Lipschitz stability for the inverse conductivity problem for a conformal class of anisotropic conductivities

Romina Gaburro[1] **and Eva Sincich**[2]

[1] Departmentof Mathematics and Statistics, University of Limerick, Castletroy, Limerick, Ireland
[2] Laboratory for Multiphase Processes, University of Nova Gorica, Vipavska 11c SI-5270 Ajdovscina, Slovenia

E-mail: romina.gaburro@ul.ie and eva.sincich@ung.si

Abstract

We consider the stability issue of the inverse conductivity problem for a conformal class of anisotropic conductivities in terms of the local Dirichlet–Neumann map. We extend here the stability result obtained by Alessandrini and Vessella (Alessandrini G and Vessella S 2005 Lipschitz stability for the inverse conductivity problem *Adv. Appl. Math.* **35** 207–241), where the authors considered the piecewise constant isotropic case.

Keywords: Lipschitz stability, anisotropic conductivity, conformal class

1. Introduction

In the present paper we study the stability issue for the inverse conductivity problem in the presence of anisotropic conductivity which is *a priori* known to depend linearly on an unknown piecewise-constant function. Let us start by recalling the basic formulation of the inverse conductivity problem.

In absence of internal sources, the electrostatic potential u in a conducting body, described by a domain $\Omega \subset \mathbb{R}^n$, is governed by the elliptic equation

$$\text{div}\,(\sigma\,\nabla u) = 0 \qquad \text{in} \quad \Omega, \tag{1.1}$$

where the symmetric, positive definite matrix $\sigma = \sigma(x)$, $x \in \Omega$ represents the (possibly anisotropic) electric conductivity. The inverse conductivity problem consists of finding σ when the so called Dirichlet–Neumann (D–N) map

$$\Lambda_\sigma\colon u\Big|_{\partial\Omega} \in H^{\frac{1}{2}}(\partial\Omega) \to \sigma\,\nabla u\cdot\nu\Big|_{\partial\Omega} \in H^{-\frac{1}{2}}(\partial\Omega)$$

is given for any $u \in H^1(\Omega)$ solution to (1.1). Here, ν denotes the unit outer normal to $\partial\Omega$. If measurements can be taken only on one portion Σ of $\partial\Omega$, then the relevant map is called the local D–N map. Let Σ be a non-empty open portion of $\partial\Omega$ and let us introduce the subspace of $H^{\frac{1}{2}}(\partial\Omega)$

$$H^{\frac{1}{2}}_{co}(\Sigma) = \left\{ f \in H^{\frac{1}{2}}(\partial\Omega) \,\Big|\, \operatorname{supp} f \subset \Sigma \right\}. \tag{1.2}$$

The local D–N map is given, in its weak formulation, as the operator Λ_σ^Σ such that

$$\left\langle \Lambda_\sigma^\Sigma\, u,\ \phi \right\rangle = \int_\Omega \sigma\ \nabla u\cdot\nabla\phi, \tag{1.3}$$

for any $u,\ \phi \in H^1(\Omega)$, $u|_{\partial\Omega}$, $\phi|_{\partial\Omega} \in H^{\frac{1}{2}}_{co}(\Sigma)$ and u is a weak solution to (1.1).

The problem of recovering the conductivity of a body by taking measurements of voltage and current on its surface has came to be known as electrical impedance tomography (EIT). Different materials display different electrical properties, so that a map of the conductivity $\sigma(x)$, $x \in \Omega$ can be used to investigate internal properties of Ω. EIT has many important applications in fields such as geophysics, medicine and non-destructive testing of materials. The first mathematical formulation of the inverse conductivity problem is due to Calderón [C], where he addressed the problem of whether it is possible to determine the (isotropic) conductivity $\sigma = \gamma I$ by the D–N map. Although Calderón studied the problem of determining σ from the knowledge of the quadratic form

$$Q_\gamma(u) = \int_\Omega \gamma|\nabla u|^2,$$

where u is a solution to (1.1), it is well known that the knowledge of Q_σ is equivalent to the knowledge of Λ_σ by

