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MANUSCRIPT

Coordinates at Small Energy and Refined Profiles for the Nonlinear Schrödinger Equation

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Abstract

In this paper we give a new and simplified proof of the theorem on selection of standing waves for small energy solutions of the nonlinear Schrödinger equations (NLS) that we gave in [6]. We consider a NLS with a Schrödinger operator with several eigenvalues, with corresponding families of small standing waves, and we show that any small energy solution converges to the orbit of a time periodic solution plus a scattering term. The novel idea is to consider the "refined profile", a quasi–periodic function in time which almost solves the NLS and encodes the discrete modes of a solution. The refined profile, obtained by elementary means, gives us directly an optimal coordinate system, avoiding the normal form arguments in [6], giving us also a better understanding of the Fermi Golden Rule.

1 Introduction

In this paper, we consider the following nonlinear Schrödinger equation (NLS):

$$\mathrm{i}\partial_t u = Hu + g(|u|^2)u, \quad (t, x) \in \mathbb{R}^{1+3}. \tag{1.1}$$

Here $H := -\Delta + V$ is a Schrödinger operator with $V \in \mathcal{S}(\mathbb{R}^3, \mathbb{R})$ (Schwartz function). For the nonlinear term we require $g \in C^\infty(\mathbb{R}, \mathbb{R})$ with $g(0) = 0$ and the growth condition:

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$$\forall n \in \mathbb{N} \cup \{0\},\ \exists C_n > 0,\ |g^{(n)}(s)| \le C_n \langle s \rangle^{2-n} \text{ where } \langle s \rangle := (1+|s|^2)^{1/2}. \tag{1.2}$$

We consider the Cauchy problem of NLS (1.1) with the initial condition $u(0) = u_0 \in H^1(\mathbb{R}^3, \mathbb{C})$. It is well known that NLS (1.1) is locally well-posed (LWP) in H^1, see e.g. [4,14].

The aim of this paper is to revisit the study of asymptotic behavior of small (in H^1) solutions when the Schrödinger operator H has several simple eigenvalues. In such situation, it have been proved that solutions decouple into a soliton and dispersive wave [6,22,24].

To state our main result precisely, we introduce some notation and several assumptions. The following two assumptions for the Schrödinger operator H hold for generic V.

Assumption 1.1 0 is neither an eigenvalue nor a resonance of H.

Assumption 1.2 There exists $N \ge 2$ s.t.

$$\sigma_d(H) = \{\omega_j \mid j = 1, \cdots, N\},\ \text{with } \omega_1 < \cdots < \omega_N < 0,$$

where $\sigma_d(H)$ is the set of discrete spectrum of H. Moreover, we assume all ω_j are simple and

$$\forall \mathbf{m} \in \mathbb{Z}^N \setminus \{0\},\ \mathbf{m} \cdot \boldsymbol{\omega} \neq 0, \tag{1.3}$$

where $\boldsymbol{\omega} := (\omega_1, \cdots, \omega_N)$. We set ϕ_j to be the eigenfunction of H associated to the eigenvalue ω_j satisfying $\|\phi_j\|_{L^2} = 1$. We also set $\boldsymbol{\phi} = (\phi_1, \cdots, \phi_N)$.

Remark 1.3 The cases $N = 0, 1$ are easier and are not treated it in this paper. Unfortunately, Assumption (1.2) excludes radial potentials $V(r)$, for $r = |x|$, where in general we should expect eigenvalues with multiplicity higher than one. In fact the symmetries imply that each eigenspace $\ker(H - \omega_j)$ is spanned by functions which in spherical coordinates are separated and are of form $\frac{1}{r} u_{j,l}(r) e^{\mathrm{i}m\theta} P_l^m(\cos(\varphi))$ for appropriate $l \in \mathbb{N} \cup \{0\}$ with P_l^m Legendre polynomials, and m taking all values between $-l$ and l, so that, if $l \ge 1$, the multiplicity is at least $2l + 1$. See p. 778 [5].

As it is well known, ϕ_j's are smooth and decays exponentially. For $s \ge 0$, $\gamma \ge 0$, we set

$$H^s_\gamma := \{u \in H^s \mid \|u\|_{H^s_\gamma} := \|\cosh(\gamma x) u\|_{H^s} < \infty\}.$$

The following is well known.

Proposition 1.4 *There exists $\gamma_0 > 0$ s.t. for all $1 \le j \le N$, we have $\phi_j \in \cap_{s \ge 0} H^s_{\gamma_0}$.*

Using $\gamma_0 > 0$, we set

$$\Sigma^s := H^s_{\gamma_0} \text{ if } s \ge 0,\ \Sigma^s := (H^{-s}_{\gamma_0})^* \text{ if } s < 0,\ \Sigma^{0-} := (\Sigma^0)^* \text{ and } \Sigma^\infty := \cap_{s \ge 0} \Sigma^s.$$

We will not consider any topology in Σ^∞ and we will only consider it as a set.

In order to introduce the notion of refined profile, we need the following combinatorial set up.
We start the following standard basis of $\mathbb{R}^N$, which we view as "non–resonant" indices:

$$\mathbf{NR}_0 := \{\mathbf{e}_j \mid j = 1, \cdots, N\},\ \mathbf{e}_j := (\delta_{1j}, \cdots, \delta_{Nj}) \in \mathbb{Z}^N,\ \delta_{ij} \text{ the Kronecker delta.} \tag{1.4}$$

More generally, the sets of resonant and non–resonant indices **R**, **NR**, are

$$\mathbf{R} := \{\mathbf{m} \in \mathbb{Z}^N \mid \sum \mathbf{m} = 1,\ \boldsymbol{\omega} \cdot \mathbf{m} > 0\}, \quad \mathbf{NR} := \{\mathbf{m} \in \mathbb{Z}^N \mid \sum \mathbf{m} = 1,\ \boldsymbol{\omega} \cdot \mathbf{m} < 0\}, \tag{1.5}$$

where $\sum \mathbf{m} := \sum_{j=1}^N m_j$ for $\mathbf{m} = (m_1, \cdots, m_N) \in \mathbb{Z}^N$.
From Assumption 1.2 it is clear that $\{\mathbf{m} \in \mathbb{Z}^N \mid \sum \mathbf{m} = 1\} = \mathbf{R} \cup \mathbf{NR}$ and $\mathbf{NR}_0 \subset \mathbf{NR}$.
For $\mathbf{m} = (m_1, \cdots, m_N) \in \mathbb{Z}^N$, we define

$$|\mathbf{m}| := (|m_1|, \cdots, |m_N|) \in \mathbb{Z}^N,\ \|\mathbf{m}\| := \sum |\mathbf{m}| = \sum_{j=1}^N |m_j|, \tag{1.6}$$

and introduce partial orders $\preceq$ and $\prec$ by

$$\mathbf{m} \preceq \mathbf{n} \ \Leftrightarrow_{\text{def}} \forall j \in \{1, \cdots, N\},\ m_j \le n_j, \quad \text{and} \quad \mathbf{m} \prec \mathbf{n} \ \Leftrightarrow_{\text{def}} \mathbf{m} \preceq \mathbf{n} \text{ and } \mathbf{m} \neq \mathbf{n}, \tag{1.7}$$

where $\mathbf{n} = (n_1, \cdots, n_N)$. We define the minimal resonant indices by

$$\mathbf{R}_{\min} := \{\mathbf{m} \in \mathbf{R} \mid \nexists \mathbf{n} \in \mathbf{R} \text{ s.t. } |\mathbf{n}| \prec |\mathbf{m}|\}. \tag{1.8}$$

We also consider $\mathbf{NR}_1$ formed by the nonresonant indices not larger than resonant indices:

$$\mathbf{NR}_1 := \{\mathbf{m} \in \mathbf{NR} \mid \forall \mathbf{n} \in \mathbf{R}_{\min},\ |\mathbf{n}| \not\prec |\mathbf{m}|\}. \tag{1.9}$$

Lemma 1.5 *Both $\mathbf{R}_{\min}$ and $\mathbf{NR}_1$ are finite sets.*

For the proof see Appendix A.

We constructively define functions $\{G_{\mathbf{m}}\}_{\mathbf{m} \in \mathbf{R}_{\min}} \subset \Sigma^\infty$ which will be important in our analysis.

For $\mathbf{m} \in \mathbf{NR}_1$, we inductively define $\widetilde{\phi}_{\mathbf{m}}(0)$ and $g_{\mathbf{m}}(0)$ by

$$\widetilde{\phi}_{\mathbf{e}_j}(0) := \phi_j,\ g_{\mathbf{e}_j}(0) = 0,\ j = 1, \cdots, N, \tag{1.10}$$

and, for $\mathbf{m} \in \mathbf{NR}_1 \setminus \mathbf{NR}_0$, by

$$\widetilde{\phi}_{\mathbf{m}}(0) := -(H - \mathbf{m} \cdot \boldsymbol{\omega})^{-1} g_{\mathbf{m}}(0), \tag{1.11}$$

$$g_{\mathbf{m}}(0) := \sum_{m=1}^{\infty} \frac{1}{m!} g^{(m)}(0) \sum_{(\mathbf{m}_1, \cdots, \mathbf{m}_{2m+1}) \in A(m, \mathbf{m})} \widetilde{\phi}_{\mathbf{m}_1}(0) \cdots \widetilde{\phi}_{\mathbf{m}_{2m+1}}(0), \tag{1.12}$$

where

$$A(m, \mathbf{m}) := \left\{ \{\mathbf{m}_j\}_{j=1}^{2m+1} \in (\mathbf{NR}_1)^{2m+1} \mid \sum_{j=0}^{m} \mathbf{m}_{2j+1} - \sum_{j=1}^{m} \mathbf{m}_{2j} = \mathbf{m},\ \sum_{j=0}^{2m+1} |\mathbf{m}_j| = |\mathbf{m}| \right\} \tag{1.13}$$

Remark 1.6 For each $m \geq 1$ and $\mathbf{m} \in \mathbf{NR}_1$, $A(m, \mathbf{m})$ is a finite set. Furthermore, for sufficiently large m, we have $A(m, \mathbf{m}) = \emptyset$. Thus, even though we are expressing $g_{\mathbf{m}}(0)$ in (1.12) by a series, the sum is finite.

For $\mathbf{m} \in \mathbf{R}_{\min}$, we define $G_{\mathbf{m}}$ by

$$G_{\mathbf{m}} := \sum_{m=1}^{\infty} \frac{1}{m!} g^{(m)}(0) \sum_{(\mathbf{m}_1, \cdots, \mathbf{m}_{2m+1}) \in A(m, \mathbf{m})} \widetilde{\phi}_{\mathbf{m}_1}(0) \cdots \widetilde{\phi}_{\mathbf{m}_{2m+1}}(0). \tag{1.14}$$

Remark 1.7 $g_{\mathbf{m}}(0)$ and $G_{\mathbf{m}}$ are defined similarly. We are using a different notation to emphasize that $g_{\mathbf{m}}(0)$ has $\mathbf{m} \in \mathbf{NR}_1$, while $G_{\mathbf{m}}$ has $\mathbf{m} \in \mathbf{R}_{\min}$.

The following is the nonlinear Fermi Golden Rule (FGR) assumption.

Assumption 1.8 For all $\mathbf{m} \in \mathbf{R}_{\min}$, we assume

$$\int_{|\zeta|^2 = \mathbf{m} \cdot \boldsymbol{\omega}} |\widehat{G}_{\mathbf{m}}(\zeta)|^2 \, dS \neq 0, \tag{1.15}$$

where $\widehat{G}_{\mathbf{m}}$ is the distorted Fourier transform associated to H.

Remark 1.9 In the case $N = 2$ and $\omega_1 + 2(\omega_2 - \omega_1) > 0$, we have $G_{\mathbf{m}} = g'(0)\phi_1\phi_2^2$, which corresponds to the condition in Tsai and Yau [25], based on the explicit formulas in Buslaev and Perelman [3] and Soffer and Weinstein [21]. These works are related to Sigal [20]. Other partial results are in [8–11]. More general situations are considered in [6], where however the $G_{\mathbf{m}}$ are obtained after a certain number of coordinate changes, so that the relation of the $G_{\mathbf{m}}$ and the ϕ_j's is not discussed in [6] and is not easy to track.

For a generic nonlinear function g the condition (1.15) is a consequence of the following simpler one, which is similar to (11.6) in Sigal [20],

$$\int_{|\zeta|^2 = \mathbf{m} \cdot \boldsymbol{\omega}} |\widehat{\phi^{\mathbf{m}}}(\zeta)|^2 \, dS \neq 0 \text{ for all } \mathbf{m} \in \mathbf{R}_{\min} \tag{1.16}$$

where $\phi^{\mathbf{m}} := \prod_{j=1,\dots,N} \phi_j^{m_j}$. Both conditions (1.15) and, even more so, (1.16) are simpler than the analogous conditions in Cuccagna and Maeda [6].

We have the following.

Proposition 1.10 *Let* $L = \sup \left\{ \frac{\|\mathbf{m}\| - 1}{2} : \mathbf{m} \in \mathbf{R}_{\min} \right\}$ *and suppose that the operator* H *satisfies condition* (1.16)*. Then there exists an open dense subset* Ω *of* $\mathbb{R}^{L-1}$ *s.t. if* $(g'(0), ..., g^{(L)}(0)) \in \Omega$ *such that Assumption 1.8 is true for* (1.1)*.*

Proof See Sect. A. □

For $\mathbf{z} = (z_1, \cdots, z_N) \in \mathbb{C}^N$, $\mathbf{m} = (m_1, \cdots, m_N) \in \mathbb{Z}^N$, we define

$$\mathbf{z}^{\mathbf{m}} := z_1^{(m_1)} \cdots z_N^{(m_N)} \in \mathbb{C}, \text{ where } z^{(m)} := \begin{cases} z^m & m \geq 0 \\ \bar{z}^{-m} & m < 0, \end{cases} \quad \text{and} \tag{1.17}$$

$$|\mathbf{z}|^k := (|z_1|^k, \cdots, |z_N|^k) \in \mathbb{R}^N, \ \|\mathbf{z}\| := \sum |\mathbf{z}| = \sum_{j=1}^{N} |z_j| \in \mathbb{R}. \tag{1.18}$$

We will use the following notation for a ball in a Banach space B:

$$\mathcal{B}_B(u, r) := \{v \in B \mid \|v - u\|_B < r\}. \tag{1.19}$$

The "refined profile" is of the form $\phi(\mathbf{z}) = \mathbf{z} \cdot \boldsymbol{\phi} + o(\|\mathbf{z}\|)$ and is defined by the following proposition.

Proposition 1.11 *[Refined Profile] For any* $s \geq 0$*, there exist* $\delta_s > 0$ *and* $C_s > 0$ *s.t.* δ_s *is nonincreasing w.r.t.* $s \geq 0$ *and there exist*

$$\{\psi_{\mathbf{m}}\}_{\mathbf{m} \in \mathbf{NR}_1} \in C^\infty(\mathcal{B}_{\mathbb{R}^N}(0, \delta_s^2), (\Sigma^s)^{\sharp \mathbf{NR}_1}), \ \boldsymbol{\varpi}(\cdot) \in C^\infty(\mathcal{B}_{\mathbb{R}^N}(0, \delta_s^2), \mathbb{R}^N)$$
$$\text{and } \mathcal{R} \in C^\infty(\mathcal{B}_{\mathbb{C}^N}(0, \delta_s), \Sigma^s),$$

s.t. $\boldsymbol{\varpi}(0, \cdots, 0) = \boldsymbol{\omega}$*,* $\psi_{\mathbf{m}}(0) = 0$ *for all* $\mathbf{m} \in \mathbf{NR}_1$ *and*

$$\|\mathcal{R}(\mathbf{z})\|_{\Sigma^s} \leq C_s \|\mathbf{z}\|^2 \sum_{\mathbf{m} \in \mathbf{R}_{\min}} |\mathbf{z}^{\mathbf{m}}|, \tag{1.20}$$

and if we set

$$\phi(\mathbf{z}) := \mathbf{z} \cdot \boldsymbol{\phi} + \sum_{\mathbf{m} \in \mathbf{NR}_1} \mathbf{z}^{\mathbf{m}} \psi_{\mathbf{m}}(|\mathbf{z}|^2) \text{ and } z_j(t) = e^{-\mathrm{i}\varpi_j(|\mathbf{z}|^2)t} z_j, \tag{1.21}$$

then, setting $\mathbf{z}(t) = (z_1(t), \cdots, z_n(t))$*, the function* $u(t) := \phi(\mathbf{z}(t))$ *satisfies*

$$\mathrm{i}\partial_t u = Hu + g(|u|^2)u - \sum_{\mathbf{m} \in \mathbf{R}_{\min}} \mathbf{z}(t)^{\mathbf{m}} G_{\mathbf{m}} - \mathcal{R}(\mathbf{z}(t)), \tag{1.22}$$

where $\{G_{\mathbf{m}}\}_{\mathbf{R}_{\min}} \subset (\Sigma^\infty)^{\sharp \mathbf{R}_{\min}}$ *is given in* (1.14)*. Finally, writing* $\psi_{\mathbf{m}} = \psi_{\mathbf{m}}^{(s)}$*,* $\boldsymbol{\varpi} = \boldsymbol{\varpi}^{(s)}$ *and* $\mathcal{R} = \mathcal{R}^{(s)}$*, for* $s_1 < s_2$ *we have* $\psi_{\mathbf{m}}^{(s_1)}(|\cdot|^2) = \psi_{\mathbf{m}}^{(s_2)}(|\cdot|^2)$*,* $\boldsymbol{\varpi}^{(s_1)}(|\cdot|^2) = \boldsymbol{\varpi}^{(s_2)}(|\cdot|^2)$ *and* $\mathcal{R}^{(s_1)} = \mathcal{R}^{(s_2)}$ *in* $\mathcal{B}_{\mathbb{R}^N}(0, \delta_{s_2})$*.*

Proof See section 4. □

The refined profile $\phi(\mathbf{z})$ contains as a special case the small standing waves bifurcating from the eigenvalues, when they are simple.

