

Topologies for the Continuous Representability of All Continuous Total Preorders

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Abstract In this paper, we present a new simple axiomatization of *useful topologies*, i.e. topologies on an arbitrary set, with respect to which every con-

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This paper is dedicated to the memory of Professor Gerhard Herden, who passed away on January 30, 2019. He was a friend and an exceptionally clever mathematician. We are deeply indebted to him.

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tinuous total preorder admits a continuous utility representation. In particular, we show that, for completely regular spaces, a topology is useful, if and only if it is separable, and every *isolated chain* of open and closed sets is countable. As a specific application to optimization theory, we characterize the continuous representability of all continuous total preorders, which admit both a maximal and a minimal element.

Keywords Useful topology · Complete Separable system · Weak topology · Completely regular space

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1 Introduction

From Herden [1], a topology on any fixed chosen set is said to be *useful* if every *continuous total preorder* has a continuous utility representation, i.e. it can be represented by a continuous real-valued order-preserving function (utility function) (see, e.g., Herden [2,3], and Herden and Pallack [4]). Continuity of the preorder means that the given topology is finer than the *order topology* induced by the preorder.

Other authors call *continuously representable* the topologies satisfying the aforementioned property (see, e.g., Candeal et al. [5], and Campi3n et al. [6–9]). In this paper, we prefer the original terminology of a useful topology, inherited from the seminal paper Herden [1], who first explicitly inaugurated and started a systematic study of this concept.

In some sense, the problem of characterizing all useful topologies on a set

can be thought as the most fundamental problem in utility theory. Indeed, the classical theorems by Eilenberg [10] (**ET**) and Debreu [11, 12] (**DT**), according to which every continuous total preorder on a *connected and separable*, and, respectively, on a *second countable* topological space, actually present sufficient conditions in order that a topology is useful.

Another very important result, in the spirit of the previous considerations, was proven by Estévez and Hervés [13], who showed that in every non-separable metric space there is a continuous total preorder, which does not admit a (continuous) utility function. This result, in combination with **DT**, can be used to state that a metrizable topology is useful if and only if it is second countable, or equivalently *separable*. By the way, we recall that the well known *Debreu Open Gap Lemma* (see Debreu [11, 12]) guarantees the equivalence, for a continuous total preorder, of admitting a utility function on one hand, and of admitting a continuous utility function on the other hand. We will refer to this latter result as Estévez-Hervés' theorem (**EHT**) (see also Candeal et al. [5, Theorem 1]).

With help of the concept of a useful topology, the fundamental theorems above can be restated as follows (see the introduction in Bosi and Herden [14]):

ET: *Every connected and separable topology is useful.*

DT: *Every second countable topology is useful.*

EHT: *A metrizable topology is useful if and only if it is separable, or equivalently second countable*

On the other hand, it is well known that second countability or separability, in general, is not necessary for t to be useful. For instance, the *Niemitzki plane*, that is extensively discussed in Steen and Seebach [15], is useful, since it is second countable, even if it is not separable.

Different characterizations of useful (or representable) topologies appear in the literature. In particular, Campi3n et al. [8, Theorem 5.1] proved that a topology is useful if and only if all its *preorderable subtopologies* are second countable, where a topology is preorderable if it is the order topology of some continuous total preorder. Bosi and Herden [14, Theorem 3.1] showed that a topology is useful if and only if the topology generated by every *complete separable system* is second countable. In particular, this latter result can be regarded as the simplest axiomatization of useful topologies, even if it, nevertheless, invokes the concept of a complete separable system (see Bosi and Zuanon [16, Definition 2.20]).

To the best of our knowledge, there is not in the literature any axiomatization of a useful topology, which directly refers to appropriate conditions relative of the considered topology, without referring to properties of some subtopologies. This paper is aimed at filling this gap. Indeed, we show that, for completely regular spaces, a topology is useful, if and only if it is separable, and every *isolated chain* of open and closed sets is countable. The concept of an isolated chain (i.e., a chain containing only *isolated sets*) of open and closed subsets is already found in Herden [4], and it is denoted by *OCCC* in the present paper.