$$Q_\gamma(u) = \left\langle \Lambda_\sigma u, u \right\rangle, \qquad \text{for every } u \in H^1(\Omega),$$

where $\sigma = \gamma I$. Here $\langle\, \cdot\,,\, \cdot\, \rangle$ denotes the dual pairing between $H^{1/2}(\partial\Omega)$ and its dual $H^{-1/2}(\partial\Omega)$, with respect to the L^2 scalar product. Reference [C] opened the way to the solution to the uniqueness issue where one is asking whether σ can be determined by the knowledge of Λ_σ (or Λ_σ^Σ in the case of local measurements). As main contributions in this respect we mention the papers by Kohn and Vogelius [K-Vo1, K-Vo2], Sylvester and Uhlmann [S-U] and Nachman [Na]. We refer to [Bo, Ch-I-N] and [U] for an overview of recent developments regarding the issues of uniqueness and reconstruction of the conductivity. Regarding the stability, Alessandrini proved in [A] that, assuming $n \geqslant 3$ and *a priori* bounds on γ of the form

$$\|\gamma\|_{H^s(\Omega)} \leqslant E, \quad \text{for some } s > \frac{n}{2} + 2, \tag{1.4}$$

γ depends continuously on Λ_σ with a modulus of continuity of logarithmic type. In [A1, A2] the same author subsequently proved that a similar stability estimate holds when the *a priori* bound (1.4) is replaced by

$$\|\gamma\|_{W^{2,\infty}(\Omega)} \leqslant E. \tag{1.5}$$

Logarithmic type stability estimates have recently been proved for $\gamma \in C^{1,\varepsilon}(\bar\Omega)$ and $n \geqslant 3$ in [Ca-G-R]. For the two-dimensional case, logarithmic type stability estimates were obtained in [B-B-R, B-F-R, Liu]. Unfortunately, all the above results share the common

inconvenient logarithmic type of stability which cannot be avoided [A3]. In fact Mandache [Ma] showed that the logarithmic stability is the best possible, in any dimension $n \geqslant 2$ if *a priori* assumptions of the form

$$\| \gamma \|_{C^k(\Omega)} \leqslant E \tag{1.6}$$

for any $k = 0, 1, 2,...$ are assumed. It seems therefore reasonable to think that, in order to restore stability in a really (Lipschitz) stable fashion, one needs to replace in some way the *a priori* assumptions expressed in terms of regularity bounds such as (1.6), with *a priori* pieces of information of a different type. Alessandrini and Vessella showed in [A-V] that γ depends in a Lipschitz continuous fashion upon the local D–N map, by assuming that γ is a function *a priori* known to be piecewise constant

$$\gamma(x) = \sum_{j=1}^{N} \gamma_j \chi_{D_j}(x), \tag{1.7}$$

where each subdomain of Ω, D_j, $j = 1,...,N$ is *given* and each number γ_j, $j = 1,...,N$ is *unknown*. From a medical imaging point of view, each D_j may represent the area occupied by different tissues or organs and one can think that the geometrical configuration of each D_j is given by means of other imaging techniques such as MRI for example. Since most tissues in the human body are anisotropic, the present authors, motivated by the work in [A-V] and its medical application, consider here the more general case of an *anisotropic* conductivity of type

$$\sigma(x) = \gamma(x)A(x),$$

where $A(x)$ is a known, matrix valued function which is Lipschitz continuous and $\gamma(x)$ is of type (1.7). Anisotropic conductivity appears in nature, for example as a homogenization limit in layered or fibrous structures such as rock stratum or muscle, as a result of crystalline structure or of deformation of an isotropic material, therefore the case treated in this paper seems to be a natural extension of [A-V] relevant to several applications.

It is well known that since Tartar's observation [K-Vo1] that any diffeomorphism of Ω which keeps the boundary points fixed has the property of leaving the D–N map unchanged, whereas σ is modified, different lines of research have been pursued. One direction has been to find the conductivity up to a diffeomorphism which keeps the boundary fixed (see [LU, Na, S, La-U]). Another direction has been the one to assume that the anisotropic conductivity is *a priori* known to depend on a restricted number of spatially-dependent parameters (see [A, A-G, A-G1,G-L, L,]). Nevertheless the problem of proving uniqueness for anisotropic conductivities still represents an open problem in dimension $n \geqslant 3$, where the two-dimensional case can be considered settled, in fact it has been proved in [A-La-P]) that an anisotropic L^∞-conductivity can be determined in dimension 2 up to a $W^{1,2}$-diffeomorphism.

