Research Article

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Quasilinear elliptic systems in divergence form associated to general nonlinearities

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Abstract: The paper is concerned with a priori estimates of positive solutions of quasilinear elliptic systems of equations or inequalities in an open set of $\Omega \subset \mathbb{R}^N$ associated to general continuous nonlinearities satisfying a local assumption near zero. As a consequence, in the case $\Omega = \mathbb{R}^N$, we obtain nonexistence theorems of positive solutions. No hypotheses on the solutions at infinity are assumed.

Keywords: Quasilinear elliptic systems, Liouville theorems, positive solutions

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1 Introduction

In this paper, we prove a priori estimates for the solutions of elliptic systems involving quasilinear operators in divergence form in an open set $\Omega \subseteq \mathbb{R}^N$. The simplest problem that we have in mind is the classical model

$$\begin{cases} -\Delta u = f(u, v) & \text{in } \Omega \subseteq \mathbb{R}^N, \\ -\Delta v = g(u, v) & \text{in } \Omega \subseteq \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $f, g\colon [0, \infty) \times [0, \infty) \to [0, \infty)$ are given nonnegative continuous functions.

More generally, we prove a priori estimates for the solutions of elliptic systems in an open set $\Omega \subseteq \mathbb{R}^N$ involving two quasilinear operators in divergence form. Specifically, we shall study problems of the type

$$\begin{cases} -\operatorname{div}(\mathscr{A}_p(x, u, \nabla u)) \geq f(x, u, v) & \text{in } \Omega, \\ -\operatorname{div}(\mathscr{A}_q(x, v, \nabla v)) \geq g(x, u, v) & \text{in } \Omega, \\ u \geq 0,\ v \geq 0 & \text{in } \Omega, \end{cases} \tag{P}$$

where $\mathscr{A}_p, \mathscr{A}_q\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ are weakly p-coercive and weakly q-coercive respectively, that is, $p > 1$, $q > 1$, and there exist $a, b > 0$ such that

$$\begin{aligned} (\mathscr{A}_p(x, t, w) \cdot w) &\geq a|\mathscr{A}_p(x, t, w)|^{p'} \quad \text{for all } (x, t, w) \in \Omega \times \mathbb{R} \times \mathbb{R}^N, \\ (\mathscr{A}_q(x, t, w) \cdot w) &\geq b|\mathscr{A}_q(x, t, w)|^{q'} \quad \text{for all } (x, t, w) \in \Omega \times \mathbb{R} \times \mathbb{R}^N, \end{aligned}$$

$f, g\colon \Omega \times [0, \infty) \times [0, \infty) \to [0, \infty)$ are Carathédory functions, and for u and v, a weak Harnack inequality holds (for further details and definitions, see Section 2).

In this setting, we prove some a priori bounds for weak solutions of system (P). We shall use some of the ideas developed in [4], where the case of scalar problems was considered.

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Our main result is Theorem 3, in which we give a sufficient condition for the nonexistence of nontrivial solutions of (P) in the case $\Omega = \mathbb{R}^N$, and the following local assumptions on $f(x, u, v) = f(u, v)$ and $g(x, u, v) = g(u, v)$, *concerning their behavior near zero*, hold. We note that this is the first attempt to study nonexistence of positive solutions for quasilinear elliptic systems in this generality. As it is well known, besides their intrinsic interest, these nonexistence theorems can be used to prove existence results for related Dirichlet problems in bounded domains via the so called blow-up technique and suitable index theorems. See, for instance, [12] and the references therein. In addition, we point out that our approach can be used to study similar quasilinear systems in the framework of Carnot groups in the same spirit as [4, 5]. For sake of brevity and in order to avoid cumbersome notations, we restrict our attention to the standard euclidean case.

Assumption 1 (Assumptions on the nonlinearities). *The functions $f, g\colon [0, \infty) \times [0, \infty) \to [0, \infty)$ are continuous and satisfy the following conditions:*
(i) *There exist $p_1 \geq 0$ and $q_1 > 0$ such that*

$$\liminf_{t+\tau \to 0} \frac{f(t, \tau)}{t^{p_1}\tau^{q_1}} > 0 \quad (\textit{possibly infinity}). \tag{f_0}$$

(ii) *There exist $p_2 > 0$ and $q_2 \geq 0$ such that*

$$\liminf_{t+\tau \to 0} \frac{g(t, \tau)}{t^{p_2}\tau^{q_2}} > 0 \quad (\textit{possibly infinity}). \tag{g_0}$$

On the possible solution (u, v) of the system, we do not require any kind of behavior at infinity. Indeed, we only assume that it belongs to a *local* Sobolev function space for which the integrals of the relevant quantities make sense. Under these hypotheses, a special case of our main nonexistence theorem applied to (1.1) reads as follows:

Theorem 1. *Suppose the functions $f, g\colon [0, \infty) \times [0, \infty) \to [0, \infty)$ are continuous and satisfy (f_0) and (g_0). Let (u, v) be a weak solution of* (1.1) *such that*

$$\operatorname*{ess\,inf}_{\mathbb{R}^N} u = \operatorname*{ess\,inf}_{\mathbb{R}^N} v = 0.$$

If $0 \leq p_1 < 1$, $0 \leq q_2 < 1$ and

$$N\Big[1 - \frac{(1-p_1)(1-q_2)}{p_2 q_1}\Big] \leq \max\Big\{2 + \frac{2q_1 + 2p_1(1-q_2)}{p_2 q_1}, 2 + \frac{2p_2 + 2q_2(1-p_1)}{p_2 q_1}\Big\}, \tag{1.2}$$

then $u = 0$ or $v = 0$ a.e. in $\mathbb{R}^N$. This result is sharp.

In [1, Theorem 5.3], a less general sufficient condition for the nonexistence has been proved in the case $f(x, u, v) = |x|^\alpha u^{p_1} v^{q_1}$ and $g(x, u, v) = |x|^\beta u^{p_2} v^{q_2}$. While, the same sufficient condition (1.2) has been considered in [3, Theorem V.3] for radial solutions of (P) and the differential system involves the (Δ_p, Δ_q) operators, in the special case $f(u, v) = u^{p_1} v^{q_1}$ and $g(u, v) = u^{p_2} v^{q_2}$. In Remark 5, we prove also that condition (1.2) is sharp, in the sense that when it does not hold, we are able to construct an explicit nontrivial solution of (P) in the special case when the system involves the same p-Laplacian operator.

We emphasize that conditions (f_0) and (g_0) allow to study problems with singular nonlinearities. For instance, dealing with $f(t, \tau) = \tau^{-1}$, it is easy to construct a function $\tilde{f}(t, \tau)$ such that $f(t, \tau) \geq \tilde{f}(t, \tau)$ and it satisfies (f_0) with $p_1 = 0$ and any $q_1 > 0$.

We also prove a nonexistence result for a nonautonomous system of inequalities, in which

$$f(x, u, v) = a(x)f(u, v) \quad \text{and} \quad g(x, u, v) = b(x)g(u, v),$$

where a, b are positive measurable functions, and $f(u, v)$, $g(u, v)$ satisfy conditions (f_0) and (g_0), respectively.

As a final remark, we note that, among others, Bourgain [2] studied a stationary Schrödinger system with critical exponents for the Bose–Einstein condensate

$$\begin{cases} -\Delta u = u^p v^q & \text{in } \mathbb{R}^N, \\ -\Delta v = v^p u^q & \text{in } \mathbb{R}^N. \end{cases}$$

For earlier results concerning nonexistence of radial positive solutions of the more general model,

$$\begin{cases} -\Delta u = u^{p_1} v^{q_1} & \text{in } \mathbb{R}^{\mathbb{N}}, \\ -\Delta v = u^{p_2} v^{q_2} & \text{in } \mathbb{R}^{\mathbb{N}}, \end{cases}$$

where $p_1, q_1, p_2, q_2 > 0$, see [6].

The paper is organized as follows: In Section 2, we give some useful definitions and preliminary results, focusing on the weak Harnack inequality and its consequences. Section 3 is totally devoted to the general a priori estimates for weak solutions of problem (P), while in Section 4, we prove our main results concerning the nonexistence of nontrivial solutions of (P) when $f(x,u,v) = f(u,v)$ and $g(x,u,v) = g(u,v)$. In Section 5, we prove a nonexistence theorem for the nonautonomous system (P) with $f(x,u,v) = a(x)f(u,v)$ and $g(x,u,v) = b(x)g(u,v)$.

2 Preliminaries

Let $\mathscr{A} : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function, that is, for each $t \in \mathbb{R}$ and $w \in \mathbb{R}^N$, $\mathscr{A}(\cdot, t, w)$ is measurable and for a.e. $x \in \mathbb{R}^N$, $\mathscr{A}(x, \cdot, \cdot)$ is continuous. We consider operators L *generated by* $\mathscr{A}$, that is,

$$L(u)(x) = \operatorname{div}(\mathscr{A}(x, u(x), \nabla u(x))).$$

Our model cases are the p-Laplace operator, the mean curvature operator and some related generalizations.

Let $\Omega \subseteq \mathbb{R}^N$ be an open set. Let $p > 1$, and let $\mathscr{A}_p : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function. The function $\mathscr{A}_p$ is called W-p-C, *weakly p-coercive*, if there exists a constant $a > 0$ such that

$$(\mathscr{A}_p(x,t,w) \cdot w) \geq a|\mathscr{A}_p(x,t,w)|^{p'} \quad \text{for all } (x,t,w) \in \Omega \times \mathbb{R} \times \mathbb{R}^N. \tag{W-p-C}$$

The function $\mathscr{A}_p$ is called S-p-C, *strongly p-coercive*, if there exist two constants $a, \tilde{a} > 0$ such that

$$(\mathscr{A}_p(x,t,w) \cdot w) \geq \tilde{a}|w|^p \geq a|\mathscr{A}_p(x,t,w)|^{p'} \quad \text{for all } (x,t,w) \in \Omega \times \mathbb{R} \times \mathbb{R}^N, \tag{S-p-C}$$

see [1, 8, 10] for details.

Example 1. Clearly, if $\mathscr{A}_p$ is S-p-C, then $\mathscr{A}_p$ is W-p-C.

Let $p > 1$. The p-Laplace operator $\Delta_p(\cdot) = \operatorname{div}(|\nabla(\cdot)|^{p-2}\nabla(\cdot))$ is generated by $\mathscr{A}_p(x,t,w) = |w|^{p-2}w$, which is S-$p$-C. In particular, when $p = 2$, the Laplace operator $\Delta(\cdot)$ is S-2-C.

The mean curvature operator

$$\operatorname{div}\left(\frac{\nabla(\cdot)}{\sqrt{1 + |\nabla(\cdot)|^2}}\right), \quad \text{generated by } \mathscr{A}_p(x,t,w) = \frac{w}{\sqrt{1 + |w|^2}},$$

is W-2-C, but not S-2-C.

For further details and comments, we refer to [4, Section 1].

In what follows, we denote by $\mathscr{A}_p$ a weakly p-coercive operator. Furthermore, B_R stands for the ball of radius $R > 0$, that is, $B_R = \{x : |x| < R\}$, and A_R is the annulus $B_{2R} \setminus \overline{B_R}$. Therefore, we have

$$|B_R| = \int_{B_R} dx = R^N \int_{|x|<1} dx = w_N R^N \quad \text{and} \quad |A_R| = w_N(2^N - 1)R^N,$$

where w_N is the measure of the unit ball B_1 in $\mathbb{R}^N$.

Consider the system of inequalities

$$\begin{cases} -\operatorname{div}(\mathscr{A}_p(x,u,\nabla u)) \geq f(x,u,v) & \text{in } \Omega, \\ -\operatorname{div}(\mathscr{A}_q(x,v,\nabla v)) \geq g(x,u,v) & \text{in } \Omega, \\ u \geq 0,\ v \geq 0 & \text{in } \Omega, \end{cases} \tag{2.1}$$

where $\Omega \subseteq \mathbb{R}^N$ is an open set, $\mathscr{A}_p, \mathscr{A}_q : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ are W-p-C and W-q-C, respectively, and

$$f, g : \Omega \times [0, \infty) \times [0, \infty) \to [0, \infty)$$

are *Carathédory functions*.

Let $p \geq 1$. Throughout the paper, we shall denote

$$W^{1,p}_{\mathrm{loc}}(\Omega) := \left\{u \in L^p_{\mathrm{loc}}(\Omega) : |\nabla u| \in L^p_{\mathrm{loc}}(\Omega)\right\}.$$

Definition 1. A pair of functions $(u, v) \in W^{1,p}_{\mathrm{loc}}(\Omega) \times W^{1,q}_{\mathrm{loc}}(\Omega)$ is a *weak solution* of (2.1) if

$$f(\cdot, u, v), g(\cdot, u, v) \in L^1_{\mathrm{loc}}(\Omega), \qquad |\mathscr{A}_p(\cdot, u, \nabla u)| \in L^{p'}_{\mathrm{loc}}(\Omega), \quad |\mathscr{A}_q(\cdot, v, \nabla v)| \in L^{q'}_{\mathrm{loc}}(\Omega),$$

and the following inequalities hold for all nonnegative functions $\phi_1, \phi_2 \in C^1_0(\Omega)$:

$$\int_\Omega (\mathscr{A}_p(x, u, \nabla u) \cdot \nabla \phi_1) \geq \int_\Omega f(x, u, v)\phi_1, \tag{2.2}$$

$$\int_\Omega (\mathscr{A}_q(x, v, \nabla v) \cdot \nabla \phi_2) \geq \int_\Omega g(x, u, v)\phi_2. \tag{2.3}$$

Moreover, we say that a weak solution (u, v) is *trivial* if $u = 0$ or $v = 0$ a.e. in $\mathbb{R}^N$.

