CANONICAL QUASICRYSTALLINE MULTILAYERED METAMATERIALS

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Summary A new class of two-phase periodic laminates with a *quasicrystalline* structure (generated by the Fibonacci substitution rule) is introduced. Recently, we found that the Floquet-Bloch spectrum of antiplane waves propagating in this particular type of layered composite has a self-similar layout which can be characterized through an invariant function of the frequency, the so-called Kohmoto invariant. Moreover, for particular ratios between the geometrical and constitutive parameters of the two constituent phases (*canonical* ratios), the spectra are periodic. We illustrate how these two unique properties can be used to design quasicrystalline lamitates providing negative refraction of an antiplane wave obliquely incident at the interface with an elastic substrate. It is shown that, beyond a certain frequency threshold, high order Fibonacci laminates allow negative refraction of a single transmitted mode at lower frequencies with respect to a periodic classical bilayer. The attained results represent an important advancement towards the realisation of multilayered quasicrystalline metamaterials with the aim to control negatively refracted elastic waves.

QUASICRYSTALLINE-GENERATED LAMINATES

We define a class of two-dimensional, two-phase quasicrystalline laminates with layering direction parallel to the axis y (see Fig. 1/(A)). Each of its elements is composed of a repeated elementary cell $\mathcal{F}_i$ where the two basic components, A and B, are arranged in series according to the standard Fibonacci sequence, which is based on the following substitution rule $A \rightarrow AB,\ B \rightarrow A$. The repetition of the fundamental cells assures global periodicity along axis x and the possibility of applying the Floquet-Bloch technique in order to study harmonic wave propagation in these systems.

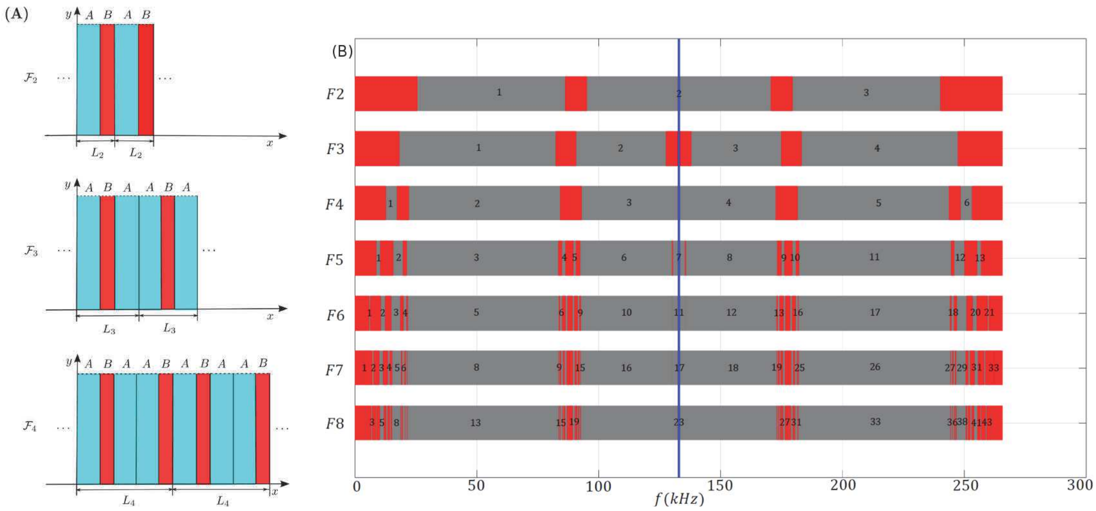

Figure 1: (A): two-dimensional laminates assembled according to $\mathcal{F}_2$, $\mathcal{F}_3$ and $\mathcal{F}_4$ Fibonacci cells. (B): Diagram reporting, in the grey zones, the number N_i^f of real solutions K_y of the dispersion relation for cells $\mathcal{F}_2$ to $\mathcal{F}_8$ as a function of the frequency. Transition zones are highlighted in red.

