STARLIKENESS FOR FUNCTIONS OF A HYPERCOMPLEX VARIABLE

ANNA GORI AND FABIO VLACCI

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Abstract. In this paper we introduce new notions of starlikeness for a class of functions of a hypercomplex variable. We then obtain equivalent formulations for starlikeness which resemble the analogous ones in the holomorphic case such as Nevanlinna's criterion. Furthermore we give a characterization of algebraic starlikeness in terms of non-vanishing of a suitable analog of the Hadamard product.

1. Introduction

The study of starlikeness has a central role in many different subjects of geometry and topology and is particularly important in geometric function theory. This very rich topic has been investigated in several papers (see [2] and the references therein) often following different approaches by many mathematicians; also in the hypercomplex setting (see [3], [4]). In the present paper we introduce a new definition of starlikeness for a class of functions of a hypercomplex variable, which is inspired by a geometric point of view and which aims at providing tools for a generalization of the usual notions in the conformal and holomorphic setting.

Definition 1.1. Assume f is an injective slice–regular function in the unit ball $B(0,1)$ of $\mathbb{H}$ such that $f(0)=0$. Then we say that f is *starlike with respect to* 0 if and only if, for any real r such that $0 \le r < 1$, then $(1-t)f(B(0,r)) \subseteq f(B(0,r))$ for any real t with $0 \le t \le 1$.

The property of starlikeness for a slice-regular function f is proved to be equivalent to the positivity of the real part of a suitable Hermitian product $\langle \cdot | \cdot \rangle$ of an expression involving the Cullen ($\partial_C f$) and the Spherical ($\partial_S f$) derivatives of f together with f as stated in the following

Theorem 1.2. *A function $f : B(0,1) \to \mathbb{H}$ is starlike with respect to 0 if and only if*

$$\Re e \left\{ q^{-1} \frac{\langle f(q) | \partial_S f(q) \rangle}{\langle \partial_C f(q) | \partial_S f(q) \rangle} \right\} \ge 0 \qquad (1.1)$$

for any $q \in B(0,1)$.

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We then introduce the notion of *algebraic starlikeness* which can be characterized by imposing positivity for the real part of an expression involving only the Cullen derivative of the slice–regular function f and f itself. The notions of starlikeness and algebraic starlikeness are proven to be equivalent for slice–preserving slice–regular functions.

Since a slice–regular function over $\mathbb{H}$ is a σ-analytic, or spherical analytic function (see [9]), it is natural to expect that these characterizations for the notions of starlikeness should extend the results obtained for complex analytic or holomorphic functions over $\mathbb{C}$: these generalizations are shown in Section 5.

Finally, a suitable application of the Hadamard product $*_H$ for hypercomplex power series allows us to extend Theorems 1 and 2 proved in [11].

Proposition 1.3. *If a function f is algebraically starlike, then for any $s \in \mathbb{R}$, I in the sphere of imaginary units of $\mathbb{H}$ and $q \in B(0,1)$ it turns out that*

$$q^{-1}[f(q) *_H [(1-q)^{-2}(q(1-sI)+q^2 sI)]] \neq 0 \tag{1.2}$$

or equivalently, if g is a primitive of $q^{-1}f(q)$ such that $g(0)=0$, then

$$q^{-1}[g(q) *_H (1-q)^{-3}(q(1-sI)+q^2(1+sI))] \neq 0. \tag{1.3}$$

Vice versa, if for a slice–regular function f and for the primitive g of $q^{-1}f(q)$ (such that $g(0)=0$) conditions (1.2) *and* (1.3) *hold, then f is algebraically starlike and g is algebraically convex.*

2. Background and preliminary results

Let V be a vector space over $\mathbb{R}$.

Definition 2.1. A subset $E \subset V$ is said to be *starlike* or *star-shaped* with respect to a point $w_0 \in E$ if the linear segment joining w_0 to every other point $w \in E$ lies entirely in E. In formula, E is starlike with respect to $w_0 \in E$ if and only if, $\forall w \in E, \quad \forall t \in \mathbb{R},\ 0 \leq t \leq 1,\ tw + (1-t)w_0$, belongs to E.

Definition 2.2. A subset $E \subset V$ is said to be *convex* if it is starlike with respect to each of its points; i.e., the linear segment joining any two points in E lies entirely in E.

Definition 2.3. Let V_1 and V_2 be real vector spaces. A function $f : U \subset V_1 \to V_2$ is

- *starlike* iff $f(U)$ is a starlike set in V_2 (with respect to a point $f(x_0)$ of V_2);
- *convex* iff $f(U)$ is a convex set in V_2.

Remark 2.4. The notion of starlikeness and convexity for sets or functions is invariant for rigid motions.

Assume that $\mathbb{K}$ is an associative, unitary real algebra with division of finite dimension (i.e. $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{H}$). If $\mathbb{S}_{\mathbb{K}}$ is the sphere of imaginary units of $\mathbb{K}$, then every non-real element q in $\mathbb{K}$ can be written in a unique way as $q = x + yI_q$, with $I_q \in \mathbb{S}_{\mathbb{K}}$ and $x, y \in \mathbb{R}, y > 0$. We will refer to $x = \Re e(q)$ as the real part of q and to $y = \Im m(q)$ as the imaginary part of q. As in the case $\mathbb{K} = \mathbb{C}$, the *conjugate* of q will be denoted by $\overline{q} := 2\Re e(q) - q$.

Each algebra $\mathbb{K}$ can be regarded as a normed topological real vector space, after introducing $|q|^2 = q\overline{q} \in \mathbb{R}$ as the norm of an element q of the algebra. Moreover if $q \neq 0$, then $q^{-1} = |q|^{-2}\overline{q}$; since the center of $\mathbb{K}$ coincides with $\mathbb{R}$ there is no ambiguity in writing q^{-1} as $|q|^{-2}\overline{q}$ or as $\overline{q}|q|^{-2}$ so we often adopt the notation $\overline{q}/|q|^2$ which will be applied in general for a fraction when the divisor is a real number (different from zero). In a similar way the real vector space $V = \mathbb{K}^N$ for $N \in \mathbb{N}$ can be equipped with the Euclidean topology.

