Low-Complexity Phase-Only Scanning by Aperiodic Antenna Arrays

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***Abstract*—This letter proposes a simple and fast method for phase-only beam-scanning of linear aperiodic arrays. The method adopts a three-step procedure, consisting in the placement of the array elements by proper distribution functions, in the synthesis of the excitation amplitudes by an extended Gaussian approach, and in the evaluation of the excitation phases by a closed-form shifting. Numerical examples and comparisons with existing approaches are presented to check the effectiveness of the method, with the final aim of confirming its satisfactory behavior in terms of tradeoff between accuracy and computational cost.**

***Index Terms*—Aperiodic arrays, Gaussian approach, pencil beam, phase-only beam-scanning, 5G antennas.**

I. Introduction

MASSIVE machine-to-machine (M2M) communication is expected to become one of the cutting-edge features that will characterize the forthcoming 5G technology. For guaranteeing the frequency resources necessary to support this kind of service, the network designers have identified the millimeter-wave (mmWave) band, which, on one hand, is underutilized, but, on the other hand, presents higher attenuations with respect to the conventional microwave band used by 2G–4G systems [1]. High-gain directive antennas, and in particular electronically steerable arrays [2], become hence fundamental to compensate the mmWave channel losses. Accordingly, many array designs for 5G antenna systems have been recently proposed, with a specific attention to linear [3]–[5] and planar [6]–[8] geometries. Within this context, aperiodic structures have been proved to offer, for a given array length, the possibility of reducing the overall number of elements, and hence the complexity of the radiating system, while providing a wider scan angle with respect to periodic structures. To exploit this potential, different methods have been developed by optimizing both the positions and the excitations of the array elements [9], [10] or, alternatively, optimizing just their positions adopting constant amplitudes [11], [12] or, again, optimizing just their excitations while *a priori* fixing their aperiodic positions [13]–[15].

Having a uniform amplitude distribution of the array excitations makes both the radiating structure and the synthesis algorithm simpler, at the expenses of a worse performance control. The degrees of freedom of the problem are better exploited when both the positions and the excitations of the elements are optimized, so the algorithms in [9] and [10] are expected to provide better results. This latter approach leads, however, to more complex solutions. In fact, in [9], the adoption of both global and local optimizations to find the positions and the excitations results computationally intensive. The two-step procedure in [10], which first evaluates the number of elements by singular value decomposition, and then their locations and excitations by the matrix pencil method, is more efficient, but not even suitable for real-time applications. Besides these complete solutions, enabling both position and excitation synthesis, a Gaussian approach has been recently developed in [16] to estimate the excitations. This algorithm has been proven to be quite simple, accurate, and fast, but is currently applicable just to periodic arrays. Its possible extension to aperiodic structures would be of specific interest for M2M communications, given the low-cost that usually characterizes the M2M devices also in terms of available computational resources. The low-cost constraint brings with it a further algorithm's requirement, consisting in the capability of enabling beam-scanning by phase-only control, in order to simplify the implementation of the array feeding network.

To address the above issues, this letter presents a fast procedure for the phase-only beam-scanning of linear aperiodic arrays. The procedure requires three separate steps: the selection of a proper distribution function to aperiodically place a limited number of array elements, the evaluation of their amplitudes through a proper extension of the Gaussian approach, and the final estimation of their phases to steer the main beam toward the intended direction. Different numerical examples and comparisons with existing algorithms are reported to illustrate the satisfactory complexity–accuracy tradeoff achieved by the proposed method. It is also proved that, besides the M2M context, the developed solution can be applied to 5G base stations. The presented method is in fact characterized by a considerable flexibility since its steps can be combined with those of other existing algorithms to generate hybrid approaches more focused on accuracy or on simplicity, depending on the specific problem's requirements.

The letter is organized as follows. Section II formulates the analyzed problem. Section III presents the synthesis procedure. Section IV discusses the numerical results. Finally, Section V summarizes the most relevant conclusions.

