Gauge transformations for twisted spectral triples

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Abstract It is extended to twisted spectral triples the fluctuations of the metric as bounded perturbations of the Dirac operator that arises when a spectral triple is exported between Morita equivalent algebras, as well as gauge transformations which are obtained by the action of the unitary endomorphisms of the module implementing the Morita equivalence. It is firstly shown that the twisted-gauged Dirac operators, previously introduced to generate an extra scalar field in the spectral description of the standard model of elementary particles, in fact follow from Morita equivalence between twisted spectral triples. The law of transformation of the gauge potentials turns out to be twisted in a natural way. In contrast with the non-twisted case, twisted fluctuations do not necessarily preserve the self-adjointness of the Dirac operator. For a self-Morita equivalence, conditions are obtained in order to maintain self-adjointness that are solved explicitly for the minimal twist of a Riemannian manifold.

Keywords Noncommutative geometry · Twisted real spectral triples · Algebra authomorphisms · Connes standard model · Gauge transformations · Morita equivalence

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1 Introduction

The gauge bosons of the standard model of elementary particles are described by (quantum) fields that, from a mathematical view-point, are connections 1-forms for a bundle over a (four dimensional) spin manifold $\mathcal{M}$, with structure (gauge) group $U(1) \times SU(2) \times SU(3)$. Noncommutative geometry provides a framework to put the Higgs field on the same footing—that is as a connection 1-form—or more precisely as the component of a connection 1-form in the noncommutative (discrete) part of the geometry. For this to make sense, one needs a notion of connection extended beyond the usual manifold case, to the noncommutative setting.

In Connes approach [9], this is done starting with a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ where $\mathcal{A}$ is an involutive algebra acting by bounded operators on a Hilbert space $\mathcal{H}$, and the *Dirac operator* D is a densely defined self-adjoint operator on $\mathcal{H}$ with compact resolvent, such that the commutator[1]

$$\delta(a) := [D, a] \tag{1.1}$$

is bounded for any a in $\mathcal{A}$ (or in a dense subalgebra). The noncommutative analog of the module of sections of a vector or tensor bundle is a $\mathcal{A}$-module $\mathcal{E}$ with some properties. Gauge fields are given by an Ω-valued connection on $\mathcal{E}$, where Ω is a $\mathcal{A}$-bimodule of 1-forms. A natural choice of these, associated with the derivation (1.1), is the $\mathcal{A}$-bimodule

$$\Omega^1_D(\mathcal{A}) := \left\{ \sum\nolimits_j a_j [D, b_j],\ \ a_j, b_j \in \mathcal{A} \right\}. \tag{1.2}$$

[1] As usual, when there is no risk of confusion, we identify an element a of $\mathcal{A}$ with its representation $\pi(a)$ as a bounded operator on $\mathcal{H}$.

The simplest choice for $\mathcal{E}$ is the algebra $\mathcal{A}$ itself. A connection is then encoded fully in a self-adjoint element ω in $\Omega^1_D(\mathcal{A})$. The later acts on the Hilbert space $\mathcal{H}$, so that $D+\omega$ makes sense as an operator on $\mathcal{H}$. By taking into account more structure, in particular the real structure J, one refines the above definition and defines the *gauged Dirac operator* as[2]

$$D_\omega := D + \omega + \epsilon' J\omega J^{-1} \tag{1.3}$$

where $\epsilon' = \pm 1$ as dictated by the KO-dimension of the spectral triple. This is an operator on $\mathcal{H}$, that has all the properties required to make $(\mathcal{A}, \mathcal{H}, D_\omega)$ a spectral triple. The substitution of D by D_ω is a *fluctuation of the metric*, the latter 'associated' with the starting D.

When applied to the spectral triple of the standard model, these fluctuations generate the gauge fields of the electroweak and strong interactions, together with the Higgs field [4]. There is, however, a part D' of the corresponding Dirac operator which does not fluctuate, that is

$$[D', a] = 0 \text{ for any } a \in \mathcal{A}. \tag{1.4}$$

This point was not relevant until the recent discovery of the Higgs boson. The prediction for its mass coming from noncommutative geometry turned out *not* to be in agreement with the experimental result. As a way out, one turns the component of D' (which was taken to be a constant parameter $\nu \in \mathbb{C}$) into a field $\sigma \in C^\infty(\mathcal{M})$. Doing so, one introduces a new scalar field in the standard model that eliminates some instability in the Higgs potential, and provides a new parameter allowing one to fit the mass of the Higgs [3].

The substitution $\nu \to \sigma$ does not follow from an ordinary fluctuation of the metric. Nevertheless, it may be obtained in a similar manner if one relaxes one of the defining condition of a spectral triple,—the first-order condition. This proposal has been developed in [5,6], and the phenomenological consequences have been investigated in [7]. An alternative approach, following the "grand symmetry model" of [12], has allowed in [13] to generate the field σ within the framework of twisted spectral triples [11]: the field σ is obtained as a twisted version of a fluctuation of the metric, with a twisted first-order condition. A twisted fluctuation of the metric comes from substituting in the forms (1.2) the commutator $[D, a]$ with a twisted commutator $[D, a]_\rho := Da - \rho(a)D$, using an automorphism ρ of $\mathcal{A}$, resulting into a bimodule

$$\Omega^1_D(\mathcal{A}, \rho) := \left\{ \sum\nolimits_j a_j [D, b_j]_\rho,\ a_j, b_j \in \mathcal{A} \right\} \tag{1.5}$$

The *twisted-gauged Dirac operator* is then defined as

$$D_{\omega_\rho} := D + \omega_\rho + \epsilon' J\omega_\rho J^{-1} \tag{1.6}$$

where $\omega_\rho \in \Omega^1_D(\mathcal{A}, \rho)$ is a twisted 1-form such that the resulting operator (1.6) is self-adjoint.

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2 Usually, one denotes by A a self-adjoint element of $\Omega^1_D(\mathcal{A})$ considered as a gauge connection. Here, we use ω instead, in order to avoid a profusion of symbols "A".

Twisted spectral triples and twisted 1-forms were introduced in [11] to deal with type III factors. In [14] we extended the construction to encompass the real structure J, and showed that many properties of metric fluctuations still make sense in the twisted case. In particular:

- Given a twisted spectral triple $(\mathcal{A}, \mathcal{H}, D; \rho)$ and a twisted-gauged Dirac operator D_{ω_ρ}, the data $(\mathcal{A}, \mathcal{H}, D_{\omega_\rho}; \rho)$ is a real twisted spectral triple with the same real structure and KO-dimension;
- Twisted fluctuations form a monoid:[3] the twisted fluctuation $D_{\omega_\rho} + \omega'_\rho + \epsilon' J \omega'_\rho J^{-1}$ of D_{ω_ρ} is the twisted fluctuation $D + \omega''_\rho + \epsilon' J \omega''_\rho J^{-1}$ of D with $\omega''_\rho = \omega_\rho + \omega'_\rho$.

However, important aspects and consequences of fluctuating the metric are yet to be understood for the twisted case. In particular:

- Usual fluctuations appear as a particular case of a general construction of exporting a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ to a Morita equivalent algebra $\mathcal{B}$. The operator (1.3) is obtained as the covariant derivative on the bundle $\mathcal{E}$ that implements a Morita equivalence of $\mathcal{A}$ with itself. The twisted fluctuations in (1.6) mimic the expression for the non-twisted case, but their possible interpretation in terms of Morita equivalence has not been addressed.
- Is there an interpretation of the bimodule $\Omega^1_D(\mathcal{A}, \rho)$ as a module of connection 1-forms?
- What is a gauge transformation in the twisted context?

In this paper, we show that Morita equivalence is directly implemented for twisted spectral triples. The twisted-gauged Dirac operator D_{ω_ρ} is—up to an endomorphism—a covariant operator associated with a connection on the algebra $\mathcal{A}$ thought of as an $\mathcal{A}$-bimodule. This result is obtained in Sect. 3 by viewing $\mathcal{A}$ first as a right $\mathcal{A}$-module (Corollary 3.6), then as a left $\mathcal{A}$-module (Corollary 3.11), and finally as a bimodule, taking into account the real structure (Proposition 3.13). In Sect. 4 we deal with gauge transformations. These are implemented as in the non-twisted case by the action of some unitary endomorphism u, the only difference being that the law of transformation of gauge potential has to be twisted (Proposition 4.3). We also show in Proposition 4.5 that the twisted-gauged Dirac operator is obtained by the twisted adjoint action of the operator $\mathrm{Ad}(u)$. This raises the question of the self-adjointness of the gauged-twisted Dirac operator, which is investigated in Sect. 5. We work out in Proposition 5.2 some conditions on the unitary u guaranteeing that this self-adjointness is preserved. These conditions are solved for the case of minimal twist of a manifold (Proposition 5.4). Interestingly, we obtain other solutions than the obvious ones (that is the unitaries u invariant under the twist). Before that, we begin in Sect. 2 with some recalling of twisted spectral triples.

In [2], there is a modified definition of a real spectral triple, in which only the reality structure is generalized while remaining in the framework of usual spectral triples (that is no twisted commutators between the Dirac operator and algebra elements). It is shown there that this allows for fluctuations of Dirac operators, which do not change the bimodule of one forms.

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[3] There is a misprint in the statement of this property in [14, Prop. 2.7]: D_ρ in (2.30) there should be D.

2 Twisted real spectral triples

This section collects well-known material on and properties of real twisted spectral triples.

A twisted spectral triple is the datum $(\mathcal{A}, \mathcal{H}, D)$ of an involutive algebra $\mathcal{A}$ acting via a representation π on a Hilbert space $\mathcal{H}$, with D an operator on $\mathcal{H}$ having compact resolvent (or with a similar condition when $\mathcal{A}$ is not unital), together with an automorphism ρ of $\mathcal{A}$, such that the twisted commutator

$$[D, a]_\rho := Da - \rho(a)D \tag{2.1}$$

is bounded for any a in $\mathcal{A}$. It is graded if there is a grading Γ of $\mathcal{H}$, that is an operator such that $\Gamma = \Gamma^*$, $\Gamma^2 = \mathbb{1}$, that commutes with $\mathcal{A}$ and anticommutes with D.

The real structure is an antilinear operator J such that

$$J^2 = \epsilon\mathbb{1}, \quad JD = \epsilon' DJ, \quad J\Gamma = \epsilon''\Gamma J \tag{2.2}$$

where the sign $\epsilon, \epsilon', \epsilon'' \in \{1, -1\}$ define the so-called KO-dimension of the spectral triple. The operator J allows one to define a bijection between $\mathcal{A}$ and the opposite algebra $\mathcal{A}^\circ$,

$$a^\circ := Ja^*J^{-1}, \tag{2.3}$$

which is used to implement a right $\mathcal{A}$-module structure on $\mathcal{H}$

$$\psi a := a^\circ\psi \quad \forall\psi \in \mathcal{H}, \quad a \in \mathcal{A}. \tag{2.4}$$

This right action of $\mathcal{A}$ is asked to commute with the left action (the order-zero condition),

$$[a, Jb^*J^{-1}] = 0 \quad \forall a, \quad b \in \mathcal{A}, \tag{2.5}$$

thus turning $\mathcal{H}$ into a $\mathcal{A}$-bimodule. In addition, one requires a twisted first-order condition [14]:

$$[[D, a]_\rho, Jb^*J^{-1}]_{\rho_0} = 0 \quad \forall a, \quad b \in \mathcal{A} \tag{2.6}$$

where ρ° is the image of ρ under the isomorphism between $\mathrm{Aut}(\mathcal{A})$ and $\mathrm{Aut}(\mathcal{A}^\circ)$ given by

$$\rho \mapsto \rho^\circ \quad \text{with} \quad \rho^\circ(a^\circ) := (\rho^{-1}(a))^\circ. \tag{2.7}$$

This choice of isomorphism is dictated by the requirement made in [11] that the twisting automorphism, rather than being a $*$-automorphism, it satisfies the condition:

$$\rho(a^*) = (\rho^{-1}(a))^*. \tag{2.8}$$

Equation (2.7) thus guarantees that "the automorphism commutes with the real structure", since one has:

$$\rho^\circ(Jb^*J^{-1}) = \rho^\circ(b^\circ) = (\rho^{-1}(b))^\circ = J(\rho^{-1}(b))^*J^{-1} = J\rho(b^*)J^{-1}. \tag{2.9}$$

Definition 2.1 A twisted spectral triple $(\mathcal{A}, \mathcal{H}, D; \rho)$ together with a grading Γ, a real structure J satisfying (2.2) as well as the order-zero condition (2.5), and the twisted first-order condition (2.6) is called a *real twisted spectral triple*.

For ρ the identity automorphism, one gets back the usual notion of a real spectral triple.

The set of twisted 1-forms is the $\mathcal{A}$-bimodule $\Omega^1_D(\mathcal{A}, \rho)$ defined in (1.5) with product

$$a \cdot \omega_\rho \cdot b = \rho(a)\, \omega_\rho b \quad \forall a, \quad b \in \mathcal{A}, \quad \omega_\rho \in \Omega^1_D(\mathcal{A}\rho). \tag{2.10}$$

The left action of $\mathcal{A}$ is twisted by ρ to guarantee the twisted commutator

$$\delta_\rho(\,\cdot\,) := [D, \,\cdot\,]_\rho \tag{2.11}$$

be a derivation of $\mathcal{A}$ in $\Omega^1_D(\mathcal{A}, \rho)$, that is (cf. [11])

$$\delta_\rho(ab) = \rho(a) \cdot \delta_\rho(b) + \delta_\rho(a) \cdot b. \tag{2.12}$$

Thus, $\Omega^1_D(\mathcal{A}, \rho)$ is the $\mathcal{A}$-bimodule generated by δ_ρ; and it acts as bounded operator on $\mathcal{H}$, since so do both $\mathcal{A}$ and $[D, \mathcal{A}]_\rho$. It is worth stressing a difference between the right and left action of $\mathcal{A}$ on 1-forms when acting on $\mathcal{H}$. By the very definition in (2.10), one has

$$(\omega_\rho \cdot a)\psi = \omega_\rho a\psi = \omega_\rho(a\psi), \tag{2.13}$$

while

$$(a \cdot \omega_\rho)\psi = \rho(a)\omega_\rho\psi \neq a(\omega_\rho\psi). \tag{2.14}$$

3 Twisted fluctuation by Morita equivalence

In the non-twisted case, the fluctuations of the metric arise as a way to export a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ to an algebra $\mathcal{B}$ which is Morita equivalent to $\mathcal{A}$, in a way compatible with the real structure. An important role is played by a connection on a module that is moved to the Hilbert space (Sect. 3.1) thus resulting into a gauged Dirac operator (Sect. 3.2). This construction is extended to the twisted situation in Sects. 3.3–3.6. The main result is Proposition 3.13, which shows that the twisted-gauged Dirac operator (1.6) is obtained by Morita equivalence, in a way similar to the one for the usual gauged Dirac operator (1.3).

3.1 Moving connections to Hilbert spaces

We recall how an Ω-valued connection on a right (or left) $\mathcal{A}$-module $\mathcal{E}$ yields a map ∇ on $\mathcal{E} \otimes_{\mathbb{C}} \mathcal{H}$ (or $\mathcal{H} \otimes_{\mathbb{C}} \mathcal{E}$), when both the $\mathcal{A}$-bimodule Ω and the algebra $\mathcal{A}$ act on $\mathcal{H}$. This map does not pass to the tensor product $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}$ (or $\mathcal{H} \otimes_{\mathcal{A}} \mathcal{E}$). We get in Propositions 3.1 and 3.2 compatibility conditions between the actions of $\mathcal{A}$ and Ω

which guarantees that this lack of $\mathcal{A}$-linearity of ∇ is captured by the derivation δ that generates Ω.

A derivation of an algebra $\mathcal{A}$ with value in a $\mathcal{A}$-bimodule Ω is a map $\delta : \mathcal{A} \to \Omega$ such that

$$\delta(ab) = \delta(a) \cdot b + a \cdot \delta(b) \tag{3.1}$$

where $\cdot$ denotes the right and left $\mathcal{A}$-module structures of Ω. An Ω-valued connection on a right $\mathcal{A}$-module $\mathcal{E}$ is a map $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega$ satisfying the Leibniz rule

$$\nabla(\eta a) - \nabla(\eta) \cdot a = \eta \otimes \delta(a) \quad \forall \eta \in \mathcal{E}, \quad a \in \mathcal{A}, \tag{3.2}$$

where the right action of $\mathcal{A}$ on $\mathcal{E} \otimes_{\mathcal{A}} \Omega$ comes from the right module structure of Ω:

$$(\eta \otimes \omega) \cdot a := \eta \otimes (\omega \cdot a) \quad \forall \eta \in \mathcal{E}, \; \omega \in \Omega. \tag{3.3}$$

When both $\mathcal{A}$ and Ω acts (on the left) on a Hilbert space $\mathcal{H}$, we use the connection ∇ to define an operator (still denoted ∇) from $\mathcal{E} \otimes_{\mathbb{C}} \mathcal{H}$ to itself. To this end, it is useful to use a Sweedler-like notation: for any $\eta \in \mathcal{E}$ we write

$$\nabla(\eta) = \eta_{(0)} \otimes \eta_{(1)} \quad \eta_{(0)} \in \mathcal{E}, \quad \eta_{(1)} \in \Omega \tag{3.4}$$

where a summation is understood. By the action of Ω on $\mathcal{H}$, there is a natural map

$$\mathcal{E} \otimes_{\mathbb{C}} \Omega \times \mathcal{H} \to \mathcal{E} \otimes_{\mathbb{C}} \mathcal{H}, \quad (\eta \otimes \omega)\psi = \eta \otimes (\omega\psi), \tag{3.5}$$

that induces a map

$$\nabla : \mathcal{E} \otimes_{\mathbb{C}} \mathcal{H} \to \mathcal{E} \otimes_{\mathbb{C}} \mathcal{H} \tag{3.6}$$

defined by

$$\nabla(\eta \otimes \psi) := \left(\eta_{(0)} \otimes \eta_{(1)}\right) \psi = \eta_{(0)} \otimes (\eta_{(1)}\psi) \quad \forall \eta \in \mathcal{E}, \quad \psi \in \mathcal{H}. \tag{3.7}$$

Somewhat abusing notation, this is often denoted as $\nabla(\eta)\psi$.