The present work is concerned about the stability issue for the Calderòn problem and it represents the first result of Lipschitz stability estimates for conductivities of anisotropic type as far as the authors know, therefore the simplest possible extension to the work done in [A-V] to anisotropic conductivities has been considered in this manuscript. The paper improves upon the results obtained in [A-V] in the sense that the global Lipschitz stability estimate obtained there is here adapted to a special anisotropic type of conductivity. The class of anisotropic conductivities considered here includes some piecewise constant anisotropic ones but not all. This is due to the fact that the construction of the fundamental solution for the conductivity equation with matrix valued constant coefficients (see (4.81)) would require a different type of study to the one carried out in [A-V] and in the present work. The precise assumptions shall be illustrated in section 2. The authors hope that the stability estimate

obtained here for this simple case of anisotropy will stimulate further study of the stability issue for anisotropic conductivities.

For related results in the anisotropic case we also refer to [Be, D-Ke-S-U, D-Ku-L-S, F-K-R, H-S, K, N-S]. We also recall [Be-Fr, Be-Fr-V, B-dH-Q] where similar Lipschitz stability results have been obtained for complex conductivity, the Lamé parameters and for a Schrödinger type of equation respectively.

For a more in-dept description and consideration of the stability issue and related open problems in the inverse conductivity problem we refer to [A3] and [A-V].

Our approach follows the one by Alessandrini and Vessella [A-V] of constructing singular solutions and studying their asymptotic behaviour when the singularity approaches the discontinuity interfaces for the conductivity. However, in order to deal with the present structure of the conductivity we had to develop original asymptotic analysis estimates and an accurate quantitative control of the error terms which represent a novel feature in the treatment of anisotropic type of conductivity.

The paper is organized as follows. Our main assumptions and our main result (theorem 2.1) are contained in section 2, where the proof of theorem 2.1 is contained in section 3. This section also lists the two main results (theorem 3.4 and proposition 3.5) needed to build the machinery for the proof of theorem 2.1. Theorem 3.4 provides original asymptotic estimates for the Green function of the conductivity equation, for conductivities belonging to a special anisotropic conformal class $\mathcal{C}$, at the interfaces between the given domains D_j, where the conductivity is discontinuous. Proposition 3.5 provides estimates of unique continuation of the solution to the conductivity equation for conductivities in $\mathcal{C}$. Section 4 is devoted to the proofs of theorem 3.4 and proposition 3.5. For the proof of theorem 3.4 we provide the explicit form of the fundamental solution for the conformal anisotropic two-phase case with flat interface. The proof of proposition 3.5 is a straight forward consequence proposition 4.3 which we state in this section. The proof of the latter is independent from the presence of anisotropy in the conductivity, therefore we refer to [A-V] for a full proof of it. In this paper we point out the main facts on which the proof is based on only.

2. Main result

2.1. Notation and definitions

In several places within this manuscript it will be useful to single out one coordinate direction. To this purpose, the following notations for points $x \in \mathbb{R}^n$ will be adopted. For $n \geqslant 3$, a point $x \in \mathbb{R}^n$ will be denoted by $x = (x', x_n)$, where $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$. Moreover, given a point $x \in \mathbb{R}^n$, $B_r(x)$, $B'_r(x)$ the open balls in $\mathbb{R}^n$, $\mathbb{R}^{n-1}$ centered at x, x', respectively with radius r and by $Q_r(x)$ the cylinder

$$Q_r(x) = B'_r(x') \times (x_n - r, x_n + r).$$

We shall also denote

$$\mathbb{R}^n_+ = \left\{ (x', x_n) \in \mathbb{R}^n \big| x_n > 0 \right\}; \quad \mathbb{R}^n_- = \left\{ (x', x_n) \in \mathbb{R}^n \big| x_n < 0 \right\};$$
$$B^+_r = B_r \cap \mathbb{R}^n_+; \quad B^-_r = B_r \cap \mathbb{R}^n_-;$$
$$Q^+_r = Q_r \cap \mathbb{R}^n_+; \quad Q^-_r = Q_r \cap \mathbb{R}^n_-.$$

where B_r, Q_r denote $B_r(0)$, $Q_r(0)$, respectively.