Lemma 1 (Weak Harnack inequality [10, 11]). *If $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N)$ is a weak solution of*

$$\begin{cases} -\operatorname{div}(\mathscr{A}_p(x, u, \nabla u)) \geq 0 & \text{in } \mathbb{R}^N, \\ u \geq 0 & \text{in } \mathbb{R}^N, \end{cases}$$

$\mathscr{A}_p$ is S-p-C and $N > p > 1$, then for any $\sigma \in (0, \frac{N(p-1)}{N-p})$, there exists a constant $c_H > 0$ independent of u such that, for all $R > 0$,

$$\left(\frac{1}{|B_R|} \int_{B_R} u^\sigma\right)^{\frac{1}{\sigma}} \leq c_H \operatorname*{ess\,inf}_{B_{R/2}} u.$$

As in [4], we introduce the following definition:

Definition 2. Let u be a weak solution of

$$\begin{cases} -\operatorname{div}(\mathscr{A}_p(x, u, \nabla u)) \geq 0 & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \end{cases}$$

where $\Omega \subseteq \mathbb{R}^N$ is an open set. We say that the *weak Harnack inequality* holds for u with exponent $\sigma > 0$ if there exists a constant $c_H > 0$ independent of u such that, for any $R > 0$ for which $B_{2R} \subset \Omega$, we have

$$\left(\frac{1}{|B_R|} \int_{B_R} u^\sigma\right)^{\frac{1}{\sigma}} \leq c_H \operatorname*{ess\,inf}_{B_{R/2}} u. \tag{WH}$$

Remark 1. Inequality (WH) implies immediately that $u \in L^\sigma_{\mathrm{loc}}(\Omega)$ and that either $u \equiv 0$ or $u > 0$ in Ω. Moreover, we point out that, by Hölder's inequality, if (WH) holds with exponent σ, it also holds with any exponent $\sigma_0 \in (0, \sigma)$.

The following is a direct consequence of (WH).

Proposition 1. *If* (WH) *holds for two nonnegative functions u and v, then* (WH) *also holds for $u + v$. Furthermore, there exists a positive constant C independent of u and v for which*

$$\operatorname*{ess\,inf}_{B_R}(u + v) \leq C(\operatorname*{ess\,inf}_{B_{R/2}} u + \operatorname*{ess\,inf}_{B_{R/2}} v)$$

for all $R > 0$ such that $B_{2R} \subset \Omega$.

Proof. Let $\sigma, \delta > 0$ be the exponents for which (WH) holds for u and v, respectively. Suppose that $\sigma \le \delta$, then (WH) holds with exponent σ for both u and v. Now, for all $R > 0$ such that $B_{2R} \subset \Omega$, we get

$$\Big(\int_{B_R}(u+v)^\sigma\Big)^{\frac{1}{\sigma}} \le c\Big[\Big(\int_{B_R}u^\sigma\Big)^{\frac{1}{\sigma}}+\Big(\int_{B_R}v^\sigma\Big)^{\frac{1}{\sigma}}\Big] \tag{2.4}$$

with $c := \max\{1, 2^{(1-\sigma)/\sigma}\}$. Indeed, if $\sigma \ge 1$, inequality (2.4) is the subadditivity of the $L^\sigma(B_R)$-norm, while if $\sigma < 1$, (2.4) follows immediately from the fact that $(u+v)^\sigma \le u^\sigma + v^\sigma$ and by the convexity of the power $(\,\cdot\,)^{1/\sigma}$. Hence, by (2.4) and (WH), on u and v, we have

$$\begin{aligned}\Big(\frac{1}{|B_R|}\int_{B_R}(u+v)^\sigma\Big)^{\frac{1}{\sigma}} &\le c\Big[\Big(\frac{1}{|B_R|}\int_{B_R}u^\sigma\Big)^{\frac{1}{\sigma}}+\Big(\frac{1}{|B_R|}\int_{B_R}v^\sigma\Big)^{\frac{1}{\sigma}}\Big]\\ &\le C_H(\operatorname*{ess\,inf}_{B_{R/2}}u+\operatorname*{ess\,inf}_{B_{R/2}}v) \le C_H \operatorname*{ess\,inf}_{B_R/2}(u+v),\end{aligned} \tag{2.5}$$

where $C_H := c \cdot c_H$. That is, (WH) holds for $u + v$.

On the other hand,

$$\Big(\frac{1}{|B_R|}\int_{B_R}(u+v)^\sigma\Big)^{\frac{1}{\sigma}} \ge \operatorname*{ess\,inf}_{B_R}(u+v),$$

thus, by (2.5), $\operatorname{ess\,inf}_{B_R}(u+v) \le C_H(\operatorname{ess\,inf}_{B_{R/2}} u + \operatorname{ess\,inf}_{B_{R/2}} v)$. □

Remark 2. Obviously, the same conclusion of Proposition 1 holds for *any finite number* of nonnegative functions verifying (WH).

3 A priori estimates

In this section, we prove some integral a priori bounds of the solutions of the system of inequalities (2.1) in which we recall that $\Omega \subseteq \mathbb{R}^N$ is an open set, $\mathscr{A}_p, \mathscr{A}_q : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ are W-p-C and W-q-C, respectively, that is, $p > 1$, $q > 1$, and there exist $a, b > 0$ such that

$$\begin{aligned}(\mathscr{A}_p(x,t,w)\cdot w) &\ge a|\mathscr{A}_p(x,t,w)|^{p'} \quad \text{for all } (x,t,w)\in\Omega\times\mathbb{R}\times\mathbb{R}^N,\\ (\mathscr{A}_q(x,t,w)\cdot w) &\ge b|\mathscr{A}_q(x,t,w)|^{q'} \quad \text{for all } (x,t,w)\in\Omega\times\mathbb{R}\times\mathbb{R}^N,\end{aligned}$$

and $f, g : \Omega \times [0,\infty) \times [0,\infty) \to [0,\infty)$ are Carathédory functions.

Theorem 2. *Let (u, v) be a weak solution of (2.1). Then, for all test functions ϕ_1, ϕ_2, every $\ell \ge 0$ and every $\alpha, \beta < 0$, we get*

$$\begin{aligned}\int_\Omega f(x,u,v)u_\ell^\alpha\phi_1 + c_1\int_\Omega(\mathscr{A}_p(x,u,\nabla u)\cdot\nabla u)u_\ell^{\alpha-1}\phi_1 &\le c_2\int_\Omega u_\ell^{\alpha-1+p}\frac{|\nabla\phi_1|^p}{\phi_1^{p-1}},\\ \int_\Omega g(x,u,v)v_\ell^\beta\phi_2 + \tilde{c}_1\int_\Omega(\mathscr{A}_q(x,v,\nabla v)\cdot\nabla v)v_\ell^{\beta-1}\phi_2 &\le \tilde{c}_2\int_\Omega v_\ell^{\beta-1+q}\frac{|\nabla\phi_2|^q}{\phi_2^{q-1}},\end{aligned} \tag{3.1}$$

where $u_\ell := u + \ell$, $v_\ell := v + \ell$, $c_1 := |\alpha| - \eta^{p'}/ap'$, $c_2 := \eta^{-p}/p$, $\eta > 0$, $\tilde{c}_1 := |\beta| - \mu^{q'}/bq'$, $\tilde{c}_2 := \mu^{-q}/q$ and $\mu > 0$.

If η, μ are so small that $c_1, \tilde{c}_1 > 0$, then, for all $\alpha, \beta < 0$ and $\ell \ge 0$,

$$\begin{aligned}\int_\Omega f(x,u,v)\phi_1 &\le c_3\Big(\int_\Omega u_\ell^{\alpha-1+p}\frac{|\nabla\phi_1|^p}{\phi_1^{p-1}}\Big)^{\frac{1}{p'}}\Big(\int_\Omega u_\ell^{(1-\alpha)(p-1)}\frac{|\nabla\phi_1|^p}{\phi_1^{p-1}}\Big)^{\frac{1}{p}},\\ \int_\Omega g(x,u,v)\phi_2 &\le \tilde{c}_3\Big(\int_\Omega v_\ell^{\beta-1+q}\frac{|\nabla\phi_2|^q}{\phi_2^{q-1}}\Big)^{\frac{1}{q'}}\Big(\int_\Omega v_\ell^{(1-\beta)(q-1)}\frac{|\nabla\phi_2|^q}{\phi_2^{q-1}}\Big)^{\frac{1}{q}},\end{aligned} \tag{3.2}$$

where $c_3 := (c_2/ac_1)^{1/p'}$ and $\tilde{c}_3 := (\tilde{c}_2/b\tilde{c}_1)^{1/q'}$.

If $u^{\alpha-1+p}, u^{(1-\alpha)(p-1)} \in L^1_{\mathrm{loc}}(A_R)$, $v^{\beta-1+q}, v^{(1-\beta)(q-1)} \in L^1_{\mathrm{loc}}(A_R)$ with $R > 0$ such that $B_{2R} \Subset \Omega$, then, for all $\alpha, \beta < 0$, there exist $c_4, \tilde{c}_4 > 0$ for which

$$\begin{aligned}
\frac{1}{|B_R|}\int_{B_R} f(x,u,v) &\le c_4 R^{-p}\left(\frac{1}{|A_R|}\int_{A_R} u^{\alpha-1+p}\right)^{\frac{1}{p'}}\left(\frac{1}{|A_R|}\int_{A_R} u^{(1-\alpha)(p-1)}\right)^{\frac{1}{p}},\\
\frac{1}{|B_R|}\int_{B_R} g(x,u,v) &\le \tilde{c}_4 R^{-q}\left(\frac{1}{|A_R|}\int_{A_R} v^{\beta-1+q}\right)^{\frac{1}{q'}}\left(\frac{1}{|A_R|}\int_{A_R} v^{(1-\beta)(q-1)}\right)^{\frac{1}{q}}.
\end{aligned} \tag{3.3}$$

If there exist $\sigma > p-1$, $\delta > q-1$ such that $u^\sigma, v^\delta \in L^1_{\mathrm{loc}}(\Omega)$, then

$$\begin{aligned}
\frac{1}{|B_R|}\int_{B_R} f(x,u,v) &\le c_4 R^{-p}\left(\frac{1}{|A_R|}\int_{A_R} u^{\sigma}\right)^{\frac{p-1}{\sigma}},\\
\frac{1}{|B_R|}\int_{B_R} g(x,u,v) &\le \tilde{c}_4 R^{-q}\left(\frac{1}{|A_R|}\int_{A_R} v^{\delta}\right)^{\frac{q-1}{\delta}}.
\end{aligned} \tag{3.4}$$

In particular, if (WH) *holds with exponent $\sigma > p-1$ for u and with exponent $\delta > q-1$ for v, then the following inequalities hold for some appropriate constants $c_5, \tilde{c}_5 > 0$:*

$$\begin{aligned}
\frac{1}{|B_R|}\int_{B_R} f(x,u,v) &\le c_5 R^{-p}(\operatorname*{ess\,inf}_{B_R} u)^{p-1},\\
\frac{1}{|B_R|}\int_{B_R} g(x,u,v) &\le \tilde{c}_5 R^{-q}(\operatorname*{ess\,inf}_{B_R} v)^{q-1}.
\end{aligned} \tag{3.5}$$

Proof. We follow essentially the proof of [4, Theorem 2.1].

Fix a test function ϕ_1, and set $r := \mathrm{dist}(\mathrm{supp}(\phi_1), \partial\Omega)$, $\Omega_r := \{y \in \Omega : \mathrm{dist}(y, \partial\Omega) > r\}$. For $\varepsilon \in (0, r)$ and $\ell > 0$, we define

$$w_\varepsilon(x) := \begin{cases} \ell + \int_{\Omega_r} D_\varepsilon(x-y)u(y)\,dy & \text{if } x \in \Omega_r,\\ 0 & \text{if } x \in \Omega \setminus \Omega_r, \end{cases}$$

where $(D_\varepsilon)_\varepsilon$ is a family of mollifiers. Thus, we can choose $w_\varepsilon^\alpha \phi_1$ as test function in (2.2). We have

$$\int_\Omega f(x,u,v)w_\varepsilon^\alpha\phi_1 + |\alpha|\int_\Omega (\mathscr{A}_p(x,u,\nabla u)\cdot\nabla w_\varepsilon)w_\varepsilon^{\alpha-1}\phi_1 \le \int_\Omega |\mathscr{A}_p(x,u,\nabla u)|\cdot|\nabla\phi_1|w_\varepsilon^\alpha.$$

Since $w_\varepsilon \to u_\ell$, $\nabla w_\varepsilon \to \nabla u$ in $L^p_{\mathrm{loc}}(\Omega_r)$ as $\varepsilon \to 0$, by Lebesgue's dominated convergence theorem and by duality, we get

$$\begin{aligned}
&\int_\Omega f(x,u,v)u_\ell^\alpha\phi_1 + |\alpha|\int_\Omega (\mathscr{A}_p(x,u,\nabla u)\cdot\nabla u)u_\ell^{\alpha-1}\phi_1\\
&\quad\le \int_\Omega |\mathscr{A}_p(x,u,\nabla u)|\cdot|\nabla\phi_1|u_\ell^\alpha = \int_\Omega |\mathscr{A}_p(x,u,\nabla u)|u_\ell^{(\alpha-1)/p'}\phi_1^{1/p'}\cdot u_\ell^{(\alpha-1+p)/p}|\nabla\phi_1|\phi_1^{-1/p'}\\
&\quad\le \frac{\eta^{p'}}{p'}\int_\Omega |\mathscr{A}_p(x,u,\nabla u)|^{p'}u_\ell^{\alpha-1}\phi_1 + \frac{1}{\eta^p p}\int_\Omega u_\ell^{\alpha-1+p}|\nabla\phi_1|^p\phi_1^{1-p}\\
&\quad\le \frac{\eta^{p'}}{ap'}\int_\Omega (\mathscr{A}_p(x,u,\nabla u)\cdot\nabla u)u_\ell^{\alpha-1}\phi_1 + \frac{1}{\eta^p p}\int_\Omega u_\ell^{\alpha-1+p}|\nabla\phi_1|^p\phi_1^{1-p},
\end{aligned}$$

where, in the last steps, we used Hölder's and Young's inequalities and the (W-p-C) condition for $\mathscr{A}_p$. This completes the proof of the first inequality in (3.1) when $\ell > 0$.