Let X and Y be open sets in $\mathbb{K}^N$ such that $0 \in X$ and $f : X \to Y$. We may always assume that $f(0) = 0$, since otherwise one can consider $f_1 = f - f(0)$. With this in mind, the condition of starlikeness for f with respect to $0 \in f(X)$ can also be summarized in the following way:

$$(1-t)f(X) \subset f(X), \quad \forall t \in [0,1], \tag{2.1}$$

which allows us to use the following notation $(1-t)f \prec f$ (commonly adopted for *subordination* relation of functions) and say that the function h, with $h(t,x) = (1-t)f(x)$, is *subordinate* to f.

If in addition we assume that $f : X \to Y$ is an injective continuous function starlike with respect to 0, then the function

$$\Phi(t,x) := f^{-1}((1-t)f(x)) \tag{2.2}$$

is well defined. With $B(0,r)$ we indicate the open disc in $\mathbb{K}^N$ centered at the origin of radius r, namely $B(0,r) = \{x \in \mathbb{K}^N \ : \ |x| < r\}$. We observe that $\Phi(0,x) = x$ for any $x \in X$ and if

$$A_r := f(X \cap B(0,r))$$

for a positive r, then $(1-t)A_r \subseteq A_r$ and hence

$$f^{-1}((1-t)A_r) \subseteq f^{-1}(A_r) = X \cap B(0,r);$$

in other words, if $|x| = r_1 < r$ and $x \in X$, then $|\Phi(t,x)| \leq r_1 = |x|$. Furthermore, since for $t \neq 0$, $(1-t)A_r \Subset A_r$, it follows that $|\Phi(t,x)| = |x|$, if and only if $t = 0$, which actually means $\Phi(0,x) = x$. Therefore we can summarize the previous considerations in the following

Lemma 2.5. *If $h(t,x) = (1-t)f(x)$ is subordinate to f and f is continuous and injective in an open set of $\mathbb{K}^N$ containing 0 and such that $f(0) = 0$, then the function $\Phi(t,x) := f^{-1}((1-t)f(x))$ is well defined and it turns out that*

$$|\Phi(t,x)| \leq |x| \tag{2.3}$$

where $|\cdot|$ is the (induced) Euclidean norm. Furthermore, equality in (2.3) holds if and only if $t = 0$; in other words $|\Phi(t,x)| = |x|$ implies $\Phi(t,x) = x$ (which can actually occur if and only if $t = 0$).

Remark 2.6. The assumption on the continuity on f is essential. Indeed it is easy to prove that, without this assumption, the function $|\Phi(t,q)|$ is decreasing in t, but it is not strictly decreasing as the following example shows: take f from $B(0,1)$ to itself, such that $f(0) = 0$ and such that a generic point $z = |z|e^{i\vartheta}$ $(0 \leq \vartheta \leq 2\pi)$ is mapped to a point $w = \dfrac{\vartheta}{2\pi}e^{i|z|}$. This function is not continuous. The function $t \mapsto |\Phi(t,q)|$ is not strictly decreasing, since it maps every line through the origin onto a circle.

3. The complex holomorphic case

In this section we'll primarily consider the case of holomorphic starlike and convex functions in the complex plane. In particular, our attention will be focused on (normalized) holomorphic starlike and convex functions which are injective in $B(0,1) = \{z \in \mathbb{C} \ : \ |z| < 1\}$. The choice of this domain of definition is not restrictive, since, by the Riemann's Mapping Theorem, any simply–connected, open and connected set of $\mathbb{C}$ different from the entire complex plane $\mathbb{C}$ is biholomorphic to $B(0,1)$.

We recall that in this setting an open and connected complex set is also called a *domain* of $\mathbb{C}$ and an injective holomorphic function is commonly said to be a *univalent* function or a *schlicht* function.

An interesting application of the classical Schwarz Lemma implies a geometric characterization for convex and starlike univalent functions, known as the Theorem of Study (see [10]).

Theorem 3.1. *A univalent function $f : B(0,1) \to \mathbb{C}$ is starlike with respect to 0 if and only if, for any real r such that $0 \le r \le 1$, $f(B(0,r))$ is a starlike set with respect to 0. A univalent function $f : B(0,1) \to \mathbb{C}$ (such that $f(0) = 0$) is convex if and only if, for any real r such that $0 \le r \le 1$, $f(B(0,r))$ is a convex set.*

Similarly, it follows directly from the Schwarz Lemma that, whenever f is a univalent function from the unit disc $B(0,1)$ to $\mathbb{C}$ and starlike with respect to 0, then $\Phi(t,z) = f^{-1}((1-t)f(z))$ is a *Schwarz function* for any t, with $0 \le t \le 1$, namely for any given t, with $0 \le t \le 1$ the function $\Phi(t,z)$ is holomorphic in $B(0,1)$ with $\Phi(t,0) = 0$ and $|\Phi(t,z)| \le |z|, \quad \forall z \in B(0,1)$. For our purposes, we can always assume that a univalent function $f : B(0,1) \to \mathbb{C}$ is such that $f(0) = 0$ and $f'(0) = 1$. Indeed, if f is univalent (hence $f'(0) \neq 0$), then so is $z \mapsto \dfrac{f(z) - f(0)}{f'(0)}$.

We then introduce the class of (normalized) univalent functions

$$\mathcal{S} := \{f : B(0,1) \to \mathbb{C},\ f \text{ injective, holomorphic and such that } f(0) = 0 \\ \text{and } f'(0) = 1\}.$$

For functions in $\mathcal{S}$, the notions of starlikeness with respect to 0 and convexity can be stated in a more analytic way, since one can apply the conformal properties of holomorphic functions to study the plane curves $t \mapsto f(\varrho e^{it})$ for suitable choices of radius $0 < \varrho < 1$. We summarize the main characterizations in the following result (see e.g. [10]), also known as *Nevanlinna's criterion.*

Theorem 3.2. *Given $f \in \mathcal{S}$, then*

- *f is starlike with respect to 0 if and only if*
$$\Re e \left\{ \frac{z f'(z)}{f(z)} \right\} > 0$$
for any $z \in B(0,1)$;
- *f is convex if and only if*
$$\Re e \left\{ 1 + \frac{z f''(z)}{f'(z)} \right\} = \Re e \left\{ \frac{(z f'(z))'}{f'(z)} \right\} > 0$$
for any $z \in B(0,1)$.