Notation: Throughout the letter, the following notation is used: j denotes the imaginary unit; $\lfloor x \rfloor$ denotes the floor function; and $\mathrm{erf}(\mathrm{x})$ denotes the error function.

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II. Problem Formulation

With reference to a Cartesian system $O(x, y, z)$, consider a linear aperiodic array of aperture L consisting of $2N+1$ isotropic elements lying on the z-axis. For $n = -N, \ldots, N$, the nth element is described by its position z_n and excitation $a_n = |a_n| \exp(j\phi_n)$, with $|a_n|$ denoting the amplitude and $\phi_n = \arg(a_n)$ denoting the phase. This array generates the pattern

$$F(\mathbf{a};\theta) = \sum_{n=-N}^{N} |a_n| \exp\left[\, j(kz_n \sin\theta + \phi_n)\right] \tag{1}$$

where $\mathbf{a} = [a_{-N}, \ldots, a_N]$ is the excitation vector; $k = 2\pi/\lambda$ is the wavenumber with λ denoting the wavelength; and θ is the angle from broadside (i.e., the elevation angle). Besides, consider a desired pencil beam pattern, described by a Gaussian function

$$F_{\mathrm{G}}(\theta) = \exp\left(-\frac{k^2 \sin^2\theta}{2\sigma^2}\right) \tag{2}$$

in which the standard deviation σ controls the beamwidth of the corresponding main beam region $\Theta \subset \Omega = [-\pi/2, \pi/2]$.

The objective is that of determining N, $z_{-N}, \ldots, z_N$, $|a_{-N}|, \ldots, |a_N|$, and $\phi_{-N}, \ldots \phi_N$ such that

$$|F(\mathbf{a};\theta)| \text{ approximates } F_{\mathrm{G}}(\theta) \text{ for } \theta \in \Theta \tag{3a}$$

$$|F(\mathbf{a};\theta)| \text{ is steered towards } \bar{\theta} \tag{3b}$$

$$d_n = z_n - z_{n-1} \geq d_{\min}, \quad n = -N+1, \ldots, N \tag{3c}$$

$$z_N = -z_{-N} = L/2 \tag{3d}$$

where $\bar{\theta}$ represents the desired pointing direction and $d_{\min}$ denotes the minimum allowed interelement distance. The synthesis procedure developed to solve this problem is described in Section III.

III. Synthesis Method

The optimal solution for the formulated problem would require the joint evaluation of the positions and of the excitations, which may result in being computationally expensive. Therefore, a suboptimal approach is here proposed, in which the three sets of variables (positions, amplitudes, and phases) are estimated in three separate steps. First, proper distribution functions are selected to aperiodically position the array elements while limiting their number. Second, the amplitudes are determined through a proper extension of the Gaussian approach. Third, the phases are evaluated in closed form to point the synthesized pencil beam toward the desired direction. Each step is described in detail in Sections III-A–III-C.

A. Step 1: Positions

Exploiting the symmetry of the desired pattern in (2), a symmetrical element distribution can be assumed, with $z_0 = 0$ and $z_{-n} = -z_n$, for $n = 1, \ldots, N$ [10]. In order to find N and $z_1, \ldots, z_N$, we first impose condition (3d) by defining the starting value $N' = \lfloor L/(2d_{\min}) \rfloor\ (\geq N)$, which refers to a periodic array of aperture L with $2N'+1$ elements spaced by $d_{\min}$. This value is used to determine N' positions $z'_1, \ldots, z'_{N'}$ such that

$$f_i(z'_n) = n/N', \quad n = 1, \ldots, N' \tag{4}$$

where $f_i(z)$ is a suitable normalized distribution function. For this latter quantity, we consider two possibilities

$$f_1(z) = (2z/L)^{\alpha}, \text{ with } 0 < \alpha \leq 1 \tag{5a}$$

$$f_2(z) = \log_{\alpha}\left[1 + 2(\alpha-1)z/L\right], \text{ with } \alpha > 1 \tag{5b}$$

in which α represents a shaping parameter. This parameter is used to control the array aperiodicity in such a way that the elements are more densely distributed at the array center and more sparse at its edges, so as to match the typical design approaches adopted for the synthesis of linear aperiodic arrays [17]. The selection of (5a) and (5b) is motivated by the satisfactory results obtained during preliminary tests. However, if desired, different distribution functions may be adopted to determine the element positions according to (4).