Analogously, it is possible to prove the second one. Indeed, fix a test function ϕ_2, and set

$$r := \operatorname{dist}(\operatorname{supp}(\phi_2), \partial\Omega), \quad \Omega_r := \{y \in \Omega : \operatorname{dist}(y, \partial\Omega) > r\}.$$

For $\varepsilon \in (0, r)$ and $\ell > 0$, define

$$\tilde{w}_\varepsilon(x) := \begin{cases} \ell + \int_{\Omega_r} D_\varepsilon(x-y)v(y)\,dy & \text{if } x \in \Omega_r, \\ 0 & \text{if } x \in \Omega \setminus \Omega_r, \end{cases}$$

use $\tilde{w}_\varepsilon^\beta \phi_2$ as test function in (2.3), and proceed as above. The case $\ell = 0$ follows immediately from the case $\ell > 0$ by an application of Beppo–Levi's theorem and letting $\ell \to 0$.

From now on, we only prove the inequalities concerning f, as an argument to obtain the other estimates in exactly the same way.

In order to prove (3.2), use (2.2), and consider $\ell > 0$. Thus, the weak p-coercivity of $\mathscr{A}_p$, Hölder's inequality and (3.1) imply

$$\begin{aligned} \int_\Omega f(x,u,v)\phi_1 &\le \int_\Omega |\mathscr{A}_p(x,u,\nabla u)|\,|\nabla\phi_1| \\ &\le \left(\int_\Omega \frac{1}{a}(\mathscr{A}_p(x,u,\nabla u)\cdot\nabla u)u_\ell^{(\alpha-1)}\phi_1\right)^{\frac{1}{p'}} \left(\int_\Omega u_\ell^{(1-\alpha)(p-1)}|\nabla\phi_1|^p\phi_1^{1-p}\right)^{\frac{1}{p}} \\ &\le c_3\left(\int_\Omega u_\ell^{\alpha-1+p}|\nabla\phi_1|^p\phi_1^{1-p}\right)^{\frac{1}{p'}} \left(\int_\Omega u_\ell^{(1-\alpha)(p-1)}|\nabla\phi_1|^p\phi_1^{1-p}\right)^{\frac{1}{p}}. \end{aligned}$$

Also here, it is enough to apply Beppo–Levi's monotone convergence theorem and/or Lebesgue's dominated convergence theorem to prove the remaining case $\ell = 0$.

Let $\phi_0 \in C_0^1(\mathbb{R})$ be such that $0 \le \phi_0 \le 1$, $c_{\phi_0} := \|\,|\phi_0'|^p/\phi_0^{p-1}\|_\infty < \infty$ and

$$\phi_0(t) = \begin{cases} 1, & \text{if } |t| < 1, \\ 0, & \text{if } |t| > 2. \end{cases}$$

Define $\phi_1(x) := \phi_0(|x/R|)$ so that

$$\frac{|\nabla\phi_1(x)|^p}{\phi_1(x)^{p-1}} = \frac{|\phi_0'(|x/R|)|^p}{\phi_0^{p-1}(|x/R|)}R^{-p} \le c_{\phi_0}R^{-p}.$$

Hence, using ϕ_1 as test function in (3.2) with $\ell = 0$, we get

$$\int_\Omega f(x,u,v)\phi_1 \le c_3\left(\int_{A_R} u^{\alpha-1+p}c_{\phi_0}R^{-p}\right)^{\frac{1}{p'}}\left(\int_{A_R} u^{(1-\alpha)(p-1)}c_{\phi_0}R^{-p}\right)^{\frac{1}{p}},$$

and so, since $|A_R| = w_N(2^N-1)R^N = (2^N-1)|B_R|$, we have

$$\frac{1}{|B_R|}\int_{B_R} f(x,u,v) \le c_3(2^N-1)c_{\phi_0}R^{-p}\left(\frac{1}{|A_R|}\int_{A_R} u^{\alpha-1+p}\right)^{\frac{1}{p'}}\left(\frac{1}{|A_R|}\int_{A_R} u^{(1-\alpha)(p-1)}\right)^{\frac{1}{p}},$$

which gives (3.3) with $c_4 := c_3(2^N-1)c_{\phi_0}$.

Estimates (3.4) follow easily from (3.3) by applying Hölder's inequality. Finally, if (WH) holds, by (3.4), we obtain

$$\frac{1}{|B_R|}\int_{B_R} f(x,u,v) \le c_4\left(1-\frac{1}{2^N}\right)^{\frac{1-p}{\sigma}} R^{-p}\left(\frac{1}{|B_{2R}|}\int_{B_{2R}} u^\sigma\right)^{\frac{p-1}{\sigma}} \le c_5R^{-p}(\operatorname*{ess\,inf}_{B_R} u)^{p-1}$$

with $c_5 := c_4(1-\frac{1}{2^N})^{(1-p)/\sigma}c_H^{p-1}$. $\square$

4 Some Liouville-type theorems

In this section, we shall prove the main results of this paper. Consider the problem

$$\begin{cases} -\operatorname{div}(\mathscr{A}_p(x,u,\nabla u)) \geq f(u,v) & \text{in } \mathbb{R}^N, \\ -\operatorname{div}(\mathscr{A}_q(x,v,\nabla v)) \geq g(u,v) & \text{in } \mathbb{R}^N, \\ u \geq 0,\ v \geq 0 & \text{in } \mathbb{R}^N. \end{cases} \tag{4.1}$$

Throughout this section, without further mentioning, we shall assume the following:

Assumption 2. *The functions $\mathscr{A}_p, \mathscr{A}_q : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ are W-p-C and W-q-C, respectively, $N > \max\{p,q\}$,* (WH) *holds for u with exponent $\sigma > p-1$ and for v with exponent $\delta > q-1$, and Assumption 1 holds.*

Example 2. Besides all the functions f such that $f(t,\tau) \geq ct^{p_1}\tau^{q_1}$ for every $(t,\tau) \in [0,\infty)\times[0,\infty)$, an example of a function satisfying condition (f_0) is given by $f(t,\tau) = \sin^2 t \sin^2 \tau$ in $[0,\infty)\times[0,\infty)$. Clearly, in this case, f satisfies (f_0) with $p_1 = q_1 = 2$.

Lemma 2 (cf. [4, Lemma 3.1]). *Let $u : \mathbb{R}^N \to [0,\infty)$ be a function such that $\operatorname{ess\,inf}_{\mathbb{R}^N} u = 0$. Assume that* (WH) *holds with exponent $\sigma > 0$. Then, for all $\varepsilon > 0$,*

$$\lim_{R\to\infty} \frac{|A_{R/2} \cap T_\varepsilon^u|}{|A_{R/2}|} = 1, \qquad \lim_{R\to\infty} \frac{|B_R \cap T_\varepsilon^u|}{|B_R|} = 1,$$

where $T_\varepsilon^u = \{x \in \mathbb{R}^N : u(x) < \varepsilon\}$ and $A_R = B_{2R} \setminus \overline{B_R}$.

Lemma 3. *Let (u,v) be a weak solution of* (4.1) *such that $\operatorname{ess\,inf}_{\mathbb{R}^N} u = \operatorname{ess\,inf}_{\mathbb{R}^N} v = 0$. If $f(u(x),v(x)) = 0$ for a.a. $x \in \mathbb{R}^N$, then $u = 0$ or $v = 0$ a.e. in $\mathbb{R}^N$. Similarly, if $g(u(x),v(x)) = 0$ for a.a. $x \in \mathbb{R}^N$, then $u = 0$ or $v = 0$ a.e. in $\mathbb{R}^N$.*

Proof. Suppose that $f(u(x),v(x)) = 0$ for a.a. $x \in \mathbb{R}^N$. Thanks to Proposition 1, we can apply Lemma 2 to the function $u+v$. Hence, by (f_0), we get

$$(\operatorname*{ess\,inf}_{B_R} u)^{p_1} (\operatorname*{ess\,inf}_{B_R} v)^{q_1} \leq \frac{1}{|A_{R/2} \cap T_\varepsilon|} \int_{A_{R/2}\cap T_\varepsilon} u^{p_1} v^{q_1} \leq c \frac{1}{|A_{R/2}\cap T_\varepsilon|} \int_{A_{R/2}} f(u,v) = 0$$

for R sufficiently large and $\varepsilon > 0$, where $T_\varepsilon = \{x \in \mathbb{R}^N : u(x) + v(x) < \varepsilon\}$. Using (WH) on u and v, we conclude that $u = 0$ or $v = 0$ a.e. in $\mathbb{R}^N$. If $g(u(x),v(x)) = 0$ for a.a. $x \in \mathbb{R}^N$, the proof is similar. □

Let us introduce the matrix

$$\mathscr{H} = \begin{pmatrix} p_1 - p + 1 & q_1 \\ p_2 & q_2 - q + 1 \end{pmatrix},$$

$$D := -\det \mathscr{H} = p_2 q_1 - (p - 1 - p_1)(q - 1 - q_2). \tag{4.2}$$

Lemma 4. *Let (u,v) be a nontrivial weak solution of* (4.1) *such that $\operatorname{ess\,inf}_{\mathbb{R}^N} u = \operatorname{ess\,inf}_{\mathbb{R}^N} v = 0$. Then there exists a constant $c > 0$ such that, for all $\varepsilon > 0$ and $R > 0$ sufficiently large, the following estimates hold:*

$$\begin{aligned} (\operatorname*{ess\,inf}_{A_{R/2}\cap T_\varepsilon} u)^{p_1-p+1} (\operatorname*{ess\,inf}_{A_{R/2}\cap T_\varepsilon} v)^{q_1} &\leq cR^{-p}, \\ (\operatorname*{ess\,inf}_{A_{R/2}\cap T_\varepsilon} u)^{p_2} (\operatorname*{ess\,inf}_{A_{R/2}\cap T_\varepsilon} v)^{q_2-q+1} &\leq cR^{-q}, \end{aligned} \tag{4.3}$$

where $T_\varepsilon = \{x \in \mathbb{R}^N : u(x) + v(x) < \varepsilon\}$,

$$\int_{B_R} f(u,v) \leq cR^{-p} |A_{R/2}| (\operatorname*{ess\,inf}_{B_R} u)^{p-1} \leq cR^{N\left[1 - \frac{p-1}{p_2} + \frac{q_2(p-1)}{p_2(q-1)}\right] - p - q\frac{q_2(p-1)}{p_2(q-1)}} \frac{\left(\int_{A_{R/2}} g(u,v)\right)^{\frac{p-1}{p_2}}}{\left(\int_{B_R} g(u,v)\right)^{\frac{q_2(p-1)}{p_2(q-1)}}}, \tag{4.4}$$

$$\int_{B_R} g(u,v) \leq cR^{-q} |A_{R/2}| (\operatorname*{ess\,inf}_{B_R} v)^{q-1} \leq cR^{N\left[1 - \frac{q-1}{q_1} + \frac{p_1(q-1)}{q_1(p-1)}\right] - q - p\frac{p_1(q-1)}{q_1(p-1)}} \frac{\left(\int_{A_{R/2}} f(u,v)\right)^{\frac{q-1}{q_1}}}{\left(\int_{B_R} f(u,v)\right)^{\frac{p_1(q-1)}{q_1(p-1)}}}. \tag{4.5}$$

In particular, if $q_2 \le q-1$, then, for R sufficiently large,

$$(\operatorname*{ess\,inf}_{A_{R/2}\cap T_\varepsilon} u)^{1-\frac{(p-1-p_1)(q-1-q_2)}{p_2 q_1}} \le cR^{-\frac{p(q-1-q_2)+qq_1}{p_2 q_1}}, \tag{4.6}$$

$$\int_{B_R} f(u,v) \le cR^{\frac{ND}{p_2 q_1}-p-\frac{qq_1(p-1)+pp_1(q-1-q_2)}{p_2 q_1}} \Big(\int_S f(u,v)\Big)^{\frac{(p-1-p_1)(q-1-q_2)}{p_2 q_1}} \tag{4.7}$$

with $S = A_{R/2}$ or $S = B_R$. If $p_1 \le p-1$, then, for R sufficiently large,

$$(\operatorname*{ess\,inf}_{A_{R/2}\cap T_\varepsilon} v)^{1-\frac{(p-1-p_1)(q-1-q_2)}{p_2 q_1}} \le cR^{-\frac{q(p-1-p_1)+pp_2}{p_2 q_1}}, \tag{4.8}$$

$$\int_{B_R} g(u,v) \le cR^{\frac{ND}{p_2 q_1}-q-\frac{pp_2(q-1)+qq_2(p-1-p_1)}{p_2 q_1}} \Big(\int_S g(u,v)\Big)^{\frac{(p-1-p_1)(q-1-q_2)}{p_2 q_1}} \tag{4.9}$$

with $S = A_{R/2}$ or $S = B_R$.

Proof. Fix $\varepsilon > 0$. By the first inequality of (3.5), we get

$$\int_{B_R} f(u,v) \le cR^{-p}|A_{R/2}|(\operatorname*{ess\,inf}_{B_R} u)^{p-1} \le cR^{-p}|A_{R/2}|(\operatorname*{ess\,inf}_{A_{R/2}\cap T_\varepsilon} u)^{p-1}.$$

On the other hand, using (f_0), we have

$$\int_{B_R} f(u,v) \ge \int_{A_{R/2}\cap T_\varepsilon} f(u,v) \ge c \int_{A_{R/2}\cap T_\varepsilon} u^{p_1} v^{q_1},$$

hence,

$$\int_{A_{R/2}\cap T_\varepsilon} u^{p_1} v^{q_1} \le cR^{-p}|A_{R/2}|(\operatorname*{ess\,inf}_{A_{R/2}\cap T_\varepsilon} u)^{p-1}.$$

Therefore,

$$(\operatorname*{ess\,inf}_{A_{R/2}\cap T_\varepsilon} u)^{p_1}(\operatorname*{ess\,inf}_{A_{R/2}\cap T_\varepsilon} v)^{q_1} \le cR^{-p}\frac{|A_{R/2}|}{|A_{R/2}\cap T_\varepsilon|}(\operatorname*{ess\,inf}_{A_{R/2}\cap T_\varepsilon} u)^{p-1},$$

and so, by Proposition 1 and by Lemma 2 applied to the function $u+v$, we obtain

$$(\operatorname*{ess\,inf}_{A_{R/2}\cap T_\varepsilon} u)^{p_1-p+1}(\operatorname*{ess\,inf}_{A_{R/2}\cap T_\varepsilon} v)^{q_1} \le cR^{-p}$$

for R sufficiently large. Similarly, from the second inequality of the system, we prove the second inequality of (4.3).