In the sequel, we shall make a repeated use of quantitative notions of smoothness for the boundaries of various domains. Let us introduce the following notation and definitions.

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Definition 2.1. Let Ω be a domain in $\mathbb{R}^n$. We say that a portion Σ of $\partial\Omega$ is of Lipschitz class with constants r_0, L if for any $P \in \Sigma$ there exists a rigid transformation of $\mathbb{R}^n$ under which we have $P \equiv 0$ and

$$\Omega \cap Q_{r_0} = \left\{ x \in Q_{r_0} : x_n > \varphi(x') \right\},$$

where φ is a Lipschitz function on B'_{r_0} satisfying

$$\varphi(0) = \left| \nabla_{x'}\varphi(0) \right| = 0; \qquad \|\varphi\|_{C^{0,1}\left(B'_{r_0}\right)} \leqslant Lr_0.$$

It is understood that $\partial\Omega$ is of Lipschitz class with constants r_0, L as a special case of Σ, with $\Sigma = \partial\Omega$. Here and in the sequel B'_{r_0} denotes $B'_{r_0}(0)$.

Definition 2.2. Let Ω be a domain in $\mathbb{R}^n$. Given α, $\alpha \in (0, 1]$, we say that a portion Σ of $\partial\Omega$ is of class $C^{1,\alpha}$ with constants r_0, M if for any $P \in \Sigma$ there exists a rigid transformation of $\mathbb{R}^n$ under which we have $P = 0$ and

$$\Omega \cap Q_{r_0} = \left\{ x \in Q_{r_0} : x_n > \varphi(x') \right\},$$

where φ is a $C^{1,\alpha}$ function on B'_{r_0} satisfying

$$\varphi(0) = \left| \nabla_{x'}\varphi(0) \right| = 0; \qquad \|\varphi\|_{C^{1,\alpha}\left(B'_{r_0}\right)} \leqslant Mr_0,$$

where we denote

$$\|\varphi\|_{C^{1,\alpha}\left(B'_{r_0}\right)} = \|\varphi\|_{L^\infty\left(B'_{r_0}\right)} + r_0 \, \|\nabla\varphi\|_{L^\infty\left(B'_{r_0}\right)} + r_0^{1+\alpha} \sup_{\substack{x,y\in B'_{r_0} \\ x \neq y}} \frac{|\nabla\varphi(x) - \nabla\varphi(y)|}{|x - y|^\alpha}.$$

Let us rigorously define the local D–N map.

Definition 2.3. Let Ω be a domain in $\mathbb{R}^n$ with Lipschitz boundary $\partial\Omega$ and Σ an open non-empty subset of $\partial\Omega$. Assume that $\sigma \in L^\infty(\Omega, Sym_n)$ satisfies the ellipticity condition

$$\lambda^{-1}|\xi|^2 \leqslant \sigma(x)\xi \cdot \xi \leqslant \lambda|\xi|^2, \qquad \text{for almost every } x \in \Omega,$$
$$\text{for every } \xi \in \mathbb{R}^n. \tag{2.1}$$

The local D–N map associated to σ and Σ is the operator

$$\Lambda_\sigma^\Sigma : H_{co}^{\frac{1}{2}}(\Sigma) \to H_{co}^{-\frac{1}{2}}(\Sigma) \tag{2.2}$$

defined by

$$<\Lambda_\sigma^\Sigma \, g, \, \eta> \; = \; \int_\Omega \sigma(x)\,\nabla u(x) \cdot \nabla\phi(x)\,\mathrm{d}x, \tag{2.3}$$

for any $g, \eta \in H_{co}^{\frac{1}{2}}(\Sigma)$, where $u \in H^1(\Omega)$ is the weak solution to

$$\begin{cases} \mathrm{div}\,(\sigma(x)\,\nabla u(x)) = 0, & \text{in} \quad \Omega, \\ u = g, & \text{on} \quad \partial\Omega, \end{cases}$$

and $\phi \in H^1(\Omega)$ is any function such that $\phi|_{\partial\Omega} = \eta$ in the trace sense. Here we denote by $<\cdot, \cdot>$ the $L^2(\partial\Omega)$-pairing between $H_{co}^{\frac{1}{2}}(\Sigma)$ and its dual $H_{co}^{-\frac{1}{2}}(\Sigma)$.