We observe that the condition of convexity for f can be equivalently obtained from the condition of starlikeness for the function $z \mapsto zf'(z)$ (this result is also known as *Alexander's Theorem*). We then introduce these classes of functions

$$\mathcal{S}^* := \{f \in \mathcal{S}, \quad f \text{ starlike with respect to } 0\}, \quad \mathcal{C} := \{f \in \mathcal{S}, \quad f \text{ convex}\}.$$

Clearly $\mathcal{C} \subset \mathcal{S}^* \subset \mathcal{S}$. Furthermore we notice that if $f \in \mathcal{S}^*$, then the function $\Phi(t, z)$ as in (2.2) turns out to be holomorphic in z and such that $\Phi(t, 0) = 0$ for any $t \in [0, 1]$.

Consider now the half–plane $H_+ := \{z \in \mathbb{C} \ : \ \Re e(z) > 0\}$; since, traditionally, a holomorphic function with positive real part is known as a *Carathéodory* function, we also introduce the class of functions

$$\mathcal{P} := \{f : B(0,1) \to H_+, \ f \text{ holomorphic and such that } f(0) = 1\}.$$

Therefore we can also say that a function f is such that

(1) $f \in \mathcal{S}^*$ if and only if $z \mapsto \frac{zf'(z)}{f(z)}$ belongs to $\mathcal{P}$,
(2) $f \in \mathcal{C}$ if and only if $z \mapsto 1 + \frac{zf''(z)}{f'(z)} = \frac{(zf'(z))'}{f'(z)}$ belongs to $\mathcal{P}$,
(3) $f \in \mathcal{C}$ if and only if $z \mapsto zf'(z)$ belongs to $\mathcal{S}^*$.

4. The analytic case

Given two analytic functions f, g whose power series are $f(w) = \sum_n w^n a_n$, $g(w) = \sum_n w^n b_n$ where a_n and b_n belong to a ring $(\mathcal{R}, +, \cdot)$, we can define their sum and their Hadamard product in the following way:

$$(f+g)(w) := \sum_n w^n (a_n + b_n); \quad f *_H g(w) := \sum_n w^n a_n \cdot b_n.$$

It turns out that the Hadamard product of analytic functions is an analytic function. In general the Hadamard product is not commutative; if the ring $\mathcal{R}$ is a unitary ring of unit $1_{\mathcal{R}}$ for $\cdot$, the function $w \mapsto \sum_n w^n \cdot 1_{\mathcal{R}}$ is the neutral element for the Hadamard product. It will be also denoted by $(1-w)^{-1}$. Furthermore if the ring is a field, then any function $f(w) = \sum_n w^n a_n$ with all a_n different from the neutral element $0_{\mathcal{R}}$ for $+$ in $\mathcal{R}$, has an inverse for the Hadamard product given by

$$w \mapsto \sum_n w^n (a_n)^{-1}.$$

In addition to that, if $f(w) = \sum_n w^n a_n$ we consider:

- $w^p f(w) = \sum_n w^{(n+p)} a_n$ with $p \in \mathbb{Z}$,
- $f'(w) = \sum_n w^{n-1} n a_n$ where $n a_n = \underbrace{a_n + a_n + \ldots + a_n}_{n-\text{times}}$,
- $af(w) = \sum_n w^n (a \cdot a_n)$ with $a \in \mathcal{R}$.

The function f' is said to be *derived* from the function f or to be the *derivative* of f.

It follows that

$$(a +_{\mathcal{R}} b) f(w) = (af + bf)(w), \text{with } a, b \in \mathcal{R}; \tag{4.1}$$

$$wf'(w) *_H g(w) = f(w) *_H wg'(w); \tag{4.2}$$

$$(wf'(w))' = (f' + w(f')')(w). \tag{4.3}$$

Therefore if $a_0 = 0_{\mathcal{R}}$ and $a_1 = 1_{\mathcal{R}}$, then

$$(wf'(w))' = (f' + w(f')')(w) = f'(w) *_H \left(\sum_n w^{n-1} n 1_{\mathcal{R}}\right). \tag{4.4}$$

Furthermore, given $f(w) = \sum_n w^n a_n$, if, for any n there exists c_n such that

$$\underbrace{c_n + c_n + \ldots + c_n}_{(n+1)-\text{times}} = a_n,$$

then $F(w) = \sum_n w^{n+1} c_n$ has the following property:

$$F'(w) = f(w)$$

and the function F is called a *primitive* of f. In this setting we can state the following result [11].

Proposition 4.1. *The analytic function f belongs to $\mathcal{S}^*$ if and only if*

$$w^{-1}\left\{f(w) *_H \left[(1-w)^{-2}\left(w + w^2\frac{x-1}{2}\right)\right]\right\} \neq 0$$

for $w \in B(0,1)$ and $|x| = 1$.

Equivalently, if g is the primitive of $w^{-1}f(w)$ such that $g(0) = 0$, then g is convex if and only if

$$w^{-1}\left\{g(w) *_H \left[(1-w)^{-3}\left(w + w^2 x\right)\right]\right\} \neq 0$$

for $w \in B(0,1)$ and $|x| = 1$.

5. The regular quaternionic case

Let $\Omega \subseteq \mathbb{H}$ be a domain.

Definition 5.1. We say that Ω is

- an *axially symmetric domain* if, for all $x + Iy \in \Omega$, with $I \in \mathbb{S}_{\mathbb{H}}$, the whole sphere $x + \mathbb{S}_{\mathbb{H}}y$ is contained in Ω;
- a *slice domain* if $\Omega \cap \mathbb{R}$ is non–empty and if given any $I \in \mathbb{S}_{\mathbb{H}}$ the complex line $L_I = \mathbb{R} + \mathbb{R}I$ intersected with Ω is a domain in L_I.

It is possible (see [9]) to introduce a notion of regularity for functions defined in any open ball $B(0,r) = \{q \in \mathbb{H} : |q| < r\}$ (and, more in general, in some axially symmetric slice domains of $\mathbb{H}$) which extends the one of holomorphicity in the complex case.

Definition 5.2. If Ω is an axially symmetric slice domain in $\mathbb{H}$, a real differentiable function $f : \Omega \to \mathbb{H}$ is said to be *slice–regular* if, for every $I \in \mathbb{S}_{\mathbb{H}}$, its restriction f_I to the complex line $L_I = \mathbb{R} + \mathbb{R}I$ passing through the origin and containing 1 and I is holomorphic on $\Omega \cap L_I$.