As previously outlined, the use of (4) with (5a) or (5b) provides N' positions, which, in general, are not uniformly distributed and do not necessarily satisfy the minimum distance constraint in (3c). This condition must be hence applied for $n = 1, \ldots, N'$. The operation is carried out by first removing the elements (if any) having $z'_n < d_{\min}$ and then replacing the possible pairs having interelement distance lower than $d_{\min}$ by a unique element positioned at $(z'_n + z'_{n-1})/2$. Once this procedure is completed, one obtains the N value and the $z_0, \ldots, z_N$ positions, which, exploiting the symmetry, then provide the $2N+1$ element positions $z_{-N}, \ldots, z_N$ satisfying conditions (3c) and (3d).

B. Step 2: Amplitudes

At the second step, the $N+1$ amplitudes $|a_0|, \ldots, |a_N|$ are calculated with the purpose of satisfying condition (3a). This step is accomplished by extending the Gaussian approach in [16] to the case of aperiodic arrays. To this aim, the variable $u' = k\sin\theta$ is introduced in (2), thus obtaining $F_{\mathrm{G}}(u') = \exp(-u'^2/2\sigma^2)$, and, subsequently, the amplitude distribution $\tilde{a}(z)$ of a continuous infinite source that exactly produces $F_{\mathrm{G}}(u')$ is evaluated using the Fourier transform relation. Then, each $|a_n|$ value is estimated as the area between the graph of $\tilde{a}(z)$ and the z-axis in the interval $[z_n - d_n/2; z_n + d_{n+1}/2]$, where, for $n = N$, the quantity $d_{N+1} = d_N$ is introduced for calculation purposes. After some manipulations, this procedure leads, for $n = 0, \ldots, N$, to the closed-form expression

$$|a_n| = \frac{1}{2}\left\{\mathrm{erf}\left[\frac{\sigma}{\sqrt{2}}\left(\mathrm{z_n} + \frac{\mathrm{d_{n+1}}}{2}\right)\right] - \mathrm{erf}\left[\frac{\sigma}{\sqrt{2}}\left(\mathrm{z_n} - \frac{\mathrm{d_n}}{2}\right)\right]\right\} \tag{6}$$

which, by again assuming symmetry, allows one to infer the remaining N amplitudes as $|a_{-n}| = |a_n|$ for $n = 1, \ldots, N$. Hence, moving from the positions evaluated at the first step, one obtains the $2N+1$ amplitudes producing the pattern $F(\mathbf{a};\theta)$ satisfying condition (3a) and directed at broadside.

C. Step 3: Phases

At the third step, we finally impose condition (3b) by introducing the variable $u = k(\sin\theta - \sin\bar{\theta})$, which is defined so that (1) can be rewritten as

$$F(\mathbf{a};u) = \sum_{n=-N}^{N} |a_n| \exp(jz_n u). \tag{7}$$

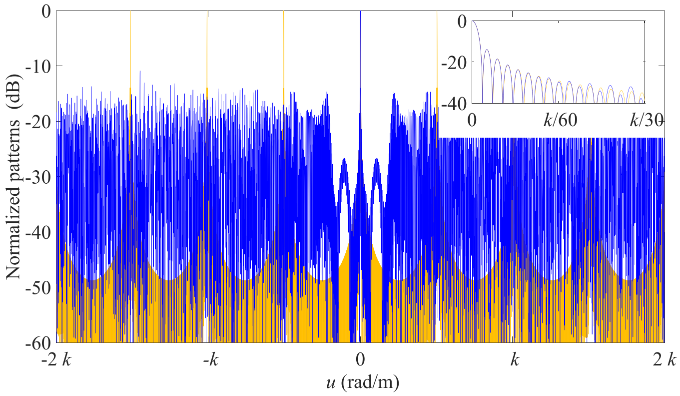

Fig. 1. First example. Normalized patterns synthesized with the proposed method (blue line) and with the method in [12] for a periodic array with the same aperture and the same number of elements (yellow line). A magnification of the patterns in the main beam region is shown in the inset.