Corollary 1.12 *Let* $s > 0$ *and* $j \in \{1, \cdots, N\}$*. Then, for* $z \in \mathcal{B}_{\mathbb{C}}(0, \delta_s)$*,* $\phi\left(z(t)\mathbf{e}_j\right)$ *solves* (1.1) *for* $z(t) = e^{-\mathrm{i}\varpi_j(|z\mathbf{e}_j|^2)t} z$*.*

Proof Since $(z\mathbf{e}_j)^{\mathbf{m}} = 0$ for $\mathbf{m} \in \mathbf{R}_{\min}$, we see that from (1.20) and (1.22) the remainder terms $\sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}(t)^{\mathbf{m}} G_{\mathbf{m}} + \mathcal{R}(\mathbf{z}(t))$ are 0 in (1.22). Therefore, we have the conclusion.

Remark 1.13 If the eigenvalues of H are not simple the above does not hold anymore in general. See Gustafson-Phan [12].

We call solitons, or standing waves, the functions

$$\phi_j(z) := \phi(z\mathbf{e}_j). \tag{1.23}$$

The main result, which have first proved in [6] is the following.

Theorem 1.14 *Under the Assumptions 1.1, 1.2 and 1.8, there exist* $\delta_0 > 0$ *and* $C > 0$ *s.t. for all* $u_0 \in H^1$ *with* $\|u_0\|_{H^1} < \delta_0$*, there exists* $j \in \{1, \cdots, N\}$*,* $z \in C^1(\mathbb{R}, \mathbb{C})$*,* $\eta_+ \in H^1$ *and* $\rho_+ \geq 0$ *s.t.*

$$\lim_{t\to\infty} \|u(t) - \phi_j(z(t)) - e^{\mathrm{i}t\Delta}\eta_+\|_{H^1} = 0,$$

and

$$\lim_{t\to\infty} |z(t)| = \rho_+, \ C^{-1}\|u_0\|_{H^1}^2 \leq \rho_+^2 + \|\eta_+\|_{H^1}^2 \leq C\|u_0\|_{H^1}^2.$$

The organization of the paper is the following. In the rest of this section, we outline the proof of the main theorem (Theorem 1.14). In Section 2, we introduce the modulation and Darboux coordinate and compute the Taylor expansion of the energy. In section 3 we prove the main theorem (Theorem 1.14). In section 4 we prove Proposition 1.11. In section 5, we state an abstract Darboux theorem with error estimate and apply it to prove Proposition 2.4. In the appendix of this paper, we prove Lemma 1.5 and Proposition 1.10.

We now outline the proof of Theorem 1.14. First of all, the fact that NLS (1.1) is Hamilton is crucial. Indeed, when we consider the symplectic form

$$\Omega_0(\cdot,\cdot) := \langle \mathrm{i}\cdot, \cdot\rangle, \ \langle u, v\rangle := \mathrm{Re}(u, \overline{v}) \text{ where } (u, v) := \int_{\mathbb{R}^3} u(x)v(x)\,dx, \tag{1.24}$$

and the energy (Hamiltonian) by

$$E(u) = \frac{1}{2}\langle Hu, u\rangle + \frac{1}{2}\int_{\mathbb{R}^3} G(|u(x)|^2)\, dx, \tag{1.25}$$

where $G(s) := \int_0^s g(s)\, ds$, we can rewrite NLS (1.1) as

$$\partial_t u = X_E^{(0)}(u).$$

Here, for $F \in C^1(H^1, \mathbb{R})$, $X_F^{(0)}$ is the Hamilton vector field of F associated to the symplectic form Ω_0 defined, for DF is the Fréchet derivative of F, by

$$\Omega_0(X_F^{(0)}, \cdot) = DF.$$

Next, as usual for the study of stability of solitons, we give a modulation coordinates in H^1 in the neighborhood of 0. In this paper, we use

$$(\mathbf{z}, \eta) \mapsto u = \phi(\mathbf{z}) + \eta, \tag{1.26}$$

while in [6] we were using

$$(\mathbf{z}, \eta) \mapsto u = \sum_{j=1,\dots,N} \phi_j(z_j) + R(\mathbf{z})\eta, \tag{1.27}$$

for specific near identity operator $R(\mathbf{z})$ which was first introduced in [13]. Here, in both (1.26) and (1.27), η is taken from the continuous component of H. That is, $P_c\eta = \eta$, where

$$P_c u := u - \sum_{j=1,\dots,N} \left(\langle u, \phi_j\rangle \phi_j + \langle u, \mathrm{i}\phi_j\rangle \mathrm{i}\phi_j \right). \tag{1.28}$$

The difference between the two coordinates (1.26) and (1.27) is that in (1.26) we are using the refined profile which takes into account the nonlinear interactions within the discrete modes. While the discrete part in (1.26) is more complicate than in (1.27), to prove Theorem 1.14 for $N > 1$ we do not need the $R(\mathbf{z})$ in front of η.

Unfortunately, even though Ω_0 is a deceptively simple symplectic form, in the coordinates (1.26) it is complicated (it is very complicated also using coordinates (1.27)). We thus introduce a new symplectic form

$$\Omega_1(\cdot, \cdot) := \Omega_0(D_{\mathbf{z}}\phi(\mathbf{z})D\mathbf{z}\,\cdot, D_{\mathbf{z}}\phi(\mathbf{z})D\mathbf{z}\,\cdot) + \Omega_0(D\eta\,\cdot, D\eta\cdot), \tag{1.29}$$

which is equal to Ω_0 at $u = 0$. Here, $D_{\mathbf{z}}$ is the Fréchet derivative w.r.t. the $\mathbf{z}$ variable.

By Darboux theorem there exists near 0 an almost identity coordinate change φ such that $\Omega_1 = \varphi^*\Omega_0$. In section 5 we give a rather simple proof of the type of Darboux theorem needed, viewing it in an abstract framework simplifying the analogous part of [6].

For $K = \varphi^* E$, the system becomes

$$\mathrm{i}\partial_t \mathbf{z} = (1 + \mathcal{O}(\|\mathbf{z}\|^2))\nabla_{\mathbf{z}} K, \quad \mathrm{i}\partial_t \eta = \nabla_\eta K,$$

where $\nabla_{\mathbf{z}}$ and ∇_η are the gradient corresponding to the Fréchet derivative w.r.t. $\mathbf{z}$ and η. In the new coordinates, the energy K expands

$$K = E(\phi(\mathbf{z})) + E(\eta) + \left\langle \widetilde{\mathcal{R}}(\mathbf{z}), \eta \right\rangle + \text{error}.$$

When using the coordinate system (1.27), in order to estimate the solutions it is necessary like in [6] to make further normal forms changes of variables. But using coordinates (1.26) we are ready for the estimates and there is no need of normal forms. First of all, we have $\widetilde{\mathcal{R}}(\mathbf{z}) = \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}} G_{\mathbf{m}} + \text{error}$, see the First Cancellation Lemma, Lemma 2.6. This implies that

$$\mathrm{i}\partial_t \eta = H\eta + P_c g(|\eta|^2)\eta + \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \left\langle \mathbf{z}^{\mathbf{m}} G_{\mathbf{m}}, \eta \right\rangle + \text{error}. \tag{1.30}$$

Thus, by the endpoint Strichartz estimate, to show that η scatters it suffices to show $\mathbf{z}^{\mathbf{m}} \in L^2(\mathbb{R})$ for $\mathbf{m} \in \mathbf{R}_{\min}$. To check this point, we consider

$$\frac{d}{dt} E(\phi(\mathbf{z})) = \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \left\{ E(\phi(\mathbf{z})), \left\langle \mathbf{z}^{\mathbf{m}} G_{\mathbf{m}}, \eta \right\rangle \right\} + \text{error},$$

where $\{\cdot,\cdot\}$ is the Poisson bracket associated to Ω_1. We obtain

$$\left\{ E(\phi(\mathbf{z})), \left\langle \mathbf{z}^{\mathbf{m}} G_{\mathbf{m}}, \eta \right\rangle \right\} = (\boldsymbol{\omega} \cdot \mathbf{m}) \left\langle \mathrm{i}\mathbf{z}^{\mathbf{m}} G_{\mathbf{m}}, \eta \right\rangle + \text{error}, \tag{1.31}$$

where, see below (3.11) and as a consequence of the Second Cancellation Lemma, Lemma 2.8,

$$|\text{error}| \lesssim |\mathbf{z}| \sum_{\mathbf{m}\in\mathbf{R}_{\min}} |\mathbf{z}^{\mathbf{m}}| \text{ for all } |\mathbf{z}| \le 1.$$

Notice that z_1^ℓ does not satisfy this inequality no matter how large we take $\ell \in \mathbb{N}$, so the error term in (1.31) is not just small, but has a specific combinatorial structure. In [6], to get the structure (1.30) and to bound $\mathbf{z}$, a painstaking normal forms argument was required, but here these fact come for free.

From this point on, the proof ends in a standard way. Since $\eta \sim -\mathbf{z}^{\mathbf{m}}(H - \omega \cdot \mathbf{m} - \mathrm{i}0)^{-1} G_{\mathbf{m}}$, where the latter is the solution of (1.30) without the nonlinear term and "error", we have, omitting errors

$$\frac{d}{dt} E(\phi(\mathbf{z})) = \sum_{\mathbf{m}\in\mathbf{R}_{\min}} (\boldsymbol{\omega} \cdot \mathbf{m}) |\mathbf{z}^{\mathbf{m}}|^2 \left\langle \mathrm{i} G_{\mathbf{m}}, (H - \omega \cdot \mathbf{m} - \mathrm{i}0)^{-1} G_{\mathbf{m}} \right\rangle.$$

Since $\langle \mathrm{i}G_{\mathbf{m}}, (H - \omega \cdot \mathbf{m} - \mathrm{i}0)^{-1} G_{\mathbf{m}} \rangle$ equals (1.15) in Assumption 1.8 which we have assumed positive, this above idealized identity yields

$$E(\phi(\mathbf{z}(t))) + \sum_{\mathbf{m} \in \mathbf{R}_{\min}} \|\mathbf{z}^{\mathbf{m}}\|^2_{L^2(0,t)} \leq E(\phi(\mathbf{z}(0))).$$

Using this, we can close estimates.

We conclude with a few comments on refined profiles, which play a central role in our proof. One of the distinctive features of our system is the existence or non existence of small quasi–periodic solutions which are not periodic. Sigal [20] stated their absence, and this follows from [6] and our analysis here. The $\mathbf{z}^{\mathbf{m}} G_{\mathbf{m}}$ terms in $\widetilde{\mathcal{R}}(\mathbf{z})$ are resonant, cannot be eliminated from the equation exactly if (1.15) holds and are an obstruction to the existence of quasi–periodic solutions. On the other hand, there are no resonant terms in the discrete NLS with $N = 2$, where quasi-periodic solutions are proved to exist in Maeda [15]. Furthermore, in Maeda [15] an equivalence is observed between being able to see quasi–periodic solutions, absence of resonant terms in the equations and, finally, existence of coordinate systems where the mixed term $\langle \widetilde{\mathcal{R}}(\mathbf{z}), \eta \rangle$, that is nonlinear degree 1 in η, is absent from the energy. Our main insight here is that, since there are no small quasi–periodic solutions, we might try to replace them with a surrogate (refined profiles), in the expectation of an equivalence, analogous to that considered in Maeda [15], between this surrogate and optimal coordinate systems. This works and, while in [6] we searched directly, and with great effort, for the coordinates, here we find, with a relatively elementary method, the refined profiles. Starting from the refined profiles we define a natural coordinate system. It turns out that these coordinates are optimal, as is seen in elementary fashion noticing that the fact that the refined profiles are approximate solutions of (1.1), specifically they solve (1.22), provides us the two Cancellation Lemmas, which in turn guarantee that our coordinates are optimal. We end remarking that refinements of the ansatz were already in the great series by Merle and Raphael [16–19], which has inspired our notion of refined profile.

2 Darboux Coordinate and Energy Expansion

We start from constructing the modulation coordinate. First, we have the following.

Lemma 2.1 *For any $s \in \mathbb{R}$ there exist $\delta_s > 0$ and $\mathbf{z} \in C^\infty(\mathcal{B}_{\Sigma^{-s}}(0, \delta_s), \mathbb{C}^N)$ s.t.*

$$u - \phi(\mathbf{z}(u)) \in P_c \Sigma^{-s},$$

where P_c is given by (1.28).

Proof This is an immediate consequence of the implicit function theorem. We consider

$$F_j(\mathbf{z}, u) = \langle \phi(\mathbf{z}) - u, \phi_j \rangle + \mathrm{i} \langle u - \phi(\mathbf{z}), \mathrm{i}\phi_j \rangle \text{ for } j = 1, ..., N.$$

We have $F := (F_1, ..., F_N) \in C^\infty(\Sigma^{-s} \times \mathcal{B}_{\mathbb{C}^N}(0, \delta_0), \mathbb{C}^N)$ for $\delta_0 > 0$ given in Proposition 1.11. Obviously $F|_{(z,u)=(0,0)} = 0$ and from $\psi_{\mathbf{m}}(0) = 0$ for all $\mathbf{m} \in \mathbf{NR}_1$, it follows $D_{\mathbf{z}} F|_{(\mathbf{z},u)=(0,0)} = \mathrm{Id}_{\mathbb{C}^N}$, where $D_{\mathbf{z}} F$ is the Fréchet derivative w.r.t. the $\mathbf{z}$ variable. By implicit function theorem we obtain the desired $\mathbf{z} \in C^\infty(\mathcal{B}_{\Sigma^{-s}}(0,\delta_s),\mathbb{C}^N)$ for some $\delta_s > 0$.

By Lemma 2.1, we have our first (modulation) coordinate.

Proposition 2.2 *For any $s \in \mathbb{R}$ there exist $\delta_s > 0$ s.t. the map*

$$\mathcal{B}_{\mathbb{C}^N}(0,\delta_s) \times \mathcal{B}_{P_c X^{-s}}(0,\delta_s) \ni (\mathbf{z},\eta) \mapsto \phi(\mathbf{z}) + \eta \in X^{-s},\ X^s = \Sigma^s \text{ or } H^s, \tag{2.1}$$

is a C^∞ local diffeomorphism. Moreover, we have

$$\|u\|_{X^s} \sim_s \|\mathbf{z}\| + \|\eta\|_{X^s}.$$

Proof It is an direct consequence of Lemma 2.1.

For Banach spaces X, Y, we set $\mathcal{L}(X,Y)$ to be the Banach space of all bounded linear operators from X to Y. Moreover, we set $\mathcal{L}(X) := \mathcal{L}(X,X)$.

For $F \in C^1(\mathcal{B}_{H^1}(0,\delta),\mathbb{R})$, we write

$$F(\mathbf{z},\eta) := F(\phi(\mathbf{z}) + \eta).$$

We define $D_\eta F(\mathbf{z},\eta) \in C(\mathcal{B}_{H^1}(0,\delta),\mathcal{L}(P_c H^1,\mathbb{R}))$ and $\nabla_\eta F(\mathbf{z},\eta) \in C(\mathcal{B}_{H^1}(0,\delta), P_c H^{-1})$ by

$$\forall Y \in P_c H^1,\ D_\eta F(\mathbf{z},\eta) Y = \left\langle \nabla_\eta F(\mathbf{z},\eta), Y \right\rangle := \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} F(\mathbf{z},\eta+\epsilon v).$$

Here, for Banach spaces A, B, $\mathcal{L}(A,B)$ is the Banach space of all bounded operators from A to B. Similarly, we define $\nabla_{\mathbf{z}} F(u) = \nabla_{\mathbf{z}} F(\mathbf{z},\eta) \in C(\mathcal{B}_{H^1}(0,\delta),\mathbb{C}^N)$ by

$$\forall \mathbf{w} \in \mathbb{C}^N,\ \langle \nabla_{\mathbf{z}} F(\mathbf{z},\eta), \mathbf{w} \rangle_{\mathbb{C}^N} := D_{\mathbf{z}} F(\mathbf{z},\eta)\mathbf{w} = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} F(\mathbf{z}+\epsilon\mathbf{w},\eta),$$

where $\langle \mathbf{w}_1, \mathbf{w}_2 \rangle_{\mathbb{C}^N} = \mathrm{Re}\sum_{j=1}^N w_{1j}\overline{w_{2j}}$ for $\mathbf{w}_k = (w_{k1},\cdots,w_{kN})$.

Using the above notations, for $u \in B_{H^1}(0,\delta)$ and $Y \in H^1$, we have

$$DF(\mathbf{z},\eta)Y = \langle \nabla_{\mathbf{z}} F(\mathbf{z},\eta), D\mathbf{z}Y \rangle_{\mathbb{C}^N} + D_\eta F(\mathbf{z},\eta) D\eta Y, \tag{2.2}$$

where $D\mathbf{z}$ and $D\eta$ are Fréchet derivatives of functions $\mathbf{z}(u)$, $\eta(u) := u - \phi(\mathbf{z}(u))$.

Notice that, since the Fréchet derivative of the identity map $u \mapsto u$ is an identity, we have

$$\mathrm{Id}_{X^s} = Du = D_{\mathbf{z}}\phi(\mathbf{z}) D\mathbf{z} + D\eta. \tag{2.3}$$

Remark 2.3 Even though $\eta = P_c\eta$, $D\eta$ is not P_c except at $u = 0$.