This map cannot be extended to the tensor product $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}$ over $\mathcal{A}$ because there is no reason that $\nabla(\eta a)\psi - \nabla(\eta)a\psi$ vanishes. However, this incompatibility is captured by the derivation δ, providing the actions of Ω and $\mathcal{A}$ on $\mathcal{H}$ are compatible.

Proposition 3.1 *If the (left) actions of Ω and $\mathcal{A}$ on $\mathcal{H}$ are such that*

$$(\omega \cdot a)\,\psi = \omega(a\psi), \tag{3.8}$$

then the map ∇ in (3.7) satisfies the Leibniz rule

$$\nabla(\eta a)\psi - \nabla(\eta)a\psi = \eta \otimes \delta(a)\psi \quad \forall \eta \in \mathcal{E}, \quad a \in \mathcal{A}, \quad \psi \in \mathcal{H}. \tag{3.9}$$

Proof In Sweedler notations, the Leibniz rule (3.2) reads

$$(\eta a)_{(0)} \otimes (\eta a)_{(1)} - \eta_{(0)} \otimes (\eta_{(1)} \cdot a) = \eta \otimes \delta(a). \tag{3.10}$$

Hence, using condition (3.8) in the second equality

$$\begin{aligned} \nabla(\eta a)\psi - \nabla(\eta)a\psi &= (\eta a)_{(0)} \otimes (\eta a)_{(1)}\psi - \eta_{(0)} \otimes \eta_{(1)}(a\psi) \\ &= (\eta a)_{(0)} \otimes (\eta a)_{(1)}\psi - \eta_{(0)} \otimes (\eta_{(1)} \cdot a)\psi \\ &= \big((\eta a)_{(0)} \otimes (\eta a)_{(1)} - \eta_{(0)} \otimes (\eta_{(1)} \cdot a)\big)\,\psi \\ &= (\eta \otimes \delta(a))\,\psi. \end{aligned}$$

Equation (3.9) follows by (3.5). □

Similarly, an Ω-valued connection on a left $\mathcal{A}$-module $\mathcal{E}$ is a map $\nabla : \mathcal{E} \to \Omega \otimes_{\mathcal{A}} \mathcal{E}$ such that

$$\nabla(a\eta) - a \cdot \nabla(\eta) = \delta(a) \otimes \eta \quad \forall a \in \mathcal{A}, \quad \eta \in \mathcal{E}, \tag{3.11}$$

with left multiplication by $\mathcal{A}$ on $\Omega \otimes_{\mathcal{A}} \mathcal{E}$ coming from the left module structure of Ω,

$$a \cdot (\omega \otimes \eta) := (a \cdot \omega) \otimes \eta \quad \forall \eta \in \mathcal{E},\ \omega \in \Omega. \tag{3.12}$$

For any $\eta \in \mathcal{E}$, we shall now write with Sweedler-like notation

$$\nabla(\eta) = \eta_{(-1)} \otimes \eta_{(0)} \quad \eta_{(0)} \in \mathcal{E}, \quad \eta_{(-1)} \in \Omega. \tag{3.13}$$

When $\mathcal{A}$ acts (on the right) and Ω acts (on the left) on a Hilbert space $\mathcal{H}$, the connection on the left module defines a map similar to the one in (3.6) with minimal changes. The map

$$\mathcal{H} \times \Omega \otimes_{\mathbb{C}} \mathcal{E} \to \mathcal{H} \otimes_{\mathbb{C}} \mathcal{E}, \qquad (\psi)(\omega \otimes \eta) = (\omega\psi) \otimes \eta, \tag{3.14}$$

induces now a map $\nabla : \mathcal{H} \otimes_{\mathbb{C}} \mathcal{E} \to \mathcal{H} \otimes_{\mathbb{C}} \mathcal{E}$

$$\nabla(\psi \otimes \eta) := (\psi)(\eta_{(-1)} \otimes \eta_{(0)}) = (\eta_{(-1)}\psi) \otimes \eta_{(0)} \qquad \forall \eta \in \mathcal{E},\ \psi \in \mathcal{H}. \tag{3.15}$$

We denote this map as $\psi\nabla(\eta)$. Again, the obstruction to extend (3.15) to $\mathcal{H} \otimes_{\mathcal{A}} \mathcal{E}$ is captured by the derivation δ, if the actions of Ω and $\mathcal{A}$ are compatible.

Proposition 3.2 *If the left action of Ω and the right action of $\mathcal{A}$ on $\mathcal{H}$ are such that*

$$(a \cdot \omega)\psi = \omega(\psi a), \tag{3.16}$$

then the map ∇ in (3.15) satisfies the Leibniz rule

$$\psi\nabla(a\eta) - \psi a\nabla(\eta) = \delta(a)\psi \otimes \eta. \tag{3.17}$$

Proof In Sweedler notations, the left Leibniz rule (3.11) becomes

$$(a\eta)_{(-1)} \otimes (a\eta)_{(0)} - (a \cdot \eta_{(-1)}) \otimes \eta_{(0)} = \delta(a) \otimes \eta. \tag{3.18}$$

Using condition (3.16) in the second equality:

$$\begin{aligned}\psi\nabla(a\eta) - \psi a \nabla(\eta) &= (a\eta)_{(-1)}\psi \otimes (a\eta)_{(0)} - \eta_{(-1)}(\psi a) \otimes \eta_{(0)} \\ &= (a\eta)_{(-1)}\psi \otimes (a\eta)_{(0)} - (a \cdot \eta_{(-1)})\psi \otimes \eta_{(0)} \\ &= (\psi)\big((a\eta)_{(-1)} \otimes (a\eta)_{(0)} - (a \cdot \eta_{(-1)}) \otimes \eta_{(0)}\big) \\ &= (\psi)\big(\delta(a) \otimes \eta\big).\end{aligned}$$

Equation (3.17) follows by (3.14). □

3.2 The non-twisted case

For completeness, the details of the construction are reported in Sect. 1, while here we recall the important steps. Following [8], a fluctuation from D to the gauged operator D_ω given in (1.3) with $\omega \in \Omega^1_D(\mathcal{A})$, is seen as a two-step process: starting with a real spectral triple $(\mathcal{A}, \mathcal{H}, D)$, J one first implements a self-Morita equivalence of $\mathcal{A}$ using as module the algebra itself, viewed as a right $\mathcal{A}$-module $\mathcal{E}_R = \mathcal{A}$. This yields a new spectral triple $(\mathcal{A}, \mathcal{H}, D + \omega)$ with $\omega \in \Omega^1_D(\mathcal{A})$. However, this is not a real spectral triple. To correct this lacking, one repeats the operation using still the algebra as a module, but this time as a left $\mathcal{A}$-module $\mathcal{E}_L = \mathcal{A}$. The iteration yields the real spectral triple $(\mathcal{A}, \mathcal{H}, D_\omega = D + \omega + J\omega J^{-1})$.

Recall that at a first level, the algebra $\mathcal{B}$ is Morita equivalent to the unital algebra $\mathcal{A}$ if it is isomorphic to the algebra of $\mathcal{A}$-linear (adjointable) endomorphisms of a finite projective (right say) $\mathcal{A}$-module $\mathcal{E}_R$, that is $\mathcal{B} \simeq \mathrm{End}_{\mathcal{A}}(\mathcal{E}_R)$. Assuming $\mathcal{E}_R$ is a hermitian module, that is it carries an $\mathcal{A}$-hermitian structure, one use this structure to make the tensor product

$$\mathcal{H}_R = \mathcal{E}_R \otimes_{\mathcal{A}} \mathcal{H}$$

into a Hilbert space (with Hilbert product recalled in (A.2)), on which the algebra $\mathcal{B}$ acts on the left in a natural manner. The "simplest" action of D on $\mathcal{H}_R$, that is

$$D_R(\eta \otimes \psi) := \eta \otimes D\psi \quad \forall \eta \in \mathcal{E}_R, \psi \in \mathcal{H} \tag{3.19}$$

is not compatible with the tensor product of $\mathcal{A}$; it needs be corrected by a connection ∇ with value in $\Omega^1_D(\mathcal{A})$. The resulting covariant derivative, $D_R := D_R + \nabla$, is well defined on $\mathcal{H}_R$. With the notation (3.4) for the connection, this operator can be written as

$$D_R(\eta \otimes \psi) = \eta \otimes D\psi + \eta_{(0)} \otimes (\eta_{(1)}\psi) \quad \forall \eta \in \mathcal{E}, \quad \psi \in \mathcal{H}. \tag{3.20}$$

When ∇ is self-adjoint, the datum $(\mathcal{B}, \mathcal{H}_R, D_R)$ is a spectral triple [1]. It could be said to be '*Morita equivalent*' to the starting $(\mathcal{A}, \mathcal{H}, D)$. However, when $(\mathcal{A}, \mathcal{H}, D)$ is a real spectral triple, its real structure J is not a real structure for $(\mathcal{B}, \mathcal{H}_R, D_R)$. To cure that, one uses the right action (2.4) of $\mathcal{A}$ on $\mathcal{H}$ to fluctuate a second time, using a left module $\mathcal{E}_L$ endowed with an $\mathcal{A}$-hermitian structure. One considers the Hilbert space

$$\mathcal{H}_L := \mathcal{H} \otimes_{\mathcal{A}} \mathcal{E}_L$$

on which the simple operator,

$$D_L(\psi \otimes \eta) := D\psi \otimes \eta, \tag{3.21}$$

is now made compatible with the tensor product thanks to a (left) connection ∇°. The resulting covariant operator $D_L + \nabla^\circ$ is well defined on $\mathcal{H}_L$, with an expression similar to that in (3.20).

Combining the two constructions, one obtains an operator $D' = D + \nabla + \nabla^\circ$ on a Hilbert space $\mathcal{H}_{RL} = \mathcal{E}_R \otimes_{\mathcal{A}} \mathcal{H} \otimes_{\mathcal{A}} \mathcal{E}_L$. The real structure requires that $\nabla = \nabla^\circ$.

For a self-Morita equivalence of $\mathcal{A}$, that is $\mathcal{B} \simeq \mathcal{A}$, one gets that D' is the gauged operator D_ω defined in (1.3), for a self-adjoint element ω in $\Omega^1_D(\mathcal{A})$. Thus, the spectral triple $(\mathcal{A}, \mathcal{H}, D_\omega)$ obtained by fluctuation of the metric is self-Morita equivalent to the starting one $(\mathcal{A}, \mathcal{H}, D)$.

3.3 Lifting automorphisms

To adapt the construction above to the twisted case, one needs some action of D on $\mathcal{H}_R$ and $\mathcal{H}_L$ whose non-compatibility with the tensor product can be corrected by derivations with value in $\Omega^1_D(\mathcal{A}, \rho)$. Such operators are obtained in Propositions 3.5 and 3.9 below, by twisting the operators D_R and D_L of (3.19) and (3.21) with a lift of the automorphism ρ to the module.

Assumption 3.3 With a right $\mathcal{A}$-module $\mathcal{E}$ (resp. a left $\mathcal{A}$-module $\mathcal{E}$), the automorphism ρ can be lifted to $\mathcal{E}$ in the sense that there is an invertible linear map $\widetilde{\rho} : \mathcal{E} \to \mathcal{E}$ such that,

$$\widetilde{\rho}(\eta a) = \eta\, \rho(a) \quad \text{resp.} \quad \widetilde{\rho}(a\eta) = \rho(a)\, \eta \quad \forall \eta \in \mathcal{E}, \quad a \in \mathcal{A}. \tag{3.22}$$

Example 3.4 With a right $\mathcal{A}$-module $\mathcal{E}_R = p\mathcal{A}^N$ for a projection $p = (p_{jk}) \in \mathrm{Mat}_N(\mathcal{A})$, which is invariant for ρ, that is $\rho(p_{jk}) = p_{jk}$, one defines the action of $\rho \in \mathrm{Aut}(\mathcal{A})$ on $\mathcal{E}$ by

$$\widetilde{\rho}(\eta) := p \begin{pmatrix} \rho(\eta_1) \\ \vdots \\ \rho(\eta_N) \end{pmatrix} \quad \text{for} \quad \eta = p \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_N \end{pmatrix} \in \mathcal{E}_R, \quad \eta_j \in \mathcal{A}. \tag{3.23}$$

Similarly, the action of ρ on a left $\mathcal{A}$-module $\mathcal{E}_L = \mathcal{A}^N p$ with an invariant projection is given by

$$\widetilde{\rho}(\eta) := (\rho(\eta_1), \ldots, \rho(\eta_N))\, p \quad \text{for} \quad \eta = (\eta_1, \ldots, \eta_N)\, p \in \mathcal{E}_L, \quad \eta_j \in \mathcal{A}. \quad (3.24)$$

In particular, for the trivial module $\mathcal{E}_R = \mathcal{E}_L \simeq \mathcal{A}$ (that is $p = \mathbb{1}$) which is the case relevant for the self-Morita equivalence, then $\widetilde{\rho}$ is simply the automorphism ρ.

3.4 Morita equivalence by right module

We first investigate the implementation of Morita equivalence for a twisted spectral triple $(\mathcal{A}, \mathcal{H}, D; \rho)$ using a hermitian finite projective right $\mathcal{A}$-module $\mathcal{E}_R$ (definitions are in Sect. 1).

Consider the Hilbert space $\mathcal{H}_R = \mathcal{E}_R \otimes_{\mathcal{A}} \mathcal{H}$. As "natural action" of D on $\mathcal{H}_R$, one considers the composition of D_R in (3.19) with the endomorphism ρ of Assumption 3.3, that is,

$$((\widetilde{\rho} \otimes \mathbb{1}) \circ D_R)(\eta \otimes \psi) = \widetilde{\rho}(\eta) \otimes D\psi \quad \forall \eta \in \mathcal{E}_R, \quad \psi \in \mathcal{H}. \quad (3.25)$$

This is not compatible with the tensor product over $\mathcal{A}$ since

$$\begin{aligned} &((\widetilde{\rho} \otimes \mathbb{1}) \circ D_R)(\eta a \otimes \psi) - ((\widetilde{\rho} \otimes \mathbb{1}) \circ D_R)(\eta \otimes a\psi) \\ &\quad = \widetilde{\rho}(\eta a) \otimes D\psi - \widetilde{\rho}(\eta) \otimes Da\psi \\ &\quad = \widetilde{\rho}(\eta)\rho(a) \otimes D\psi - \widetilde{\rho}(\eta) \otimes Da\psi \\ &\quad = \widetilde{\rho}(\eta) \otimes \rho(a)D\psi - \widetilde{\rho}(\eta) \otimes Da\psi \\ &\quad = -\widetilde{\rho}(\eta) \otimes [D, a]_\rho \psi \end{aligned} \quad (3.26)$$

has no reason to vanish. The r.h.s. of (3.26) is—up to a twist—the action on $\mathcal{H}_R$ of the derivation (2.11). So to turn (3.25) into a well-defined operator on $\mathcal{H}_R$, one should proceed as in the non-twisted case and add the action of a connection.

Proposition 3.5 *Let ∇ be an $\Omega^1_D(\mathcal{A}, \rho)$-valued connection on $\mathcal{E}_R$. Then, the operator*

$$\widetilde{D}_R := (\widetilde{\rho} \otimes \mathbb{1}) \circ (D_R + \nabla) \quad (3.27)$$

is well defined on $\mathcal{H}_R$, with ∇ the map on $\mathcal{E}_R \otimes \mathcal{H}$ induced by the connection, as in (3.7).