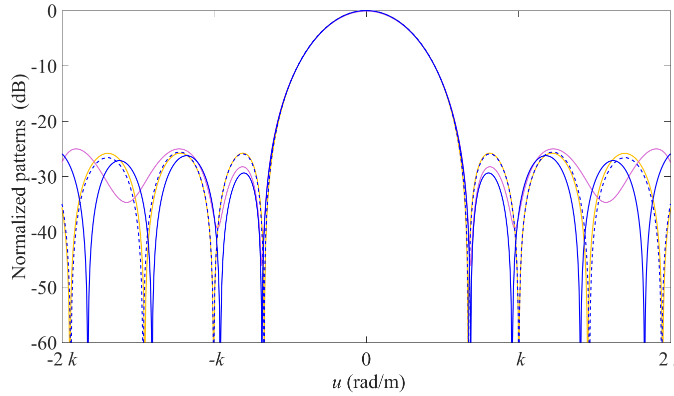

Fig. 2. Second example. Normalized patterns synthesized with the proposed method (blue solid line), with the method in [10] (yellow solid line), with the method in [19] (purple solid line), and with the proposed method for the sole excitations using the positions given in [10] (blue dashed line).

Hence, it directly follows from (1) and (7) that the phases

$$\phi_n = -kz_n \sin\bar{\theta}, \quad n = -N, \ldots, N \tag{8}$$

make the amplitude of the resulting pattern maximum in the desired direction $\bar{\theta}$. Thus, condition (3b) is finally satisfied. Observe that, once steps 1 and 2 are accomplished, the modification of $\bar{\theta}$ requires the sole recalculation of (8), which is immediate. Hence, beam-scanning can be realized by phase-only control in real time. One may also notice that this third step of the algorithm may recall the progressive phase method for the beam-scanning of periodic arrays with uniform spacing d and positions $z_n = nd$ ($n = -N, ..., N$) [18], but with the relevant difference that the phase shifting in (8) holds also for the much more general case of aperiodic linear structures.

IV. Results

This section presents three numerical examples to validate the developed method. The results are obtained using MATLAB R2018b on a laptop with an Intel Core i5-5300U CPU@2.30 GHz processor and 8 GB RAM.

A. Example 1

The first example considers an array having as requirements $\sigma = 0.0019$ rad/m, $L = 500\lambda$, and $d_{\min} = \lambda/2$ (leading to $N' = 500$). The solution is derived by using $f_1(z)$ in (5a) with $\alpha = 0.1$, which generates an aperiodic array with minimum distance 0.52λ and $2N+1 = 253$ elements. Note that this latter value is approximately one fourth of that corresponding to a periodic array with the same aperture and $\lambda/2$-spaced elements. Fig. 1 reports the pattern synthesized using the proposed method (blue line) and that obtained employing the algorithm in [12] (yellow line). Since [12] operates on periodic structures, the second pattern is obtained considering a periodic array with the same number of elements $2N+1$ and the same aperture L of the synthesized aperiodic array.

Concerning the beam-scanning capabilities of the two compared arrays, it is worth remarking that (7) holds for $-\infty < u < +\infty$, but, for a given $\bar{\theta}$ value, only the range $[k(-1 - \sin\bar{\theta}); k(1 - \sin\bar{\theta})]$ identifies the visible pattern, so that the interval $[-2k; 2k]$ certainly contains all the possible visible regions for any $\bar{\theta} \in \Omega$. Therefore, with reference to Fig. 1, we may observe that the pattern obtained from the periodic array presents a grating lobe already in the visible region $[-k; k]$, corresponding to that of the broadside direction $\bar{\theta} = 0°$, since the constant interelement distance $L/(2N+1) \cong 1.97\lambda$ is too large. For the pattern synthesized by the proposed method, instead, a maximum sidelobe level equal to -10.90 dB is obtained for the entire interval $[-2k; 2k]$. Thus, the synthesized aperiodic array is able to provide a satisfactory phase-only beam-scanning in the entire angular region Ω. Furthermore, this solution has been achieved in a very low CPU time since the estimation of all the positions and excitations has been completed in just 1 ms.