By (2.3), we have

$$\Omega_0 = \Omega_0(D_{\mathbf{z}}\phi(\mathbf{z})D\mathbf{z}, D_{\mathbf{z}}\phi(\mathbf{z})D\mathbf{z}) + \Omega_0(D\eta, D\eta) + \Omega_0(D_{\mathbf{z}}\phi(\mathbf{z})D\mathbf{z}, D\eta) + \Omega_0(D\eta, D_{\mathbf{z}}\phi(\mathbf{z})D\mathbf{z}).$$

Therefore, removing the cross terms (the latter two terms), we have the symplectic form Ω_1 given in (1.29). Given $F \in C^1(\mathcal{B}_{H^1}(0,\delta), \mathbb{R})$, the Hamilton vector field $X_F^{(1)}$ associated to the symplectic form Ω_1 is defined by $\Omega_1(X_F^{(1)}, \cdot) = DF$. Thus, by (2.2), we have

$$\left\langle \mathrm{i}D_{\mathbf{z}}\phi(\mathbf{z})D\mathbf{z}X_F^{(1)}, D_{\mathbf{z}}\phi(\mathbf{z})D\mathbf{z}Y \right\rangle + \left\langle \mathrm{i}D\eta X_F^{(1)}, D\eta Y \right\rangle = \langle \nabla_{\mathbf{z}}F, D\mathbf{z}Y\rangle_{\mathbb{C}^N} + D_\eta F D\eta Y. \tag{2.4}$$

In particular, we have

$$\mathrm{i}D\eta X_F^{(1)} = \nabla_\eta F. \tag{2.5}$$

We turn to $\mathbf{z}$. Setting $\psi(\mathbf{z}) := \sum_{\mathbf{m}\in\mathbf{NR}_1} \mathbf{z}^{\mathbf{m}}\psi_{\mathbf{m}}(|\mathbf{z}|^2)$, we have $\phi(\mathbf{z}) = \mathbf{z}\cdot\boldsymbol{\phi} + \psi(\mathbf{z})$ with $\|\psi(\mathbf{z})\|_{\Sigma^s} \lesssim_s \|\mathbf{z}\|^3$. Then, since $\nabla_{\mathbf{z}}\phi(\mathbf{z})\mathbf{w} = \mathbf{w}\cdot\boldsymbol{\phi} + \mathcal{O}_{\mathcal{L}(\mathbb{C}^N,\Sigma^s)}(\|\mathbf{z}\|^2)\mathbf{w}$, $\langle \mathrm{i}\mathbf{w}_1\cdot\boldsymbol{\phi}, \mathbf{w}_2\cdot\boldsymbol{\phi}\rangle = \langle \mathrm{i}\mathbf{w}_1, \mathbf{w}_2\rangle_{\mathbb{C}^N}$ and $\mathcal{L}(\mathbb{C}^N\times\mathbb{C}^N, \mathbb{R}) \simeq \mathcal{L}(\mathbb{C}^N)$, we see there exists $\widetilde{A} \in C^\infty(\mathcal{B}_{\mathbb{C}^N}(0,\delta_0), \mathcal{L}(\mathbb{C}^N))$ s.t.

$$\left\langle \mathrm{i}D_{\mathbf{z}}\phi(\mathbf{z})D\mathbf{z}X_F^{(1)}, D_{\mathbf{z}}\phi(\mathbf{z})D\mathbf{z}Y \right\rangle = \left\langle \mathrm{i}\left(1+\widetilde{A}(\mathbf{z})\right)D\mathbf{z}X_F^{(1)}, D\mathbf{z}Y \right\rangle_{\mathbb{C}^N},$$

with $\|\widetilde{A}(\mathbf{z})\|_{\mathcal{L}(\mathbb{C}^N)} \lesssim \|\mathbf{z}\|^2$. Thus, setting $A \in C^\infty(\mathcal{B}_{\mathbb{C}^N}(0,\delta_0), \mathcal{L}(\mathbb{C}^N))$ by $1+A(\mathbf{z}) = (1+\widetilde{A}(\mathbf{z}))^{-1}$, we have $\|A(\mathbf{z})\|_{\mathcal{L}(\mathbb{C}^N)} \lesssim \|\mathbf{z}\|^2$ and

$$\mathrm{i}D\mathbf{z}X_F^{(1)} = (1+A(\mathbf{z}))\nabla_{\mathbf{z}}F. \tag{2.6}$$

The following proposition allows us to move to the "diagonalized" symplectic form Ω_1.

Proposition 2.4 *For any $s > 0$ there exists $\delta_s > 0$ and $\varphi \in C^\infty(\mathcal{B}_{\Sigma^{-s}}(0,\delta_s), \Sigma^{-s})$ satisfying*

$$\|\varphi(u) - u\|_{\Sigma^s} \le C_s\|\mathbf{z}(u)\|^2\|\eta(u)\|_{\Sigma^{-s}} \tag{2.7}$$

which is a local diffeomorphism and such that

$$\varphi^*\Omega_0 = \Omega_1.$$

We give the proof of Proposition 2.4 in section 5. It will be a direct consequence of an abstract Darboux theorem with error estimate (Proposition 5.8).

We study the dynamics of $u^*=\varphi^{-1}(u)$, where u is the solution of NLS (1.1) with $\|u(0)\|_{H^1}\ll 1$, which reduces to the study of the dynamics of $\mathbf{z}(u^*)$ and $\eta(u^*)$. Since $u(t)$ is the integral curve of the Hamilton vector field $X_E^{(0)}$, $u^*(t)$ is the integral curve of the Hamilton vector field $X_K^{(1)}$, where $K:=\varphi^*E=E(\varphi(\cdot))$. By (2.5), (2.6), we have

$$\mathrm{i}\partial_t\eta=\nabla_\eta K(\mathbf{z},\eta),\quad \mathrm{i}\partial_t\mathbf{z}=(1+A(\mathbf{z}))\nabla_{\mathbf{z}}K(\mathbf{z},\eta). \tag{2.8}$$

To compute the r.h.s. of (2.8), we expand K. Before going into the expansion, we prepare a notation to denote some reminder terms.

Definition 2.5 Let $F\in C^1(\mathcal{B}_{H^1}(0,\delta),\mathbb{R})$ for some $\delta>0$. We write $F=\mathcal{R}_1$ if, for $s\geq 0$, there exists $\delta_s>0$ s.t. for $\|u\|_{H^1}<\delta_s$ we have

$$\|\nabla_\eta F(u)\|_{\Sigma^s}+|\langle\nabla_{\mathbf{z}}F(u),\mathrm{i}\alpha(\mathbf{z})\rangle|\lesssim_s\|u\|_{H^1}^2\left(\|\eta\|_{\Sigma^{-s}}^2+\sum_{\mathbf{m}\in\mathbf{R}_{\min}}|\mathbf{z}^{\mathbf{m}}|^2\right), \tag{2.9}$$

where $\alpha(\mathbf{z})=\mathbf{z}$ or $\|\alpha(\mathbf{z})\|\lesssim\sum_{\mathbf{m}\in\mathbf{R}_{\min}}|\mathbf{z}^{\mathbf{m}}|$. In our notation, if $F=\mathcal{R}_1$ and $G=\mathcal{R}_1$, we will have $F+G=\mathcal{R}_1$. So, an equation like $F+\mathcal{R}_1=\mathcal{R}_1$ will not mean $F=0$ but only $F=\mathcal{R}_1$. This rule will also be applied to $\mathcal{R}_2$ below.

By Taylor expanding $F(s,t)=K(s\mathbf{z},t\eta)$, we have

$$\begin{aligned}K(\mathbf{z},\eta)&=K(0,\eta)+K(\mathbf{z},0)+\int_0^1\partial_s\partial_tK(s\mathbf{z},0)\,ds\\&\quad+\int_0^1\int_0^1(1-t)\partial_s\partial_t^2K(s\mathbf{z},t\eta)\,dtds.\end{aligned} \tag{2.10}$$

Since $\varphi(\eta)=\eta$ by (2.7), we have $K(0,\eta)=E(\eta)$. Similarly, since $\varphi(\phi(\mathbf{z}))=\phi(\mathbf{z})$, we have $K(\mathbf{z},0)=E(\phi(\mathbf{z}))$. The third term of the r.h.s. of (2.10) is

$$\int_0^1\partial_s\partial_tK(s\mathbf{z},0)\,ds=\partial_tK(\mathbf{z},0)=\left\langle\nabla_\eta K(\mathbf{z},0),\eta\right\rangle,$$

because $D_\eta K(0,0)=0$. The following lemma is the crux of this paper.

Lemma 2.6 (First Cancellation Lemma) *We have, near the origin,*

$$\nabla_\eta K(\mathbf{z},0)=P_cD\varphi(\phi(\mathbf{z}))^*\left(\sum_{\mathbf{m}\in\mathbf{R}_{\min}}\mathbf{z}^{\mathbf{m}}G_{\mathbf{m}}+\mathcal{R}(\mathbf{z})\right). \tag{2.11}$$

Proof We fix arbitrary $\mathbf{z}_0=(z_{01},\cdots,z_{0N})\in\mathcal{B}_{\mathbb{C}^N}(0,\delta_0)$ with δ_0 sufficiently small. It is enough to prove (2.11) with $\mathbf{z}=\mathbf{z}_0$. We set $\mathbf{z}_0(t)=(z_{01}(t),\cdots,z_{0N}(t))\in\mathbb{C}^N$ with

$$z_{0j}(t)=e^{-\mathrm{i}\varpi_j(|\mathbf{z}_0|^2)t}z_{0j},$$

where ϖ_j is also given in Proposition 1.11. Consider the non-autonomous Hamiltonian

$$E_{\mathbf{z}_0}(u,t) := E(u) - \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \left\langle \mathbf{z}_0(t)^{\mathbf{m}} G_{\mathbf{m}}, u \right\rangle - \langle \mathcal{R}(\mathbf{z}_0(t)), u\rangle .$$

Then, the Hamilton vector field $X^{(0)}_{E_{\mathbf{z}_0}}(u,t)$ of $E_{\mathbf{z}_0}(u,t)$ associated with the symplectic form Ω_0 is

$$\mathrm{i}X^{(0)}_{E_{\mathbf{z}_0}}(u,t) = Hu + g(|u|^2)u - \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}_0(t)^{\mathbf{m}} G_{\mathbf{m}} - \mathcal{R}(\mathbf{z}_0(t)) .$$

Thus, by Proposition 1.11, $\phi(\mathbf{z}_0(t))$ is the integral curve of this flow with initial value $\phi(\mathbf{z}_0)$.
Consider now the pullback of $E_{\mathbf{z}_0}(u,t)$ by the φ of Proposition 2.4. By Taylor expansion we get

$$\varphi^* E_{\mathbf{z}_0}(u,t) = K(u) - \left\langle \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}_0(t)^{\mathbf{m}} G_{\mathbf{m}} + \mathcal{R}(\mathbf{z}_0(t)), \varphi(u) \right\rangle =$$
$$K(u) - \left\langle \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}_0(t)^{\mathbf{m}} G_{\mathbf{m}} + \mathcal{R}(\mathbf{z}_0(t)), \phi(\mathbf{z}) + D\varphi(\phi(\mathbf{z}))\eta + \int_0^1 (1-s) D^2\varphi(\phi(\mathbf{z}+s\eta))(\eta,\eta) \right\rangle .$$

Differentiating in η at $\eta = 0$, yields

$$\nabla_\eta \left(\varphi^* E_{\mathbf{z}_0}(t)\right)\big|_{\eta=0} = \nabla_\eta K\big|_{\eta=0} - P_c(D\varphi(\phi(\mathbf{z})))^* \left(\sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}_0(t)^{\mathbf{m}} G_{\mathbf{m}} + \mathcal{R}(\mathbf{z}_0(t)) \right) .$$

Because of (2.7), we know that $\varphi^{-1}(\phi(\mathbf{z})) = \phi(\mathbf{z})$ for all $\mathbf{z}$. Then, $\phi(\mathbf{z}_0)$ is an integral trajectory also for $\varphi^* E_{\mathbf{z}_0}(u,t)$. But since, in $(\mathbf{z},\eta)$, integral trajectories satisfy $\mathrm{i}\dot\eta = \nabla_\eta \left(\varphi^* E_{\mathbf{z}_0}(t)\right)$, from $\eta \equiv 0$ and, thus, from $\dot\eta \equiv 0$, it follows that $\nabla_\eta \left(\varphi^* E_{\mathbf{z}_0}(t)\right)\big|_{\eta=0} \equiv 0$. So, for $t = 0$, we obtain (2.11).

By Proposition 2.4, Definition 2.5 and (2.11), we have

$$\left\langle \nabla_\eta K(\mathbf{z},0), \eta \right\rangle = \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \left\langle \mathbf{z}^{\mathbf{m}} G_{\mathbf{m}}, \eta \right\rangle + \mathcal{R}_1 . \tag{2.12}$$

We next study the last term in r.h.s. of (2.10). By direct computation, for the linear part of the energy we have

$$\begin{aligned}
&\partial_s \partial_t^2 \left\langle H\varphi(s\phi(\mathbf{z})+t\eta), \varphi(s\phi(\mathbf{z})+t\eta)\right\rangle \\
&\quad = 4\left\langle HD^2\varphi(s\phi(\mathbf{z})+t\eta)(\phi,\eta), D\varphi(s\phi(\mathbf{z})+t\eta)\eta\right\rangle \\
&\qquad + 2\left\langle HD^2\varphi(s\phi(\mathbf{z})+t\eta)(\eta,\eta), D\varphi(s\phi(\mathbf{z})+t\eta)\phi\right\rangle \\
&\qquad + 2\left\langle HD^3\varphi(s\phi(\mathbf{z})+t\eta)(\phi,\eta,\eta), \varphi(s\phi(\mathbf{z})+\eta)\right\rangle.
\end{aligned}$$

Thus,

$$\frac{1}{2}\int_0^1\int_0^1 (1-t)\partial_s\partial_t^2 \left\langle H\varphi(s\phi(\mathbf{z})+t\eta), \varphi(s\phi(\mathbf{z})+t\eta)\right\rangle\, dtds = \mathcal{R}_1. \tag{2.13}$$

For the nonlinear part of the energy, we have

$$\begin{aligned}
&\partial_s\partial_t^2 \int_{\mathbb{R}^3} G(|u_{t,s}|^2)\, dx = 4\left\langle 2g'' u_{t,s}\left(\mathrm{Re}\left(\overline{u_{t,s}}\,\widetilde{\eta}\right)\right)^2 + 2g'\widetilde{\eta}\mathrm{Re}\left(\overline{u_{t,s}}\,\widetilde{\eta}\right) + g' u_{t,s}|\widetilde{\eta}|^2, \widetilde{\phi}\right\rangle \\
&\quad + 2\left\langle 2g' u_{t,s}\mathrm{Re}\left(u_{t,s}\overline{D^2\varphi(\eta,\eta)}\right) + gD^2\varphi(\eta,\eta), \widetilde{\phi}\right\rangle \\
&\quad + 4\left\langle 2g' u_{t,s}\mathrm{Re}\left(\overline{u_{t,s}}\,\widetilde{\eta}\right) + g\widetilde{\eta}, D^2\varphi(\phi(\mathbf{z}),\eta)\right\rangle \\
&\quad + 2\left\langle g u_{t,s}, D^3\varphi(\phi(\mathbf{z}),\eta,\eta)\right\rangle.
\end{aligned} \tag{2.14}$$

where $u_{t,s} := \varphi(s\phi(\mathbf{z})+t\eta)$, $\widetilde{\eta} = D\varphi(s\phi(\mathbf{z})+t\eta)\eta$, $\widetilde{\phi} = D\varphi(s\phi(\mathbf{z}+t\eta))\phi(\mathbf{z})$, $g^{(k)} = g^{(k)}(|u_{t,s}|^2)$ and $D^{k+1}\varphi = D^{k+1}\varphi(s\phi(\mathbf{z})+t\eta)$ for $k=0,1,2$.

To handle these terms, we introduce another notation of error terms.

Definition 2.7 Let $\delta > 0$ and $F \in C^3(\mathcal{B}_{H^1}(0,\delta),\mathbb{R})$. We write $F = \mathcal{R}_2$ if F is a linear combination of functions of the form

$$\int_0^1\int_0^1 (1-t)\left\langle f(u_{t,s}), \mathfrak{f}(u_{t,s})(\phi,\eta,\eta)\right\rangle\, dtds,$$

where $f(u)(x) = \tilde{f}(\mathrm{Re}\, u(x), \mathrm{Im}\, u)$ with $\tilde{f} \in C^\infty(\mathbb{R}^2,\mathbb{C})$ and where either one or the other of the following two conditions are satisfied:

(I) $|\tilde{f}(s_1,s_2)| \lesssim |s|\,\langle s\rangle^2$, $|\partial_{s_j}\tilde{f}(s_1,s_2)| \lesssim \langle s\rangle^2$ $(j=1,2)$, $|\partial_{s_j}\partial_{s_k}\tilde{f}(s_1,s_2)| \lesssim \langle s\rangle$ $(j,k=1,2)$ and $\mathfrak{f}(u)(\phi,\eta,\eta) := (D\varphi(u)\phi)\,(D\varphi(u)\eta)^2$;

(II) $|\tilde{f}(s_1,s_2)| \lesssim |s|^2\,\langle s\rangle^2$, $|\partial_{s_j}\tilde{f}(s_1,s_2)| \lesssim |s|\,\langle s\rangle^2$ $(j=1,2)$, $|\partial_{s_j}\partial_{s_k}\tilde{f}(s_1,s_2)| \lesssim \langle s\rangle^2$ $(j,k=1,2)$ and $\mathfrak{f}(u)(\phi,\eta,\eta) := (D\varphi(u)\phi)\, D^2\varphi(u)(\eta,\eta)$ or $D\varphi(u)\eta D^2\varphi(u)(\phi,\eta)$ or $D^3\varphi(u)(\phi,\eta,\eta)$.