Proof The module law (2.10) guarantees that $(\omega_\rho \cdot a)\psi = \omega_\rho(a\psi)$, so that by Proposition 3.1 the map ∇ satisfies the Leibniz rule

$$\nabla(\eta a)\psi - \nabla(\eta)a\psi = \nabla(\eta a \otimes \psi) - \nabla(\eta \otimes a\psi) = \eta \otimes \delta_\rho(a)\psi. \quad (3.28)$$

Therefore,

$$((\widetilde{\rho} \otimes \mathbb{1}) \circ \nabla)(\eta a \otimes \psi) - ((\widetilde{\rho} \otimes \mathbb{1}) \circ \nabla)(\eta \otimes a\psi) = \widetilde{\rho}(\eta) \otimes \delta_\rho(a)\psi. \quad (3.29)$$

Putting this together with (3.26), one obtains

$$\widetilde{D}_R(\eta a \otimes \psi) - \widetilde{D}_R(\eta \otimes a\psi) = 0, \quad \forall a \in \mathcal{A}, \quad \eta \in \mathcal{E}_R, \quad \psi \in \mathcal{H}. \tag{3.30}$$

Hence the result. □

The explicit form of $\widetilde{D}_R$, with the Sweedler-like notation of (3.7), is

$$\widetilde{D}_R(\eta \otimes \psi) := \widetilde{\rho}(\eta) \otimes D\psi + \widetilde{\rho}(\eta_{(0)}) \otimes (\eta_{(1)}\psi), \quad \forall \eta \in \mathcal{E}_R, \quad \psi \in \mathcal{H}. \tag{3.31}$$

For the case of a self-Morita equivalence, that is $\mathcal{B} = \mathcal{E}_R = \mathcal{A}$, this operator reduces to a bounded perturbation of D by elements in $\Omega^1_D(\mathcal{A}, \rho)$.

Corollary 3.6 *In case $\mathcal{E}_R$ is the algebra $\mathcal{A}$ itself, then $\widetilde{D}_R = D + \omega_\rho$, with $\omega_\rho \in \Omega^1_D(\mathcal{A}, \rho)$.*

Proof Clearly now $\widetilde{\rho} = \rho$. With $\delta_\rho(\,\cdot\,) := [D, \cdot]_\rho$, as for the non-twisted case recalled in Sect. 1, any connection ∇ on $\mathcal{E}_R = \mathcal{A}$ decomposes as

$$\nabla = \nabla_0 + \boldsymbol{\omega}_\rho \quad \text{where} \quad \begin{cases} \nabla_0(a) = \mathbb{1} \otimes \delta_\rho(a) & \text{is the Grassmann connection,} \\ \boldsymbol{\omega}_\rho(a) = \mathbb{1} \otimes \omega_\rho\, a & \text{with } \omega_\rho \in \Omega^1_D(\mathcal{A}, \rho). \end{cases} \tag{3.32}$$

Hence

$$\begin{aligned} \widetilde{D}_R(a \otimes \psi) &:= (\rho \otimes \mathbb{1})\big(a \otimes D\psi + \mathbb{1} \otimes \delta_\rho(a)\psi + \mathbb{1} \otimes \omega_\rho a\psi\big), \\ &= \rho(a) \otimes D\psi + \mathbb{1} \otimes \delta_\rho(a)\psi + \mathbb{1} \otimes \omega_\rho a\psi \\ &= \mathbb{1} \otimes (D + \omega_\rho)a\psi. \end{aligned} \tag{3.33}$$

Identifying $a \otimes \psi = \mathbb{1} \otimes a\psi$ with $a\psi$ and $\mathbb{1} \otimes (D + \omega_\rho)a\psi$ with $(D + \omega_\rho)a\psi$, one gets that $\widetilde{D}_R$ acts on $\mathcal{H} \simeq \mathcal{A} \otimes_{\mathcal{A}} \mathcal{H}$ as $D + \omega_\rho$. □

The operator $D + \omega_\rho$ has a compact resolvent, being a bounded perturbation of an operator with compact resolvent; and $[D + \omega_\rho, a]_\rho = [D, a]_\rho + [\omega_\rho, a]_\rho$ is bounded for any $a \in \mathcal{A}$, since ω_ρ is bounded. Furthermore, any grading Γ of $(\mathcal{A}, \mathcal{H}, D)$ will anticommute with ω_ρ, hence with $D + \omega_\rho$. Thus, as soon as ω_ρ is self-adjoint, one gets a twisted spectral triple

$$(\mathcal{A}, \mathcal{H}, D + \omega_\rho; \rho). \tag{3.34}$$

However, and as it happens for the non-twisted case, a priori a real structure J of $(\mathcal{A}, \mathcal{H}, D; \rho)$ needs not be a real structure for (3.34). Indeed, $J(D + \omega_\rho) = \epsilon'(D + \omega_\rho)J$ if and only if $\omega_\rho = J\omega_\rho J^{-1}$ which has no reason to be true due to the following lemma.

Lemma 3.7 *Let $(\mathcal{A}, \mathcal{H}, D; \rho)$ together with J be a real twisted spectral triple. With*

$$\omega_\rho = \sum\nolimits_j a_j\, [D, b_j]_\rho \in \Omega^1_D(\mathcal{A}, \rho), \tag{3.35}$$

one has

$$J\omega_\rho J^{-1} = \epsilon' \sum\nolimits_j (a_j^*)^\circ [D, (b_j^*)^\circ]_{\rho^\circ}. \tag{3.36}$$

Proof Without loss of generality, we may take $\omega_\rho = a[D, b]_\rho$. Then

$$\begin{aligned} J\omega_\rho J^{-1} = Ja[D, b]_\rho J^{-1} = JaJ^{-1}J[D, b]_\rho J^{-1} &= (a^*)^\circ J[D, b]_\rho J^{-1} \\ &= (a^*)^\circ (JDbJ^{-1} - J\rho(b)DJ^{-1}) \\ &= \epsilon'(a^*)^\circ (DJbJ^{-1} - J\rho(b)J^{-1}D) \\ &= \epsilon'(a^*)^\circ (DJbJ^{-1} - \rho^\circ(JbJ^{-1})D \\ &= \epsilon'(a^*)^\circ [D, (b^*)^\circ]_{\rho^\circ}, \end{aligned} \tag{3.37}$$

where we used (2.9) in the fourth line. □

To implement the self-Morita equivalence of $\mathcal{A}$ in a way which is compatible with the real structure, one proceeds as in the non-twisted case, and fluctuates the triple (3.34) using also a left-module structure thus considering altogether an $\mathcal{A}$-bimodule $\mathcal{E}$.

3.5 Morita equivalence by left module

Let $(\mathcal{A}, \mathcal{H}, D; \rho)$, J be a real twisted spectral triple. Given a left $\mathcal{A}$-module $\mathcal{E}_L$, the right $\mathcal{A}$-module structure (2.4) of $\mathcal{H}$ allows one to define the Hilbert space $\mathcal{H}_L = \mathcal{H} \otimes_{\mathcal{A}} \mathcal{E}_L$ with Hilbert product recalled in (A.17). As an action of D on $\mathcal{H}$, we consider the twist of the action (3.21) by the endomorphism $\widetilde{\rho}^{-1}$, following Assumption 3.3:

$$(\mathbb{1} \otimes \widetilde{\rho}^{-1}) \circ D_L : \mathcal{H}_L \to \mathcal{H}_L, \quad \psi \otimes \eta \to D\psi \otimes \widetilde{\rho}^{-1}(\eta). \tag{3.38}$$

As before, this is not compatible with the tensor product since

$$\begin{aligned} &\big((\mathbb{1} \otimes \widetilde{\rho}^{-1}) \circ D_L\big)(\psi \otimes a\eta) - (\mathbb{1} \otimes \widetilde{\rho}^{-1}) \circ D_L(\psi a \otimes \eta) \\ &\quad = D\psi \otimes \widetilde{\rho}^{-1}(a\eta) - D(\psi a) \otimes \widetilde{\rho}^{-1}(\eta) \\ &\quad = (D\psi)\rho^{-1}(a) \otimes \widetilde{\rho}^{-1}(\eta) - Da^\circ \psi \otimes \widetilde{\rho}^{-1}(\eta) \\ &\quad = (\rho^{-1}(a))^\circ D\psi \otimes \widetilde{\rho}^{-1}(\eta) - Da^\circ \psi \otimes \widetilde{\rho}^{-1}(\eta) \\ &\quad = -([D, a^\circ]_{\rho^\circ} \psi) \otimes \widetilde{\rho}^{-1}(\eta), \end{aligned} \tag{3.39}$$

where in the last line we used (2.7). Again, equation (3.39) has no reason to vanish. In order to correct it via a connection, one needs to check that $[D, a^\circ]_{\rho^\circ}$ is actually a derivation.

Lemma 3.8 *The twisted commutator*

$$\delta_\rho^\circ(a) := [D, a^\circ]_{\rho^\circ} \tag{3.40}$$

is a derivation of $\mathcal{A}$ in the $\mathcal{A}$-bimodule

$$\Omega^1_D(\mathcal{A}^\circ, \rho^\circ) := \left\{ \sum\nolimits_j a_j^\circ [D, b_j^\circ]_{\rho^\circ}, \quad a_j^\circ, b_j^\circ \in \mathcal{A}^\circ \right\}, \tag{3.41}$$

with product law

$$a \cdot \omega_\rho^\circ \cdot b := \rho^\circ(b^\circ)\, \omega_\rho^\circ\, a^\circ \quad \forall a, b \in \mathcal{A}, \quad \omega_\rho^\circ \in \Omega^1_D(\mathcal{A}^\circ, \rho). \tag{3.42}$$

Proof By explicit computation of the twisted commutator, one has

$$\delta_\rho^\circ(ab) = [D, b^\circ a^\circ]_{\rho^\circ} = \rho^\circ(b^\circ)[D, a^\circ]_{\rho^\circ} + [D, b^\circ]_{\rho^\circ} a^\circ = \delta_\rho^\circ(a^\circ) \cdot b + a \cdot \delta_\rho^\circ(b). \tag{3.43}$$

To check that (3.41) is a $\mathcal{A}$-bimodule, first notice that by construction it is stable under the left multiplication by $\mathcal{A}^\circ$, hence under the right multiplication by $\mathcal{A}$ defined by (3.42). In addition,

$$\omega^\circ \cdot (ab) = \rho^\circ((ab)^\circ)\, \omega^\circ = \rho^\circ(b^\circ)\rho^\circ(a^\circ)\omega^\circ = (\omega^\circ \cdot a) \cdot b, \tag{3.44}$$

showing that $\Omega^1_D(\mathcal{A}^\circ, \rho^\circ)$ is a right $\mathcal{A}$-module. Stability for the left multiplication by $\mathcal{A}$ follows from (3.43):

$$a \cdot [D, b^\circ]_{\rho^\circ} = [D, b^\circ]_{\rho^\circ} a^\circ = [D, (ab)^\circ]_{\rho^\circ} - [D, a]_\rho^\circ \cdot b. \tag{3.45}$$

The left $\mathcal{A}$-module structure is obtained checking that

$$(ab) \cdot \omega^\circ = w^\circ (ab)^\circ = w^\circ b^\circ a^\circ = (b \cdot \omega^\circ) a^\circ = a \cdot (b \cdot \omega^\circ)). \tag{3.46}$$

Finally, the bimodule structure follows from

$$(a \cdot \omega^\circ) \cdot b = (\omega_\rho^\circ a^\circ) \cdot b = \rho^\circ(b^\circ)\, \omega_\rho^\circ\, a^\circ = (\omega_\rho^\circ \cdot b) a^\circ = a \cdot (\omega_\rho^\circ \cdot b). \tag{3.47}$$

This finishes the proof. □

Therefore, the r.h.s. of (3.39) is—up to a twist—the action on $\mathcal{H}_L$ of the derivation δ_ρ°. And once again, in order to define a linear operator on $\mathcal{H}_L$ using D, one needs to correct the action (3.38) with a connection, this time with value in $\Omega^1_D(\mathcal{A}^\circ, \rho^\circ)$.

Proposition 3.9 *Let ∇° be an $\Omega^1_D(\mathcal{A}^\circ, \rho^\circ)$-valued connection on the module $\mathcal{E}_L$. Then the following operator is well defined on $\mathcal{H}_L$,*

$$\widetilde{D}_L := (\mathbb{1} \otimes \widetilde{\rho}^{-1}) \circ (D_L + \nabla^\circ), \tag{3.48}$$

where ∇° denotes the map induced on $\mathcal{H} \otimes_{\mathbb{C}} \mathcal{E}_L$ by the connection, as in (3.15).

Proof By (3.42), the actions of $\Omega^1_D(\mathcal{A}^\circ, \rho)$ and $\mathcal{A}$ on $\mathcal{H}_L$ are compatible as in (3.16), that is,

$$(a \cdot \omega^\circ_\rho)\psi = \omega^\circ_\rho a^\circ \psi = \omega^\circ_\rho(\psi a). \tag{3.49}$$

Hence by Proposition 3.2 the connection ∇° satisfies the Leibniz rule

$$\nabla^\circ(\psi \otimes a\eta) - \nabla^\circ(\psi a \otimes \eta) = \delta^\circ_\rho(a)\psi \otimes \eta. \tag{3.50}$$

Therefore

$$\left((\mathbb{1} \otimes \widetilde{\rho}^{-1}) \circ \nabla^\circ\right)(\psi \otimes a\eta) - \left((\mathbb{1} \otimes \widetilde{\rho}^{-1}) \circ \widetilde{\nabla}\right)(\psi a \otimes \eta) = \delta^\circ_\rho(a)\psi \otimes \rho^{-1}(\eta). \tag{3.51}$$

Together with (3.39) this yields $\widetilde{D}_R(\psi \otimes a\eta) - \widetilde{D}_R(\psi a \otimes \eta) = 0$, hence the result. □

With the Sweedler-like notations of (3.15), the explicit form of $\widetilde{D}_L$ is

$$\widetilde{D}_L(\psi \otimes \eta) := D\psi \otimes \widetilde{\rho}^{-1}(\eta) + (\eta_{(-1)}\psi) \otimes \widetilde{\rho}^{-1}(\eta_{(0)}). \tag{3.52}$$

To get the more friendly $\widetilde{D}_L$ for a self-Morita equivalence, one needs a relation between $\Omega^1_D(\mathcal{A}, \rho)$ and $\Omega^1_D(\mathcal{A}^\circ, \rho^\circ)$ similar to the one between $\Omega^1_D(\mathcal{A})$ and $\Omega^1_D(\mathcal{A}^\circ)$ given in Lemma A.3.

Lemma 3.10 *Any $\omega^\circ_\rho = \sum_j a^\circ_j[D, b^\circ_j]_{\rho^\circ}$ in $\Omega^1_D(\mathcal{A}^\circ, \rho^\circ)$ acts on $\mathcal{H}$ as*

$$\omega^\circ_\rho = \epsilon' J\omega_\rho J^{-1} \tag{3.53}$$

for $\omega_\rho = \sum_j a^*_j[D, b^*_j]_\rho \in \Omega^1_D(\mathcal{A}, \rho)$.

Proof Without loss of generality, we may take $\omega^\circ_\rho = a^\circ[D, b^\circ]_{\rho^\circ}$. Using equation (2.9) one gets

$$\begin{aligned} a^\circ[D, b^\circ]_{\rho^\circ} &= a^\circ Db^\circ - a^\circ\rho^\circ(b^\circ)D = Ja^*J^{-1}DJb^*J^{-1} - Ja^*J^{-1}J\rho(b^*)J^{-1}D, \\ &= \epsilon'\left(Ja^*Db^*J^{-1} - Ja^*\rho(b^*)DJ^{-1}\right) = J\omega_\rho J^{-1} \end{aligned} \tag{3.54}$$

where $\omega_\rho := a^*[D, b^*]_\rho \in \Omega^1_D(\mathcal{A})$. □

In case of a self-Morita equivalence $\mathcal{B} = \mathcal{E}_L = \mathcal{A}$, then $\widetilde{D}_L$ is just a bounded perturbation of D by $\Omega^1_D(\mathcal{A}^\circ, \rho)$, similarly to the right module case of Corollary 3.6.