B. Example 2

The second example considers an array having as requirements $\sigma = 1.6904$ rad/m, $L = 1.8169\lambda$, and $d_{\min} = 0.27820\lambda$ (leading to $N' = 3$). These very specific values are selected according to the results in [10] for example 5, with the aim of enabling a comparison between the here-proposed method, that developed in [10] and that proposed in [19]. The results of this comparison are reported in Fig. 2, which shows four patterns. The first pattern (blue solid line) is obtained from the proposed method by using the distribution function $f_2(z)$ in (5b) with $\alpha = 1.2$, which generates an aperiodic array with $2N+1 = 7$ elements and minimum distance 0.28461. Thus, in this case, no elements are removed at the first step, while the given $d_{\min}$ requirement is anyway satisfied. The second pattern (yellow solid line) is obtained implementing the method in [10] by sampling a -25 dB Chebyshev pattern and selecting a threshold 10^{-3}. The third pattern (purple solid line) is derived implementing the algorithm in [19] with a maximum sidelobe level of -25 dB and selecting, for the function to be approximated in the main beam region, that obtained with the method proposed here. The last pattern (blue dashed line) is derived from an hybrid approach, which is obtained by inserting the positions $z_{-N}, ..., z_N$ given in [10] into step 2 of the here presented algorithm, that is, into (6), to evaluate the amplitude of each excitation.

Two main considerations may be formulated from Fig. 2. First, all four methods provide a close and satisfactory performance, since the maximum sidelobe level is always not higher

TABLE I
SECOND EXAMPLE: POSITIONS/EXCITATIONS OBTAINED USING THE PROPOSED METHOD AND PERFORMANCE COMPARISON WITH EXISTING ALGORITHMS

n	z_n/λ	$\lvert a_n \rvert$
0	0	1
± 1	± 0.28461	0.91693
± 2	± 0.58706	0.67084
± 3	± 0.90845	0.36328

Performance comparison			
	Proposed method	Method in [10]	Method in [19]
Number of elements	7	7	7
DRR	2.75267	3.59518	6.26374
$\min\{d_n/\lambda\}$	0.28461	0.27820	0.31372
CPU time [ms]	1.51720	19.64650	675031.25

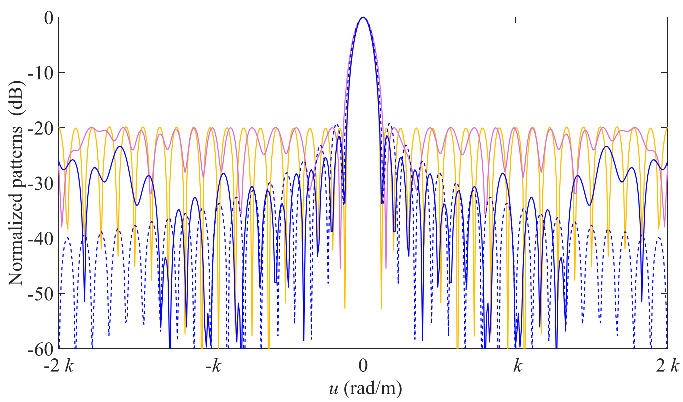

Fig. 3. Third example. Normalized patterns synthesized with the proposed method (blue solid line), with the method in [10] (yellow solid line), with the method in [19] (purple solid line), and with the proposed method for the sole excitations using the positions given in [19] (blue dashed line).