Remark 5.3. It can be proved that a function $f : B(0,r) \to \mathbb{H}$ is *slice–regular* in $B(0,r) \subset \mathbb{H}$ if and only if there exists a converging power series $\sum_n q^n a_n$ in $B(0,r)$, with $a_n \in \mathbb{H}$ for any $n \in \mathbb{N}$, such that $f(q) = \sum_n q^n a_n$ with $q \in B(0,r)$.

As a direct computation on the real components of a slice–regular function, one immediately obtains (see [9])

Lemma 5.4. *If f is a slice–regular function on an axially symmetric slice domain $\Omega \subset \mathbb{H}$, then for every $I \in \mathbb{S}_{\mathbb{H}}$ and for any $J \in \mathbb{S}_{\mathbb{H}}$, $J \perp I$, there exist two holomorphic functions $F_1, F_2 : \Omega \cap L_I \to L_I$ such that $f_I(z) = F_1(z) + F_2(z)J$ with $z = x + Iy$.*

For the sequel it will be important to recall a natural notion of product of polynomials (then extended to power series) which turns out to provide a "regular" multiplication of slice–regular functions when represented by converging regular power series.

Definition 5.5. Let $f(q) = \sum_{n=0}^{+\infty} q^n a_n$ and $g(q) = \sum_{n=0}^{+\infty} q^n b_n$ be given power series with coefficients in $\mathbb{H}$ whose radii of convergence are greater than r. We define the *regular product* of f and g as the series $f * g(q) = \sum_{n=0}^{+\infty} q^n c_n$, where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ for all n, which is convergent in $B(0, r)$.

It is not difficult to see that $f*g$ is a slice–regular function defined in the open ball $B(0, r)$. Furthermore, the regular product is extended for slice–regular functions defined on a general axially symmetric domain Ω in the following way:

$$f * g(q) = \begin{cases} 0 & \text{if } f(q) = 0, \\ f(q)g(f(q)^{-1}qf(q)) & \text{otherwise.} \end{cases} \tag{5.1}$$

Definition 5.6. For a slice–regular function $f : B(0, r) \to \mathbb{H}$, with $f(q) = \sum_{n=0}^{+\infty} q^n a_n$, we define the *regular conjugate* f^c and the *symmetrized* f^s of f as $f^c(q) = \sum_{n=0}^{+\infty} q^n \bar{a}_n$, and $f^s(q) = f * f^c(q) = f^c * f(q) = \sum_{n=0}^{+\infty} q^n r_n$ with $r_n = \sum_{k=0}^{n} a_k \bar{a}_{n-k} \in \mathbb{R}$. Finally if $f \not\equiv 0$, then the *regular reciprocal* of f is the (slice–regular) function $f^{-*} = \frac{1}{f^s(q)} f^c$ defined on $B(0, r) \setminus Z(f^s)$, where $Z(f^s)$ is the zero set of the symmetrized function f^s.

Regular reciprocals are well–defined slice–regular functions whose power series expansions are converging in their domains of definition.

In the spirit of Gateaux, a notion of derivative is well defined for slice–regular functions, namely (see [9])

Definition 5.7. Let Ω be an axially symmetric slice domain in $\mathbb{H}$ and let $f : \Omega \to \mathbb{H}$ be a slice–regular function. For any $I \in \mathbb{S}_{\mathbb{H}}$ and any point $q = x + yI$ in Ω (with $x = \Re e q$ and $y = \Im m q$) we define the *Cullen derivative* of f at q as

$$\partial_C f(x + yI) = f'(x + yI) := \frac{1}{2}\left(\frac{\partial}{\partial x} - I\frac{\partial}{\partial y}\right) f_I(x + yI).$$

Since in $\mathbb{H}$ one can choose different imaginary units, it is also worth considering the following

Definition 5.8. Let Ω be an axially symmetric slice domain in $\mathbb{H}$ and let $f : \Omega \to \mathbb{H}$ be a slice–regular function. We define the *spherical derivative* of f at q as

$$\partial_S f(q) := (q - \overline{q})^{-1}[f(q) - f(\overline{q})].$$

Recently, some attempts to generalize the notion of convexity and starlikeness from the holomorphic case to the class of quaternionic valued functions have appeared (see [3], [4], [5]). The strategy of imitating directly the approach and definitions of the holomorphic setting fails almost immediately. As pointed out, in the holomorphic case, the function $\Phi(t, z) = f^{-1}((1 - t)f(z))$ for any real t with

$0 \le t \le 1$ turns out to be automatically holomorphic in $B(0,1)$ when $f \in \mathcal{S}^*$ and this fact is crucial to obtain Theorem 3.1. Indeed the property of starlikeness of f in $B(0,1)$ is proved to be equivalent to starlikeness of f in any ball $B(0,r)$ with $0 < r < 1$. We remark that this characterization is essential to prove the Nevanlinna's criterion (Theorem 3.2). Despite the similarity with the complex holomorphic case, in the hypercomplex setting, the assumption of geometric starlikeness allows us only to define the function $\Phi(t,q) = f^{-1}((1-t)f(q))$ for any real t with $0 \le t \le 1$ when $q \in B(0,1)$, but in general one cannot prove any regularity (besides continuity) for $\Phi(t,q)$ in q (for a given t, with $0 \le t \le 1$) even though f is assumed to be a slice–regular function. In order to obtain inequality conditions for starlikeness which generalize the Nevanlinna's criterion for hypercomplex functions, in this paper we characterize starlikeness of slice–regular functions in the following way.

Definition 5.9. Assume f is an injective slice–regular function in the unit ball $B(0,1)$ of $\mathbb{H}$ such that $f(0) = 0$. Then we say that f is starlike with respect to 0 if and only if, for any real r such that $0 \le r < 1$, then $(1-t)f(B(0,r)) \subseteq f(B(0,r))$ for any real t with $0 \le t \le 1$.

In analogy with the complex holomorphic case we introduce the following classes of slice–regular functions:

$$\mathcal{S}_{\mathbb{H}} := \{f : B(0,1) \to \mathbb{H},\ f \text{ injective, slice-regular such that } f(0) = 0 \text{ and } f'(0) = 1\},$$

$$\mathcal{S}^*_{\mathbb{H}} := \{f \in \mathcal{S}_{\mathbb{H}},\ \text{starlike with respect to } 0\},$$

$$\mathcal{P}_{\mathbb{H}} := \{f : B(0,1) \to \mathbb{H},\ f \text{ slice-regular such that } \Re e f(q) > 0 \text{ and } f(0) = 1\}.$$

5.1. **Inequality conditions for starlikeness.** We begin by proving a result which generalizes Lemma 1 in [12] for holomorphic functions.