The dispersion relation for Floquet-Bloch harmonic antiplane waves propagating in Fibonacci laminates as those illustrated in Fig. 1/(A) assumes the form

$$\cos\left(K_x L_i\right) = \frac{1}{2}\mathrm{tr}\mathbf{M}_i(f, K_y), \tag{1}$$

where $\mathbf{M}_i$ is the trasmission matrix of the ith-order cell $\mathcal{F}_i$, L_i is the total length of the cell, $f = \omega/2\pi$ is the wave frequency, K_x and K_y are the components of the wave vector directed along $x-$ and $y-$ axis, respectively. $\mathbf{M}_i$ is unimodular, i.e. $\det\mathbf{M}_i = 1$, and follows the recursion rule $\mathbf{M}_{i+1} = \mathbf{M}_{i-1}\mathbf{M}_i$, with $\mathbf{M}_0 = \mathbf{M}_B$ and $\mathbf{M}_1 = \mathbf{M}_A$ [1]. Assuming a given wave frequency f, for any real value of K_x we found a finite number N_i^f of real and an infinite number of imaginary solutions K_y of the dispersion relation (1) [2]. This means that for any real K_x we have N_i^f propogating waves and infinite evanescent modes along the $y-$direction. The numerical solution of (1), obtained for a determinate set of frequencies by varying K_x along the intervals $0 \le K_x \le m\pi/L_i$, $m \in \mathbb{N}$ (*Brillouin zones*), shows that: i) assuming given values of f and K_x, N_i^f increases for high order Fibonacci cells $\mathcal{F}_i$ [3]; ii) if f belongs to some particular frequency ranges, N_i^f depends on the value of K_x and we have $N_i^f = t-1$ for $0 \le K_x \le \overline{K}_x$ and $N_i^f = t$ for $\overline{K}_x \le K_x \le m\pi/L_i$. We denote these frequency ranges as *transition zones*. The t-th transition zone is that where N_i^f switches from $t-1$ to t.

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SELF-SIMILAR TRANSITION ZONE LAYOUT AND CANONICAL RATIO

An example of transition zone layout for laminates designed according to cells $\mathcal{F}_2$ to $\mathcal{F}_8$ is reported in Fig. 1/(B) as a function of the frequency. We note that they are disposed according to a self-similar pattern. The analysis of the dispersion curves obtained assuming $K_y = 0$ (transverse wave propagation in the laminate) reveals that for any cell $\mathcal{F}_i$ the position of pass bands coincide with that of transition zones. In this specific case, the transfer matrix $\mathbf{M}_i$ and the dispersion relation (1) become identical to those defined in [1] and [5]. Consequently, the self similarity is governed by a local scaling whose factor is analogous to that used in those papers for studying the spectrum of quasicrystalline-generated structural waveguides:

$$\kappa_0^+(f) = \frac{1}{4}\left(\sqrt{4 + (4 + I_0(f))^2} + (4 + I_0(f))\right)^2, \tag{2}$$

where $I_0(f)$ is the so-called Kohmoto's invariant [4]. A transition zone $f_i^B - f_i^A$ of the cell $\mathcal{F}_i$ centred at a frequency f^* is related to the transition zone $f_{i+6}^B - f_{i+6}^A$ of $\mathcal{F}_{i+6}$ centred almost about the same value f^* by the scaling law $f_{i+6}^B - f_{i+6}^A \approx (f_i^B - f_i^A)/\kappa$, where $\kappa = \kappa_0^+(f^*)$. Similarly, $f_{i+3}^B - f_{i+3}^A \approx (f_i^B - f_i^A)/\lambda$, with $\lambda = \sqrt{\kappa}$.
By observing Fig. 1/(B), we also note that for any cell $\mathcal{F}_i$, the arrangement of the transition zones is periodic. This is achieved for rational values of the ratio $\beta = c_A h_B / c_B h_A$, where h_A and h_B are the thicknesses of phases A and B, c_A and c_B the shear wave speed in materials A and B, respectively. We denote β as *canonical ratio*, and the laminates characterized by $\beta \in \mathbb{Q}$ as *canonical laminates* (for the case shown in Fig.1/(B), $\beta = 2$).

