UNIVERSITÀ DEGLI STUDI DI TRIESTE

DIPARTIMENTO DI FISICA

DOTTORATO DI RICERCA IN FISICA
XXXIII CICLO

Special Relativity and Spontaneous Collapse Models

Settore scientifico-disciplinare FIS/02

DOTTORANDO:
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COORDINATORE:
Prof. Francesco Longo (Univ. Trieste)

SUPERVISORE:
Prof. Angelo Bassi (Univ. Trieste)

ANNO ACCADEMICO 2019/2020

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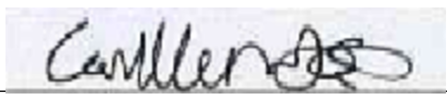
Declaration of Authorship

I, Caitlin Jones, declare that this thesis titled, "Special Relativity and Spontaneous Collapse models" and the work presented in it are my own. I confirm that:

- This work was done wholly while in candidature for a research degree at this University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

Signed: 

Date: June 28, 2021

University of Trieste

Abstract

Department of Physics

Doctor of Philosophy

Special Relativity and Spontaneous Collapse models

by Caitlin Jones

The work of this PhD investigates spontaneous collapse models, which are a proposed modification of quantum mechanics, and their consistency with special relativity. Soon after they were first proposed in 1976 there have been many attempts to construct a relativistic collapse model, but none of these have yet been satisfactory. This thesis considers what it means for a collapse model to be consistent with special relativity and reviews the existing proposals and their issues. The key results are a demonstration of why a many particle relativistic spontaneous collapse model with point-like collapses is not possible and that a relativistic generalisation of one of the most studied collapse models, Continuous Spontaneous Localisation, is not possible.

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Notation and Constants

- **Natural Units** - In this thesis we will use natural units and hence $c = \hbar = 1$.
- **Vectors** - Three vectors will be written in bold font so that as $\mathbf{x} = (x_1, x_2, x_3)$ whereas four vectors will be written in plain text, hence $x = (x_0, x_1, x_2, x_3)$.
- **Set Notation** - Sets will be written in blackboard bold font, for example the set of positive real numbers is $\mathbb{R}^+$.
- **Operators** - All operators in Hilbert spaces will be noted by hats, for example a system Hamiltonian is written $\hat{H}$.
- Any other mathematical notation will be described were it is introduced in the text.

Publications

The work of this thesis has contributed to the following publications:

- C. Jones, T. Guaita, and A. Bassi. Impossibility of extending the ghirardi-rimini-weber model to relativistic particles. *Phys. Rev. A*, 103:042216, Apr 2021
- Caitlin Jones, Giulio Gasbarri, and Angelo Bassi. Mass-coupled relativistic spontaneous collapse models. *Journal of Physics A: Mathematical and Theoretical*, 2021.

Chapter 1

Introduction

It has never been clear to even the creators of quantum mechanics how to interpret the mathematical framework of the theory. Almost since its inception the biggest issue that has plagued the interpretation of quantum mechanics is the measurement problem [16]. This is the issue that quantum mechanics has two forms of evolution but no set rule for when to use which form. The first type is unitary evolution, which is time reversible and preserves superpositions. This describes the evolution of isolated systems, and is given by the Schrödinger equation. The second type is the evolution described by positive operator valued measures (POVMs) which is irreversible, destroys superpositions and describes a quantum system under going a measurement.

When describing an idealised experimental set-up these two forms of evolution allow results to be predicted. Suppose there is a system made up of a quantum system and classical measurement apparatus controlled by an external observer. The quantum system evolves under unitary evolution which may result in the state being in a superposition in a particular basis. When the observer decides to make a measurement that basis, the system interacts with the apparatus, which collapses the state [63, chapter 6]. This collapse occurs because the measurement apparatus is classical and so must must a definite measurement result as it cannot be in a superposition[1]. The 'problem' part of the measurement problem is that the universe is not divided neatly up into quantum systems and classical observers performing measurements. If quantum mechanics is to be a fundamental theory which seeks to describe the behaviour of the whole universe then it must be able to include scenarios where there are no observers performing measurement, e.g.

[1] One could describe the measurement apparatus as quantum as well, so that it enters a superposition after interacting, but then one must describe how *that* superposition collapses to give a definite result.

in the very early universe. A solution to the measurement problem must therefore offer an explanation to how and why the state collapses.

Many physicists do argue that quantum mechanics (and relativistic quantum field theory) is a low energy limit of a more fundamental theory which is also compatible with general relativity [22]. Addressing the measurement problem is an important endeavor as any candidate theory will have to answer the question of what causes the macroscopic world to appear classical when the microscopic world is not.

Two of the most famous suggestions that reinterpret quantum mechanics to solve the measurement problem are the many worlds interpretation [29, 62] and Bohmian mechanics [17, 18, 55]. Both of these give exactly the same predictions for the results of experiments as quantum mechanics, which makes them unattractive as theories to some as they are not falsifiable.

In contrast, spontaneous collapse models are a proposed alteration to quantum mechanics, and hence offer different predictions to conventional quantum mechanics. Spontaneous collapse models are a class of quantum dynamics where, in addition to the usual unitary evolution of the state given by the Hamiltonian, there is a non-unitary stochastic part of the evolution which causes the state to spontaneously collapse. Chapters 2 and 3 will introduce in detail two of the most important collapse models, the Ghirardi-Rimini-Weber model and continuous spontaneous localisation, but here we will give a brief, non-technical general overview of collapse models in to prepare the reader for what is to come.

Spontaneous collapse models, first introduced by Ghirardi-Rimini-Weber [36] and Pearle [45], solve the measurement problem by replacing the two forms of evolution in conventional quantum mechanics with a single form of non-linear evolution. This evolution is a combination of the normal unitary part and a stochastic part, which has the effect of driving the system to an eigenstate of a particular eigenbasis, hence destroying superpositions in that basis.

As there is a single form of evolution there is no need to include the idea of an observer performing a measurement to interpret the model, states of isolated systems collapse on their own. Of course, it is known experimentally that microscopic systems can remain in superpositions and do not immediately collapse, so the effect of the stochastic dynamics must be weak enough to be negligible for a single particle (or a low number of particles), i.e. the rate of collapse for a single particle must be extremely low. Collapse models have one or more free parameters that characterise the frequency of the collapse and the spread of the state after collapse.

However, macroscopic systems are never observed in superpositions, for a large number of particles the dynamics must virtually guarantee that the system collapses. Spontaneous collapse models successfully predict this using properties of entanglement. Systems with large numbers of particles become entangled through interaction, then any single particle spontaneously collapsing will collapse all particles it is entangled with. This effectively increases the rate of collapse for systems with high numbers of particles, such that macroscopic bodies are localised on extremely short time scales. This is often called the amplification mechanism and it ensures macroscopic classicality.

We have stated that collapse models cause collapse in one particular basis; since the intended purpose of spontaneous collapse models is to explain why macroscopic objects are only in one place at a time, the most common choice is the number density or mass density basis since these ensure that superpositions in position are destroyed.

Of course, the measurement problem extends beyond just spatial measurements, it is necessary to explain how superpositions in all eigenbases are destroyed upon measurement. The collapse model solution to this issue is to note that all measurements involve coupling the degree of freedom of interest to the position of a macroscopic object. For example the spin of a particle is measured by that particle interacting with a macroscopic 'pointer' which is made up of a large number of particles, whose different positions corresponds to the different outcome of measurements. After interaction the particle and pointer will be entangled and if the particle is in a spin superposition then the pointer will be in a position superposition, see [11] for a demonstration of this As the pointer is made up of a large number of particles then the amplification mechanism acts on the pointer and hence the particle collapses almost immediately, giving a definite measurement result for the value of the spin.

However there are several issues that must be solved for a spontaneous collapse model to be taken seriously as a replacement for conventional quantum mechanics. The first is that most collapse models do not conserve energy and predict that the energy of a system goes to infinity in the infinite time limit [10]. Some models [52], known as dissipative models, do not experience this issue but these are not Galilei covariant,[32].

A second issue is that many people view collapse models as phenomenological models [9], meaning that there is not physical justification for additional stochastic terms. On the one hand this could be viewed as a strength as it means that, if verified empirically, collapse models leave space and act as a clue for an as-yet-undiscovered more fundamental theory. On the other hand, this lack of underlying theory means that the constants which characterise these models are not set, making it difficult to verify the theory as the parameter space is large.

The third issue, which is the subject of this thesis, is that as originally formulated spontaneous collapse models are not in agreement with special relativity. For collapse models to replace quantum mechanics there must be a relativistic collapse model, analogous to relativistic quantum field theory for quantum mechanics.

If it can be shown that no successful relativistic collapse model exists, then this could imply that at best a spontaneous collapse model could be the low energy

limit of some other theory. This would still pose a problem as high energy systems suffer from the measurement problem. The other possibility is that special relativity itself is incorrect and therefore is not an appropriate condition to apply.
.

This thesis seeks to answer the question of if a relativistic collapse model is possible.

In chapter 2 the Ghirardi-Rimini-Weber (GRW) model is introduced, in chapter 3 the continuous spontaneous localisation (CSL) model is introduced. In chapter 4 we consider what it means for spontaneous collapse models to be consistent with special relativity and review the existing proposed relativistic models and their issues. In chapter 5 we show that a relativistic generalisation of a many particle GRW type collapse model is not possible. In chapter 6 we demonstrate that a relativistic generalisation of CSL is not possible. Finally, chapter 7 gives the conclusions and reviews the limitations of the present work before offering some outlook.

Chapter 2

The Ghirardi-Rimini-Weber Model

The first consistent spontaneous collapse model to be proposed was the Ghirardi-Rimini-Weber (GRW) Model [36]. This is a non-relativistic model describing distinguishable particles where the spontaneous collapse occurs via discreet jumps of the state from an un-collapsed state to a collapsed state.

Here we will present the GRW model for a single particle before expanding to the case of N distinguishable particles. As in conventional quantum mechanics, a single particle system at time t is described by a wave function $\psi_t(\mathbf{y})$. For the purposes of this explanation we will consider scalar particles and neglect spin, hence the Hilbert space of the system is $\mathcal{H} = \mathcal{L}^2(\mathbf{R}^3)$. The system has a Hamiltonian, $\hat{H}$ which governs the unitary evolution.

The evolution has two components. The first is the spontaneous collapse, where the state undergoes collapse at random times given by a stochastic process. The time interval, Δt, between two collapses of the state is a value in the realisation of a stochastic process labelled X, see figure 2.1. For GRW X is a Markovian Poisson point process with mean time τ, so the probability density function for Δt is:

$$\mathbb{P}(\Delta t) = \frac{1}{\tau} \exp(-\Delta t/\tau) \tag{2.1}$$

where τ is a free parameter of the model that sets the rate of collapse.

When the state collapses, it collapses about a point $\mathbf{x}$ and undergoes an instantaneous change:

$$\psi_t(\mathbf{y}) \to \psi_t^{(c)}(\mathbf{y}) = \frac{\hat{L}(\mathbf{x})\psi_t(\mathbf{y})}{\|\hat{L}(\mathbf{x})\psi_t(\mathbf{y})\|} \tag{2.2}$$

where $\|\cdot\|$ is the norm and $\hat{L}(\mathbf{x})$ is the collapse operator:

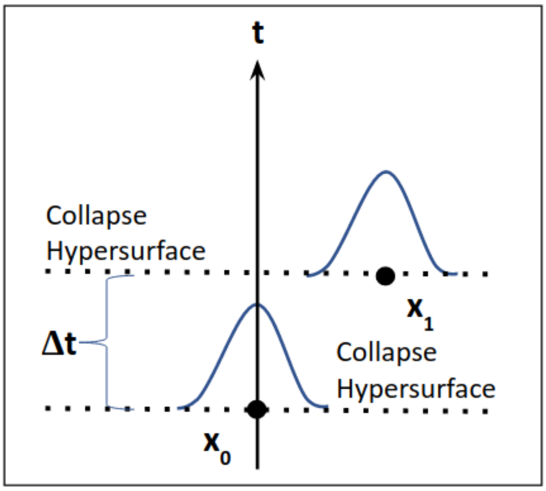


Figure 2.1: Schematic GRW model for a single particle. The dotted black lines show the hypersurfaces where collapses occur. The solid blue lines show the amplitude of the state immediately after collapse. Here the initial collapse, x_0, is at the origin and the next collapse will occur at time $t = \Delta t$.

$$\hat{L}(\mathbf{x}) := \left(\frac{1}{r_c^2 \pi}\right)^{\frac{3}{4}} \exp\left[-\frac{(\mathbf{x}-\hat{\mathbf{q}})^2}{2r_c^2}\right] \tag{2.3}$$

where r_c is a free parameter of the model and $\hat{\mathbf{q}}$ is the position operator of the particle. The probability that the state collapses about a point $\mathbf{x}$ is given by:

$$P(\mathbf{x}|\psi_t(\mathbf{y})) = \|\hat{L}(\mathbf{x})\psi_t(\mathbf{y})\|^2 \tag{2.4}$$

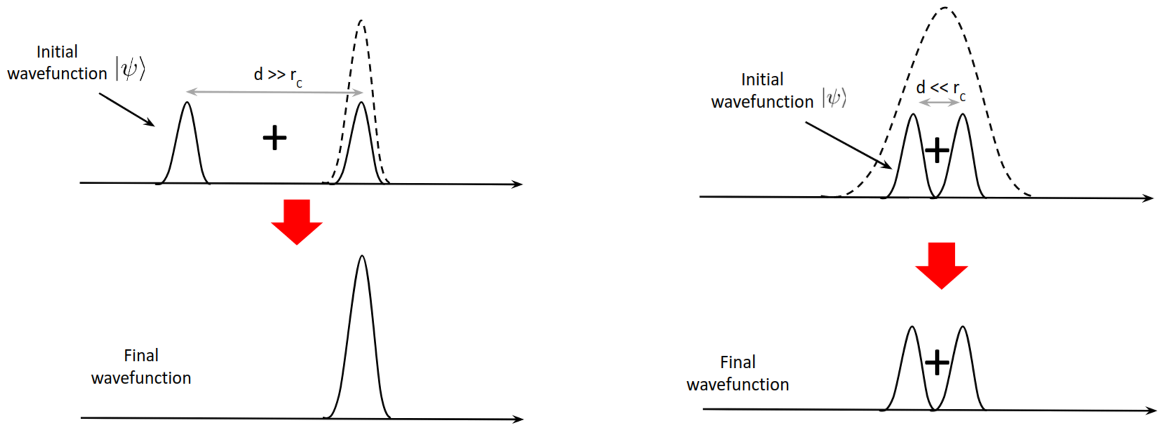
The form of $\hat{L}$ guarantees that the probably distribution is normalised:

$$\int d^3\mathbf{x}\, P(\mathbf{x}|\psi_t(\mathbf{y})) = 1. \tag{2.5}$$

The second component of the evolution is unitary evolution that occurs between the collapses which is given by the standard Schrödinger equation:

$$i\frac{d|\psi_t\rangle}{dt} = \hat{H}|\psi_t\rangle. \tag{2.6}$$

where $\psi_t|\mathbf{y}\rangle \to |\psi_t\rangle$. The model is initialised at time $t = t_0$ with the state $\psi_{t_0}(\mathbf{y})$, the first collapse occurs at time $t_1 = t_0 + \Delta t$. The initial state is evolved to t_1 with unitary evolution then the state collapses about a point $\mathbf{x}_1$ following equation Eq. (2.2). The probability distribution for the position of $\mathbf{x}_1$ is given by Eq. (2.4). This procedure then repeats itself for every value of the stochastic process.



(a) "Large superposition" **(b)** "Small superposition"

Figure 2.2: Schematic diagram showing how r_c affects the collapse. If the separation between peaks of the state d is much greater than r_c the state collapses to a single peak but if $r_c \gg d$ the state is left in a superposition.

To see how this procedure destroys spatial superpositions let us consider a simple example. Suppose at $t = t_0$ the initial state of the particle is in a superposition:

$$\psi_{t_0}(\mathbf{y}) = C(f_{\mathbf{z}_1}(\mathbf{y}) + f_{\mathbf{z}_2}(\mathbf{y})) \tag{2.7}$$

where C is a normalisation constant and f is a family of functions parameterised by the coordinate $\mathbf{z}$:

$$f_{\mathbf{z}}(\mathbf{y}) = \frac{1}{\sqrt{\sigma\sqrt{2\pi}}} e^{\frac{-(\mathbf{y}-\mathbf{z})^2}{4\sigma^2}} \tag{2.8}$$

i.e. it is a Gaussian function centred on $\mathbf{z}$ with width σ. We take $\sigma \ll |\mathbf{z}_1 - \mathbf{z}_2|$ such that the tails of the two functions in $\psi_{t_0}(\mathbf{y})$ do not overlap significantly. Additionally, we assume that $r_c \ll |\mathbf{z}_1 - \mathbf{z}_2|$, otherwise the state would not be affected by the collapse. The GRW model tells us that after time Δt there will be a collapse around $\mathbf{x}$. It is simple to see by substituting Eq. (2.7) into Eq. (2.4) that the probability distribution for $\mathbf{x}$ is maximised at $\mathbf{z}_1$ and $\mathbf{z}_2$[1].

For example, let us assume that the collapse occurs at $\mathbf{z}_1$, then Eq. (2.2) gives:

$$\psi^{(c)}_{t_1}(\mathbf{y}) = \frac{\hat{L}(\mathbf{z}_1)\psi_{t_1}(\mathbf{y})}{\|\hat{L}(\mathbf{z}_1)\psi_{t_1}(\mathbf{y})\|} \tag{2.9}$$

Due to the exponential weighting of Eq. (2.3) the operator $\hat{L}(\mathbf{z}_1)$ will suppress the amplitude of the state far away from $\mathbf{z}_1$. This means that the part of the amplitude of the state peaked about $\mathbf{z}_2$ will become negligible, leaving only a peak at $\mathbf{z}_1$ (see figure 2.2a). This leaves the system in the state:

$$\psi^{(c)}_{t_1}(\mathbf{y}) \approx f_{\mathbf{z}_1}(\mathbf{y}) \tag{2.10}$$

[1]We assume that the unitary evolution between t_0 and t_1 has had a negligible effect on the spread of the state.

The effect the collapse given by Eq. (2.2) is that after the collapse the state is localised about $\mathbf{x}$, which destroys spatial superpositions. The parameter r_c specifies the resolution of the collapse, i.e. only superpositions larger than r_c will be affected by the spontaneous collapse (see figure 2.2a) while smaller superpostions will not be collapsed (see 2.2b). Together, the two free parameters τ and r_c characterise GRW.

2.1 The GRW Model for Distinguishable Particles

The single particle model can easily be extended to the N distinguishable particle case. Here, the state is $\psi_t(\mathbf{y}_1, \mathbf{y}_2...)$ in a Hilbert space $\mathcal{H} = \mathcal{L}^2(\mathbf{R}^{3N})$. Instead of there being a single stochastic process there are N independent stochastic processes, one for each of the particles (although τ is the same for all of these processes).

The collapse operator for the n^{th} particle is:

$$\hat{L}^n(\mathbf{x}) := \left(\frac{1}{r_c^2\pi}\right)^{\frac{3}{4}} \exp\left[-\frac{(\mathbf{x}-\hat{\mathbf{q}}_n)^2}{2r_c^2}\right] \tag{2.11}$$

where $\hat{\mathbf{q}}_n$ is the position operator of the n^{th} particle. The effect of the collapse and the probability of collapse is found in the same way as for the single particle case.

The key difference between the single and multiple particle case is the presence of the amplification mechanism. A many particle system can have particles that are entangled with each other. This entanglement means that if one of the particles undergoes a spontaneous collapse then all the particles it is entangled with are also affected and also undergo collapse. Since each particle has the same rate of collapse, the entanglement means that the overall rate of collapse is proportional to the number of particles, i.e. if the rate for a single particle is $\frac{1}{\tau}$ then for N particles it is $\frac{N}{\tau}$, see [10] for further explanation of this.

This increased collapsed rate with many particles is called the amplification mechanism and it is what causes macroscopic bodies to behave classically; since the effective rate of collapse is very high, they are never found in superpostions. In this sense the parameters of GRW can be thought of as defining the transition between the microscopic quantum and macroscopic classical realms, as r_c gives the maximum length scale superposition can persist for and τ fixes the number of particles needed to collapse a system in a negligibly short time.

To see how the amplification mechanism works we can consider a 2 particle system that is initially entangled, such that each particle has an amplitude peaked at two points. The two particle system has the Hilbert space $\mathcal{H} = \mathcal{L}^2(\mathbf{R}^3) \otimes \mathcal{L}^2(\mathbf{R}^3)$. The initial state is:

$$\psi_{t_0}(\mathbf{y}_1, \mathbf{y}_2) = D\Big(f_{\mathbf{z}_1+\delta}(\mathbf{y}_1) \otimes f_{\mathbf{z}_1-\delta}(\mathbf{y}_2) + f_{\mathbf{z}_2+\delta}(\mathbf{y}_1) \otimes f_{\mathbf{z}_2-\delta}(\mathbf{y}_2)\Big) \tag{2.12}$$

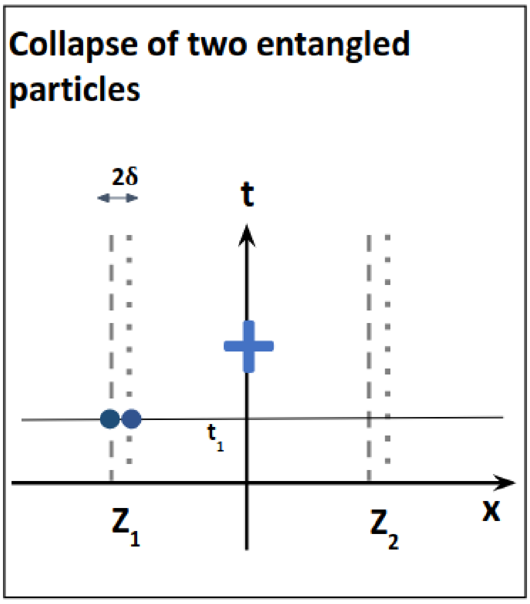


Figure 2.3: Schematic diagram showing two the support of two initially entangled particles, the dotted line is the support of particle one and the dashed line is the support of particle two if they did not undergo collapse. If, at t_1, particle one collapses at $z_1 + \delta$ then particle two will collapse at $z_1 - \delta$.

where D is a normalisation constant and δ is a small distance ($|\mathbf{z}_1 - \mathbf{z}_2| \gg \delta$) included in the state so that the two particles are not in exactly the same location, see figure 2.3. We assume that $\mathbf{z}_1$ and $\mathbf{z}_2$ are sufficiently far apart that the overlap between the tails of the functions $f_{\mathbf{z}_1}$ and $f_{\mathbf{z}_2}$ is negligible. For the sake of this example, assume that the first particle to undergo collapse is particle one which is localised to $\mathbf{z}_1 + \delta$. Particle two will automatically be localised to $\mathbf{z}_1 - \delta$ as well i.e. after collapse the state will be:

$$\psi_{t_1}(\mathbf{y}_1, \mathbf{y}_2) \approx f_{\mathbf{z}_1+\delta}(\mathbf{y}_1) \otimes f_{\mathbf{z}_1-\delta}(\mathbf{y}_2) \tag{2.13}$$

So although the collapse rate for each particle is $\frac{1}{\tau}$ the entanglement means that the rate is effectively doubled. This demonstrates the amplification mechanism.

2.2 The evolution of the density operator

So far we have only considered the evolution of the system in terms of the state ψ_t. However, if we wished to consider the time evolution of physical quantities then it is convenient to know how the density operator $\hat{\rho}$ evolves.

For this section we will consider the single particle case, see [10] for the N particle case.

It has been shown in [10] that this time evolution is given by:

$$\frac{d\rho(\hat{t})}{dt} = -i[\hat{H}, \rho(\hat{t})] - \frac{1}{\tau}(\rho(t) - T[\rho(t)]) \tag{2.14}$$

where $\hat{H}$ is the normal system Hamiltonian and $T[\cdot]$ is:

$$\int d^3\mathbf{x}\, \hat{L}(\mathbf{x})|\psi_t\rangle\langle\psi_t|\hat{L}(\mathbf{x}) \tag{2.15}$$

Equation (2.14) is referred to as the master equation. It is easy to see how the master equation preserves the normalisation of $\hat{\rho}(t)$ as it is trace preserving [43].

From here we can compare the expectation and variance of the position and momentum to the conventional quantum mechanics case. For simplicity we will work in one dimensional [10]:

$$\begin{aligned}
\langle\hat{\mathbf{q}}\rangle &= \langle\hat{\mathbf{q}}\rangle_0 \\
\langle\hat{\mathbf{p}}\rangle &= \langle\hat{\mathbf{p}}\rangle_0 \\
\text{Var}(\hat{\mathbf{q}}) &= \text{Var}(\hat{\mathbf{q}})_0 + \frac{t^3}{6r_c\tau m^2} \\
\text{Var}(\hat{\mathbf{p}}) &= \text{Var}(\hat{\mathbf{p}})_0 + \frac{t}{2r_c\tau}
\end{aligned} \tag{2.16}$$

where $\hat{\mathbf{p}}$ is the momentum operator, m is the mass of the particle, the subscript 0 is the standard quantum case. The expectation values are unchanged, which makes physical sense as this means the dynamics do not cause a drift which would break Galilei covariance. However the master equation predicts that the variance of the position and momentum increase with time with respect the usual case.