Lemma 5.10. *With the above given notation (and assumptions) for $f \in \mathcal{S}^*_{\mathbb{H}}$ and $\Phi(t,q) = f^{-1}((1-t)f(q))$ for any real t with $0 \le t \le 1$ when $q \in B(0,1)$, assume there exists a positive ϱ such that the limit*

$$\lim_{t\to 0^+} \frac{\Phi(0,q) - \Phi(t,q)}{t^\varrho} \tag{5.2}$$

exists; call it $\omega_\varrho(q)$. Then, for $q \ne 0$,

$$\Re e(q^{-1}\omega_\varrho(q)) \ge 0. \tag{5.3}$$

Proof. Define, for $q \ne 0$, $\Psi(t,q) := 2q(q+\Phi(t,q))^{-1}(q-\Phi(t,q))$; it follows from Lemma 2.5 that the function Ψ is well defined and

$$q^{-1}\Psi(t,q) = 2(q+\Phi(t,q))^{-1}(q-\Phi(t,q));$$

hence

$$\begin{aligned}\Re e(q^{-1}\Psi(t,q)) &= 2\Re e((q+\Phi(t,q))^{-1}(q-\Phi(t,q)))\\ &= 2\Re e((q(1+q^{-1}\Phi(t,q)))^{-1}(q(1-q^{-1}\Phi(t,q)))\\ &= 2\Re e((1+q^{-1}\Phi(t,q))^{-1}(1-q^{-1}\Phi(t,q))).\end{aligned}$$

Put $\beta := q^{-1}\Phi(t,q)$; since

$$\Re e(q^{-1}\Psi(t,q)) = 2\Re e((1+\beta)^{-1}(1-\beta))$$
$$= \frac{(1+\overline{\beta})(1-\beta)+(1+\beta)(1-\overline{\beta})}{|1+\beta|^2} = 2\frac{1-|\beta|^2}{|1+\beta|^2}$$

from (2.3) we conclude that

$$\Re e(q^{-1}\Psi(t,q)) \geq 0. \tag{5.4}$$

Finally we have

$$\lim_{t\to 0^+}\frac{q^{-1}\Psi(t,q)}{t^{\varrho}} = \lim_{t\to 0^+}\frac{q^{-1}2q(q+\Phi(t,q))^{-1}(q-\Phi(t,q))}{t^{\varrho}}$$
$$= \lim_{t\to 0^+}\left[2(q+\Phi(t,q))^{-1}\frac{(q-\Phi(t,q))}{t^{\varrho}}\right];$$

we recall that $\Phi(0,q) = q$ for any q, therefore, using the assumption on the existence of $\lim_{t\to 0^+}\frac{q-\Phi(t,q)}{t^{\varrho}} = \omega_{\varrho}(q)$ and (5.4) we obtain that $\Re e(q^{-1}\omega_{\varrho}(q)) \geq 0$. □

Definition 5.11. Let Ω be an axially symmetric slice domain in $\mathbb{H}$ and let $f : \Omega \to \mathbb{H}$ be a slice–regular function. If $q_0 = x_0 + y_0 I_0 \in \Omega$, we consider the slice–regular function $R_{q_0} f : \Omega \to \mathbb{H}$ to be defined as follows:

$$R_{q_0} f(q) := (q-q_0)^{-*}[f(q) - f(q_0)].$$

We have this result (see [8])

Proposition 5.12. *Let Ω be an axially symmetric slice domain in $\mathbb{H}$ and let $f : \Omega \to \mathbb{H}$ be a slice–regular function. If $q_0 = x_0 + y_0 I_0 \in \Omega$, then, for any $q \in \Omega$,*

$$f(q) = f(q_0) + (q-q_0)R_{q_0} f(\overline{q_0}) + [(q-x_0)^2 + y_0^2]R_{\overline{q_0}}R_{q_0} f(q_0).$$

Remark 5.13. It turns out that

$$R_{q_0} f(q_0) = \partial_C f(q_0), \quad R_{q_0} f(\overline{q_0}) = \partial_S f(q_0).$$

We recall also (see [7])

Theorem 5.14. *Let Ω be an axially symmetric slice domain in $\mathbb{H}$ and let $f : \Omega \to \mathbb{H}$ be a slice-regular function. If $q_0 = x_0 + y_0 J \in \Omega$, $J \perp I_0$ and $R_{q_0} f(q)$ splits as $R_{q_0} f(q) = R_{1,q_0}(q) + R_{2,q_0}(q)J$ with R_{1,q_0}, R_{2,q_0} holomorphic in L_{I_0}, then the complex Jacobian of f at q_0 is*

$$Df_{q_0} = \begin{pmatrix} R_{1,q_0}(q_0) & -\overline{R_{2,q_0}(\overline{q_0})} \\ R_{2,q_0}(q_0) & \overline{R_{1,q_0}(\overline{q_0})} \end{pmatrix}.$$

We can proceed and prove

Lemma 5.15. *Let Ω be an axially symmetric slice domain in $\mathbb{H}$ and $f : \Omega \to \mathbb{H}$ be a slice-regular function. If $q_0 = x_0 + y_0 I_0 \in \Omega$, $J \perp I_0$, $v = v_1 + v_2 J$ with $v_1, v_2 \in L_{I_0}$, then*

$$(Df_{q_0})^{-1}[v]$$
$$= \frac{1}{\langle \partial_C f(q_0) | \partial_S f(q_0)\rangle}\left\{v_1(\overline{R_{1,q_0}(\overline{q_0})} - R_{2,q_0}(q_0)J) + v_2 J(\overline{R_{1,q_0}(q_0)} - R_{2,q_0}(\overline{q_0})J)\right\}.$$