Here, $s=(s_1,s_2)$ and $|s| = (s_1^2+s_2^2)^{1/2}$, $\langle s\rangle = (1+s_1^2+s_2^2)^{1/2}$.

Thus, we have

$$\frac{1}{2}\int_0^1\int_0^1 (1-t)\partial_s\partial_t^2 \int_{\mathbb{R}^3} G(|\varphi(s\phi(\mathbf{z})+\eta)|^2)\,dx = \mathcal{R}_2. \tag{2.15}$$

We record that under the assumption $\|u\|_{H^1} \lesssim 1$, we have

$$\|\nabla_{\mathbf{z}}\mathcal{R}_2\| \lesssim \|u\|_{H^1}\|\eta\|_{L^6}^2. \tag{2.16}$$

Summarizing, (2.10), (2.12), (2.13) and (2.15) we have

$$K(u) = E(\phi(\mathbf{z})) + E(\eta) + \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \langle \mathbf{z}^{\mathbf{m}} G_{\mathbf{m}}, \eta\rangle + \mathcal{R}_1 + \mathcal{R}_2. \tag{2.17}$$

We can study the structure of $E(\phi(\mathbf{z}))$ by an argument similar to the proof of Lemma 2.6.

Lemma 2.8 (Second Cancellation Lemma) *We have*

$$(1 + A(\mathbf{z}))\nabla_{\mathbf{z}} E(\phi(\mathbf{z})) = \Lambda(|\mathbf{z}|^2)\mathbf{z} + B(\mathbf{z}), \tag{2.18}$$

where, $\Lambda(|\mathbf{z}|^2)\mathbf{w} := (\varpi_1(|\mathbf{z}|^2)w_1, \cdots, \varpi_N(|\mathbf{z}|^2)w_N)$ *and* $\|B(\mathbf{z})\| \lesssim \sum_{\mathbf{m}\in\mathbf{R}_{\min}} |\mathbf{z}^{\mathbf{m}}|$.

Proof Fix $\mathbf{z}_0 \in \mathcal{B}_{\mathbb{C}^N}(0,\delta_0)$ and consider $\mathbf{z}_0(t)$ and $E_{\mathbf{z}_0}(u,t)$ as in the proof of Lemma 2.6. Then $(\mathbf{z}_0(t), 0)$ is an integral curve of $\varphi^* E_{\mathbf{z}_0}(u,t)$ and for $t = 0$ we have

$$\begin{aligned}\Lambda(|\mathbf{z}_0|^2)\mathbf{z}_0 &= (1 + A(\mathbf{z}_0))\ \nabla_{\mathbf{z}}|_{\mathbf{z}=\mathbf{z}_0,\eta=0,t=0} \left(\varphi^* E_{\mathbf{z}_0}(u,t)\right)\\ &= (1 + A(\mathbf{z}_0))\ \nabla_{\mathbf{z}}|_{\mathbf{z}=\mathbf{z}_0} \left(E(\phi(\mathbf{z})) - \left\langle \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}_0^{\mathbf{m}} G_{\mathbf{m}} + \mathcal{R}(\mathbf{z}_0), \phi(\mathbf{z})\right\rangle\right).\end{aligned}$$

This yields the equality (2.18) at $\mathbf{z} = \mathbf{z}_0$ with the desired bound on the remainder term, thanks to

$$\begin{aligned}&\left\| \nabla_{\mathbf{z}}|_{\mathbf{z}=\mathbf{z}_0} \left\langle \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}_0^{\mathbf{m}} G_{\mathbf{m}} + \mathcal{R}(\mathbf{z}_0), \phi(\mathbf{z})\right\rangle\right\| = \left\| \left\langle \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}_0^{\mathbf{m}} G_{\mathbf{m}} + \mathcal{R}(\mathbf{z}_0),\ \nabla_{\mathbf{z}}|_{\mathbf{z}=\mathbf{z}_0}\phi(\mathbf{z})\right\rangle\right\|\\ &\le \left\| \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}_0^{\mathbf{m}} G_{\mathbf{m}} + \mathcal{R}(\mathbf{z}_0)\right\|_{L^2(\mathbb{R}^3)} \|\ \nabla_{\mathbf{z}}|_{\mathbf{z}=\mathbf{z}_0}\phi(\mathbf{z})\|_{L^2(\mathbb{R}^3)} \lesssim \sum_{\mathbf{m}\in\mathbf{R}_{\min}} |\mathbf{z}_0^{\mathbf{m}}|.\end{aligned}$$

3 Proof of the Main Theorem

Given an interval $I \subseteq \mathbb{R}$ we set

$$\mathrm{Stz}^j(I) := L_t^\infty H^j(I) \cap L_t^2 W^{j,6}(I), \quad \mathrm{Stz}^{*j}(I) := L_t^1 H^j(I) + L_t^2 W^{j,6/5}(I),\ \ j = 0, 1,$$

where $H^0 = L^2$ and $W^{0,p} = L^p$. We will be using the Strichartz inequality, see [26]:

$$\|e^{-\mathrm{i}tH} P_c v\|_{\mathrm{Stz}^j} \lesssim \|v\|_{H^j}, \ \|\int_0^t e^{-\mathrm{i}(t-s)H} f(s)\, ds\|_{\mathrm{Stz}^j} \lesssim \|f\|_{\mathrm{Stz}^{*j}}, \ j = 0, 1.$$

We now consider the Hamiltonian system in the $(\mathbf{z}, \eta)$ with Hamiltonian K and symplectic form Ω_1. Then we have the following.

Theorem 3.1 (Main Estimates) *There exist $\delta_0 > 0$ and $C_0 > 0$ s.t. if the constant $\|u_0\|_{H^1} < \delta_0$ for $I = [0, \infty)$ and $C = C_0$ we have:*

$$\|\eta\|_{\mathrm{Stz}^1(I)} + \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \|\mathbf{z}^{\mathbf{m}}\|_{L_t^2(I)} \le C\|u_0\|_{H^1}, \tag{3.1}$$

$$\|\mathbf{z}\|_{W_t^{1,\infty}(I)} \le C\|u_0\|_{H^1}. \tag{3.2}$$

Furthermore, there exists $\rho_+ \in [0, \infty)^N$ s.t. there exist a j_0 with $\rho_{+j} = 0$ for $j \ne j_0$, and there exists $\eta_+ \in H^1$ with $\|\eta_+\|_{H^1} \le C\epsilon$ for $C = C_0$, such that

$$\lim_{t\to+\infty} \|\eta(t) - e^{\mathrm{i}t\Delta}\eta_+\|_{H^1} = 0 \quad , \quad \lim_{t\to+\infty} |z_j(t)| = \rho_{+j}. \tag{3.3}$$

Note that from the energy and mass conservation, Definitions 2.5 and 2.7, (2.16), (2.17) and Lemma 2.8 and we have the apriori bound

$$\|\mathbf{z}\|_{W_t^{1,\infty}(\mathbb{R})} + \|\eta\|_{L_t^\infty H^1(\mathbb{R})} \lesssim \|u_0\|_{H^1}.$$

The proof that Theorem 3.1 implies Theorem 1.14 is like in [6]. Furthermore, by completely routine arguments discussed in [6], (3.1) for $I = [0, \infty)$ is a consequence of the following Proposition.

Proposition 3.2 *There exists a constant $c_0 > 0$ s.t. for any $C_0 > c_0$ there is a value $\delta_0 = \delta_0(C_0)$ s.t. if (3.1) holds for $I = [0, T]$ for some $T > 0$, for $C = C_0$ and for $u_0 \in B_{H^1}(0, \delta_0)$, then in fact for $I = [0, T]$ the inequalities (3.1) holds for $C = C_0/2$.*

The rest of this section is devoted to the proof of Proposition 3.2. In the following, we always assume (3.1) holds for $C = C_0$ and the integration w.r.t. t is always be over I.

We fist estimate the contribution of $\mathcal{R}_j$, $j = 1, 2$.

Lemma 3.3 *Under the assumption of Proposition 3.2, there is a constant $C(C_0)$ such that*

$$\|\nabla_\eta \mathcal{R}_j\|_{\mathrm{Stz}^{*1}} \le C(C_0)\|u_0\|_{H^1}^3, \quad j = 1, 2.$$

Proof For $\mathcal{R}_1$, we have

$$\|\nabla_\eta \mathcal{R}_1\|_{\mathrm{Stz}^{*1}} \le \|\nabla_\eta \mathcal{R}_1\|_{L_t^2\Sigma^1(I)} \lesssim \|u_0\|_{H^1}^2 \left(\|\eta\|_{\mathrm{Stz}^1(I)} + \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \|\mathbf{z}^{\mathbf{m}}\|_{L_t^2(I)} \right) \lesssim C_0\|u_0\|_{H^1}^3.$$

We next estimate type (I) of $\mathcal{R}_2$. Ignoring the integral w.r.t. t and s and the complex conjugate, which are irrelevant in the estimate, we have

$$D_\eta \mathcal{R}_2 w = \left\langle f'(u)w, \widetilde{\phi}\widetilde{\eta}^2\right\rangle + \left\langle f(u), D^2\varphi(u)(\phi, w)\widetilde{\eta}^2 + 2\widetilde{\phi}\widetilde{\eta}D^2\varphi(u)(\eta, w) + 2\widetilde{\phi}\widetilde{\eta}w\right\rangle. \quad (3.4)$$

where $f'(u)w = \partial_R f(u)\,\mathrm{Re}\,w + \partial_I f(u)\,\mathrm{Im}\,w$ and $\widetilde{\phi}$, $\widetilde{\eta}$ are defined in (2.14). The contribution of the first term in the r.h.s. of (3.4) can be estimated as

$$\|f'(u)\widetilde{\phi}\widetilde{\eta}^2\|_{L_t^2L^{6/5}} \lesssim \|\mathbf{z}\|_{L_t^\infty}\|\eta\|_{L_t^\infty L^6}\|\eta\|_{L_t^2L^6} \lesssim C_0^3\|u_0\|_{H^1}^3, \quad (3.5)$$

where we have used $\|\langle u\rangle\|_{L^\infty+L^6} \lesssim 1$ and the Sobolev embedding $H^1 \hookrightarrow L^6$. Furthermore,

$$\begin{aligned}\|\nabla_x\left(f'(u)\widetilde{\phi}\widetilde{\eta}^2\right)\|_{L_t^2L^{6/5}} &\lesssim \|f''(u)\nabla_x u\widetilde{\phi}\widetilde{\eta}^2\|_{L_t^2L^{6/5}} + \|f'(u)\nabla_x\widetilde{\phi}\widetilde{\eta}^2\|_{L_t^2L^{6/5}} \\ &+ \|f'(u)\widetilde{\phi}\widetilde{\eta}\nabla_x\widetilde{\eta}\|_{L_t^2L^{6/5}}, \end{aligned} \quad (3.6)$$

and, using Sobolev's embedding $W^{1,6} \hookrightarrow L^\infty$,

$$\|f''(u)\nabla_x u\widetilde{\phi}\widetilde{\eta}^2\|_{L_t^2L^{6/5}} \lesssim \|\langle u\rangle\widetilde{\phi}\|_{L_t^\infty L^6}\|\nabla_x u\|_{L_t^\infty L^2}\|\eta\|_{L_t^2L^\infty}\|\eta\|_{L_t^\infty L^6} \lesssim C_0^3\|u_0\|_{H^1}^3.$$

Similar estimates hold for the other two terms in (3.6).

Turning to the contribution of the second term in (3.4), we have

$$\begin{aligned}\sup_{\|w\|_{(W^{1,6/5})^*}\le 1}\left|\left\langle f(u), D^2\varphi(u)(\phi, w)\widetilde{\eta}^2\right\rangle\right| &\lesssim \|f(u)\eta^2\|_{\Sigma^{-1}}\|\phi\|_{\Sigma^{-1}}\|w\|_{\Sigma^{-1}} \lesssim \|\mathbf{z}\|\|f(u)\eta^2\|_{L^{6/5}} \\ &\lesssim \|\mathbf{z}\|\|u\|_{L^2\cap L^6}\|\langle u\rangle\|_{L^6+L^\infty}^2\|\eta\|_{L^6}^2.\end{aligned}$$

where we have used $(W^{1,6/5})^* \hookrightarrow \Sigma^{-1}$ and $L^{6/5} \hookrightarrow \Sigma^{-1}$ which hold by duality. Thus, we have the estimate $\lesssim C_0\|u_0\|_{H^1}^3$ for this term too. The third term in (3.4) can be estimated just as the second term and the fourth term can be estimated just as the first term.

The estimates of the type (II) terms in $\mathcal{R}_2$ is similar, easier and is omitted.

From

$$\mathrm{i}\partial_t\eta = \nabla_\eta K(u) = P_c\left(H\eta + g(|\eta|^2)\eta + \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}}G_{\mathbf{m}} + \nabla_\eta\mathcal{R}_1 + \nabla_\eta\mathcal{R}_2\right),$$

by Lemma 3.3 we obtain

$$\|\eta\|_{\mathrm{Stz}^1} \lesssim \|u_0\|_{H^1} + C(C_0)\|u_0\|_{H^1}^3 + \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \|\mathbf{z}^{\mathbf{m}}\|_{L_t^2}. \tag{3.7}$$

We need bounds on $\mathbf{z}$. We set $Z := \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}} R_+(\mathbf{m}\cdot\boldsymbol{\omega})P_c G_{\mathbf{m}}$ and $\xi := \eta + Z$, where $R_+(\lambda) := (H-\lambda-\mathrm{i}0)^{-1}$. Then,

$$\mathrm{i}\partial_t \xi = P_c\left(H\xi + g(|\eta|^2)\eta + \nabla_\eta \mathcal{R}_1 + \nabla_\eta \mathcal{R}_2 + \mathcal{R}_3\right),$$

where $\mathcal{R}_3 := \mathrm{i}\partial_t Z - HZ + \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}} P_c G_{\mathbf{m}}$, which satisfies

$$\mathcal{R}_3 = \sum_{\mathbf{m}\in\mathbf{R}_{\min}} a_{\mathbf{m}} R_+(\mathbf{m}\cdot\boldsymbol{\omega})P_c G_{\mathbf{m}}, \text{ where } a_{\mathbf{m}} := \mathrm{i}\partial_t(\mathbf{z}^{\mathbf{m}}) - (\mathbf{m}\cdot\boldsymbol{\omega})\mathbf{z}^{\mathbf{m}}.$$

Lemma 3.4 *Under the assumption of Proposition 3.2, there is a constant $C(C_0)$ such that*

$$\|a_{\mathbf{m}}\|_{L_t^2(I)} \le C(C_0)\|u_0\|_{H^1}^3. \tag{3.8}$$

Proof We have $a_{\mathbf{m}} = \sum_{j=1}^N a_{\mathbf{m},j}$ with

$$a_{\mathbf{m},j} = \begin{cases} m_j(\mathrm{i}\partial_t z_j - \omega_j z_j)\frac{\mathbf{z}^{\mathbf{m}}}{z_j} & \text{if } m_j > 0 \\ m_j\overline{(\mathrm{i}\partial_t z_j - \omega_j z_j)}\frac{\mathbf{z}^{\mathbf{m}}}{\overline{z_j}} & \text{if } m_j < 0. \end{cases} \tag{3.9}$$

By (2.6), (2.17) and Lemma 2.8, we have

$$\begin{aligned} &\mathrm{i}\partial_t z_j - \omega_j z_j = (\mathrm{i}\partial_t \mathbf{z} - \Lambda(0)\mathbf{z})\cdot \mathbf{e}_j \\ &= \left(\left(\Lambda(|\mathbf{z}|^2) - \Lambda(0)\right)\mathbf{z} + B(\mathbf{z}) + (1+A(\mathbf{z}))\nabla_{\mathbf{z}}\left(\sum_{\mathbf{R}_{\min}}\langle \mathbf{z}^{\mathbf{m}} G_{\mathbf{m}}, \eta\rangle + \mathcal{R}_1 + \mathcal{R}_2\right)\right)\cdot \mathbf{e}_j. \end{aligned} \tag{3.10}$$

We estimate each $a_{\mathbf{m},j}$ by distinguishing the contribution coming from the terms in the last line in (3.10).
Using $\left(\Lambda(|\mathbf{z}|^2) - \Lambda(0)\right)\mathbf{z}\cdot\mathbf{e}_j = (\varpi_j(|\mathbf{z}|^2) - \omega_j)z_j$, for the first term we have

$$\begin{aligned} &m_j\left\| \left(\Lambda(|\mathbf{z}|^2) - \Lambda(0)\right)\mathbf{z}\cdot\mathbf{e}_j \frac{\mathbf{z}^{\mathbf{m}}}{z_j}\right\|_{L_t^2} = m_j\|(\varpi_j(|\mathbf{z}|^2) - \omega_j)\mathbf{z}^{\mathbf{m}}\|_{L_t^2} \\ &\lesssim \|\mathbf{z}\|_{L_t^\infty}^2 \|\mathbf{z}^{\mathbf{m}}\|_{L_t^2} \le C_0^3\|u_0\|_{H^1}^3. \end{aligned}$$

Similarly, by Lemma 2.8,

$$m_j \|B(\mathbf{z}) \cdot \mathbf{e}_j \frac{\mathbf{z}^{\mathbf{m}}}{z_j}\|_{L^2_t} \lesssim \sum_{\mathbf{n}\in\mathbf{R}_{\min}} m_j \|\mathbf{z}^{\mathbf{n}} \frac{\mathbf{z}^{\mathbf{m}}}{z_j}\|_{L^2_t} \le \sum_{\mathbf{n}\in\mathbf{R}_{\min}} m_j \|\mathbf{z}^{\mathbf{n}}\|_{L^2_t} \|\frac{\mathbf{z}^{\mathbf{m}}}{z_j}\|_{L^\infty_t} \lesssim C_0^3 \|u_0\|^3_{H^1},$$

from the fact that $\mathbf{m} \in \mathbf{R}_{\min}$implies $\|\mathbf{m}\| \ge 3$.
For $\mathbf{n} \in \mathbf{R}_{\min}$, we have

$$m_j \|(1 + A(\mathbf{z}))\nabla_{\mathbf{z}} \langle \mathbf{z}^{\mathbf{n}} G_{\mathbf{m}}, \eta\rangle \cdot \mathbf{e}_j \frac{\mathbf{z}^{\mathbf{m}}}{z_j}\|_{L^2_t} \lesssim m_j \|\eta\|_{L^2_t L^6_x} \|\nabla_{\mathbf{z}} \mathbf{z}^{\mathbf{n}}\|_{L^\infty_t} \|\frac{\mathbf{z}^{\mathbf{m}}}{z_j}\|_{L^\infty_t} \lesssim C_0^5 \|u_0\|^5_{H^1}.$$

Similar estimates using (2.9) and (2.16) can be obtained for the terms with $\mathcal{R}_1$ and $\mathcal{R}_2$.