Note that, by (2.3), it is easily verified that Λ_σ^Σ is selfadjoint. We shall denote by $\|\cdot\|_*$ the norm on the Banach space of bounded linear operators between $H_{co}^{\frac{1}{2}}(\Sigma)$ and $H_{co}^{-\frac{1}{2}}(\Sigma)$.

2.2. Our assumptions

We give here the precise assumptions for the domain Ω under investigation and its conductivity σ. The dimension of the space for Ω is denoted by n and for sake of simplicity we only consider $n \geqslant 3$.

2.2.1. Assumptions about the domain Ω.

(1) We assume that Ω is a domain in $\mathbb{R}^n$ satisfying

$$|\Omega| \leqslant N r_0^n, \qquad (2.4)$$

where $|\Omega|$ denotes the Lebesgue measure of Ω.

(2) We assume that $\partial\Omega$ is of Lipschitz class with constants r_0, L.

(3) We fix an open non-empty subset Σ of $\partial\Omega$ (where the measurements in terms of the local D–N map are taken).

(4)

$$\bar{\Omega} = \bigcup_{j=1}^{N} \bar{D}_j,$$

where D_j, $j = 1,...,N$ are known open sets of $\mathbb{R}^n$, satisfying the conditions below.

(a) D_j, $j = 1,...,N$ are connected and pairwise nonoverlapping.

(b) ∂D_j, $j = 1,...,N$ are of Lipschitz class with constants r_0, L.

(c) There exists one region, say D_1, such that $\partial D_1 \cap \Sigma$ contains a $C^{1,\alpha}$ portion Σ_1 with constants r_0, M.

(d) For every $i \in \{2...,N\}$ there exists $j_1,...,j_K \in \{1,...,N\}$ such that

$$D_{j_1} = D_1, \qquad D_{j_K} = D_i. \qquad (2.5)$$

In addition we assume that, for every $k = 1,...,K$, $\partial D_{j_k} \cap \partial D_{j_{k-1}}$ contains a $C^{1,\alpha}$ portion Σ_k (here we agree that $D_{j_0} = \mathbb{R}^n \backslash \Omega$), such that

$$\Sigma_1 \subset \Sigma,$$

$$\Sigma_k \subset \Omega, \quad \text{for every } k = 2,...,K,$$

and, for every $k = 1,...,K$, there exists $P_k \in \Sigma_k$ and a rigid transformation of coordinates under which we have $P_k = 0$ and

$$\Sigma_k \cap Q_{r_0/3} = \left\{ x \in Q_{r_0/3} \,\middle|\, x_n = \phi_k(x') \right\}$$

$$D_{j_k} \cap Q_{r_0/3} = \left\{ x \in Q_{r_0/3} \,\middle|\, x_n > \phi_k(x') \right\}$$

$$D_{j_{k-1}} \cap Q_{r_0/3} = \left\{ x \in Q_{r_0/3} \,\middle|\, x_n < \phi_k(x') \right\}, \qquad (2.6)$$

where ϕ_k is a $C^{1,\alpha}$ function on $B'_{r_0/3}$ satisfying

$$\phi_k(0) = \left| \nabla \phi_k(0) \right| = 0$$

and

$$\| \phi_k \|_{C^{1,\alpha}\left(B'_{r_0} \right)} \leqslant M r_0.$$

Definition 2.4. We shall say that $\sigma \in C$ if σ is of type

$$\sigma(x) = \sigma_A(x) = \sum_{j=1}^{N} \gamma_j \, A(x) \chi_{D_j}(x), \qquad x \in \Omega, \tag{2.7}$$

where γ_j are unknown real numbers, D_j, $j = 1,..., N$ are the given subdomains introduced in section 2.2.1 and

$$\bar{\gamma} \leqslant \gamma_j \leqslant \bar{\gamma}^{-1}, \qquad \text{for any } j = 1,... N. \tag{2.8}$$