This has the consequence that the energy of the system is not conserved:

$$\langle E\rangle = \langle E\rangle_0 + \frac{t}{4r_c\tau m} \tag{2.17}$$

This non-conversation of energy presents a big issue for GRW, as conservation of energy is normally taken to be a requirement of any fundamental theory. GRW only describes distinguishable particles, so in any case cannot be fundamental, but non-conservation of energy is an issue for any type of spontaneous collapse model. There is a dissipative extension [53] to GRW which does not guarantee that the rate of change of energy is zero but does ensure that at $t \to \infty$ the energy of any state remains finite.

2.3 Ontology of the GRW Model

As discussed in the introduction, the motivation for developing spontaneous collapse models was to have a clear ontology, unlike quantum mechanics. In their history different authors have suggested different underlying ontologies for the GRW Model,which we will briefly list here for the interested reader.

In 1989 Bell proposed the 'flash' ontology in which it is the flashes, the points in spacetime where spontaneous collapses occur, that describe physical reality, not the value of the observables or the state itself, [23]. Bell suggested this approach as it has the advantage that the flashes are local and exist in the real world.

Another possible ontology of GRW was suggested in [34] which proposed an appropriately averaged mass density function associated to the state. This was suggested because unlike conventional quantum mechanics, GRW provides a consistent value of the mass density over every point in space time and for macroscopic objects. Additionally, mass density makes intuitive sense as it describes the continuous distribution of mass in 3D space, which matches our experience of the world.

In [59] a version of GRW for indistinguishable particles was developed, in which the collapse operator the mass density field at each point. In this theory it is the value of the mass density field at each point that is the beable.

2.4 Conclusion

In this chapter we have looked at the GRW model and how it offers a mechanism for the state to localise in a consistent way that preserves normalisation of the state and guarantees that macroscopic superpostions are collapsed. The defining characteristics of GRW are that However the GRW model is limited to describing distinguishable non-relativistic particles which does not describe reality.

- The model is defined via a conditional probability distribution for the position of a spontaneous collapse given the position of previous collapses
- The stochastic process is a piece-wise, meaning that there is a collapse at random time intervals.
- The model is Markovian, the conditional probability for a collapse only depends on the most recent collapse, not the whole prior series of collapses.
- The collapses occur in the spatial basis
- For any $N > 1$ system of particles it must have an amplification mechanistic to ensure emergence of macroscopic classicality.

These conditions have been inferred by the author from the existing literature [36, 59, 60] and will be used as the definition of 'a GRW model' for the remainder of this thesis[2]

The other form of collapse models are those with continuous stochastic process, the most famous of these is Continuous Spontaneous Localisation (CSL) which is the subject of the next chapter.

[2] A recent proposal [61] for a relativistic GRW model for interacting particles is non-Markovian and so does not fit within this definition.

Chapter 3

The Continuous Spontaneous Localisation Model

3.1 Introduction

In the previous chapter we introduced one of the earliest spontaneous collapse models, the GRW model, which incorporates a (piecewise) stochastic process. The other type of spontaneous collapse model are those that use a continuous stochastic process [21]; the most well studied of these is the Continuous Spontaneous Localisation model [35] (CSL). This model describes non-relativistic indistinguishable particles which evolve according to a non-linear stochastic modification of the Schrödinger equation.

Like GRW, the CSL model (and its modified versions [3, 52]) successfully accounts for why microscopic objects remain in spatial superpostions whilst macroscopic objects do not. It offers different experimental predictions to standard quantum mechanics.

We will introduce the stochastic Schrödinger equation of CSL, explain how it causes localisation of the state with reference to the model's two free parameters. We will also discuss the proposed ontologies of the model, the various modifications of CSL and the experiments seeking to validate or rule out CSL.

3.2 Evolution of the state

Let us consider a system of indistinguishable scalar particles each with mass m, with Hilbert space $\mathcal{H} = \mathcal{F}_{\pm}$, a Fock space where $\pm$ references to symmetric or anti-symmetric particles[1]. According to CSL, the state vector $|\psi_t\rangle$ in this space evolves with:

$$\begin{aligned} d|\psi_t\rangle = \Big[& - i\hat{H}dt + \frac{\sqrt{\gamma}}{m_0}\int d\mathbf{x}\int d\mathbf{y}(\hat{m}(\mathbf{y}) - \langle\hat{m}(\mathbf{y})\rangle_t)dW_t(\mathbf{x}) \\ & - \frac{\gamma}{2m_0^2}\int d\mathbf{x}\int d\mathbf{y}D(\mathbf{x},\mathbf{y})\big(\hat{m}(\mathbf{x}) - \langle\hat{m}(\mathbf{x})\rangle_t\big)\big(\hat{m}(\mathbf{y}) - \langle\hat{m}(\mathbf{y})\rangle_t\big)dt\Big]|\psi_t\rangle \end{aligned} \tag{3.1}$$

where $\hat{H}$ is the Hamiltonian associated with the unitary evolution, γ is a coupling constant and is one of the free parameters of the model and m_0 is a reference mass

[1] It is possible to construct CSL for particles with different masses and values of spin [10], we only use the simplest example here for ease of explanation

(usually the nucleon mass). The function g is a Gaussian spread:

$$D(\mathbf{x},\mathbf{y}) = \frac{1}{(\sqrt{2\pi}r_c)^3}\exp[-(\mathbf{x}-\mathbf{y})^2/2r_c^2] \tag{3.2}$$

where r_c is the other free parameter of the model. $dW_t(\mathbf{x})$ is an infinitesimal element of a family of Wiener processes [21] where there is a process at each point $\mathbf{x}$.

The family of Wiener processes is characterised by having a expectation value of zero and having the two point correlation function:

$$\mathbb{E}_P[dW_t(\mathbf{x})dW_t(\mathbf{y})] = D(\mathbf{x},\mathbf{y})dt \tag{3.3}$$

Where $E_{\mathbb{P}}$ is the stochastic average with respect to the probability measure of the Wiener process. The operator $\hat{m}(\mathbf{x})$ is the mass density operator defined as:

$$\hat{m}(\mathbf{x}) = m\,\hat{a}^{\dagger}(\mathbf{x})\hat{a}(\mathbf{x}) \tag{3.4}$$

where $\hat{a}^{\dagger}(\mathbf{x})$ is the creation operator and $\hat{a}(\mathbf{x})$ is the annihilation operator for the particle field at point $\mathbf{y}$ and m is the particle mass. Finally we use the notation: $\langle\hat{m}(\mathbf{x})\rangle_t = \langle\psi_t|\hat{m}(\mathbf{x})|\psi_t\rangle$. The presence these terms mean that the evolution is non-linear.

The overall structure of Eq. (3.1) is such that the equation for the evolution of the density operator has a Lindblad form [38, 43] and that it causes spatial superpositions to spontaneously collapse, as we will see in the next section.

3.3 Evolution of the Density Operator

To see how both of these requirements are fulfilled it is simpler to work with the statistical operator $\hat{\rho}_t$ rather than the state. The statistical operator is defined as the stochastic average of the outer product of the state:

$$\hat{\rho}_t \equiv E_{\mathbb{P}}[|\psi_t\rangle\langle\psi_t|] \tag{3.5}$$

This stochastic average means that the statistical operator does not describe a single run but instead describes the behaviour averaged over many instances of the stochastic process [10] .

Through Eq. (3.5) and Eq. (3.1) the evolution of $\hat{\rho}_t$ can be written as [21]:

$$\frac{d\hat{\rho}_t}{dt} = -i[\hat{H},\hat{\rho}_t] - \frac{\gamma}{2m_0^2}\int d\mathbf{x}\int d\mathbf{y} D(\mathbf{x},\mathbf{y})[\hat{m}(\mathbf{x}),[\hat{m}(\mathbf{x}),\hat{\rho}_t]] \tag{3.6}$$

where $\hat{m}(\mathbf{x})$ has the same definition as in the state equation. Equation (3.6) is trace-preserving and completely positive[2] [25, 38, 43]. This guarantees that the norm of $\hat{\rho}_t$ is preserved and that the evolution is completely positive, meaning that the dynamics will never predict nonphysical negative probabilities.

The other requirement for CSL to be a successful spontaneous collapse model is to check that the evolution causes the spatial superpositions to be destroyed, i.e if the model localises particles in space.

To check for particle localisation, the variance of the mass density operator should go to zero in the infinite time limit when the unitary Hamiltonian is neglected. This condition implies that the non-unitary dynamics eventually fixes the position of the particles, as the mass density operator represents position for an indistinguishable particle system.

The condition for the model to localise particles is thus:

$$\lim_{t\to\infty} \mathbb{E}_P(Var([\hat{m}(\mathbf{x})]_t)) = 0 \tag{3.7}$$

where:

$$Var([\hat{m}(\mathbf{x})]_t) = \langle \hat{m}^2(\mathbf{x})\rangle_t - \langle \hat{m}(\mathbf{x})\rangle_t^2 \tag{3.8}$$

and $\hat{m}(\mathbf{x})$ is defined as in (3.4).

Note that the condition Eq. (3.7) requires that the variance goes to zero in the infinite time limit only at the level of the stochastic average. However we shall see that in fact it is possible to conclude that for CSL the variance is zero for every realisation of the stochastic process, except for a subset of measure zero i.e. a subset of realisations of the stochastic process which have zero probability of occurring.

Using the definition of the the density operator we have:

$$\mathbb{E}_{\mathbb{P}}[\langle \hat{m}^2(\mathbf{x})\rangle_t] = Tr(\hat{m}^2(\mathbf{x})\rho_t) \tag{3.9}$$

due to the fact that $\hat{m}(\mathbf{x})$ commutes with itself and $\hat{H}$ is set to zero we have:

$$\frac{d}{dt}\mathbb{E}_{\mathbb{P}}[\langle \hat{m}^2(\mathbf{x})\rangle_t] = \frac{d}{dt}Tr(\hat{m}^2(\mathbf{x})\rho_t) = 0 \tag{3.10}$$

This implies that $\langle \hat{m}^2(\mathbf{x})\rangle_t = \langle \hat{m}^2(\mathbf{x})\rangle_0$ for all values of t. This is true for both CSL and standard quantum mechanics. The important difference comes from the term $\frac{d}{dt}\mathbb{E}_{\mathbb{P}}[\langle \hat{m}(\mathbf{x}\rangle_t^2]$. It has been shown in [3] and [33] that a stochastic differential equation with this form implies that:

$$\frac{d}{dt}E_{\mathbb{P}}[\langle \hat{m}(\mathbf{x})\rangle_t^2] > 0 \tag{3.11}$$

[2] In the Markovian case this is the well known Lindblad equation.

Since $\frac{d}{dt}[\langle\hat{m}^2(\mathbf{x})\rangle_t] = 0$ and $\frac{d}{dt}E_{\mathbb{P}}[\langle\hat{m}(\mathbf{x})\rangle_t^2] > 0$, then the expectation of the variance must decrease with time, going to zero in the infinite time limit. Since $\langle\hat{m}^2(\mathbf{x})\rangle_t = \langle\hat{m}^2(\mathbf{x})\rangle_0$ this then implies that that in the infinite time limit the variance must go to zero for all realisations, except for a subset of measure zero.

Hence CSL predicts that particles are always localised in the long time limit. The standard CSL model is Galilei invariant [10], guaranteeing that, in a non-relativistic setting, the dynamical laws are the same in all inertial frames.

3.4 Ontologies of CSL

Similarly to the GRW model the CSL model was originally proposed to resolve issues with the interpretation of standard quantum mechanics. For continuous processes there are no 'flashes' as there are in GRW so there cannot be a flash ontology.

However CSL (and other continuous spontaneous collapse models) have a consistently defined locally averaged mass density distribution which can be interpreted as the 'beable' of the theory, which constitutes physical reality [15]. The mass density distribution is averaged over a volume proportional to r_c^3 [10]:

$$M(\mathbf{x}) = \langle\psi|\int d\mathbf{y}\; D(\mathbf{x},\mathbf{y})\hat{m}(\mathbf{y})|\psi\rangle_t. \tag{3.12}$$

The reason that it must be averaged is that the collapse dynamics will not guarantee that spatial superpositions are collapsed below the r_c scale. If the superpositions remain then one cannot state that the mass density field definitely has a certain value at a certain point, which is what is required of a beable.

3.5 General form of a continuous spontaneous collapse model

What we have discussed so far is the form and outcomes of CSL. However this is not the only possible continuous spontaneous collapse model. There is also coloured CSL [3] (so called because the stochastic noise field is no longer a white noise), dissipative CSL [52] which has a mechanism that dissipates the energy such that in the infinite time limit it does not go to infinity, and models in which the collapse is caused by coupling to a gravitational field [24, 33, 48]. There is a general form of collapse models that encompasses all these forms, which we will give here in order to review other proposed collapse models and as any attempt to make a relativistic generalisation of a continuous spontaneous must have this form.

As in the case of CSL, the general form can be expressed as an evolution equation for the state ψ_t (as in [33]) or as a map for the evolution of the density operator. Since in general the latter is more compact and easier to deal with, that is the one we will use here. We will also switch to using 4D notation from now onwards as this will make further discussion of the relativistic case simpler.

Independently of particle type we can write the evolution of $\hat{\rho}$ as [3, 31, 33]:

$$\hat{\rho}_t = \mathcal{U}_t^0[\mathcal{M}_t[\hat{\rho}_0]] \tag{3.13}$$

where $\mathcal{U}_t^0[\cdot] = e^{-i\hat{H}t} \cdot e^{i\hat{H}t}$ is the standard quantum evolution, with $\hat{H}$ as the standard unitary Hamiltonian, and $\mathcal{M}_t$ is the contribution due to the presence of the stochastic noise and has the following expression:

$$\mathcal{M}_t = \overleftarrow{\mathrm{T}} \exp\left(\frac{\gamma}{m_0^2}\mathcal{L}_t\right) \tag{3.14}$$

with

$$\mathcal{L}_t = \int_{\Omega_t} d^4x \int_{\Omega_t} d^4y D(x,y) \left[\hat{Q}^L(y)\hat{Q}^R(x) - \theta(x^0-y^0)\hat{Q}^L(x)\hat{Q}^L(y) - \theta(y^0-x^0)\hat{Q}^R(y)\hat{Q}^R(x)\right] \tag{3.15}$$

$\hat{Q}(\mathbf{x})$ is an arbitrary self-adjoint operator, $\hat{Q}(x) = \mathcal{U}_{x^0}^{0\dagger}[\hat{Q}(\mathbf{x})]$, $\Omega_t = \{x = (\mathbf{x}, x^0) \,|\, x^0 \in [0,t],\ \mathbf{x} \in \mathbb{R}^3\}$, the superscript L (R) denotes the operator acting on the statistical operator $\hat{\rho}$ from left (right), i.e. $\hat{Q}^R(x)\hat{Q}^L(x)\hat{\rho} = \hat{Q}(y)\hat{\rho}\hat{Q}(y)$ and $\overleftarrow{\mathrm{T}}$ is the chronological time ordering acting on the L/R operators and is defined as:

$$\overleftarrow{\mathrm{T}} Q^{L/R}(y)Q^{L/R}(x) = \begin{cases} Q^{L/R}(x)Q^{L/R}(y) \text{ if } x^0 > y^0 \\ Q^{L/R}(y)Q^{L/R}(x) \text{ if } y^0 > x^0 \end{cases} \tag{3.16}$$

Equation (3.14) suppresses the off diagonal elements of $\hat{\rho}$ in the basis of the eigenstates of $\hat{Q}(\mathbf{x})$. Due to the structure of eq. (3.13) we have that $\mathcal{M}_t$ and hence $\mathcal{L}_t$ are in the interaction picture.

For collapse models this is a consequence of the collapse of the state, which again occurs in the eigenbasis of $\hat{Q}(\mathbf{x})$ when the unitary Hamiltonian dynamics is neglected, see [33] for a proof. It is for this reason that many collapse models have $\hat{Q}(\mathbf{x})$ as the mass density or number density operator, since this means macroscopic objects do not remain in spatial superposition. Note that eq. (3.15) is a non-unitary map, meaning that it is irreversible and can only describe forward evolution in time.

As one can see from Eq. (3.15), all collapse models are defined by specifying $D(x,y)$, $\hat{Q}(x)$, and the coupling γ. For example the average map of the coloured CSL model for equal mass scalar particles is characterized by the above equation with:

$$\begin{aligned} D(x,y) &= \frac{1}{(4\pi r_c^2)^{3/2}} \exp[-(\mathbf{x}-\mathbf{y})^2/2r_c^2]F(t-s) \\ \hat{Q}(x) &= \hat{m}(x) \end{aligned} \tag{3.17}$$

where $F(t-s)$ is an arbitrary positive function of time, $\hat{m}(x)$ is the non-relativistic mass density operator, m_0 a reference mass (usually the nucleon mass), and r_c is a parameter of the model which determines the length scale of the collapse. Other choices of $D(x, y)$, $\hat{Q}(x)$, and the coupling γ have also been proposed [24, 33, 48, 52].

Original CSL, described by (3.6) is recovered when $F(t-s) = \delta(t-s)$.

3.6 Conclusion

In this chapter we reviewed the CSL model, introducing its form in terms of evolution of the state and the density operator. It was shown that due to the structure of CSL's time evolution equation the state is normalised and can still be used to find the expectation value of operators. It was also shown how CSL implies that particles are localised in space in the infinite time limit. Finally the proposed ontology for CSL was briefly discussed. The defining characteristics of any CSL type model are that the dynamics has the structure of eq. (3.14), the collapse causes localisation in the position basis i.e. the condition given in eq. (3.7) and that the rate of change of energy is not divergent, guaranteed by the Gaussian spread of the correlation function.

So far we reviewed both discrete (GRW) and continuous (CSL) non-relativistic spontaneous collapse models, outlining their mathematical structure and the properties they must have to be successful at solving the measurement problem. This has set the stage for discussing what is required for a successful relativistic model which will be the topic of the next chapters.

Chapter 4

Relativity and Spontaneous Collapse Models

4.1 Introduction

In this chapter we consider the question, 'what is it that is required to make a spontaneous collapse model consistent with special relativity?'. The reason that this question is difficult to answer is that normal demonstrations of how quantum field theory (QFT) is consistent with special relativity rely on the fact that QFT is both unitary and deterministic whereas spontaneous collapse models are both non-unitary and stochastic.

First, we will consider what kind of beable is suitable for forming a relativistic condition for quantum mechanics. Then we show that when applied to spontaneous collapse models this implies that the following two conditions have to be satisfied:

- initial conditions must be comparable in different inertial frames,
- the dynamics must be Poincaré covariant.

The implications of these two requirements are then discussed in more detail. For the continuous model we reach the conclusion that in order for a relativistic spontaneous collapse model to be viable one must accept the assumptions that:

- one may only consider systems whose initial state may be compared to the initial state in a different inertial frame via a unitary Poincaré transformation,
- the appropriate condition for Poincaré covariance is that the evolution map transforms under the unitary dynamics as defined in eq. (4.13).

If these assumptions are not accepted then it is not possible to construct a continuous spontaneous collapse model that describes many particles and meets the relativistic condition given. The conclusions for GRW type piece-wise models will be postponed until the next chapter.

For all the analysis and discussion in this chapter we we do not need to specialise to a particular Hilbert space as the analysis applies to all particle types.

In the light of these conclusions we will then review the existing literature on relativistic collapse models and make comparisons between them on the basis of what particle types they describe, what relativistic conditions they are using and whether they are appropriate. We will also note what drawbacks such theories suffer from, in order to avoid repeating such issues when attempting to construct a relativistic version of GRW in chapter 5 and CSL in chapter 6.

The chapter concludes with a summary of what is required of a successful relativistic collapse model. These requirements will then be applied in the rest of the work to evaluate if a relativistic GRW or CSL model is possible. The novel work in this chapter is the explicit demonstration of why Lorentz boosts for non-unitary dynamics cannot be constructed, and recognition and discussion of the fact that there are two different definitions of Lorentz covariant dynamics in the collapse model literature.

4.2 Consistency with special relativity for quantum mechanical theories and the choice of relativistic beables

It is well known that special relativity implies that the laws of physics should be independent of the inertial (non-accelerating) frame. Mathematically, one inertial frame $\mathcal{F}$ with coordinates labelled x can be compared to another $\mathcal{F}'$ (with x') via a coordinate transform:

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + a^{\mu} \tag{4.1}$$

where Λ is an element of the proper orthochronous Lorentz group and a is a 4 vector. In this thesis we will consider passive Lorentz transformations [51], meaning that it is the coordinates that transform between frames, and the fields and vectors do not change but only refer to a different coordinate basis.

If a field ϕ is invariant under a passive transformation then in frame $\mathcal{F}$ if the field has value ϕ_0 at a point $\bar{x}$ i.e. $\phi(\bar{x}^{\nu}) = \phi_0$ then in a different inertial frame $\mathcal{F}'$ with a different coordinate system x' where x and x' are related to each other by eq. (4.1) then the field in the primed frame is

$$\phi'(\bar{x}'^{\mu}) = \phi'(\Lambda^{\mu}{}_{\nu}\bar{x}^{\nu} + a^{\mu}) = \phi_0. \tag{4.2}$$

Conceptually, passive transforms are equivalent to describing a single system (for example the value of the field ϕ) using two different coordinate systems to describe a field at the same point.

This is in contrast to active transformations, where it is the vectors and fields that are different and the coordinate systems remain the same in the two frames. So for active transformations we have that for frame $\mathcal{F}$ with coordinate system x and a field ϕ; if the field at point $\bar{x}^{\nu}$ is ϕ_0: $\phi(\bar{x}^{\nu}) = \phi_0$ then for frame $\mathcal{F}'$ with the same coordinate system x and a field ϕ' then the field is invariant under active transformation if:

$$\phi'(\bar{x}^{\nu}) = \phi(\Lambda^{\nu}{}_{\mu}(\bar{x}^{\mu} - a^{\mu})) = \phi_0 \tag{4.3}$$

This is conceptually the same as having one coordinate system and two different fields which describe the same physics.

For any candidate theory of physics to agree with special relativity it must offer predictions which are frame invariant. Since quantum mechanics is probabilistic then the predictions are given in the form of conditional probability distributions. But predictions for what? If one takes an instrumentalist viewpoint then the predictions are the outcomes measurements; from now on we will call this viewpoint 'quantum mechanics with measurements'. However, if one wishes to avoid the concept of observers, as this leads to the measurement problem, then the events must be taken to be the value of beables. Beables are a concept introduced in [15], to replace the language of measurements. Beables are what a candidate quantity of physics claims to actually exist, and knowing the value of beables describes the state of physical reality. For example, in classical Newtonian physics, beables are the positions of point like particles (see sections 2.3 and 3.4 for a discussion of proposed beables for collapse models).

Due to the importance of the state, one might be tempted to treat it as a beable of quantum mechanics, and therefore ask that it is a function of 4-dimensional spacetime that is Poincaré covariant. However in [4] Albert and Aharanov show that, due to the non-local properties of quantum mechanics and the non-unitary effect of measurements, it is not possible to have a state that both collapses instantaneously (due to measurements or otherwise) and is Poincaré covariant. Instantaneous collapses are required to ensure that non-local observables (for example momentum or total charge) are conserved [4, 6]. Equally importantly, instantaneous collapse of the state vector is required to ensure that Bell's inequalities are violated. This is because in order for the outcomes of the Bell experiment to be perfectly correlated even though the results are from space-like measurements, the two wings of the experiment must affect each other instantly, hence the state must collapse instantly[1].

If a state collapses instantaneously in one frame, then it will not collapse instantly in another. This implies that the state will not be normalised on a constant time hyperplane in some inertial frames, see Figure 4.1. This is not consistent with special relativity as a preferred frame is selected, the one where the collapse occurs instantaneously.

So instantaneous collapses (which are required for consistency with experimental evidence e.g. the violation of Bell's inequality) prevent the state from being used as a beable for a relativistic theory. Instead the beables must be local, meaning they are defined at single points in space-time. For example in standard quantum mechanics the beables could be the results of local measurements, for the GRW model it may be points of collapse, etc. At this stage we do not need to further specify the nature of the beables as our analysis does not require the exact specification.

[1]Hence proposals like [39] where the collapse only effects the future light cone fail as they do not predict non-local correlations between outcomes of experiments. Since these non-local correlations are observed in nature then any successful theory must predict them.

As quantum mechanical theories (including spontaneous collapse models) are probabilistic, the relativistic condition therefore must be that the quantity:

$$P(A_a(x_1), B_b(x_2)....|C_c(y_1), D_d(y_2)...) \tag{4.4}$$

must be invariant under Poincaré transformations. Here $P(A_a(x_1))$ is the probability that a beable A will have value a at point $x_1 \in \mathbb{R}^4$ and so on. The meaning of this condition is that eq. (4.4) must have the same value in every inertial frame where x_1 etc. are fixed points which are relabelled in each frame's coordinate system.