When we seek for the nonlinear effect of the radiation η on the $\mathbf{z}$, we think of Z as the main term and of ξ as a remainder term. We first estimate ξ.

Lemma 3.5 *Under the assumption of Proposition 3.2, there is a constant $C(C_0)$ such that*

$$\|\xi\|_{L^2_t \Sigma^{0-}} \lesssim \|u_0\|_{H^1} + C(C_0)\|u_0\|^3_{H^1}.$$

Here, the the key difference from (3.7) is that the last summation in the r.h.s. of (3.7) has been eliminated. This because the formula $\xi = \eta + Z$ is a normal form expansion designed exactly to eliminate that summation from the equation of ξ.

Proof Since $\xi = \eta + Z$, we have

$$\begin{aligned}\|\xi\|_{L^2_t \Sigma^{0-}} \lesssim & \|e^{-\mathrm{i}tH}\eta(0)\|_{\mathrm{Stz}^0} + \|e^{-\mathrm{i}tH} Z(0)\|_{L^2_t \Sigma^{0-}} + \|g(|\eta|^2)\eta\|_{\mathrm{Stz}^{*0}} + \|\nabla_\eta \mathcal{R}_1\|_{\mathrm{Stz}^{*0}} \\ & + \|\nabla_\eta \mathcal{R}_2\|_{\mathrm{Stz}^{*0}} + \|\int_0^t e^{-\mathrm{i}(t-s)H} P_c \mathcal{R}_3\, ds\|_{L^2_t \Sigma^{0-}}.\end{aligned}$$

Using the estimate $\|e^{-\mathrm{i}tH} R_+(\mathbf{m}\cdot\boldsymbol{\omega}) P_c f\|_{\Sigma^{0-}} \lesssim \langle t\rangle^{-3/2} \|f\|_{\Sigma^0}$ for $\mathbf{m} \in \mathbf{R}_{\min}$, we have

$$\|e^{-\mathrm{i}tH} Z(0)\|_{L^2_t \Sigma^{0-}} \lesssim \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \|\mathbf{z}(0)\|^{\|\mathbf{m}\|} \lesssim \|u_0\|^3_{H^1} \text{ and}$$

$$\left\|\int_0^t e^{-\mathrm{i}(t-s)H} P_c \mathcal{R}_3\, ds\right\|_{L^2_t \Sigma^{0-}} \lesssim \|a_{\mathbf{m}}\|_{L^2_t} \lesssim C(C_0)\|u_0\|^3_{H^1}$$

Therefore, we have the conclusion.

We recall that for $F, G \in C^1(B_{H^1}(0,\delta), \mathbb{R})$ we have the Poisson brackets given by

$$\{F, G\} := DF X^{(1)}_G = \Omega_1(X^{(1)}_F, X^{(1)}_G).$$

Obviously $\{F, G\} = -\{G, F\}$. The relevance here is that, if $u(t)$ is an integral curve of the Hamilton vector field $X_G^{(1)}$, then $\frac{d}{dt} F(u(t)) = \{F, G\}$. Therefore

$$\frac{d}{dt} E(\phi(\mathbf{z})) = \{E(\phi(\mathbf{z})), K(\mathbf{z}, \eta)\} = \left\{ E(\phi(\mathbf{z})), \sum_{\mathbf{m} \in \mathbf{R}_{\min}} \left\langle \mathbf{z}^{\mathbf{m}} G_{\mathbf{m}}, \eta \right\rangle + \mathcal{R}_1 + \mathcal{R}_2 \right\}, \tag{3.11}$$

where we used that $\{E(\phi(\mathbf{z})), E(\phi(\mathbf{z}))\} = \{E(\phi(\mathbf{z})), E(\eta)\} = 0$ because Poisson brackets are anti-symmetric and the symplectic form is diagonal w.r.t. $\mathbf{z}$ and η. For the main Poisson bracket in the r.h.s. (3.11) we claim

$$\begin{aligned} \sum_{\mathbf{m} \in \mathbf{R}_{\min}} \left\{ E(\phi(\mathbf{z})), \left\langle \mathbf{z}^{\mathbf{m}} G_{\mathbf{m}}, \eta \right\rangle \right\} &= - \sum_{\mathbf{m} \in \mathbf{R}_{\min}} \left\langle \nabla_{\mathbf{z}} \left\langle \mathbf{z}^{\mathbf{m}} G_{\mathbf{m}}, \eta \right\rangle, D\mathbf{z} X^{(1)}_{E(\phi(\mathbf{z}))} \right\rangle_{\mathbb{C}^N} \\ &= \sum_{\mathbf{m} \in \mathbf{R}_{\min}} \left\langle \mathrm{i}(\boldsymbol{\omega} \cdot \mathbf{m}) \mathbf{z}^{\mathbf{m}} G_{\mathbf{m}}, \eta \right\rangle + \mathcal{R}_4, \end{aligned} \tag{3.12}$$

where $\mathcal{R}_4 = \sum_{\mathbf{m} \in \mathbf{R}_{\min}} \langle \mathrm{i} \widetilde{a_{\mathbf{m}}} G_{\mathbf{m}}, \eta \rangle$ with $\widetilde{a_{\mathbf{m}}} = D_{\mathbf{z}}(\mathbf{z}^{\mathbf{m}}) \left((\Lambda(|\mathbf{z}|^2) - \Lambda(0))\mathbf{z} + B(\mathbf{z}) \right)$. To prove formula (3.12), using Lemma 2.8 we compute

$$\begin{aligned} &\left\{ E(\phi(\mathbf{z})), \left\langle \mathbf{z}^{\mathbf{m}} G_{\mathbf{m}}, \eta \right\rangle \right\} = - \left\langle \nabla_{\mathbf{z}} \left\langle \mathbf{z}^{\mathbf{m}} G_{\mathbf{m}}, \eta \right\rangle, D\mathbf{z} X^{(1)}_{E(\phi(\mathbf{z}))} \right\rangle_{\mathbb{C}^N} \\ &= \left\langle \nabla_{\mathbf{z}} \left\langle \mathbf{z}^{\mathbf{m}} G_{\mathbf{m}}, \eta \right\rangle, \mathrm{i}(1 + A(z) E(\phi(\mathbf{z})) \right\rangle_{\mathbb{C}^N} = \left\langle \nabla_{\mathbf{z}} \left\langle \mathbf{z}^{\mathbf{m}} G_{\mathbf{m}}, \eta \right\rangle, \mathrm{i}\Lambda(0)\mathbf{z} \right\rangle_{\mathbb{C}^N} \\ &+ \left\langle \nabla_{\mathbf{z}} \left\langle \mathbf{z}^{\mathbf{m}} G_{\mathbf{m}}, \eta \right\rangle, \mathrm{i} \left((\Lambda(|\mathbf{z}|^2) - \Lambda(0))\mathbf{z} + B(\mathbf{z}) \right) \right\rangle_{\mathbb{C}^N}. \end{aligned}$$

By elementary computations, we have the following, which completes the proof of (3.12):

$$\begin{aligned} &\left\langle \nabla_{\mathbf{z}} \left\langle \mathbf{z}^{\mathbf{m}} G_{\mathbf{m}}, \eta \right\rangle, \mathrm{i}\Lambda(0)\mathbf{z} \right\rangle_{\mathbb{C}^N} = 2^{-1} \sum_{j=1,\dots,N} \left\langle \nabla_{\mathbf{z}} \mathbf{z}^{\mathbf{m}} (G_{\mathbf{m}}, \overline{\eta}) + \nabla_{\mathbf{z}} \overline{\mathbf{z}^{\mathbf{m}}} (\overline{G}_{\mathbf{m}}, \eta), \mathrm{i}\omega_j z_j \mathbf{e}_j \right\rangle_{\mathbb{C}^N} \\ &= 2^{-1} \sum_{j=1,\dots,N} \left[\partial_{z_j} \left(\mathbf{z}^{\mathbf{m}} (G_{\mathbf{m}}, \overline{\eta}) + \overline{\mathbf{z}^{\mathbf{m}}} (\overline{G}_{\mathbf{m}}, \eta) \right) \mathrm{i}\omega_j z_j - \partial_{\overline{z}_j} \left(\mathbf{z}^{\mathbf{m}} (G_{\mathbf{m}}, \overline{\eta}) + \overline{\mathbf{z}^{\mathbf{m}}} (\overline{G}_{\mathbf{m}}, \eta) \right) \mathrm{i}\omega_j \overline{z}_j \right] \\ &= 2^{-1} \mathrm{i}(\boldsymbol{\omega} \cdot \mathbf{m}) \mathbf{z}^{\mathbf{m}} (G_{\mathbf{m}}, \overline{\eta}) - 2^{-1} \mathrm{i}(\boldsymbol{\omega} \cdot \mathbf{m}) \overline{\mathbf{z}^{\mathbf{m}}} (\overline{G}_{\mathbf{m}}, \eta) = \left\langle \mathrm{i}(\boldsymbol{\omega} \cdot \mathbf{m}) \mathbf{z}^{\mathbf{m}} G_{\mathbf{m}}, \eta \right\rangle. \end{aligned}$$

Proceeding as in Lemma 3.4 we have

$$\| \widetilde{a_{\mathbf{m}}} \|_{L^2_t(I)} \le C(C_0) \| u_0 \|^3_{H^1}. \tag{3.13}$$

Entering the expansion $\eta = -Z + \xi$, we obtain

$$\sum_{\mathbf{m} \in \mathbf{R}_{\min}} \left\langle \mathrm{i}(\boldsymbol{\omega} \cdot \mathbf{m}) \mathbf{z}^{\mathbf{m}} G_{\mathbf{m}}, \eta \right\rangle = - \sum_{\mathbf{m} \in \mathbf{R}_{\min}} (\boldsymbol{\omega} \cdot \mathbf{m}) |\mathbf{z}|^{2|\mathbf{m}|} \langle \mathrm{i} P_c G_{\mathbf{m}}, R_+(\mathbf{m} \cdot \boldsymbol{\omega}) P_c G_{\mathbf{m}} \rangle + \mathcal{R}_5 + \mathcal{R}_6, \tag{3.14}$$

where

$$\mathcal{R}_5 = - \sum_{\mathbf{m},\mathbf{n}\in\mathbf{R}_{\min},\ \mathbf{m}\neq\mathbf{n}} \left\langle \mathrm{i}(\boldsymbol{\omega}\cdot\mathbf{m})\mathbf{z}^{\mathbf{m}}G_{\mathbf{m}}, \mathbf{z}^{\mathbf{n}}R_{+}(\mathbf{m}\cdot\boldsymbol{\omega})P_cG_{\mathbf{n}}\right\rangle,$$

$$\mathcal{R}_6 = \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \left\langle \mathrm{i}(\boldsymbol{\omega}\cdot\mathbf{m})\mathbf{z}^{\mathbf{m}}G_{\mathbf{m}}, \xi\right\rangle.$$

Lemma 3.6 *We have, for a fixed constant* c_0

$$\sum_{j=1,2} \| \left\langle \nabla_{\mathbf{z}}\mathcal{R}_j, D\mathbf{z}X^{(1)}_{E(\phi(\mathbf{z}))}\right\rangle \|_{L^1_t(I)} + \sum_{j=4,6} \|\mathcal{R}_j\|_{L^1_t(I)} \leq c_0C_0\|u_0\|^2_{H^1} + C(C_0)\left(\|u_0\|^3_{H^1} + \|u_0\|^4_{H^1}\right). \tag{3.15}$$

Here the crucial point is that in the quadratic term we have C_0 instead of C_0^2, while the exact dependence in C_0 of $C(C_0)$ is immaterial.

Proof The main bound is the following, using Lemma 3.5 and the a priori estimate (3.1),

$$\|\mathcal{R}_6\|_{L^1_t(I)} \lesssim \|\xi\|_{L^2_t\Sigma^{0-}} \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \|\mathbf{z}^{\mathbf{m}}\|_{L^2_t(I)} \lesssim C_0\left(\|u_0\|_{H^1} + C(C_0)\|u_0\|^3_{H^1}\right)\|u_0\|_{H^1}.$$

Turning to the remainders, for $j=1$ the upper bound $\leq C(C_0)\|u_0\|^3_{H^1}$ follows from (2.9) combined with Lemma 2.8 and for $j=2$ follows from (2.16) and the a priori estimates (3.1). The upper bound $\leq C(C_0)\|u_0\|^4_{H^1}$ for $j=4$ follows from (3.13).

Lemma 3.7 *We have*

$$\left|\int_I \mathcal{R}_5\, dt\right| \lesssim C_0^2\|u_0\|^4_{H^1}. \tag{3.16}$$

Proof Let $\mathbf{m}\neq\mathbf{n}$. By (2.8), (2.17) and Lemma 2.8, we have

$$\mathrm{i}\partial_t(\mathbf{z}^{\mathbf{m}}\mathbf{z}^{-\mathbf{n}}) = (\mathbf{m}-\mathbf{n})\cdot\boldsymbol{\omega}\mathbf{z}^{\mathbf{m}}\mathbf{z}^{-\mathbf{n}} + \mathcal{R}_7,$$

where

$$\begin{aligned}\mathcal{R}_7 =&(\mathbf{m}-\mathbf{n})\cdot(\boldsymbol{\varpi}(|\mathbf{z}|^2)-\boldsymbol{\omega})\mathbf{z}^{\mathbf{m}}\mathbf{z}^{-\mathbf{n}}\\ &+ D_{\mathbf{z}}(\mathbf{z}^{\mathbf{m}}\mathbf{z}^{-\mathbf{n}})\left(B(\mathbf{z}) + (1+A(\mathbf{z}))\left(\nabla_{\mathbf{z}}\left(\sum_{\mathbf{R}_{\min}}\left\langle \mathbf{z}^{\mathbf{m}}G_{\mathbf{m}}, \eta\right\rangle + \mathcal{R}_1 + \mathcal{R}_2\right)\right)\right).\end{aligned}$$

Then, we have

$$\|\mathcal{R}_7\|_{L^2} \lesssim C_0\|u_0\|^4_{H^1}.$$

Therefore, since

$$\left\langle \mathrm{i}(\boldsymbol{\omega}\cdot\mathbf{m})\mathbf{z}^{\mathbf{m}}G_{\mathbf{m}},\mathbf{z}^{\mathbf{n}}R_{+}(\boldsymbol{\omega}\cdot\mathbf{n})G_{\mathbf{n}}\right\rangle$$
$$=-\frac{\boldsymbol{\omega}\cdot\mathbf{m}}{(\mathbf{m}-\mathbf{n})\cdot\boldsymbol{\omega}}\left(\partial_t\left\langle\mathbf{z}^{\mathbf{m}}G_{\mathbf{m}},\mathbf{z}^{\mathbf{n}}R_{+}(\boldsymbol{\omega}\cdot\mathbf{n})G_{\mathbf{n}}\right\rangle+\langle\mathcal{R}_7G_{\mathbf{m}},R_{+}(\boldsymbol{\omega}\cdot\mathbf{n})G_{\mathbf{n}}\rangle\right),$$

integrating the above equation over I, we have (3.16).