The paper is structured as follows. Section 2 presents the main definitions and the preliminary results. Section 3 contains the main new results concerning the formulation of, so to say, purely topological characterizations of useful topologies. In particular, we prove a fundamental result, generalizing the aforementioned Estévez and Hervés' [13] theorem, according to which a *completely regular* useful topology is separable (see Theorem 3.1, point 3). We show that the Eilenberg and, respectively, the Debreu utility representation theorem are immediate corollaries of our result. Finally, Section 4 is devoted to applications to optimization theory. In particular, Theorem 4.2 below shows that a topology is useful, and every continuous total preorder has both a minimal and a maximal element, if and only if the weak topology of continuous functions is compact, separable, and satisfies *OCCC*.

2 Notation and Preliminary Results

A *preorder* \preceq on a nonempty set X is a *reflexive* and *transitive* binary relation on X . A preorder \preceq on X is said to be *total* if, for all $x, y \in X$, either $x \preceq y$ or $y \preceq x$. The *strict part* (or *asymmetric part*) of a preorder \preceq on X is defined as follows, for all $x, y \in X$: $x \prec y$ if and only if $(x \preceq y)$ and *not* $(y \preceq x)$. Further, the *symmetric part* \sim of a preorder \preceq on X is defined as follows, for all $x, y \in X$: $x \sim y$ if and only if $(x \preceq y)$ and $(y \preceq x)$. We have that \sim is an *equivalence* on X , and we denote by X_{\sim} the *quotient set*, made up by the equivalence classes $[x] = \{z \in X : z \sim x\}$ ($x \in X$).

An *order* \preceq on X is a preorder which in addition is *antisymmetric* (i.e., for

all $x, y \in X$, $(x \preceq y)$ and $(y \preceq x)$ implies that $x = y$). An order \preceq on a set X is said to be *linear* (or *complete*) if, for all $x \neq y$ ($x, y \in X$), either $x \preceq y$ or $y \preceq x$. If \succsim is a preorder on X , then the *quotient order* $\succsim_{|\sim}$ on the quotient set $X_{|\sim}$ is defined as follows, for all $x, y \in X$: $[x] \succsim_{|\sim} [y] \Leftrightarrow x \succsim y$. If \succsim is a total preorder on X , then $\preceq := \succsim_{|\sim}$ is a linear order on $X_{|\sim}$.

A subset D of a *preordered set* (X, \succsim) is said to be *decreasing* if $(x \in D)$ and $(z \succsim x)$ imply $z \in D$, for all $z \in X$.

If t is a *topology* on X , then a family $\mathcal{B}' \subset t$ is said to be a *subbasis* of t , if the family \mathcal{B} consisting of all possible intersections of finitely many elements of \mathcal{B}' is a *basis* of t (i.e., every set $O \in t$ is the union of some sets of \mathcal{B}).

A topology t on X is said to be *second countable* if there is a countable basis $\mathcal{B} = \{B_n : n \in \mathbb{N}^+\}$ for t . Further, a topology t on X is said to be *separable*, if there exists a countable subset D of X such that $D \cap O \neq \emptyset$ for every $O \in t$.

We recall that, for two topologies t', t on set X , t' is said to be *coarser* (respectively, *finer*) than t if it happens that $t' \subset t$ ($t \subset t'$). If t' is coarser than t , we also say that t' is a *subtopology* of t .

Further, a topological space (X, t) is said to be

- (i) T_1 , if all singleton sets $\{x\}$ are closed;
- (ii) *Hausdorff*, if, given any two points $x, y \in X$ with $x \neq y$, there exist two open disjoint sets U, V such that $x \in U$ and $y \in V$;

- (iii) *completely regular*, if it is T_1 and, for every $x \in X$, and every closed set $F \subseteq X$ not containing x , there exists a continuous function $f : X \rightarrow [0, 1]$ such that $f(x) = 0$ and $f(y) = 1$ for $y \in F$;
- (iv) *normal*, if it is T_1 and, for every pair (A, B) of disjoint closed subsets of X , there exist two open disjoint sets U, V such that $A \subseteq U, B \subseteq V$.