$A(x)$ is a known Lipschitz matrix valued function satisfying

$$\|A\|_{C^{0,1}(\Omega)} \leqslant \bar{A}, \tag{2.9}$$

where $\bar{A} > 0$ is a constant and

$$\lambda^{-1} |\xi|^2 \leqslant A(x)\xi \cdot \xi \leqslant \lambda |\xi|^2, \qquad \begin{aligned} &\text{for almost every } x \in \Omega, \\ &\text{for every } \xi \in \mathbb{R}^n. \end{aligned} \tag{2.10}$$

Definition 2.5. Let N, r_0, L, M, α, λ, $\bar{\gamma}$, $\bar{A}$ be given positive numbers with $N \in \mathbb{N}$ and $\alpha \in (0, 1]$. We will refer to this set of numbers, along with the space dimension n, as to the *a priori* data. Our main result is the following.

Theorem 2.1. *Let Ω, D_j, $j = 1,..., N$ and Σ be a domain, N subdomains of Ω and a portion of $\partial\Omega$ as in section 2.2.1 respectively. If $\sigma_A^{(i)} \in C$, i = 1, 2 are two conductivities of type*

$$\sigma_A^{(i)}(x) = \sum_{j=1}^{N} \gamma_j^{(i)} \, A(x) \chi_{D_j}(x) \qquad x \in \Omega, \; i = 1, 2, \tag{2.11}$$

then we have

$$\|\sigma_A^{(1)} - \sigma_A^{(2)}\|_{L^\infty(\Omega)} \leqslant C \, \|\Lambda_{\sigma_A^{(1)}}^{\Sigma} - \Lambda_{\sigma_A^{(2)}}^{\Sigma}\|_*, \tag{2.12}$$

where C is a positive constant that depends on the a priori data only.

3. Proof of the main result

The proof of our main result (theorem 2.1) is based on an argument that combines asymptotic type of estimates for the Green's function of the operator

$$L = \text{div}(\sigma(x)\nabla) \qquad \text{in} \quad \Omega, \tag{3.1}$$

(theorem 3.4), with $\sigma \in C$, together with a result of unique continuation (proposition 3.5) for solutions to

$$Lu = 0, \qquad \text{in} \quad \Omega.$$

We shall give the precise formulation of these results in what follows.

3.1. Measurable conductivity σ

We shall start with some general considerations about the Green's function $G(x, y)$ associated to the operator (3.1), where σ is merely a measurable matrix valued function satisfying the ellipticity condition (2.1).

3.1.1. Green's function. If L is the operator given in (3.1), then for every $y \in \Omega$, the Green's function $G(\cdot, y)$ is the weak solution to the Dirichlet problem

$$\begin{cases} \operatorname{div}(\sigma \nabla G(\cdot, y)) = -\delta(\cdot -y), & \text{in} \quad \Omega, \\ G(\cdot, y) = 0, & \text{on} \quad \partial\Omega, \end{cases} \tag{3.1}$$

where $\delta(\cdot -y)$ is the Dirac measure at y. We recall that G satisfies the properties ([Lit-St-W])

$$G(x, y) = G(y, x), \quad \text{for every } x, y \in \Omega, \quad x \neq y, \tag{3.2}$$

$$0 < G(x, y) < |x - y|^{2-n}, \quad \text{for every } x, y \in \Omega, \quad x \neq y. \tag{3.3}$$

Moreover, the following result holds true.

Proposition 3.1. *For any $y \in \Omega$ and every $r > 0$ we have that*

$$\int_{\Omega \setminus B_r(y)} |\nabla G(\cdot, y)|^2 \leqslant Cr^{2-n}, \tag{3.4}$$

where $C > 0$ depends on λ and n only.