Corollary 3.11 *In case $\mathcal{E}_L$ is the algebra itself, then*

$$\widetilde{D}_L = D + \epsilon' J\omega_\rho J^{-1} \quad \textit{with} \quad \omega_\rho \in \Omega^1_D(\mathcal{A}, \rho). \tag{3.55}$$

Proof Any $\Omega^1_D(\mathcal{A}^\circ, \rho)$-valued connection $\widetilde{\nabla}^\circ$ on $\mathcal{E}_L = \mathcal{A}$ decomposes as

$$\nabla^\circ = \nabla^\circ_0 + \boldsymbol{\omega}^\circ_\rho \quad \text{where} \quad \begin{cases} \nabla^\circ_0(a) = \delta^\circ_\rho(a) \otimes \mathbb{1} & \text{is the Grassmann connection,} \\ \boldsymbol{\omega}^\circ_\rho(a) = (\omega^\circ_\rho a^\circ) \otimes \mathbb{1} & \text{where } \omega^\circ_\rho \in \Omega^1_D(\mathcal{A}^\circ, \rho). \end{cases} \tag{3.56}$$

Hence

$$\begin{aligned} \widetilde{D}_L(\psi \otimes a) &= D\psi \otimes \rho^{-1}(a) + \delta^\circ_\rho(a)\psi \otimes \mathbb{1} + \omega^\circ_\rho a^\circ \psi \otimes \mathbb{1}, \\ &= (D\psi)\rho^{-1}(a) \otimes \mathbb{1} + (Da^\circ - \rho^\circ(a^\circ)D)\psi \otimes \mathbb{1} + \omega^\circ_\rho a^\circ \psi \otimes \mathbb{1}, \\ &= (\rho^{-1}(a))^\circ D\psi \otimes \mathbb{1} + (Da^\circ - \rho^\circ(a^\circ)D)\psi \otimes \mathbb{1} + \omega^\circ_\rho a^\circ \psi \otimes \mathbb{1} \\ &= Da^\circ \psi \otimes \mathbb{1} + \omega^\circ_\rho a^\circ \psi \otimes \mathbb{1}, \end{aligned} \tag{3.57}$$

where in the last line we used (2.7). By identifying $\mathcal{H} \otimes_{\mathcal{A}} \mathcal{A} \simeq \mathcal{H}$, that is $\psi \otimes a = a^\circ \psi \otimes \mathbb{1}$ with ψ and similarly for $(Da^\circ\psi) \otimes \mathbb{1}$ and $(\omega^\circ_\rho a^\circ \psi) \otimes \mathbb{1}$, one gets that $\widetilde{D}_L$ is the operator $D + \omega^\circ_\rho$. The results follows from Lemma 3.10, which states that ω°_ρ acts as $\epsilon' J \omega_\rho J^{-1}$ for some $\omega_\rho \in \Omega^1_D(\mathcal{A}, \rho)$. □

For reasons similar to those of the right module case, and explained below Corollary 3.6, for a self-adjoint ω_ρ one has that the triple $(\mathcal{A}, \mathcal{H}, D + \epsilon' J\omega_\rho J^{-1})$ is a (graded) twisted spectral triple failing to admit J as a real structure, thus the need of a bimodule.

Remark 3.12 In (3.38), we have used ρ^{-1} rather than ρ, so that the failure of linearity is captured by δ°_ρ. Twisting by ρ, one would arrive at $\delta^\circ_{\rho^{-1}}$. Alternatively one may require that ρ is a $*$-automorphism: equation (2.8) then implies $\rho^{-1} = \rho$.

3.6 Bimodule and the real structure

To make the real structure compatible with Morita equivalence of twisted spectral triples, one combines the two constructions above in a way similar to the non-twisted case. Firstly, fluctuate the real twisted spectral triple $(\mathcal{A}, \mathcal{H}, D; \rho)$, J using the right module $\mathcal{E}_R = \mathcal{A}$, then fluctuate the resulting triple (3.34) via the left module $\mathcal{E}_L = \mathcal{A}$. This yields the triple $(\mathcal{A}, \mathcal{H}, D')$ where

$$D' = D + \omega^L_\rho + \epsilon' J \omega^R_\rho J^{-1} \tag{3.58}$$

with ω^R_ρ and ω^L_ρ two elements of $\Omega^1_D(\mathcal{A}, \rho)$ that are a priori distinct.

Proposition 3.13 *It holds that $D'J = \epsilon' D'J$ if and only if there exists an element ω_ρ in $\Omega^1_D(\mathcal{A}, \rho)$ such that*

$$D' = D + \omega_\rho + \epsilon' J \omega_\rho J^{-1}. \tag{3.59}$$

Proof From (3.58), one finds that $JD' = \epsilon' D'J$ if and only if

$$(\omega^R_\rho - \omega^L_\rho) - \epsilon' J(\omega^R_\rho - \omega^L_\rho)J^{-1} = 0. \tag{3.60}$$

Adding half of this expression to the r.h.s. of (3.58), one gets

$$D' = D + \tfrac{1}{2}(\omega_\rho^R + \omega_\rho^L) + \epsilon' J \tfrac{1}{2}(\omega_\rho^R + \omega_\rho^L) J^{-1}. \tag{3.61}$$

Hence the result with $\omega_\rho := \frac{1}{2}(\omega_\rho^R + \omega_\rho^L)$. $\square$

Proposition 3.13 shows that Morita equivalence together with the real structure yields the twisted fluctuation (1.6). This answers the first question raised in the introduction, and puts the twisted-gauged Dirac operator $D' = D_{\omega_\rho}$ on the same footing as the covariant operator D_ω, namely as a covariant derivative associated with a connection. The only difference is that, in the twisted case, the action $D_{R,L}$ of the Dirac operator on $\mathcal{H}_{R,L}$ and the action of the $\Omega^1_D(\mathcal{A}, \rho)$-valued connection have to be twisted by $(\mathbb{1} \otimes \widetilde{\rho})$ and $(\mathbb{1} \otimes \widetilde{\rho}^{-1})$.

Remark 3.14 It is worth stressing that fluctuations by Morita equivalence translate to the twisted case because the conditions (3.8) and (3.16), that allow one to pass the Leibniz rule from the connection, as a map on $\mathcal{E}$, to the connection as a map on $\mathcal{E}_R \otimes \mathcal{H}$ or $\mathcal{H} \otimes \mathcal{E}_L$, are still valid in the twisted case, that is it holds that

$$(\omega_\rho \cdot a)\psi = \omega_\rho a \psi = \omega_\rho(a\psi), \quad \psi(a \cdot \omega_\rho^\circ) = \omega_\rho^\circ a^\circ \psi = (\psi a)\omega_\rho^\circ. \tag{3.62}$$

Remark 3.15 By choosing the Grassmann connection, that is $\omega = 0$ in Corollaries 3.6 and 3.11, one gets $\widetilde{D}_L = \widetilde{D}_R = D$, so that $D' = D$ in (3.58). In other terms—and as in the non-twisted case (see Remark A.6)—implementing the self-Morita equivalence of $\mathcal{A}$ in a twisted spectral triple with the Grassmann connection yields no fluctuation $D_{\omega_\rho} = D$.

4 Twisted gauge transformation

A *gauge transformation* on a module $\mathcal{E}$ is the action of a unitary endomorphism u of $\mathcal{E}$ on a Ω-valued connection ∇ on the module (see Sect. A.2 for details),

$$\nabla \to \nabla^u := u \nabla u^* \quad u \in \mathcal{U}(\mathcal{E}). \tag{4.1}$$

Given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, with $\mathcal{E} = \mathcal{A}$ and $\Omega = \Omega^1_D(\mathcal{A})$, a gauge transformation by $u = u^*$ a unitary element of $\mathcal{A}$, amounts to substituting the gauged Dirac operator $D + \omega + J\,\omega\,J^{-1}$ with $D + \omega^u + J\,\omega^u\,J^{-1}$ where

$$\omega^u = u[D, u^*] + u\,\omega\,u^*. \tag{4.2}$$

(see (A.52)). This transformation maps a self-adjoint $\omega \in \Omega^1_D(\mathcal{A})$ to a self-adjoint $\omega^u \in \Omega^1_D(\mathcal{A})$, and gives the usual transformation rule of the gauge potential when applied to almost commutative geometry (that is the product of a manifold by a finite dimensional spectral triple).

It is clear that (4.2) cannot be valid in the twisted case, when one considers a connection with value in the bimodule of twisted 1-forms. Indeed, given $\omega_\rho \in \Omega^1_D(\mathcal{A}, \rho)$, there is no reason for $u[D, u^*] + u\,\omega_\rho\, u^*$ to be in $\Omega^1_D(\mathcal{A}, \rho)$, because $[D, u^*]$ has no reason to be in $\Omega^1_D(\mathcal{A}, \rho)$ (let alone to be a bounded operator). We show in Sect. 4.1 that a gauge transformation (4.1) in fact substitutes ω_ρ in the twisted-gauged Dirac operator $D_{\omega_\rho} = D + \omega_\rho + \epsilon' J \omega_\rho J^{-1}$ with

$$\omega^u_\rho := \rho(u)[D, u^*]_\rho + \rho(u)\omega_\rho u^*. \tag{4.3}$$

Furthermore, we show in Sect. 4.2 that a gauge transformation is equivalent to the twisted conjugate action on the Dirac operator of the adjoint representation (A.55) of the unitaries of $\mathcal{A}$, that is,

$$D_{\omega^u_\rho} = \rho(U) D_{\omega_\rho} U^* \quad \text{for } \; U = \mathrm{Ad}(u), \quad u \in \mathcal{U}(\mathcal{A}). \tag{4.4}$$

4.1 Transformation of the gauge potential

In all this section, $(\mathcal{A}, \mathcal{H}, D; \rho)$, J is a real twisted spectral triple, $\mathcal{E}$ a hermitian $\mathcal{A}$-module and $\mathcal{U}(\mathcal{E})$ its group of unitary endomorphisms.

Lemma 4.1 *Let ∇ be a $\Omega^1_D(\mathcal{A}, \rho)$-valued connection on $\mathcal{E}$. Then, for any $u \in \mathcal{U}(\mathcal{E})$, one has*

$$(\widetilde{\rho} \otimes \mathbb{1})\, \nabla^u = (\widetilde{\rho} \circ u) \nabla u^* \quad \textit{for a right module}, \tag{4.5}$$
$$(\mathbb{1} \otimes \widetilde{\rho})\, \nabla^u = (\widetilde{\rho} \circ u) \nabla u^* \quad \textit{for a left module}, \tag{4.6}$$

with ∇^u the gauge transformation (4.1) and $\widetilde{\rho}$ the endomorphism of $\mathcal{E}$ in the Assumption 3.3. In particular, taking for $\mathcal{E}$ the algebra itself, one gets

$$(\rho \otimes \mathbb{1})\nabla^u(a) = \rho(u) \otimes \delta_\rho(u^* a) + \rho(u) \otimes \omega_\rho u^* a \qquad \textit{for } \mathcal{E} = \mathcal{E}_R = \mathcal{A} \textit{ as right module},$$
$$(\mathbb{1} \otimes \rho)\nabla^u(a) = \delta^\circ_\rho(au) \otimes \rho(u^*) + \omega^\circ_\rho(au)^\circ \otimes \rho(u^*) \qquad \textit{for } \mathcal{E} = \mathcal{E}_L = \mathcal{A} \textit{ as left module},$$

where now u is a unitary element of $\mathcal{A}$, while $\omega_\rho \in \Omega^1_D(\mathcal{A}, \rho)$ and $\omega^\circ_\rho \in \Omega^1_D(\mathcal{A}^\circ, \rho^\circ)$ are the 1-forms associated with ∇ as defined in Corollaries 3.6 and 3.11.

Proof Assume $\mathcal{E}$ is a right $\mathcal{A}$-module. For any $\eta \in \mathcal{E}$ and $u \in \mathcal{U}(\mathcal{E})$, write $\nabla(u^*(\eta)) = \eta^u_{(0)} \otimes \eta^u_{(1)}$ with $\eta^u_{(0)} \in \mathcal{E}$ and $\eta^u_{(1)} \in \Omega^1_D(\mathcal{A}, \rho)$ (with an implicit sum). By (A.40), one gets on the one hand

$$\big((\widetilde{\rho} \otimes \mathbb{1})(u \nabla u^*)\big)(\eta) = (\widetilde{\rho} \otimes \mathbb{1}) \left(u(\eta^u_{(0)}) \otimes \eta^u_{(1)}\right) = \widetilde{\rho}(u(\eta^u_{(0)})) \otimes \eta^u_{(1)}, \tag{4.7}$$

while on the other hand

$$\big((\widetilde{\rho} \circ u) \nabla u^*\big)(\eta) = (\widetilde{\rho} \circ u)(\eta^u_{(0)} \otimes \eta^u_{(1)}) = \left(\widetilde{\rho} \circ u(\eta^u_{(0)})\right) \otimes \eta^u_{(1)} = \widetilde{\rho}(u(\eta^u_{(0)})) \otimes \eta^u_{(1)}. \tag{4.8}$$

Hence (4.5). The proof is similar for a left $\mathcal{A}$-module.

For the second part of the lemma, for any $a \in \mathcal{E}_R \simeq \mathcal{A}$ with $\nabla = \nabla_0 + \boldsymbol{\omega}_\rho$, by (A.47) and (3.32) one writes the r.h.s. of (4.5) as

$$\begin{aligned}
\left((\rho \circ u)\nabla u^*\right)(a) = ((\rho \circ u)\nabla)(u^* a) &= (\rho \circ u)\left(\nabla_0(u^* a) + \boldsymbol{\omega}_\rho(u^* a)\right), \\
&= (\rho \circ u)(\mathbb{1} \otimes \delta_\rho(u^* a) + \mathbb{1} \otimes \omega_\rho u^* a) \\
&= \rho(u) \otimes \delta_\rho(u^* a) + \rho(u) \otimes \omega_\rho u^* a.
\end{aligned} \tag{4.9}$$

Similarly, for $a \in \mathcal{E}_L \simeq \mathcal{A}$ with $\nabla = \nabla_0^\circ + \boldsymbol{\omega}_0^\circ$, by (A.47) and (3.56), the r.h.s. of (4.6) reads

$$\begin{aligned}
\left((\rho \circ u)\nabla u^*\right)(a) = ((\rho \circ u)\nabla)(au) &= (\rho \circ u)\left(\nabla_0^\circ(au) + \boldsymbol{\omega}_\rho^\circ(au)\right), \\
&= (\rho \circ u)(\delta_\rho^\circ(au) \otimes \mathbb{1} + \omega_\rho^\circ(au)^\circ \otimes \mathbb{1}) \\
&= \delta_\rho^\circ(au) \otimes \rho(u^*) + \omega_\rho^\circ(au)^\circ \otimes \rho(u^*).
\end{aligned} \tag{4.10}$$

Hence the result. □

A gauge transformation (4.1) amounts to substituting $(\widetilde{\rho} \otimes \mathbb{1}) \circ \nabla$ with $(\widetilde{\rho} \otimes \mathbb{1}) \circ \nabla^u$ in the definition (3.27) of $\widetilde{D}_R$, and $(\mathbb{1} \otimes \widetilde{\rho}^{-1}) \circ \nabla^\circ$ with $(\mathbb{1} \otimes \widetilde{\rho}^{-1}) \circ \nabla^{\circ u}$ in the definition (3.48) of $\widetilde{D}_L$. For the cases $\mathcal{E} = \mathcal{A}$, one obtains the following explicit formulas.

Proposition 4.2 *For a gauge transformation with a unitary $u \in \mathcal{A}$, the operators $\widetilde{D}_R = D + \omega_\rho$ and $\widetilde{D}_L = D + \omega_\rho^\circ$ of Corollaries 3.6 and 3.11 are mapped to $\widetilde{D}_R^u = D + \omega_\rho^u$ and $\widetilde{D}_L^u = D + \omega_\rho^{\circ u}$ where the transformed twisted 1-forms are given by*

$$\omega_\rho^u := \rho(u)[D, u^*]_\rho + \rho(u)\, \omega_\rho\, u^* \tag{4.11}$$

$$\omega_\rho^{\circ u} := \rho^\circ(u^{*\circ})[D, u^\circ]_{\rho^\circ} + \rho^\circ(u^{*\circ})\, \omega_\rho^\circ\, u^\circ. \tag{4.12}$$

Proof By Lemma 4.1, substituting ∇ with ∇^u in (3.33) yields the operator

$$\begin{aligned}
\widetilde{D}_R^u(a \otimes \psi) &= \rho(a) \otimes D\psi + \rho(u) \otimes \delta_\rho(u^* a)\psi + \rho(u) \otimes \omega_\rho u^* a\psi, \\
&= \mathbb{1} \otimes \left(\rho(a)D + \rho(u)[D, u^* a]_\rho\right)\psi + \mathbb{1} \otimes \rho(u)\omega_\rho u^* a\psi, \\
&= \mathbb{1} \otimes \left(D + \rho(u)[D, u^*]_\rho\right) a\psi + \mathbb{1} \otimes \rho(u)\omega_\rho u^* a\psi,
\end{aligned} \tag{4.13}$$

where in the last line we used

$$\rho(a)D + \rho(u)[D, u^* a]_\rho = \rho(u)Du^* a = (D + \rho(u)[D, u^*]_\rho)a. \tag{4.14}$$

Identifying $a \otimes \psi = \mathbb{1} \otimes a\psi$ with $a\psi$ in $\mathcal{A} \otimes_{\mathcal{A}} \mathcal{H} \simeq \mathcal{H}$, Eq. (4.13) shows that $\widetilde{D}_R^u$ acts on $\mathcal{H}$ as $D + \omega_\rho^u$ with ω_ρ^u as in (4.11).
Similarly, substituting ∇° with ∇° in (3.57) yields the operator

$$\begin{aligned}
&\widetilde{D}_L^u(\psi \otimes a) \\
&\quad = D\psi \otimes \rho^{-1}(a) + \delta^\circ(au)\psi \otimes \rho^{-1}(u^*) + \omega_\rho^\circ(au)^\circ\psi \otimes \rho^{-1}(u^*),
\end{aligned}$$

$$
\begin{aligned}
&= \Big((\rho^{-1}(a))^\circ D + (\rho^{-1}(u^*))^\circ [D, (au)^\circ]_{\rho^\circ}\Big)\, \psi \otimes \mathbb{1} + (\rho^{-1}(u^*))^\circ \omega_\rho^\circ (au)^\circ \psi \otimes \mathbb{1},\\
&= \left(\rho^\circ(a^\circ) D + \rho^\circ(u^{*\circ})[D, (au)^\circ]_{\rho^\circ}\right) \psi \otimes \mathbb{1} + \rho^\circ(u^{*\circ})\omega_\rho^\circ (au)^\circ \psi \otimes \mathbb{1},\\
&= \Big(D + \rho^\circ(u^{*^\circ})[D, u^\circ]_{\rho^\circ}\Big)\, a^\circ \psi \otimes \mathbb{1} + \rho^\circ(u^{*\circ})\omega_\rho^\circ u^\circ a^\circ \psi \otimes \mathbb{1},
\end{aligned}
\tag{4.15}
$$

where we used (2.7) and, in the last line,

$$
\rho^\circ(a^\circ) D + \rho^\circ(u^{*\circ})[D, (au)^\circ]_{\rho^\circ} = \rho^\circ(u^{*\circ}) D (au)^\circ = \left(D + \rho^\circ(u^{*\circ})[D, u^\circ]_{\rho^\circ}\right) a^\circ. \tag{4.16}
$$

Identifying $\psi \otimes a = a^\circ \psi \otimes \mathbb{1}$ with $a^\circ \psi$ in $\mathcal{H} \otimes_{\mathcal{A}} \mathcal{A} \simeq \mathcal{H}$ Eq. (4.15) shows that $\widetilde{D}_L^u$ acts on $\mathcal{H}$ as $D + \omega_\rho^{\circ\, u}$, with $\omega_\rho^{\circ\, u}$ as defined in (4.12). $\square$

Proposition 4.3 *In a twisted spectral triple* $(\mathcal{A}, \mathcal{H}, D; \rho)$, *the law of transformation of the gauge potential is* $\omega_\rho \to \omega_\rho^u$, *with* ω_ρ^u *given in* (4.3).