By (3.5) and (g_0), for R sufficiently large, it follows that

$$\begin{aligned}
\int_{B_R} f(u,v) &\le cR^{-p}|A_{R/2}|(\operatorname*{ess\,inf}_{B_R} u)^{p-1} \le cR^{-p}|A_{R/2}|(\operatorname*{ess\,inf}_{A_{R/2}\cap T_\varepsilon} u)^{p-1}\\
&\le cR^{-p}|A_{R/2}|(\operatorname*{ess\,inf}_{A_{R/2}\cap T_\varepsilon} v)^{-\frac{q_2(p-1)}{p_2}}\Big(\frac{1}{|A_{R/2}\cap T_\varepsilon|}\int_{A_{R/2}\cap T_\varepsilon} u^{p_2}v^{q_2}\Big)^{\frac{p-1}{p_2}}\\
&\le c\frac{R^{-p}|A_{R/2}|}{|A_{R/2}\cap T_\varepsilon|^{\frac{p-1}{p_2}}}(\operatorname*{ess\,inf}_{B_R} v)^{-\frac{q_2(p-1)}{p_2}}\Big(\int_{A_{R/2}\cap T_\varepsilon} g(u,v)\Big)^{\frac{p-1}{p_2}}\\
&\le cR^{-p}|A_{R/2}|^{1-\frac{p-1}{p_2}}\Big(\frac{R^{-q}|B_R|}{\int_{B_R} g(u,v)}\Big)^{\frac{q_2(p-1)}{p_2(q-1)}}\Big(\int_{A_{R/2}} g(u,v)\Big)^{\frac{p-1}{p_2}},
\end{aligned}$$

where, in the last step, we have applied Lemma 2 to the function $u+v$, which, thanks to Proposition 1, satisfies all the required assumptions. Similarly, working on the second inequality of (4.1), we obtain (4.5).

Combining the two inequalities in (4.3) and using the assumption $q_2 \le q-1$, we immediately get (4.6),

$$(\operatorname*{ess\,inf}_{A_{R/2}\cap T_\varepsilon} u)^{1-\frac{(q-1-q_2)(p-1-p_1)}{p_2 q_1}} \le cR^{-\frac{p(q-1-q_2)+qq_1}{p_2 q_1}}$$

for R sufficiently large.

From (4.4) and (4.5), we obtain

$$\int_{B_R} f(u,v) \le cR^{N\left[1-\frac{p-1}{p_2}+\frac{q_2(p-1)}{p_2(q-1)}\right]-p-q\frac{q_2(p-1)}{p_2(q-1)}}\left(\int_S g(u,v)\right)^{\frac{p-1}{p_2}\left(1-\frac{q_2}{q-1}\right)},$$

$$\int_{B_R} g(u,v) \le cR^{N\left[1-\frac{q-1}{q_1}+\frac{p_1(q-1)}{q_1(p-1)}\right]-q-p\frac{p_1(q-1)}{q_1(p-1)}}\left(\int_S f(u,v)\right)^{\frac{q-1}{q_1}\left(1-\frac{p_1}{p-1}\right)}$$

with $S = A_{R/2}$ or $S = B_R$, being f and g nonnegative and $A_{R/2} \subset B_R$. Since $q_2 \le q-1$, these two inequalities imply

$$\int_{B_R} f(u,v) \le cR^{-p-q\frac{p-1}{p_2}+p\frac{p_1(q_2-q+1)}{q_1 p_2}}|A_{R/2}|^{1-\frac{(p_1-p+1)(q_2-q+1)}{p_2 q_1}}\left(\int_S f(u,v)\right)^{\frac{(q-1)(p-1)}{q_1 p_2}\left(1-\frac{p_1}{p-1}\right)\left(1-\frac{q_2}{q-1}\right)}.$$

Similarly, under the assumption $p_1 \le p-1$, we can prove (4.8) and (4.9). □

Theorem 3. *Let $p_1 < p-1$, $q_2 < q-1$ and*

$$N \ge \min\Big\{\Big(\frac{N-p}{p-1}p_2+\frac{N-q}{q-1}q_2\Big)\frac{q_1}{q_1-q_2+q-1}+\Big(\frac{N-p}{p-1}p_1+\frac{N-q}{q-1}q_1\Big)\frac{q-1-q_2}{q_1-q_2+q-1},$$
$$\Big(\frac{N-p}{p-1}p_1+\frac{N-q}{q-1}q_1\Big)\frac{p_2}{p_2-p_1+p-1}+\Big(\frac{N-p}{p-1}p_2+\frac{N-q}{q-1}q_2\Big)\frac{p-1-p_1}{p_2-p_1+p-1}\Big\}. \quad (4.10)$$

If (u,v) is a weak solution of (4.1) *such that* $\operatorname{ess\,inf}_{\mathbb{R}^N} u = \operatorname{ess\,inf}_{\mathbb{R}^N} v = 0$, *then either $u = 0$ or $v = 0$ a.e. in $\mathbb{R}^N$.*

We note that (4.10) is equivalent to

$$N\Big[1-\frac{(p-1-p_1)(q-1-q_2)}{p_2 q_1}\Big] \le \max\Big\{p+\frac{qq_1(p-1)+pp_1(q-1-q_2)}{p_2 q_1},$$
$$q+\frac{pp_2(q-1)+qq_2(p-1-p_1)}{p_2 q_1}\Big\} \quad (4.11)$$

when $p_1 < p-1$ and $q_2 < q-1$. Indeed, starting from (4.10), when the minimum is the first quantity in the brackets, we get

$$\begin{aligned} N &\ge \frac{N-p}{p-1}\cdot\frac{p_2q_1+p_1(q-1-q_2)}{q_1-q_2+q-1}+N\frac{q_1}{q_1-q_2+q-1}-\frac{qq_1}{q_1-q_2+q-1}\\ &= \frac{q-1-q_2}{q_1-q_2+q-1}\Big\{\frac{N-p}{p-1}\Big(\frac{p_2q_1}{q-1-q_2}+p_1\Big)-\frac{qq_1}{q-1-q_2}\Big\}+N\frac{q_1}{q_1-q_2+q-1}, \end{aligned}$$

that is,

$$N\frac{q-1-q_2}{q_1-q_2+q-1} \ge \frac{q-1-q_2}{q_1-q_2+q-1}\Big\{\frac{N-p}{p-1}\Big(\frac{p_2q_1}{q-1-q_2}+p_1\Big)-\frac{qq_1}{q-1-q_2}\Big\}.$$

Now, since $q_2 < q-1$, we get

$$N \ge \Big\{\frac{N-p}{p-1}\Big(\frac{p_2q_1}{q-1-q_2}+p_1\Big)-\frac{qq_1}{q-1-q_2}\Big\},$$

Multiplying both sides by $\frac{(p-1)(q-1-q_2)}{p_2q_1}$, we have

$$(N-p)\Big[1+\frac{p_1(q-1-q_2)}{p_2q_1}\Big]-\frac{qq_1(p-1)}{p_2q_1} \le N\frac{(p-1)(q-1-q_2)}{p_2q_1},$$

namely,

$$N\Big[1-\frac{(p-1-p_1)(q-1-q_2)}{p_2q_1}\Big] \le p+\frac{qq_1(p-1)+pp_1(q-1-q_2)}{p_2q_1}.$$

Similarly, we can easily prove the second part of the equivalence.

Proof of Theorem 3. We shall distinguish two cases depending on whether the constant D defined in (4.2), as well as the left side of (4.11), is positive or nonpositive.

Case $D > 0$. Suppose that

$$p + \frac{qq_1(p-1) + pp_1(q-1-q_2)}{p_2q_1} \ge q + \frac{pp_2(q-1) + qq_2(p-1-p_1)}{p_2q_1},$$

the remaining case being analogous. Without loss of generality, we prove the theorem only when

$$N\left[1 - \frac{(p-1-p_1)(q-1-q_2)}{p_2q_1}\right] = p + \frac{qq_1(p-1) + pp_1(q-1-q_2)}{p_2q_1}.$$

Suppose, by contradiction, that both $u > 0$ and $v > 0$ in $\mathbb{R}^N$. By (4.7), we have

$$\int_{B_R} f(u,v) \le c\Big(\int_{A_{R/2}} f(u,v)\Big)^{\frac{(p-1-p_1)(q-1-q_2)}{p_2q_1}} \le c\Big(\int_{B_R} f(u,v)\Big)^{\frac{(p-1-p_1)(q-1-q_2)}{p_2q_1}}, \tag{4.12}$$

hence $f(u,v) \in L^1(\mathbb{R}^N)$. Thus, by the first inequality of (4.12), letting $R \to \infty$, we get $f(u,v) = 0$ a.e. in $\mathbb{R}^N$. By Lemma 3, we conclude that either $u = 0$ or $v = 0$ a.e. in $\mathbb{R}^N$. This contradiction proves the claim.

Case $D \le 0$. Note that, in this case, condition (4.11) is trivially satisfied. Suppose, by contradiction, that both $u > 0$ and $v > 0$. Clearly, $p(q_2 - q + 1) - qq_1 < 0$ and $q(p_1 - p + 1) - pp_2 < 0$, since $p_1 < p - 1$ and $q_2 < q - 1$.

Hence, if $D < 0$, by (4.6) and (4.8), R large and $\varepsilon > 0$, we get

$$\operatorname*{ess\,inf}_{A_{R/2}\cap T_\varepsilon} u \ge cR^{\frac{-p(q-1-q_2)-qq_1}{p_2q_1-(p-1-p_1)(q-1-q_2)}}, \quad \operatorname*{ess\,inf}_{A_{R/2}\cap T_\varepsilon} v \ge cR^{\frac{-q(p-1-p_1)-pp_2}{p_2q_1-(p-1-p_1)(q-1-q_2)}}.$$

Therefore,

$$\lim_{R\to\infty} \operatorname*{ess\,inf}_{A_{R/2}\cap T_\varepsilon} u \ge \infty, \quad \lim_{R\to\infty} \operatorname*{ess\,inf}_{A_{R/2}\cap T_\varepsilon} v \ge \infty,$$

which is impossible.

Next, if $D = 0$, then, by (4.6) and R large, it follows that

$$1 \le cR^{-p(q-1-q_2)-qq_1}.$$

Clearly, by letting $R \to \infty$, we reach a contradiction. □

Remark 3. In Theorem 3, as well as in all the nonexistence theorems of this paper, we require that the solutions of the system have an essential infimum on $\mathbb{R}^N$ equal to zero. If, for instance, $f(u,v) \ge cu^{p_1}v^{q_1}$ in all of $\mathbb{R}^N$, the assumption on the essential infimum of u and v is quite natural. Indeed, if $\operatorname{ess\,inf}_{\mathbb{R}^N} u > 0$ and $\operatorname{ess\,inf}_{\mathbb{R}^N} v > 0$, then every solution (u,v) of (4.1) is also a solution of

$$\begin{cases} -\operatorname{div}(\mathscr{A}_p(x,u,\nabla u)) \ge \text{const.} > 0 & \text{in } \mathbb{R}^N, \\ -\operatorname{div}(\mathscr{A}_q(x,v,\nabla v)) \ge g(u,v) & \text{in } \mathbb{R}^N, \\ u \ge 0,\ v \ge 0 & \text{in } \mathbb{R}^N. \end{cases} \tag{4.13}$$

The first inequality of (4.13) does not have any weak solutions (see e.g. [4, Corollary 2.4]), therefore also system (4.1) has no weak solutions.

Furthermore, if $\operatorname{ess\,inf}_{\mathbb{R}^N} v = 0$ and $\mathscr{A}_p$ does not depend explicitly on u, we have the following result.

Corollary 1. *Let (u,v) be a weak solution of the problem*

$$\begin{cases} -\operatorname{div}(\mathscr{A}_p(x,\nabla u)) \ge u^{p_1}v^{q_1} & \textit{in } \mathbb{R}^N, \\ -\operatorname{div}(\mathscr{A}_q(x,v,\nabla v)) \ge u^{p_2}v^{q_2} & \textit{in } \mathbb{R}^N, \\ u \ge 0,\ v \ge 0 & \textit{in } \mathbb{R}^N \end{cases} \tag{4.14}$$

with $q_2 < q - 1$. If $\operatorname{ess\,inf}_{\mathbb{R}^N} v = 0$ and $\operatorname{ess\,inf}_{\mathbb{R}^N} u > 0$, then $v = 0$ a.e. in $\mathbb{R}^N$.

Proof. Put $u_0 := \operatorname{ess\,inf}_{\mathbb{R}^N} u > 0$ and $\tilde{u} := u - u_0$. Then $(\tilde{u}, v)$ solves the problem

$$\begin{cases} -\operatorname{div}(\mathscr{A}_p(x, \nabla\tilde{u})) \geq (\tilde{u} + u_0)^{p_1} v^{q_1} & \text{in } \mathbb{R}^N, \\ -\operatorname{div}(\mathscr{A}_q(x, v, \nabla v)) \geq (\tilde{u} + u_0)^{p_2} v^{q_2} & \text{in } \mathbb{R}^N, \\ \tilde{u} \geq 0,\ v \geq 0 & \text{in } \mathbb{R}^N. \end{cases} \tag{4.15}$$

Consider the functions $f, g\colon [0, \infty) \times [0, \infty) \to [0, \infty)$ defined by

$$f(t, \tau) = (t + u_0)^{p_1} \tau^{q_1} \quad \text{and} \quad g(t, \tau) = (t + u_0)^{p_2} \tau^{q_2}$$

for all $(t, \tau) \in [0, \infty) \times [0, \infty)$. It follows that

$$\liminf_{t+\tau\to 0} \frac{f(t, \tau)}{t^{\tilde{p}_1}\tau^{q_1}} = \liminf_{t+\tau\to 0} \frac{(t + u_0)^{p_1}\tau^{q_1}}{t^{\tilde{p}_1}\tau^{q_1}} = +\infty > 0 \quad \text{for all } \tilde{p}_1 > 0,$$

$$\liminf_{t+\tau\to 0} \frac{g(t, \tau)}{t^{\tilde{p}_2}\tau^{q_2}} = \liminf_{t+\tau\to 0} \frac{(t + u_0)^{p_2}\tau^{q_2}}{t^{\tilde{p}_2}\tau^{q_2}} = +\infty > 0 \quad \text{for all } \tilde{p}_2 > 0,$$

that is, f and g satisfy (f_0) and (g_0) with exponents $\tilde{p}_1, q_1, \tilde{p}_2, q_2$. Next, by choosing $\tilde{p}_1$ and $\tilde{p}_2$ so small so that $\tilde{p}_1 < p - 1$ and $\tilde{p}_2 q_1 < (p - 1 - \tilde{p}_1)(q - 1 - q_2)$, we see that we can apply Theorem 3 to problem (4.15). Consequently, $u - u_0 = 0$ or $v = 0$ a.e. in $\mathbb{R}^N$. If $v = 0$ a.e. in $\mathbb{R}^N$, we are done. On the other hand, if $u = u_0$ a.e. in $\mathbb{R}^N$, then, by the first inequality of (4.14), it follows that $v = 0$ a.e. in $\mathbb{R}^N$. ☐

Obviously, an analogous result as above can be obtained when $\operatorname{ess\,inf}_{\mathbb{R}^N} u = 0$, $\operatorname{ess\,inf}_{\mathbb{R}^N} v > 0$, $p_1 < p - 1$, and $\mathscr{A}_q$ does not depend explicitly on v.