Proof. The proof can be obtained by combining Caccioppoli inequality with (3.3) ([A-V], proposition 3.1). $\qquad\square$

3.1.2. Integral solutions of L. Let $\sigma^{(i)}$, $i = 1, 2$ be two measurable matrix valued functions satisfying the ellipticity condition (2.1) and let $G_i(x, y)$ be the Green's functions associated to the operators

$$L_i = \operatorname{div}\left(\sigma^{(i)}(x) \nabla\right) \quad \text{in} \quad \Omega, \quad i = 1, 2. \tag{3.5}$$

Let $\mathcal{U}$ be an open subset of Ω and $\mathcal{W} = \Omega \setminus \overline{\mathcal{U}}$. For any $y, z \in \mathcal{W}$ we define

$$S_{\mathcal{U}}(y, z) = \int_{\mathcal{U}} \left(\sigma^{(1)}(x) - \sigma^{(2)}(x)\right) \nabla_x G_1(x, y) \cdot \nabla_x G_2(z, x) dx. \tag{3.6}$$

Remark 3.2.

$$\left|S_{\mathcal{U}}(y, z)\right| \leqslant C \|\sigma^{(1)} - \sigma^{(2)}\|_{L^\infty(\Omega)} \ (\mathrm{d}(y)\mathrm{d}(z))^{1-\frac{n}{2}}, \quad \text{for every } y, z \in \mathcal{W}, \tag{3.7}$$

where $\mathrm{d}(y) = \operatorname{dist}(y, \mathcal{U})$ and C is a positive constant depending on λ and n only.

Observe that (3.7) is a straightforward consequence of Hölder inequality and proposition 3.1. We constructed in this way an integral function $S_{\mathcal{U}}(\cdot, \cdot)$ on $\mathcal{W} \times \mathcal{W}$, which is written in terms of the two Green's functions $G_1(\cdot, y)$, $G_2(\cdot, z)$ of L_1, L_2 respectively; $S_{\mathcal{U}}(\cdot, z)$, $S_{\mathcal{U}}(y, \cdot)$ are in turn solutions for L_1, L_2 respectively on the complement part of $\mathcal{U}$ in Ω. More precisely we have

Proposition 3.3. *For every $y, z \in W$ we have that $S_{\mathcal{U}}(\cdot, z), S_U(y, \cdot) \in H^1_{loc}(W)$ are weak solutions to*

$$\text{div}\left(\sigma^{(1)}(\cdot)\,\nabla S_{\mathcal{U}}(\cdot, z)\right) = 0, \quad \text{div}\left(\sigma^{(2)}(\cdot)\,\nabla S_{\mathcal{U}}(y, \cdot)\right) = 0, \quad \text{in } \mathcal{W}. \quad (3.8)$$

Proof. The proof relies on differentiation under the integral sign arguments and the symmetry of G_i, $i = 1, 2$. □

3.2. Conductivity $\sigma \in C$

We shall denote with

$$\Gamma(x, y) = \frac{1}{(n-2)\omega_n}\,|x - y|^{2-n}, \quad (3.9)$$

the fundamental solution of the Laplace operator (here ω_n/n denotes the volume of the unit ball in $\mathbb{R}^n$). If D_i, $i = 1,\dots,N$ are the domains introduced in section 2.2.1 and L is the operator given by (3.1), with $\sigma \in C$, we shall give asymptotic estimates for the Green's function of L, with respect to (3.9) at the interfaces between the domains D_i, $i = 1,\dots N$. These estimates are given below. In what follows let G be the Green's function associated to the operator L in Ω.

3.2.1. Green's function

Theorem 3.4. *(Asymptotic estimates) For every $l \in \{1,\dots,K-1\}$, let $\nu(P_{l+1})$ denote the unit exterior normal to $D_{j_{l+1}}$ at the point P_{l+1}. There exist constants $\beta \in (0, \alpha)$ and $\bar{C} > 1$ depending on $\bar{\gamma}, \lambda, M, \alpha$ and n only such that the following inequalities hold true for every $\bar{x} \in B_{\frac{r_0}{\bar{C}}}(P_{l+1}) \cap D_{j_{l+1}}$ and every $\bar{y} = P_{l+1} + r\nu(P_{l+1})$, where $r \in (0, \frac{r_0}{\bar{C}})$*