This condition begs the question how are the Poincaré transformation of probabilities calculated? In this work we will not transform the probabilities directly but instead transform the operators and states that are used to find the probabilities. We postpone the discussion of how these transformation operators are constructed to section 4.4 for continuous models and chapter 5 for GRW type models.

In quantum mechanics probabilities are calculated via states:

$$P(A_a(x_2)|B_b(x_1)) = \|\hat{A}_a(x_2)\hat{U}(t_2, t_1)\psi_1\|^2 \tag{4.5}$$

where ψ_1 is the state at $t_1 = x_1^0$, the form of which is known from the fact that $B_b(x_1)$ has occurred, $\hat{A}$ is the operator corresponding to the beable A, and $\hat{U}(t_2, t_1)$ is the operator which evolves the state from time t_1 to time t_2. Due to this, even though states cannot be considered beables in a relativistic theory, it is still crucial to know how to deal with them in a relativistic context, this is what we will cover in the next subsection.

4.2.1 The state on hypersurfaces

In order to offer a frame independent description of instantaneous collapse of the state, Aharonov and Albert proposed an alternative way of describing the collapse when a measurement is performed [5], in which the state collapses instantaneously in *every* inertial frame.

To allow the state to collapse instantaneously in every frame, it must be defined not on the 4D manifold but on space-like 3D hypersurfaces which make up the manifold. Then the state, and hence normalised states in a Hilbert space can be defined on each hypersurface. Here we will describe how this can be mathematically expressed for for a single particle, although the concept can be applied to many particle systems.

For states are on space-like hypersurfaces, if we label a hypersurface as ω then we can write a state on it as $\psi_\omega(x)$. The coordinate x here labels the coordinates of the 3D surface ω, but is a four vector $x \in \mathbb{M}^4$ as ω is understood to be embedded in 4D spacetime. So then every inertial observer has a state defined on their constant time 3D hypersurface. However, each state may have different values at the

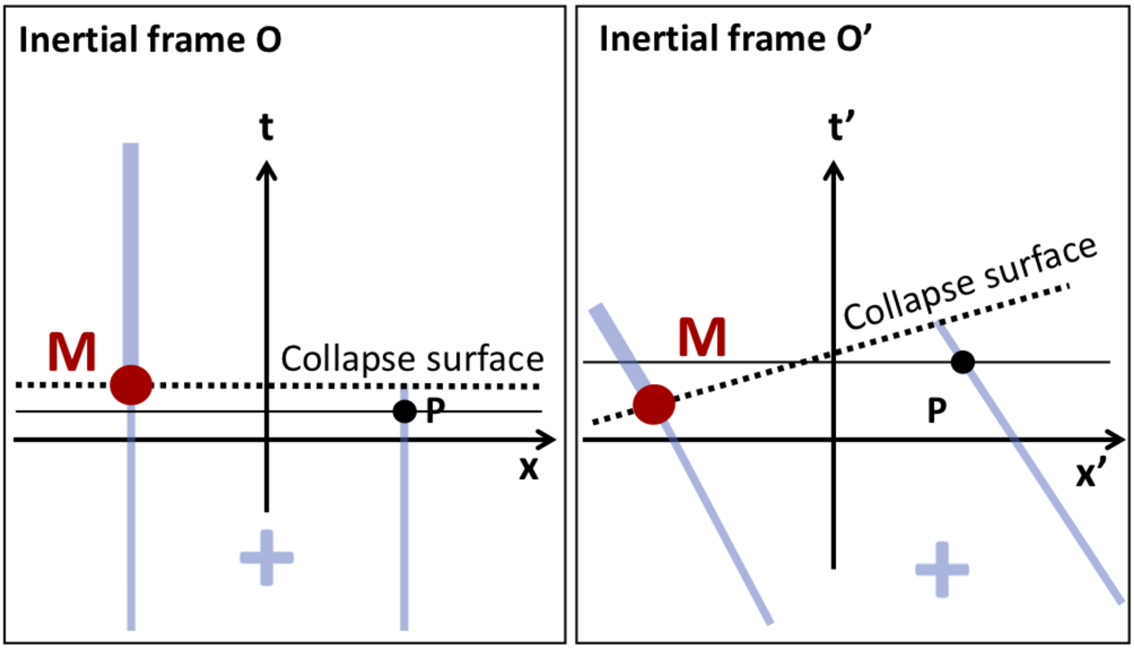


Figure 4.1: A spacetime diagram showing the support of the state before and after a measurement M where the state is a function over all of spacetime. The support is the shaded line with the amplitude proportional to the thickness of the line. The point P is a spacetime point of interest. Suppose in one frame (left figure) the state is initially in a spatial superposition (as seen in that the support is present in two places and as denoted with the plus), then M occurs and the state collapses along a specific constant time hypersurface (dotted line). The state on the surface intersecting point P (thin black line) is normalised. However in a different inertial frame (right figure) if the collapse occurs along the *same* hypersurface (dotted line), then the state on the constant time hyperplane intersecting point P in the new frame (thin black line) is not normalised.

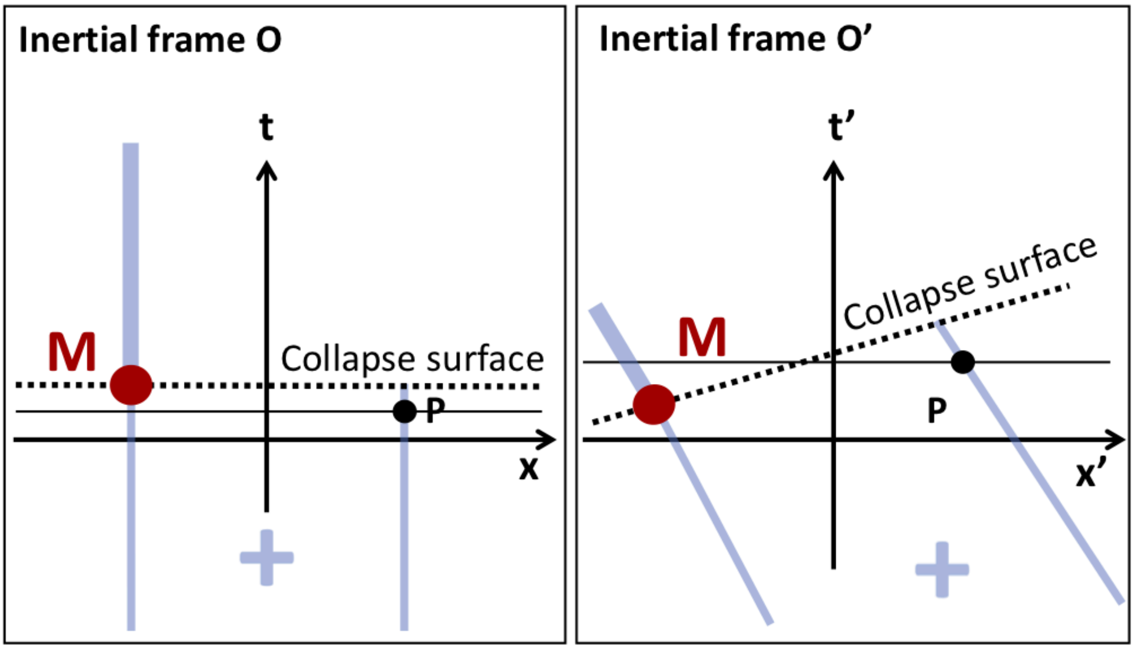


Figure 4.2: A spacetime diagram showing the support of the state before and after a measurement M. Here, in every inertial frame the state collapses on a constant time hypersurface (dotted line) so that the state is always normalised for all observers. Note that the amplitude of the state at P in different frames differs, this is a consequence of treating the state as a function on a 3D hypersurface.

same specific space-time point X so that $\psi_\omega(X) \neq \psi_{\omega'}(X')$ where X and X' are the same point in two different inertial frames. This allows states to be normalised in every frame, as shown in Figure 4.2. This is acceptable because the state in this framework has no ontological meaning, it is not a beable, it is simply a tool for calculating probabilities for the value of beables. As originally formulated the Albert-Aharonov framework was applied to the case of observers performing measurements, but for this we will adopt it for all situations where the state undergoes collapse, as the essential point that the state must remain normalised is the same in both cases.

In this framework every inertial observer can describe the time evolution of their system in terms of states on parallel constant time hypersurfaces within their frame using the Tomogana-Schwinger formalism. We will introduce this formalism in chapter 5 and show that if collapses are excluded, then this description is Lorentz covariant if it is integrable.

So how can it be that states like ψ_ω which are not frame independent can be used to calculate conditional probabilities which must be Poincaré invariant? It is because although the initial state used in the calculation is frame dependant [2], the information used to construct and evolve the state (i.e. the right hand side of Eq. (4.5)) is frame independent [5], as the local beables are point like events. Any relativistic candidate quantum theory must therefore be initialised by point-like

[2]For instance if the initial state is the state at the time of the earliest observable, then as 'earliest' is a frame dependant quantity the state would differ in different frames

events alone, as states cannot transform between frames and hence are not viable candidates for beables.

Initial Conditions and Passive and Active transformations

For comparing initial conditions there is a difference between active and passive transformations. For passive transformations there is one system, and the initial condition can be written in terms of the coordinates of each initial frame, e.g. frame $\mathcal{F}$ could have the initial condition $\phi(x)$ and $\mathcal{F}'$ would have $\phi'(x')$. Then then system is invariant if the description of the initial condition and dynamics in the primed frame is the same as the the unprimed frame.

For active transformations there are two systems, each with a frame of coordinates, e.g. $\mathcal{F}$ and $\mathcal{F}'$, where one system is the transformed (e.g. Lorentz boosted, translated etc.) version of the other. The dynamics are invariant if the transformed initial condition in $\mathcal{F}'$ follows the same equations of motion as the initial conditions in frame $\mathcal{F}$ do.

The conceptual difference between these approaches is if the initial conditions are actually describing the same system, and hence if it is possible to be able to construct a map that directly relates the two. As we are asking for invariance in the passive sense we are requiring that it is possible to make such a map.

The motivation for this choice is that we hold the view that coordinate systems and inertial frames are mathematical tools for describing the same underlying system and the passive view captures this. Therefore comparing initial conditions is a reasonable requirement. In contrast, the concept of the two systems that is employed by the active view is a abstract tool that does not hold meaning.

Of course, whilst we find this argument convincing so prefer the passive over the active view and demand that it is possible to transform between initial conditions for consistency with special relativity, not all readers may agree. In fact, some people studying relativistic spontaneous collapse models [10, 46] take the opposing view. In order to accommodate this, in this thesis we endeavor to make it clear whenever conclusions wheaher or not they depend on this choice.

Beyond constructing the initial state, the other ingredients for calculating if the conditional probability distribution, Eq. 4.4, is Poincaré invariant is how the time translation operators $\hat{U}(t_2, t_1)$ and the operators corresponding to values of beables e.g. $\hat{A}_a(x_2)$ transform under the Poincaré group.

So, for a theory of quantum mechanics to be consistent with special relativity the initial conditions, which are local beables of the system, must be agreed upon by all observers, and the dynamics must be Poincaré covariant. We will see how these requirements can be applied to non-unitary dynamics in the following sections.

We will spilt the discussion into two parts, one for GRW type models and one for models with continuous collapses. The reason of this is due to the fact that

GRW models have instant collapse and the normal unitary time translation operator. In contrast for continuous collapse models the collapse is due to the time translation operator itself so the question of how to represent coordinates on the Hilbert space differs, as will become clear. We will leave the discussion of relativistic GRW models for chapter 5 and concentrate only on continuous models for the remainder of this chapter.

4.3 Continuous non-unitary dynamics

In this section we will discuss the relativistic condition for a continuous spontaneous collapse model in two parts: first how to compare initial conditions between frames, and second how to ensure the dynamics is Poincaré covariant. Note here that as the purpose of collapse models is to remove the need for observers performing measurements, for the rest of this chapter we will not be considering instantaneous collapses of the state due to a local measurement.

We have stated that a relativistic dynamics must be one in which initial conditions between frames can be compared, and that in order to be consistent with relativity those conditions must be given by local beables, not the non-local state (or density operator).

Non-relativistic continuous spontaneous collapse models where originally proposed with the state as the initial condition. However, for all the reasons given in the above section we insist that in order to be consistent with special relativity the initial state must be constructed from the values of local beables.

For a system with space-like initial conditions comparing them between inertial frames is a significant problem for creating a relativistic continuous spontaneous collapse model, as the evolution is non-unitary.

In the next section we will demonstrate this by attempting to to relate initial conditions between inertial frames by using an irreversible time evolution operator, and show that this is not possible.

4.3.1 Relating initial conditions in different inertial frames with non-unitary dynamics

We have already discussed that for unitary quantum mechanics with point like events it is not possible to relate states to each other in different frames and hence we must consider initial conditions given by local beables. It was shown by [4] that for quantum mechanics with point like measurements, a state on a constant time hyperplane in one frame in general cannot be mapped to the state of an constant time hyperplane in a different inertial frame with a unitary operator.

So:

$$\psi'_{\sigma_{0'}} \neq \hat{U}\psi_{\sigma_0} \tag{4.6}$$

where σ_0 is a constant time hyperplane in one frame, $\sigma_{0'}$ is a constant time hyperplane in another inertial frame and $\hat{U}$ is a unitary operator which is not the appropriate map if there are point-like collapses.

But as spontaneous collapse models are non-unitary, is it possible that states could be related between frames? We shall see that the answer is no.

By definition non-unitary dynamics are irreversible, meaning that given an initial state ψ_{σ_0} there is a time translation map $\psi_{\sigma_t} = \Gamma_t[\psi_{\sigma_0}]$ but there is no map $\Gamma_t^\dagger$ such that $\psi_{\sigma_0} = \Gamma_t^\dagger[\psi_{\sigma_t}]$.

When considering only time translations this is acceptable, however when considering all possible Poincaré transformations, the irreversibility implies that the Lorentz boost operator must also be irreversible as the Lorentz boosts mix space and time.

This comes about because time translations can be performed via a combination of Lorentz boosts and position translations. Suppose that time translation map is Γ_t, the Lorentz boost map is $B_{\mathbf{v}}$ where $\mathbf{v}$ is the relative velocity between frames, and the position translation map is $P_{\mathbf{x}}$, where $\mathbf{x}$ is a three vector which denotes the distance translated. If position and Lorentz boost transformations are both reversible then $P_{-\mathbf{x}}$ and $B_{-\mathbf{v}}$ must both exist. Then, as shown in figure 4.3 for an initial state ψ_{σ_0}, a time translation can be written as:

$$\Gamma_T[\psi_{\sigma_0}] = P_{-\mathbf{x}}[B_{-\mathbf{v}}[P_{\mathbf{x}}[B_{\mathbf{v}}[\psi_{\sigma_0}]]]] \tag{4.7}$$

where $\mathbf{v}$ and $\mathbf{x}$ are such that they give the correct value of T. As every operator on the right hand side of this expression is reversible it follows that:

$$\Gamma_{-T}[\psi_{\sigma_0}] = B_{\mathbf{v}}[P_{\mathbf{x}}[B_{-\mathbf{v}}[P_{-\mathbf{x}}[\psi_{\sigma_0}]]]] \tag{4.8}$$

This proves that if both $B_{\mathbf{v}}$ and $P_{\mathbf{x}}$ are reversible Γ_T must also be reversible. So, for an irreversible Γ_T, either $B_{\mathbf{v}}$ or $P_{\mathbf{x}}$ must also be irreversible. Both of these options are physically unacceptable, as the former implies that position translations are only possible in one direction and the latter implies that an inertial frame can be related to another frame with a relative velocity $\mathbf{v}$ but not one with $-\mathbf{v}$.

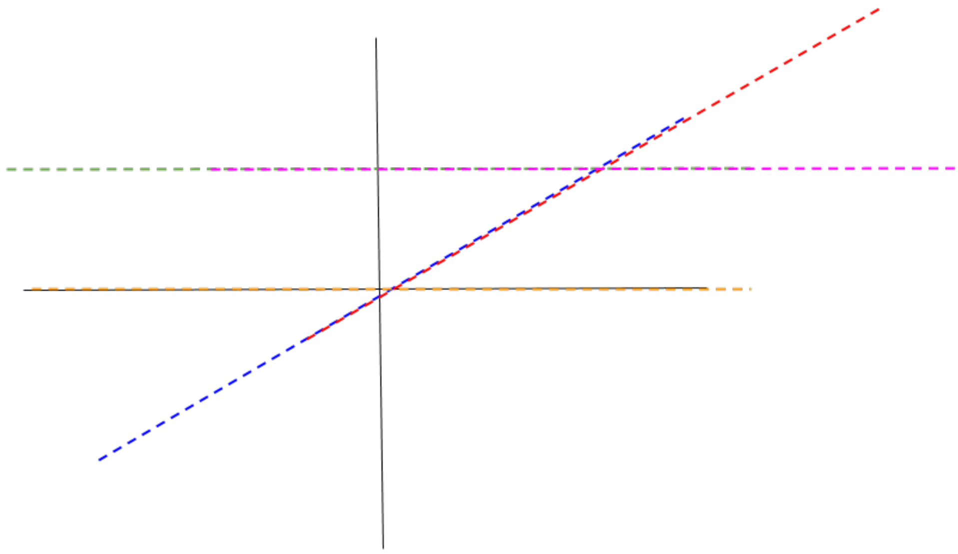


Figure 4.3: A graphic showing that a Lorentz boost (orange to blue) followed by a position translation (blue to red) followed by a Lorentz boost (red to pink) and finally a position translation (pink to green) are equivalent to a time translation (orange to green).

Therefore, for non-unitary dynamics, Lorentz boost transformations of a state cannot be defined consistently with all the other transformations of the Poincaré group.

This means that for non-unitary dynamics, as is the case for standard quantum mechanics (see section 4.2) a state on a constant time hypersurface cannot be mapped to the state on a constant time hypersurface in a different inertial frame using unitary transformations.

So, for a relativistic continuous spontaneous collapse model, we must both have that the initial conditions are given entirely by local beables and that from just those beables it is possible to construct an initial state in any inertial frame which can be mapped to any other frame. But as we will see in the below example this is not always possible.

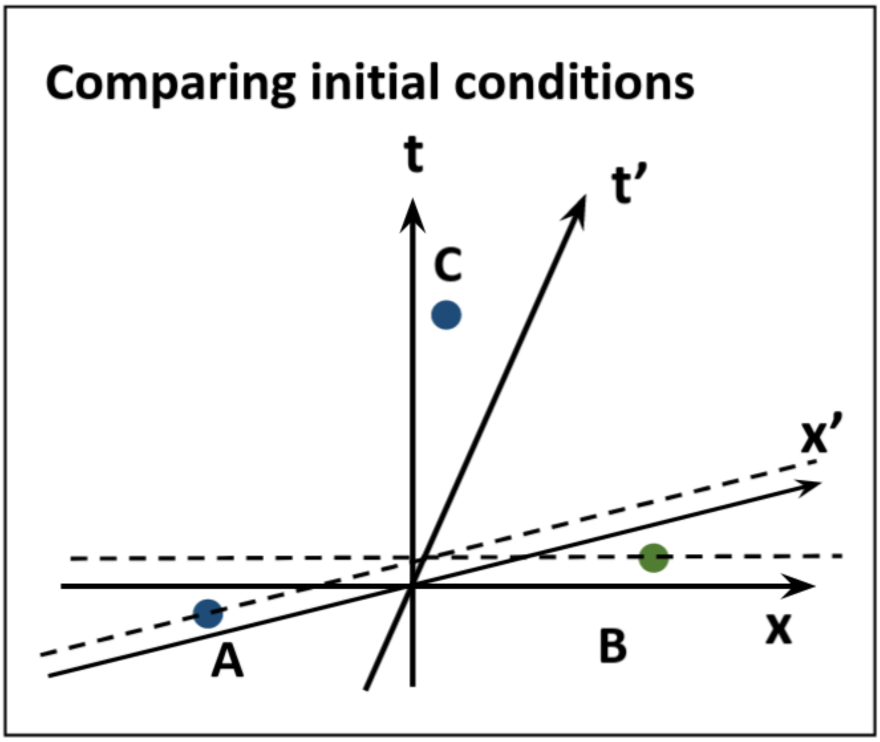


Figure 4.4: A graphic showing three space time points A, B and C. In the unprimed frame $\mathcal{F}$ the joint conditional probably for an event at C is found from a density operator defined on the dashed line intersecting with B whereas for the primed frame $\mathcal{F}'$ it is found from a density operator defined on the dashed line intersecting with A.

Consider the situation shown in figure 4.4 where the initial conditions are given by the values of local beables at points A and B, which are space-like to each other. We label these values as a_A and b_B where the subscript indicates the location. Suppose we are interested in calculating the joint conditional probability of a beable having the value c at point C, i.e. $P(c_C|a_A, b_B)$. In frame $\mathcal{F}$ this can be calculated via:

$$P(c_C|a_A, b_B) = Tr(\hat{C}_c \mathcal{M}_t[\hat{\rho}_B]) \tag{4.9}$$

where $\hat{C}_c$ is projection operator associated to the beable c, $\mathcal{M}_t$ is the generic time evolution map for continuous collapse model as defined in eq. (3.13) and $\hat{\rho}_B$ is the density operator on the constant time hypersurface that intersects the point B. This is found by constructing the state on the constant time hypersurface that intersects the point A and then time evolving it to t_B, then acting with the projection operator $\hat{B}_b$ and renormalising. This procedure of updating the state by acting with the operator associated with the beable and then renormalising is standard for spontaneous collapse models [10].

In a different initial frame $\mathcal{F}'$, the time order of points A and B is reversed, hence the joint conditional probability is:

$$P(c'_C|a'_A, b'_B) = Tr(\hat{C}'_c \mathcal{M}_{t'}[\hat{\rho}'_A]) \tag{4.10}$$

where $\hat{C}_c^{'}$ and $\mathcal{M}_{t'}$ are Lorentz transformed and $\hat{\rho}'_A$ is the density operator on the constant time hypersurface in $\mathcal{F}'$, see figure 4.4. In this frame, $\hat{\rho}'_A$ is found by constructing the state on the constant time hypersurface that intersects the point B and then time evolving it to t'_A, then acting with the projection operator $\hat{A}_a^{'}$ and renormalising.

In order to be consistent with special relativity, eq. 4.9 and 4.10 must be equal. In the next subsection we will discuss how to construct Poincaré transformation operators for non-unitary dynamics and hence how $\hat{C}_c$ and $\mathcal{M}_t$ transform, but first we must consider how to compare $\hat{\rho}_B$ and $\hat{\rho}'_A$. As these density operators are defined on non-parallel hypersurfaces it is not possible to use a time evolution operator to relate them. Instead, a Lorentz boost operator for non-unitary dynamics must be used. However, as shown previously, such an operator leads to inconsistencies and cannot exist. Therefore, if the time evolution between the local beables that are taken as initial conditions is stochastic is it not possible to compare probabilities between different inertial frames.

An alternative is to only consider scenarios where the initial state in one inertial frame may be related to that of a different frame via a unitary Poincaré transformation.

4.3.2 Comparing initial states using unitary dynamics

A possibility for overcoming this issue of comparing two different inertial frames is to only consider situations where initially the dynamics of the system is unitary such that an initial state constructed from the beables can be compared between two different inertial frames. This follows the ethos of how the Poincaré transformations of initial and final states in relativistic quantum field theory are dealt with.

In quantum field theory, the paradigmatic situation considered is one in which particles are initially far apart and non-interacting, then they travel close to one another and interact before finally separating again [64].

This means that the initial and final fields (in and out states) must be solutions of the free (i.e. non-interacting) equations of motion, and they must belong to the same Hilbert space.

For this prescription to be applicable it must be the case where there is only free dynamics as $t \to \pm\infty$ and the complete interacting dynamics at some intermediate time.

Suppose that a similar restriction is applied to continuous collapse models, where instead of the interacting part of the Hamiltonian only taking effect for an finite period of time, it is the stochastic non-unitary part of the dynamics that is only in effect for a finite period of time. Then the same conclusion as for relativistic quantum field theory can be drawn: that for this case it is possible to compare initial conditions using unitary Poincaré operators.

This does not apply to all initial conditions in a situation where non-unitary stochastic effect is present at all times, as would be the case if the dynamics were fundamentally non-unitary, as this would affect the dynamics over all of spacetime.

Hence consistency with special relativity in the in-out formalism is not sufficient to show that a fundamental model, like a spontaneous collapse model can be consistent with special relativity.

One caveat to make here is that so far we have been assuming that at fundamental theory must be in agreement with special relativity. However it is well known, [30], that special relativity only describes nature in the limiting case of a flat spacetime, described by the Minkowski metric, and that a more fundamental theory must be consistent with general relativity, which describes curved spacetime. This is relevant to the discussion here as when considering long timescales, as implied by $t \to \pm\infty$, its possible that a flat space time is no longer appropriate for describing reality.

A full consideration of the consistency between general relativity and spontaneous collapse models is outside the scope of this work. We do however argue that studying the relationship between special relativity and collapse models might be used as a stepping stone for developing a general relativistic theory, analogously to the way that special relativistic quantum field theory is used to attempt to construct a theory of quantum gravity.

We now turn to the other element of consistency with special relativity, the Poincaré covariance of the dynamics.

