n(y) 2^{-n} m_\eta(x_0 - y)\, dy > 0.$$

4 Applications to uniqueness of solutions

In this section, we consider weakly elliptic linear differential operators of the form

$$Lu := \operatorname{div}(B(x)\nabla u) = \operatorname{div}_L(\nabla_L u),$$

and the associated uniqueness problem for the semilinear equation

$$Lu = f(u) + h \quad \text{on } \Omega.$$

Notice that since $Lu = \operatorname{div}(B(x)\nabla u)$, where B is a positive semidefinite matrix, by writing B as $B = \mu^T \cdot \mu$ and defining $\operatorname{div}_L = \operatorname{div}(\mu^T \cdot)$ and $\nabla_L = \mu\nabla$, it follows that

$$Lu = \operatorname{div}(\mu^T \cdot \mu\nabla u) = \operatorname{div}_L(\nabla_L u).$$

This means that a Kato inequality holds for L; see [8].

Definition 4.1. Let $f \in \mathscr{C}(\mathbb{R})$ and $h \in L^1_{\mathrm{loc}}(\Omega)$. A weak solution of

$$Lu \geq f(u) + h \quad \text{on } \Omega, \tag{4.1}$$

is a function

$$u \in W^{1,2}_{L,loc}(\Omega) := \{u \in L^2_{\mathrm{loc}} : |\nabla_L u| \in L^2_{\mathrm{loc}}\}$$

with $f(u) \in L^1_{\mathrm{loc}}(\Omega)$, such that for any nonnegative $\phi \in \mathscr{C}^1_0(\Omega)$ we have

$$-\int_\Omega \nabla_L u \cdot \nabla_L \phi \geq \int_\Omega (f(u) + h)\phi.$$

If $L = \Delta_G$ is a sub-Laplacian on a Carnot group, then a distributional solution of (4.1) is a function $u \in L^1_{\mathrm{loc}}(\Omega)$ such that $f(u) \in L^1_{\mathrm{loc}}(\Omega)$, and for any nonnegative $\phi \in \mathscr{C}^2_0(\Omega)$ we have

$$\int_\Omega u \cdot \Delta_G \phi \geq \int_\Omega (f(u) + h)\phi.$$

Theorem 4.2. *Let X be a subspace of $L^1_{\mathrm{loc}}(\Omega)$ such that if $u \in X$, then $u^+ \in X$. Let $b : [0, \infty[\to [0, +\infty[$ be a continuous function such that $b(0) = 0$ and the problem*

$$Lv \geq b(v) \quad [\Delta_G u \geq b(v)], \qquad v \geq 0, \quad \text{on } \Omega, \tag{4.2}$$

has no nontrivial weak [distributional] solution belonging to X.

Let $h \in L^1_{\mathrm{loc}}(\Omega)$ and let $f \in \mathscr{C}(\mathbb{R})$ be such that

$$f(t) - f(s) \geq b(t - s) \quad \text{for any } t > s.$$

Then the equation

$$Lv = f(v) + h \quad [\Delta_G v = f(v) + h] \quad \text{on } \Omega \tag{4.3}$$

has at most one weak [distributional] solution belonging to X.

Proof. Let $h \in L^1_{\mathrm{loc}}(\Omega)$ and let $u, v \in X$ be solutions of (4.3). The function $u - v \in X$ is a weak solution of

$$L(u - v) = f(u) - f(v) \quad \text{on } \Omega.$$

An application of the appropriate Kato inequality (3.1) or [8, Theorem 2.1] yields

$$L((u - v)^+) \geq \operatorname{sign}^+(u - v)(f(u) - f(v)) \quad \text{on } \Omega,$$

which in turn implies that the function $w := (u - v)^+$ is a weak (or distributional) solution of

$$Lw \geq \operatorname{sign}^+(u - v)(f(u) - f(v)) \geq \operatorname{sign}^+(u - v) b(u - v) = b(w) \quad \text{on } \Omega.$$

In other words, w solves (4.2). Hence $w \equiv 0$ a.e. on Ω, that is, $u \leq v$ a.e. on Ω. Inverting the role of u and v, the claim follows. □

A concrete application of Theorem 4.2 is contained in the following result.