Let (X, \preceq) be an arbitrarily chosen *preordered set*. We define, for every point $x \in X$, the following subsets of X :

$$d_{\preceq}(x) := \{z \in X : z \preceq x\}, \quad i_{\preceq}(x) := \{z \in X : x \preceq z\},$$

$$l_{\preceq}(x) := \{z \in X : z \prec x\}, \quad r_{\preceq}(x) := \{z \in X : x \prec z\}.$$

For any pair $(x, y) \in X \times X$ such that $(x, y) \in \prec$, we shall denote by $]x, y[_{\preceq}$ the (maybe empty) *open interval* defined as $]x, y[_{\preceq} := r_{\preceq}(x) \cap l_{\preceq}(y)$.

A pair $(x, y) \in \prec$ is said to be a *jump* in (X, \preceq) if $]x, y[_{\preceq} = \emptyset$.

All the previous definitions can be found, for example, in Herden [2].

A total preorder \preceq on the topological space (X, t) is said to be *continuous* if the sets $l_{\preceq}(x) = \{z \in X : z \prec x\}$ and $r_{\preceq}(x) = \{z \in X : x \prec z\}$ are open subsets of X for every $x \in X$. Equivalently, this is the case when t is *finer* than the *order topology* t^{\preceq} on X associated to \preceq , which is precisely the topology *generated* by the family $\{l_{\preceq}(x) : x \in X\} \cup \{r_{\preceq}(x) : x \in X\}$ (i.e., $\{l_{\preceq}(x) : x \in X\} \cup \{r_{\preceq}(x) : x \in X\}$ is a subbasis of t). In other words, t^{\preceq} is the coarsest topology on X such that the sets $l_{\preceq}(x)$ and $r_{\preceq}(x)$ are open for every $x \in X$.

If t is a topology on X , and X' is any nonempty subset of X , then the *relativized topology* $t|_{X'}$ on X' is defined as follows: $t|_{X'} := \{O \cap X' : O \in t\}$.

A real-valued function u on a preordered set (X, \preceq) is said to be

1. *increasing*, if, for all $x, y \in X$,

$$x \preceq y \Rightarrow u(x) \leq u(y);$$

2. *order-preserving*, if u is increasing and, for all $x, y \in X$,

$$x \prec y \Rightarrow u(x) < u(y).$$

A topology t on X is said to be *useful* (see Herden [1]), if every continuous total preorder on the topological space (X, t) has a *continuous utility representation* (*order-preserving function*) u , i.e., there exists a continuous real-valued function u on the *preordered topological space* (X, \preceq, t) , such that $x \preceq y$ is equivalent to $u(x) \leq u(y)$, for all $x, y \in X$.

We shall denote by t_{nat} the *natural* (*interval*) topology on the real line \mathbb{R} .

The following definition is found in Herden and Pallack [17].

Definition 2.1 A preorder \preceq on a topological space (X, t) is said to be *weakly continuous* if, for every pair $(x, y) \in \prec$, there exists a continuous and increasing real-valued function u_{xy} on X , such that $u_{xy}(x) < u_{xy}(y)$.

We now recall the definition of a *complete separable system* on a topological space (X, t) .

The following definition is found in Bosi and Herden [14].

Definition 2.2 Let a topology t on X be given. A family \mathcal{E} of open subsets of the topological space (X, t) , such that $\bigcup_{E \in \mathcal{E}} E = X$, is said to be a *complete separable system* on (X, t) if it satisfies the following conditions:

S1 : There exist sets $E_1 \in \mathcal{E}$ and $E_2 \in \mathcal{E}$, such that $\overline{E_1} \subset E_2$.

S2 : For all sets $E_1 \in \mathcal{E}$ and $E_2 \in \mathcal{E}$ such that $\overline{E_1} \subset E_2$, there exists some set

$E_3 \in \mathcal{E}$ such that $\overline{E_1} \subset E_3 \subset \overline{E_3} \subset E_2$.

S3 : For all sets $E \in \mathcal{E}$ and $E' \in \mathcal{E}$, at least one of the following conditions

$E = E'$, or $\overline{E} \subset E'$, or $\overline{E'} \subset E$ holds.

In the case when \preceq is a preorder on X , a complete separable system \mathcal{E} on (X, t) is said to be a *complete decreasing separable system* on (X, \preceq, t) as soon as every set $E \in \mathcal{E}$ is required to be decreasing.