Proof. Following the notation as in [8]

$$(Df_{q_0})^{-1}[v] = \begin{pmatrix} R_{1,q_0}(q_0) & -\overline{R_{2,q_0}(\overline{q_0})} \\ R_{2,q_0}(q_0) & \overline{R_{1,q_0}(\overline{q_0})} \end{pmatrix}^{-1} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$= \frac{1}{R_{1,q_0}(q_0) \cdot \overline{R_{1,q_0}(\overline{q_0})} + \overline{R_{2,q_0}(\overline{q_0})} \cdot R_{2,q_0}(q_0)} \begin{pmatrix} \overline{R_{1,q_0}(\overline{q_0})} \cdot v_1 + \overline{R_{2,q_0}(\overline{q_0})} \cdot v_2 \\ -R_{2,q_0}(q_0) \cdot v_1 + R_{1,q_0}(q_0) \cdot v_2 \end{pmatrix}.$$

Using the (standard) Hermitian product $\langle \cdot | \cdot \rangle$ in $\mathbb{C}^2$, applied to the splitting, we can write

$$R_{1,q_0}(q_0) \cdot \overline{R_{1,q_0}(\overline{q_0})} + \overline{R_{2,q_0}(\overline{q_0})} \cdot R_{2,q_0}(q_0) = \langle \partial_C f(q_0) | \partial_S f(q_0) \rangle$$

and the previous equation becomes

$$\frac{1}{\langle \partial_C f(q_0) | \partial_S f(q_0) \rangle} \Big\{ \overline{R_{1,q_0}(\overline{q_0})} \cdot v_1 + \overline{R_{2,q_0}(\overline{q_0})} \cdot v_2 \\ + [-R_{2,q_0}(q_0) \cdot v_1 + R_{1,q_0}(q_0) \cdot v_2] J \}$$

$$= \frac{1}{\langle \partial_C f(q_0) | \partial_S f(q_0) \rangle} \Big\{ v_1 (\overline{R_{1,q_0}(\overline{q_0})} - R_{2,q_0}(q_0) J) + v_2 J (\overline{R_{1,q_0}(q_0)} - R_{2,q_0}(\overline{q_0}) J) \Big\}.$$

□

Remark 5.16. Note that both the functions $\overline{R_{1,q_0}(\overline{q})} - R_{2,q_0}(q) J$ and $\overline{R_{1,q_0}(q)} - R_{2,q_0}(\overline{q}) J$ are not holomorphic.

Let $H(t,q) := (1-t) f(q)$; then $H(t,q) - H(0,q) = -t f(q)$. Therefore, $H(t,q) - H(0,q)$ is a slice–regular function for any t and

$$\frac{H(t,q) - H(0,q)}{t} = -f(q).$$

Now $f(\Phi(t,q)) = H(t,q)$ and $f(\Phi(t,q)) - f(q) = H(t,q) - H(0,q)$. On the other hand

$$-f(q) = \frac{1}{t}[f(\Phi(t,q)) - f(q)] = [f(\Phi(t,q)) - f(q)][\Phi(t,q) - q]^{-1} \frac{[\Phi(t,q) - q]}{t}.$$

Now as $t \to 0^+$ the limit

$$\lim_{t \to 0^+} \frac{\Phi(t,q) - q}{t}$$

exists and is finite; we call it $-\omega_1(q)$ and we get $f(q) = Df_q[\omega_1(q)]$ so that $\omega_1(q) = (Df_q)^{-1}[f(q)]$. Following the notation of Lemma 5.15 with $v = f(q) = F_1(q) + F_2(q) J$ we get

$$\omega_1(q) = \frac{1}{\langle \partial_C f(q) | \partial_S f(q) \rangle} \begin{pmatrix} \overline{R_{1,q}(\overline{q})} \cdot F_1(q) + \overline{R_{2,q}(\overline{q})} \cdot F_2(q) \\ -R_{2,q}(q) \cdot F_1(q) + R_{1,q}(q) \cdot F_2(q) \end{pmatrix}. \tag{5.5}$$

Applying Lemma 5.10, we get

$$\Re e \{ q^{-1} (Df_q)^{-1} [f(q)] \} \geq 0. \tag{5.6}$$

Now observe that the real part of $q^{-1}\omega_1(q)$ is given by the real part of q^{-1} times the first row of the previous matrix, so

$$\Re e \left\{ q^{-1} \frac{\overline{R_{1,q}(\overline{q})} \cdot F_1(q) + \overline{R_{2,q}(\overline{q})} \cdot F_2(q)}{\langle \partial_C f(q) | \partial_S f(q) \rangle} \right\} = \Re e \left\{ q^{-1} \frac{\langle f(q) | \partial_S f(q) \rangle}{\langle \partial_C f(q) | \partial_S f(q) \rangle} \right\} \geq 0.$$

Notice that, also, the inverse of $q^{-1} \frac{\langle f(q) | \partial_S f(q) \rangle}{\langle \partial_C f(q) | \partial_S f(q) \rangle}$ has positive real part; then

Theorem 5.17. *Assume $f : B(0,1) \to \mathbb{H}$ is starlike with respect to 0. Then it turns out that*

(5.7)

$$\Re e\left\{q^{-1}\frac{\langle f(q)|\partial_S f(q)\rangle}{\langle \partial_C f(q)|\partial_S f(q)\rangle}\right\} \geq 0 \text{ or analogously } \Re e\left\{q\frac{\langle \partial_C f(q)|\partial_S f(q)\rangle}{\langle f(q)|\partial_S f(q)\rangle}\right\} \geq 0$$

for any $q \in B(0,1)$.

We can also prove the converse of the previous proposition, namely

Proposition 5.18. *Assume that f is in $\mathcal{S}_{\mathbb{H}}$ and is such that*

$$\Re e\left\{q^{-1}\frac{\langle f(q)|\partial_S f(q)\rangle}{\langle \partial_C f(q)|\partial_S f(q)\rangle}\right\} \geq 0. \tag{5.8}$$

Then f belongs to $\mathcal{S}^*_{\mathbb{H}}$.