than -25 dB in the interval $[-2k; 2k]$, and hence the phase-only beam-scanning can be realized in the entire angular region Ω. Second, the hybrid approach reveals the flexibility of the conceived method, whose steps may be combined with those of other algorithms. Beside the satisfactory behavior observed in Fig. 2, the main advantage of the proposed method may be inferred from Table I, which reports the positions and amplitudes estimated with the presented algorithm, and, for each of the three compared methods, the obtained number of elements, the dynamic range ratio (DRR) of the excitations, the minimum normalized interelement distance, and the CPU time. Indeed this latter quantity reveals that, thanks to the fast evolution allowed by the closed-form expressions available at each step, the proposed algorithm requires a CPU time lower than one tenth of that required by [10], and several order of magnitudes lower than that required by [19].

C. *Example 3*

The third example considers an array having as requirements $\sigma = 0.28697$ rad/m, $L = 9\lambda$, and $d_{\min} = \lambda/4$ (leading to $N' = 18$). The solution is derived by using $f_1(z)$ in (5a) with $\alpha = 0.7$, which generates an aperiodic array with minimum distance 0.2715λ and $2N+1 = 27$ elements. Also in this case, the problem is solved with the method in [10] and with that in [19],

TABLE II
THIRD EXAMPLE: POSITIONS/EXCITATIONS OBTAINED USING THE PROPOSED METHOD AND PERFORMANCE COMPARISON WITH EXISTING ALGORITHMS

n	z_n/λ	$\lvert a_n \rvert$	n	z_n/λ	$\lvert a_n \rvert$
0	0	1	± 7	$\pm$ 2.52147	0.02642
± 1	$\pm$ 0.43643	0.04708	± 8	$\pm$ 2.82692	0.02557
± 2	$\pm$ 0.82933	0.04741	± 9	$\pm$ 3.14262	0.02442
± 3	$\pm$ 1.29016	0.04508	± 10	$\pm$ 3.46814	0.02302
± 4	$\pm$ 1.67174	0.03343	± 11	$\pm$ 3.80310	0.02141
± 5	$\pm$ 1.94329	0.02717	± 12	$\pm$ 4.14715	0.01964
± 6	$\pm$ 2.22674	0.02696	± 13	$\pm$ 4.50000	0.01775

Performance comparison			
	Proposed method	Method in [10]	Method in [19]
Number of elements	27	21	18
DRR	2.81151	2.41574	4.49354
$\min\{d_n/\lambda\}$	0.27155	0.42515	0.44000
CPU time [ms]	5.42020	40.48990	585718.75

for which a maximum sidelobe level of -20 dB is imposed. The results are shown in Fig. 3, which also includes the hybrid solution derived by inserting the positions calculated through [19] into step 2 of the here developed method. A joint observation of this figure and of the performance comparison reported in Table II reveals that, on one hand, the methods in [10] and [19] provide arrays with a lower number of elements and a larger minimum inter-element distance, but, on the other hand, the pattern obtained by the proposed algorithm presents a lower sidelobe level and, more importantly, it is obtained in a significantly lower CPU time. Finally, the satisfactory combining of the presented method with that in [10] (example 2) and with that in [19] (example 3) confirms the considerable versatility of the proposed approach.

V. CONCLUSION

A fast method for the phase-only beam-scanning of linear aperiodic arrays has been presented. The method adopts a sub-optimal approach to separately estimate the positions of the array elements; their excitation amplitudes and their phases, according to specific constraints on the pattern shape; the array aperture; and the minimum allowed interelement distance. Appropriate distribution functions are combined with an extended Gaussian approach to obtain the positions and the excitations in closed form, thus avoiding possible iterative procedures. Comparisons with existing algorithms have shown the satisfactory radiation performance achievable by the proposed method, further revealing its additional advantage in terms of computational cost, with synthesis times in the order of few milliseconds. The numerical results have hence proved the interesting tradeoff between accuracy and complexity achieved by the conceived method, whose steps may be also flexibly combined with those of other suitable algorithms to develop possible hybrid synthesis approaches.

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