Theorem 4.3. *Let $f \in \mathscr{C}(\mathbb{R})$ be such that*

$$f(t) - f(s) \geq b(t - s) \quad \text{for any } t > s, \tag{4.4}$$

where $b : [0, +\infty[\to [0, +\infty[$ is a continuous function satisfying the following assumptions:

(i) *$b(0) = 0$, $b(t) > 0$ for $t > 0$;*

(ii) *it holds that*

$$\int_1^{+\infty} \left(\int_1^t b(s)\, ds \right)^{-\frac{1}{2}} dt < +\infty; \tag{4.5}$$

(iii) *b is convex.*

Let $h \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$. Then the problem

$$\Delta_G u = f(u) + h \quad \text{in } \mathcal{D}'(\mathbb{R}^N)$$

has at most one distributional solution $u \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$. Moreover, if $h \geq 0$, then $u \leq 0$ a.e. on $\mathbb{R}^N$.

Proof. The obvious idea is to apply Theorem 4.2. To this end, it is enough to check that the inequality

$$\Delta_G v \geq b(v), \quad v \geq 0, \quad \text{in } \mathcal{D}'(\mathbb{R}^N) \tag{4.6}$$

has only the trivial solution. Indeed, let us assume that $v \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ is a solution of (4.6). By a mollification argument (as in the proof of Lemma 3.3) we have

$$\Delta_G(v_\eta) \geq (b(v))_\eta.$$

Next, by the convexity of b and the Jensen inequality, it follows that

$$\Delta_G(v_\eta) \geq b(v_\eta), \quad v_\eta \geq 0, \quad \text{on } \mathbb{R}^N. \tag{4.7}$$

Now v_η is smooth and solves (4.7) with the function b nondecreasing (indeed, it satisfies (i) and it is convex) and satisfying (4.5), thus we are in the position to apply [7, Theorem 3.10] (by changing $u := -v_\eta$), so we deduce that $v_\eta \equiv 0$. Thus, by letting $\eta \to 0$ we obtain $v \equiv 0$. □

Remark 4.4. When dealing with $\mathscr{C}^1$ solutions, hypothesis (iii) can be relaxed by assuming that b is nonincreasing; see [7].

Corollary 4.5. *Let $q > 1$ and let $h \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$. The problem*

$$\Delta_G u = |u|^{q-1}u + h \quad \text{in } \mathcal{D}'(\mathbb{R}^N)$$

has at most one solution $u \in L^q_{\mathrm{loc}}(\mathbb{R}^N)$. Moreover, if $h \geq 0$, then $u \leq 0$ a.e. on $\mathbb{R}^N$.

Remark 4.6. The above result, as far it is concerned with *uniqueness and nonpositivity* of the possible solutions, is the analog on Carnot groups of [3, Theorem 2].

Remark 4.7. All the above results still hold when one replaces the function $h \in L^1_{\mathrm{loc}}$ with a distribution $h \in \mathcal{D}'$.

Theorem 4.3 allows us to generalize Corollary 4.5 to a more general class of nonlinearities, as the following example shows.

Example 4.8. Let f be defined by

$$f(t) := \begin{cases} t^{q_1} & \text{if } t \geq 0, \\ -|t|^{q_2} & \text{if } t < 0, \end{cases}$$

where $q_1, q_2 > 1$. Theorem 4.3 applies to such f. Indeed, for $t \geq 0$ define $g(t) := \min\{t^{q_1}, t^{q_2}\}$. The function b that we need is the convexification of cg for a small constant $c > 0$.