NEGATIVE REFRACTION USING QUASICRYSTALLINE LAMINATES

We now use the quasicrystalline laminates to obtain negative refraction of an antiplane wave across an interface with an elastic substrate (schematic of the problem reported in Fig.2/(A)). For each value of K_x corresponding to a given frequency and an arbitrary angle of incidence in the interval $0 \leq \theta \leq \pi/2$, we have N_i^f real solutions of the dispersion relation (1). These real solutions correspond to propagating modes transmitted at the interface. In order to have only one single negatively refracted mode (*pure negative refraction*), the incident wave frequency should belong to the interval $f_i^{min} \leq f \leq \tilde{f}_i$ [3], where $f_i^{min} = \sqrt{\mu_0}/(2L_i\rho_0)$ (with μ_0 and ρ_0 shear modulus and mass density of the substrate, respectively), and $\tilde{f}_i$ is the highest frequency of the second transition zone. Remembering the spectrum analysis reported in the first section, we can calculate $\tilde{f}_i$ for any Fibonacci cell $\mathcal{F}_i$. Moreover, using the scaling factor (2) together with the associated scaling relationships and the condition of periodicity $\beta \in \mathbb{Q}$, we can design canonical laminates providing pure negative refraction in several ranges of frequencies considering different elastic substrates. The results illustrated in Fig.2/(B) for a PMMA-steel laminate bonded to a polyethylene substrate show that, by considering the same angle of incidence ($\theta = 20°$) high-order Fibonacci cells yield single negatively refracted modes at lower frequencies with respect to standard two-phase periodic laminates (represented by cell $\mathcal{F}_2$).

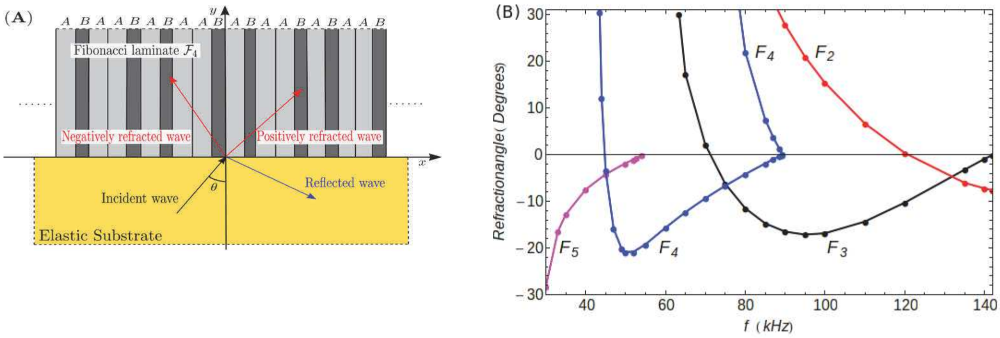

Figure 2: (A): Schematic of the problem of an antiplane wave approaching the interface between an elastic substrate and a Fibonacci laminate $\mathcal{F}_4$. (B): Angles of refraction corresponding to an incident angle $\theta = 20°$ plotted versus the frequency for cells $\mathcal{F}_2$ to $\mathcal{F}_5$.

CONCLUSIONS

The propagation of antiplane shear waves in two-phase periodic quasicrystalline-generated laminates following the Fibonacci substitution rules has been studied. The Floquet-Bloch analysis reveals that particular ranges of frequencies, called transition zones, are disposed in the dispersion diagrams according to a self-similar pattern. Numerical results show that these self-similarity properties together with the condition for the periodicity of the spectrum can be used to design high order Fibonacci laminates allowing negative wave refraction of a single transmitted mode at lower frequencies with respect to a classical periodic bilayer.

References

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