4.4 Unitary Quantum mechanics and Poincaré Covariance

As usually constructed the Poincaré covariance of quantum mechanics is closely connected to its unitarity.

For unitary quantum mechanics, a theory is consistent with special relativity if its Lagrangian, $\mathcal{L}$, is a Poincaré scalar quantity. It is then possible to use $\mathcal{L}$ to define a representation of all the transformations of the Poincaré group in a specific Hilbert space. In essence for a Hilbert space $\mathcal{H}$ and density operator $\hat{\rho}_\alpha$ (where α labels all indices and coordinates), then $\mathcal{L}$ defines the set of Poincaré transformation operators $\hat{S}(\Lambda, a)$ such that:

$$\hat{\rho}_{\alpha'} = \hat{S}(\Lambda, a)\hat{\rho}_\alpha \hat{S}^{-1}(\Lambda, a) \tag{4.11}$$

where α' denotes the primed coordinates defined in Eq. (4.1). Here we are clearly in the Schrödinger picture. See [64, pg. 58] for a detailed derivation of how the operators $\hat{S}(\Lambda, a)$ are constructed. Since a time translation is included in the

Poincaré transformations then the time evolution operator is:

$$\hat{U}(t,0) \equiv \hat{S}(\mathbb{I}^3,(t,0,0,0)) \tag{4.12}$$

where $\mathbb{I}^3$ is a three by three identity matrix. Therefore, unitary mechanics is automatically Poincaré covariant as $\hat{S}(\Lambda)$ is a self-consistent representation of the transformations of the Poincaré group, i.e. the transformation operators are themselves covariant.

However, for non-unitary quantum mechanics, the time translation operator for a system acting on states is non-Hermitian and not derived directly from a deterministic $\mathcal{L}$ but either by starting from a stochastic Lagrangian [54] or by defining the theory starting from the time evolution operator for the density operator (as is the case with spontaneous collapse models).

How can the representation of the Poincaré group on the Hilbert space of the system be found if not from a deterministic Lagrangian? There are two approaches in the literature; the first is to use the representation corresponding to the unitary part of the dynamics and the second is to use the representation corresponding to the full non-unitary dynamics but only onto the semi-group of the future light cone of a specific point. Both of these will be discussed in the following sections.

4.5 Covariance under the unitary dynamics

In this section we will discuss Poincaré covariance under the unitary part of the dynamics for non-unitary dynamics. By Poincaré covariance under the unitary dynamics, we mean that the time evolution map $\mathcal{M}_t$ is covariant with respect to Poincaré transformation operators which correspond to the unitary Lagrangian. The paradigmatic situation that this could be applied to is where there is a unitary dynamics and a weakly coupled non-unitary dynamics. This is the case for all continuous spontaneous collapse models as the rate of collapse must be low to be in agreement with experiment [10].

So for the coordinate transform given in Eq. (4.1) and given the unitary Lagrangian $\mathcal{L}_0$ there are associated Poincaré transformation operators $\hat{S}_0(\Lambda,a)$. A map that is covariant under the unitary dynamics satisfies:

$$\mathcal{S}_0^{-1}(\Lambda)\mathcal{M}_t\mathcal{S}_0(\Lambda) = \mathcal{M}_{\Lambda(t)}, \tag{4.13}$$

where $\mathcal{S}_0(\Lambda,a) = \hat{S}_0^L(\Lambda,a)\cdot\hat{S}_0^{\dagger,R}(\Lambda,a)$.

It is obviously correct to use this representation of Poincaré transformations to check covariance when the dynamics is given by a Poincaré scalar $\mathcal{L}_0$. However, spontaneous collapse models cannot be described from this Lagrangian dynamics due to their stochasticity and non-linearity. So this definition of covariance is perturbative as there is not an underlying reason why this is the appropriate definition.

This condition for covariance can be used to describe non-Markovian dynamics and has been used as a condition in multiple proposed relativistic collapse models, [13, 44, 46, 47]. We turn now to the other definition of Poincaré covariance for non-unitary dynamics.

4.6 Covariance under the complete dynamics

Poincaré covariance under the complete dynamics means that the operators which represent the Poincaré group transformations onto the Hilbert space of states corresponding to the non-unitary dynamics, not only the unitary dynamics. There are three proposals in the literature for this kind of dynamics. In [46] the differential equations for the evolution of the state vector under the action of a position translation, a time translation and a Lorentz boost are given. In [7] the Poincaré group is represented onto Weyl operators and from this a time evolution map, $\mathcal{M}_t$, for the density operator with the correct symmetries is constructed. Then in [19, 20] the Poincaré transformations are represented onto the probability distributions for the value of the state vector. As each of these approaches is for irreversible dynamics they only define forward time evolution. This tells us that each of the representations cannot be for the full Poincaré group, but of a subgroup, where Λ is an element of the proper orthochronous Lorentz group and $a^0 > 0$, a group we label $\mathbb{R}^+$.

As explained in section 4.3.1, it is not possible find a consistent set of Poincaré transformation operators acting on states for non-unitary dynamics.

The way this is handled in [7, 19, 20] is that only transformations into the future light cone of a point are permitted. In essence, given the density operator at a point x, $\hat{\rho}(x)$, then the relation $\hat{\rho}(x') = \hat{S}(\Lambda, a)\hat{\rho}(x)\hat{S}^{-1}(\Lambda, a)$ only holds when x' is in the future light cone of x. This solves the issue as it restricts the translations possible so that no inconsistency can be formulated so the issue described in section 4.3.1 is thus avoided. Formally, the Poincaré group is represented onto a sub-semi-group of $\mathbb{R}^4$ which we will label $\mathbb{F}$, see figure 4.5 for an illustration of the difference between the different regions the dynamics can describe.

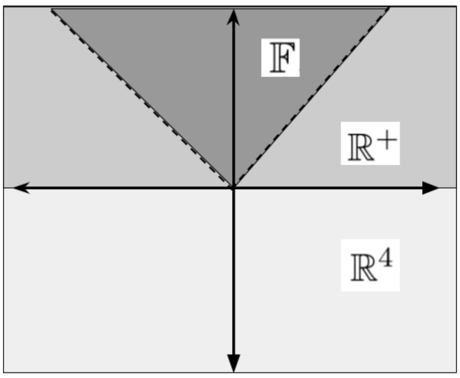


Figure 4.5: A diagram illustrating $\mathbb{R}^4$ and some sub-groups of interest, $\mathbb{R}^4$ includes the whole diagram, $\mathbb{R}^+$ encompasses the entire diagram with the time coordinate positive and $\mathbb{F}$ is the future light cone of the origin.

Both [7, 19] are stochastic models at the level of the state. Unlike non-relativistic models where the dynamics is defined with an initial time and the stochastic process is parameterised by the time coordinate, these models have dynamics defined by initial space-time point x_0 and a stochastic process parameterised by the proper time from the initial point. The value of the state at a point x_1 depends on the four-distance $(x_0 - x_1)^2$, see figure 4.6. This means that the stochastic process is defined in a frame independent way, at the cost of selecting a special initial point.

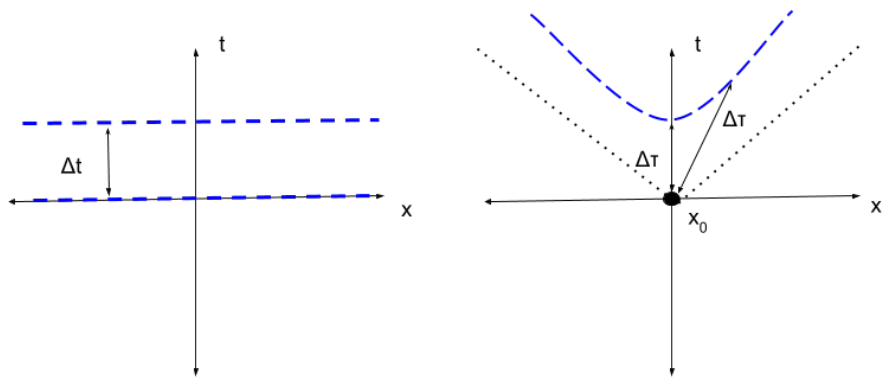


Figure 4.6: A diagram illustrating how a stochastic process X is linked to time intervals for non relativistic and relativistic processes. The left diagram shows the non relativistic case stochastic process is parameterised by the time coordinate t so given an initial state at time $t = 0$ then after Δt, the state depends on the value of $X(\Delta t)$. Trivially $X(t)$ has the same value at every point of equal time, on each point along the blue dotted line X is constant. The right diagram shows the relativistic case where the stochastic process is parameterised by the proper time τ from an origin point x_0, so the state at a point x_1 depends on the value of $X(\Delta\tau)$ where τ is the proper time between x_1 and x_0. So the stochastic process has the same value at every point on the blue dotted hyperboloid.

A dynamics like this is sufficient for giving the transition probability between time-like points in a frame-independent way. However by definition the dynamics cannot do this for space-like events. This limits the physical situations the theory is able to describe, as it cannot describe space-like initial conditions.

The way that the issue of consistently defining Lorentz transforms for irreversible dynamics is handled in [46] is different from the other two proposals, as the authors do not select a special point. Instead the set of possible Lorentz boost transformations is limited, such that only forward time evolution of each point is permitted. The authors argue that this is satisfactory as only active Poincaré transformations need to be considered, meaning that only the dynamics must be covariant, and so the initial conditions do not need to be compared. This is not the position we hold in this work, for the reasons given in section 4.2.

4.7 Proposals for a relativistic Spontaneous Collapse model

We will subdivide this section into GRW-like models and continuous models and within these subsections we will consider them chronologically. For each model we discuss the method that they use for the definition of relativistic, what situation they describe and any other problems that may rule them out.

4.7.1 GRW-like collapse models

Dove and Squires, 1996

In [26, 27] a Markovian local model of collapse is proposed where, after a point of collapse, only the state in the future light cone is collapsed. It is relativistic in the sense that the collapse operator is defined using Lorentz scalar quantities. However, as the model is local it fails to predict the correct results for the Bell experiment and does not have non-local correlations and hence is not viable. The authors note in the abstract of [27]:

> "It is shown that, although incompatible collapses, e.g. on opposite sides of an EPR-type of experiment, can occur, they will not persist in time and that eventually only compatible results will be obtained. "

The model takes as an initial condition a state on a constant time hypersurface and so the issue of how to compare initial conditions between inertial frames also applies for this model.

Dowker, 2004

In [28] a collapse model on a $1 + 1$ (one space and one time) lattice is presented. The model is a GRW type model which is successful in predicting the collapse of the state, where the state is defined on a surface on the lattice. Since the model is on a $1 + 1$ lattice, the definitions of relativistic given in this chapter cannot apply. Instead, the authors show that the probability distribution for the value of observables is independent of the sequence of surfaces on the lattice. The model is a toy model which does not describe the real world but the authors suggest that it could be used as a guide and that it may be relativistic in the continuum limit.

Tumulka, 2006

In [59] a Markovian relativistic generalisation of GRW model is proposed for non-interacting, distinguishable particles. This theory is the subject of chapter 5 but we will briefly review it here.

Like GRW it has point like collapses, where the position of each collapse is found from the previous collapse. The definition of relativistic that the model uses is that the transition probability distribution for the position of collapses is calculated in a Lorentz invariant way. That is, given a point of collapse, the rule for finding the next point is frame independent.

The main limitation of this model is the limitation in the types of particles that it can describe, only a few physical situations can be approximated by non-interacting distinguishable particles. Chapter 5 will consider models of this form in greater detail showing how it is not possible for there to be a relativistic extension of GRW to describe realistic particles.

Tumulka, 2020

The model in [61] is a development of the model from [60] to account for interacting (but still distinguishable particles). The definition of relativistic that is used is the same as that of the 2006 model. The interaction is dealt with by making the form of the time evolution operator conditional on the relative position of the collapses. The model again does not describe indistinguishable particles. As noted in the conclusion of the paper, it is not clear if superluminal signalling is prohibited by the model as the value of local beables can depend on the value of the state in their future. The author claims that this effect is negligible due to the parameters of the theory, however this is not explicitly shown.

4.7.2 Continuous collapse models

Pearle, 1990

In chapter 13 of [46], a model for a fermionic field coupled to a real scalar meson field is suggested. The model is Poincaré covariant under the complete dynamics, however as discussed in secton 6.4 of [46] it suffers from a divergent change in energy, meaning that it predicts all particles immediately gain infinite energy and hence is non-physical.

Pearle, 1999

In [47], a non-Markovian continuous model for a Dirac fermionic field is proposed, which is relativistic in the sense that it is covariant under the unitary dynamics. It avoids the issue of divergent energy, but at the cost of introducing a tachyonic field. The tachyons have the effect of predicting that some massive particles are accelerated superluminally, which is nonphysical.

Breuer and Petruccione, 1998

In [19, 20], a Markovian collapse model for a Dirac field in the one particle sector is proposed. The model successfully[3] predicts that a single particle collapses in

[3] Successful in the sense that they show that the model predicts collapse of spatial superpositions, does not have nonphysical energy divergences and has a completely positive trace-preserving dynamics.

the relativistic regime and is Poincaré covariant under the complete dynamics. However, as in the case of [7], the model only describes the evolution of the state in the future light-cone of a single point, taken by the authors of [19, 20] to be an initial measurement. Therefore the model cannot describe multiple interacting entangled particles, see section 4.6.

Nicrosini and Rimini, 2003

In [44] an interacting non-Markovian model for multiple particles defined using a Tomonaga-Swinger equation is present. However, since the model is non-Markovian the authors point out that it is not clear that the model is integrable, that it is unknown if the dynamics are independent of the choice of the path in the manifold of space-like surfaces. The problem with the integrability is due to the fact that the stochastic term and the interaction term in the model do not commute, meaning that the microcausality condition is not satisfied.

Bedingham, 2011

In [14], a novel idea for relativistic collapse model is proposed where a 'pointer field' is introduced. We will describe this model in more detail in order to more clearly describe an issue with normalisation in the definition of the model.

The model is defined by giving a stochastic Tomonaga-Schwinger equation for the evolution of the state vector $|\Psi_\sigma\rangle$, where $|\Psi_\sigma\rangle$ describes both the state of the matter field ψ_m (which is the normal quantum system) and the pointer field ψ_p on the hypersurface σ. The pointer field is a quantised field at every space-time point (unlike in normal quantum field theory where a field is quantised on a space-like hypersurface).

The pointer field mediates between the quantum matter field and that classical stochastic field to prevent energy divergences whilst maintaining Lorentz covariance. This works by the matter field and pointer field interacting and becoming entangled, and then the pointer field and the classical stochastic field interacting, which collapses the pointer field, which in turn collapses the matter field. The novel trick is the intersection between the matter field and pointer field is via non-local smeared out operators, which prevent energy divergences.

The model is relativistic in the sense that it is covariant under the unitary dynamics.

This model is interesting, however the mathematical definition of the model remains a bit unclear with regards to the inner product of the state vector $|\Psi_\sigma\rangle$. The state vector is defined only on σ but includes the pointer field ψ_p which is defined over all of space time. It then seems that $\langle\Psi_\sigma|\Psi_\sigma\rangle$ must contain an integral over all of space time which would not be possible as the evolution is stochastic. If not, and $\langle\Psi_\sigma|\Psi_\sigma\rangle$ only has an integral over the points in σ, then the normalisation of the state vector is ill defined.

In either case it is not clear to us how the pointer field can fulfill the role it needs to in this model whilst leaving $|\Psi_\sigma\rangle$ with the required properties for a state vector.

Bedingham et al., 2014

In [12], a proposal for which events should be defined in a frame independent way for spontaneous collapse models is given. The authors propose the quantity of interest is the mass density and they give a frame independent way of finding the mass density at each point. This is from within the Tomonaga-Swinger formalism where the mass density at each point x is found from the state ψ on the past lightcone of x. The proposal assumes that an initial state ψ_0 is given and that the evolution of ψ uses a non-unitary collapse model dynamics, for example [14] or [59]. This work can be understood as an additional proposal for giving the dynamics of a particular quantity, the mass density, built on top of previous collapse models, and is consistent with special relativity insofar as the dynamics for ψ are consistent with special relativity .

Tilloy, 2017

[57] is a relativistic collapse model derived from an interacting quantum field theory. It finds a non-Markovian collapse model for fermions by starting from a Yukawa theory of fermions and tracing out the bosons. The model is therefore relativistic at the level of the closed system (the entire boson and fermion system), not at the level of the collapse equation of the fermions alone. As described in section 4A of [57], an issue with the model is that the collapse is transitory, the model predicts that collapses of fermionic superpositions after a certain time scale but as $t \to \infty$ the state returns to a superposition.

Bedingham, 2019

In [13], a Markovian model for Lorentz scalar particles is presented, and it is shown that such a model predicts the collapse of the state. The model is Poincaré covariant under the unitary dynamics, but has energy divergences as shown in section 3 of [13] and in [37, 46] so it is nonphysical.

4.8 Conclusion

In this chapter we have studied what it means for a spontaneous collapse model to be consistent with special relativity. We briefly reviewed Poincaré covariance and active and passive transformations and motivated our choice of using passive transformations for this thesis.

By considering the work of Albert and Aharonov [4, 5] it was concluded that for a collapse model to be consistent with special relativity it must be able to predict the conditional probabilities of the values of local beables in a frame invariant way.

The discussion was then spilt into two parts: how the initial conditions are related to each other in different inertial frames and how to construct Poincaré transformation operators for the non-unitary dynamics, i.e how to check that the dynamics is Poincaré covariance.

This section focused on continuous spontaneous collapse models, the discussion of a relativistic condition for discrete models is postponed to chapter 5.

It was found that it is not possible to construct a Lorentz boost operator for continuous non-unitary dynamics without creating paradoxes. Hence, the only way that initial conditions between frames can be compared is if initial conditions can be related to each other using unitary dynamics.

Concerning Poincaré covariance two potential definitions were discussed: covariance under the unitary dynamics and covariance under the complete dynamics. It was found that the former is able to describe non-Markovian dynamics but lacks a deep explanation for why it is the appropriate condition. The later can only describe Markovian models and can only describe the behaviour of the future light cone of a particular point, and hence cannot describe the behaviour of multi-particle systems with space-like beables as initial conditions.

Taking all of these conclusions together either one accepts that a relativistic continuous spontaneous collapse model is not possible or that we only consider systems which initially evolve under unitary dynamics and that covariance under the unitary dynamics is the appropriate definition for Poincaré covariance. We will operate under these assumptions in chapter 6 which attempts to find if a relativistic version of CSL is possible.

In the light of these conclusions the existing literature on spontaneous collapse models was reviewed and each was assessed in terms of what definition of relativistic was applied and how successful the model was in solving the measurement problem.

Chapter 5

Relativistic GRW Models

5.1 Introduction

As we have seen in chapter 2 in its original formulation the GRW model was not relativistic and described distinguishable particles with discrete points of localisation.

In this chapter we will consider models which attempt to extend GRW to be consistent with special relativity. We will present a condition that a relativistic GRW model must meet for three cases: for a single particle, for N distinguishable particles, and for N indistinguishable particles. We will only consider scalar particles as this is sufficient to draw conclusions from the analysis. We will then show that this relativistic condition implies that one can have a relativistic GRW model for a single particle or for distinguishable non-interacting, non-entangled particles, but not otherwise.

We discuss how an existing model, [60], fits into this framework .

As we have seen in chapter 4 for quantum mechanics with instantaneous collapses it is not possible to treat the state as a function over 4D spacetime in a relativistic setting. Instead, states are defined on 3D hypersurfaces. Therefore, in this chapter we will work within the Tomogana-Schwinger formalism which describes states evolving between hypersurfaces.

This chapter is organised as follows: in section 5.1.1 the Tomogana-Schwinger formalism for quantum mechanics with and without collapses is introduced. Then, in section 5.2, the requirements for a relativistic theory from chapter 4 are applied to GRW type models to find relativistic conditions for single, distinguishable and indistinguishable particles. Finally, as anticipated in section 5.2.3 for the indistinguishable case it is shown that a such a model is either not relativistic or does not achieve macroscopic classicality. In novel work in this chapter is the precise definition of what is needed for GRW to be consistent with special relativity , in particular the fact that the intervals between collapses must be defined on an invariant interval. Additionally demonstration of why an indistinguishable model with space-like collapses is not viable is novel work.

5.1.1 The Tomogana-Schwinger formalism

In this framework, every inertial observer can describe the time evolution of their system in terms of states on parallel constant time hypersurfaces within their frame. We will introduce this formalism and show that if collapses are excluded, then this description is Lorentz covariant if it is integrable. For the case of quantum mechanics with measurements, we will then derive a condition on the measurement operator for Lorentz covariance in this framework.

The Tomogana-Schwinger formalism [50, 58] describes unitary evolution as maps between states defined on arbitrary space-like hypersurfaces without collapses. First, we will introduce some additional notation for hypersurfaces. Let ω signify any generic space-like 3-dimensional hypersurface, let σ_t denote a constant time hyperplane at time t in a inertial frame $\mathcal{F}$ and thus let $\sigma'_{t'}$ represent a constant time hyperplane in a different inertial frame $\mathcal{F}'$. Then suppose the state is defined on an ω in the manifold $\mathbb{M}^4$. In this article we restrict ourselves to considering Minkowski spacetime $\mathbb{M}^4$ as it is sufficient to see the relevant Lorentz transformation properties of the probability distributions. Then, in inertial frame $\mathcal{F}$ which has coordinates x on a hypersurface ω, the state is $\psi_\omega(x)$. In another inertial frame $\mathcal{F}'$ with coordinates x', then on the same hyperplane ω the state is written $\psi'_\omega(x')$. A state under a Lorentz boost transforms as:

$$\psi_\omega(x) \to \psi'_\omega(x'^\mu) = \psi_\omega(\Lambda^\mu_\nu x^\nu). \tag{5.1}$$

Where the coordinates are related to each other by (4.1). In this chapter from now on we will suppress the indices for ease of reading we will not consider translations hence consider the Lorentz group not the Poincaré group. So eq. (5.1) tells us that, on the same hypersurface the states are equivalent up to a Lorentz transform.

Analogously to the Schrödinger equation, Tomogana and Schwinger defined the evolution of a state as it evolves between hypersurfaces, if there are no measurements between those surfaces as:

$$\frac{\delta}{\delta\omega(x)}\psi_\omega(x) = -i\mathcal{H}(x)\psi_\omega(x) \tag{5.2}$$

where $\frac{\delta}{\delta\omega(x)}$ is the functional derivative with respect to ω and $\mathcal{H}(x)$ is the Hamiltonian density. The functional derivative can be understood to be the variation in $\psi_\omega(x)$ with respect to a infinitesimal variation of ω about point x, see figure 5.1. The integrability condition for this system is that $[\mathcal{H}(x), \mathcal{H}(y)] = 0$ if x and y are space-like separated. Equation (5.2) gives rise to an unitary evolution operator which relates two hypersurfaces:

$$U^{\omega_2}_{\omega_1} = T\ \exp\left[-i\int_{\omega_1}^{\omega_2} d^4x\mathcal{H}(x)\right] \tag{5.3}$$

such that $\psi_{\omega_2}(x) = U^{\omega_2}_{\omega_1}\psi_{\omega_1}(x)$, where T means time ordering with respect to the frame $\mathcal{F}$. This operator is frame independent even though $\mathcal{H}(x)$ is not Lorentz invariant, the only frame dependant terms from the time ordering are zero due

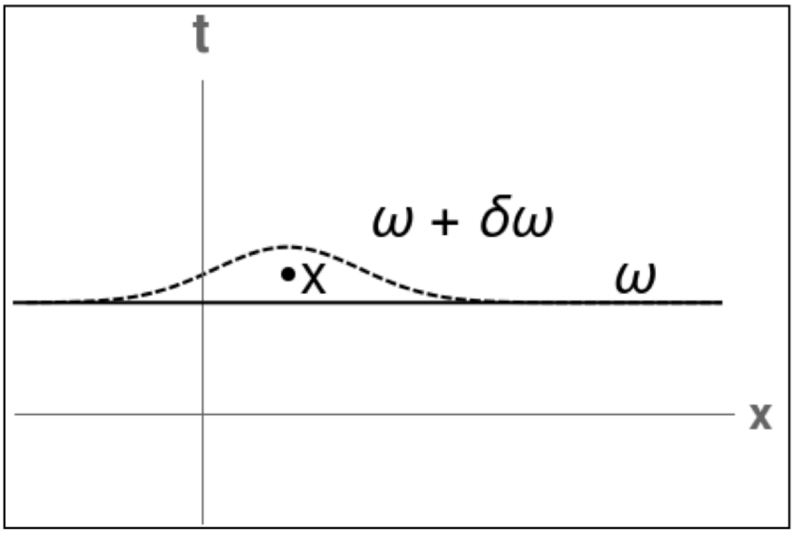


Figure 5.1: A diagram showing the infinitesimal variation, $\delta\omega$, of the hypersurface ω about the point x

to the integrability condition [19, 42]. Therefore, we have that for a frame $\mathcal{F}'$:

$$U'^{\omega_2}_{\omega_1} = T' \ \exp\left[-i\int_{\omega_1}^{\omega_2} d^4x'\mathcal{H}'(x')\right] = \hat{S}^{\dagger}(\Lambda)U^{\omega_2}_{\omega_1}\hat{S}(\Lambda) \tag{5.4}$$

where $\hat{S}(\Lambda)$ is the Lorentz transformation operator corresponding to the dynamics given by $\mathcal{H}$ and the dagger superscript is its Hermitian conjugate, see [64] for details about how this operator is constructed.