We claim that there exists a constant $c > 0$ such that for any $t > s$ we have

$$f(t) - f(s) \geq cg(t - s).$$

Assume that $q_1 \leq q_2$. By the well-known inequality

$$t^p - s^p \geq c_p(t - s)^p \quad \text{for } t > s \text{ and } p > 1,$$

we have the following three cases:

(i) Let $t > s > 0$. Then

$$f(t) - f(s) = t^{q_1} - s^{q_1} \geq c_{q_1}(t - s)^{q_1} \geq c_{q_1} g(t - s).$$

(ii) Let $0 > t > s$. Then

$$f(t) - f(s) = -|t|^{q_2} + |s|^{q_2} \geq c_{q_2}(|s| - |t|)^{q_2} \geq c_{q_2} g(|s| - |t|) = c_{q_2} g(t - s).$$

(iii) Let $t > 0 > -s$. The proof of the claim will follow if we prove that

$$t^{q_1} + s^{q_2} \geq cg(t+s) \quad \text{for any } t, s > 0.$$

By using the inequality

$$a^p + b^p \geq 2^{1-p}(a+b)^p \quad \text{for } a, b > 0 \text{ and } p > 1$$

and distinguishing three different cases, we have the following:
(a) Let $s \geq 1$ and $t > 0$. Then

$$t^{q_1} + s^{q_2} \geq t^{q_1} + s^{q_1} \geq 2^{1-q_1}(t+s)^{q_1} \geq 2^{1-q_1} g(t+s).$$

(b) Let $1 > t > 0$ and $1 > s > 0$. Then

$$t^{q_1} + s^{q_2} \geq t^{q_2} + s^{q_2} \geq 2^{1-q_2}(t+s)^{q_2} \geq 2^{1-q_2} g(t+s).$$

(c) Let $t \geq 1$ and $1 > s > 0$. Then

$$t^{q_1} + s^{q_2} \geq t^{q_1} \geq 2^{-q_1}(t+1)^{q_1} \geq 2^{-q_1}(t+s)^{q_1} \geq 2^{-q_1} g(t+s).$$

Next, by choosing

$$c := \min\{c_{q_1}, c_{q_2}, 2^{1-q_1}, 2^{1-q_2}, 2^{-q_1}\},$$

we get the claim.

By defining $b := \operatorname{conv}(cg)$, it follows that assumptions (4.4) and Theorem 4.3 (i) and (iii) are fulfilled. Notice that Theorem 4.3 (ii) is satisfied since at infinity the function b behaves like t^{q_1} with $q_1 > 1$.

We point out that f does not satisfy the Brezis condition $f'(t) \geq |t|^{q-1}$ for any $t \in \mathbb{R}$ unless $q_1 = q_2$. The interested reader may compare this with [3].

5 Some applications to a class of semilinear systems

In this section, as in the previous Section 4, we consider weakly elliptic linear differential operators of the form $Lu = \operatorname{div}_L(\nabla_L u)$. We refer to Definition 4.1 for the appropriate notion of solutions.

Theorem 5.1. *Let X be a subspace of $L^1_{\mathrm{loc}}(\Omega)$ such that if $u \in X$, then $u^+ \in X$. Let $b : [0, \infty[\to [0, +\infty[$ be a continuous function such that $b(0) = 0$ and the problem*

$$Lv \geq b(v) \quad [\Delta_G v \geq b(v)], \quad v \geq 0 \text{ on } \Omega \tag{5.1}$$

has no nontrivial weak [distributional] solutions belonging to X. Let $f \in \mathscr{C}(\mathbb{R})$ be such that

$$f(t) + f(s) \geq b(t+s) \quad \text{for any } t > -s. \tag{5.2}$$

Let $(u, v) \in X \times X$ be a weak [distributional] solution of the system of inequalities

$$\begin{cases} Lv \geq f(u), \\ Lu \geq f(v) \end{cases} \quad \left[\begin{cases} \Delta_G v \geq f(u), \\ \Delta_G u \geq f(v) \end{cases} \right] \quad \text{on } \Omega. \tag{5.3}$$