The following theorem holds, presenting different conditions all equivalent to the continuity of a total preorder on a topological space (see Bosi and Zuanon [16, Theorem 2.23]).

Theorem 2.1 *Let (X, \preceq, t) be a totally preordered topological space. Then, the following conditions are equivalent:*

- (i) \preceq is continuous;
- (ii) The order topology t^{\preceq} is coarser than t ;
- (iii) $d_{\preceq}(x) = \{y \in X : y \preceq x\}$ is a closed subset of X , and $l_{\preceq}(x) = \{y \in X : y \prec x\}$ is an open subset of X , for every point $x \in X$;

(iv) $i_{\preceq}(x) = \{z \in X : x \preceq z\}$ is a closed subset of X , and $r_{\preceq}(x) = \{z \in X : x \prec z\}$ is an open subset of X , for every point $x \in X$;

(v) \preceq is weakly continuous;

(vi) For every pair $(x, y) \in \prec$, a complete decreasing separable system \mathcal{E}_{xy} on X can be chosen, in such a way that there exist sets $E \subset \overline{E} \subset E'$ in \mathcal{E}_{xy} such that $x \in E$ and $y \notin E'$.

The following theorem holds true, which provides a simple characterization of useful topologies (see Bosi and Herden [14, Theorem 3.1]).

Theorem 2.2 *The following assertions are equivalent on a topology t on set X :*

- (i) t is useful.
- (ii) For every complete separable system \mathcal{E} on (X, t) , the topology $t_{\mathcal{E}}$ generated by \mathcal{E} is second countable.

The following proposition is found in Herden [2, Lemma 3.2].

Proposition 2.1 *Let \preceq be a total preorder on a set X . Then, the following conditions are equivalent:*

- (i) There exists a utility function u on (X, \preceq) ;
- (ii) The following conditions are verified:
 - (a) The order topology t^{\preceq} on X is separable;

(b) *There are only countable many jumps in $(X_{|\sim}, \lesssim_{|\sim})$.*

The easy proof of the following Proposition is left to the reader.

Proposition 2.2 *Let (X, \lesssim) be a totally preordered set. Then, the following conditions are equivalent for all $x, y \in X$:*

(i) *A pair $([x], [y]) \in (X_{|\sim}, X_{|\sim})$ is a jump in $(X_{|\sim}, \lesssim_{|\sim})$;*

(ii) $l_{\lesssim}(y) = d_{\lesssim}(x)$.

3 Continuous Representability of All Continuous Total Preorders

We are going to furnish a very simple and direct characterization of useful topologies. To this aim, starting from a topological space (X, t) , denote by $\mathbb{O}\mathbb{C}(t)$ the set of all open and closed non-empty subsets of X , and let, in addition, \mathfrak{C} be set of all *chains* in $\mathbb{O}\mathbb{C}(t)$ (i.e., an element \mathcal{O} of \mathfrak{C} is a subset of $\mathbb{O}\mathbb{C}(t)$, which is totally ordered by set inclusion).

Two more definitions are needed.

Definition 3.1 Let \mathcal{O} be a chain of open and closed subsets of a topological space (X, t) . Then a set $O \in \mathcal{O}$ is said to be *isolated* if

$$\bigcup_{O' \subsetneq O, O' \in \mathcal{O}} O' \subsetneq O \subsetneq \bigcap_{O \subsetneq O'', O'' \in \mathcal{O}} O''.$$

Definition 3.2 A chain \mathcal{O} of open and closed subsets of a topological space (X, t) is said to be *isolated* if it only consists of isolated sets.

We now present the main definition of *open and closed countable chain condition* (OCCC), which is originally due to Herden [4].

Definition 3.3 We say that a topology t on a nonempty set X satisfies the *open and closed countable chain condition* (OCCC), if every isolated chain \mathcal{O} of open and closed subsets of X is countable.

The simple proof of the following proposition, which is based on Proposition 2.2, is left to the reader.

Proposition 3.1 *Let \preceq a continuous total preorder on a topological space (X, t) , and assume that there exists some jump $]x, y[$ in (X, \preceq) . Then, the family*

$$\mathcal{O} := \{l_{\preceq}(y) : y \text{ is the right endpoint of some jump }]x, y[\text{ in } (X, \preceq)\}$$

is an isolated family of open and closed subsets of X .