Remark 4. In the case $p = q$, $p_1 < p - 1$, $q_2 < p - 1$ and $D > 0$, condition (4.11) is *sharp* also for systems of equations. Indeed, if

$$N\left[1 - \frac{(p - 1 - p_1)(p - 1 - q_2)}{p_2 q_1}\right] > \max\left\{p + \frac{pq_1(p - 1) + pp_1(p - 1 - q_2)}{p_2 q_1},\right.$$
$$\left. p + \frac{pp_2(p - 1) + pq_2(p - 1 - p_1)}{p_2 q_1}\right\},$$

then we can construct an explicit nontrivial solution of the problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f(u, v) & \text{in } \mathbb{R}^N, \\ -\operatorname{div}(|\nabla v|^{p-2}\nabla v) = g(u, v) & \text{in } \mathbb{R}^N, \\ u \geq 0,\ v \geq 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{4.16}$$

where $f, g\colon [0, \infty) \times [0, \infty) \to [0, \infty)$ are continuous and such that, for all $(t, \tau) \in [0, 1] \times [0, 1]$,

$$f = f(t, \tau) = \left(\frac{\alpha p}{p - 1}\right)^{p-1} \tau^{(\alpha+1)(p-1)/\beta}\{N - (\alpha + 1)p + (\alpha + 1)pt^{1/\alpha}\},$$

$$g = g(t, \tau) = \left(\frac{\beta p}{p - 1}\right)^{p-1} t^{(\beta+1)(p-1)/\alpha}\{N - (\beta + 1)p + (\beta + 1)p\tau^{1/\beta}\},$$

where

$$\alpha := \frac{(p - 1)(q_1 + p - 1)}{p_2 q_1 - (p - 1)^2}, \quad \beta := \frac{(p - 1)(p_2 + p - 1)}{p_2 q_1 - (p - 1)^2}.$$

Hence f and g satisfy (f_0) and (g_0) with exponents $p_1 = 0$, $q_1 = (\alpha + 1)(p - 1)/\beta$, $p_2 = (\beta + 1)(p - 1)/\alpha$, $q_2 = 0$. By straightforward calculation, it follows that the functions defined by

$$u(x) = \frac{1}{(1 + |x|^{p/(p-1)})^\alpha}, \quad v(x) = \frac{1}{(1 + |x|^{p/(p-1)})^\beta}$$

are weak solutions of (4.16).

Remark 5. In the case $p_1 < p-1$, $q_2 < q-1$ and $D > 0$, condition (4.11) is *sharp* for systems of inequalities. Indeed, if

$$N\Big[1 - \frac{(p-1-p_1)(q-1-q_2)}{p_2 q_1}\Big] > \max\Big\{p + \frac{qq_1(p-1) + pp_1(q-1-q_2)}{p_2 q_1}, \\ q + \frac{pp_2(q-1) + qq_2(p-1-p_1)}{p_2 q_1}\Big\}, \tag{4.17}$$

then (4.1) has a nontrivial solution. Indeed, if (4.17) holds, then we can construct an explicit solution of the problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) \geq f(u,v) & \text{in } \mathbb{R}^N, \\ -\operatorname{div}(|\nabla v|^{q-2}\nabla v) \geq g(u,v) & \text{in } \mathbb{R}^N, \\ u \geq 0,\ v \geq 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{4.18}$$

where f and g satisfy (f_0) and (g_0), respectively. Consider the functions defined by

$$u(x) = \frac{1}{(1+|x|^{p/(p-1)})^{\alpha}}, \quad \alpha := \frac{p-1}{p} \cdot \frac{qq_1 + p(q-1-q_2)}{p_2 q_1 - (p-1-p_1)(q-1-q_2)},$$
$$v(x) = \frac{1}{(1+|x|^{q/(q-1)})^{\beta}}, \quad \beta := \frac{q-1}{q} \cdot \frac{pp_2 + q(p-1-p_1)}{p_2 q_1 - (p-1-p_1)(q-1-q_2)}.$$

Denoting $\varrho := |x|$, an easy computation shows that

$$\frac{-\Delta_p u}{u^{p_1} v^{q_1}} = \Big(\frac{\alpha p}{p-1}\Big)^{p-1} (1+\varrho^{p/(p-1)})^{\alpha p_1 - (\alpha+1)(p-1)-1} (1+\varrho^{q/(q-1)})^{\beta q_1} \{[N-(\alpha+1)p]\varrho^{p/(p-1)} + N\},$$
$$\frac{-\Delta_q v}{u^{p_2} v^{q_2}} = \Big(\frac{\beta q}{q-1}\Big)^{q-1} (1+\varrho^{q/(q-1)})^{\beta q_2 - (\beta+1)(q-1)-1} (1+\varrho^{p/(p-1)})^{\alpha p_2} \{[N-(\beta+1)q]\varrho^{q/(q-1)} + N\}.$$

By (4.17) and our assumptions $p_1 < p-1$ and $q_2 < q-1$, it follows that $N > (\alpha+1)p$ and $N > (\beta+1)q$. Hence, if we denote

$$h_1(\varrho) := \Big(\frac{\alpha p}{p-1}\Big)^{p-1} (1+\varrho^{p/(p-1)})^{\alpha p_1 - (\alpha+1)(p-1)-1} (1+\varrho^{q/(q-1)})^{\beta q_1} \{[N-(\alpha+1)p]\varrho^{p/(p-1)} + N\},$$
$$h_2(\varrho) := \Big(\frac{\beta q}{q-1}\Big)^{q-1} (1+\varrho^{q/(q-1)})^{\beta q_2 - (\beta+1)(p-1)-1} (1+\varrho^{p/(p-1)})^{\alpha p_2} \{[N-(\beta+1)q]\varrho^{q/(q-1)} + N\},$$

it follows that $h_1(\varrho) > 0$ and $h_2(\varrho) > 0$ for all $\varrho \geq 0$. Moreover, by the definitions of α and β, we get

$$\frac{p}{p-1}[\alpha p_1 - (\alpha+1)(p-1)] + \frac{q}{q-1}\beta q_1 = 0,$$
$$\frac{p}{p-1}\alpha p_2 + \frac{q}{q-1}[\beta q_2 - (\beta+1)(q-1)] = 0,$$

hence,

$$\lim_{\varrho\to\infty} h_1(\varrho) = \Big(\frac{\alpha p}{p-1}\Big)^{p-1} \cdot [N-(\alpha+1)p] > 0,$$

and similarly,

$$\lim_{\varrho\to\infty} h_2(\varrho) = \Big(\frac{\beta q}{q-1}\Big)^{q-1} \cdot [N-(\beta+1)q] > 0.$$

Therefore, since h_1 and h_2 are continuous functions, there are two positive constants C_1 and C_2 such that $h_1(\varrho) \geq C_1$ and $h_2(\varrho) \geq C_2$ for all $\varrho \geq 0$. Thus, we have, for all $x \in \mathbb{R}^N$,

$$\frac{-\Delta_p u}{u^{p_1} v^{q_1}} \geq C_1 > 0 \quad \text{and} \quad \frac{-\Delta_q v}{u^{p_2} v^{q_2}} \geq C_2 > 0,$$

that is, (u,v) is a nontrivial solution of (4.18) with $f(u,v) = C_1 u^{p_1} v^{q_1}$ and $g(u,v) = C_2 u^{p_2} v^{q_2}$.

By our construction, it follows that $0 < u(x) \leq 1$ and $0 < v(x) \leq 1$ for all $x \in \mathbb{R}^N$. Hence this counterexample works also for all continuous functions $f, g\colon [0,\infty)\times[0,\infty) \to [0,\infty)$ such that $f(t,\tau) = C_1 t^{p_1}\tau^{q_1}$ and $g(t,\tau) = C_2 t^{p_2}\tau^{q_2}$ for all $(t,\tau) \in [0,1]\times[0,1]$ and nonnegative elsewhere.

Corollary 2. *Let* (u, v) *be a weak solution of the system*

$$\begin{cases} -\operatorname{div}(\mathscr{A}_p(x, u, \nabla u)) \geq f(v) & \text{in } \mathbb{R}^N, \\ -\operatorname{div}(\mathscr{A}_q(x, v, \nabla v)) \geq g(u) & \text{in } \mathbb{R}^N, \\ u \geq 0,\ v \geq 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{4.19}$$

where $f, g\colon [0, \infty) \to [0, \infty)$ *are continuous functions satisfying the following conditions:*

(i) *There exists* $q_1 > 0$ *such that*

$$\liminf_{t \to 0^+} \frac{f(t)}{t^{q_1}} > 0 \quad (\textit{possibly infinity}). \tag{$(f_0)'$}$$

(ii) *There exists* $p_2 > 0$ *such that*

$$\liminf_{t \to 0^+} \frac{g(t)}{t^{p_2}} > 0 \quad (\textit{possibly infinity}). \tag{$(g_0)'$}$$

If

$$N\Big[1 - \frac{(p-1)(q-1)}{p_2 q_1}\Big] \leq \max\Big\{p + \frac{q(p-1)}{p_2},\, q + \frac{p(q-1)}{q_1}\Big\} \tag{4.20}$$

and $\operatorname{ess\,inf}_{\mathbb{R}^N} u = \operatorname{ess\,inf}_{\mathbb{R}^N} v = 0$, *then* $u \equiv v \equiv 0$ *a.e. in* $\mathbb{R}^N$.

Proof. By Theorem 3, with $p_1 = q_2 = 0$, we have that $u = 0$ or $v = 0$ a.e. in $\mathbb{R}^N$. If $v \equiv 0$, by W-q-C, for $\mathscr{A}_q$, we have $\mathscr{A}_q(\cdot, v, \nabla v) = 0$ a.e. in $\mathbb{R}^N$, and in turn, $g(u) = 0$ a.e. in $\mathbb{R}^N$. Thus, by (WH) on the first inequality of system (4.19), $(g_0)'$ and Lemma 2, we obtain, for R large,

$$\Big(\frac{1}{|B_{2R}|}\int_{B_{2R}} u^\sigma\Big)^{\frac{1}{\sigma}} \leq c_H \operatorname*{ess\,inf}_{B_R} u \leq cR^{-N/p_2}\Big(\int_{A_{R/2}} g(u)\Big)^{\frac{1}{p_2}} = 0,$$

that is, $u = 0$ a.e. in $\mathbb{R}^N$. □

Remark 6. Note that condition (4.20) is equivalent to

$$\max\Big\{\frac{qq_1 + p(q-1)}{p_2 q_1 - (p-1)(q-1)} - \frac{N-p}{p-1},\, \frac{pp_2 + q(p-1)}{p_2 q_1 - (p-1)(q-1)} - \frac{N-q}{q-1}\Big\} \geq 0. \tag{4.21}$$

This is the assumption required in [7, Theorem 2.1], when $f(v) = v^{q_1}$ and $g(u) = u^{p_2}$. In [7, Section 3], the authors prove also that the nonexistence result is sharp, in the sense that if (4.21) is not valid, they are able to construct a solution $(u, v) \neq (0, 0)$ of (4.19). Corollary 2 in a more general setting has been studied in [5].

Remark 7. Consider the problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) \geq u^{p_1} v^{q_1} & \text{in } \mathbb{R}^N, \\ -\operatorname{div}(|\nabla v|^{q-2}\nabla v) \geq u^{p_2} v^{q_2} & \text{in } \mathbb{R}^N, \\ u \geq 0,\ v \geq 0 & \text{in } \mathbb{R}^N \end{cases} \tag{4.22}$$

with $p, q > 1$, $p_1, q_2 \geq 0$ and $p_2, q_1 > 0$. As pointed out in [1, Remark 5.1], it is possible to obtain a *nonoptimal* sufficient condition of nonexistence for (4.22), as a consequence of Corollary 2. Since $-\Delta_p$ and $-\Delta_q$ are S-p-C and S-q-C, respectively, inequality (WH) holds for both u and v. Hence, by Remark 1, either $u \equiv 0$ or $u > 0$ in $\mathbb{R}^N$, and analogously, either $v \equiv 0$ or $v > 0$ in $\mathbb{R}^N$. Therefore, with a change of variables, we can obtain, from problem (4.22), a system of the type (4.19). More precisely, let $\theta, \tau \in (0, 1)$. Set $w := u^\theta$, $z := v^\tau$. Then

$$\begin{cases} -\Delta_p w \geq C w^{[p_1 - (1-\theta)(p-1)]/\theta} z^{q_1/\tau} & \text{in } \mathbb{R}^N, \\ -\Delta_q z \geq C w^{p_2/\theta} z^{[q_2 - (1-\tau)(q-1)]/\tau} & \text{in } \mathbb{R}^N, \end{cases}$$

where $C > 0$. When $p_1 < p - 1$ and $q_2 < q - 1$, we can choose $\theta = 1 - \frac{p_1}{p-1}$, $\tau = 1 - \frac{q_2}{q-1}$ and find

$$\begin{cases} -\Delta_p w \geq C z^{q_1/\tau} & \text{in } \mathbb{R}^N, \\ -\Delta_q z \geq C w^{p_2/\theta} & \text{in } \mathbb{R}^N. \end{cases}$$

Hence, if $\frac{q_1}{\tau} > q-1$, $\frac{p_2}{\theta} > p-1$ (i.e. $q_1 > q-1-q_2$ and $p_2 > p-1-p_1$), and if we require condition (4.20) with $\frac{q_1}{\tau}$ in place of q_1 and $\frac{p_2}{\theta}$ in place of p_2, that is,

$$N\left[1-\frac{(p-1-p_1)(q-1-q_2)}{p_2q_1}\right] \le \max\left\{p+\frac{q(p-1-p_1)}{p_2}, q+\frac{p(q-1-q_2)}{q_1}\right\}, \tag{4.23}$$

then problem (4.22) has no nontrivial solutions by Corollary 2. Nevertheless, condition (4.23) is not sharp, as Theorem 3 proves.