Then the following assertions hold:
(i) *$u + v \leq 0$ a.e. on Ω.*
(ii) *Let $C \geq 1$ and assume that the function $\overline{f}(t) := -Cf(-t)$ satisfies (5.2). Let $(u, v) \in X \times X$ be a weak [distributional] solution of the system*

$$\begin{cases} Cf(u) \geq Lv \geq f(u), \\ Cf(v) \geq Lu \geq f(v) \end{cases} \quad \left[\begin{cases} Cf(u) \geq \Delta_G v \geq f(u), \\ Cf(v) \geq \Delta_G u \geq f(v) \end{cases} \right] \quad \text{on } \Omega. \tag{5.4}$$

Then $u = -v$ a.e. on Ω. Therefore, u satisfies

$$Cf(u) \geq -Lu \geq f(u) \quad [Cf(u) \geq -\Delta_G u \geq f(u)] \qquad \text{on } \Omega, \tag{5.5}$$

and the function f must be odd on the range of u, that is, for any $t \in u(\Omega)$ the condition $f(t) = -f(-t)$ holds.

Proof. Let $(u, v) \in X \times X$ be a solution of (5.3). The function $u + v \in X$ solves

$$L(u + v) \geq f(v) + f(u) \quad \text{on } \Omega.$$

An application of the Kato inequality yields

$$L((u + v)^+) \geq \operatorname{sign}^+(u + v)(f(v) + f(u)) \quad \text{on } \Omega,$$

which in turn implies that the function $w := (u + v)^+$ is a weak solution of

$$Lw \geq \operatorname{sign}^+(u + v)(f(u) + f(v)) \geq \operatorname{sign}^+(u + v)b(u + v) = b(w) \quad \text{on } \Omega,$$

that is, w solves (5.1). Hence $w \equiv 0$ a.e. on Ω, that is, $u + v \leq 0$ a.e. on Ω. This proves case (i).

(ii) The functions $\bar{u} := -u$ and $\bar{v} := -v$ satisfy also the inequalities

$$L\bar{u} \geq -Cf(v) = \bar{f}(\bar{v}) \quad \textit{and} \quad L\bar{v} \geq \bar{f}(\bar{u}).$$

Since condition (5.2) is satisfied by $\bar{f}$, from (i) we have $\bar{u} + \bar{v} \leq 0$, that is, $u = -v$.

From the first inequality in (5.4) it follows that u solves (5.5). Adding (5.5) and the second inequality of (5.4) (and taking into account that $v = -u$), we obtain

$$C(f(u) + f(-u)) \geq 0 \geq f(u) + f(-u).$$

This last chain of inequalities implies that $f(u) = -f(-u)$, completing the proof. □

Remark 5.2. (i) If f is odd and (5.2) holds, then the function $\bar{f}$ in statement (ii) satisfies condition (5.2) as well.
(ii) If f is odd and (5.2) holds, then f is nondecreasing.
(iii) If f is odd, then (5.2) is equivalent to (4.4).

A concrete application of Theorem 5.1 is given by the following result.

Theorem 5.3. *Let $f \in \mathscr{C}(\mathbb{R})$ satisfy (5.2), where $b : [0, +\infty[\to [0, +\infty[$ is a continuous function such that*
(i) $b(0) = 0$, $b(t) > 0$ *for* $t > 0$;
(ii) *it holds*

$$\int_1^{+\infty} \Big(\int_1^t b(s)\, ds \Big)^{-\frac{1}{2}} dt < +\infty;$$

(iii) *b is convex.*
Let (u, v) be a distributional solution of the problem

$$\begin{cases} \Delta_G v \geq f(u), \\ \Delta_G u \geq f(v) \end{cases} \quad \textit{in } \mathcal{D}'(\mathbb{R}^N).$$

Then the conclusions of Theorem 5.1 hold.

Proof. It is enough to check that the inequality

$$\Delta_G w \geq b(w), \quad w \geq 0, \qquad \text{in } \mathcal{D}'(\mathbb{R}^N),$$

has only the trivial solution. This follows from the proof of Theorem 4.3. □

Remark 5.4. Dealing with $\mathscr{C}^1$ solutions, hypothesis (iii) can be weakened, assuming that b is nonincreasing.