Proof By Lemma 3.10, we substitute ω_ρ° in (4.12) with $\epsilon' J \omega_\rho^L J^{-1}$, with $\omega_\rho^L \in \Omega_D^1(\mathcal{A}, \rho)$. Explicitly, using (2.7) and (2.8) to write

$$
\rho^\circ(u^{*\circ}) = (\rho^{-1}(u^*))^\circ = (\rho(u)^*)^\circ = J \rho(u) J^{-1}, \tag{4.17}
$$

one obtains

$$
\begin{aligned}
\omega_\rho^{\circ\, u} &= \rho^\circ(u^{*\circ})[D, u^\circ]_{\rho^\circ} + \rho^\circ(u^{*\circ})\omega_\rho^\circ u^\circ\\
&= J\rho(u)J^{-1}[D, u^\circ]_{\rho^\circ} + J\rho(u)J^{-1}\omega_\rho^\circ u^\circ\\
&= \epsilon' J \Big(\rho(u)[D, u^*]_\rho + \rho(u)\, \omega_\rho^L\, u^*\Big)\, J^{-1} = \epsilon' J\, (\omega_\rho^L)^u\, J^{-1},
\end{aligned}
\tag{4.18}
$$

where in the third line we used again (4.17) to write

$$
[D, u^\circ]_{\rho^\circ} = D u^\circ - \rho^\circ(u^\circ) D = D J u^* J^{-1} - J \rho(u^*) J^{-1} D = \epsilon' J [D, u^*]_\rho J^{-1}. \tag{4.19}
$$

Therefore, with the notation of Proposition 3.13, one has that $\omega_\rho = \frac{1}{2}(\omega_\rho^R + \omega_\rho^L)$ is mapped under a gauge transformation to

$$
\begin{aligned}
\tfrac{1}{2}((\omega_\rho^R)^u + (\omega_\rho^L)^u) &= \rho(u)[D, u^*]_\rho + \tfrac{1}{2}\rho(u)(\omega_\rho^R + \omega_\rho^L)u^*,\\
&= \rho(u)[D, u^*]_\rho + \rho(u)\omega_\rho u^*,
\end{aligned}
\tag{4.20}
$$

that is ω_ρ^u as defined in (4.3). $\square$

The transformation of the gauge potential of a twisted spectral triple is thus the usual gauge transformation (A.52), in which the left action of u and the commutator have been twisted by the automorphism. This suggests that a twisted fluctuation may also be obtained by twisting the left action of $\mathrm{Ad}(u)$ in (A.56).

4.2 Twisted action of unitaries

Let $\mathrm{Ad}(u) = uJuJ^{-1}$ denote the adjoint action on $\mathcal{H}$ of a unitary $u \in \mathcal{A}$ as recalled in (A.55):

$$\mathrm{Ad}(u)\psi := u\,\psi\,u^* = uJuJ^{-1}\psi \quad \forall\psi \in \mathcal{H}. \tag{4.21}$$

We show in Proposition 4.5 that a twisted conjugation by $\mathrm{Ad}(u)$ of a twisted-gauged Dirac operator D_{ω_ρ} yields the gauge transformation of Proposition 4.3. Preliminarily, we begin by proving a twisted version of (A.56).

Lemma 4.4 *Let $(\mathcal{A}, \mathcal{H}, D; \rho)$ be a real twisted spectral triple; for any $u \in \mathcal{U}(\mathcal{A})$ define*

$$\rho(\mathrm{Ad}(u)) := \mathrm{Ad}(\rho(u)) = \rho(u)J\rho(u)J^{-1}. \tag{4.22}$$

Then, it holds that

$$\rho(\mathrm{Ad}(u))\,D\,\mathrm{Ad}(u)^{-1} = D + \rho(u)[D, u^*]_\rho + \epsilon' J\,\rho(u)[D, u^*]_\rho\,J^{-1}. \tag{4.23}$$

Proof Let $v := JuJ^{-1} \in \mathcal{A}^\circ$. By (4.17) one has $\rho^\circ(v) = \rho^\circ(u^{*\circ}) = J\rho(u)J^{-1}$, so that

$$\mathrm{Ad}(u) = uv = vu, \qquad \rho(\mathrm{Ad}(u)) = \rho(u)\rho^\circ(v) = \rho^\circ(v)\rho(u) \tag{4.24}$$

by the order 0 condition. Using the twisted first-order condition (2.6) one computes:

$$\begin{aligned}
\rho(\mathrm{Ad}(u))\,D\,\mathrm{Ad}(u)^{-1} &= \rho^\circ(v)\,(\rho(u)Du^{-1})\,v^{-1} = \rho^\circ(v)(D + \rho(u)[D, u^{-1}]_\rho)v^{-1}\\
&= \rho^\circ(v)Dv^{-1} + \rho^\circ(v)\rho(u)[D, u^{-1}]_\rho v^{-1}\\
&= D + \rho^\circ(v)[D, v^{-1}]_{\rho^\circ} + \rho(u)[D, u^{-1}]_\rho.
\end{aligned} \tag{4.25}$$

By (2.2), one has

$$\begin{aligned}
\rho^\circ(v)[D, v^{-1}]_{\rho^\circ} &= \rho^\circ(v)Dv^{-1} - D = \epsilon' J\rho(u)Du^{-1}J^{-1} - D,\\
&= \epsilon' J\left(D + \rho(u)[D, u^{-1}]_\rho\right)J^{-1} - D,\\
&= (\epsilon')^2 D + J\epsilon'\rho(u)[D, u^{-1}]_\rho J^{-1} - D = \epsilon' J\rho(u)[D, u^{-1}]_\rho J^{-1}.
\end{aligned} \tag{4.26}$$

Plugged into (4.25), one gets (4.23). □

Proposition 4.5 *Let $(\mathcal{A}, \mathcal{H}, D; \rho)$, J be a real twisted spectral triple and consider a twisted-gauged Dirac operator $D_{\omega_\rho} = D + \omega_\rho + \epsilon' J\omega_\rho J^{-1}$ as in (3.45). Then, for any $u \in \mathcal{U}(\mathcal{A})$, one has*

$$\rho(\mathrm{Ad}(u))\,D_{\omega_\rho}\,\mathrm{Ad}(u)^{-1} = D + \omega^u_\rho + \epsilon'\,J\,\omega^u_\rho\,J^{-1} \tag{4.27}$$

with transformed ω^u_ρ given in (4.3).

Proof For $\omega_\rho = a[D,b]_\rho$ (without loss of generality), one needs to compute $\rho(\mathrm{Ad}(u))\,\omega_\rho\,\mathrm{Ad}(u)^{-1}$ and $\rho(\mathrm{Ad}(u))\,J\omega_\rho J^{-1}\,\mathrm{Ad}(u)^{-1}$. By the twisted first-order condition, one gets

$$\begin{aligned}\rho(\mathrm{Ad}(u))\,\omega_\rho\,\mathrm{Ad}(u)^{-1} &= \rho(u)\,\left(\rho^\circ(v)a[D,b]_\rho v^{-1}\right)\,u^{-1},\\ &= \rho(u)\left(a[D,b]_\rho\right)u^{-1} = \rho(u)\omega_\rho u^{-1}.\end{aligned} \tag{4.28}$$

In order to compute $\rho(\mathrm{Ad}(u))\,J\omega_\rho J^{-1}\,\mathrm{Ad}(u)^{-1}$, one uses on the one hand,

$$\begin{aligned}\rho(u)\,J\omega_\rho J^{-1}\,u^{-1} &= \rho(u)\,Ja[D,b]_\rho J^{-1}\,u^{-1} = J\,(J^{-1}\rho(u)J)\,a[D,b]_\rho J^{-1}\,u^{-1},\\ &= Ja[D,b]_\rho\,(J^{-1}uJ)\,J^{-1}u^{-1} = Ja[D,b]_\rho J^{-1} = J\omega_\rho J^{-1},\end{aligned} \tag{4.29}$$

and on the other hand

$$\rho^\circ(v)\,J\omega_\rho J^{-1}\,v^{-1} = J\rho(u)J^{-1}\,J\omega_\rho J^{-1}\,Ju^{-1}J^{-1} = J\rho(u)\omega_\rho u^{-1}J^{-1}, \tag{4.30}$$

so that

$$\rho(\mathrm{Ad}(u))J\omega_\rho J^{-1}\,\mathrm{Ad}(u)^{-1} = \rho^\circ(v)\,\left(\rho(u)\,J\omega_\rho J^{-1}\,u^{-1}\right)\,v^{-1} = J\rho(u)\omega_\rho u^* J^{-1}. \tag{4.31}$$

Collecting (4.31) and (4.28) one gets

$$\rho(\mathrm{Ad}(u))\left(\omega_\rho + \epsilon' J\omega_\rho J^{-1}\right)\mathrm{Ad}(u)^{-1} = \rho(u)\omega_\rho u^{-1} + \epsilon' J\rho(u)\omega_\rho u^{-1}J^{-1}. \tag{4.32}$$

Together with (4.23), this yields the result. □

5 Self-adjointness

In the non-twisted case, a gauge transformation preserves the self-adjointness of the Dirac operator. The transformed operator

$$D_{\omega^u} = \mathrm{Ad}(u))\,D_\omega\,\mathrm{Ad}(u)^{-1} \tag{5.1}$$

is self-adjoint if and only if so is D_ω, since $\mathrm{Ad}(u)$ is unitary (see Lemma 5.1 below). Thus, starting with a spectral triple $(\mathcal{A}, \mathcal{H}, D_\omega)$, a gauge transformation yields a spectral triple $(\mathcal{A}, \mathcal{H}, D_{\omega^u})$, which is unitary equivalent to the former [10]. This result is no longer true in the twisted case: by Proposition 4.5 the gauge transformed of the twisted-gauged Dirac operator D_{ω_ρ} is

$$D_{\omega_\rho^u} = \rho(\mathrm{Ad}(u))\,D_{\omega_\rho}\,\mathrm{Ad}(u)^*, \tag{5.2}$$

which has no reason to be self-adjoint, even if D_{ω_ρ} is self-adjoint.

We next work out conditions on the unitary element u to guarantee that the operator $D_{\omega_\rho^u}$ be self-adjoint. A simple condition would be that u is invariant for the twist: $\rho(u) = u$. We show, for the example of the minimal twist of a spin manifold constructed in [14], that there exists other solutions than this trivial one.

5.1 Conditions for self-adjointness

Let us begin with recalling some properties of antilinear operators. The adjoint of an antilinear operator C on a Hilbert space $\mathcal{H}$ is the antilinear operator C^* such that

$$\langle C\xi, \zeta\rangle = \overline{\langle \xi, C^*\zeta\rangle}, \quad \forall \xi, \zeta \in \mathcal{H}. \tag{5.3}$$

Such an operator is antiunitary if

$$\langle C\xi, C\zeta\rangle = \overline{\langle \xi, \zeta\rangle}, \quad \text{that is} \quad C^*C = CC^* = \mathbb{1}. \tag{5.4}$$

Hence $C^* = C^{-1}$, as for linear unitary operators. However, one has to be careful that the usual rule for the adjoint holds for the product of two antilinear operators C, C',

$$\langle CC'\xi, \zeta\rangle = \overline{\langle C'\xi, C^*\zeta\rangle} = \langle \xi, C'^*C\zeta\rangle \quad \text{so that} \quad (CC')^* = C'^*C^* \tag{5.5}$$

but not for the product of an antilinear C with a linear T, for

$$\langle CT\xi, \zeta\rangle = \overline{\langle T\xi, C^*\zeta\rangle} = \overline{\langle \xi, T^*C\zeta\rangle}. \tag{5.6}$$

On the other hand, the usual rule for the adjoint holds for any product involving an even number of antilinear operators, such as JTJ^{-1} with T linear, that often appear in this work. This is shown explicitly in the following lemma for $T = u$ an unitary element.

Lemma 5.1 *Let $(\mathcal{A}, \mathcal{H}, D)$ be a real spectral triple with real structure (the antilinear operator) J. Then, for any unitary $u \in \mathcal{A}$, one has that* $\mathrm{Ad}(u) = uJuJ^{-1}$ *is unitary.*

Proof The operator JuJ^{-1} is linear, hence $\mathrm{Ad}(u)^* = (JuJ^{-1})^*u^*$. A direct computation yields

$$\langle JuJ^{-1}\xi, \zeta\rangle = \overline{\langle uJ^{-1}\xi, J^*\zeta\rangle} = \overline{\langle J^{-1}\xi, u^*J^*\zeta\rangle} = \langle \xi, (J^{-1})^*u^*J^*\zeta\rangle \tag{5.7}$$

that is, using $J^* = J^{-1}$,

$$\left(JuJ^{-1}\right)^* = Ju^*J^{-1}. \tag{5.8}$$

Hence, $\mathrm{Ad}(u)^* = Ju^*J^{-1}u^*$, so that $\mathrm{Ad}(u)^*\,\mathrm{Ad}(u) = \mathrm{Ad}(u)\,\mathrm{Ad}(u)^* = \mathbb{1}$. □

We now work out a condition on a unitary element u which is equivalent to $D_{\omega_\rho^u}$ being self-adjoint. Taking advantage of the two formulas for $D_{\omega_\rho^u}$ [the two sides of (4.23)], we actually exhibit two conditions which are equivalent.

Proposition 5.2 *Let $(\mathcal{A}, \mathcal{H}, D; \rho)$, J be a real twisted spectral triple, D_{ω_ρ} a twisted-gauged Dirac operator and u a unitary element of $\mathcal{A}$. Then, the gauge transformed operator $D_{\omega_\rho^u}$ in (5.2) is self-adjoint if and only if*

$$J\omega(u)J^{-1} = -\epsilon'\omega(u), \tag{5.9}$$

for

$$\omega(u) = u^\circ\,[D, \rho(u)^* u]_\rho\, u^{*\circ} \quad \textit{or} \quad \omega(u) = u\,[D, \rho(u)^* u]_\rho u^*, \tag{5.10}$$

the two choices being equivalent.