Theorem 4. *Assume that $p_1 \le p-1$ and $q_2 \le q-1$. If (u,v) is a weak solution of* (4.1) *such that* $\operatorname{ess\,inf}_{\mathbb{R}^N} u = \operatorname{ess\,inf}_{\mathbb{R}^N} v = 0$ *and*

$$\begin{aligned} N\left[1-\frac{(p-1-p_1)(q-1-q_2)}{p_2q_1}\right] < \max\Big\{&p+\frac{qq_1(p-1)+pp_1(q-1-q_2)}{p_2q_1}, \\ &q+\frac{pp_2(q-1)+qq_2(p-1-p_1)}{p_2q_1}\Big\}, \end{aligned} \tag{4.24}$$

then $u=0$ or $v=0$ a.e. in $\mathbb{R}^N$.

Proof. From Theorem 3, if $p_1 < p-1$ and $q_2 < q-1$, we already know a stronger result. Therefore, we prove this result only when $p_1 = p-1$ and $q_2 \le q-1$, and we omit the similar proof in the case $p_1 \le p-1$ and $q_2 = q-1$. Suppose, by contradiction, that problem (4.1) admits a nontrivial solution (u,v). By (4.7) and (4.9), we have, for R sufficiently large,

$$\int_{B_R} f(u,v) \le cR^{N-p-p_1\frac{qq_1+p(q-1-q_2)}{p_2q_1}}, \qquad \int_{B_R} g(u,v) \le cR^{N-q-\frac{p(q-1)}{q_1}}.$$

By hypothesis (4.24) and letting $R\to\infty$, we get $f(u,v)=0$ or $g(u,v)=0$ a.e. in $\mathbb{R}^N$. We complete the proof by using Lemma 3. □

Lemma 5. *Let (u,v) be a weak solution of* (4.1) *such that* $\operatorname{ess\,inf}_{\mathbb{R}^N} u = \operatorname{ess\,inf}_{\mathbb{R}^N} v = 0$. *If there exists $z\in[0,1]$ such that*

$$\begin{cases} \dfrac{p_2}{p-1}z+\left(\dfrac{p_1}{p-1}-1\right)(1-z) \ge 0, \\ \left(\dfrac{q_2}{q-1}-1\right)z+\dfrac{q_1}{q-1}(1-z) \ge 0, \\ N > \left(\dfrac{N-p}{p-1}p_1+\dfrac{N-q}{q-1}q_1\right)(1-z)+\left(\dfrac{N-p}{p-1}p_2+\dfrac{N-q}{q-1}q_2\right)z, \end{cases} \tag{4.25}$$

then $u=0$ or $v=0$ a.e. in $\mathbb{R}^N$.

Proof. By contradiction, if $u>0$ and $v>0$, from (4.4) and (4.5), we have, for R large and for all $z\in[0,1]$,

$$\left(\int_{B_R} f(u,v)\right)^{\frac{p_2}{p-1}z}\left(\int_{B_R} g(u,v)\right)^{\left(\frac{q_2}{q-1}-1\right)z} \le cR^{N\left(\frac{p_2}{p-1}+\frac{q_2}{q-1}-1\right)z-p\frac{p_2}{p-1}z-q\frac{q_2}{q-1}z},$$

$$\left(\int_{B_R} f(u,v)\right)^{\left(\frac{p_1}{p-1}-1\right)(1-z)}\left(\int_{B_R} g(u,v)\right)^{\frac{q_1}{q-1}(1-z)} \le cR^{N\left(\frac{q_1}{q-1}+\frac{p_1}{p-1}-1\right)(1-z)-q\frac{q_1}{q-1}(1-z)-p\frac{p_1}{p-1}(1-z)},$$

and so

$$\left(\int_{B_R} f(u,v)\right)^{\alpha}\left(\int_{B_R} g(u,v)\right)^{\beta} \le cR^{-\gamma}, \tag{4.26}$$

where

$$\alpha := \frac{p_2}{p-1}z+\left(\frac{p_1}{p-1}-1\right)(1-z), \quad \beta := \left(\frac{q_2}{q-1}-1\right)z+\frac{q_1}{q-1}(1-z),$$

$$\gamma := N-\left(\frac{N-p}{p-1}p_2+\frac{N-q}{q-1}q_2\right)z-\left(\frac{N-p}{p-1}p_1+\frac{N-q}{q-1}q_1\right)(1-z).$$

By (4.25), $\alpha\ge0$, $\beta\ge0$ and $\gamma>0$, hence, by (4.26), $f(u,v)=0$ or $g(u,v)=0$ a.e. in $\mathbb{R}^N$, and so, by Lemma 3, we have that $u=0$ or $v=0$ a.e. in $\mathbb{R}^N$. This completes the proof. □

Theorem 5. *Let (u, v) be a weak solution of (4.1) such that* $\operatorname{ess\,inf}_{\mathbb{R}^N} u = \operatorname{ess\,inf}_{\mathbb{R}^N} v = 0$.

(i) *If $p_1 \ge p-1$, $q_2 \le q-1$ and*

$$N > \min\left\{\frac{N-p}{p-1}p_1 + \frac{N-q}{q-1}q_1, \left(\frac{N-p}{p-1}p_2 + \frac{N-q}{q-1}q_2\right)\frac{q_1}{q_1-q_2+q-1} + \left(\frac{N-p}{p-1}p_1 + \frac{N-q}{q-1}q_1\right)\frac{q-1-q_2}{q_1-q_2+q-1}\right\}, \tag{4.27}$$

then $u = 0$ or $v = 0$ a.e. in $\mathbb{R}^N$.

(ii) *If $p_1 \le p-1$, $q_2 \ge q-1$ and*

$$N > \min\left\{\frac{N-p}{p-1}p_2 + \frac{N-q}{q-1}q_2, \left(\frac{N-p}{p-1}p_1 + \frac{N-q}{q-1}q_1\right)\frac{p_2}{p_2-p_1+p-1} + \left(\frac{N-p}{p-1}p_2 + \frac{N-q}{q-1}q_2\right)\frac{p-1-p_1}{p_2-p_1+p-1}\right\}, \tag{4.28}$$

then $u = 0$ or $v = 0$ a.e. in $\mathbb{R}^N$.

(iii) *If $p_1 \ge p-1$, $q_2 \ge q-1$ and*

$$N > \min\left\{\frac{N-p}{p-1}p_1 + \frac{N-q}{q-1}q_1, \frac{N-p}{p-1}p_2 + \frac{N-q}{q-1}q_2\right\}, \tag{4.29}$$

then $u = 0$ or $v = 0$ a.e. in $\mathbb{R}^N$.

Proof. (i) Let (u, v) be a weak solution of (4.1). By Lemma 5, for all $z \in [0, 1]$ satisfying (4.25), we have that (u, v) is trivial. Now, system (4.25) is equivalent to

$$\begin{cases} z \le \dfrac{q_1}{q_1+q-1-q_2}, \\ N > \left(\dfrac{N-p}{p-1}p_1 + \dfrac{N-q}{q-1}q_1\right)(1-z) + \left(\dfrac{N-p}{p-1}p_2 + \dfrac{N-q}{q-1}q_2\right)z, \end{cases} \tag{4.30}$$

since $p_1 \ge p-1$ and $q_2 \le q-1$. Put

$$\varphi(z) := \left(\frac{N-p}{p-1}p_1 + \frac{N-q}{q-1}q_1\right)(1-z) + \left(\frac{N-p}{p-1}p_2 + \frac{N-q}{q-1}q_2\right)z.$$

If

$$\frac{N-p}{p-1}p_1 + \frac{N-q}{q-1}q_1 \le \frac{N-p}{p-1}p_2 + \frac{N-q}{q-1}q_2,$$

then φ is nondecreasing, and we obtain the best condition taking $z = 0$ in the second inequality of (4.30), namely,

$$N > \frac{N-p}{p-1}p_1 + \frac{N-q}{q-1}q_1.$$

While, if

$$\frac{N-p}{p-1}p_1 + \frac{N-q}{q-1}q_1 > \frac{N-p}{p-1}p_2 + \frac{N-q}{q-1}q_2,$$

then we have the best condition taking $z = \frac{q_1}{q_1+q-1-q_2}$ in second inequality of (4.30), that is

$$N > \left(\frac{N-p}{p-1}p_2 + \frac{N-q}{q-1}q_2\right)\frac{q_1}{q_1-q_2+q-1} + \left(\frac{N-p}{p-1}p_1 + \frac{N-q}{q-1}q_1\right)\frac{q-1-q_2}{q_1-q_2+q-1}.$$

Finally, by an easy calculation, we see that

$$\left(\frac{N-p}{p-1}p_2 + \frac{N-q}{q-1}q_2\right)\frac{q_1}{q_1-q_2+q-1} + \left(\frac{N-p}{p-1}p_1 + \frac{N-q}{q-1}q_1\right)\frac{q-1-q_2}{q_1-q_2+q-1} < \frac{N-p}{p-1}p_1 + \frac{N-q}{q-1}q_1$$

if and only if

$$\frac{N-p}{p-1}p_2 + \frac{N-q}{q-1}q_2 < \frac{N-p}{p-1}p_1 + \frac{N-q}{q-1}q_1.$$

This completes the proof of the first part of the theorem.

(ii) The proof is similar to the proof of (i), and it is omitted.

(iii) Let (u, v) be as in the statement. By Lemma 5, for all $z \in [0, 1]$ satisfying (4.25), we have that (u, v) is trivial. Now, system (4.25) is equivalent to

$$N > \left(\frac{N-p}{p-1}p_1 + \frac{N-q}{q-1}q_1\right)(1-z) + \left(\frac{N-p}{p-1}p_2 + \frac{N-q}{q-1}q_2\right)z, \tag{4.31}$$

since $p_1 \geq p - 1$ and $q_2 \geq q - 1$. Now, if

$$\frac{N-p}{p-1}p_1 + \frac{N-q}{q-1}q_1 \leq \frac{N-p}{p-1}p_2 + \frac{N-q}{q-1}q_2,$$

then the function φ defined in part (i) is nondecreasing, and we obtain the best condition taking $z = 0$ in (4.31), namely,

$$N > \frac{N-p}{p-1}p_1 + \frac{N-q}{q-1}q_1.$$

While, if

$$\frac{N-p}{p-1}p_1 + \frac{N-q}{q-1}q_1 > \frac{N-p}{p-1}p_2 + \frac{N-q}{q-1}q_2,$$

then we have the best condition taking $z = 1$ in (4.31), that is,

$$N > \frac{N-p}{p-1}p_2 + \frac{N-q}{q-1}q_2.$$

□

Remark 8. If $p_1 = p - 1$ and $q_2 = q - 1$, then (4.27) jointly with (4.28) give the same condition as (4.29). Moreover, in this case, this curve is equivalent to condition (4.24).

Theorem 6. *Let (u, v) be a weak solution of (4.1) with $\operatorname{ess\,inf}_{\mathbb{R}^N} u = \operatorname{ess\,inf}_{\mathbb{R}^N} v = 0$.*

(i) *If $q_2 < q - 1$, $D > 0$ and*

$$N \geq \left(\frac{N-p}{p-1}p_2 + \frac{N-q}{q-1}q_2\right)\frac{q_1}{q_1 - q_2 + q - 1} + \left(\frac{N-p}{p-1}p_1 + \frac{N-q}{q-1}q_1\right)\frac{q-1-q_2}{q_1 - q_2 + q - 1}, \tag{4.32}$$

then $u = 0$ or $v = 0$ a.e. in $\mathbb{R}^N$. In particular, if $q_2 < q - 1$, $p_1 \geq p - 1$ and (4.32) holds, then (u, v) is trivial.

(ii) *If $p_1 < p - 1$, $D > 0$ and*

$$N \geq \left(\frac{N-p}{p-1}p_1 + \frac{N-q}{q-1}q_1\right)\frac{p_2}{p_2 - p_1 + p - 1} + \left(\frac{N-p}{p-1}p_2 + \frac{N-q}{q-1}q_2\right)\frac{p-1-p_1}{p_2 - p_1 + p - 1}, \tag{4.33}$$

then $u = 0$ or $v = 0$ a.e. in $\mathbb{R}^N$. In particular, if $p_1 < p - 1$, $q_2 \geq q - 1$ and (4.33) holds, then (u, v) is trivial.