Corollary 5.5. *Let $q > 1$. Let (u, v) be a distributional solution of the problem*

$$\begin{cases} \Delta_G v = |u|^{q-1} u, \\ \Delta_G u = |v|^{q-1} v \end{cases} \quad \text{in } \mathcal{D}'(\mathbb{R}^N).$$

Then $u = -v$ a.e. on $\mathbb{R}^N$ and

$$-\Delta_G u = |u|^{q-1} u \quad \text{in } \mathcal{D}'(\mathbb{R}^N).$$

An immediate consequence is the following corollary.

Corollary 5.6. *Let $q > 1$. Let (u, v) be a distributional solution of the problem*

$$\begin{cases} -\Delta_G v = |u|^{q-1} u, \\ -\Delta_G u = |v|^{q-1} v \end{cases} \quad \text{in } \mathcal{D}'(\mathbb{R}^N).$$

Then $u = v$ a.e. on $\mathbb{R}^N$.

The above results improve some theorems obtained in [4].

6 A note on the complex case

In this section, we shall prove a complex version of some results stated in Section 3 and [8] in the framework of Carnot groups. For the Euclidean case, see [13, 14].

Theorem 6.1 (Kato's inequality: The complex case). *Let $u, f \in L^1_{\mathrm{loc}}(\Omega; \mathbb{C})$ be such that*

$$\Delta_G u = f \quad \text{in } \mathcal{D}'(\Omega).$$

Then

$$\Delta_G |u| \geq \mathbb{R}\Big(\frac{\bar{u}}{|u|} f\Big) \quad \text{in } \mathcal{D}'(\Omega). \tag{6.1}$$

The proof is based on the following lemma.

Lemma 6.2. *Let $\gamma \in \mathscr{C}^2(\mathbb{R}^2)$ be a convex function with bounded first derivatives. Let $u, f \in L^1_{\mathrm{loc}}(\Omega; \mathbb{C})$ be such that*

$$\Delta_G u = f \quad \text{in } \mathcal{D}'(\Omega).$$

Then $\gamma(u) \in L^1_{\mathrm{loc}}(\Omega)$ and

$$\Delta_G \gamma(u) \geq \mathbb{R}\Big(2 \frac{\partial \gamma}{\partial z}(u) f\Big),$$

where $\frac{\partial \gamma}{\partial z}$ is the Wirtinger operator defined by

$$\frac{\partial \gamma}{\partial z}(x, y) = \frac{1}{2}\Big(\frac{\partial \gamma}{\partial x} - i \frac{\partial \gamma}{\partial y}\Big).$$

Proof. We shall use the same notations as in the proof of Lemma 3.3. Without loss of generality, we assume that u and f are smooth (if this is not the case we can use a mollification process as in the proof of Lemma 3.3).

Let $u := s + it$. By computation it follows

$$\Delta_G \gamma(u) = \gamma_{xx} |\nabla_L s|^2 + 2\gamma_{xy} \nabla_L s \cdot \nabla_L t + \gamma_{yy} |\nabla_L t|^2 + \gamma_x \Delta_G s + \gamma_y \Delta_G t.$$

We claim that

$$\Delta_G \gamma(u) \geq \gamma_x \Delta_G s + \gamma_y \Delta_G t.$$

Indeed, taking into account that γ is convex and writing $\alpha_1 e_1 := \nabla_L s$ and $\alpha_2 e_2 := \nabla_L t$ with unitary vectors e_i and real numbers α_i, we have

$$\begin{aligned}\gamma_{xx}|\nabla_L s|^2 + 2\gamma_{xy}\nabla_L s\cdot\nabla_L t + \gamma_{yy}|\nabla_L t|^2 &= \gamma_{xx}\alpha_1^2 + \epsilon 2\gamma_{xy}\alpha_1\alpha_2 + \gamma_{yy}\alpha_2^2 + 2\gamma_{xy}\alpha_1\alpha_2[e_1\cdot e_2 - \epsilon]\\ &\geq 2\gamma_{xy}\alpha_1\alpha_2[e_1\cdot e_2-\epsilon],\end{aligned} \tag{6.2}$$

where $\epsilon \in \{1,-1\}$. By a suitable choice of ϵ, the right-hand side of inequality (6.2) becomes nonnegative, and we get the claim.