Proof We write $D = D_{\omega_\rho}$, taken to be self-adjoint. Then $D_{\omega_\rho^u} = \rho(\mathrm{Ad}(u))D\,\mathrm{Ad}(u)^*$ is self-adjoint by Lemma 5.1 if and only if $\rho(\mathrm{Ad}(u))D\,\mathrm{Ad}(u)^* = \mathrm{Ad}(u)D\rho(\mathrm{Ad}(u))^*$ or, equivalently

$$\mathrm{Ad}(u)^*\rho(\mathrm{Ad}(u))D = D\rho(\mathrm{Ad}(u))^*\,\mathrm{Ad}(u). \tag{5.11}$$

By (2.8) and (2.5),

$$\rho(\mathrm{Ad}(u))^* = \rho(u)^* J\rho(u)^* J^{-1} = \rho^{-1}(u^*)\,J\,\rho^{-1}(u^*)J^{-1} = \rho^{-1}(\mathrm{Ad}(u)^*), \tag{5.12}$$

so that $\rho\,(\rho(\mathrm{Ad}(u))^*\,\mathrm{Ad}(u)) = \mathrm{Ad}(u)^*\rho(\mathrm{Ad}(u))$. Hence, condition (5.11) becomes

$$[D, \rho(\mathrm{Ad}(u))^*\,\mathrm{Ad}(u)]_\rho = 0. \tag{5.13}$$

By the order-zero condition, one has

$$\rho(\mathrm{Ad}(u))^*\,\mathrm{Ad}(u) = \rho(u)^* J\rho(u)^* J^{-1}\,uJuJ^{-1} = \rho(u)^* u\;J\rho(u)^* uJ^{-1} = \mathfrak{u}J\mathfrak{u}J^{-1} \tag{5.14}$$

where $\mathfrak{u} := \rho(u)^* u$. Therefore

$$\begin{aligned}[D, \rho(\mathrm{Ad}(u))^*\,\mathrm{Ad}(u)]_\rho &= [D, \mathfrak{u}J\mathfrak{u}J^{-1}]_\rho \\ &= \rho(\mathfrak{u})[D, J\mathfrak{u}J^{-1}]_\rho + [D, \mathfrak{u}]_\rho J\mathfrak{u}J^{-1} = \epsilon' J\omega(\mathfrak{u})J^{-1} + \omega(\mathfrak{u}),\end{aligned} \tag{5.15}$$

with

$$\omega(\mathfrak{u}) := J\rho(\mathfrak{u})J^{-1}[D, \mathfrak{u}]_\rho, \tag{5.16}$$

where we used the twisted first-order condition as well as

$$\begin{aligned}\rho(\mathfrak{u})[D, J\mathfrak{u}J^{-1}]_\rho &= J^{-1}\left(J\rho(\mathfrak{u})J^{-1}\,J[D, J\mathfrak{u}J^{-1}]_\rho\right) \\ &= \epsilon' J^{-1}\left(J\rho(\mathfrak{u})J^{-1}\left(DJ^2\mathfrak{u} - J^2\rho(\mathfrak{u})D\right)\right)J^{-1}, \\ &= \epsilon'\epsilon'' J^{-1}\left(J\rho(\mathfrak{u})J^{-1}[D, \mathfrak{u}]_\rho\right)J^{-1} \\ &= \epsilon' J\left(J\rho(\mathfrak{u})J^{-1}[D, \mathfrak{u}]_\rho\right)J^{-1} = \epsilon' J\omega(\mathfrak{u})J^{-1}.\end{aligned}$$

The first part of the proposition follows from (5.15), noticing that

$$\rho(\mathfrak{u}) = \rho(\rho(u)^* u) = \rho(\rho(u)^*)\rho(u) = u^* \rho(u), \tag{5.17}$$

so that

$$\begin{aligned}\omega(\mathfrak{u}) &= J u^* \rho(u) J^{-1} [D, \rho(u)^* u]_\rho = J u^* J^{-1} [D, \rho(u)^* u]_\rho J u J^{-1} \\ &= u^\circ\, [D, \rho(u)^* u]_\rho\, u^{*\circ}.\end{aligned}$$

The second part of the proposition is obtained turning back to the definition of $D_{\omega_\rho^u}$ that is the right-hand side of (4.23). One has that $D_{\omega_\rho^u}$ is self-adjoint if and only if

$$\left(\omega_\rho^u - (\omega_\rho^u)^*\right) + \epsilon' J \left(\omega_\rho^u - (\omega_\rho^u)^*\right) J^{-1} = 0. \tag{5.18}$$

By hypothesis $D_{\omega_\rho} = D + \omega_\rho + J\omega_\rho J^{-1}$ is self-adjoint, so that $(\omega_\rho - \omega_\rho^*) + \epsilon' J \left(\omega_\rho - \omega_\rho^*\right) J^{-1} = 0$. Therefore, from the definition (4.3) of ω_ρ^u, equation (5.18) becomes $\omega(u) + \epsilon' J \omega(u) J^{-1} = 0$ with

$$\omega(u) := \rho(u)[D, u^*]_\rho - (\rho(u)[D, u^*]_\rho)^*. \tag{5.19}$$

The result follows remembering that $\rho(u)[D, u^*]_\rho = \rho(u) D u^* - D$, so that

$$\omega(u) = \rho(u) D u^* - u D \rho(u)^* = u \left(u^* \rho(u) D - D\rho(u)^* u\right) u^* = -u[D, \rho(u)^* u]_\rho\, u^*, \tag{5.20}$$

where we used (5.17). □

Remark 5.3 One may check directly the equivalence of the two choices for $\omega(u)$ in (5.10). Writing $\omega := [D, \rho(u)^* u]_\rho$, one gets that for $\omega(u) = u^\circ\, [D, \rho(u)^* u]_\rho\, u^{*\circ} = u^\circ \omega\, u^{*\circ}$, equation (5.9) is equivalent to

$$\omega = -\epsilon' (u^\circ)^* \left(J u^\circ \omega\, u^{*\circ}\, J^{-1}\right) (u^{*\circ})^* = -\epsilon'\, u^* J u\; \omega\; u^* J^{-1} u, \tag{5.21}$$

where we use that u° is unitary, with $(u^\circ)^* = (u^*)^\circ$ so that $(u^{*\circ})^* = u^\circ$, as well as

$$(u^\circ)^* J u^\circ = J u J^{-1} J J u^* J^{-1} = \epsilon'' J u J^{-1} u^* J^{-1} = \epsilon'' u^* J u (J^{-1})^2 = u^* J u, \tag{5.22}$$

and similarly $u^{*\circ}\, J^{-1}\, u^\circ = u^* J^{-1} u$. On the other hand, for $\omega(u) = u\, [D, \rho(u)^* u]_\rho\, u^* = u \omega u^*$, Eq. (5.9) is equivalent to

$$\omega = -\epsilon'\, u^* J u\; \omega\; u^* J^{-1} u, \tag{5.23}$$

which is precisely the r.h.s. of (5.21).

An obvious solution to (5.9) is that $\rho(u)^* u$ twist-commutes with D. This happens in particular when u is invariant under the twist, $\rho(u) = u$, so that $\rho(u)^* u = \mathbb{1}$. An extensive study of (5.9) and its solutions will be undertaken elsewhere. Here, we just solve it in the simple example of the minimal twist of manifold.

5.2 Minimal twist of a manifold

The minimal twist of a closed spin manifold $\mathcal{M}$ of even dimension $2m$, $m \in \mathbb{N}$, has been defined in [14] as the real, graded, twisted spectral triple

$$\mathcal{A} = C^\infty(\mathcal{M}) \otimes \mathbb{C}^2, \quad \mathcal{H} = L^2(\mathcal{M}, S), \quad D = \partial\!\!\!/, \quad J, \quad \rho \tag{5.24}$$

where $C^\infty(\mathcal{M})$ is the algebra of smooth functions on $\mathcal{M}$, the Hilbert space $L^2(\mathcal{M}, S)$ is that of square integrable spinors, with usual Dirac operator

$$\partial\!\!\!/ = -\mathrm{i}\sum_{\mu=1}^{2m} \gamma^\mu \nabla_\mu \quad \text{and} \quad \nabla_\mu = \partial_\mu + \omega^\mu \tag{5.25}$$

(γ^μ are the Dirac matrices of size 2^m, ω^μ is the spin connection), the real structure J is the charge conjugation operator composed with complex conjugation, and the automorphism ρ

$$\rho(f, g) = (g, f) \quad \forall (f, g) \in \mathcal{A} \simeq C^\infty(\mathcal{M}) \oplus C^\infty(\mathcal{M}). \tag{5.26}$$

is the flip. The grading Γ (the product of all the Dirac matrices) splits $\mathcal{H}$ in two orthogonal subspaces $\mathcal{H}_\pm$, on which each copy of $C^\infty(\mathcal{M})$ acts independently (by point-wise multiplication). The representation π of $\mathcal{A}$ on $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ is

$$\pi(a) = \begin{pmatrix} f\mathbb{1}_{2^{m-1}} & 0 \\ 0 & g\mathbb{1}_{2^{m-1}} \end{pmatrix} \quad \forall a = (f, g) \in \mathcal{A} \text{ with } f, g \in C^\infty(\mathcal{M}). \tag{5.27}$$

Finally, the KO-dimension of the twisted spectral triple (5.24) is $2m \mod 8$.

Proposition 5.4 *In KO-dim* $= 0, 4$, *any unitary of* $\mathcal{A}$ *is a solution of* (5.9). *On the other hand, in KO-dim* $= 2, 6$, *the only solutions are the trivial one* $\omega(u) = 0$.

Proof A unitary u of $\mathcal{A}$ is (omitting the representation symbol and the identity operator)

$$u = \begin{pmatrix} e^{\mathrm{i}\theta_1} & 0 \\ 0 & e^{\mathrm{i}\theta_2} \end{pmatrix}, \tag{5.28}$$

where θ_1, θ_2 are smooth real functions on $\mathcal{M}$. Hence,

$$\rho(u)^* u = \begin{pmatrix} e^{-\mathrm{i}\theta_2} & 0 \\ 0 & e^{-\mathrm{i}\theta_1} \end{pmatrix} \begin{pmatrix} e^{\mathrm{i}\theta_1} & 0 \\ 0 & e^{\mathrm{i}\theta_2} \end{pmatrix} = \begin{pmatrix} e^{\mathrm{i}\varphi} & 0 \\ 0 & e^{-\mathrm{i}\varphi} \end{pmatrix} \quad \text{with} \quad \varphi := \theta_1 - \theta_2. \tag{5.29}$$

For any $a \in \mathcal{A}$, one has [14, eq. (5.9)]

$$[\partial\!\!\!/, a]_\rho = -\mathrm{i}\gamma^\mu(\partial_\mu a), \tag{5.30}$$

so that

$$[\partial\!\!\!/, \rho(u)^* u]_\rho = -\,\mathrm{i}\gamma^\mu \begin{pmatrix} \partial_\mu e^{\mathrm{i}\varphi} & 0 \\ 0 & \partial_\mu e^{-\mathrm{i}\varphi} \end{pmatrix} = -\mathrm{i}\gamma^\mu \begin{pmatrix} \mathrm{i}(\partial_\mu \varphi) e^{\mathrm{i}\varphi} & 0 \\ 0 & -\mathrm{i}(\partial_\mu \varphi) e^{-\mathrm{i}\varphi} \end{pmatrix}. \tag{5.31}$$

In addition [14, eq.(5.10)],

$$\gamma^\mu u = \rho(u)\gamma^\mu, \tag{5.32}$$

so by an easy calculation

$$u[\partial\!\!\!/, \rho(u)^* u]_\rho u^* = -\mathrm{i}\gamma^\mu \begin{pmatrix} \mathrm{i}\partial_\mu \varphi & 0 \\ 0 & -\mathrm{i}\partial_\mu \varphi \end{pmatrix}. \tag{5.33}$$

Furthermore, by [14, Lem. 5.2], for $\omega_\rho = -\mathrm{i}\gamma^\mu \rho(a)\partial_\mu b$, one has

$$J\omega_\rho J^{-1} = \begin{cases} -\mathrm{i}\gamma^\mu \rho(a^*)\partial_\mu b^* & \text{if } KO\text{-dim} = 0, 4, \\ -\mathrm{i}\gamma^\mu a^* \partial_\mu \rho(b^*) & \text{if } KO\text{-dim} = 2, 6. \end{cases} \tag{5.34}$$

Therefore, for KO-dim $= 0, 4$, one obtains (remembering that φ is a real function)

$$\begin{aligned} Ju[\partial\!\!\!/, \rho(u)^* u]_\rho u^* J^{-1} &= -\,\mathrm{i}\gamma^\mu \begin{pmatrix} \partial_\mu (\mathrm{i}\varphi)^* & 0 \\ 0 & \partial_\mu(-\mathrm{i}\varphi)^* \end{pmatrix} = -\,\mathrm{i}\gamma^\mu \begin{pmatrix} -\mathrm{i}\partial_\mu \varphi & 0 \\ 0 & \mathrm{i}\partial_\mu \varphi \end{pmatrix} \\ &= -\,u[\partial\!\!\!/, \rho(u)^* u]_\rho u^*, \end{aligned} \tag{5.35}$$

whereas for KO-dim $= 2, 6$ one has

$$\begin{aligned} Ju[\partial\!\!\!/, \rho(u)^* u]_\rho u^* J^{-1} &= -\,\mathrm{i}\gamma^\mu \begin{pmatrix} \partial_\mu (-\mathrm{i}\varphi)^* & 0 \\ 0 & \partial_\mu(\mathrm{i}\varphi)^* \end{pmatrix} = -\,\mathrm{i}\gamma^\mu \begin{pmatrix} \mathrm{i}\partial_\mu \varphi & 0 \\ 0 & -\mathrm{i}\partial_\mu \varphi \end{pmatrix} \\ &= u[\partial\!\!\!/, \rho(u)^* u]_\rho u^*. \end{aligned} \tag{5.36}$$

The result follows noticing that in even dimension, one has the sign $\epsilon' = 1$, so that (5.35) is solution to (5.9) for any u, while (5.36) is solution only when $\omega(u) = 0$. □

This simple example exhibits two interesting cases: the unitaries that preserve the self-adjointness of the Dirac operator are either the whole group $C^\infty(\mathcal{M}, U(1)\times U(1))$ of unitaries of $\mathcal{A}$, or the trivial solution to (5.9). Intriguingly, the group $C^\infty(\mathcal{M}, U(1))$ of unitaries which are invariant under the twist is of no particular importance.

To understand why this is the case, recall from [14, Lemma 5.1] that in KO-dimension 0, 4, one has $JuJ^{-1} = u^*$, so that $\mathrm{Ad}(u) = \mathbb{1}$. Therefore, the Dirac operator is invariant under any gauge transformation, no matter whether u is invariant under the twist or not. Moreover, the fact that the action of $\mathrm{Ad}(u)$ is trivial indicates that the twisted fluctuations, are *not* generated by the action of a unitary. This fact can be checked explicitly, computing $\omega_\rho = \rho(a)[D, a']_\rho$ for $a = (f, g)$, $a' = (f', g')$: one gets from (5.30) and (5.32)

$$\omega_\rho = \rho(a)[D, a']_\rho = -\mathrm{i}\gamma^\mu a\, \partial_\mu a' = -\mathrm{i}\gamma^\mu \begin{pmatrix} f\, \partial_\mu f' & 0 \\ 0 & g\, \partial_\mu g' \end{pmatrix}, \tag{5.37}$$

and by (5.34)

$$J\omega_\rho J^{-1} = -\mathrm{i}\gamma^\mu a^*\partial_\mu a'^* = -\mathrm{i}\gamma^\mu \begin{pmatrix} \bar{f}\,\partial_\mu \bar{f}' & 0 \\ 0 & \bar{g}\,\partial_\mu \bar{g}' \end{pmatrix}, \tag{5.38}$$

so that

$$\omega_\rho + J\omega_\rho J^{-1} = -\mathrm{i}\gamma^\mu \begin{pmatrix} f_\mu & 0 \\ 0 & g_\mu \end{pmatrix} \tag{5.39}$$

with $f_\mu = f\,\partial_\mu f' + \bar{f}\,\partial_\mu \bar{f}'$ and $g_\mu = g\,\partial_\mu g' + \bar{g}\,\partial_\mu \bar{g}'$ real function on $\mathcal{M}$. The r.h.s. of (5.39) is self-adjoint if and only if

$$\begin{aligned} 0 &= -\,\mathrm{i}\gamma^\mu \begin{pmatrix} f_\mu & 0 \\ 0 & g_\mu \end{pmatrix} - \left(-\mathrm{i}\gamma^\mu \begin{pmatrix} f_\mu & 0 \\ 0 & g_\mu \end{pmatrix}\right)^* \\ &= -\,\mathrm{i}\gamma^\mu \begin{pmatrix} f_\mu & 0 \\ 0 & g_\mu \end{pmatrix} - \mathrm{i}\gamma^\mu \begin{pmatrix} g_\mu & 0 \\ 0 & f_\mu \end{pmatrix} \\ &= -\,\mathrm{i}(f_\mu + g_\mu)\gamma^\mu, \end{aligned} \tag{5.40}$$

that is if and only if $f_\mu = -g_\mu$. In that case, (5.39) yields $\partial\!\!\!/_{\omega_\rho} = \partial\!\!\!/ - \mathrm{i}\,f_\mu\gamma^\mu\Gamma$, as already shown in [14]. The point is that such a fluctuation cannot be obtained with $a = u$ a unitary and $a' = u^*$, that is for $f = e^{\mathrm{i}\theta_1}$, $g = e^{\mathrm{i}\theta_2}$, $f' = e^{-i\theta_1}$, $g' = e^{-i\theta_2}$, since this would give $f_\mu = g_\mu = 0$.