5.1.2 The Tomogana-Schwinger formalism with collapses

Now we wish to extend this formalism to include collapses of the state. In this thesis we will consider only collapses in the spatial basis as this is sufficient to explain the values of any experiment performed, as any observable can be coupled to position [11].

In a frame $\mathcal{F}$ the spatial collapse of the state at $x \in \mathbb{M}^4$ is described though an operator $\hat{L}_{\omega}(x)$ defined on the Hilbert space on a space-like hypersurface ω passing through x. Following Albert and Aharanov we consider the that the collapse occurs on the constant time hyperplane intersecting x, labelled σ_t where $t = x_0$. This means that the collapse is described as occurring instantaneously in $\mathcal{F}$.

$\hat{L}_{\sigma_t}(x)$ localises the particle it acts on about x (if the state is not already localised). The properties of this operator are model dependant however in general it is not unitary. In a different frame $\mathcal{F}'$, the collapse operator $\hat{L}'_{\sigma'_{t'}}(x')$ is defined on a constant time hypersurface $\sigma'_{t'}$.

To illustrate evolution with collapses, consider in a frame $\mathcal{F}$ two hypersurfaces σ_0 and σ_f before and after a collapse at a point x, see figure 5.2. The state ψ_{σ_f} is found by evolving the state to a hyperplane of collapse, applying the collapse

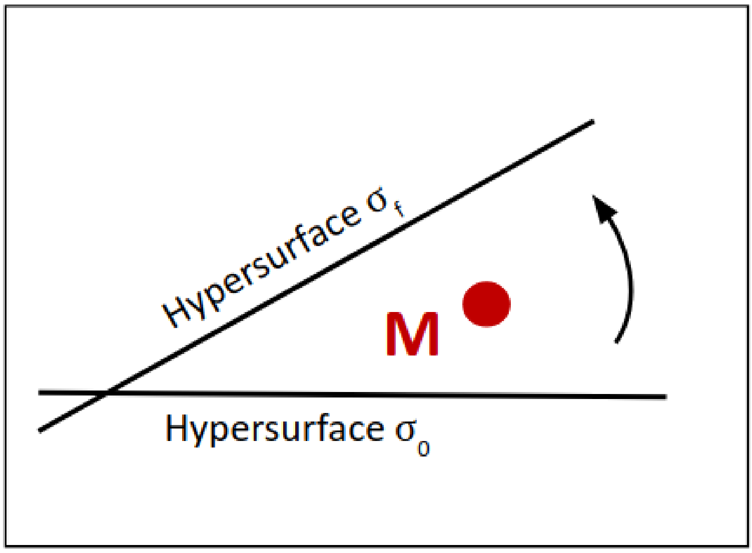


Figure 5.2: A schematic showing a measurement at a point M between two hypersurfaces σ_0 and σ_f. It is not possible to relate states on σ_0 and σ_f without knowing the if there are measurement between them.

operator and normalising, then evolving to σ_f:

$$\psi_{\sigma_0} \to \psi_{\sigma_f} = \frac{U_{\sigma_t}^{\sigma_f} \hat{L}_{\sigma_t}(x) U_{\sigma_0}^{\sigma_t} \psi_{\sigma_0}}{\|\hat{L}_{\sigma_t}(x) U_{\sigma_0}^{\sigma_t} \psi_{\sigma_0}\|}. \tag{5.5}$$

It is necessary that *all points of collapse between σ_0 and σ_f are known in order to construct such a map between them*, as in general $\hat{L}_{\sigma_t}(x)\psi_\omega \neq \psi_\omega$ for any ω. Therefore, in order to relate states in different frames on their respective constant time hypersurfaces all collapses between those hypersurfaces must be known.

To find the condition on $\hat{L}_\sigma(x)$ for consistency with relativity we consider the probability $P(x_1, x_2|\psi_{\sigma_1})$ which is Eq. (4.4) applied to the case of two measurements at space-time points x_1 and x_2 given an initial state ψ_{σ_1}[1] . σ_1 is a constant time hypersurface intersecting the point x_1 in $\mathcal{F}$. For quantum mechanics with measurements the state ψ_{σ_1} can be assumed to be specified by measurements in the past of σ_1. Then special relativity implies that:

$$P(x_1, x_2|\psi_{\sigma_1}) = P(x_1', x_2'|\psi'_{\sigma_1'}), \tag{5.6}$$

If the points x_1 and x_2 are time-like to each other and x_1 occurs before x_2 in all frames the conditional probability for one frame is given by:

$$P(x_1, x_2|\psi_{\sigma_1}) = \|\hat{L}_{\sigma_2}(x_2) U_{\sigma_1}^{\sigma_2} \hat{L}_{\sigma_1}(x_1)\psi_{\sigma_1}\|^2 \tag{5.7}$$

[1]To keep notation simple and to highlight the invariance requirements we write $P_{t_1}(\mathbf{x}_1|\psi_{\sigma_1})$ as $P(x_1|\psi_{\sigma_1})$ however as x_1 is a space-time point of measurement the equation below should be understood in the same way Eq. 4.4 is.

To compare the two sides of Eq. (5.6) the relationship between the states ψ_{σ_1} and $\psi'_{\sigma'_{1'}}$ must be specified. If there are no measurements (hence no collapses) between the two hypersurfaces then they can be related by:

$$\psi'_{\sigma'_{1'}} = \hat{S}^{\dagger}(\Lambda) U_{\sigma_1}^{\sigma'_{1'}} \psi_{\sigma_1}. \tag{5.8}$$

Here the operator $U_{\sigma_1}^{\sigma'_{1'}}$ is transforming the state between the two hyperplanes and $\hat{S}^{\dagger}(\Lambda)$ is transforming the coordinates so expressing the coordinate system explicitly we have:

$$\psi'_{\sigma'_{1'}}(x') = \hat{S}^{\dagger}(\Lambda) \psi_{\sigma'_{1'}}(x). \tag{5.9}$$

If there are measurements between ψ_{σ_1} and $\psi'_{\sigma'_{1'}}$ then the states can be related with Eq. (5.5) when a measurement occurs and Eq. (5.8) for subsequent evolution, using the appropriate positions and outcomes of measurements. In standard quantum mechanics this is acceptable as it includes the concept of observers performing measurements and recording the results. Therefore all measurements between the two surfaces can be compared between two frames. Assuming that the Hamiltonian is covariant so that Eq. (5.10) holds, then the right hand side of Eq. (5.6) can be written as:

$$\begin{aligned}
&P(x'_1, x'_2 | \psi'_{\sigma'_{1'}}) = \\
&\| \hat{L}'_{\sigma'_{2'}}(x'_2) U'^{\sigma'_{2'}}_{\sigma'_{1'}} \hat{L}'_{\sigma'_{1'}}(x'_1) \psi'_{\sigma'_{1'}} \|^2 = \\
&\| \hat{L}'_{\sigma'_{2'}}(x'_2) \hat{S}^{\dagger}(\Lambda) U_{\sigma_2}^{\sigma'_{2'}} U_{\sigma_1}^{\sigma_2} U_{\sigma'_{1'}}^{\sigma_1} \hat{S}(\Lambda) \hat{L}'_{\sigma'_{1'}}(x'_1) \hat{S}^{\dagger}(\Lambda) U_{\sigma_1}^{\sigma'_1} \psi_{\sigma'_1} \|^2,
\end{aligned} \tag{5.10}$$

where Eq. (5.10) has been used to transform the unitary operators and $\sigma'_{1'}$ and $\sigma'_{2'}$ are hypersurfaces of collapse intersecting x_1 and x_2 in frame $\mathcal{F}'$ and x' is the same spacetime point in a different coordinate system. By inspection the condition required for Eq. (5.6) to hold is:

$$\hat{L}'_{\sigma'_{t'}}(x') = \hat{S}^{\dagger}(\Lambda) U_{\sigma_t}^{\sigma'_{t'}} \hat{L}_{\sigma_t}(x) U_{\sigma'_{t'}}^{\sigma_t} \hat{S}(\Lambda). \tag{5.11}$$

Eq. (5.11) requires that the collapse operator transforms covariantly and that the collapse can be described by an operator acting on any space-like hypersurface intersecting x. This is equivalent to requiring that the collapse happens instantaneously in all inertial frames.

If instead x_1 and x_2 are space-like to each other, then in some frames their time ordering may be reversed. In this case, if in $\mathcal{F}'$ x_1 precedes x_2, then Eq. (5.7) holds and in the primed frame we have:

$$P(x'_1, x'_2 | \psi'_{\sigma'_{1'}}) = \| \hat{L}'_{\sigma'_{1'}}(x'_1) U'^{\sigma'_{1'}}_{\sigma'_{2'}} \hat{L}'_{\sigma'_{2'}}(x'_2) U'^{\sigma'_{2'}}_{\sigma'_{1'}} \psi'_{\sigma'_{1'}} \|^2. \tag{5.12}$$

Substituting in Eq. (5.8) and Eq. (5.11) it is found that for Eq. (5.6) to be satisfied:

$$[\hat{L}_{\sigma_1}(x_1), U^{\sigma_1}_{\sigma_2}\hat{L}_{\sigma_2}(x_2)U^{\sigma_2}_{\sigma_1}] = 0 \tag{5.13}$$

which is met if $\hat{L}_\sigma(x)$ satisfies the microcausality condition:

$$[\hat{L}_\sigma(x_1), \hat{L}_\sigma(x_2)] = 0 \quad \forall\sigma\text{and} \quad x_1, x_2 \in \sigma \tag{5.14}$$

As discussed in chapter 4 the state is a tool to calculate probabilities and in order to be consistent with special relativity the state must collapse instantaneously in every inertial frame. Therefore, although we have written the collapse operator as acting on a constant time hypersurface in a particular frame, it could be written as a collapse operator acting on the Hilbert space of any space-like surface using the relationship:

$$\hat{L}_{\sigma_t}(x) = U^{\sigma_t}_{\omega}\hat{Q}_\omega(x)U^{\omega}_{\sigma_t} \tag{5.15}$$

where ω is any arbitrary space-like hypersurface intersecting the point x, and $\hat{Q}_\omega(x)$ is a collapse operator like $\hat{L}_\omega(x)$ that also satisfies Eq. (5.11) and Eq. (5.13).

In order to check that Eq. (5.6) is satisfied it has been implicitly assumed that in any one frame the time ordering between x_1 and x_2 is known. Otherwise it would not have been possible to write the explicit expressions of Eq. (5.7), (5.10), and (5.12).

As mentioned already, standard quantum mechanics has the concept of observers comparing results, which means that the order of measurements can be known between frames. If in one frame observer A measured x_1 to be before x_2 and in another frame observer B measures the inverse, then A and B can reconcile their conditional probability distributions and check consistency with special relativity . In this section we have found the condition for relativistic collapse using the conditional probability for two collapses, however it can be easily shown that this applies to any number of collapses.

5.2 GRW and Special Relativity

In chapter 2 the GRW model was introduced and its defining characteristics were listed in 2.4.

In order for a model that has these features to be consistent with special relativity it the conditional probability distribution for the position of collapses must be Poincaré covariant in the sense discussed in chapter 4 and non-physical effects such as superluminal signalling and macroscopic superpostions must be be predicted.

In this section we will consider what these requirements imply for the form of a relativistic spontaneous collapse model for a single particle, distinguishable particles and indistinguishable particles.

5.2.1 Relativistic condition for a single particle

We consider a relativistic GRW model for a single particle, where there is a single series of collapses. In analogy with the original GRW model, it may be tempting to define the times at which, in a given frame, collapses occur via a Poissonian distribution with average time τ, but then one is faced with the fact that due to time dilation this prescription is not Lorentz invariant. In order to overcome this difficulty, the time intervals between collapses have to be defined in terms of Lorentz invariant time-like four-distances. This seems to be the only way to ensure that the time intervals are defined in a frame independent way. The four-distances have to be time-like not only because we are seeking a sequence of time intervals, but also because this prescription allows us to define a time-ordered sequence of collapses. This approach is followed in [60].

Consider then, a Poissonian point process with average τ, with initial value 0. Let ΔT_i be the distance between the i-th and $(i-1)$-th point of the process. Then define the times at which collapses occur as follows. Given the initial point of collapse $x_0 = (\mathbf{x}_0, x_0^0)$, the next point of collapse $x_1 = (\mathbf{x}_1, x_1^0)$ will occur at four-distance ΔT_1 from x_0, therefore x_1 will be on the future hyperboloid defined by all points with same *time-like* four-distance $\Delta T_1 = |x_1 - x_0|$ from x_0. The following point of collapse $x_2 = (\mathbf{x}_2, x_2^0)$ will lie in the future hyperboloid defined by all points with same time-like four-distance $\Delta T_2 = |x_2 - x_1|$ from x_1, and so on. See Figure 5.3.

The four-distances among consecutive collapses have an interesting physical interpretation. Consider a particle whose state is well-localised in an inertial reference frame O where the particle is at rest, for simplicity we can defined this point at the origin. As we shall see, with an appropriately defined collapse operator then in that frame, collapses are likely to occur only about the origin (where the state is non-zero), and the four-distances ΔT_i between consecutive collapses correspond to the *coordinate* time intervals Δt_i between collapses. In a different inertial frame O', the particle will be moving and while the four-distances among the collapses do not change, the coordinate time intervals $\Delta t_i'$ are dilated. The opposite would be true for a well-localised particle at rest with respect to O', thus in motion with respect to O. Therefore, the four-distances ΔT_i roughly correspond to the coordinate time intervals in the frame at rest with respect to the particle; in all other frames, the coordinate time intervals between collapses undergo time dilation. So observers measure different rates of collapse in different frames due to time dilation, but the overall prescription of the rate of collapse is frame independent. This is analogous to the situation in particle physics where a particle with a half life λ, for example a muon, appears to have a longer half life when it is travelling at a high velocity inside a particle accelerator.

The prescription above defines, in a relativistic invariant way, *when* a collapse occurs. The model must also define *where* on the hyperboloid the collapse occurs, i.e. give a normalized probability distribution for the position of the collapse on that hypersurface, such as done in [60]. In the spirit of GRW, this probability distribution must be equal to $P_\Sigma(x) = \|\hat{L}_\Sigma(x)\psi_\Sigma\|^2$ in order to avoid superluminal

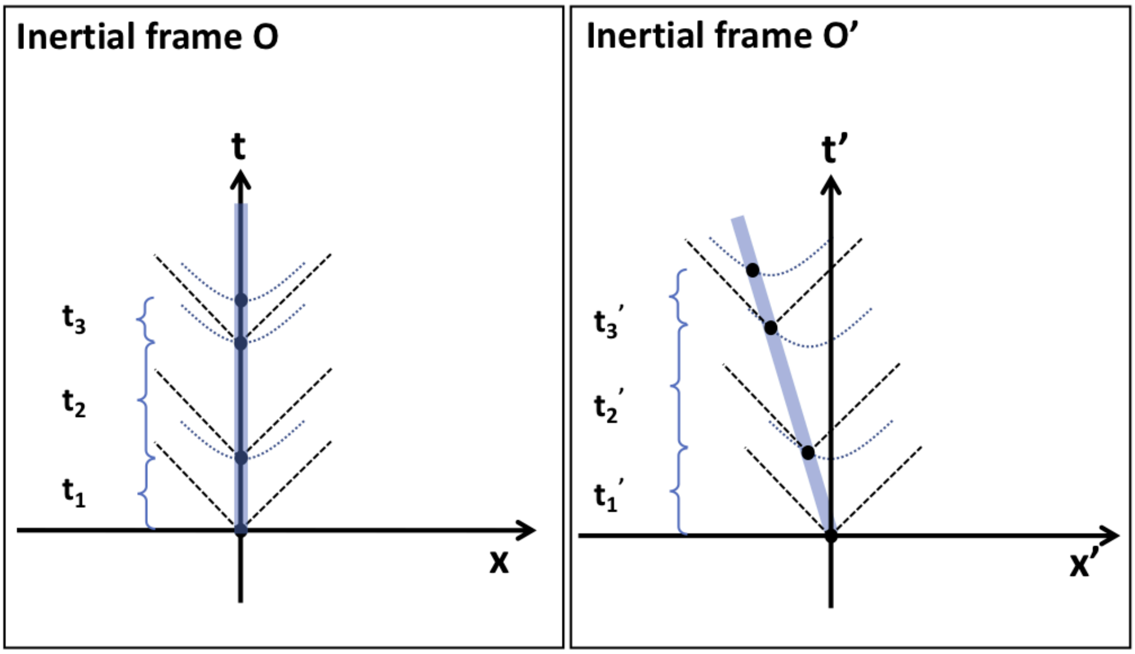


Figure 5.3: A schematic diagram showing how the stochastic process defines intervals between collapses. The shaded area shows the maximum of the state's density, straight dotted lines show the future light cone of each point of collapse and the curved dotted lines show the surfaces of constant 4-distance from the previous point of collapse. The left diagram shows a frame where each collapse occurs at the same spatial point so the coordinate time and the 4-distance coincide, $\Delta T_i = t_i$. The right diagram shows a different inertial frame where the 4-distance between each point of collapse is still ΔT_i but the coordinate time is different t_i'.

signaling, where $\hat{L}_\Sigma(x)$ is the collapse operator centred around the point of collapse x, defined on the hyperboloid Σ. We can leave $\hat{L}_\Sigma(x)$ unspecified, but it has to be chosen in such a way that it localises the state, is Lorentz covariant and that the probability is correctly normalized. However once it is specified it defines the collapse operator on all space-like hypersurfaces through Eq. (5.15).

The last ingredient is *how* a collapse occurs, i.e. how the state changes due to a sudden collapse at x. [60] assumes that the wave function collapses along the hyperboloid previously introduced; this is mathematically implemented by applying $\hat{L}_\Sigma(x)$ to ψ_Σ, and then normalizing the collapsed state. In fact, the collapses can be carried out with respect to *any* space-like hypersurface containing the point of collapse as the two prescriptions can be related by a unitary transformation. Specifically, suppose the collapse is defined to occur along a space-like hypersurface ω_1 according to the prescription:

$$\psi_{\omega_1} \to \psi^{(c)}_{\omega_1} = \frac{\hat{L}_{\omega_1}(x)\psi_{\omega_1}}{\|\hat{L}_{\omega_1}(x)\psi_{\omega_1}\|}, \tag{5.16}$$

where x is the point of collapse. Given a second space-like hypersurface ω_2 containing the point of collapse x, since $\psi_{\omega_2} = U^{\omega_2}_{\omega_1}\psi_{\omega_1}$ for the state prior to the collapse, and $\psi^{(c)}_{\omega_2} = U^{\omega_2}_{\omega_1}\psi^{(c)}_{\omega_1}$ for the state after the collapse (because by construction there are no collapses in between ω_1 and ω_2 apart from x, since all collapses are assumed to be time-like separated with respect to each other), then Eq. (5.16) can be equivalently rewritten as:

$$\psi_{\omega_2} \to \psi^{(c)}_{\omega_2} = \frac{\hat{L}_{\omega_2}(x)\psi_{\omega_2}}{\|\hat{L}_{\omega_2}(x)\psi_{\omega_2}\|}, \tag{5.17}$$

with $\hat{L}_{\omega_2}(x) = U^{\omega_2}_{\omega_1}\hat{L}_{\omega_1}(x)U^{\omega_1}_{\omega_2}$. See figure 5.4. Also, the probability distribution $P_\Sigma(x)$ previously defined can be computed along any space-like hypersurface passing through x, since

$$P_\Sigma(x) = \|\hat{L}_\Sigma(x)\psi_\Sigma\|^2 = P_\omega(x) = \|\hat{L}_\omega(x)\psi_\omega\|^2 \equiv P(x), \tag{5.18}$$

as one can easily check. It is in this sense that we can say that the collapse can be described consistently in all frames.

Therefore we are precisely in the same situations envisaged by Albert and Aharonov: *a collapse occurs instantaneously along all space-like hyper-surfaces intersecting the point of collapse*, with the only (important) difference that there, the collapses are triggered by measurements, while here they are part of the dynamical law. As pointed out by Albert and Aharonov, this is necessary so that every inertial observer can provide a normalized state both before and after the collapse on their constant-time hyperplanes. Constant-time hyperplanes are important because these are the hypersurfaces where observers describe their physics.

It is for this reason that the model presented in [26] is not a successful relativistic model, as this model has the state collapse only in the future light cone of the point of collapse. This means that the state is not normalised along different constant time hyperplanes and hence the theory does not give normalised probability distributions for systems with entangled particles.

We argue that the only consistent way to understand the model in [60] is that the state collapses on every hypersurface intersecting the point of collapse, i.e. it collapses instantly in every frame. This is the only consistent way to interpret the model as otherwise the states on hyperplanes after a point of collapse would be ill defined. This in agreement with Eq. 37 of [60] which gives the state on a constant time hypersurface for given foliation of spacetime.

To see how an inconsistency would arise otherwise, consider a collapse at point x and three hypersurfaces of interest, a hyperboloid Σ intersecting x, a hyperplane σ_{t_1} intersecting the point x and a hyperplane σ_{t_2} a short time in the future of σ_{t_1}. The state ψ_Σ will be the collapsed state. Suppose that collapses only occur on hyperboloids then the state on σ_{t_1} would be uncollapsed. Now the question is: what is the state on σ_{t_2}? As there is unitary dynamics everywhere except on the hyperboloid then the state can either be written as $\psi_{\sigma_{t_2}} = U^{\sigma_2}_{\sigma_1}\psi_{\sigma_{t_1}}$, meaning $\psi_{\sigma_{t_2}}$ would be an uncollapsed state, or as $\psi_{\sigma_{t_2}} = U^{\sigma_2}_{\Sigma}\psi_\Sigma$, meaning $\psi_{\sigma_{t_2}}$ would be collapsed. The only way to resolve this inconsistency and still have unitary dynamics is to have the collapse occur along σ_{t_1} as well.

Now we are in the position to assess whether this framework for a relativistic GRW model is consistent with special relativity. Given the initial state ψ_{σ_0} defined on a space-like hyperplane σ_0 and the initial point of collapse x_0 on σ_0, the probability for the next collapse to occur at x is given by

$$P(x_1|x_0,\psi_{\sigma_0}) = \|\hat{L}_\omega(x_1)U^\omega_{\sigma_0}\psi_{\sigma_0}\|^2 \tag{5.19}$$

where ω is a surface intersecting x_1. This conditional probability distribution is analogous to Eq. (4.4) where ψ_{σ_0} gives all the possible information about the system at (x_0,t_0) based on the position of previous collapses.

For a Lorentz transformed inertial frame $\mathcal{F}'$ with coordinates x' the initial conditions are the point of last collapse x'_0 and the state on the hyperplane $\sigma'_{0'}$. Therefore special relativity requires that:

$$P(x_1|x_0,\psi_{\sigma_0}) = P(x'_1|x'_0,\psi'_{\sigma'_{0'}}). \tag{5.20}$$

Here the condition of Eq. (4.4) is applied to spontaneous collapse models where the measurements are replaced by points of spontaneous collapse[2]. For a single particle the state is specified by the point of last collapse x_0. As described in chapter 4, in order to check this condition one must be able to *compare the initial conditions* (here the state and position of the previous collapse) *between different inertial frames*, as noted in [19]. This has consequences when considering collapse models for multiple particles, as we will see in sections 5.2.2 and 5.2.3.

[2]We will keep the notation of section 5.1.2 with the understanding that now the time coordinate of x is now probabilistic.

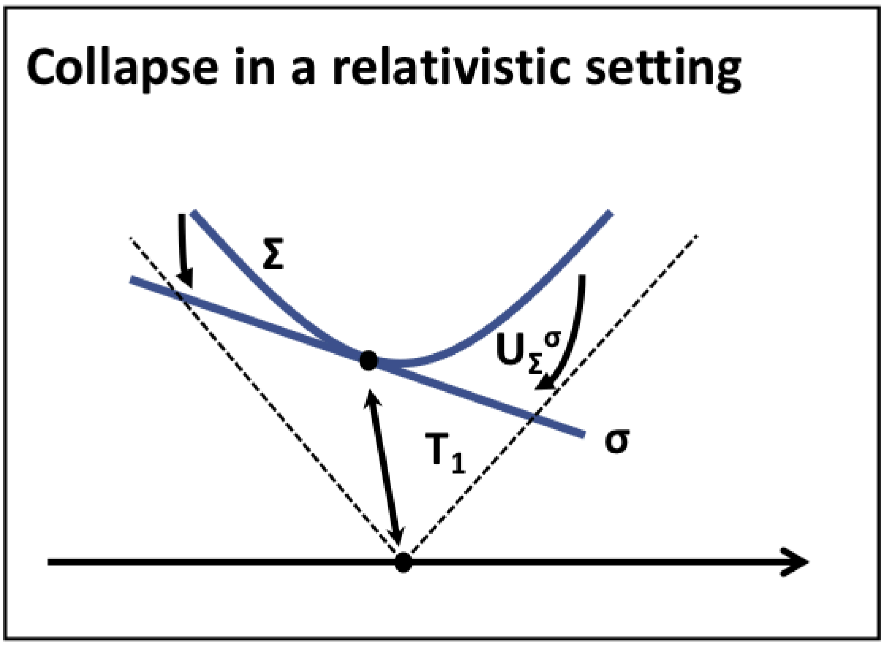


Figure 5.4: Schematic diagram showing possible surfaces of collapse in relativistic GRW. The curved thick line labeled Σ is a hyperboloid of made up of points 4-distance ΔT_1 from the previous point of collapse. The collapse operator can be defined on the surface Σ or equivalently on the hyperplane σ via the operator U_Σ^σ The straight thick line labeled σ.