We are now ready to state and prove a very general result, characterizing useful topologies, at least in the completely regular case.

Theorem 3.1 *Let t be a topology on a nonempty set X . Then, the following statements hold true:*

1. *If t is separable and satisfies OCCC, then t is useful;*
2. *If t is useful, then t satisfies OCCC;*
3. *If t is a completely regular useful topology, then t is separable.*

Proof 1. In order to show that t is useful, if t is separable, and satisfies OCCC, consider any continuous total preorder \preceq on (X, t) . Since t is separable, we

have that the order topology $t^{\lesssim} (\subset t)$ is also separable. Further, by Proposition 3.1, there are only countably many jumps in $(X|_{\sim}, \lesssim|_{\sim})$, and the continuous total preorder \lesssim on (X, t) has a continuous utility representation by Proposition 2.1. Indeed, the aforementioned *Debreu Open Gap Lemma* (see Debreu [11, 12]) guarantees that a continuous total preorder, which admits a utility function, also admits a continuous utility function. This consideration completes this part of the proof.

2. In order to show that a useful topology t satisfies *OCCC*, assume, by contradiction, that there exists an uncountable isolated chain \mathcal{O} of open and closed sets. Consider the continuous total preorder $\lesssim_{\mathcal{O}}$ on (X, t) , defined to be, for all $x, y \in X$,

$$x \lesssim_{\mathcal{O}} y \Leftrightarrow \forall O \in \mathcal{O} (y \in O \Rightarrow x \in O).$$

A jump in $(X|_{\sim_{\mathcal{O}}}, \lesssim|_{\sim_{\mathcal{O}}})$ is associated to every isolated set $O \in \mathcal{O}$. Indeed, if O is an isolated set of \mathcal{O} , then, for every $x \in O \setminus \bigcup_{O' \subsetneq O, O' \in \mathcal{O}} O'$, and for every $y \in \bigcap_{O'' \not\subset O, O'' \in \mathcal{O}} O'' \setminus O$, the pair $([x]_{|_{\sim_{\mathcal{O}}}}, [y]_{|_{\sim_{\mathcal{O}}}})$ is a jump in $(X|_{\sim_{\mathcal{O}}}, \lesssim|_{\sim_{\mathcal{O}}})$. Since there are uncountably many jumps in $(X|_{\sim_{\mathcal{O}}}, \lesssim|_{\sim_{\mathcal{O}}})$, the continuous total preorder $\lesssim_{\mathcal{O}}$ doesn't admit a (continuous) utility representation by Proposition 2.1. Hence, t is not useful.

3. The proof of the fact that a completely regular and useful topology is separable is based upon some straightforward transfinite induction argument on all countable ordinal numbers $0 \leq \alpha < \Omega$, where Ω is the first uncountable ordinal number. In particular, at the generic step $\alpha - 1$, we are going to guarantee the existence of a countable set $D_{\alpha-1} \subset X$, thought of as endowed with the

relativized topology $t|_{D_{\alpha-1}}$, and of a continuous total preorder $\lesssim_{\alpha-1}$ on $D_{\alpha-1}$, in such a way that either $\overline{D_{\alpha-1}} = X$, or $D_{\alpha-1} \subsetneq D_\alpha$, and $(D|_{\sim_\alpha}, \lesssim_\alpha|_{\sim_\alpha})$ has more jumps than $(D|_{\sim_{\alpha-1}}, \lesssim_{\alpha-1}|_{\sim_{\alpha-1}})$.

$$\lesssim_0 := \{(x_0, x_0)\} \text{ and } D_0 := (\{x_0\}, \lesssim_0).$$

$0 < \alpha < \Omega$ is not a limit ordinal: Now the induction hypothesis allows us to assume that $\lesssim_{\alpha-1}$ is a continuous total preorder on the set $D_{\alpha-1}$, endowed with the relativized topology $t|_{D_{\alpha-1}}$. Then, we distinguish between the following two cases.

Case 1: $\overline{D_{\alpha-1}} = X$. In this case we finish the induction procedure.