In KO-dimension 2, 6, one has that $\omega(u) = 0$ if and only if

$$[\partial\!\!\!/, \rho(u)^* u]_\rho = 0. \tag{5.41}$$

By (5.31), this mean that $u = (e^{\mathrm{i}\theta_1}, e^{\mathrm{i}\theta_2})$ with $\theta_1 - \theta_2$ a constant function. Notice that this is a bigger set than the unitaries invariant under the twist (for which the constant is zero). However, in any case, such unitaries do not generate a fluctuation. Indeed, ω_ρ is still given by (5.37), but

$$J\omega_\rho J^{-1} = -\,\mathrm{i}\gamma^\mu \begin{pmatrix} \bar{g}\,\partial_\mu \bar{g}' & 0 \\ 0 & \bar{f}\,\partial_\mu \bar{f}' \end{pmatrix}. \tag{5.42}$$

Thus $\omega_\rho + J\omega_\rho J^{-1}$ is given by (5.39) with

$$f_\mu = f\,\partial_\mu f' + \bar{g}\,\partial_\mu \bar{g}', \quad g_\mu = \bar{f}_\mu. \tag{5.43}$$

With $f = e^{\mathrm{i}\theta_1}$, $g = e^{\mathrm{i}\theta_2}$, $f' = e^{-i\theta_1}$, $g = e^{-i\theta_2}$, one gets $f_\mu = \mathrm{i}\,\partial_\mu(\theta_1 - \theta_2)$, which vanishes when $\theta_1 - \theta_2$ is constant. More generally, one finds back the result of [14] noticing that for arbitrary f, f' and g, g', a computation similar to (5.40) yields that $\omega_\rho + J\omega_\rho J^{-1}$ is self-adjoint if and only if $f_\mu = g_\mu = 0$.

To summarize, one has the following result.

Proposition 5.5 *In KO-dimension $0, 4$, the operator $\partial\!\!\!/$ has non-zero twisted self-adjoint fluctuations given by*

$$\partial\!\!\!/_{\omega_\rho} = \partial\!\!\!/ - \mathrm{i} f_\mu \gamma^\mu \Gamma, \qquad f_\mu \in C^\infty(\mathcal{M}, \mathbb{R}). \tag{5.44}$$

They are invariant under a gauge transformation, but are not generated by the action of unitaries. In KO-dimension $2, 6$, there is no non-zero self-adjoint fluctuations.

A. The non-twisted case

The material in this Appendix is well known and taken mainly from [10] and [8].

A.1 Fluctuations and Morita equivalence

Recall that a finitely generated, projective (right, say) $\mathcal{A}$-module $\mathcal{E}$ is hermitian if it comes equipped with an $\mathcal{A}$-valued inner product, that is a sesquilinear map $\langle \cdot, \cdot \rangle_{\bullet} : \mathcal{E} \times \mathcal{E} \to \mathcal{A}$ such that $\langle \xi, \xi \rangle_{\bullet} \geq 0$ for any $\xi \in \mathcal{E}$, $(\langle \xi, \eta \rangle_{\bullet})^* = \langle \eta, \xi \rangle_{\bullet}$ and $\langle \xi a, \eta b \rangle_{\bullet} = a^* \langle \xi, \eta \rangle_{\bullet} b$, for all $\xi, \eta \in \mathcal{E}$ and $a, b \in \mathcal{A}$. A similar notion goes for left modules with a sesquilinear map ${}_{\bullet}\langle \cdot, \cdot \rangle : \mathcal{E} \times \mathcal{E} \to \mathcal{A}$ which is now linear in the first entry (and anti-linear on the second). The module $\mathcal{E}$ is taken to be self-dual for the $\mathcal{A}$-valued hermitian structure [16, Prop. 7.3], in the sense that for any $\varphi \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{A})$ there exists a unique $\zeta_\varphi \in \mathcal{E}$ such that $\varphi(\xi) = \langle \zeta_\varphi, \xi \rangle_{\bullet}$, for all $\xi \in \mathcal{E}$.

In the crudest version [16], the algebra $\mathcal{B}$ is Morita equivalent to the (unital) algebra $\mathcal{A}$ if there exists a hermitian finite projective $\mathcal{A}$-module $\mathcal{E}$ such that $\mathcal{B}$ is isomorphic to the algebra $\mathrm{End}_{\mathcal{A}}(\mathcal{E})$ of $\mathcal{A}$-linear endomorphisms of $\mathcal{E}$ which are adjointable (with respect to the hermitian structure of $\mathcal{E}$). In particular, an algebra is Morita equivalent to itself. In that case, the module $\mathcal{E}$ can be taken to be the algebra itself, with hermitian map $\langle a, b \rangle_{\bullet} = a^* b$ or ${}_{\bullet}\langle a, b \rangle = ab^*$.

A.1.1 Morita equivalence by right module

Let us assume that the module implementing the Morita equivalence between $\mathcal{A}$ and $\mathcal{B}$ is a right $\mathcal{A}$-module $\mathcal{E}_R$ with $\mathcal{A}$-valued inner product $\langle \cdot, \cdot \rangle_{\bullet}$. The action of $\mathcal{B} \simeq \mathrm{End}_{\mathcal{A}}(\mathcal{E})$ on $\mathcal{E}_R$ is not suitable to build a spectral triple, for $\mathcal{E}_R$ is not an Hilbert space. However, the tensor product

$$\mathcal{H}_R := \mathcal{E}_R \otimes_{\mathcal{A}} \mathcal{H} \tag{A.1}$$

is an Hilbert space for the inner product [9, p. 562]

$$\langle \eta_1 \otimes \psi_1, \eta_2 \otimes \psi_2 \rangle_{\mathcal{H}_R} = \langle \psi_1, \langle \eta_1, \eta_2 \rangle_{\bullet} \psi_2 \rangle_{\mathcal{H}} \quad \forall \eta_1, \eta_2 \in \mathcal{E}_R, \quad \psi_1, \quad \psi_2 \in \mathcal{H}, \tag{A.2}$$

with $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ the inner product of $\mathcal{H}$. The action of $\mathcal{B} \simeq \mathrm{End}_{\mathcal{A}}(\mathcal{E})$ is then extended to $\mathcal{H}_R$ as

$$\pi_R(b)(\eta \otimes \psi) := b\eta \otimes \psi \quad \forall b \in \mathcal{B}. \tag{A.3}$$

To make D act on $\mathcal{H}_R$, the simplest guess,

$$D_R(\eta \otimes \psi) := \eta \otimes D\psi, \tag{A.4}$$

is not compatible with the tensor product over $\mathcal{A}$ [8, p. 204] since

$$D_R(\eta a \otimes \psi) - D_R(\eta \otimes a\psi) = \eta a \otimes D\psi - \eta \otimes Da\psi = -\eta \otimes [D, a]\psi \tag{A.5}$$

has no reason to vanish. To correct this, one uses the derivation $\delta = [D, \cdot\,]$ of $\mathcal{A}$ in the $\mathcal{A}$-bimodule $\Omega^1_D(\mathcal{A})$ of 1-forms as defined in (1.2). Since both $\Omega^1_D(\mathcal{A})$ and $\mathcal{A}$ act on $\mathcal{H}$ as bounded operators in a compatible way (3.8), the r.h.s. of (A.5), viewed as $-(\eta \otimes \delta(a))\psi$, is made zero by adding to D_R an $\Omega^1_D(\mathcal{A})$-valued connection ∇ on $\mathcal{E}$. One thus defines the gauged operator

$$D_R(\eta \otimes \psi) := \eta \otimes D\psi + (\nabla\eta)\psi \quad \forall \eta \in \mathcal{E}_R, \psi \in \mathcal{H}, \tag{A.6}$$

and checks by Proposition 3.1 that this is linear, since

$$\begin{aligned} D_R(\eta a \otimes \psi) - D_R(\eta \otimes a\psi) &= D_R(\eta a \otimes \psi - \eta \otimes a\psi) + \nabla(\eta a)\psi - (\nabla\eta)a\psi, \\ &= -\eta \otimes [D, a]\psi + \eta \otimes \delta(a)\psi = 0. \end{aligned} \tag{A.7}$$

If the right $\mathcal{A}$-module $\mathcal{E}_R$ is finite projective thus of the type $\mathcal{E}_R = p\mathcal{A}^N$ for some $N \in \mathbb{N}$, with p a self-adjoint matrix in $M_N(\mathcal{A})$ such that $p^2 = p$. Moreover, given a derivation δ of $\mathcal{A}$ in a $\mathcal{A}$-bimodule Ω, any Ω-valued connection is of the form

$$\nabla = \nabla_0 + \boldsymbol{\omega} \tag{A.8}$$

where

$$\nabla_0\, \eta = p \begin{pmatrix} \delta(\eta_1) \\ \vdots \\ \delta(\eta_N) \end{pmatrix} \quad \forall\, \eta = p \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_N \end{pmatrix} \in \mathcal{E}_R, \quad \eta_j \in \mathcal{A}, \tag{A.9}$$

is the Grassmann connection, while $\boldsymbol{\omega}$ is an $\mathcal{A}$-linear map $\mathcal{E}_R \to \mathcal{E}_R \otimes \Omega$, that is

$$\boldsymbol{\omega}(\eta a) = \boldsymbol{\omega}(\eta) \cdot a \quad \forall \eta \in \mathcal{E}_R, a \in \mathcal{A}. \tag{A.10}$$

In particular, for a self-Morita equivalence, the operator D_R has a friendlier form.

Proposition A.1 *For $\mathcal{B} = \mathcal{A}$ and $\mathcal{E}_R = \mathcal{A}$, one obtains*

$$D_R = D + \omega \quad \textit{for some} \quad \omega \in \Omega^1_D(\mathcal{A}). \tag{A.11}$$

Proof Any connection ∇ on $\mathcal{E}_R = \mathcal{A}$ is written as $\nabla = \delta + \omega$ for a 1-form $\omega \in \Omega^1_D(\mathcal{A})$. Then,

$$D_R(a \otimes \psi) = a \otimes D\psi + \mathbb{1} \otimes (\delta(a) + \omega a)\psi. \tag{A.12}$$

Identifying $a \otimes \psi \in \mathcal{H}_R$ with $a\psi \in \mathcal{H}$, one rewrites (A.12) as

$$D_R(a\psi) = aD\psi + \delta(a)\psi + \omega a\psi = aD\psi + (Da - aD)\psi + \omega a\psi = (D+\omega)(a\psi), \tag{A.13}$$

meaning that the action of D_R on $\mathcal{H}$ coincides with the operator $D + \omega$. □

Since ω is bounded, the operator D_R has a compact resolvent and bounded commutator with $\mathcal{A}$. Consequently, for a self-adjoint ω one gets that

$$(\mathcal{A}, \mathcal{H}, D + \omega) \tag{A.14}$$

is a spectral triple [1], *Morita equivalent* to $(\mathcal{A}, \mathcal{H}, D)$. Furthermore, any grading Γ of $(\mathcal{A}, \mathcal{H}, D)$, since anticommutes with any $a[D, b]$, hence with ω, thus with D_R, is also a grading of $(\mathcal{A}, \mathcal{H}, D_R)$.

However, if $(\mathcal{A}, \mathcal{H}, D)$ is a real spectral triple with real structure J, the later is not necessarily a real structure for (A.14). Indeed, $J(D+\omega) = \epsilon'(D+\omega)J$ if and only if $\omega = \epsilon' J\omega J^{-1}$. This has no reason to be true, because of the following lemma [whose proof follows from (2.2), (2.3)].

Lemma A.2 *Let* $(\mathcal{A}, \mathcal{H}, D)$, J *be a real spectral triple, and* $\omega = \sum_j a_j[D, b_j] \in \Omega^1_D(\mathcal{A})$. *Then*

$$J\omega J^{-1} = \epsilon' \left(\sum_j (a_j^*)^\circ [D, (b_j^*)^\circ] \right). \tag{A.15}$$

A.1.2 Morita equivalence by left module

To implement $\mathcal{A}$ self-Morita equivalence in a way compatible with the real structure, one uses $\mathcal{A}$ not only as a right $\mathcal{A}$-module $\mathcal{E}_R$, but also as a left $\mathcal{A}$-module $\mathcal{E}_L$ (as explained in this section), then as a $\mathcal{A}$-bimodule $\mathcal{E}$ (this is the content of Sect. A.1.3).

In defining the Hilbert space $\mathcal{H}_R$ in (A.1), one takes advantage of the left $\mathcal{A}$-module structure of $\mathcal{H}$ induced by the representation π. Alternatively, one has available the right $\mathcal{A}$-module structure (2.4) of $\mathcal{H}$, $\psi a = a^\circ \psi$ for $\psi \in \mathcal{H}, a \in \mathcal{A}$, which offers a possibility to implement the Morita equivalence between $\mathcal{A}$ and $\mathcal{B}$ thanks to a hermitian finite projective *left* $\mathcal{A}$-module $\mathcal{E}_L$, with $\mathcal{A}$-valued inner product ${}_\bullet\langle \cdot, \cdot \rangle$. One thus considers the Hilbert space

$$\mathcal{H}_L := \mathcal{H} \otimes_{\mathcal{A}} \mathcal{E}_L, \tag{A.16}$$

with inner product

$$\langle \psi_1 \otimes \eta_1, \psi_2 \otimes \eta_2 \rangle_{\mathcal{H}_L} = \langle \psi_1 \, {}_\bullet\langle \eta_1, \eta_2 \rangle \, , \psi_2 \rangle_{\mathcal{H}}. \tag{A.17}$$

The right action of $\mathcal{B} \simeq \mathrm{End}_{\mathcal{A}}(\mathcal{E})$ on $\mathcal{E}_L$ is extended to $\mathcal{H}_L$ as

$$(\psi \otimes \eta)\, b := \psi \otimes \eta b. \tag{A.18}$$

Again, the natural action

$$D_L(\psi \otimes \eta) := D\psi \otimes \eta \tag{A.19}$$

of D on $\mathcal{H}_L$ is not compatible with the tensor product over $\mathcal{A}$ because

$$\begin{aligned} D_L(\psi \otimes a\eta) - D_L(\psi a \otimes \eta) &= (D\psi) \otimes a\eta - D(\psi a) \otimes \eta = (D\psi)a \otimes \eta - D(\psi a) \otimes \eta, \\ &= a^\circ(D\psi) \otimes \eta - D(a^\circ\psi) \otimes \eta = -[D, a^\circ]\psi \otimes \eta \end{aligned} \tag{A.20}$$

does not vanish. To correct this, one uses a connection ∇° on $\mathcal{E}_L$ with value in the $\mathcal{A}$-bimodule

$$\Omega^1_D(\mathcal{A}^\circ) = \left\{ \sum\nolimits_j a_j^\circ[D, b_j^\circ], \quad a_j^\circ, b_j^\circ \in \mathcal{A}^\circ \right\} \tag{A.21}$$

generated by the derivation

$$\delta^\circ(a) := [D, a^\circ], \tag{A.22}$$

with bimodule law

$$a \cdot \omega^\circ \cdot b := b^\circ \omega^\circ a^\circ. \tag{A.23}$$

This law guarantees that (A.21) is indeed a bimodule over $\mathcal{A}$ and δ° a derivation of $\mathcal{A}$ (not of $\mathcal{A}^\circ$) with values in $\Omega^1_D(\mathcal{A}^\circ)$. The relation between $\Omega^1_D(\mathcal{A}^\circ)$ and $\Omega^1_D(\mathcal{A})$ is given by the following lemma, whose proof follows from (2.2) and (2.3).

Lemma A.3 *Any $\omega^\circ = \sum_j a_j^\circ[D, b_j^\circ]$ in $\Omega^1_D(\mathcal{A}^\circ)$ acts on the left on $\mathcal{H}$ as the bounded operator*

$$\omega^\circ = \epsilon' J \omega J^{-1} \tag{A.24}$$

for $\omega = \sum_j a_j^*[D, b_j^*] \in \Omega^1_D(\mathcal{A})$.

The right action of $\mathcal{A}$ and the left action of $\Omega^1_D(\mathcal{A}^\circ)$ on $\mathcal{H}$ (corresponding to a right action of $\Omega^1_D(\mathcal{A})$) are compatible in the sense of condition (3.16):

$$(a \cdot \omega^\circ)\psi = (\omega^\circ a^\circ)\psi = \omega^\circ(\psi a). \tag{A.25}$$

The connection ∇° thus defines an operator $\mathcal{H}_L \to \mathcal{H}_L$ which satisfies the Leibniz rule (3.17); therefore, the following is a well-defined operator on $\mathcal{H}_L$,

$$D_L(\psi \otimes \eta) := D\psi \otimes \eta + \nabla^\circ(\psi \otimes \eta). \tag{A.26}$$

For a left module $\mathcal{E}_L \simeq A^N p$ with $p = p^2 \in M_N(\mathcal{A})$, the connection decomposes as

$$\nabla^\circ = \nabla^\circ_0 + \boldsymbol{\omega}^\circ \tag{A.27}$$

with Grassmann connection

$$\nabla^\circ_0 \eta = (\delta^\circ(\eta_1), \ldots, \delta^\circ(\eta_N))\, p \quad \forall \eta = (\eta_1, \ldots, \eta_N) \in \mathcal{E}_L, \quad \eta_j \in \mathcal{A}, \tag{A.28}$$

while $\boldsymbol{\omega}^\circ$ is a map $\mathcal{E}_L \to \Omega^1_D(\mathcal{A}^\circ) \otimes_{\mathcal{A}} \mathcal{E}_L$ which is $\mathcal{A}$-linear in the sense that

$$\boldsymbol{\omega}^\circ(a\eta) = a \cdot \boldsymbol{\omega}^\circ(\eta). \tag{A.29}$$

We use this to get a more tractable expression for D_L, for a self-Morita equivalence.