In order to verify Eq. (5.20) the map between ψ_{σ_0} defined on the constant-time hyperplane σ_0 for O, and $\psi'_{\sigma'_{0'}}$ defined on the constant-time hyperplane $\sigma'_{0'}$ for O' must be known, and in order to do this, positions of all collapses between those surfaces must be known. For a series of *time-like* collapses this condition is met as there can be no collapses between σ_0 and $\sigma'_{0'}$, see figure 5.5, hence the two hyperplanes are related by Eq. (5.8). By the same argument presented in section 5.1.2, the collapse operator $\hat{L}_{\sigma_t}(x)$ must transform as in Eq. (5.11) and obey Eq. (5.13).

If these conditions are met these spatial collapses which are time-like to one another may be described in a way that is consistent with special relativity for a single particle. The model proposed by [60] for a single particle meets these conditions.

On the contrary, for a single particle theory with collapses which are *space-like* to each other (we do not discuss how such a model could be formulated), the initial state in different inertial frames can no longer be related to each other by Eq. (5.8), as there might be collapses in the region enclosed between σ_0 and $\sigma'_{0'}$, as shown in figure 5.5. Then to verify Eq. (5.20) the position of all collapses in this region must be known; since this region includes points which are in the future of x_0 in $\mathcal{F}$, this is not possible.

One should notice the difference between standard quantum mechanics and spontaneous collapse models. Standard quantum mechanics has space-like collapses.

However as discussed in section 5.1.2 this is consistent with relativity due to the position of collapses being given. In spontaneous collapse models the position of collapses are probabilistic and are not know *a priori* and hence it cannot be taken for granted that initial conditions in two different inertial frames can be related. For space-like spontaneous collapses comparing initial conditions between two inertial frames is equivalent to requiring knowledge of future points of collapse in one of the inertial frames, as there might be collapses between two constant time hypersurfaces, see figure 5.5. Stochastic theories cannot meet this requirement, as the points of collapse are a single realisation of a random process and hence cannot be determined with certainty.

The convention that the initial collapse occurs at the origin has been taken. Since this is just a choice of coordinate system one would expect that the results discussed hold regardless of the choice of origin. Since in two different inertial frames $\mathcal{F}$ ($\mathcal{F}'$) the initial conditions are an initial point of collapse x (x') displaced from the origin and a state ψ_0 ($\psi_{0'}$) on the hypersurface intersecting it, then the same rules for relating the two initial conditions as in the case of collapse at the origin can be applied.

An additional reason that space-time collapses are not permitted is that, as stated before, for a relativistic GRW model the intervals between collapses must be a function of an Lorentz invariant interval. If space-like separated collapses are permitted in the theory then the Lorentz invariance interval must be permitted to have negative values i.e. the Poissonian point process must also produce negative values as these are space-like four-distances. This of course means that some intervals will have a negative time component meaning that there is a negative time interval between collapses, i.e. the future is predicting the past. This would make the theory unworkable, as it would not be possible to iteratively construct the probability distribution for the location of each collapse, i.e. eq. (5.19). This reasoning holds true even if one does not require that initial conditions are comparable.

Therefore single particle spontaneous collapse models can meet the condition in Eq. (5.20) when collapses are time-like to each other, but for space-like collapses the initial condition for observers in two frames cannot be compared and the condition is not satisfied. For the model proposed in [60] for a single particle the collapses are time-like and hence it is a viable relativistic GRW model. Of course one does not expect space-like collapses for single particles as this would imply superluminal velocities, however this observation is relevant for multi-particle spontaneous collapse models.

5.2.2 Relativistic condition for N distinguishable particles

The natural generalization of the previous model to the N distinguishable particle case is to assume that there are N series of collapses and hence N realisations of the stochastic process. The i^{th} realisation is: $S_i = \{T_{i1}, T_{i2}...\}$. For each realization, the construction of the collapse process—where they occur and how they change the state—is the same as for the single particle case. Note that in general, points of

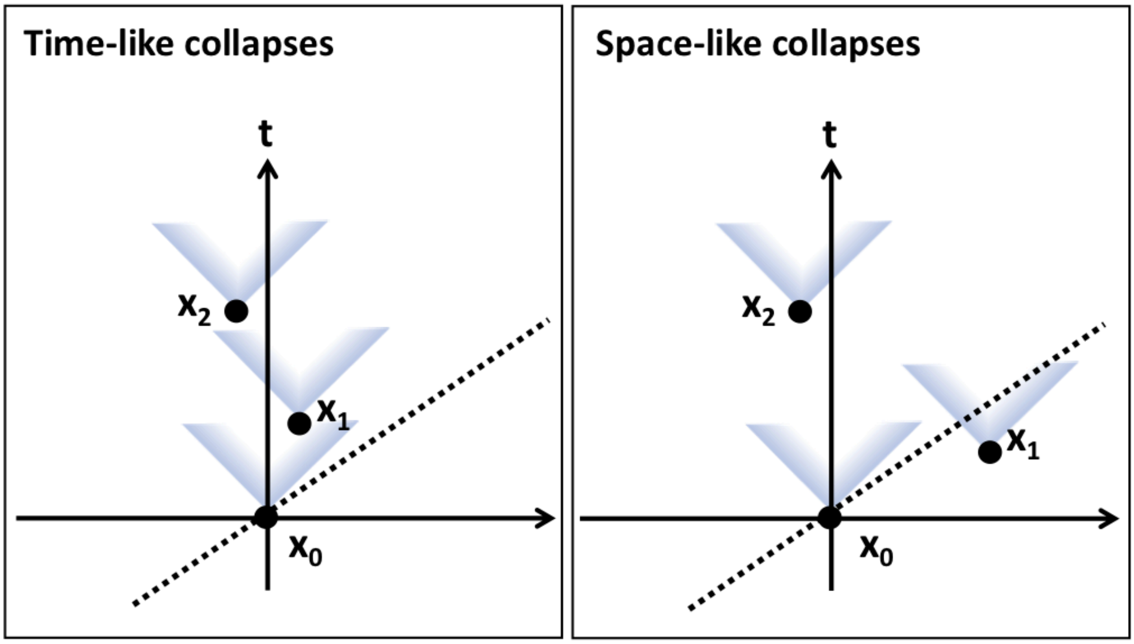


Figure 5.5: For time-like collapses (left) initial conditions between two inertial frames can always be related by unitary evolution as, for an initial collapse x_1, there can be no collapses between constant time hypersurfaces intersecting x_1, σ_0 and $\sigma'_{0'}$. For space-like collapses (right) then initial conditions between frames may not be related since there may be points of collapse between σ_0 and $\sigma'_{0'}$ e.g. x_1.

collapse associated to different particles can be space-like separated, while points of collapse associated to the same particle are always time-like to each other in a series. The space-like collapses here do not present an issue with the intervals of the stochastic process as each realisation of the stochastic process S_i still only needs to contain four-distances with positive time intervals. The Hilbert space for N distinguishable particles is given by:

$$H = \underbrace{H_1 \otimes H_2 ... \otimes H_N}_{N}, \tag{5.21}$$

where H_i is a single particle Hilbert space for the i^{th} particle. The state on a hypersurface may be written as $\Psi_\sigma \in H$.

Then in this case, the condition for consistency with special relativity is that:

$$P(x_{11}, ..., x_{i1}, ...x_{N1} | x_{10}, ..., x_{i0}, ..., x_{N0}, \Psi_{\sigma_0}) = \\ P(x'_{11}, ..., x'_{i1}, ...x'_{N1} | x'_{10}, ..., x'_{i0}, ..., x'_{N0}, \Psi'_{\sigma'_{0'}}) \tag{5.22}$$

where for the i^{th} series of collapses the collapse at x_{i0} is followed by a collapse at x_{i1} and Ψ_{σ_0} is the multiparticle state on a constant time hyperplane at the initial time. In the single particle case the initial state was defined on a hyperplane intersecting the initial point of collapse. As for multiple particles there are many initial points of collapse, it is not immediately obvious which hypersurface the initial state should be defined on. The model can be defined consistently if in frame $\mathcal{O}$ the initial hypersurface intersects the earliest point of collapse in that frame for that generation of collapses, in this case the earliest point within the group $\{x_{10}, ..., x_{i0}, ..., x_{N0}\}$. So Ψ_{σ_0} is the state on the hyperplane intersecting the earliest x_{i0}. The relation Eq. (5.22) should hold true for every value of j, not only when $j = 1$, however, since the model is Markovian, the relation can be easily iterated and checked for any pair of consecutive collapses.

A necessary requirement for Lorentz invariance of the probability distribution is that the distance between each x_{ij} and x_{ij+1} is a time-like 4-distance given by T_{ij}.

As in the case of the single particle sector, in order for eq. (5.22) to be satisfied then it must be possible to relate the initial states Ψ_{σ_0} and $\Psi'_{\sigma'_{0'}}$ in any two inertial frames. Again we note that $\sigma'_{0'}$ is a different hypersurface than σ_0 so the states Ψ_{σ_0} and $\Psi'_{\sigma'_{0'}}$ are not just one state expressed in different coordinate systems. For a completely generic initial state Ψ_{σ_0}, which may be entangled, then collapses for any particle may affect the probability for the collapse of another particle via entanglement. As collapses for different particles may be space-like to each other then the initial state in one frame Ψ_{σ_t} cannot in general be related to $\Psi'_{\sigma'_{t'}}$, as is shown in section IV of [4].

We will now give a simple illustrative example of an entangled initial state where two observers cannot compare initial conditions, but for a more rigorous explanation see [4]. Consider the situation shown in figure 5.6 with a system of two distinguishable particles. At time $t = 0$ there are two collapses, at X_1 for particle 1 and at X_2 for particle 2. We assume by fiat that immediately after this the system is in the entangled state:

$$|\psi_{\sigma_0}\rangle = \frac{1}{\sqrt{2}}\Big(|L\rangle_1|L\rangle_2 + |R\rangle_1|R\rangle_2\Big) \tag{5.23}$$

where the subscripts refer to the particle number and $|L\rangle$ is a localised state centred on the left and $|R\rangle$ is a localised state centred on the right. We assume that their centres are sufficiently far apart such that $\langle L|R\rangle \approx 0$. The entanglement ensures that any further collapse will localise both particles, for example if particle 2 collapses to $|R\rangle$ then particle 1 will also collapse to $|R\rangle$. This is a similar situation to the well known Bell locality scenario.

For a GRW type model in frame $\mathcal{F}$, the probability of particle 1 collapsing at Y_1 can be given by knowing the state $|\psi_{\sigma_0}\rangle$ and the position of the previous collapse X_1. However in frame $\mathcal{F}'$, the initial state must be given on a constant time hypersurface $\sigma'_{0'}$ in that frame. As can be seen from figure 5.6 particle 2 may have already collapsed in $\mathcal{F}'$, for example at Y_2, which would also affect particle 1.

So, in order to compare the initial states on σ_0 and $\sigma'_{0'}$ the position of collapses between them (in this case Y_2) must be known, which would include collapse in the future for frame $\mathcal{F}$. Since in principle it is not possible to specify the position of future collapses for a stochastic theory, the two initial conditions cannot be compared.

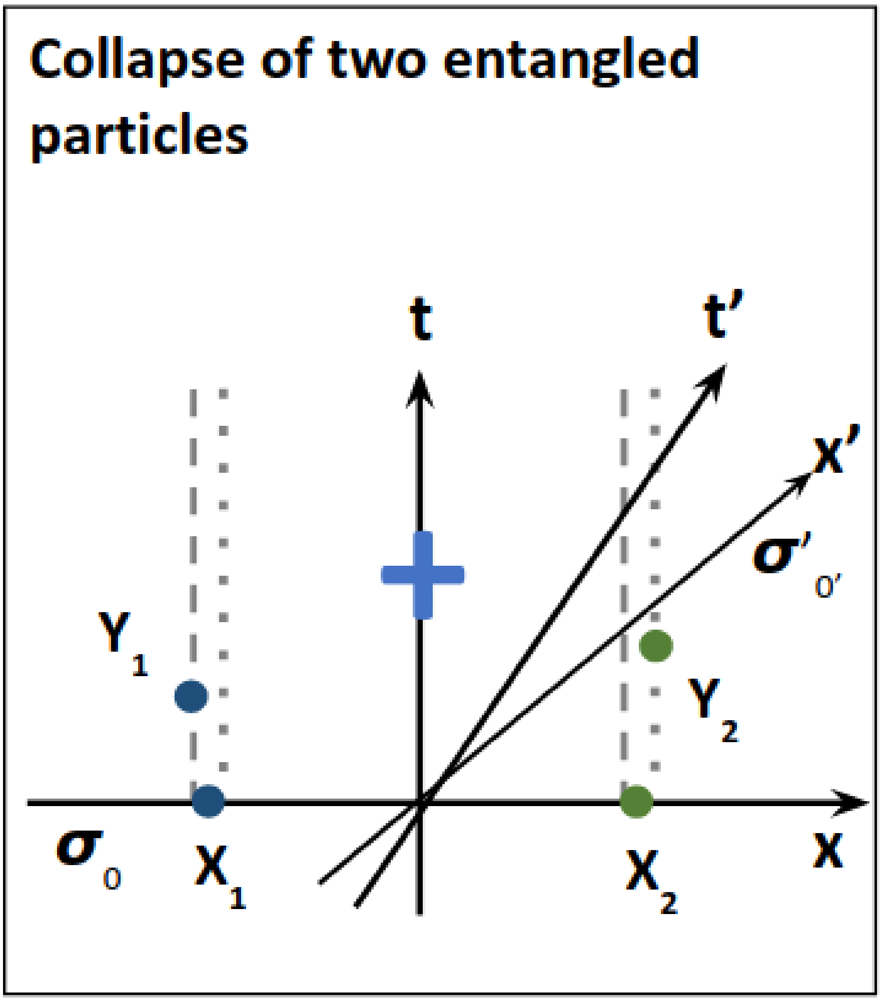


Figure 5.6: Schematic diagram showing two entangled distinguishable particles. The time and space axis are shown for two different frames. The dashed and dotted lines show the support for each part of the state without collapses (see Eq. (5.23)). Points X_1 and X_2 are the known initial points of collapse and Y_1 and Y_2 are possible future points of collapse.

A system described by conventional quantum mechanics (with observers performing measurements) with initially entangled states and interacting dynamics can be consistent with special relativity and have its initial conditions compared between frames, so why is it not possible for a GRW type model? The reason is the same one explained in section 5.2.1 for single particles. We have argued that to be consistent with special relativity it is necessary to be able to specify the position of each particle's initial point of collapse. However, this is equivalent to requiring knowledge of future points of collapse in some inertial frames, which is not permitted in stochastic models. In conventional quantum mechanics the position of the initial collapses is specified, as these are the positions of the external observer's measurements, but for a stochastic model like the GRW model this assumption cannot be made.

In conclusion, there cannot be a special relativistic GRW model for entangled distinguishable particles. For the special case of non-interacting particles in a separable state it is possible to have a relativistic GRW. We now discuss the two cases more in detail.

Non-interacting separable particles

For non-interacting particles the unitary evolution operator between two surfaces, ω_1 ω_2, may be written:

$$W_{\omega_1}^{\omega_2} = \underbrace{U_{\omega_1,1}^{\omega_2} \otimes U_{\omega_1,2}^{\omega_2} \otimes U_{\omega_1,i}^{\omega_2} \ldots}_{\text{N terms}} \tag{5.24}$$

Where $U_{\omega_1,i}^{\omega_2}$ is the unitary operator for the i^{th} particle. The collapse operator for the the i^{th} particle is:

$$\hat{L}_{\omega,i}(x) = \underbrace{\mathbb{I} \otimes \mathbb{I} \ldots}_{\text{i terms}} \otimes \hat{L}_{\omega}(x) \otimes \underbrace{\ldots \mathbb{I} \otimes \mathbb{I}}_{\text{N-i-1 terms}} \tag{5.25}$$

where $\hat{L}_{\omega}(x)$ is the collapse operator for a single particle (here for simplicity we assume that the form of each collapse operator is the same for every particle). There are N of such operators.

For a *separable* initial condition:

$$\Psi_{\sigma_0} = \psi_{\sigma_0,1} \otimes \psi_{\sigma_0,2} \ldots \otimes \psi_{\sigma_0,N} \tag{5.26}$$

then each side of Eq. 5.22 can be factorised into N distributions of the form:

$$P(x_{i1}, |x_{i0}, \psi_{\sigma_0,i}) = \|\hat{L}_{\sigma_{i1}}(x_{i1}) U_{\sigma_0,i}^{\sigma_{i1}} \psi_{\sigma_0,i}\|^2. \tag{5.27}$$

where σ_{ij} is the hyperplane intersecting the point x_{ij}. For consistency with special relativity each $P(x_{i1}, |x_{i0}, \psi_{\sigma_0,i})$ must satisfy Eq. (5.20). If each particle has a series of collapses that are time-like to each other, and the collapse operator $\hat{L}_{\omega,i}(x)$

transforms as in Eq. (5.11), then the model is consistent with special relativity. The model presented in [60] with a separable initial state meets this condition.

If the initial state Ψ_{σ_0} is *not separable* then Eq. (5.22) will not be factorable. If it is not factorable then the initial state Ψ_{σ_1} cannot be specified in a frame independent way and hence the model cannot be consistent with special relativity.

Interacting particles

If the particles are interacting and the state is initially separable, then the unitary operator cannot be decomposed as in Eq. (5.24). In this case the condition for Eq. (5.22) to be factorable is:

$$[W_{\omega_1}^{\omega_2}, \hat{L}_{\omega,i}(x)] = 0 \implies [\hat{H}, \hat{L}_{\omega,i}(x)] = 0 \tag{5.28}$$

where $\hat{H}$ is the Hamiltonian for the system (both the free and interacting parts). As is well known, if an operator commutes with the Hamiltonian then it corresponds to a globally conserved quantity. Therefore, if the condition of Eq. (5.28) holds then $\hat{L}_{\omega,i}(x)$ is a global operator and hence cannot be a local function of the fields. In this case, it has been shown that the dynamics does not result in a successful collapse model [2]. For example if the collapse operator is $\hat{H}$ then the collapse rate is proportional to the distance between the energy eigenvalues of the system, see Eq. 21 of [1]. For systems in spatial superpositions but with degenerate energy eigenstates, then the model would not predict any collapse. This would fail to solve the measurement problem as it would not lead to a reduction in the state for situations where we observe that the state collapses.

Therefore, since Eq. (5.22) is not factorable, then one is faced with the same problem as the non-separable state, the initial state Ψ_{σ_0} will be different in different frames due to the interaction. Hence it is not possible to have a special relativistic GRW model for interacting distinguishable particles.

5.2.3 Relativistic condition for indistinguishable particles

A relativistic GRW model for indistinguishable particles must only have a single collapse operator, $\hat{L}_\sigma(x)$, which acts over every particle, to preserve the particle interchange symmetry or anti-symmetry for bosons and fermions respectively. Due to this, indistinguishable particles have the same relativistic condition as a single particle, namely that the stochastic process gives the 4-distance between points of collapse and Eq. (5.20) holds, where ψ_σ is an element of an N particle Fock space. From this, the requirements Eq.(5.11) and Eq. (5.13) are derived. If the collapses are time-like to each other then Eq. (5.8) can be used as the initial condition in one frame, the position of last collapse x_0 and the state ψ_{σ_0} can be related to the initial condition in a different frame.

Conversely ,if the collapses are space-like to each other then Eq. (5.8) does not hold and also it is possible for the time-interval between some points becomes negative, as described in section 5.2.2. So such a model is not consistent with special relativity.

5.2.4 Emergence of macroscopicality for indistinguishable particles

As has been discussed in section 5.2.3 a relativistic GRW model for indistinguishable particles is possible if each collapse is time-like to the previous one. However such a model has an issue. If, given a point of collapse, x_0, the only region that the subsequent collapse can occur in is the future light cone of x_0 then macroscopic classicality is not recovered. This can be seen with a simple example with two macroscopic objects. Suppose there is a system made up of a large number of indistinguishable particles N, where N is an even number. The initial state of the system is two macroscopic objects, i.e. two areas with high densities of particles, with a large distance separation between the centre of mass of these two areas, labelled $2d$, as in figure 5.7. Assume that initially each object is in a spatial superposition, separated by a distance $2r$, where $r \ll d$. For simplicity we will work in one dimension but the argument can be extended to three dimensions. We will work in the framework of second quantization.

The initial state of the system on a constant time hypersurface σ_0 is:

$$|\Psi_{\sigma_0}\rangle = \frac{1}{2}(\hat{A}_1 + \hat{A}_2)(\hat{B}_1 + \hat{B}_2)|0\rangle \tag{5.29}$$

where $|0\rangle$ is the vacuum of a N particle anti-symmetric Fock space and:

$$\hat{A}_1 = \prod_{n=0}^{N/2} \hat{g}(-d-r,n) \tag{5.30a}$$

$$\hat{A}_2 = \prod_{n=0}^{N/2} \hat{g}(-d+r,n) \tag{5.30b}$$

$$\hat{B}_1 = \prod_{n=0}^{N/2} \hat{g}(d-r,n) \tag{5.30c}$$

$$\hat{B}_2 = \prod_{n=0}^{N/2} \hat{g}(d+r,n) \tag{5.30d}$$

where:

$$\hat{g}(x,n) = \hat{a}^{\dagger}(x - N\epsilon/4 + n\epsilon) \tag{5.31}$$

were ϵ is a distance such that $N\epsilon/2 \ll r \ll d$. Additionally assume that the distance scale of the collapse is much less than the size of the superposition: $1/\alpha \ll r$. So the operator $\hat{A}_1$ acting on the vacuum creates $N/2$ fermions, each displaced a distance ϵ from each other, centred about the point $-d-r$ see figure 5.8, and similarly for $\hat{A}_2$, $\hat{B}_1$ and $\hat{B}_2$.

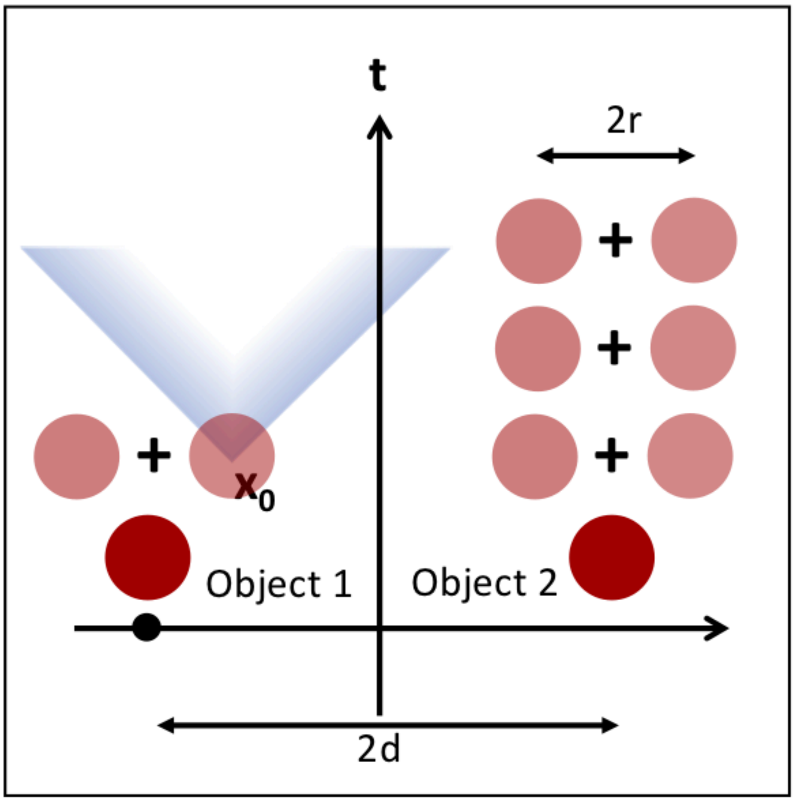


Figure 5.7: Schematic spacetime diagram showing the evolution of a pair of space-like separated macroscopic objects separated by distance $2d$. For time-like collapses if there is a collapse at point x_0 the next collapse must occur in the future light cone of x_0 (shaded grey area), and therefore the object on the right will stay in a superposition.