From (3.11), (3.12), (3.14), Lemmas 3.6 and 3.7, and

$$\left\langle \mathrm{i}G_{\mathbf{m}},(H-\boldsymbol{\omega}\cdot\mathbf{m}-\mathrm{i}0)^{-1}G_{\mathbf{m}}\right\rangle=\frac{1}{16\pi\sqrt{\boldsymbol{\omega}\cdot\mathbf{m}}}\int_{|\zeta|^2=\boldsymbol{\omega}\cdot\mathbf{m}}|\widehat{G_{\mathbf{m}}}(\zeta)|\,d\zeta\gtrsim 1,$$

(for the latter see $(H-\boldsymbol{\omega}\cdot\mathbf{m}-\mathrm{i}0)^{-1}=\mathrm{P.V.}\frac{1}{H-\boldsymbol{\omega}\cdot\mathbf{m}}+\mathrm{i}\pi\delta(H-\boldsymbol{\omega}\cdot\mathbf{m})$ and formula (2.5) p. 156 [23]) and Assumption 1.8, we have

$$\sum_{\mathbf{m}\in\mathbf{R}_{\min}}\|\mathbf{z}^{\mathbf{m}}\|_{L^2}^2\lesssim C_0\|u_0\|_{H^1}^2+C(C_0)\|u_0\|_{L^2}^3. \tag{3.17}$$

By taking $\|u_0\|_{H^1}<\delta_0$ with $\delta_0>0$ sufficiently small, the l.h.s. in (3.17) is smaller than $c_0^2C_0\|u_0\|_{H^1}^2$ for a fixed c_0. Adjusting the constant and using (3.7) we conclude that (3.1) with $C=C_0$ implies

$$\|\eta\|_{\mathrm{Stz}^1(I)}+\sum_{\mathbf{m}\in\mathbf{R}_{\min}}\|\mathbf{z}^{\mathbf{m}}\|_{L_t^2(I)}\le c_0\sqrt{C_0}\|u_0\|_{H^1}<\frac{C_0}{2}\|u_0\|_{H^1}$$

where c_0 is a fixed constant and we are free to choose $C_0>4c_0^2$, so that the last inequality is true. This completes the proof of Proposition 3.2. □

4 Soliton and Refined Profile

In this section, we prove Proposition 1.11. We first note that due to our notation (1.17), $\mathbf{z}^{\mathbf{m}_1}\mathbf{z}^{\mathbf{m}_2}$ is not $\mathbf{z}^{\mathbf{m}_1+\mathbf{m}_2}$ in general. In fact, we have the following elementary lemma.

Lemma 4.1 *Let* $\mathbf{m}_1,\mathbf{m}_2\in\mathbb{Z}^N$ *and* $\mathbf{z}\in\mathbb{C}^N$. *Then,*

$$\mathbf{z}^{\mathbf{m}_1}\mathbf{z}^{\mathbf{m}_2}=|\mathbf{z}|^{|\mathbf{m}_1|+|\mathbf{m}_2|-|\mathbf{m}_1+\mathbf{m}_2|}\mathbf{z}^{\mathbf{m}_1+\mathbf{m}_2}.$$

Proof It suffices to consider $N=1$, where $\mathbf{m}_1,\mathbf{m}_2\in\mathbb{Z}$. If they are both ≥ 0 or ≤ 0, then $|\mathbf{m}_1|+|\mathbf{m}_2|-|\mathbf{m}_1+\mathbf{m}_2|=0$ and it is immediate from (1.17) that $\mathbf{z}^{\mathbf{m}_1}\mathbf{z}^{\mathbf{m}_2}=\mathbf{z}^{\mathbf{m}_1+\mathbf{m}_2}$. Otherwise, we reduce to $\mathbf{m}_1>0>\mathbf{m}_2$. Then $|\mathbf{m}_1|+|\mathbf{m}_2|-|\mathbf{m}_1+\mathbf{m}_2|=2|\mathbf{m}_{j_0}|$ with $|\mathbf{m}_{j_0}|=\min_j|\mathbf{m}_j|$. If $j_0=2$, we have $\mathbf{z}^{\mathbf{m}_1}\mathbf{z}^{\mathbf{m}_2}=\mathbf{z}^{\mathbf{m}_1}\bar{\mathbf{z}}^{|\mathbf{m}_2|}=|\mathbf{z}|^{2|\mathbf{m}_2|}\mathbf{z}^{\mathbf{m}_1+\mathbf{m}_2}$, which is the desired formula. If $j_0=1$, then $\mathbf{z}^{\mathbf{m}_1}\mathbf{z}^{\mathbf{m}_2}=\mathbf{z}^{\mathbf{m}_1}\bar{\mathbf{z}}^{|\mathbf{m}_2|}=|\mathbf{z}|^{2\mathbf{m}_1}\bar{\mathbf{z}}^{|\mathbf{m}_2|-\mathbf{m}_1}=|\mathbf{z}|^{2\mathbf{m}_1}\mathbf{z}^{\mathbf{m}_1+\mathbf{m}_2}$, which again is the desired formula.

Remark 4.2 Each component of $|\mathbf{m}_1|+|\mathbf{m}_2|-|\mathbf{m}_1+\mathbf{m}_2|$ are nonnegative and even integers.

Proof of Proposition 1.11 Recall $\boldsymbol{\phi}=(\phi_1,\cdots,\phi_N)\in(\Sigma^\infty)^N$ are the eigenvectors of H given in Assumption 1.2. We look for an approximate solution of (1.1) of form $u=\phi(\mathbf{z}(t))$ for appropriate

$$\phi(\mathbf{z}):=\mathbf{z}\cdot\boldsymbol{\phi}+\sum_{\mathbf{m}\in\mathbf{NR}_1}\mathbf{z}^{\mathbf{m}}\psi_{\mathbf{m}}(|\mathbf{z}|^2), \tag{4.1}$$

with real valued $\psi_{\mathbf{m}}$ and orthogonality conditions $\langle\psi_{\mathbf{e}_j},\phi_j\rangle=0$ for all $j\in\{1,\cdots,N\}$. We set

$$\widetilde{\phi}_{\mathbf{m}}(|\mathbf{z}|^2):=\begin{cases}\phi_j+\psi_{\mathbf{e}_j}(|\mathbf{z}|^2) & \text{if } \mathbf{m}=\mathbf{e}_j,\\ \psi_{\mathbf{m}}(|\mathbf{z}|^2) & \text{if } \mathbf{m}\in\mathbf{NR}_1\setminus\mathbf{NR}_0.\end{cases} \tag{4.2}$$

Remark 4.3 We will show that $\widetilde{\phi}_{\mathbf{m}}(|\mathbf{z}|^2)$ for $\mathbf{z}=0$ are equal to the $\widetilde{\phi}_{\mathbf{m}}(0)$ given in (1.10) and (1.11).

Assuming $z_j(t)=e^{-\mathrm{i}\varpi_j(|\mathbf{z}|^2)t}z_j$, with ϖ_j to be determined, from $\frac{d}{dt}|z_j(t)|^2=0$ we have

$$\mathrm{i}\partial_t\phi(\mathbf{z})=\sum_{\mathbf{m}\in\mathbf{NR}_1}\mathbf{z}^{\mathbf{m}}\,(\boldsymbol{\varpi}\cdot\mathbf{m})\,\widetilde{\phi}_{\mathbf{m}}. \tag{4.3}$$

Next, we have

$$H\phi(\mathbf{z})=\sum_{\mathbf{m}\in\mathbf{NR}_1}\mathbf{z}^{\mathbf{m}}H\widetilde{\phi}_{\mathbf{m}}. \tag{4.4}$$

We need to Taylor expand the nonlinearity g till the remainder becomes sufficiently small. We will expand now $g(|\phi(\mathbf{z})|^2)\phi(\mathbf{z})=\sum_{\mathbf{m}\in\mathbf{NR}_1}\mathbf{z}^{\mathbf{m}}g_{\mathbf{m}}+\widetilde{R}$ with $\|\widetilde{R}\|_{\Sigma^s}\lesssim_s\|\mathbf{z}\|^2\sum_{\mathbf{R}_{\min}}|\mathbf{z}^{\mathbf{m}}|$. We start with

$$\begin{aligned}|\phi(\mathbf{z})|^2&=\left(\sum_{\mathbf{m}_1\in\mathbf{NR}_1}\mathbf{z}^{\mathbf{m}_1}\widetilde{\phi}_{\mathbf{m}_1}\right)\left(\sum_{\mathbf{m}_2\in\mathbf{NR}_1}\mathbf{z}^{-\mathbf{m}_2}\widetilde{\phi}_{\mathbf{m}_2}\right)\\&=\sum_{\mathbf{m}\in\mathbf{NR}_1}|\mathbf{z}|^{2|\mathbf{m}|}\widetilde{\phi}_{\mathbf{m}}^2+\sum_{\substack{\mathbf{m}_1,\mathbf{m}_2\in\mathbf{NR}_1\\ \mathbf{m}_1\neq\mathbf{m}_2}}\mathbf{z}^{\mathbf{m}_1}\mathbf{z}^{-\mathbf{m}_2}\widetilde{\phi}_{\mathbf{m}_1}\widetilde{\phi}_{\mathbf{m}_2}.\end{aligned}$$

Claim 4.4 Assume $\|\widetilde{\phi}_{\mathbf{m}}\|_{\Sigma^s} \lesssim_s 1$ for all $\mathbf{m} \in \mathbf{NR}_1$. Then, there exists $M > 0$ s.t. for all $\mathbf{z} \in \mathbb{C}^N$ with $\|\mathbf{z}\| \leq 1$,

$$\left\| \left(\sum_{\mathbf{m}_1 \neq \mathbf{m}_2} \mathbf{z}^{\mathbf{m}_1} \mathbf{z}^{-\mathbf{m}_2} \widetilde{\phi}_{\mathbf{m}_1} \widetilde{\phi}_{\mathbf{m}_2} \right)^{M+1} \left(\sum_{\mathbf{m}_3 \in \mathbf{NR}_1} \mathbf{z}^{\mathbf{m}_3} \widetilde{\phi}_{\mathbf{m}_3} \right) \right\|_{\Sigma^s} \lesssim_s \|\mathbf{z}\|^2 \sum_{\mathbf{m} \in \mathbf{R}_{\min}} |\mathbf{z}^{\mathbf{m}}|. \tag{4.5}$$

Proof An $M \in \mathbb{N}$ such that $\omega_1 + M \min_{1 \leq j \leq N-1} \left(\omega_{j+1} - \omega_j \right) > 0$ will work. To begin, we remark that for $\|\mathbf{z}\| \leq 1$ we have $|\mathbf{z}^{\mathbf{m}_1} \mathbf{z}^{-\mathbf{m}_2}| \leq \|\mathbf{z}\|^2$ for $\mathbf{m}_1 \neq \mathbf{m}_2$. Indeed, by Lemma 4.1 this can only fail if $|\mathbf{m}_1| + |\mathbf{m}_2| - |\mathbf{m}_1 - \mathbf{m}_2| = 0$. This implies $m_{1j} m_{2j} \geq 0$ for all $j = 1, ..., N$. Furthermore, if the inequality fails, we can reduce to the case $|\mathbf{m}_1| - |\mathbf{m}_2| = \mathbf{e}_{j_0}$ for an index j_0. So $m_{1j} = m_{2j}$ for all $j \neq j_0$, and $m_{1 j_0} = m_{2 j_0} \pm 1$. This is incompatible with $\sum \mathbf{m}_1 = \sum \mathbf{m}_2 = 1$.
With the above remark, we can take one of the factors of the $M + 1$–th power in (4.5) bounding it with $\|\mathbf{z}\|^2$, concluding that to prove (4.5) it suffices to show that for $\mathbf{m}_{1j}, \mathbf{m}_{2j}, \mathbf{m}_3 \in \mathbf{NR}_1$ with $\mathbf{m}_{1j} \neq \mathbf{m}_{2j}$, there exists $\mathbf{m} \in \mathbf{R}_{\min}$ s.t.

$$|\mathbf{z}^{\mathbf{m}_3}| \prod_{j=1}^{M} |\mathbf{z}^{\mathbf{m}_{1j}} \mathbf{z}^{-\mathbf{m}_{2j}}| \leq |\mathbf{z}^{\mathbf{m}_3}| \prod_{j=1}^{M} |\mathbf{z}^{\mathbf{m}_{1j} - \mathbf{m}_{2j}}| \leq |\mathbf{z}^{\mathbf{m}}| \text{ when } \|\mathbf{z}\| \leq 1, \tag{4.6}$$

where the first inequality follows from Lemma 4.1. Noticing that complex conjugation does not change absolute value, we conclude that each factor $|\mathbf{z}^{\mathbf{m}_{1j} - \mathbf{m}_{2j}}|$ has at least one factor $|z_{a_j} \overline{z}_{b_j}|$ with $a_j > b_j$. There is a nonzero component $m_{3k} \neq 0$ of $\mathbf{m}_3$. Set

$$\mathbf{n} := \mathbf{e}_k + \sum_{j=1}^{M} \left(\mathbf{e}_{a_j} - \mathbf{e}_{b_j} \right).$$

Obviously $\sum \mathbf{n} = 1$. Moreover, $\mathbf{n} \in \mathbf{R}$, since, by our choice of M,

$$\boldsymbol{\omega} \cdot \mathbf{n} = \omega_k + \sum_{j=1}^{M} \left(\omega_{a_j} - \omega_{b_j} \right) \geq \omega_1 + M \min_{1 \leq j \leq N-1} \left(\omega_{j+1} - \omega_j \right) > 0.$$

But for any $\mathbf{n} \in \mathbf{R}$ there exists an $\mathbf{m} \in \mathbf{R}_{\min}$ s.t. $|\mathbf{m}| \preceq |\mathbf{n}|$. Obviously, all the factors of the l.h.s. of (4.6) which we ignored are ≤ 1. This proves (4.6) and completes the proof of Claim 4.4.

We consider a Taylor expansion

$$
\begin{aligned}
& g(|\phi(\mathbf{z})|^2)\phi(\mathbf{z}) \\
&= \left(\sum_{m=0}^{M} \frac{1}{m!} g^{(m)} \left(\sum_{\mathbf{m}\in\mathbf{NR}_1} |\mathbf{z}|^{2|\mathbf{m}|} \widetilde{\phi}_{\mathbf{m}}^2 \right) \left(\sum_{\mathbf{m}_1\neq\mathbf{m}_2} |\mathbf{z}|^{|\mathbf{m}_1|+|\mathbf{m}_2|-|\mathbf{m}_1-\mathbf{m}_2|} \mathbf{z}^{\mathbf{m}_1-\mathbf{m}_2} \widetilde{\phi}_{\mathbf{m}_1} \widetilde{\phi}_{\mathbf{m}_2} \right)^m \right) \\
&\quad \times \left(\sum_{\mathbf{m}_3\in\mathbf{NR}_1} \mathbf{z}^{\mathbf{m}_3} \widetilde{\phi}_{\mathbf{m}_3} \right) + \widetilde{\mathcal{R}},
\end{aligned} \tag{4.7}
$$

where $\widetilde{\mathcal{R}} = \mathcal{O}\left(\|\mathbf{z}\|^2 \sum_{\mathbf{m}\in\mathbf{R}_{\min}} |\mathbf{z}^{\mathbf{m}}|\right)$, by Claim 4.4, and so can be absorbed in the $\mathcal{R}(\mathbf{z}(t))$ in (1.22). Thus, we only have to consider the contribution of the summation. For $0 \le m \le M$, we have

$$
\begin{aligned}
& \left(\sum_{\mathbf{m}_1\neq\mathbf{m}_2} |\mathbf{z}|^{|\mathbf{m}_1|+|\mathbf{m}_2|-|\mathbf{m}_1-\mathbf{m}_2|} \mathbf{z}^{\mathbf{m}_1-\mathbf{m}_2} \widetilde{\phi}_{\mathbf{m}_1} \widetilde{\phi}_{\mathbf{m}_2} \right)^m \left(\sum_{\mathbf{m}_3\in\mathbf{NR}_1} \mathbf{z}^{\mathbf{m}_3} \widetilde{\phi}_{\mathbf{m}_3} \right) \\
&= \sum_{\substack{\mathbf{m}_{1j}\neq\mathbf{m}_{2j} \\ \mathbf{m}_3}} |\mathbf{z}|^{\sum_{j=1}^m (|\mathbf{m}_{1j}|+|\mathbf{m}_{2j}|)+|\mathbf{m}_3|-|\sum_{j=1}^m (\mathbf{m}_{1j}-\mathbf{m}_{2j})+\mathbf{m}_3|} \\
&\quad \times \left(\prod_{j=1}^m \widetilde{\phi}_{\mathbf{m}_{1j}} \widetilde{\phi}_{\mathbf{m}_{2j}} \right) \widetilde{\phi}_{\mathbf{m}_3} \mathbf{z}^{\sum_{j=1}^m (\mathbf{m}_{1j}-\mathbf{m}_{2j})+\mathbf{m}_3}
\end{aligned}
$$

Thus, if for each $\mathbf{m} \in \mathbb{Z}^N$, we set

$$
\begin{aligned}
g_{\mathbf{m}} &:= g_{\mathbf{m}}(|\mathbf{z}|^2, \{\psi_{\mathbf{m}}\}_{\mathbf{m}\in\mathbf{NR}_1}) := \sum_{m=0}^{M} \frac{1}{m!} g^{(m)} \left(\sum_{\mathbf{n}\in\mathbf{NR}_1} |\mathbf{z}|^{2|\mathbf{n}|} \widetilde{\phi}_{\mathbf{n}}^2 \right) \\
&\quad \times \sum_{\substack{\mathbf{m}_3, \mathbf{m}_{kj}\in\mathbf{NR}_1,\ k=1,2,\ j=1,\cdots,m \\ \sum_{j=1}^m (\mathbf{m}_{1j}-\mathbf{m}_{2j})+\mathbf{m}_3=\mathbf{m} \\ \mathbf{m}_{1j}\neq\mathbf{m}_{2j}}} |\mathbf{z}|^{\sum_{j=1}^m (|\mathbf{m}_{1j}|+|\mathbf{m}_{2j}|)+|\mathbf{m}_3|-|\mathbf{m}|} \left(\prod_{j=1}^m \widetilde{\phi}_{\mathbf{m}_{1j}} \widetilde{\phi}_{\mathbf{m}_{2j}} \right) \widetilde{\phi}_{\mathbf{m}_3},
\end{aligned}
$$

for $\widetilde{\mathcal{R}}$ the term given in (4.7) we obtain

$$
g(|\phi(\mathbf{z})|^2)\phi(\mathbf{z}) = \sum_{\mathbf{m}\in\mathbf{NR}_1} \mathbf{z}^{\mathbf{m}} g_{\mathbf{m}} + \sum_{\mathbf{m}\notin\mathbf{NR}_1} \mathbf{z}^{\mathbf{m}} g_{\mathbf{m}} + \widetilde{\mathcal{R}}. \tag{4.8}
$$

Remark 4.5 Notice that $g_{\mathbf{m}}(|\mathbf{z}|^2, \{\psi_{\mathbf{m}}(|\mathbf{z}|^2)\}_{\mathbf{m}\in\mathbf{NR}_1})\big|_{\mathbf{z}=0}$ coincides with the $g_{\mathbf{m}}(0)$ in (1.10) and (1.12).