Proposition A.4 *For $\mathcal{B} = \mathcal{A}$ and $\mathcal{E}_L = \mathcal{A}$, the construction above yields*

$$D_L = D + \omega^\circ = D + \epsilon' J \omega J^{-1} \tag{A.30}$$

for some $\omega^\circ = \epsilon' J \omega J^{-1} \in \Omega^1_D(\mathcal{A}^\circ)$, with $\omega \in \Omega^1_D(\mathcal{A})$.

Proof The decomposition (A.27) will now read $\nabla^\circ = \delta^\circ + \omega^\circ$ with the form ω° be such that

$$\boldsymbol{\omega}^\circ(a) = a \cdot \omega^\circ = (a \cdot \omega^\circ) = \omega^\circ a^\circ. \tag{A.31}$$

Therefore;

$$\begin{aligned} D_L(\psi \otimes a) &= D\psi \otimes a + \delta^\circ(a)\psi \otimes \mathbb{1} + \omega^\circ a^\circ \psi \otimes \mathbb{1}, \\ &= (D\psi)a \otimes \mathbb{1} + (Da^\circ - a^\circ D)\psi \otimes \mathbb{1} + \omega^\circ a^\circ \psi \otimes \mathbb{1}, \\ &= a^\circ D\psi \otimes \mathbb{1} + (Da^\circ - a^\circ D)\psi \otimes \mathbb{1} + \omega^\circ a^\circ \psi \otimes \mathbb{1} \\ &= D(a^\circ \psi) \otimes \mathbb{1} + \omega^\circ a^\circ \psi \otimes \mathbb{1}. \end{aligned} \tag{A.32}$$

Identifying $a^\circ \psi \otimes \mathbb{1} = \psi \otimes a = \psi a \otimes \mathbb{1}$ in $\mathcal{H}_L$ with $a^\circ \psi \in \mathcal{H}$, one obtains that D_L acts as $D + \omega^\circ$. The rest of the result follows from Lemma A.2. □

As in the right module case, when ω is self-adjoint the datum

$$(\mathcal{A}, \mathcal{H}, D + \epsilon J \omega J^{-1}) \tag{A.33}$$

is a spectral triple, admitting as grading any grading of $(\mathcal{A}, \mathcal{H}, D)$. However, it is not a real spectral triple for the real structure J, because $J(D + \epsilon' J \omega J^{-1}) = (D + \epsilon' J \omega J^{-1})J$ if and only if $\omega = \epsilon' J \omega J^{-1}$. This has no reason to be true, by Lemma A.2.

A.1.3 Morita equivalence by bimodule and the real structure

To make the real structure compatible with Morita equivalence of spectral triples, one needs to combine the two constructions above. Explicitly, given a real spectral triple $(\mathcal{A}, \mathcal{H}, D)$, one first implements the self-Morita equivalence of $\mathcal{A}$ by using the right module $\mathcal{E}_R = \mathcal{A}$—thus obtaining the spectral triple (A.14), and then uses this with the left module $\mathcal{E}_L = \mathcal{A}$. This yields the Morita equivalent spectral triple $(\mathcal{A}, \mathcal{H}, D')$ where

$$D' = D + \omega_R + \epsilon' J \omega_L J^{-1} \tag{A.34}$$

with ω_R, ω_L two self-adjoint elements of $\Omega^1_D(\mathcal{A})$, a priori distinct. It is the real structure that forces these two 1-forms to be equal.

Proposition A.5 *The real structure J of $(\mathcal{A}, \mathcal{H}, D)$ is a real structure for the Morita equivalent spectral triple $(\mathcal{A}, \mathcal{H}, D')$ if and only if there exists $\omega \in \Omega^1_D(\mathcal{A})$ such that*

$$D' = D_\omega := D + \omega + \epsilon' J \omega J^{-1}. \tag{A.35}$$

Proof By an easy computation, one finds that $JD' = \epsilon' DJ$ if and only if

$$(\omega_L - \omega_R) - \epsilon' J(\omega_L - \omega_R)J^{-1} = 0. \tag{A.36}$$

Adding half of this expression to the r.h.s. of (A.34), one gets

$$D' = D + \tfrac{1}{2}(\omega_R + \omega_L) + \epsilon' J \tfrac{1}{2}(\omega_R + \omega_L)J^{-1}. \tag{A.37}$$

Hence the result with $\omega := \frac{1}{2}(\omega_R + \omega_L)$. □

Remark A.6 Taking as a connection the Grassmann connection in the definition (A.6) of D_R (i.e. $\nabla = \nabla_0$), one finds that D_R coincides with D. Similarly, taking $\nabla^\circ = \nabla_0^\circ$ in (A.26) yields $D_L = D$. Then D' in (A.34) coincide with D as well. In other terms, given a real spectral triple $(\mathcal{A}, \mathcal{H}, D)$, implementing the self-Morita equivalence of $\mathcal{A}$ using the Grassmann connection on the $\mathcal{A}$-bimodule $\mathcal{A}$ leaves the Dirac operator invariant (i.e., it fluctuates with $\omega = 0$).

A.2 Gauge transformations

Also the material in this section is well known and mainly taken from [10] and [8].

A.2.1 Gauge transformations on a hermitian module

An endomorphisms $u \in \mathrm{End}_{\mathcal{A}}(\mathcal{E})$ of a hermitian $\mathcal{A}$-module $\mathcal{E}$ is unitary if $u^*u = uu^* = \mathrm{id}_{\mathcal{E}}$, where the adjoint of an operator is defined using the hermitian structure by

$$\langle T^*\eta, \xi\rangle := \langle \eta, T\xi\rangle, \quad \forall T \in \mathrm{End}_{\mathcal{A}}(\mathcal{E}), \quad \xi, \eta \in \mathcal{E}. \tag{A.38}$$

Unitary endomorphisms form a group $\mathcal{U}(\mathcal{E})$, acting on Ω-valued connections on $\mathcal{E}$ as

$$\nabla^u := u\nabla u^* \quad \forall u \in \mathcal{U}(\mathcal{E}), \tag{A.39}$$

where $\mathcal{U}(\mathcal{E})$ acts on $\mathcal{E} \otimes \Omega$ (if $\mathcal{E}$ is a right $\mathcal{A}$-module) or $\Omega \otimes \mathcal{E}$ (if $\mathcal{E}$ is a left $\mathcal{A}$-module) as

$$u \otimes \mathrm{id}_\Omega, \quad \text{or} \quad \mathrm{id}_\Omega \otimes u. \tag{A.40}$$

Not surprisingly, such an action is a *gauge transformation*.

Proposition A.7 *The operator ∇^u is a connection, for any $u \in \mathcal{U}(\mathcal{E})$ and connection ∇.*

Proof In case $\mathcal{E}$ is a right $\mathcal{A}$-module, one gets from (A.40) and (3.3) that

$$u(\nabla(\eta) \cdot a) = (u\nabla(\eta)) \cdot a\,. \tag{A.41}$$

Hence,

$$\nabla^u(\eta a) = u\nabla(u^*(\eta a)) = u\nabla(u^*(\eta)a) = u\left(\nabla(u^*(\eta)).a + u^*(\eta) \otimes \delta(a)\right),$$

$$= (u\nabla(u^*(\eta))).a + \eta \otimes \delta(a) = \nabla^u(\eta).a + \eta \otimes \delta(a),$$

showing that ∇^u is a connection. For a left $\mathcal{A}$-module $\mathcal{E}$ one has from (3.12)

$$u(a \cdot \nabla(\eta)) = a \cdot u(\nabla(\eta)), \tag{A.42}$$

so that

$$\begin{aligned} \nabla^u(a\eta) &= u\nabla(u^*(a\eta)) = u\nabla(au^*(\eta)) = u\left(a \cdot \nabla(u^*(\eta)) + \delta(a) \otimes u^*(\eta)\right), \\ &= a \cdot u\nabla(u^*(\eta)) + \delta(a) \otimes \eta = \nabla^u(\eta).a + \delta(a) \otimes \eta. \end{aligned}$$

Hence the result. □

With ∇_0 the Grassmann connections an $\boldsymbol{\omega}$ defined in (A.8) or (A.27), any connection

$$\nabla = \nabla_0 + \boldsymbol{\omega} \tag{A.43}$$

is mapped under a gauge transformation to

$$\nabla^u = \nabla_0 + \boldsymbol{\omega}^u. \tag{A.44}$$

with the gauge transformation fully encoded in the law of transformation of the *gauge potential*

$$\boldsymbol{\omega} \to \boldsymbol{\omega}^u. \tag{A.45}$$

Explicitly, given a right (or left) $\mathcal{A}$-module $\mathcal{E}_R = p\mathcal{A}^N$ (or $\mathcal{E}_L = \mathcal{A}^N p$), a unitary endomorphism is a unitary matrix in $M_N(\mathcal{A})$ that commutes with p,

$$\mathcal{U}(\mathcal{E}_{L,R}) := \left\{u \in M_N(\mathcal{A}),\ [u, p] = 0,\ u^*u = \mathrm{id}_{\mathcal{E}}\right\}, \tag{A.46}$$

and acts by ordinary matrix multiplication

$$u(\eta) := p(u\eta) \ \text{ for } \ \eta \in \mathcal{E}_R, \quad u(\eta) := (\eta u^*)p \ \text{ for } \ \eta \in \mathcal{E}_L. \tag{A.47}$$

The choice to act with u^* instead of u in the left-module case is discussed in Remark A.9.

Given a derivation δ of $\mathcal{A}$, we denote by $\delta(u)$, $\delta(u^*)$ the elements of $M_N(\Omega)$ with components $\delta(u_{ij})$ or $\delta(u^*_{ij}) \in \Omega$, $1 \leq i, j \leq N$, where $u_{ij}, u^*_{ij} \in \mathcal{A}$ are the components of u, u^*.

Proposition A.8 *The gauge transformations on right and left modules are given by*

$$\boldsymbol{\omega}^u(\eta) := p\,u \cdot \delta(u^*) \cdot \eta + u(\boldsymbol{\omega}(u^*(\eta))) \qquad \forall \eta \in \mathcal{E}_R, \tag{A.48}$$

$$\boldsymbol{\omega}^u(\eta) := \eta \cdot \delta(u) \cdot u^*\,p + u(\boldsymbol{\omega}(u^*(\eta))) \qquad \forall \eta \in \mathcal{E}_L. \tag{A.49}$$

Proof For $\eta = p(\eta_j) \in \mathcal{E}_R$ (with $\eta_j \in \mathcal{A}$), using that p commutes with u^* and $p\eta = \eta$, one gets

$$\begin{aligned}\nabla_0(u^*(\eta)) &= \nabla_0(pu^*\eta) = \nabla_0(u^*p\eta) = \nabla_0(u^*\eta)\\ &= p\begin{pmatrix}\delta(u^*_{1j}\eta_j)\\ \vdots\\ \delta(u^*_{Nj}\eta_j)\end{pmatrix} = p\begin{pmatrix}\delta(u^*_{1j})\cdot\eta_j + u^*_{1j}\cdot\delta(\eta_j)\\ \vdots\\ \delta(u^*_{Nj})\cdot\eta_j + u^*_{Nj}\cdot\delta(\eta_j)\end{pmatrix}\\ &= p\delta(u^*)\cdot\eta + u^*\nabla_0(\eta),\end{aligned}$$

with summation on the index $j = 1, \ldots, N$. Acting with u on the left, one gets

$$\nabla_0^u = \nabla_0 + p\,u\cdot\delta(u^*). \tag{A.50}$$

the result follows from (A.43), (A.44). Similarly, for $\eta \in \mathcal{E}_L$, one has

$$\begin{aligned}\nabla_0(u^*(\eta)) &= \big(\delta(\eta_j u_{j1}), \ldots, \delta(\eta_j u_{jN})\big)\,p\\ &= \big(\delta(\eta_j)\cdot u_{j1} + \eta_j\cdot\delta(u_{j1}), \ldots, \delta(\eta_j)\cdot u_{jN} + \eta_j\cdot\delta(u_{jN})\big)\,p\\ &= \nabla_0(\eta)\cdot u + \eta\cdot\delta(u)p.\end{aligned}$$

Acting with the endomorphism u on the left, which by (A.47) amounts to multiply by the matrix u^* on the right, one obtains

$$\nabla_0^u(\eta) = \nabla_0(\eta) + \eta\cdot\delta(u)\cdot u^*\,p. \tag{A.51}$$

Hence the result. □

A.2.2 Gauge transformation for a spectral triple

Let $(\mathcal{A}, \mathcal{H}, D)$ be a real spectral triple, and consider the right $\mathcal{A}$-module $\mathcal{E}_R = \mathcal{A}$, with derivation $\delta(\cdot) = [D, \cdot]$ in the $\mathcal{A}$-bimodule $\Omega^1_D(\mathcal{A})$ defined in (1.2). The equation (A.48) yields the usual law of transformation of the gauge potential,

$$\omega^u = u[D, u^*] + u\,\omega\,u^*. \tag{A.52}$$

Under a gauge transformation, the gauged Dirac operator D_ω in (1.3) is thus mapped to

$$D_{\omega^u} = D + \omega^u + \epsilon' J\omega^u J^{-1}. \tag{A.53}$$

Remark A.9 To write (A.53), one applies the gauge transformation $\omega \to \omega^u$ on the operator D_ω obtained in Proposition A.5, that is once ω_L and ω_R have been identified.

For the sake of coherence, let us check that the same result follows by applying the gauge transformation on ω_L and ω_R independently. Consider the left module $\mathcal{E}_L = \mathcal{A}$ with derivation $\delta^\circ(a) = [D, a^\circ]$ in $\Omega^1_D(\mathcal{A}^\circ)$ defined in (A.21). By Lemma A.3, a gauge potential in $\Omega^1_D(\mathcal{A}^\circ)$ is $\omega^\circ = \epsilon' J\omega_L J^{-1}$ with $\omega_L \in \Omega^1_D(\mathcal{A})$. The law of transformation (A.49) reads

$$\begin{aligned}\omega^{\circ u} &= \delta^\circ(u)\cdot u^* + u\cdot\omega^\circ\cdot u^* = u^{*\circ}\delta^\circ(u) + u^{*\circ}\omega^\circ u^\circ,\\ &= u^{*\circ}[D, u^\circ] + u^{*\circ}\omega^\circ u^\circ = \epsilon' Ju[D,u^*]J^{-1} + \epsilon' J\, u\,\omega_L u^* J^{-1} = \epsilon' J\,\omega_L^u\, J^{-1}.\end{aligned}$$

Thus, the operator $D + \omega_R + \epsilon' J\omega_L J^{-1}$ in Proposition A.5 is mapped under a gauge transformation to $D + \omega_R^u + \epsilon' J\omega_L^u J^{-1}$, meaning that $\omega = \frac{1}{2}(\omega_R + \omega_L)$ is mapped to

$$\tfrac{1}{2}(\omega_R^u + \omega_L^u) = u[D,u^*] + u\frac{1}{2}(\omega_R + \omega_L)u^* = u[D,u^*] + u\omega u^*. \tag{A.54}$$

One thus finds back (A.52), as expected.

Remarkably [10], the gauge transformation $D_\omega \to D_{\omega^u}$ can be retrieved from the adjoint action on $\mathcal{H}$ of the unitary group of $\mathcal{A}$, defined by using the real structure. That is, for any unitary element $u \in \mathcal{A}$, $u^*u = uu^* = \mathbb{1}$, one defines

$$\mathrm{Ad}(u)\psi := u\,\psi\,u^* = uJuJ^{-1}\psi \quad \forall\psi \in \mathcal{H}. \tag{A.55}$$

Under this action, the Dirac operator is mapped to $\mathrm{Ad}(u)\, D\, \mathrm{Ad}(u)^{-1}$. By the order-zero and the first-order conditions, one shows that [8, Prop. 1.141]

$$\mathrm{Ad}(u)\, D\, \mathrm{Ad}(u)^{-1} = D + u[D,u^*] + \epsilon' Ju[D,u^*]J^{-1}, \tag{A.56}$$

which is nothing but the operator D_{ω^u} of (A.53) obtained for $\omega = 0$ so that $\omega^u = u[D,u^*]$ from (A.52). More generally, for a gauged Dirac operator

$$D_\omega = D + \omega + \epsilon'\, J\omega J^{-1} \tag{A.57}$$

where ω is an arbitrary self-adjoint element of $\Omega^1_D(\mathcal{A})$, one has [8, Prop. 1.141])

$$\mathrm{Ad}(u)\, D_\omega\, \mathrm{Ad}(u)^{-1} = D_{\omega^u} \tag{A.58}$$

with ω^u defined in (A.52).

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