Proof. (i) By contradiction, let (u, v) be a nontrivial weak solution of (4.1). By (4.4) and (4.5), for R sufficiently large, we have

$$\left(\int_{B_R} f(u,v)\right)^{\frac{p_2}{p-1}}\left(\int_{B_R} g(u,v)\right)^{\frac{q_2}{q-1}} \leq cR^{N(\frac{p_2}{p-1}-1+\frac{q_2}{q-1})-p\frac{p_2}{p-1}-q\frac{q_2}{q-1}} \int_{A_{R/2}} g(u,v) \tag{4.34}$$

and

$$\left(\int_{B_R} g(u,v)\right)^{\frac{q_1}{q-1}}\left(\int_{B_R} f(u,v)\right)^{\frac{p_1}{p-1}} \leq cR^{N(\frac{q_1}{q-1}-1+\frac{p_1}{p-1})-q\frac{q_1}{q-1}-p\frac{p_1}{p-1}} \int_{A_{R/2}} f(u,v). \tag{4.35}$$

By (4.34),

$$\int_{B_R} g(u,v) \geq c\left(R^{p\frac{p_2}{p-1}+q\frac{q_2}{q-1}-N(\frac{p_2}{p-1}-1+\frac{q_2}{q-1})}\right)^{\frac{q-1}{q-1-q_2}}\left(\int_{B_R} f(u,v)\right)^{\frac{p_2(q-1)}{(p-1)(q-1-q_2)}}$$

since $q_2 < q - 1$. Combining this last inequality with (4.35), we get

$$\int_{A_{R/2}} f(u,v) \geq cR^{\frac{pp_1}{p-1}+\frac{qq_1}{q-1}-N(\frac{p_1}{p-1}+\frac{q_1}{q-1}-1)+[\frac{pp_2}{p-1}+\frac{qq_2}{q-1}-N(\frac{p_2}{p-1}+\frac{q_2}{q-1}-1)]\frac{q_1}{q-1-q_2}}\left(\int_{B_R} f(u,v)\right)^{\frac{p_1}{p-1}+\frac{p_2q_1}{(p-1)(q-1-q_2)}}, \tag{4.36}$$

and so

$$\left(\int_{B_R} f(u,v)\right)^{\frac{D}{(p-1)(q-1-q_2)}} \le cR^{-\gamma} \tag{4.37}$$

for R large, where

$$\gamma := \frac{pp_1}{p-1} + \frac{qq_1}{q-1} - N\left(\frac{p_1}{p-1} + \frac{q_1}{q-1} - 1\right) + \left[\frac{pp_2}{p-1} + \frac{qq_2}{q-1} - N\left(\frac{p_2}{p-1} + \frac{q_2}{q-1} - 1\right)\right]\frac{q_1}{q-1-q_2}.$$

By hypothesis (i), $\gamma \ge 0$. Now, by (4.37), $\int_{\mathbb{R}^N} f(u,v) < \infty$, since $\frac{D}{(p-1)(q-1-q_2)} > 0$, so that by (4.36), $f(u,v) = 0$ a.e. in $\mathbb{R}^N$. The contradiction follows by Lemma 3. For the second part of the statement, it is enough to note that if $q_2 < q-1$ and $p_1 \ge p-1$, then $D > 0$.

(ii) The proof is analogous to the proof of (i). □

Remark 9. Note that, in the case $p_1 = p-1$ and $q_2 = q-1$, condition (4.25) is equivalent to (4.24). Moreover, when

$$q_2 < q-1, \quad p_1 \ge p-1, \quad \frac{N-p}{p-1}p_1 + \frac{N-q}{q-1}q_1 \le \frac{N-p}{p-1}p_2 + \frac{N-q}{q-1}q_2, \tag{4.38}$$

then condition (4.32) is stronger than (4.27). Similarly, if

$$p_1 < p-1, \quad q_2 \ge q-1, \quad \frac{N-p}{p-1}p_2 + \frac{N-q}{q-1}q_2 \le \frac{N-p}{p-1}p_1 + \frac{N-q}{q-1}q_1, \tag{4.39}$$

then (4.33) is stronger than (4.28).

Now we prove that conditions (4.32) and (4.33) are sharp at least when (4.38) and (4.39) hold, respectively.

For simplicity, we show a counterexample for (4.32) when $p = q$.

Let $q_2 < p-1 \le p_1$, $p_1 + q_1 \le p_2 + q_2$ and

$$\frac{N(p-1)}{N-p} < (p_2+q_2)\frac{q_1}{q_1-q_2+q-1} + (p_1+q_1)\frac{q-1-q_2}{q_1-q_2+q-1}. \tag{4.40}$$

We prove that, under these assumptions, the system

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) \ge f(u,v) & \text{in } \mathbb{R}^N, \\ -\operatorname{div}(|\nabla v|^{p-2}\nabla v) \ge g(u,v) & \text{in } \mathbb{R}^N, \\ u \ge 0,\ v \ge 0 & \text{in } \mathbb{R}^N, \end{cases}$$

where f and g satisfy (f_0) and (g_0), admits a nontrivial solution. Consider the functions

$$u(x) = \frac{1}{(1+|x|^{p/(p-1)})^{\alpha}}, \quad \alpha := \frac{(p-1)(q_1+p-1-q_2)}{p_2q_1-(p-1-p_1)(p-1-q_2)},$$
$$v(x) = \frac{1}{(1+|x|^{p/(p-1)})^{\beta}}, \quad \beta := \frac{(p-1)(p_2+p-1-p_1)}{p_2q_1-(p-1-p_1)(p-1-q_2)},$$

and denote $\varrho := |x|$.

By straightforward computation, we know that

$$\frac{-\Delta_p u}{u^{p_1}v^{q_1}} = h_1(\varrho), \quad \frac{-\Delta_q v}{u^{p_2}v^{q_2}} = h_2(\varrho),$$

where

$$h_1(\varrho) := \left(\frac{\alpha p}{p-1}\right)^{p-1}(1+\varrho^{p/(p-1)})^{\alpha p_1+\beta q_1-(\alpha+1)(p-1)-1}\{[N-(\alpha+1)p]\varrho^{p/(p-1)}+N\},$$
$$h_2(\varrho) := \left(\frac{\beta q}{q-1}\right)^{q-1}(1+\varrho^{p/(p-1)})^{\alpha p_2+\beta q_2-(\beta+1)(p-1)-1}\{[N-(\beta+1)p]\varrho^{p/(p-1)}+N\}.$$

The exponents α and β are such that

$$\begin{cases} \alpha p_1 + \beta q_1 - (\alpha+1)(p-1) - 1 = 0, \\ \alpha p_2 + \beta q_2 - (\beta+1)(p-1) - 1 = 0, \end{cases}$$

	$p_1 < p-1$			$p_1 = p-1$			$p_1 > p-1$	
$q_2 < p-1$	(4.41)			$A_1 < A_2$ (4.42)	$A_1 = A_2$ (4.46)	$A_1 > A_2$ (4.43)	$A_2 \le A_1$ (4.46)	$A_2 > A_1$ (4.43)
$q_2 = p-1$	$A_1 < A_2$ (4.42)	$A_1 = A_2$ (4.47)	$A_1 > A_2$ (4.44)	(4.42) $\Leftrightarrow$ (4.45)			(4.45)	
$q_2 > p-1$	$A_1 \le A_2$ (4.47)		$A_1 > A_2$ (4.44)	(4.45)			(4.45)	

Table 1: Conditions implying that the solutions of (4.1) are trivial.

hence, the expressions of h_1 and h_2 become simply

$$h_1(\varrho) := \left(\frac{\alpha p}{p-1}\right)^{p-1}\{[N-(\alpha+1)p]\varrho^{p/(p-1)}+N\},$$
$$h_2(\varrho) := \left(\frac{\beta q}{q-1}\right)^{q-1}\{[N-(\beta+1)p]\varrho^{p/(p-1)}+N\}.$$

By (4.40) and our assumptions $q_2 < p-1 \le p_1$ and $p_1+q_1 \le p_2+q_2$, it follows that $N > (\alpha+1)p$ and $N > (\beta+1)q$. Hence, $h_1(\varrho) > 0$ and $h_2(\varrho) > 0$ for all $\varrho \ge 0$.

For simplicity, we summarize the results obtained in Theorems 3, 4 and 5 for $p = q$.

Corollary 3. *Consider system* (4.1) *with $p = q$. Let (u, v) be a weak solution of this problem such that*

$$\operatorname*{ess\,inf}_{\mathbb{R}^N} u = \operatorname*{ess\,inf}_{\mathbb{R}^N} v = 0.$$

Denote $A_1 := p_1 + q_1$, $A_2 := p_2 + q_2$, $\alpha = \frac{p_2}{p_2-p_1+p-1}$, $\beta = \frac{q_1}{q_1-q_2+p-1}$ and

$$\frac{N(p-1)}{N-p} \ge \min\{A_1\alpha + A_2(1-\alpha),\ A_2\beta + A_1(1-\beta)\}, \tag{4.41}$$
$$\frac{N(p-1)}{N-p} > \min\{A_1\alpha + A_2(1-\alpha),\ A_2\beta + A_1(1-\beta)\}, \tag{4.42}$$
$$\frac{N(p-1)}{N-p} > \min\{A_1,\ A_2\beta + A_1(1-\beta)\}, \tag{4.43}$$
$$\frac{N(p-1)}{N-p} > \min\{A_2,\ A_1\alpha + A_2(1-\alpha)\}, \tag{4.44}$$
$$\frac{N(p-1)}{N-p} > \min\{A_1,\ A_2\}, \tag{4.45}$$
$$\frac{N(p-1)}{N-p} \ge A_2\beta + A_1(1-\beta), \tag{4.46}$$
$$\frac{N(p-1)}{N-p} \ge A_1\alpha + A_2(1-\alpha). \tag{4.47}$$

Then, under the assumptions described in Table 1, it follows that either $u = 0$ or $v = 0$ a.e. in $\mathbb{R}^N$.

In problem (4.1), we have excluded the cases $q_1 = 0$ or $p_2 = 0$. In this final part of the section, we would like to show that these cases can be treated essentially with the tools used in [4] for the inequalities. For simplicity, we consider now problem (4.1) with $p = q$. Moreover, we require that the functions $f, g\colon [0,\infty) \times [0,\infty) \to [0,\infty)$ are *continuous* and satisfy conditions (f_0) and (g_0), introduced in Section 1, *with $q_1 = 0$ or $p_2 = 0$.*

Theorem 7. *Consider system* (4.1). *Let $q_1 = 0$ or $p_2 = 0$, and suppose that $p_1 > p-1$, $q_2 > p-1$ and*

$$\frac{N(p-1)}{N-p} \ge \min\{p_1+q_1, p_2+q_2\}. \tag{4.48}$$

If (u, v) is a weak solution of (4.1) *such that* $\operatorname{ess\,inf}_{\mathbb{R}^N} u = \operatorname{ess\,inf}_{\mathbb{R}^N} v = 0$, *then $u = 0$ or $v = 0$ a.e. in $\mathbb{R}^N$.*

Proof. Suppose that $q_1 = 0$. If $p_1 \le p_2 + q_2$, we can proceed as in the proofs of [4, Theorems 3.3 and 3.4]. Indeed, for R large and for all $\varepsilon > 0$, we have, by (3.5) and (f_0),

$$\begin{aligned}\int_{B_R} f(u,v) &\le cR^{-p}|A_{R/2}|(\operatorname*{ess\,inf}_{B_R} u)^{p-1}\\ &\le cR^{-p}|A_{R/2}|\left(\frac{1}{|A_{R/2}\cap T_\varepsilon|}\int_{A_{R/2}\cap T_\varepsilon} u^{p_1}\right)^{\frac{p-1}{p_1}}\\ &\le cR^{-p}|A_{R/2}|^{1-\frac{p-1}{p_1}}\left(\int_{A_{R/2}} f(u,v)\right)^{\frac{p-1}{p_1}}\\ &\le cR^{N-p-\frac{N(p-1)}{p_1}}\left(\int_{A_{R/2}} \tilde{f}(u,v)\right)^{\frac{p-1}{p_1}}.\end{aligned}$$

Hence, $f(u,v) = 0$ a.e. in $\mathbb{R}^N$ by (4.48), since $p_1 > p-1$. Therefore, (u,v) is trivial.

If $p_1 > p_2 + q_2$, we distinguish two cases. If $u = 0$ a.e. in $\mathbb{R}^N$, then we are done. If $u > 0$ a.e. in $\mathbb{R}^N$, then, by (3.5), (g_0) and Lemma 2, we obtain, for R large enough and for all $\varepsilon > 0$,

$$\begin{aligned}\int_{B_R} g(u,v) &\le \frac{cR^{-p}|A_{R/2}|}{(\operatorname{ess\,inf}_{A_{R/2}\cap T_\varepsilon} u)^{\frac{p_2(p-1)}{q_2}}}\left(\frac{1}{|A_{R/2}\cap T_\varepsilon|}\int_{A_{R/2}\cap T_\varepsilon} u^{p_2}v^{q_2}\right)^{\frac{p-1}{q_2}}\\ &\le cR^{-p+N\left(1-\frac{p-1}{q_2}\right)+\frac{(N-p)p_2}{q_2}}\frac{\left(\int_{A_{R/2}} g(u,v)\right)^{\frac{p-1}{q_2}}}{\left(\int_{B_R} f(u,v)\right)^{\frac{p_2}{q_2}}},\end{aligned} \tag{4.49}$$

that is,

$$\begin{aligned}\left(\int_{B_R} f(u,v)\right)^{\frac{p_2}{q_2}}\int_{B_R} g(u,v) &\le cR^{-p+N\left(1-\frac{p-1}{q_2}\right)+\frac{(N-p)p_2}{q_2}}\left(\int_{A_{R/2}} g(u,v)\right)^{\frac{p-1}{q_2}}\\ &\le cR^{-p+N\left(1-\frac{p-1}{q_2}\right)+\frac{(N-p)p_2}{q_2}}\left(\int_{B_R} g(u,v)\right)^{\frac{p-1}{q_2}}.\end{aligned}$$

Now, by (4.48), we have that the exponent of R in the right side is nonpositive. Without loss of generality, we consider only the case for which the exponent of R is equal to 0, i.e., when (4.48) holds with the equality sign. Thus, since $q_2 > p-1$, we can have three cases. Either $\int_{B_R} g(u,v) \to 0$ and $\int_{B_R} f(u,v) \to \infty$ as $R \to \infty$ or vice versa. In both cases, we conclude that $g(u,v) = 0$ or $f(u,v) = 0$ a.e. in $\mathbb{R}^N$, and we are done. In the third case, $\int_{B_R} g(u,v) \to$ const. and $\int_{B_R} f(u,v) \to$ const. as $R \to \infty$. Hence, in particular, $g(u,v) \in L^1(\mathbb{R}^N)$, and so $\int_{A_{R/2}} g(u,v) \to 0$ as $R \to \infty$. Hence, we conclude that $g(u,v) = 0$ a.e. in $\mathbb{R}^N$ by (4.49). In the case $p_2 = 0$, the proof is analogous. □

5 A nonautonomous system of inequalities

In this section, we consider the problem

$$\begin{cases} -\operatorname{div}(\mathscr{A}_p(x,u,\nabla u)) \ge a(x)f(u,v) & \text{in } \mathbb{R}^N,\\ -\operatorname{div}(\mathscr{A}_q(x,v,\nabla v)) \ge b(x)g(u,v) & \text{in } \mathbb{R}^N,\\ u \ge 0,\ v \ge 0 & \text{in } \mathbb{R}^N,\end{cases} \tag{5.1}$$

where we assume the following:

Assumption 3. *All conditions of Assumption 2 hold. Moreover, $a, b\colon \mathbb{R}^N \to \mathbb{R}_0^+$ are nonnegative measurable functions.*

Theorem 8. *Let $p_1 < p-1$, $q_2 < q-1$, $p_2 q_1 > (p-1-p_1)(q-1-q_2)$, and suppose there exist $r, s > 0$ such that $a^{-r} \in L^1_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$ and $b^{-s} \in L^1_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$. Let (u, v) be a weak solution of (5.1) such that $\operatorname{ess\,inf}_{\mathbb{R}^N} u = \operatorname{ess\,inf}_{\mathbb{R}^N} v = 0$. If one of the conditions*

$$\lim_{R\to\infty} R^{N\left[1-\frac{(p-1-p_1)(q-1-q_2)}{p_2 q_1}\right]-q\frac{p-1}{p_2}-p\left[1+\frac{p_1(q-1-q_2)}{p_2 q_1}\right]}\left(\fint_{A_{R/2}} a^{-r}\right)^{\frac{(p-1)(q-1-q_2)}{r p_2 q_1}}\left(\fint_{A_{R/2}} b^{-s}\right)^{\frac{p-1}{s p_2}} < \infty, \tag{5.2a}$$

$$\lim_{R\to\infty} R^{N\left[1-\frac{(p-1-p_1)(q-1-q_2)}{p_2 q_1}\right]-p\frac{q-1}{q_1}-q\left[1+\frac{q_2(p-1-p_1)}{p_2 q_1}\right]}\left(\fint_{A_{R/2}} a^{-r}\right)^{\frac{q-1}{r q_1}}\left(\fint_{A_{R/2}} b^{-s}\right)^{\frac{(p-1-p_1)(q-1)}{s p_2 q_1}} < \infty \tag{5.2b}$$

holds, then $u = 0$ or $v = 0$ a.e. in $\mathbb{R}^N$.