The number operator for the whole system is:

$$\hat{N}_T = \int_{-\infty}^{\infty} dx\, \hat{a}^\dagger(x)\hat{a}(x). \tag{5.32}$$

The number operator for the left part of the system is:

$$\hat{N}_A = \int_{-\infty}^{0} dx\, \hat{a}^\dagger(x)\hat{a}(x), \tag{5.33}$$

With a the equivalent definition for $\hat{N}_B$. Finally the number operator for the region to the left of $-d$ is:

$$\hat{N}_{A_1} = \int_{-\infty}^{-d} dx\, \hat{a}^\dagger(x)\hat{a}(x). \tag{5.34}$$

The initial state $|\Psi_{\sigma_0}\rangle$ is an eigenstate of the number operator for the total system:

$$\begin{aligned}
&\hat{N}_T|\Psi_{\sigma_0}\rangle \\
&= \int_{-\infty}^{\infty} dx\, \hat{a}^\dagger(x)\hat{a}(x)\frac{1}{2}(\hat{A}_1 + \hat{A}_2)(\hat{B}_1 + \hat{B}_2)|0\rangle \\
&= \frac{1}{2}(N\hat{A}_1\hat{B}_1 + N\hat{A}_1\hat{B}_2 + N\hat{A}_2\hat{B}_1 + N\hat{A}_2\hat{B}_2)|0\rangle \\
&= N|\Psi_{\sigma_0}\rangle.
\end{aligned}$$

The initial state is also an eigenstate of the number operator for the left part of the system:

$$\begin{aligned}
\hat{N}_A|\Psi_{\sigma_0}\rangle &= \int_{-\infty}^{0} dx\, \hat{a}^\dagger(x)\hat{a}(x)\frac{1}{2}(\hat{A}_1 + \hat{A}_2)(\hat{B}_1 + \hat{B}_2)|0\rangle \\
&= \frac{N}{2}|\Psi_{\sigma_0}\rangle
\end{aligned}$$

and similarly for the right part of the system $\hat{N}_B|\Psi_{\sigma_0}\rangle = N/2|\Psi_{\sigma_0}\rangle$. However, the initial state is not in an eigenstate of the number operator for the region to the left of $-d$:

$$\begin{aligned}
\hat{N}_{A_1}|\Psi_{\sigma_0}\rangle &= \int_{-\infty}^{-d} dx\, \hat{a}^\dagger(x)\hat{a}(x)(\hat{A}_1\hat{B}_1 + \hat{A}_1\hat{B}_2 + \hat{A}_2\hat{B}_1 + \hat{A}_2\hat{B}_2)|0\rangle \\
&=\frac{1}{2}(\frac{N}{2}\hat{A}_1\hat{B}_1 + \frac{N}{2}\hat{A}_1\hat{B}_2 + \mathbb{I} + \mathbb{I})|0\rangle
\end{aligned} \tag{5.36}$$

which is not proportional to $|\Psi_{\sigma_0}\rangle$. $|\Psi_{\sigma_0}\rangle$ is also not an eigenstate of $\hat{N}_{A_2}$, $\hat{N}_{B_1}$ or $\hat{N}_{B_2}$. This implies there are two objects, each in a superposition over two areas, rather than one object in a superposition over four areas or four objects each in a localised position.

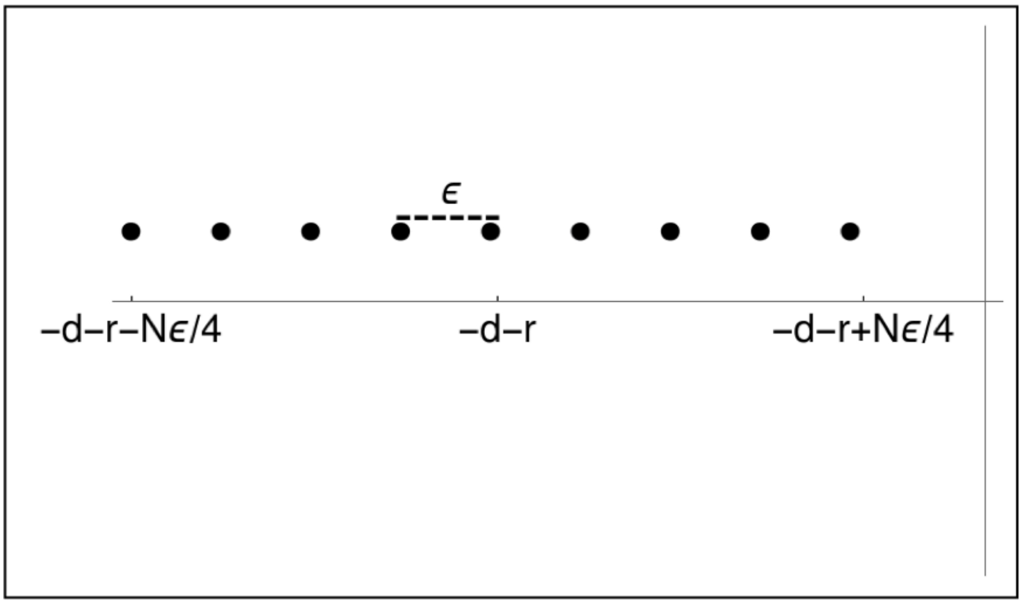


Figure 5.8: Diagram showing the action of the operator $\hat{A}_1$ on the vacuum, were particles are created separated by distance ϵ.

The amplification mechanism will cause a collapse of one of the objects almost immediately. Suppose that the collapse is at spacetime point $(t, -d+r)$, where t is so small that $U_{\sigma_0}^{\sigma_t} \approx \mathbb{I}$. Then following Eq. (5.5) we find the state immediately after the collapse, on the constant time hypersurface σ_t, to be:

$$|\Psi_{\sigma_t}\rangle = \frac{\hat{J}_{\sigma_t}(-d+r)|\Psi_{\sigma_0}\rangle}{\|\hat{J}_{\sigma_t}(-d+r)|\Psi_{\sigma_0}\rangle\|^2} \tag{5.37}$$

where $\hat{J}_{\sigma_t}(x)$ is an approximation for the form of a relativistic collapse operator $\hat{L}_{\sigma_t}(x)$ in the limit of low velocity particles. The form of $\hat{J}_{\sigma_t}(x)$ is:

$$\hat{J}_{\sigma_t}(x)|\Psi_{\sigma_0}\rangle = \int_{-\infty}^{\infty} dy\, K(y) f_\alpha(x-y)\hat{a}^\dagger(y)\hat{a}(y)|\Psi_{\sigma_0}\rangle \tag{5.38}$$

where $f_\alpha(x)$ is a function sharply peaked about $x=0$ with a width proportional to $1/\alpha$ and $K(y)$ is a normalisation function. This form ensures that particles are localised about the point of collapse. To evaluate Eq. (5.37) consider just the term:

$$\begin{aligned} &\hat{J}_{\sigma_t}(-d+r)\hat{A}_1\hat{B}_1|0\rangle \\ &= \frac{1}{2}\int_{-\infty}^{\infty} dy\, K(y) f_\alpha(-d+r-y) \times \hat{a}^\dagger(y)\hat{a}(y)\prod_{n=0}^{N/2}\prod_{m=0}^{N/2}\hat{g}(-d-r,n)\hat{g}(d-r,m)|0\rangle \end{aligned} \tag{5.39}$$

is needed. The contributions from the $\hat{g}(-d-r,n)$ and $\hat{g}(d-r,m)$ operators are weighted by factors of $f_\alpha(-2r+n\epsilon/4)$ and $f_\alpha(2d-2r+n\epsilon/4)$ respectively. As

$-2r + n\epsilon/4 \gg 1/\alpha$ and $2d - 2r + n\epsilon/4 \gg 1/\alpha$ then $f_\alpha(-2r + n\epsilon/4) \approx 0$ and $f_\alpha(2d - 2r + n\epsilon/4) \approx 0$. Hence :

$$\hat{J}_{\sigma_t}(-d + r)\hat{A}_1\hat{B}_1|0\rangle \approx 0. \tag{5.40}$$

A similar suppression occurs for $\hat{J}_{\sigma_t}(-d+r)\hat{A}_1\hat{B}_2|0\rangle$. However, the terms $\hat{A}_2\hat{B}_1 + \hat{A}_2\hat{B}_2$ are not suppressed in this case as the integral f_α is approximately 1 for the part of the state centred on $-d + r$. Therefore, we are left with:

$$\hat{J}_{\sigma_t}(-d + r)|\Psi_{\sigma_0}\rangle \approx \frac{N}{4}\hat{A}_2(\hat{B}_1 + \hat{B}_2)|0\rangle \tag{5.41}$$

and therefore:

$$|\Psi_{\sigma_t}\rangle \approx \frac{1}{\sqrt{2}}\hat{A}_2(\hat{B}_1 + \hat{B}_2)|0\rangle \tag{5.42}$$

This describes how object 1 has been collapsed but object 2 remains in a superposition. Object 2 will be left in a superposition for approximately $2d/c$ seconds, where c is the speed of light, shown in figure 5.7. If d is sufficiently large then one of the macroscopic objects will remain in a spatial superposition for an arbitrarily long time, in violation of what we observe in nature.

To avoid this problem for macroscopic objects then a collapse model must permit space-like points of collapse. If there are space-like collapse points, then the position of the initial collapse does not limit the region of possible collapses. Hence any region with a high-average number density of particles is almost certain to have a collapse occur within it in a short time interval. However, as discussed in section 5.2.3, if one attempts to include space-like collapses into the indistinguishable particle model suggested here, then the model is not consistent with special relativity.

5.3 Conclusion

In this chapter we have considered the GRW model and its consistency with special relativity. We have emphasised that for a model to be consistent with special relativity the dynamics must be Lorentz covariant and initial conditions in different inertial frames must be able to be related; we have then applied these requirements for the case of relativistic GRW models. We have also shown that the interval between points in a series of collapses in GRW models must be given by a Lorentz invariant space-time interval. From this we concluded that these points in a series must then be time-like to each other to prevent negative time intervals between collapes, which would render the theory inconsistent.

In table 5.1 we summarise the conclusions of this work, showing for which cases a relativistic GRW model is possible.

We have shown that a relativistic GRW is possible for single particles and non-interacting, non-entangled distinguishable particles, as due to the fact that the collapses for each particle are time-like to each other, the initial conditions can be related in different inertial frames. However, for entangled, non-interacting, distinguishable particles, as entanglement implies that space-like collapses for one particle can affect the probability of collapse of another particle the initial conditions in different inertial frames cannot be related. For indistinguishable particles, either the collapses are space-like and hence not compatible with special relativity, or the collapses are time-like and the recovery of macroscopic classicality is not for guaranteed hence such a model is not a viable collapse model. For interacting particles as the interactions can entangle the particles, the initial conditions in two frames cannot be related, and therefore there is not a relativistic GRW model for interacting particles.

One thing to note is that this chapter only considers the fixed particle sector, and a completely relativistic collapse model must also describe changes in particle number. This limitation will be further discussed in the conclusion.

Summary of Conclusions		
Particle Type	Separable State	Entangled State
Single	Yes	N/A
N distinguishable non-interacting	Yes	No
N indistinguishable non-interacting	No	No
Interacting	No	No

Table 5.1: A table showing the regimes where a relativistic GRW model is possible.

Chapter 6

An Attempted Relativistic CSL Model

6.1 Introduction

In this chapter we will attempt to construct a relativistic extension of the CSL model. The motivation for doing this is that CSL, by some measures, is the most studied continuous spontaneous collapse model.

As we have seen in the previous chapter it is not possible to find a relativistic extension to GRW, a model with a discrete stochastic process, we therefore ask if it is possible to have one for a continuous process. This is done by starting from the most general form of a continuous time spontaneous collapse model and then applying a set of minimal requirements for a successful relativistic generalization. We will consider the case of scalar bosons but the results of this work are not expected to differ for fermions. These requirements are: that the dynamical map collapses in the mass density basis covariantly, that the rate of change of energy is finite, and that microcausality is respected. It is found that it is not possible for all the requirements to be simultaneously satisfied.

This work is laid out as follows: in section 6.2 the requirements for a continuous model to be consistent with special relativity are applied to the general non-Markovian form of the continuous spontaneous collapse model. It is shown that the requirements for this map to be Lorentz covariant are that the two point correlation function is a function of the invariant 4-distance and that the collapse operator is a Lorentz scalar.

The collapse operator is specified in section 6.3. In section 6.4 it is shown that the rate of change of the energy under the map is only finite if the two point correlation function tends to zero for high momentum. Finally in section 6.5 it is shown how under the above requirements such a dynamics permits superluminal signalling.

In novel content in this chapter is the selection of the minimal set of requirements that relativistic CSL must satisfy and all calculations; including the checks for Poincaré covariance, the finite rate of energy change, and microcausality.

6.2 Lorentz covariant CSL

As discussed in chapter 4, in order for a continuous model to be consistent with special relativity we may only consider systems whose initial conditions can be compared via unitary Poincaré transformations and where the map transforms under the unitary evolution, see eq. (4.13). It is this second requirement that is relevant here, as we wish to restrict the general form of the collapse model.

Since the map $\mathcal{U}_t^0$ describing the standard unitary quantum evolution is Lorentz covariant, we are left to verify the covariance of the collapse contribution $\mathcal{M}_t$. Therefore, in the rest of this work we will work with the map $\mathcal{M}_t$ and therefore be in the interaction picture.

With the help of Eq. (3.14) and Eq. (3.15) and by exploiting the unitarity of $\mathcal{S}_0(\Lambda)$, it is straightforward to show that Eq. (4.13) is satisfied if and only if the following condition is satisfied:

$$\begin{aligned}
&\overleftarrow{\mathrm{T}}\exp\Bigg\{\int_{\Omega_t} d^4x \int_{\Omega_t} d^4y D(x,y)\Big[\mathcal{S}_0(\Lambda)[\hat{Q}^L(y)]\mathcal{S}_0(\Lambda)[\hat{Q}^R(x)] \\
&\qquad -\theta(x^0-y^0)\mathcal{S}_0(\Lambda)[\hat{Q}^L(x)]\mathcal{S}_0(\Lambda)[\hat{Q}^L(y)]-\theta(y^0-x^0)\mathcal{S}_0(\Lambda)[\hat{Q}^R(y)]\mathcal{S}_0(\Lambda)[\hat{Q}^R(x)]\Big]\Bigg\} \\
&=\overleftarrow{\mathrm{T}}\exp\Bigg\{\int_{\Lambda(\Omega_t)} d^4x \int_{\Lambda(\Omega_t)} d^4y D(x,y) \\
&\qquad \Big[\hat{Q}^L(y)\hat{Q}^R(x)-\theta(x^0-y^0)\hat{Q}^L(x)\hat{Q}^L(y)-\theta(y^0-x^0)\hat{Q}^R(y)\hat{Q}^R(x)\Big]\Bigg\} \qquad (6.1)
\end{aligned}$$

Let us consider the case when the operator $\hat{Q}$ is a Lorentz scalar:

$$\mathcal{S}_0(\Lambda)[\hat{Q}(y)]=\hat{Q}(\Lambda(y)). \qquad (6.2)$$

In this case the condition in Eq. (6.1) is met only if $D(x,y)=D(|x-y|)$ [56], in which case $D(x',y')=D(x,y)$ where $x'=\Lambda(x)$.

Finally, we note that as $D(x,y)$ is a function of the 4-distance, then it is symmetric under exchange of x and y. This allows Eq. (3.15) to be simplified to:

$$\mathcal{L}_t=\int_{\Omega_t} d^4x \int_{\Omega_{x^0}} d^4y\, D(x,y)\,[\hat{Q}^L(y)\hat{Q}^R(x)+\hat{Q}^L(x)\hat{Q}^R(y)-\hat{Q}^L(x)\hat{Q}^L(y)-\hat{Q}^R(x)\hat{Q}^R(y)] \qquad (6.3)$$

which is a form which makes calculations easier to perform.

6.3 Choice of the collapse operator

As discussed in Section II, the choice of the operator $\hat{Q}(x)$ determines the basis that spontaneous collapse occurs in. We seek a model that is relativistically invariant and at the same time is able to reproduce the CSL dynamics in the non relativistic limit. A natural choice is to select $\hat{Q}(x)$ to be the relativistic mass density operator, i.e.

$$\hat{Q}(x) = m\hat{a}^{\dagger}(x)\hat{a}(x) \tag{6.4}$$

where $\hat{a}^{\dagger}(x)$ is the positive energy part of the Klein-Gorden field operator. From now on we take Eq. (6.4) to be the collapse operator.

6.3.1 Ontology of relativistic CSL

As in the non-relativistic case (see section 3.4), only may wish to proposed an ontology for relativistic CSL. The beable proposed for the non-relativistic case is the averaged density, eq. (3.12), however this beable is an average over space but not time, so is clearly not Lorentz covariant. However the reason for the averaging, to ensure that the beable cannot be in superpostions, is still required, so one cannot simply use the raw mass density eq. 6.4. One suggestion for a relativistic beable has been proposed in [12], see section 3.12, however there has not been an conclusive discussion of this. So currently the existence of an appropriate relativistic beable for CSL is still an open question.

Previous attempts to derive relativistic extensions of CSL model have encountered the problem of an infinite energy rate in the dynamics. In the next section we will find a necessary condition to avoid this unpleasant feature in a relativistic collapse model.

6.4 Finiteness of rate of change of energy

An important physical requirement for the dynamics to satisfy is that the rate of change of energy is finite. Some previous attempts to find a relativistic collapse model have suffered from this divergence [10, 13]. Here we show that the dynamics given by Eq. (3.15) with the choice of Eq. (6.4) as the collapse operator does not lead to a divergent rate of change of energy provided that the correlator prevents large transfers of momentum between momentum eigenstates. We show this for a single particle system in the weak coupling regime.

The rate of change of the energy is given by:

$$\frac{d}{dt}\mathrm{Tr}(\hat{H}\mathcal{M}_t[\hat{\rho}]) = \frac{d}{dt}\mathrm{Tr}(\hat{H}\ \overleftarrow{\mathrm{T}}\exp\lambda\mathcal{L}_t[\hat{\rho}]), \tag{6.5}$$

where $\hat{H}$ is the Hamiltonian for the unitary dynamics [49]:

$$\hat{H} = \int \frac{d\mathbf{p}}{(2\pi)^3}\, E_{\mathbf{p}}\hat{a}^{\dagger}_{\mathbf{p}}\hat{a}_{\mathbf{p}} \tag{6.6}$$

where $E_{\mathbf{p}} = \sqrt{m^2 + \mathbf{p}^2}$, $\mathbf{p}$ is the three momentum and $\hat{a}_{\mathbf{p}}$ and $\hat{a}^\dagger_{\mathbf{p}}$ are the creation and annihilation operators in momentum eigenbasis which satisfy the following commutation relation:

$$[\hat{a}_{\mathbf{p}}, \hat{a}^\dagger_{\mathbf{q}}] = (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{q}) \tag{6.7}$$

We expand the map up to the first order in the coupling γ, to obtain:

$$\overleftarrow{\mathrm{T}} \exp \gamma \mathcal{L}_t[\hat{\rho}] = \mathbb{I} + \gamma \mathcal{L}_t[\hat{\rho}] + \mathcal{O}(\gamma^2). \tag{6.8}$$

Exploiting this and dropping terms of orders higher than γ gives:

$$\frac{d}{dt}\mathrm{Tr}(\hat{H}\mathcal{M}_t[\hat{\rho}]) \approx \gamma \mathrm{Tr}(\hat{H}\mathcal{L}_t[\hat{\rho}]) \tag{6.9}$$

We evaluate Eq. (6.9) in the single particle sector, where the state of the system can be expressed as:

$$\hat{\rho} = \int d\mathbf{p} \int d\mathbf{q}\, A(\mathbf{p}, \mathbf{q})\, \hat{a}^\dagger_{\mathbf{p}} |0\rangle\langle 0| \hat{a}_{\mathbf{q}} \tag{6.10}$$

where $A(\mathbf{p}, \mathbf{q})$ is an arbitrary function such that:

$$\mathrm{Tr}(\hat{\rho}) = \int d\mathbf{p}\, A(\mathbf{p}, \mathbf{p}) = 1 \tag{6.11}$$

In order to calculate Eq. (6.9) we expand the correlation function and the collapse operator in Fourier components, i.e.

$$\begin{aligned} D(\mathbf{x}, t) &= \frac{1}{(2\pi)^3} \int d\mathbf{q}\, e^{i\mathbf{x}\cdot\mathbf{q}} \tilde{D}(\mathbf{q}, t) \\ \hat{Q}(\mathbf{x}, t) &= \frac{m}{(2\pi)^6} \int d\mathbf{q} \int d\mathbf{p}\, \frac{1}{2\sqrt{E_{\mathbf{p}} E_{\mathbf{q}}}} e^{i\mathbf{x}\cdot(\mathbf{q}-\mathbf{p})} e^{-i(E_{\mathbf{q}} - E_{\mathbf{p}})t} \hat{a}^\dagger_{\mathbf{q}} \hat{a}_{\mathbf{p}} \end{aligned} \tag{6.12}$$

substituting these into Eq. (6.3) with $D(x, y) = D(|x - y|)$ and then integrating over the spatial variables $d\mathbf{x}$ and $d\mathbf{y}$ allows $\mathcal{L}_t$ be written as:

$$\begin{aligned} \mathcal{L}_t = \frac{1}{(2\pi)^{12}} \int d\mathbf{q} \int_0^t ds \int_0^s d\tau\, \tilde{D}(\mathbf{q}, s - \tau) \{ &\hat{K}^L(\mathbf{q}, s) \hat{K}^{\dagger, L}(\mathbf{q}, \tau) + \hat{K}^R(\mathbf{q}, s) \hat{K}^{\dagger, R}(\mathbf{q}, \tau) \\ &- \hat{K}^{\dagger, L}(\mathbf{q}, \tau) \hat{K}^R(\mathbf{q}, s) - \hat{K}^L(\mathbf{q}, s) \hat{K}^{\dagger, R}(\mathbf{q}, \tau) \} \end{aligned} \tag{6.13}$$

where:

$$\hat{K}(\mathbf{q}, t) = m \int \frac{d\mathbf{p}}{2\sqrt{E_{\mathbf{p}} E_{\mathbf{p}-\mathbf{q}}}} e^{i\Delta E(\mathbf{p}, \mathbf{q})t} \hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{p}-\mathbf{q}} \tag{6.14}$$

with $\Delta E(\mathbf{p}, \mathbf{q}) = E_{\mathbf{p}-\mathbf{q}} - E_{\mathbf{p}}$. Note that $\hat{K}(\mathbf{q}, t) = \hat{K}^\dagger(-\mathbf{q}, -t)$.

Then using Eq. (6.13) and Eq. (6.12) we have that Eq. (6.9) evaluates to:

$$\mathrm{Tr}(\hat{H}\mathcal{L}_t[\hat{\rho}]) =$$
$$-\frac{m^2}{(2\pi)^3}\int d\mathbf{q}\int d\mathbf{p}\int_0^t ds\int_0^s d\tau \tilde{D}(\mathbf{q}, s-\tau)\cos(\Delta E(\mathbf{p},\mathbf{q})(s-\tau))A(\mathbf{p},\mathbf{p})\left(\frac{1}{2E_{\mathbf{p}-\mathbf{q}}}-\frac{1}{2E_{\mathbf{p}}}\right) \tag{6.15}$$

We are interested to see if this integral is finite. The integrals over both $\mathbf{q}$ and $\mathbf{p}$ are over an infinite range, but if the system is assumed to initially have a finite energy then the integral over $\mathbf{p}$ will automatically converge to a finite value. Then in order for the integral over $\mathbf{q}$ to converge a necessary condition is that:

$$\lim_{|\mathbf{q}|\to\infty} \tilde{D}(|\mathbf{q}|, s-\tau) = 0. \tag{6.16}$$

Note here that if the noise is white, i.e. if $D(x, y) = \delta^4(x-y)$ then:

$$\tilde{D}(|\mathbf{q}|, s-\tau) = (2\pi)^3\delta(s-\tau) \tag{6.17}$$

and the energy rate in Eq. (6.15) diverges. Therefore, relativistic white-noise models are physically inconsistent, as already noticed in chapter 13 of [10] and [13]. Through a similar calculation it can be seen that if the correlation function $D(x-y)$ further satisfies the following condition

$$\tilde{D}(|\mathbf{q}|, t) \approx 0 \quad \text{for } \mathbf{q} > \kappa m \tag{6.18}$$

where $\kappa \gg 1$ is a fixed constant, then the map described by Eq. (3.15) is well behaved in the non-relativistic sector, i.e. it leaves non-relativistic particles non-relativistic.

Notice that condition Eq. (6.16) is a weaker condition than Eq.(6.18), hence if the model has a non-relativistic limit then it also has a finite energy rate.

6.5 Superluminal Signalling and Micro-causality

We have given the minimal requirements for the map in Eq. (6.3) to guarantee that the state collapses and has a finite energy rate. Here we check the conditions under which the model does not allow superluminal signalling by checking if the model satisfies the microcausality condition. We will show that given the requirements from sections 6.2, 6.3 and 6.4, the dynamics described by Eq. (6.3) violates microcausality. For this section we will work in the Heisenberg picture, therefore operators evolve with the dual map $\mathcal{M}_t^*[\mathcal{U}_t^{0*}[\,\cdot\,]]$.