Case 2: $\overline{D_{\alpha-1}} \subsetneq X$. In this situation we arbitrarily choose some point $x_\alpha \in X \setminus \overline{D_{\alpha-1}}$ in order to set $D_\alpha := D_{\alpha-1} \cup \{x_\alpha\}$. Since t is assumed to be completely regular, there exists some function $f_\alpha \in C(X, [0, 1])$ such that $f_\alpha(\overline{D_{\alpha-1}}) = \{0\}$ and $f_\alpha(x_\alpha) = 1$. Then, the preorder

$$\lesssim_\alpha := \{(z, z) : z \in D_\alpha\} \cup \lesssim_{\alpha-1} \cup \{(z, x_\alpha) : z \in D_{\alpha-1}\}$$

is a continuous total preorder on D_α , since f_α is a continuous increasing function with respect to \lesssim_α , and $\lesssim_{\alpha-1}$ is a continuous total preorder (see Theorem 2.1, (v)). Now we may continue the transfinite induction procedure.

$0 < \alpha < \Omega$ is a limit ordinal: In this situation we set $D_\alpha = \bigcup_{\beta < \alpha} D_\beta$. Then, we go the same way as in the case $\alpha < \Omega$ to not being a limit ordinal. This means that we finish the induction procedure if $\overline{D_\alpha} = X$. Otherwise, we consider the totally preordered set

$$(D_\alpha, \lesssim_\alpha) := \left(\bigcup_{\beta < \alpha} D_\beta, \bigcup_{\beta < \alpha} \lesssim_\alpha^{(\beta)} \right),$$

where $\lesssim_\alpha^{(\beta)}$ is a continuous total preorder on D_α , which is obtained by “extending” to D_α the continuous total preorder \lesssim_β on D_β by using complete regularity of t , as follows. Define, for every $\beta < \alpha$, the total preorder $\lesssim_\alpha^{(\beta)}$ on D_α by setting:

$$\lesssim_\alpha^{(\beta)} := \{(z, z) : z \in D_\alpha\} \cup \lesssim_\beta \cup \{(z, x_\alpha^{(\beta)}) : (z \in D_\beta) \text{ and } (x_\alpha^{(\beta)} \in D_\alpha \setminus D_\beta)\}.$$

In order to show that $\lesssim_\alpha^{(\beta)}$ is actually a continuous total preordered on D_α for every $\beta < \alpha$, consider that, for every $z \in D_\beta$, and for every $x_\alpha^{(\beta)} \in D_\alpha \setminus D_\beta$ there exists $f_\alpha^{(\beta)} \in C(X, [0, 1])$ such that $f_\alpha^{(\beta)}(z) = 0$ and $f_\alpha^{(\beta)}(x_\alpha^{(\beta)}) = 1$, where, in addition, $f_\alpha^{(\beta)}$ is identically equal to 0 on $\overline{D_\beta}$. We have that $f_\alpha^{(\beta)}$ is a continuous increasing function on the totally preordered set $(D_\alpha, \lesssim_\alpha^{(\beta)})$, so that $\lesssim_\alpha^{(\beta)}$ is actually a continuous total preorder on D_α (see, again, Theorem 2.1, (v)).

We notice that D_α is a countable subset of X for every ordinal number $0 \leq \alpha < \Omega$. Indeed, either we set $D_\alpha := D_{\alpha-1} \cup \{x_\alpha\}$, or $D_\alpha = \bigcup_{\beta < \alpha} D_\beta$, and clearly there are countably many ordinal numbers $\alpha' \in [0, \alpha]$ for every $0 \leq \alpha < \Omega$.

It still remains to verify that the transfinite induction procedure described above stops at some countable ordinal number $\alpha < \Omega$. Let us, therefore, assume, in contrast, that the procedure does not stop. Then, we consider the preorder \lesssim^* on X that is defined by setting

$$\lesssim^* := \{(z, z) : z \in X\} \cup \bigcup_{\alpha < \Omega} \lesssim_\alpha.$$

The transfinite induction process implies that \lesssim^* is a continuous total preorder on X that cannot be representable by a continuous utility function. Indeed, there are uncountable many jumps in $(X|_{\sim^*}, \lesssim^*_{|\sim^*})$. This contradiction completes the proof of the last statement of the theorem. \square

Remark 3.1 Since every metrizable topology is completely regular, condition 3 of Theorem 3.1 generalizes Estévez and Hervés' theorem [13], according to which a non-separable metric topology is not useful.