Summing up, we obtain the following (where in the 2nd line we have a finite sum)

$$
\begin{aligned}
\mathrm{i}\partial_t\phi(\mathbf{z}) - H\phi(\mathbf{z}) - g(|\phi(\mathbf{z})|^2)\phi(\mathbf{z}) = & \sum_{\mathbf{m}\in\mathbf{NR}_1} \mathbf{z}^{\mathbf{m}} \left((\boldsymbol{\varpi}\cdot\mathbf{m})\widetilde{\phi}_{\mathbf{m}} - H\widetilde{\phi}_{\mathbf{m}} - g_{\mathbf{m}}\right) \\
& - \sum_{\mathbf{m}\notin\mathbf{NR}_1} \mathbf{z}^{\mathbf{m}} g_{\mathbf{m}} - \widetilde{\mathcal{R}}.
\end{aligned} \tag{4.9}
$$

Notice that, by the definition of $\mathbf{NR}_1$ and $\mathbf{R}_{\min}$, we have

$$
\| \sum_{\mathbf{m}\notin\mathbf{NR}_1\cup\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}} g_{\mathbf{m}} \|_{\Sigma^s} \lesssim \|\mathbf{z}\|^2 \sum_{\mathbf{m}\in\mathbf{R}_{\min}} |\mathbf{z}^{\mathbf{m}}|.
$$

Thus, entering $G_{\mathbf{m}}$ for $\mathbf{m}\in\mathbf{R}_{\min}$ defined in (1.14) and

$$
\mathcal{R}(\mathbf{z}) := \sum_{\mathbf{m}\notin\mathbf{NR}_1\cup\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}} g_{\mathbf{m}} + \sum_{\mathbf{m}\in\mathbf{R}_{\min}} \mathbf{z}^{\mathbf{m}} \left(g_{\mathbf{m}}(|\mathbf{z}|^2) - G_{\mathbf{m}}\right) + \widetilde{\mathcal{R}},
$$

we have the estimate (1.20). Thus the proof of Proposition 1.11 follows if the 1st summation in the r.h.s. of (4.9) cancels out, that is, if we solve the system

$$
(\boldsymbol{\varpi}\cdot\mathbf{m})\widetilde{\phi}_{\mathbf{m}} = H\widetilde{\phi}_{\mathbf{m}} + g_{\mathbf{m}},\ \ \mathbf{m}\in\mathbf{NR}_1. \tag{4.10}
$$

Here the unknowns are $\boldsymbol{\varpi}$ and $\psi_{\mathbf{m}}$, since the latter determines $\widetilde{\phi}_{\mathbf{m}}$ by (4.2), while $g_{\mathbf{m}}$ are given functions of both the variables $|\mathbf{z}|^2$ and $\{\psi_{\mathbf{m}}\}_{\mathbf{m}\in\mathbf{NR}_1}$. We will later determine $\{\psi_{\mathbf{m}}\}_{\mathbf{m}\in\mathbf{NR}_1}$ as a function of $|\mathbf{z}|^2$, and so at the end $\boldsymbol{\varpi}$ and $g_{\mathbf{m}}$ will depend only on $|\mathbf{z}|^2$. We first focus on (4.10) for $\mathbf{m}=\mathbf{e}_j$ splitting in the direction parallel to ϕ_j and the space orthogonal to ϕ_j. In the direction parallel to ϕ_j, that is taking inner product with ϕ_j (and recalling assumption $\langle\psi_{\mathbf{e}_j},\phi_j\rangle=0$), we have

$$
\varpi_j(|\mathbf{z}|^2, \{\psi_{\mathbf{m}}\}_{\mathbf{m}\in\mathbf{NR}_1}) = \omega_j + \left\langle g_{\mathbf{e}_j}(|\mathbf{z}|^2, \{\psi_{\mathbf{m}}\}_{\mathbf{m}\in\mathbf{NR}_1}), \phi_j\right\rangle. \tag{4.11}
$$

This determines $\boldsymbol{\varpi}$ as a function of $|\mathbf{z}|^2$ and $\{\psi_{\mathbf{m}}\}_{\mathbf{m}\in\mathbf{NR}_1}$. Later we will determine $\{\psi_{\mathbf{m}}\}_{\mathbf{m}\in\mathbf{NR}_1}$ as a function of $|\mathbf{z}|^2$, so in the end $\boldsymbol{\varpi}$ will be a function of $|\mathbf{z}|^2$. Notice also that $\varpi_j(0,\{\psi_{\mathbf{m}}\}_{\mathbf{m}\in\mathbf{NR}_1})=\omega_j$ because $g_{\mathbf{e}_j}(0,\{\psi_{\mathbf{m}}\}_{\mathbf{m}\in\mathbf{NR}_1})=0$, as can be seen from the definition of $g_{\mathbf{m}}$.
Next, set

$$
A_{\mathbf{m}} := \begin{cases} \left((H-\omega_j)\big|_{\{\phi_j\}^\perp}\right)^{-1} & \mathbf{m}=\mathbf{e}_j\in\mathbf{NR}_0, \\ (H-\mathbf{m}\cdot\boldsymbol{\omega})^{-1} & \mathbf{m}\in\mathbf{NR}_1\setminus\mathbf{NR}_0. \end{cases}
$$

The following lemma is standard and we skip the proof.

Lemma 4.6 *For all* $\mathbf{m}\in\mathbf{NR}_1$ *and any* $s\in\mathbb{R}$ *we have* $\|A_{\mathbf{m}}\|_{\Sigma^s\to\Sigma^{s+2}}\lesssim_s 1$.

It is elementary that (4.10) holds if and only if both (4.11) and the following system hold:

$$F_{\mathbf{m}}(|\mathbf{z}|^2, \{\psi_{\mathbf{n}}\}_{\mathbf{n}\in\mathbf{NR}_1}) := \psi_{\mathbf{m}} - A_{\mathbf{m}}\left((\boldsymbol{\varpi} - \boldsymbol{\omega})\cdot \mathbf{m}\psi_{\mathbf{m}} - g_{\mathbf{m}}\right) = 0,\ \mathbf{m} \in \mathbf{NR}_1. \tag{4.12}$$

We have $\{F_{\mathbf{m}}\}_{\mathbf{m}\in\mathbf{NR}_1} \in C^\infty(\mathbb{R}^N \times \Sigma^s_{\mathbf{NR}_1}, \Sigma^s_{\mathbf{NR}_1})$, for

$$\Sigma^s_{\mathbf{NR}_1} = \Sigma^s_{\mathbf{NR}_1}(\mathbb{R}^3, \mathbb{R}) := \left\{\{\psi_{\mathbf{m}}\}_{\mathbf{m}\in\mathbf{NR}_1} \in (\Sigma^s(\mathbb{R}^3, \mathbb{R}))^{\sharp\mathbf{NR}_1} \mid \left\langle \psi_{\mathbf{e}_j}, \phi_j\right\rangle = 0,\ j = 1, \cdots, N\right\}.$$

Since, for $D_{\{\psi_{\mathbf{m}}\}_{\mathbf{m}\in\mathbf{NR}_1}}F$ the Fréchet derivative of F w.r.t. the $\{\psi_{\mathbf{m}}\}_{\mathbf{m}\in\mathbf{NR}_1}$,

$$\{F_{\mathbf{m}}(0,0)\}_{\mathbf{m}\in\mathbf{NR}_1} = 0 \text{ and } D_{\{\psi_{\mathbf{m}}\}_{\mathbf{m}\in\mathbf{NR}_1}}F(0,0) = \mathrm{Id}_{\Sigma^s_{\mathbf{NR}_1}},$$

by implicit function theorem there exist $\delta_s > 0$ and $\{\psi_{\mathbf{m}}(\cdot)\}_{\mathbf{m}\in\mathbf{NR}_1} \in C^\infty(B_{\mathbb{R}^N}(0, \delta_s), \Sigma^s_{\mathbf{NR}_1})$ s.t.

$$F_{\mathbf{m}}(|\mathbf{z}|^2, \{\psi_{\mathbf{n}}(|\mathbf{z}|^2)\}_{\mathbf{n}\in\mathbf{NR}_1}) = 0,\ \mathbf{m} \in \mathbf{NR}_1.$$

Setting $\boldsymbol{\varpi}(|\mathbf{z}|^2) := \boldsymbol{\varpi}(|\mathbf{z}|^2, \{\psi_{\mathbf{n}}(|\mathbf{z}|^2)\}_{\mathbf{n}\in\mathbf{NR}_1})$, $g_{\mathbf{m}}(|\mathbf{z}|^2) := g_{\mathbf{m}}(|\mathbf{z}|^2, \{\psi_{\mathbf{n}}(|\mathbf{z}|^2)\}_{\mathbf{n}\in\mathbf{NR}_1})$, and $u(t) = \phi(\mathbf{z}(t))$ with $z_j(t) = e^{-\varpi_j(|\mathbf{z}|^2)t}z_j$ and $\boldsymbol{\phi}$ defined in (4.1), we obtain the conclusions of Proposition1.11.

Remark 4.7 From (4.12) we have $\psi_{\mathbf{e}_j}(0) = 0$, since $g_{\mathbf{e}_j}(0) = 0$ and $\varpi_j(\mathbf{z}, \{\psi_{\mathbf{m}}\}_{\mathbf{m} \in \mathbf{NR}_1})\Big|_{\mathbf{z}=0} = \omega_j$, as we remarked under (4.11).

5 Darboux Theorem and Proof of Proposition 2.4

In this section, we will always assume $B \subset_{\text{dense}} H$ (i.e. B is a dense subset of H) where B is a reflexive Banach space and H is a Hilbert space. We further always identify H^* with H by the isometric isomorphism $H \ni u \mapsto \langle u, \cdot\rangle \in H^*$, where $\langle \cdot, \cdot\rangle$ is the inner product of H. We will also denote the coupling between B^* and B by $\langle f, u\rangle$.

When we have $B \subset_{\text{dense}} H \subset_{\text{dense}} B^*$, we think B as a "regular" subspace of H and B^*. We introduce several notation.

Definition 5.1 Let $U \subset B^*$. Let φ be a C^∞-diffeomorphism from $U \subset B^*$ to $\varphi(U) \subset B^*$. We call φ a (B-)almost identity if $\varphi(u) - u \in C^\infty(U, B)$.

Definition 5.2 Let $U \subset B^*$. We define B-regularizing vector fields $\mathfrak{X}_\sharp(U)$ and regularizing 1–forms $\Omega^1_\sharp(U)$ by

$$\mathfrak{X}_\sharp(U) := C^\infty(U, B) \subset \mathfrak{X}(U) := C^\infty(U, B^*) \text{ and } \Omega^1_\sharp(U) := \Omega^1(U).$$

Here, for a Banach space B_1 and an open subset $U_1 \subset B_1$, the space of k-forms is given by $\Omega^k(U_1, \mathcal{L}_a^k(B_1, \mathbb{R}))$ where $\mathcal{L}_a^k(B_1, \mathbb{R})$ is the Banach space of anti-symmetric k-linear operators.

Remark 5.3 B-regularizing 1–forms are mere 1–forms on B^*. However, if we think "standard" 1–forms as differential forms defined on H, B-regularizing 1–forms are more regular than "standard" 1–forms because they make sense with more "rough" vectors which are in B^* and not in H. We further remark

$$\Omega^1_\sharp(U) = C^\infty(U, \mathcal{L}(B^*, \mathbb{R})) \simeq C^\infty(U, B) = \mathfrak{X}_\sharp(U).$$

Definition 5.4 (Symplectic forms) Let $U \subset H$. We say that $\Omega \in \Omega^2(U)$ is a symplectic form on U if $d\Omega = 0$ and if for each $u \in U$ the following map is an isomorphism:

$$H \ni v \mapsto \Omega(u)(v, \cdot) \in H^* \simeq H \tag{5.1}$$

Following [7], we call the map in (5.1) symplector, denoting it by $J(u)$. The symplector satisfies $J \in C^\infty(U, \mathcal{L}(H))$. If there exists an open set $V \subset B^*$ s.t. $U \subset V$ and we can extend J and J^{-1} on V so that both are in $C^\infty(V, \mathcal{L}(B))$, we say that Ω is a B-compatible symplectic form.

Remark 5.5 Our symplectic form is the strong symplectic form of [1].

Let Ω be a symplectic form and J be the associated symplector. Then, we have

$$\Omega(u)(X, Y) = \langle J(u)X, Y\rangle, \ X, Y \in H. \tag{5.2}$$

Moreover, if Ω is a B-compatible symplectic form, then for $X \in B$ and $Y \in B^*$, we can define $\Omega(u)(X, Y)$ by (5.2). Of course we can also define $\Omega(u)(Y, X)$ by $\Omega(u)(Y, X) := -\Omega(u)(X, Y)$.

The symplector corresponding to the symplectic form Ω_0 given in (1.24) is $J(u) =$ i. Obviously, this symplectic form is B compatible for any Banach space $B \subset_{\text{dense}} H$ satisfying the property $f \in B \Rightarrow \mathrm{i}f \in B$. In particular, if $H = L^2(\mathbb{R}^3, \mathbb{C})$ and $B = H^s_\gamma(\mathbb{R}^3, \mathbb{C})$, Ω_0 is a $H^s_\gamma(\mathbb{R}^3, \mathbb{C})$-compatible symplectic form.

We next consider a small perturbation of a B-compatible symplectic form.

Lemma 5.6 *Let $U_1 \subset H$ and $U_2 \subset B^*$ with $U_1 \subset U_2$, Ω be a B-compatible symplectic form and $F \in \Omega^1_\sharp(U_2)$. Let $u_0 \in U_1$ and assume $F(u_0) = 0$ and $DF(u_0) = 0$. Then, there exists an open set $V \subset U_1$ in H s.t. $\Omega + dF$ is an B-compatible symplectic form on V, where $dF \in \Omega^2(U)$ is the exterior derivative of F.*

Remark 5.7 Since $F \in \Omega^1_\sharp(U) = C^\infty(U, \mathcal{L}(B^*, \mathbb{R}))$, we have $DF \in C^\infty(U, \mathcal{L}^2(B^*, \mathbb{R}))$. If $DF(u_0) = 0$, then from the definition of exterior derivative, we have $dF(u_0) = 0$ too.

Proof We identify $\mathcal{L}^2(B^*, \mathbb{R})$ with $\mathcal{L}(B^*, \mathcal{L}(B^*, \mathbb{R})) = \mathcal{L}(B^*, B^{**}) \simeq \mathcal{L}(B^*, B)$. In this case, we can write $DF(u)(X, Y) = \langle DF(u)X, Y\rangle$ for $X, Y \in B^*$. Therefore, we have

$$dF(u)(X, Y) = \langle DF(u)X, Y\rangle - \langle DF(u)Y, X\rangle = \left\langle\left(DF(u) - (DF(u))^*\right) X, Y\right\rangle,$$

where $(DF(u))^* \in \mathcal{L}(B^*, B)$ is the adjoint of $DF(u) \in \mathcal{L}(B^*, B)$. Thus, for $J \in C^\infty(U_1, \mathcal{L}(H))$ the symplector of Ω, we have

$$\Omega(u)(X, Y) + dF(u)(X, Y) = \left\langle\left(J(u) + DF(u) - (DF(u))^*\right) X, Y\right\rangle. \tag{5.3}$$

Since $DF(u_0) = 0$, there exists an open neighborhood V of u_0 in B^* s.t. $J(u) + DF(u) - (DF(u))^*$ is invertible for all $u \in V$. Hence $\Omega + dF$ is a symplectic form with symplector $J(u) + DF(u) - (DF(u))^*$. Since $DF(u) - (DF(u))^* \in \mathcal{L}(B^*, B)$, the restriction of $DF(u) - (DF(u))^*$ to B is in $\mathcal{L}(B)$. Therefore, we have the conclusion.

We are now in the position to prove a Darboux theorem with appropriate error estimates.

Proposition 5.8 (Darboux theorem) *Let $U_1 \subset H$ and $U_2 \subset B^*$ be open sets with $U_1 \subset U_2$ and let Ω_1 be a B-compatible symplectic form and $F \in \Omega^1_\sharp(U_2)$. Let $u_0 \in U_1$ and assume $F(u_0) = 0$ and $DF(u_0) = 0$. Set $\Omega_2 := \Omega_1 + dF$. Then, there exists an open neighborhood $V \subset U_2$ of u_0 in B^* and a map $\varphi \in C^\infty(V, B^*)$ s.t. $\varphi^*\Omega_2 = \Omega_1$, and*

$$\forall u \in V, \ \|\varphi(u) - u\|_B \lesssim \|F(u)\|_B. \tag{5.4}$$

Proof We first set

$$\Omega_{s+1} := \Omega_1 + s(\Omega_2 - \Omega_1) = \Omega_1 + sdF,$$

and look for a vector field $\mathcal{X}_{s+1}$ that satisfies $i_{\mathcal{X}_{s+1}}\Omega_{s+1} := \Omega_{s+1}(\mathcal{X}_{s+1}, \cdot) = -F$.