Proof. Suppose that (5.2b) holds, the remaining case being analogous. If $v = 0$ a.e. in $\mathbb{R}^N$, we are done. Otherwise, put $\tilde{r} = 1 + 1/r$ and $\tilde{s} = 1 + 1/s$. By the first inequality of (3.5), Lemma 2 and (g_0), we get, for R sufficiently large,

$$\begin{aligned}
\int_{B_R} a(x) f(u, v) &\le c R^{-p} |A_{R/2}| (\operatorname*{ess\,inf}_{B_R} u)^{p-1} \le c R^{-p} |A_{R/2}|^{1-\frac{\tilde{s}(p-1)}{p_2}} \left(\int_{A_{R/2}\cap T_\varepsilon} u^{p_2/\tilde{s}}\right)^{\frac{\tilde{s}(p-1)}{p_2}} \\
&\le c R^{-p} |A_{R/2}|^{1-\frac{\tilde{s}(p-1)}{p_2}} (\operatorname*{ess\,inf}_{A_{R/2}\cap T_\varepsilon} v)^{-\frac{q_2(p-1)}{p_2}} \left(\int_{A_{R/2}\cap T_\varepsilon} u^{p_2/\tilde{s}} v^{q_2/\tilde{s}}\right)^{\frac{\tilde{s}(p-1)}{p_2}} \\
&\le c R^{-p} |A_{R/2}|^{1-\frac{\tilde{s}(p-1)}{p_2}} (\operatorname*{ess\,inf}_{A_{R/2}} v)^{-\frac{q_2(p-1)}{p_2}} \left(\int_{A_{R/2}} b(x)^{-1/\tilde{s}} (b(x) g(u, v))^{1/\tilde{s}}\right)^{\frac{\tilde{s}(p-1)}{p_2}} \\
&\le c R^{-p} |A_{R/2}|^{1-\frac{\tilde{s}(p-1)}{p_2}} (\operatorname*{ess\,inf}_{A_{R/2}} v)^{-\frac{q_2(p-1)}{p_2}} \left(\int_{A_{R/2}} b^{-(\tilde{s}'-1)}\right)^{\frac{p-1}{p_2(\tilde{s}'-1)}} \left(\int_{A_{R/2}} b(x) g(u, v)\right)^{\frac{p-1}{p_2}},
\end{aligned} \tag{5.3}$$

where, in the last step, we have used Hölder's inequality. By the second inequality of (3.5),

$$\int_{B_R} b(x) g(u, v) \le c R^{-q} |A_{R/2}| (\operatorname*{ess\,inf}_{B_R} v)^{q-1} \le c R^{-q} |A_{R/2}| (\operatorname*{ess\,inf}_{A_{R/2}\cap T_\varepsilon} v)^{q-1},$$

thus

$$\operatorname*{ess\,inf}_{A_{R/2}\cap T_\varepsilon} v \ge c\left(\frac{R^q}{|A_{R/2}|}\int_{B_R} b(x) g(u, v)\right)^{\frac{1}{q-1}} \ge c\left(\frac{R^q}{|A_{R/2}|}\int_{A_{R/2}} b(x) g(u, v)\right)^{\frac{1}{q-1}}. \tag{5.4}$$

Combining (5.3) and (5.4), we have

$$\begin{aligned}
\int_{B_R} a(x) f(u, v) &\le c R^{-p} |A_{R/2}| (\operatorname*{ess\,inf}_{B_R} u)^{p-1} \\
&\le c R^{-p-q\frac{q_2(p-1)}{p_2(q-1)}} |A_{R/2}|^{1-\frac{(p-1)\tilde{s}}{p_2}+\frac{q_2(p-1)}{p_2(q-1)}} \left(\int_{A_{R/2}} b(x)^{-(\tilde{s}'-1)}\right)^{\frac{p-1}{p_2(\tilde{s}'-1)}} \left(\int_{A_{R/2}} b(x) g(u, v)\right)^{\frac{p-1}{p_2}-\frac{q_2(p-1)}{p_2(q-1)}}.
\end{aligned} \tag{5.5}$$

Similarly, for g, we obtain

$$\begin{aligned}
\int_{B_R} b(x) g(u, v) &\le c R^{-q} |A_{R/2}| (\operatorname*{ess\,inf}_{B_R} v)^{q-1} \\
&\le c R^{-q-p\frac{p_1(q-1)}{q_1(p-1)}} |A_{R/2}|^{1-\frac{(q-1)\tilde{r}}{q_1}+\frac{p_1(q-1)}{q_1(p-1)}} \left(\int_{A_{R/2}} a(x)^{-(\tilde{r}'-1)}\right)^{\frac{q-1}{q_1(\tilde{r}'-1)}} \left(\int_{A_{R/2}} a(x) f(u, v)\right)^{\frac{q-1}{q_1}-\frac{p_1(q-1)}{q_1(p-1)}}.
\end{aligned}$$

Hence, by (5.5), we obtain

$$\int_{B_R} b(x)g(u,v) \le cR^{N\left[1-\frac{(p-1-p_1)(q-1-q_2)}{p_2q_1}\right]-p\frac{q-1}{q_1}-q\left[1+\frac{q_2(p-1-p_1)}{p_2q_1}\right]} \cdot \Big(\fint_{A_{R/2}} a^{-(\tilde{r}'-1)}\Big)^{\frac{q-1}{q_1(\tilde{r}'-1)}} \Big(\fint_{A_{R/2}} b^{-(\tilde{s}'-1)}\Big)^{\frac{(p-1-p_1)(q-1)}{p_2q_1(\tilde{s}'-1)}} \Big(\int_{A_{R/2}} b(x)g(u,v)\Big)^{\frac{(p-1-p_1)(q-1-q_2)}{p_2q_1}}.$$

Therefore, by hypothesis (5.2b), it follows that $b(x)g(u,v) = 0$ for a.a. $x \in \mathbb{R}^N$, since

$$p_2q_1 > (p-1-p_1)(q-1-q_2).$$

Using (WH) on the first inequality of the system and by (5.5), we get, for R sufficiently large,

$$\begin{aligned}\Big(\frac{1}{|B_{2R}|}\int_{B_{2R}} u^\sigma\Big)^{\frac{p-1}{\sigma}} &\le c(\operatorname*{ess\,inf}_{B_R} u)^{p-1} \\ &\le cR^{-q\frac{q_2(p-1)}{p_2(q-1)}} |A_{R/2}|^{-\frac{(p-1)\tilde{s}}{p_2}+\frac{q_2(p-1)}{p_2(q-1)}} \Big(\int_{A_{R/2}} b(x)^{-(\tilde{s}'-1)}\Big)^{\frac{p-1}{p_2(\tilde{s}'-1)}} \Big(\int_{A_R/2} b(x)g(u,v)\Big)^{\frac{p-1}{p_2}-\frac{q_2(p-1)}{p_2(q-1)}} = 0.\end{aligned}$$

Hence, $u = 0$ a.e. in $\mathbb{R}^N$. □

Remark 10. It is worth pointing out that, in the case $a \equiv b \equiv 1$, requesting the validity of condition (5.2a) or (5.2b) of Theorem 8 is equivalent to hypothesis (4.11) of Theorem 3.

Furthermore, if a and b are nonnegative, periodic, continuous functions, then there exists $M > 0$ such that

$$\fint_{A_{R/2}} a(x)^{-r}\,dx < M, \qquad \fint_{A_{R/2}} b(x)^{-s}\,dx < M,$$

see [4, Theorem 3.23] for further details. Hence, in this setting, the condition (5.2a) or (5.2b) of Theorem 8 reduces to

$$N\Big[1-\frac{(p-1-p_1)(q-1-q_2)}{p_2q_1}\Big] \le \max\Big\{p+\frac{qq_1(p-1)+pp_1(q-1-q_2)}{p_2q_1},\; q+\frac{pp_2(q-1)+qq_2(p-1-p_1)}{p_2q_1}\Big\},$$

namely, we find again the same sufficient condition (4.11) of Theorem 3.

Corollary 4. *Let* $p_1 < p-1$, $q_2 < q-1$, $p_2q_1 > (p-1-p_1)(q-1-q_2)$, *and suppose that* $a(x) = |x|^\theta$ *and* $b(x) = |x|^\eta$. *Let* (u,v) *be a weak solution of* (5.1) *such that* $\operatorname{ess\,inf}_{\mathbb{R}^N} u = \operatorname{ess\,inf}_{\mathbb{R}^N} v = 0$. *Put*

$$D = p_2q_1 - (p-1-p_1)(q-1-q_2),$$

as usual. If the condition

$$\begin{aligned}N \le \frac{1}{D}\max\{&qq_1(p-1)+p[p_2q_1+p_1(q-1-q_2)]+\theta(p-1)(q-1-q_2)+\eta q_1(p-1), \\ &pp_2(q-1)+q[p_2q_1+q_2(p-1-p_1)]+\theta p_2(q-1)+\eta(q-1)(p-1-p_1)\}\end{aligned}$$

holds, then $u = 0$ *or* $v = 0$ *a.e. in* $\mathbb{R}^N$.

Proof. It is enough to apply Theorem 8 by taking into account that, for any $r > 0$ such that $N - \theta r > 0$ and $s > 0$ such that $N - \eta s > 0$, it follows that

$$\fint_{A_{R/2}} a(x)^{-r}\,dx = \frac{1}{|A_{R/2}|}\int_{A_{R/2}} |x|^{-\theta r}\,dx = cR^{-N}\frac{R^{N-\theta r}-(R/2)^{N-\theta r}}{N-\theta r} = cR^{-\theta r},$$

$$\fint_{A_{R/2}} b(x)^{-s}\,dx = \frac{1}{|A_{R/2}|}\int_{A_{R/2}} |x|^{-\eta s}\,dx = cR^{-N}\frac{R^{N-\eta s}-(R/2)^{N-\eta s}}{N-\eta s} = cR^{-\eta s},$$

where c denotes a positive constant. □

References

[1] M. F. Bidaut-Véron and S. I. Pohozaev, Nonexistence results and estimates for some nonlinear elliptic problems, *J. Anal. Math.* **84** (2001), 1–49.

[2] J. Bourgain, *Global Solutions of Nonlinear Schrödinger Equations*, Amer. Math. Soc. Colloq. Publ. 46, American Mathematical Society, Providence, 1999.

[3] P. Clément, J. Fleckinger, E. Mitidieri and F. de Thélin, Existence of positive solutions for a nonvariational quasilinear elliptic system, *J. Differential Equations* **166** (2000), 455–477.

[4] L. D'Ambrosio and E. Mitidieri, A priori estimates, positivity results, and nonexistence theorems for quasilinear degenerate elliptic inequalities, *Adv. Math.* **224** (2010), 967–1020.

[5] L. D'Ambrosio and E. Mitidieri, Entire solutions of quasilinear elliptic systems on Carnot groups, *Proc. Steklov Inst. Math.* **283** (2013), 3–19.

[6] E. Mitidieri, Nonexistence of positive solutions of semilinear elliptic systems in $\mathbb{R}^N$, *Differential Integral Equations* **9** (1996), 465–479.

[7] E. Mitidieri and S. I. Pohozaev, Nonexistence of positive solutions for a system of quasilinear elliptic equations and inequalities in $\mathbb{R}^N$, *Dokl. Math.* **59** (1999), 351–355.

[8] E. Mitidieri and S. I. Pohozaev, A priori estimates and the absence of solutions of nonlinear partial differential equations and inequalities, *Proc. Steklov Inst. Math.* **234** (2001), 1–362.

[9] P. Quittner, A priori estimates, existence and Liouville theorems for semilinear elliptic systems with power nonlinearities, *Nonlinear Anal.* **102** (2014), 144–158.

[10] J. Serrin, Local behavior of solutions of quasi–linear equations, *Acta Math.* **111** (1964), 247–302.

[11] N. S. Trudinger, On Harnack type inequalities and thier application to quasilinear elliptic equations, *Comm. Pure Appl. Math.* **20** (1967), 721–747.

[12] H. H. Zou, A priori estimates and existence for quasi-linear elliptic equations, *Calc. Var. Partial Differential Equations* **33** (2008), 417–437.