Remark 3.2 The fact that a useful topology t on X satisfies *OCCC*, without requiring complete regularity of t , can be seen in another way by using Theorem 2.2. Indeed, it is immediate to check that the topology generated by an uncountable isolated chain \mathcal{O} of open and closed subsets of X is not second countable.

We recall that, when we consider the space $C(X, t, \mathbb{R})$ of all continuous real-valued function on some topological space (X, t) , the *weak topology* on X , $\sigma(X, C(X, t, \mathbb{R}))$, is the coarsest topology on X satisfying the property that every continuous real-valued function on (X, t) remains being continuous. Two points $x, y \in X$ are considered as being *equivalent* if $f(x) = f(y)$ for all functions $f \in C(X, t, \mathbb{R})$. For two equivalent points $x, y \in X$, we write $x \sim_{C(X, t, \mathbb{R})} y$ ($x \sim_C y$ for the sake of brevity).

It is well known that $(X, \sigma(X, C(X, t, \mathbb{R})))$ is a completely regular space (cf., for instance, Cigler and Reichel [18, Satz 10, page 101], and Aliprantis and Border [19, Theorem 2.55 and Corollary 2.56]), and $(X|_{\sim_C}, \sigma(X, C(X, t, \mathbb{R}))|_{\sim_C})$

is a completely regular Hausdorff-space, where $(X|_{\sim_C}, \sigma(X, C(X, t, \mathbb{R}))|_{\sim_C})$ is the quotient space of $\sigma(X, C(X, t, \mathbb{R}))$ that is induced by the equivalence relation \sim_C .

The following lemma holds true.

Lemma 3.1 *The coarsest topology on X satisfying the property that all continuous total preorders on (X, t) remain being continuous is $\sigma(X, C(X, t, \mathbb{R}))$. (Of course, this assertion is equivalent to the statement that a total preorder \lesssim on (X, t) is continuous if and only if it is continuous with respect to $\sigma(X, C(X, t, \mathbb{R}))$).*

Proof Although the validity of this assertion appears somewhat surprisingly, its trueness is trivial. Indeed, since continuity of a preorder \lesssim on (X, t) is described by continuous (increasing) real-valued functions, its validity is immediate (see the equivalence “(i) \Leftrightarrow (v)” in Theorem 2.1). \square

The previous considerations guarantee that complete regularity is a necessary condition for a topology t on X to be useful, when we further assume that t is the coarsest topology on X such that every continuous preorder remains continuous.

Corollary 3.1 *Let t be a topology on a nonempty set X . The the following conditions are equivalent:*

- (i) t is useful;
- (ii) $\sigma(X, C(X, t, \mathbb{R}))$ is separable and satisfies OCCC.

The classical theorems by Eilenberg [10] and Debreu [11,12] are now derived immediately as corollaries of Theorem 3.1. statement 1.

Theorem 3.2 (Eilenberg theorem) *Every connected and separable topology is useful.*

Proof Just consider that t is separable, and it obviously satisfies $OCCC$. \square

Lemma 3.2 *Let t be a second countable topology on a set X , Then, every linearly ordered subtopology t^l of t is also second countable.*

Proof Let, therefore, t be a second countable topology on X , and consider a subtopology t^l of t which is linearly ordered by set inclusion. Then, we choose a countable base \mathcal{B} of t and consider the countable subset

$$\begin{aligned} \mathcal{B}^l = & \{O \in t^l : \exists B \in \mathcal{B} (B \subset O \wedge \forall O' \in t^l (O' \subsetneq O \Rightarrow B \not\subset O'))\} \\ & \cup \{O \in t^l : \exists B \in \mathcal{B} (O = \bigcup_{B \subset O' \in t^l} O')\} \cup \{\emptyset, X\} \end{aligned}$$

of t^l , in order to easily verify that \mathcal{B}^l is a base of t^l . Indeed, consider any element $x \in X$ such that $x \in O'' \in t^l$ for some set $O'' \in t^l$. Then, there exists $B \in \mathcal{B}$ such that $x \in B \subset O''$. If for no $(t^l \ni) O' \subsetneq O''$ it happens that $B \subset O'$ then $O'' \in \mathcal{B}^l$ and we are done. Otherwise, there exist $O' \in t^l$, and $B' \in \mathcal{B}$ such that $x \in B \subset O' \subsetneq O''$, $B' \not\subset O'$, so that $O' \in \mathcal{B}^l$. This consideration completes the proof. \square