Proof. Take $q \in B(0,r)$ and consider $(1-t)f(q)$. For $|t| < \varepsilon$, with ε sufficiently small, there exists $\Phi(t,q) \in B(0,1)$ such that $f(\Phi(t,q)) = (1-t)f(q)$ since f is a local diffeomorphism. Observe that $\Phi(0,q) = q$ from the univalence of f. Now we have

$$-f(q) = \frac{f(\Phi(t,q)) - f(\Phi(0,q))}{t} = [f(\Phi(t,q)) - f(q)][\Phi(t,q) - q]^{-1}\frac{[\Phi(t,q) - q]}{t}.$$

As $t \to 0^+$ we get $f(q) = Df_q[\omega_1(q)]$ and $\omega_1(q) = (Df_q)^{-1}[f(q)]$, thus $\Phi(t,q) = \Phi(0,q) + t\omega_1(q) + g(t)$ where $\lim_{t\to 0^+}\dfrac{g(t)}{t} = 0$. Using the assumption

$$\Re e\left\{q^{-1}\frac{\langle f(q)|\partial_S f(q)\rangle}{\langle \partial_C f(q)|\partial_S f(q)\rangle}\right\} \geq 0$$

we get that $|\Phi(t,q)|$ is decreasing in t in a right open neighborhood of 0. In fact

$$\begin{aligned}
&\lim_{t\to 0^+}\frac{|\Phi(t,q)|^2 - |\Phi(0,q)|^2}{t}\\
&\quad= \lim_{t\to 0^+}\frac{(\overline{\Phi(0,q)} - \overline{\omega_1(q)}t + \overline{g(t)})(\Phi(0,q) - \omega_1 t + g(t))) - |q|^2}{t}\\
&\quad= -2\Re e\,\overline{q}\omega_1(q) = -2|q|^2\Re e\,q^{-1}\omega_1(q) = -2\Re e\left\{q^{-1}\frac{\langle f(q)|\partial_S f(q)\rangle}{\langle \partial_C f(q)|\partial_S f(q)\rangle}\right\} \leq 0.
\end{aligned}$$

Hence for any $0 < t \leq 1$ and $q \in B(0,r)$ with $0 < r < 1$ there exists a point $\Phi(t,q) \in B(0,r)$ such that $(1-t)f(q) = f(\Phi(t,q))$. We conclude that, for any $r < 1$, $f(B(0,r))$ is starlike with respect to 0, so f is starlike. □

Remark 5.19. *If $q \in \mathbb{R}$, then f is quaternion-differentiable at q and its derivative $f'(q)$ coincides with $\partial_C f(q)$ so, following Theorem* 5.17, *we get that $\Re e\, q^{-1}f(q) \cdot f'(q)^{-1} = \Re e\, q^{-1}f(q) * \partial_C f(q)^{-*} \geq 0$.*

It becomes interesting to study those functions in $\mathcal{S}_{\mathbb{H}}$ such that the previous condition holds for $q \notin \mathbb{R}$. We give the following

Definition 5.20. We say that a function $f \in \mathcal{S}_{\mathbb{H}}$ is *algebraically starlike* (and we write $f \in \mathcal{AS}^*_{\mathbb{H}}$) iff $q^{-1}f(q) * \partial_C f(q)^{-*} \in \mathcal{P}_{\mathbb{H}}$.

Notice that since $q^{-1}f(q)*\partial_C f(q)^{-*}$ is slice-regular, then the maximum principle on the real part of it (Theorem 7.13 in [8]) implies that for $f \in \mathcal{AS}^*_{\mathbb{H}}$ necessarily $\Re e\, q^{-1}f(q) * \partial_C f(q)^{-*} > 0$. We can also assert that a function f is in $\mathcal{AS}^*_{\mathbb{H}}$ iff

$$\Re e\, [(q\partial_C f(q)) * f(q)^{-*}] > 0. \tag{5.9}$$

The previous formula follows from the next lemma which is more general and is in the spirit of the results proved in [6].

Lemma 5.21. *Let Ω be an axially symmetric domain in $\mathbb{H}$ and let $F : \Omega \to \mathbb{H}$ be a slice–regular function. If $\Re e F > 0$ in Ω, then $\Re e(F^{-*}) > 0$ in $\Omega' := \Omega \setminus Z(F^s)$.*

Proof. The function F^{-*} is well defined in Ω', where the following formula holds: $F^{-*}(q) = F(T(q))^{-1}$ with $T(q) := F^c(q)^{-1}qF^c(q)$ which maps Ω' to itself. Now from $q^{-1} = |q|^{-2}\overline{q}$, we have

$$\Re e\, (F^{-*}(q)) = \Re e\, (F(T(q))^{-1}) = |F(T(q))|^{-2}\Re e\, (F(T(q))) > 0.$$

□

Notice that in our case $h^s(q) = q^{-2}f^s(q) * {f'^s(q)}^{-*}$ and $Z(h^s) = \varnothing$. Indeed $Z(h^s) = Z(q^{-1}f(q) * q^{-1}f^c(q))$.

Furthermore, if $q_0 = x_0 + y_0 I_0 \in \Omega$, $J \perp I_0$ and $q \in L_{I_0}$, thanks to Lemma 5.4, one can write $f_{I_0}(q) = F_1(q) + F_2(q)J$ and $\partial_C f(q) = F_1'(q) + F_2'(q)J$. Using the expressions (1.16), (1.19) and (1.20) as in [8], we get

$$[(\partial_C f(q))^{-*}]_{|I_0} = \left[F_1'(q)\overline{F_1'(\overline{q})} + F_2'(q)\overline{F_2'(\overline{q})}\right]^{-1} \cdot [\overline{F_1'(\overline{q})} - F_2'(q)J]$$

and so the real part of $\left\{[q^{-1}f(q) * \partial_C f(q)^{-*}]_{|I_0}\right\}$ is

$$\Re e \left\{ q^{-1}[F_1'(q)\overline{F_1'(\overline{q})} + F_2'(q)\overline{F_2'(\overline{q})}]^{-1} \cdot [F_1(q)\overline{F_1'(\overline{q})} + F_2(q)\overline{F_2'(\overline{q})}] \right\}.$$

Remark 5.22. It is not difficult to observe that for slice-preserving functions (i.e. slice-regular functions that map each slice L_I to itself) the condition of starlikeness is equivalent to the one of algebraic starlikeness. Indeed, the Cullen and Spherical derivatives of f are also slice-preserving functions so that the complex Jacobian of f at q_0 is diagonal. It then turns out that $(Df_{q_0})^{-1}[f(q_0)] = f(q_0) \cdot \partial_C f(q_0)^{-1}$, and therefore $0 < \Re e[qf(q) * \partial_C f(q)^{-*}] = \Re e[qf(q) \cdot \partial_C f(q)^{-1}]$, since from (5.1) for slice-preserving functions the $*$–product coincides with the usual one (see Lemma 1.30 in [8]).