Since

$$2f\frac{\partial\gamma}{\partial z} = (\Delta_G s + i\Delta_G t)\Big(\frac{\partial\gamma}{\partial x} - i\frac{\partial\gamma}{\partial y}\Big) = \gamma_x\Delta_G s + \gamma_y\Delta_G t + i(\gamma_x\Delta_G t - \gamma_y\Delta_G s),$$

we complete the proof. ☐

Proof of Theorem 6.1. Apply Lemma 6.2 to the convex function $\gamma(x,y) := \sqrt{\epsilon^2+x^2+y^2}$ and let $\epsilon\to 0$. We leave the remaining details to the interested reader. ☐

As an application of Theorem 6.1 we have the following result.

Theorem 6.3 (Reduction principle: Complex case). *Let $\Omega\subset\mathbb{R}^N$ be an open set and let $f:\Omega\times\mathbb{R}\to\mathbb{R}$ be a Caratheodory function. Let $X\subset L^1_{\mathrm{loc}}(\Omega)$. Assume that the problem*

$$\Delta_G v \geq f(x,v),\quad v\geq 0,\qquad \text{in } \mathcal{D}'(\Omega)\cap X,$$

has no nontrivial distributional solutions. If $u\in L^1_{\mathrm{loc}}(\Omega;\mathbb{C})$ is a complex distributional solution of

$$\Delta_G u = f(x,|u|)\frac{u}{|u|}\quad \text{in } \mathcal{D}'(\Omega)$$

such that $|u|\in X$, then $u\equiv 0$ a.e. on Ω.

Proof. By (6.1) it follows that the function $|u|$ is a nonnegative distributional solution of

$$\Delta_G|u| \geq f(x,|u|)\quad \text{in } \mathcal{D}'(\Omega)\cap X.$$

By assumption it follows that $|u|\equiv 0$ a.e. on Ω. ☐

We end this section with easy consequences that follow from the proof of Theorem 4.3.

Theorem 6.4. *Let $f\in\mathscr{C}(\mathbb{R})$ be such that*

$$-f(-t), f(t)\geq b(t) > 0\quad \text{for any } t>0,$$

where $b:[0,+\infty[\to[0,+\infty[$ is a continuous convex function satisfying (4.5). If $u\in L^1_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{C})$ is a complex distributional solution of

$$\Delta_G u = f(x,|u|)\frac{u}{|u|}\quad \text{in } \mathcal{D}'(\mathbb{R}^N),$$

then $u\equiv 0$ a.e. on $\mathbb{R}^N$.

Corollary 6.5. *Let $q>1$ and $h\in L^1_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{C})$. Then the problem*

$$\Delta_G u = |u|^{q-1}u + h\quad \text{in } \mathcal{D}'(\mathbb{R}^N) \tag{6.3}$$

has at most one distributional solution $u\in L^q_{\mathrm{loc}}(\mathbb{R}^N;\mathbb{C})$. Moreover, if there exists $\theta\in\mathbb{R}$ such that $e^{i\theta}h\in\mathbb{R}$, then $e^{i\theta}u\in\mathbb{R}$.