Claim 5.9 There exists an open neighborhood V_1 of u_0 in B^* s.t. there exists

$$\mathcal{X}_{\cdot+1} \in C^\infty((-2, 2) \times V_1, B) \text{ satisfying } i_{\mathcal{X}_{s+1}}\Omega_{s+1} = -F \tag{5.5}$$

and such that there exists a $C_1 > 0$ s.t.

$$\sup_{s\in(-2,2)} \|\mathcal{X}_{s+1}(u)\|_B \le C_1\|F(u)\|_B \text{ for all } u \in V_1. \tag{5.6}$$

Proof of Claim 5.9 Since $F \in \Omega^1_\sharp(U_2) = C^\infty(U_2, \mathcal{L}(B^*, \mathbb{R})) \simeq C^\infty(U_2, B)$, we can express $F(u)X = \langle F(u), X\rangle$ for any $X \in B^*$. Therefore, using also (5.3), (5.5) can be expressed as

$$(J_1(u) + s\left(DF(u) - (DF(u))^*\right))\mathcal{X}_{s+1} = -F(u),$$

where $J_1 \in C^\infty(U, \mathcal{L}(H))$ is the symplector of Ω_1. Since Ω_1 is a B-compatible symplectic form, we can extend J_1 to $J_1 \in C^\infty(V', \mathcal{L}(B))$ for some $U \subset V'$ with $V' \subset B^*$.

Take now $\delta_1 > 0$ sufficiently small, so that $\overline{\mathcal{B}_{B^*}(u_0, \delta_1)} \subset V'$ and

$$\sup_{u \in \mathcal{B}_{B^*}(u_0,\delta_1)} \|DF(u)\|_{\mathcal{L}(B^*,B)} \le (8 \sup_{u \in \mathcal{B}_{B^*}(u_0,\delta_1)} \|J_1(u)^{-1}\|_{\mathcal{L}(B)})^{-1}.$$

Then, we have

$$\forall u \in \mathcal{B}_{B^*}(u_0, \delta_1),\ \|J_1(u)^{-1}\left(DF(u) - (DF(u))^*\right)\|_{\mathcal{L}(B^*,B)} \le \frac{1}{4}$$

Thus, by Neumann series we have

$$\mathcal{X}_{s+1}(u) = -\sum_{n=0}^{\infty} \left(sJ_1(u)^{-1}\left(DF(u) - (DF(u))^*\right)\right)^n J_1(u)^{-1}F(u),$$

where the r.h.s. absolutely converges uniformly for $s \in (-2, 2)$. Therefore, setting $V_1 = \mathcal{B}_{B^*}(u_0, \delta_1)$, we have the conclusion. □

Claim 5.10 There exit an open neighborhood $V_2 \subset V_1$ of u_0 in B^* and a map $\widetilde{\varphi}_\cdot \in C^\infty((-2, 2) \times V_2, B)$ s.t.

$$\frac{d}{ds}\widetilde{\varphi}_s(u) = \mathcal{X}_{s+1}(u + \widetilde{\varphi}_s(u)), \quad \widetilde{\varphi}_0(u) = 0 \tag{5.7}$$

and

$$\sup_{s\in[0,1]} \|\widetilde{\varphi}_s(u)\|_B \le 2C_1\|F(u)\|_B \text{ for all } u \in V_2. \tag{5.8}$$

Proof of Claim 5.10 The existence of $\widetilde{\varphi}$ satisfying (5.7) is standard so we concentrate on the estimate (5.8). First, by the assumption $F(u_0) = 0$ and $DF(u_0) = 0$, there exists $\delta_2 \in (0, \delta_1]$ s.t. for $u \in \overline{\mathcal{B}_{B^*}(u_0, \delta_2)}$, we have $\|F(u)\|_B \le \frac{1}{4C_1}\|u - u_0\|_{B^*}$, where $C_1 > 0$ is the constant (5.6). We define $s^*(u) \in [0, 1]$ by

$$s^*(u) := \min(\inf\{s \in (0, 2) \mid \varphi_s(u) \notin \overline{\mathcal{B}_{B^*}(u_0, \delta_2/2)}\},\ 1).$$

Then, since $\widetilde{\varphi}_s$ is continuous, we have $s^*(u) > 0$ for $u \in \mathcal{B}_{B^*}(u_0, \delta_2/2)$. Furthermore,

$$\begin{aligned}\sup_{s\in[0,s^*(u)]} \|\widetilde{\varphi}_s(u)\|_{B^*} &\le \sup_{s\in[0,s^*(u)]} \int_0^s \|\mathcal{X}_{s'+1}(u + \widetilde{\varphi}_{s'}(u))\|_{B^*}\, ds' \\ &\le C_1 \sup_{s\in[0,1]} \|F(u + \widetilde{\varphi}_s(u))\|_B \le \frac{1}{4}\left(\|u - u_0\|_{B^*} + \sup_{s\in[0,1]} \|\widetilde{\varphi}_s(u)\|_{B^*}\right).\end{aligned}$$

Thus, we conclude that $s^*(u) = 1$ from

$$\sup_{s\in[0,s^*(u)]} \|\widetilde{\varphi}_s(u)\|_{B^*} \le \frac{1}{3}\|u-u_0\|_{B^*} \le \frac{1}{6}\delta_2 < \frac{1}{2}\delta_2. \tag{5.9}$$

Since $B \subset H \subset B^*$, there exists $C_2 \ge 1$ s.t. for all $u \in B$, $\|u\|_{B^*} \le C_2\|u\|_B$. Now, take $\delta_3 \in (0, \delta_2]$ s.t. if $u \in \overline{\mathcal{B}_{B^*}(u_0, \delta_3)}$, then $\|DF(u)\|_{\mathcal{L}(B^*,B)} \le (2C_1C_2)^{-1}$. Then, for all $u \in \overline{\mathcal{B}_{B^*}(u_0, \delta_3/2)}$ and all $s, s_1 \in [0, 1]$, we have $u + s_1\widetilde{\varphi}_s(u) \in \overline{\mathcal{B}_{B^*}(u_0, \delta_3)}$ by (5.9). Therefore, by Taylor expansion and (5.5),

$$\begin{aligned}\sup_{s\in[0,1]} \|\widetilde{\varphi}_s(u)\|_B &\le C_1\|F(u)\|_B + C_1 \sup_{s_1\in[0,1]} \|DF(u+s_1\widetilde{\varphi}_s(u))\|_{\mathcal{L}(B^*,B)}\|\widetilde{\varphi}(u)\|_{B^*}\\ &\le C_1\|F(u)\|_B + \frac{1}{2}\sup_{s\in[0,1]}\|\widetilde{\varphi}_s(u)\|_B.\end{aligned}$$

This completes the proof of Claim 5.10. □

We set $\varphi_s(u) := u + \widetilde{\varphi}_s(u)$. Then, by Cartan's formula, see (7.4.6) [1], we have

$$\frac{d}{ds}\varphi_s^*\Omega_{s+1} = \varphi_s^*\left(\mathcal{L}_{\mathcal{X}_{s+1}}\Omega_{s+1} + dF\right) = \varphi_s^*\left(\left(di_{\mathcal{X}_{s+1}} + i_{\mathcal{X}_{s+1}}d\right)\Omega_{s+1} + dF\right) = 0. \tag{5.10}$$

Therefore, since $\varphi_0 = \mathrm{id}$, we have

$$\Omega_2 = \varphi_1^*\Omega_1.$$

Setting $\varphi := \varphi_1$, we have the conclusion.

We show now that Proposition 2.4 follows from Proposition 5.8.

Proof of Proposition 2.4 Let $B = \Sigma^s$ and $H = L^2$. Set $F := 2^{-1}\Omega_0(D_{\mathbf{z}}\phi(\mathbf{z})D\mathbf{z}, \eta)$. Then, by $F \in \Omega^1_\sharp(\mathcal{B}_{\Sigma^{-s}}(0, \delta_s))$ and by the identification $\Omega^1_\sharp(\mathcal{B}_{\Sigma^{-s}}(0, \delta_s)) \simeq C^\infty(\mathcal{B}_{\Sigma^{-s}}(0, \delta_s), \Sigma^s)$, we have $F(u) = -\mathrm{i}2^{-1}(D_{\mathbf{z}}\phi(\mathbf{z})D\mathbf{z})^*\mathrm{i}\eta$. Notice the cancellation

$$(D_{\mathbf{z}}\phi(\mathbf{z})D\mathbf{z})^*\mathrm{i}\eta = (D_{\mathbf{z}}\phi(\mathbf{z})D\mathbf{z} - D_{\mathbf{z}}(\mathbf{z}\boldsymbol{\phi})D\mathbf{z})^*\mathrm{i}\eta.$$

So from (4.1) and by $\|(D_{\mathbf{z}}(\mathbf{z}^{\mathbf{m}}\psi_{\mathbf{m}}(|\mathbf{z}|^2))D\mathbf{z})^*\|_{\mathcal{L}(\Sigma^{-s},\Sigma^s)} = \|D(\mathbf{z}^{\mathbf{m}}\psi_{\mathbf{m}}(|\mathbf{z}|^2))D\mathbf{z}\|_{\mathcal{L}(\Sigma^{-s},\Sigma^s)}$, we have

$$\begin{aligned}\|F(u)\|_{\Sigma^s} &\le \sum_{\mathbf{m}\in\mathbf{NR}_1} \|(D_{\mathbf{z}}(\mathbf{z}^{\mathbf{m}}\psi_{\mathbf{m}}(|\mathbf{z}|^2))D\mathbf{z})^*\mathrm{i}\eta\|_{\Sigma^s}\\ &\le \sum_{\mathbf{m}\in\mathbf{NR}_1} \|D(\mathbf{z}^{\mathbf{m}}\psi_{\mathbf{m}}(|\mathbf{z}|^2))D\mathbf{z}\|_{\mathcal{L}(\Sigma^{-s},\Sigma^s)}\|\eta\|_{\Sigma^{-s}}.\end{aligned}$$

Next we use the fact that, for $\mathbf{m} \in \mathbf{NR}_1$, we have

$$\|D_{\mathbf{z}}(\mathbf{z}^{\mathbf{m}}\psi_{\mathbf{m}}(|\mathbf{z}|^2))D\mathbf{z}\|_{\mathcal{L}(\Sigma^{-s},\Sigma^s)} \leq \|D_{\mathbf{z}}(\mathbf{z}^{\mathbf{m}}\psi_{\mathbf{m}}(|\mathbf{z}|^2))\|_{\Sigma^s}\|D\mathbf{z}\|_{\mathcal{L}(\Sigma^{-s},\mathbb{C}^N)} \leq C_s\|\mathbf{z}\|^2,$$

where for $\mathbf{m} \in \mathbf{NR}_0$, $\|D_{\mathbf{z}}(\mathbf{z}^{\mathbf{m}}\psi_{\mathbf{m}}(|\mathbf{z}|^2))\|_{\Sigma^s} \leq C\|\mathbf{z}\|^2$ follows from Remark 4.7, and for $\mathbf{m} \in \mathbf{NR}_1\backslash\mathbf{NR}_0$ it follows from $|\mathbf{z}^{\mathbf{m}}| \leq \|\mathbf{z}\|^3$, since $\mathbf{z}^{\mathbf{m}}$ has an odd number of factors.
Summing up, we have proved

$$\|F(u)\|_{\Sigma^s} \leq C_s\|\mathbf{z}\|^2\|\eta\|_{\Sigma^{-s}}. \tag{5.11}$$

Then the statement of Proposition 2.4 is a consequence of $\Omega_0 = \Omega_1 + dF$ and of Proposition 5.8.

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A Proofs of Lemma 1.5 and of Proposition 1.10

Proof of Lemma 1.5 For $j, k \in \{1, \cdots, N\}$, $j < k$, set n_{jk} to be the smallest integer satisfying $n_{jk}(\omega_k - \omega_j) + \omega_k > 0$. Then, for $\mathbf{m}^{(jk)} = (m_1^{(jk)}, \cdots, m_N^{(jk)})$ defined by

$$m_j^{(jk)} = -n_{jk},\ m_k^{(jk)} = n_{jk} + 1 \text{ and } m_l^{(jk)} = 0\ (l \neq j, k), \tag{A.1}$$

we have $\mathbf{m}^{(jk)} \in \mathbf{R}_{\min}$. Suppose $\mathbf{R}_{\min}$ is an infinite set. Then, there exists $j \in \{1, \cdots, N\}$ and $\{\mathbf{m}_k\}_{k=1}^{\infty} \subset \mathbf{R}_{\min}$ s.t. $|m_{kj}| \xrightarrow{k\to\infty} \infty$. If there exists $M > 0$ s.t. for all $l \neq j$, $|m_{kj}| \leq M$, then $\mathbf{m}_k$ cannot satisfy $\sum \mathbf{m}_k = 1$. Therefore, if necessary taking a subsequence, there exists $l \neq j$ s.t. $|m_{kl}| \xrightarrow{k\to\infty} \infty$. However, for k sufficiently large, we have $|\mathbf{m}^{(jl)}| \prec |\mathbf{m}_k|$ with $\mathbf{m}^{(jl)} \in \mathbf{R}_{\min}$ defined by (A.1). This, by the definition of $\mathbf{R}_{\min}$ in (1.8), implies $\mathbf{m}_k \notin \mathbf{R}_{\min}$, contradicting the hypothesis $\mathbf{m}_k \in \mathbf{R}_{\min}$.

Let $\mathbf{m} \in \mathbf{NR}_1$. It is elementary, by the definition of $\mathbf{NR}_1$ (1.9), that for all $\mathbf{n} \in \mathbf{R}_{\min}$, either there exists j s.t. $|n_j| > |m_j|$ or $|\mathbf{n}| = |\mathbf{m}|$. So, for $\mathbf{n} = \mathbf{m}^{(jk)}$ in (A.1), we have either $|\mathbf{m}| = |\mathbf{m}^{(jk)}|$ or $|m_l| < |m_l^{(jk)}|$ for $l = j$ or k. Since there are finitely many $\mathbf{m} \in \mathbf{NR}_1$ s.t. $|\mathbf{m}| = |\mathbf{m}^{(jk)}|$, we can assume $|\mathbf{m}| \neq |\mathbf{m}^{(jk)}|$ for all $j < k$. Thus, for all $j < k$, we have $|m_l| < m_l^{(jk)}$ for at least one of $l \in \{j, k\}$. It is easy to conclude that $|m_j| \leq \max_{1\leq k<l\leq N}(|n_{kl}| + 1)$ for all j except for at most one. However, from $\sum \mathbf{m} = 1$ it is immediate that this special j must satisfy $|m_j| \leq N \max_{1\leq k<l\leq N}(|n_{kl}| + 1)$. Thus, $\mathbf{m}$ is in a fixed bounded set. Hence $\mathbf{NR}_1$ is a finite set.

Proof of Proposition 1.10 The simple proof is analogous to Bambusi and Cuccagna [2, p.1444]. For $\mathbf{m} \in \mathbf{R}_{\min}$ set $\mathbb{N} \ni L_{\mathbf{m}} := \frac{\|\mathbf{m}\|-1}{2}$. Then from (1.13)–(1.14), for any $\mathbf{m} \in \mathbf{R}_{\min}$ we have

$$G_{\mathbf{m}} = N_{\mathbf{m}} \frac{g^{(L_{\mathbf{m}})}(0)}{L_{\mathbf{m}}!} \phi^{\mathbf{m}} + K_{\mathbf{m}},$$

where $N_{\mathbf{m}} \in \mathbb{N}$ is the number of elements of $A(L_{\mathbf{m}}, \mathbf{m})$, which in this particular case is given by the set

$$A(L_{\mathbf{m}}, \mathbf{m}) = \left\{ \{\mathbf{e}_{\ell_j}\}_{j=1}^{\|\mathbf{m}\|} \in (\mathbf{NR}_0)^{\|\mathbf{m}\|} \mid \sum_{j=0}^{L_{\mathbf{m}}} \mathbf{e}_{\ell_{2j+1}} - \sum_{j=1}^{L_{\mathbf{m}}} \mathbf{e}_{\ell_{2j}} = \mathbf{m} \right\},$$

and where

$$K_{\mathbf{m}} := \sum_{1 \le m < L_{\mathbf{m}}} \frac{1}{m!} g^{(m)}(0) \sum_{(\mathbf{m}_1, \cdots, \mathbf{m}_{2m+1}) \in A(m, \mathbf{m})} \widetilde{\phi}_{\mathbf{m}_1}(0) \cdots \widetilde{\phi}_{\mathbf{m}_{2m+1}}(0).$$

So, expanding we have on the sphere $S_{\mathbf{m}} = \{\xi : |\xi|^2 = \mathbf{m} \cdot \boldsymbol{\omega}\}$ we obtain

$$\begin{aligned} \|\widehat{G}_{\mathbf{m}}\|^2_{L^2(S_{\mathbf{m}})} &= \left(N_{\mathbf{m}} \frac{g^{(L_{\mathbf{m}})}(0)}{L_{\mathbf{m}}!} \right)^2 \|\widehat{\phi^{\mathbf{m}}}\|^2_{L^2(S_{\mathbf{m}})} \\ &\quad + 2N_{\mathbf{m}} \frac{g^{(L_{\mathbf{m}})}(0)}{L_{\mathbf{m}}!} \langle \widehat{\phi^{\mathbf{m}}}, \widehat{K_{\mathbf{m}}} \rangle_{L^2(S_{\mathbf{m}})} + \|\widehat{K}_{\mathbf{m}}\|^2_{L^2(S_{\mathbf{m}})}. \end{aligned}$$

Equating the above to 0 we obtain, in view of (1.16), a quadratic equation for $g^{(L_{\mathbf{m}})}(0)$ which expresses it in terms of $(g'(0), ..., g^{(L_{\mathbf{m}}-1)}(0))$. This proves Proposition 1.10. □

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