5.2. **Hadamard conditions for algebraic starlikeness.** Assume f is in $\mathcal{AS}^*_{\mathbb{H}}$. Under this hypothesis, the (slice–regular) function $q^{-1}f(q)$ is well defined, and if $f(q) = \sum_{n\geq 1} q^n a_n$, then $q^{-1}f(q) = \sum_{n\geq 1} q^{n-1}a_n$. Let g be a primitive of $q^{-1}f(q)$ such that $g(0) = 0$; in other words, if $g(q) = \sum_{n\geq 1} q^n b_n$, then $\partial_C g(q) = q^{-1}f(q)$. This condition implies that $nb_n = a_n$ for any $n \geq 0$. In particular, since $a_1 = 1$, then $b_1 = 1$.

Definition 5.23. We say that a slice–regular function g in the unit ball, such that $g(0) = 0$ and $\partial_C g(0) = 1$ is *algebraically convex* iff $\Re e\, \{\partial_C(q\partial_C g)(q) * [\partial_C g(q)]^{-*}\} > 0$.

Thanks to the previous definition we obtain this analog of Alexander's Theorem

Lemma 5.24. *If a function f is in $\mathcal{AS}^*_{\mathbb{H}}$, then the primitive g of $q^{-1}f(q)$ such that $g(0) = 0$ is algebraically convex.*

Proof. Since

$$\Re e\left(\partial_C(q\partial_C g)(q) * \partial_C g(q)^{-*}\right) = \Re e\left(1 + q\partial_C(\partial_C g(q)) * \partial_C g(q)^{-*}\right)$$

$$= \Re e\left(1 + (q\partial_C f(q) - f(q)) * f(q)^{-*}\right) = \Re e\left(q\partial_C f(q) * f(q)^{-*}\right) > 0$$

we conclude that g is algebraically convex. □

We have the following results which generalize Proposition 4.1 (see also [11]).

Proposition 5.25. *If a function f is in $\mathcal{AS}^*_{\mathbb{H}}$, then for any $s \in \mathbb{R}$, $I \in \mathbb{S}_{\mathbb{H}}$ and $q \in B(0,1)$ it turns out that*

$$q^{-1}[f(q) *_H [(1-q)^{-2}(q(1-sI) + q^2 sI)]] \neq 0 \tag{5.10}$$

or equivalently, if g is a primitive of $q^{-1}f(q)$ such that $g(0) = 0$, then

$$q^{-1}[g(q) *_H (1-q)^{-3}(q(1-sI) + q^2(1+sI))] \neq 0. \tag{5.11}$$

Vice versa, if for a slice–regular function f and for the primitive g of $q^{-1}f(q)$ (such that $g(0) = 0$) conditions (5.10) and (5.11) hold, then $f \in \mathcal{AS}^*_{\mathbb{H}}$ and g is algebraically convex.

Proof. Note that for $q \in B(0,1)$

$$\begin{aligned}
& q^{-1}[g(q) *_H (1-q)^{-3}(q(1-sI) + q^2(1+sI))] \\
= \; & q^{-1}[g(q) *_H q(1-q)^{-3}(2q(1+qsI-sI) + (1-q)(1-sI+2qsI))] \\
= \; & q^{-1}[g(q) *_H q(1-q)^{-3}(2q(1+qsI-sI)] \\
& + q^{-1}[g(q) *_H q(1-q)^{-2}(1-sI+2qsI)] \\
= \; & q^{-1}[g(q) *_H q\partial_C[(1-q)^{-2}(q(1-sI)+q^2 sI)]].
\end{aligned}$$

Using equation (4.2) it then follows that

$$\begin{aligned}
& q^{-1}[g(q) *_H q\partial_C[(1-q)^{-2}(q(1-sI)+q^2 sI)]] \\
= \; & q^{-1}[f(q) *_H [(1-q)^{-2}(q(1-sI)+q^2 sI)]],
\end{aligned}$$

so we conclude that condition (5.10) is equivalent to condition (5.11).

Now from the algebraic starlikeness of f it turns out that g is algebraically convex; then $\Re e[\partial_C(q\partial_C g)(q) * (\partial_C g(q))^{-*}] > 0$. From the assumptions $f(0) = 0$ and $\partial_C f(0) = 1$ it also follows that

$$\partial_C(q\partial_C g)(q) * (\partial_C g(q))^{-*}|_{q=0} = 1$$

and from $g(q) = q + \sum_{n\geq 2} q^n b_n$ we have

$$\partial_C(q\partial_C g)(q) = 1 + \sum_{n\geq 2} n^2 q^{n-1} b_n = \partial_C g(q) *_H \sum_{n\geq 1} n q^{n-1} = \partial_C g(q) *_H (1-q)^{-2}.$$

Hence $\Re e[\partial_C(q\partial_C g)(q) * (\partial_C g(q))^{-*}] > 0$ is equivalent to $\partial_C(q\partial_C g)(q) * (\partial_C g(q))^{-*} \neq sI$ for any $s \in \mathbb{R}$ and for any $I \in \mathbb{S}_{\mathbb{H}}$ or to

$$\partial_C(q\partial_C g)(q) - \partial_C g(q) sI \neq 0$$

which can be also written as $\partial_C g(q) *_H [(1-q)^{-2} - (1-q)^{-1} sI] \neq 0$ thanks to the associativity property of the Hadamard product and the notation adopted for the neutral element of the Hadamard product. Finally, from

$$\partial_C g(q) *_H [(1-q)^{-2} - (1-q)^{-1} sI] = \partial_C g(q) *_H q^{-1}[(1-q)^{-2}(q(1-sI) + q^2 sI)]$$

we conclude our proof. □

The authors are also looking for possible new characterizations of other classes of functions of a hypercomplex variable and for their applications to the proof of other statements. In this sense, a version of the solution of the Bieberbach conjecture for starlike functions of a hypercomplex variable will be given in a forthcoming paper.

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Dipartimento di Matematica, Università di Milano, Via Saldini 50, 20133 Milano, Italy

E-mail address: anna.gori@unimi.it

Dipartimento di Matematica e Informatica "U. Dini", Università di Firenze Viale Morgagni 67/A, 50134 Firenze, Italy

E-mail address: vlacci@math.unifi.it