Theorem 3.3 (Debreu theorem) *Every second countable topology is useful.*

Proof Clearly, every second countable topology is separable. Further, from Lemma 3.2, every second countable topology must satisfy $OCCC$, since a chain

of open and closed sets which contains uncountably many isolated sets generates a linearly ordered subtopology, which is not second countable. Therefore, the thesis follows from Theorem 3.1, point 1. \square

The following corollary concerns the case of connected topologies. The immediate proof is omitted.

Corollary 3.2 *Let t be a connected completely regular topology on a set X . Then, t is useful, if and only if t is separable.*

4 Application to Optimization of All Continuous Total Preorders

We apply the results of the previous section to optimization of all continuous total preorders on a topological space.

A total preorder on a topological space (X, t) is said to be *upper semicontinuous* (respectively, *lower semicontinuous*) if $l_{\preceq}(x) := \{z \in X : z \prec x\}$ (respectively $r_{\preceq}(x) := \{z \in X : x \prec z\}$) is an open subset of X for every $x \in X$ (see, e.g., Bosi and Zuanon [16]).

We recall that, if (X, \preceq) is any preordered set, an element $x_0 \in X$ is said to be *maximal* (respectively, *minimal*), if for no $x \in X$ it occurs that $x_0 \prec x$ (respectively, for no $x \in X$ it occurs that $x \prec x_0$). Clearly, an element $x_0 \in X$ is maximal (minimal) for a totally preordered set (X, \preceq) if and only if $x \preceq x_0$ for every $x \in X$ (respectively, $x_0 \preceq x$ for every $x \in X$).

Compact spaces are obviously of particular interest when searching maximal/minimal elements for total preorders representable by continuous utilities.

We recall that a topology t on a set X is said to be *compact* if every *open cover* of X admits a finite subcover (see e.g. Gutiérrez [20]).

The following proposition may be viewed as a corollary of Theorem 3.1, given the fact that every compact Hausdorff space is normal, and therefore completely regular (see, e.g., Aliprantis and Border [19, Theorem 2.48]).

Proposition 4.1 *The following conditions are equivalent on a compact Hausdorff topology t on a nonempty set X .*

- (i) t is useful;
- (ii) t is separable and satisfies OCCC.

Gutiérrez [20, Theorem 2.1] proved the following interesting result concerning the existence of maximal elements for all continuous total preorders on a topological space.

Theorem 4.1 *A topology t on a nonempty set X is compact, provided that every upper semicontinuous total preorder \succsim on (X, t) admits a maximal element.*

We are now able to characterize all the topologies such that every continuous total preorder is continuously representable, and, in addition, admits both a maximal and minimal element.

Theorem 4.2 *The following conditions are equivalent on a topology t on a nonempty set X .*

- (i) *Every continuous total preorder \succsim on (X, t) is representable by a continuous utility function, and has both a maximal and a minimal element;*

- (ii) *Every continuous total preorder \lesssim on $(X, \sigma(X, C(X, t, \mathbb{R})))$ is representable by a continuous utility function, and has both a maximal and a minimal element;*
- (iii) *$\sigma(X, C(X, t, \mathbb{R}))$ is compact, separable, and satisfies *OCCC*.*

Proof (i) \Rightarrow (ii). See Lemma 3.1.

(ii) \Rightarrow (iii). Since $\sigma(X, C(X, t, \mathbb{R}))$ is completely regular and useful, it is separable and satisfies *OCCC* by Corollary 3.1. Further, t is compact by Theorem 4.1.

(iii) \Rightarrow (i). This implication is an immediate consequence of Corollary 3.1, and of the consideration according to which the maximization/minimization of a continuous utility function for a total preorder on a compact space (see the famous Weierstrass Theorem) leads to maximal/minimal elements for the total preorder. □

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