In standard quantum mechanics where the dynamics $\mathcal{U}_t$ is unitary the microcausality condition reads

$$[\hat{A}(z_1), \hat{B}(z_2)] = 0 \qquad\qquad \forall \quad |z_1 - z_2| < 0 \tag{6.19}$$

where $\hat{A}(z_1) = \mathcal{U}_{t_1}^{0*}\hat{A}[(\mathbf{z}_1, 0)]$ and $\hat{B}(z_2) = \mathcal{U}_{t_2}^{0*}[\hat{B}(\mathbf{z}_2, 0)]$ are local operators, and z_1 and z_2 are two points in spacetime such that $z_1 = (\mathbf{z}_1, t_1)$ and $z_2 = (\mathbf{z}_2, t_2)$ where $\mathbf{z}_1$ and $\mathbf{z}_2$ are two points in 3D space. The above condition is synonymous with no superluminal signalling, because it guarantees that local measurements at space-like separated points do not influence each other. For the case of a non-unitary dynamics it is not possible to evaluate Eq. (6.19) because:

$$\mathcal{M}_{t_1}^*[\mathcal{U}_{t_1}^{0*}\hat{A}(\mathbf{z}_1, 0)\mathcal{U}_{t_1}^{0*}\hat{B}(\mathbf{z}_2, 0)] \neq \mathcal{M}_{t_1}^*[\mathcal{U}_{t_1}^{0*}\hat{A}(\mathbf{z}_1, 0)]\mathcal{M}_{t_1}^*[\mathcal{U}_{t_1}^{0*}\hat{B}(\mathbf{z}_2, 0)]. \tag{6.20}$$

In other words, for non-unitary dynamics the evolution of the product of two operators $\hat{A}$ and $\hat{B}$ cannot be described by the product of the independently evolved operators.

One way around this problem is if the non-unitary map admits a unitary stochastic unravelling $\tilde{\mathcal{U}}_t$, i.e. $\mathbb{E}(\tilde{\mathcal{U}}_t) = \mathcal{U}_t^0\mathcal{M}_t$ as in the case of the map in Eq. (6.3) (for an explicit expression of the unitary stochastic unravelling see [3, 10]).
In this case one could define a microcausality condition as:

$$\mathbb{E}([\hat{A}(z_1), \hat{B}(z_2)]) = 0 \qquad\qquad \forall \;\; |z_1 - z_2| < 0 \tag{6.21}$$

where $\hat{A}(z_1) \equiv \tilde{\mathcal{U}}_{t_1}^*[\hat{A}(\mathbf{z}_1, 0)]$ and $\hat{B}(z_2) \equiv \tilde{\mathcal{U}}_{t_2}^*[\hat{B}(\mathbf{z}_2, 0)]$ are local operators.

However the presence of the stochastic average makes the equation hard to evaluate in full generality, but for our purpose it is sufficient to restrict the study to the less general case $\hat{A} = \hat{B} = \hat{\phi}(x)$, $t_1 = t$ and $t_2 = 0$. Under these assumptions Eq. (6.21) reduces to:

$$\mathbb{E}([\tilde{\mathcal{U}}_t^*[\hat{\phi}(\mathbf{z}_1, 0)], \hat{\phi}(\mathbf{z}_2, 0)]) = [\mathcal{M}_t^*[\hat{\phi}(\mathbf{z}_1, t)], \hat{\phi}(\mathbf{z}_2, 0)] = 0 \tag{6.22}$$

where $\hat{\phi}(\mathbf{z}_1, t) = \mathcal{U}_t^{0*}[\hat{\phi}(\mathbf{z}_1, 0)]$ is the field operator evolved under the standard unitary evolution dynamics. We expand the map $\mathcal{M}_t^*$ to first order in γ. By substituting the Fourier expansion for $\hat{\phi}(x)$ (see 8.0.2) we arrive at:

$$\begin{aligned}
&[\mathcal{M}_t^*[\hat{\phi}(\mathbf{z}_1, t)], \hat{\phi}(\mathbf{z}_2, 0)] \approx [\hat{\phi}(\mathbf{z}_1, t) + \gamma\mathcal{L}_t^*[\hat{\phi}(\mathbf{z}_1, t)], \hat{\phi}(\mathbf{z}_2, 0)] \\
&\quad = [\hat{\phi}(\mathbf{z}_1, t), \hat{\phi}(\mathbf{z}_2, 0)] + \gamma \int_{\Omega_t} d^4x \int_{\Omega_{x0}} d^4y\, D(x-y)[[\hat{Q}(y), [\hat{\phi}(\mathbf{z}_1, t), \hat{Q}(x)]], \hat{\phi}(\mathbf{z}_2, 0)] \\
&\quad = [\hat{\phi}(\mathbf{z}_1, t), \hat{\phi}(\mathbf{z}_2, 0)] + \gamma m^2 \int_0^t d\tau \int d\mathbf{x} D(\mathbf{x}, \tau)\left\{F(\mathbf{z}_1 - \mathbf{z}_2, \mathbf{x}, t, \tau) - F(\mathbf{z}_2 - \mathbf{z}_1, -\mathbf{x}, t, \tau)\right\}
\end{aligned} \tag{6.23}$$

where

$$F(\mathbf{z}_1 - \mathbf{z}_2, \mathbf{x}, t, \tau) = \int \frac{d\mathbf{k}}{2E_\mathbf{k}} \int \frac{d\mathbf{k}'}{4E_{\mathbf{k}'}^2} e^{i[\mathbf{k}\cdot(\mathbf{z}_1 - \mathbf{z}_2) - E_\mathbf{k} t]} e^{-i[(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x} - (E_\mathbf{k} - E_{\mathbf{k}'})\tau]} \tag{6.24}$$

The first term is the microcausality condition for standard quantum fields while the second term is due to the non-unitary evolution. From this equation one immediately notices that the microcausality condition could be satisfied only if:

- $D(x) = 0$, i.e. the dynamics is unitary and no collapse mechanism is present, but this is not the working hypothesis of this paper
- $F(\mathbf{z}_1 - \mathbf{z}_2, \mathbf{x}, t, \tau) = F(\mathbf{z}_2 - \mathbf{z}_1, -\mathbf{x}, t, \tau)$, a simple inspection at Eq. (6.24) shows that in general this is not true
- $D(x - y) = \delta^{(4)}(x - y)$,[1].

In this case the model produces an infinite energy rate and cannot be reduced to the CSL model in the non relativistic limit, as shown in Section 6.4.

6.6 Conclusion

In this chapter we have analysed the possibility of a relativistic extension of the CSL model. The study was done by constructing a candidate relativistic generalization of the CSL model.

Starting from the prototypical structure of a generic continuous collapse model it was required that the model was Lorentz covariant, had a well defined non-relativistic sector, did not have a divergent energy rate, reduced to a mass coupled spontaneous collapse model in the non-relativistic limit and prohibits superluminal signalling. It was found that it is not possible to construct a model that fulfils all of these requirements. This is because the need for a finite rate of change of energy implies that the stochastic noise associated with the model must be coloured, however this requirement implies that the microcausality condition is violated.

[1] exploiting the invariance of integral measure $\int \frac{d\mathbf{k}}{2E_{\mathbf{k}}}$ it is not difficult to show that $F(\mathbf{z}, 0, t, \tau) = F(0, 0, t, \tau)$ for any value of z

Chapter 7

Discussion and Conclusions

In this thesis we asked the question: is a relativistic spontaneous collapse model possible? In order to answer this we first introduced the motivation behind the spontaneous collapse models and gave a brief non-technical introduction to them in chapter 1. In chapters 2 and 3, we review two of the most studied spontaneous collapse models, the GRW model and the CSL model respectively were introduced in detail. In chapter 4 we studied how to define consistency with special relativity for different spontaneous collapse models. Here we first considered special relativity and conventional quantum mechanics and then moved on to the more involved question of how collapse models which are non-unitary fit into this picture.

Our starting point for considering quantum mechanics and special relativity together was the seminal works by Albert and Aharonov [4, 5]. The key takeaway from these works is that in a relativistic setting it is not the state vector that should transform covariantly between inertial frames, but the conditional probability of local beables. This is because quantum mechanics is irreversible due to the collapses of the state and that collapses may occur at space-like locations, so their order may be frame dependant. These facts together imply that there is not a consistent way to assign the evolution of the state in every inertial frame.

It is therefore the probability for local beables, not states, that must be Poincaré covariant in different frames for agreement between quantum mechanics and special relativity.

The remainder of chapter 4 specialised to continuous spontaneous collapse models (like CSL), and considered two factors required for consistency with special relativity. The first is that initial conditions between different inertial frames must be comparable and the second is that the dynamics must be Poincaré covariant.

It was shown that it is not possible to construct a set of Poincaré transformation operators that are self consistent for non-unitary dynamics. Hence, it is not possible to compare initial states on two different hyperplanes using non-unitary dynamics. We conclude that the other option is to take inspiration from the approach in relativistic quantum field theory and to consider situations where the initial state evolves under unitary evolution.

Two approaches for extending covariance to continuous non-unitary dynamics were then given; covariance under the unitary dynamics and covariance under the complete dynamics. Covariance under the unitary dynamics can be applied but there is no underlying justification for why this would be the appropriate check for covariance.

The other approach, covariance under the complete dynamics was considered. It was shown that, as models with this dynamics require the introduction of a special initial spacetime point (rather than a initial time) from which the stochastic process is defined, there is an issue when specifying joint probabilities of space-like events. The issue is that since the time-ordering between space-like events is frame dependant, irreversible evolution between events can only be defined in some frames. This implies that single particles and non-interacting, distinguishable particles can be described using this method, but neither interacting particles nor indistinguishable particles can be.

From this evaluation we conclude that a relativistic continuous spontaneous collapse model can exist only if one considers systems that initially evolve under the unitary evolution and the definition of covariance used is covariance under the unitary dynamics. These were the assumptions made in chapter 6 where we attempted to construct a relativistic generalisation of coloured CSL.

In chapter 6 it was found that no model that did not have immediate energy divergence and did not permit superluminal signaling exists for a collapse model with the mass density operator as the collapse operator. This is in line with the spirit of result given in [8], although they consider a Markovian model and the mathematical condition they use for causality is different.

The limitations to this analysis is that it is still, in principle, possible that there exists another choice of collapse operator for this model that avoids non-physical predictions of energy divergence and faster-than-light signalling, and also has coloured CSL as its non-relativistic limit. The obvious extension to this work is to attempt to prove whether or not such an operator exists.

An additional limitation is that currently there is no candidate for a beable for relativistic CSL that has been shown to be able to give a compelling description of nature.

In chapter 5 we considered GRW-type collapse models and to what extent they can be consistent with special relativity. As in the continuous case, the condition for this is if the conditional probabilities of the relevant beables are Lorentz invariant. For GRW type models, the relevant beables are for example the locations of the points of collapse themselves.

We showed how such models can be consistent with special relativity for a series of time-like collapses, as the interval between each collapse can be specified in a Lorentz invariant way and it is possible to compare the initial state between two different inertial frames. With this framework it is possible to describe single particles. However, any GRW type model for multiple particles must have space-like collapses, in order to recover macroscopic classicality. If there are space-like collapses then the time ordering of these events are frame dependant. This causes problems because the definition of the model relies on initial points of collapse being specified and hence, if initial points (and therefore states) are frame dependant, the model can't be initialised in a frame independent way. Additionally, if there is a series of space-like collapses, it is not possible to define the intervals between collapses in a Lorentz invariant way.

Putting this all together, we see that for both discrete and continuous spontaneous collapse models there is an issue for many particle systems were the initial conditions are described by space-like beables. This is because in this case, comparing initial conditions between frames is not possible due to the irreversibly of the dynamics.

It is worth noting that a way out of this issue is to just assume by fiat that the initial conditions between two inertial frames are equivalent, and hence only require that the dynamics are Lorentz covariant, which would be considering only active transformations. We do not support this view, as discussed in section 4.2.1.

The conclusions reached in this thesis do not reach the level of a proof that a relativistic collapse model is not possible, as at least for the continuous case; if one accepts the caveats required for covariance there is still the potential for a different choice of the collapse operator and correlation function combination which destroy spatial superpositions and do not have physically unacceptable predictions. However, given the findings of this thesis, we believe that there is not much hope for such a path.

Chapter 8

Appendices

8.0.1 Existence of the Non-relativistic Sector

In this appendix we will provide necessary conditions under which the map in Eq. (6.3) does not produce relativistic phenomena such as creation or annihilation of NR particles, nor accelerate NR particles to relativistic velocities. We restrict our analysis to the one particle sector and we will work in the interaction picture.

For single particle system $\hat{\rho}$, has the form given in Eq. (6.10). We consider a one particle system to be in a non-relativistic state $\hat{\rho}_{NR}$ if the state satisfies the following condition:

$$\langle \mathbf{p}_L | \hat{\rho}_{NL} | \mathbf{p}_R \rangle \simeq 0 \quad \text{if} \quad |\mathbf{p}_L| > \kappa m \quad \text{or} \quad |\mathbf{p}_R| > \kappa m \tag{8.1}$$

where $|\mathbf{p}\rangle$ is a one particle state with 3-momentum $\mathbf{p}$ and $\kappa \in \mathcal{R}^+$ that acts as a momentum cut off $\kappa \ll 1$. In other words, the system is characterised by momentum much less than the rest energy m.

Substituting Eq. (6.10) into Eq. (8.1) shows that condition Eq. (8.1) is met if:

$$A(\mathbf{p}_L, \mathbf{p}_R) = 0 \quad \text{if} \quad |\mathbf{p}_L| > \kappa m \quad \text{or} \quad |\mathbf{p}_R| > \kappa m \tag{8.2}$$

With this in hand we can say that the map $\mathcal{M}_t$ is "well behaved" in the NR scenario if the following conditions are met:

$$\langle \mathbf{p}_L | \mathcal{M}_t[\hat{\rho}_{NL}] | \mathbf{p}_R \rangle \simeq 0 \quad \text{if} \quad |\mathbf{p}_R| > \kappa\, m \text{ or } |\mathbf{p}_L| > \kappa\, m \tag{8.3}$$

and

$$\langle \mathbf{p}_L, \mathbf{q}_L | \mathcal{M}_t[\hat{\rho}_{NL}] | \mathbf{p}_R, \mathbf{q}_R \rangle \approx 0 \tag{8.4}$$

Where $|\mathbf{p}_R, \mathbf{q}_R\rangle$ is a two particle state. Equation (8.3) guarantees that a NR one particle state is not driven to a relativistic states by the map $\mathcal{M}_t$. Equation (8.4) forbids the creation of particles in the NR regime.

We check these conditions by expanding in γ using Eq. (6.8), which simplifies eqs. (8.3) and (8.4) to:

$$\langle \mathbf{p}_L | \mathcal{L}_t[\hat{\rho}_{NL}] | \mathbf{p}_R \rangle \simeq 0 \text{ if } \quad |\mathbf{p}_L| > \kappa\, m \text{ or } |\mathbf{p}_R| > \kappa\, m \tag{8.5}$$

and

$$\langle \mathbf{p}_L, \mathbf{q}_L | \mathcal{L}_t[\hat{\rho}_{NL}] | \mathbf{q}_R, \mathbf{p}_R \rangle \approx 0 \tag{8.6}$$

With this equation in hand it is straightforward to verify that condition Eq. (8.6) is always satisfied [1]. To evaluate condition Eq. (8.5) we use Eq. (6.13), Eq. (6.12) and Eq. (6.10) and find that:

$$\langle \mathbf{p}_L|\mathcal{L}_t[\hat{\rho}_{NL}]|\mathbf{p}_R\rangle = -\frac{m^2}{4}\int d\mathbf{q}\int_0^t ds\int_0^s d\tau \tilde{D}(\mathbf{q}, s-\tau)$$
$$\left\{ A(\mathbf{p}_L, \mathbf{p}_R)\left(\frac{e^{i\Delta E(\mathbf{p}_L,\mathbf{q})(x_0-y_0)}}{E_{\mathbf{p}_L}E_{\mathbf{p}_L-\mathbf{q}}} + \frac{e^{-i\Delta E(\mathbf{p}_R,\mathbf{q})(s-\tau)}}{E_{\mathbf{p}_R}E_{\mathbf{p}_R-\mathbf{q}}}\right) - \right.$$
$$\left. \frac{A(\mathbf{p}_R-\mathbf{q}, \mathbf{p}_L-\mathbf{q})}{\sqrt{E_{\mathbf{p}_L}E_{\mathbf{p}_R}E_{\mathbf{p}_R-\mathbf{q}}E_{\mathbf{p}_L-\mathbf{q}}}}\left(e^{-i\Delta E(\mathbf{p}_R,\mathbf{q})s}e^{i\Delta E(\mathbf{p}_L,\mathbf{q})\tau} + e^{i\Delta E(\mathbf{p}_L,\mathbf{q})s}e^{-i\Delta E(\mathbf{p}_R,\mathbf{q})\tau}\right)\right\} \tag{8.7}$$

In order for the map to have a well behaved NR sector this quantity must be negligible when $\mathbf{p}_L$ or $\mathbf{p}_R$ are relativistic, i.e. when $|\mathbf{p}_R| > \kappa m$ or $|\mathbf{p}_L > \kappa m$. For the term in the first curly brackets, this condition is automatically met due to Eq. (8.2) which ensures that $A(\mathbf{p}_L, \mathbf{p}_R) \approx 0$ for a non-relativistic state ρ_{NL}. However, the condition in Eq. (8.2) is not enough to guarantee that the term in the second pair of curly brackets vanishes for $\mathbf{p}_R > \kappa m$ or $\mathbf{p}_L > \kappa m$, due to the explicit dependency of $A(\mathbf{p}_L - \mathbf{q}, \mathbf{p}_R - \mathbf{q})$ on the momentum transfer $\mathbf{q}$ which can be only bounded by the form of the correlation function $\tilde{D}(\mathbf{q}, s-\tau)$. This means that assuming that $\tilde{D}(\mathbf{q}, s-\tau)$ vanishes for relativistic momentum transfer

$$\tilde{D}(\mathbf{q}, s-\tau) \approx 0 \quad \text{for } \mathbf{q} > \kappa m. \tag{8.8}$$

will be sufficient to guarantee that Eq. (8.7) vanishes when when $|\mathbf{p}_L| > \kappa m$ or $|\mathbf{p}_R > \kappa m$. To summarise, a sufficient condition for the map specified by Eq. (6.3) to be well behaved in the NR regime is if $\tilde{D}(\mathbf{q}, t)$ satisfies Eq. (8.8), or in other words,s if the Fourier transform of the noise correlation function is only characterized by non-relativistic momentum transfer.

8.0.2 Calculation details for checking necessary condition for microcausality

Here we evaluate Eq. (6.22). We expand the map $\mathcal{M}_t^*$ to first order in γ to obtain:

$$[\mathcal{M}_t^*[\hat{\phi}(\mathbf{z}_1, t)], \hat{\phi}(\mathbf{z}_2, 0)] \approx [\hat{\phi}(\mathbf{z}_1, t) + \mathcal{L}_t[\hat{\phi}(\mathbf{z}_1, t)]), \hat{\phi}(\mathbf{z}_2, 0)]$$
$$= [\hat{\phi}(\mathbf{z}_1, t), \hat{\phi}(\mathbf{z}_2, 0)] + \gamma\int_{\Omega_t} ds d\mathbf{x}\int_{\Omega_s} d\tau d\mathbf{y} D(s, \tau, \mathbf{x}, \mathbf{y})[[\hat{Q}(\mathbf{y}, \tau), [\hat{\phi}(\mathbf{z}_1, t), \hat{Q}(\mathbf{x}, s)]], \hat{\phi}(\mathbf{z}_2, 0)] \tag{8.9}$$

[1]Indeed writing $|\mathbf{p}_1, \mathbf{p}_2 \ldots, \mathbf{p}_n\rangle = \hat{a}_{\mathbf{p}_1}, \hat{a}_{\mathbf{p}_2} \ldots \hat{a}_{\mathbf{p}_n}|0\rangle$ one immediately notices that the expression will always contain an odd number of creation and annihilation operators.

The first term is the normal microcausality condition that we will leave as it is and only consider the second term.

We evaluate this expression from the inner commutator outwards. Recalling that

$$\begin{aligned}\hat{\phi}(\mathbf{x},t) &= a(\mathbf{x},t) + a^\dagger(\mathbf{x},t) \\ &= \frac{1}{(2\pi)^3}\int \frac{d\mathbf{k}}{\sqrt{2E_k}}\left(e^{i(\mathbf{k}\cdot\mathbf{x}-E_k t)}\hat{a}_{\mathbf{k}} + e^{-i(\mathbf{k}\cdot\mathbf{x}-E_k t)}\hat{a}^\dagger_{\mathbf{k}}\right)\end{aligned} \tag{8.10}$$

and exploiting commutation relations in Eq. (6.7) its easy to obtain

$$[\hat{\phi}(\mathbf{z}_1,t),\hat{Q}(\mathbf{x},s)] = m\int \frac{d\mathbf{k}}{2E_{\mathbf{k}}}\left(e^{i[\mathbf{k}\cdot(\mathbf{z}_1-\mathbf{x})-E_k(t-s)]}\hat{a}^\dagger(\mathbf{x},s) - h.c.\right) \tag{8.11}$$

This expression can be used to find that

$$\begin{aligned}&[[\hat{Q}(\mathbf{y},\tau),[\hat{\phi}(\mathbf{z}_1,t),\hat{Q}(\mathbf{x},s)]],\hat{\phi}(\mathbf{z}_2,0)] = \\ &= m^2\int\frac{d\mathbf{k}}{2E_{\mathbf{k}}}\int\frac{d\mathbf{k}'}{2E_{\mathbf{k}'}}\int\frac{d\mathbf{k}''}{2E_{\mathbf{k}''}}\Big(e^{i[\mathbf{k}\cdot(\mathbf{z}_1-\mathbf{x})-E_{\mathbf{k}}(t-s)]}e^{-i[\mathbf{k}'\cdot(\mathbf{x}-\mathbf{y})-E_{\mathbf{k}'}(\tau-s)]}e^{i[\mathbf{k}''\cdot(\mathbf{y}-\mathbf{z}_2)-E_{\mathbf{k}''}(\tau)]} - h.c.\Big)\end{aligned} \tag{8.12}$$

Finally, substituting this expression back into Eq. (8.9), sending $\mathbf{x}\to\mathbf{x}+\mathbf{y}$, $\tau\to s+\tau$ and integrating over $d\mathbf{y}$ gives:

$$[\mathcal{L}_t[\hat{\phi}(\mathbf{z}_1,t)],\hat{\phi}(\mathbf{z}_2,0)] \tag{8.13}$$

$$\begin{aligned}&= \gamma m^2\int_{\Omega_t} ds\,d\mathbf{x}\int_{\Omega_s} d\tau\,d\mathbf{y}\,D(\mathbf{x}-\mathbf{y},s-\tau)[[\hat{Q}(\mathbf{y},\tau),[\hat{\phi}(\mathbf{z}_1,t),\hat{Q}(\mathbf{x},s)]],\hat{\phi}(\mathbf{z}_2,0)] \\ &= \gamma m^2\int_0^t ds\int_0^s d\tau\int d\mathbf{x}\,D(\mathbf{x},\tau)\left\{F(\mathbf{z}_1-\mathbf{z}_2,t,\tau,\mathbf{x}) - F(\mathbf{z}_1-\mathbf{z}_2,-t,-\tau,\mathbf{x})\right\}\end{aligned} \tag{8.14}$$

with

$$F(\mathbf{z}_1-\mathbf{z}_2,t,\tau,\mathbf{x}) = \int\frac{d\mathbf{k}}{2E_{\mathbf{k}}}\int\frac{d\mathbf{k}'}{4E^2_{\mathbf{k}'}}e^{i[\mathbf{k}\cdot(\mathbf{z}_1-\mathbf{z}_2)-E_{\mathbf{k}}t]}e^{-i[(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}-(E_{\mathbf{k}}-E_{\mathbf{k}'})\tau]} \tag{8.15}$$

Further making the transformation $(\mathbf{x},\tau)\to(\mathbf{x},-\tau)$ in the second term of Eq. (8.13) and making use of the symmetry $D(\mathbf{x},\tau) = D(-\mathbf{x},\tau)$ and $F(\mathbf{z}_2-\mathbf{z}_1,t,\tau,-\mathbf{x}) = F(\mathbf{z}_1-\mathbf{z}_2,-t,-\tau,\mathbf{x})$ we get

$$\begin{aligned}&= \gamma m^2\int_{\Omega_t} ds\,d\mathbf{x}\int_{\Omega_s} d\tau\,d\mathbf{y}\,D(\mathbf{x}-\mathbf{y},s-\tau)[[\hat{Q}(\mathbf{y},\tau),[\hat{\phi}(\mathbf{z}_1,t),\hat{Q}(\mathbf{x},s)]],\hat{\phi}(\mathbf{z}_2,0)] \\ &= \gamma m^2\int_0^t ds\int_0^s d\tau\int d\mathbf{x}\,D(\mathbf{x},\tau)\left\{F(\mathbf{z}_1-\mathbf{z}_2,t,\tau,\mathbf{x}) - F(\mathbf{z}_2-\mathbf{z}_1,t,\tau,-\mathbf{x})\right\}.\end{aligned} \tag{8.16}$$

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