Proof. Let u and v be distributional solutions of (6.3) and set $w := u - v$. The function w satisfies

$$\Delta_G w = |u|^{q-1}u - |v|^{q-1}v.$$

Hence, by the Kato inequality (6.1) we have

$$\Delta_G|w| \geq \mathbb{R}\Big((|u|^{q-1}u - |v|^{q-1}v) \cdot \frac{\overline{w}}{|w|}\Big).$$

Now, by a well-known inequality (see for example [15]) it follows that

$$\mathbb{R}\Big(\frac{(|u|^{q-1}u - |v|^{q-1}v) \cdot (\overline{u} - \overline{v})}{|u-v|}\Big) = \frac{(|u|^{q-1}u - |v|^{q-1}v) \cdot (u-v)}{|u-v|} \geq 2^{1-q}|u-v|^q.$$

Thus the uniqueness follows from the fact that

$$\Delta_G|w| \geq 2^{1-q}|w|^q \implies w = 0 \quad \text{a.e. on } \mathbb{R}^N.$$

The second claim is a consequence of the uniqueness property. Indeed, if $\theta = 0$, that is, if h is a real function, since u and $\overline{u}$ are solutions of (6.3), it follows that $u = \overline{u}$. This proves the claim for $\theta = 0$. If $\theta \neq 0$ it suffices to multiply (6.3) by $e^{i\theta}$ and apply the uniqueness property. □

Funding: The authors are supported by the MIUR Bando PRIN 2015 2015KB9WPT_001. The first author is a member of GNAMPA. The second author acknowledges the support from FRA 2015: Equazioni differenziali: teoria qualitativa e computazionale, Università degli Studi di Trieste.

References

[1] A. Baldi, A non-existence problem for degenerate elliptic PDEs, *Comm. Partial Differential Equations* **25** (2000), no. 7–8, 1371–1398.
[2] A. Bonfiglioli, E. Lanconelli and F. Uguzzoni, *Stratified Lie Groups and Potential Theory for Their sub-Laplacians*, Springer Monogr. Math., Springer, Berlin, 2007.
[3] H. Brezis, Semilinear equations in $\mathbf{R}^N$ without condition at infinity, *Appl. Math. Optim.* **12** (1984), no. 3, 271–282.
[4] L. D'Ambrosio, A new critical curve for a class of quasilinear elliptic systems, *Nonlinear Anal.* **78** (2013), 62–78.
[5] L. D'Ambrosio, A. Farina, E. Mitidieri and J. Serrin, Comparison principles, uniqueness and symmetry results of solutions of quasilinear elliptic equations and inequalities, *Nonlinear Anal.* **90** (2013), 135–158.
[6] L. D'Ambrosio and E. Mitidieri, A priori estimates, positivity results, and nonexistence theorems for quasilinear degenerate elliptic inequalities, *Adv. Math.* **224** (2010), no. 3, 967–1020.
[7] L. D'Ambrosio and E. Mitidieri, Nonnegative solutions of some quasilinear elliptic inequalities and their applications, *Mat. Sb.* **201** (2010), no. 6, 75–92.
[8] L. D'Ambrosio and E. Mitidieri, A priori estimates and reduction principles for quasilinear elliptic problems and applications, *Adv. Differential Equations* **17** (2012), no. 9–10, 935–1000.
[9] L. D'Ambrosio and E. Mitidieri, Liouville theorems for elliptic systems and applications, *J. Math. Anal. Appl.* **413** (2014), no. 1, 121–138.
[10] A. Farina and J. Serrin, Entire solutions of completely coercive quasilinear elliptic equations, *J. Differential Equations* **250** (2011), no. 12, 4367–4408.
[11] A. Farina and J. Serrin, Entire solutions of completely coercive quasilinear elliptic equations. II, *J. Differential Equations* **250** (2011), no. 12, 4409–4436.
[12] G. B. Folland and E. M. Stein, *Hardy Spaces on Homogeneous Groups*, Math. Notes 28, Princeton University Press, Princeton, 1982.
[13] B. Helffer, *Spectral Theory and Its Applications*, Cambridge Stud. Adv. Math. 139, Cambridge University Press, Cambridge, 2013.
[14] T. Kato, Schrödinger operators with singular potentials, *Israel J. Math.* **13** (1972), 135–148.
[15] P. Lindqvist, Notes on the p-Laplace equation, Report 102, University of Jyväskylä, Jyväskylä, 2006.
[16] P. Pucci and J. Serrin, A remark on entire solutions of quasilinear elliptic equations, *J. Differential Equations* **250** (2011), no. 2, 675–689.