$$\left| G(\bar{x}, \bar{y}) - \frac{2}{\gamma_{j_l} + \gamma_{j_l+1}}\Gamma\left(J(\bar{x}), J(\bar{y})\right) \right| \leqslant \frac{\bar{C}}{r_0^\beta}\,\left|\bar{x} - \bar{y}\right|^{\beta+2-n}, \quad (3.10)$$

$$\left| \nabla_x G(\bar{x}, \bar{y}) - \frac{2}{\gamma_{j_l} + \gamma_{j_l+1}}\nabla_x\Gamma\left(J(\bar{x}), J(\bar{y})\right) \right| \leqslant \frac{\bar{C}}{r_0^\beta}\,\left|\bar{x} - \bar{y}\right|^{\beta+1-n}, \quad (3.11)$$

where J is the positive definite matrix such that $J = \sqrt{A(P_{l+1})^{-1}}$.

3.2.2. Integral solutions of L: unique continuation.

We recall that up to a rigid transformation of coordinates we can assume that

$$P_1 = 0 \quad ; \quad \left(\mathbb{R}^n \backslash \Omega\right) \cap B_{r_0} = \left\{(x', x_n) \in B_{r_0} \,\big|\, x_n < \varphi(x')\right\},$$

where φ is a Lipschitz function such that

$$\varphi(0) = 0 \quad \text{and} \quad \|\varphi\|_{C^{0,1}\left(B'_{r_0}\right)} \leqslant Lr_0.$$

Denoting by

$$D_0 = \left\{ x \in \left(\mathbb{R}^n \backslash \Omega \right) \cap B_{r_0} \;\middle|\; |x_i| < \frac{2}{3} r_0, \; i = 1,..., n-1, \; \left| x_n - \frac{r_0}{6} \right| < \frac{5}{6} r_0 \right\},$$

it turns out that the augmented domain $\Omega_0 = \Omega \cup D_0$ is of Lipschitz class with constants $\frac{r_0}{3}$ and $\tilde{L}$, where $\tilde{L}$ depends on L only. We consider the operator L_i given by (3.5) and extend $\sigma^{(i)} \in C$ to $\tilde{\sigma}^{(i)} = \tilde{\gamma}^{(i)} \tilde{A}$ on Ω_0, by setting $\tilde{\gamma}^{(i)}|_{D_0} = 1$, and extending A to $\tilde{A} \in C^{0,1}(\Omega_0)$ with Lipschitz constant L, for $i = 1, 2$. We denote by $\tilde{G}_i$ the Green function associated to $\tilde{L}_i = \operatorname{div}(\tilde{\sigma}^{(i)}(x) \nabla \cdot)$ in Ω_0, for $i = 1, 2$. For any number $r \in \left(0, \frac{2}{3} r_0 \right)$ we also denote

$$\left(D_0 \right)_r = \left\{ x \in D_0 \;\middle|\; \operatorname{dist}(x, \Omega) > r \right\}.$$

Let us fix $k \in \{2,... N\}$ and recall that there exist $j_1,... j_K \in \{1,... N\}$ such that

$$D_{j_1} = D_1,... D_{j_K} = D_k.$$

We denote

$$\mathcal{W}_K = \bigcup_{i=0}^{K} D_{j_i}, \qquad \mathcal{U}_k = \Omega_0 \backslash \overline{\mathcal{W}_K}, \qquad \text{when } k \geqslant 0$$

$(D_{j_0} = D_0)$ and for any $y, z \in \mathcal{W}_K$

$$\tilde{S}_{\mathcal{U}_K}(y, z) = \int_{\mathcal{U}_K} \left(\tilde{\sigma}_A^{(1)}(\cdot) - \tilde{\sigma}_A^{(2)}(\cdot) \right) \nabla \tilde{G}_1(\cdot , y) \cdot \nabla \tilde{G}_2(\cdot , z), \qquad \text{when } k \geqslant 0.$$

We introduce for any number $b > 0$ as in [A-V], the concave non-decreasing function $\omega_b(t)$, defined on $(0, +\infty)$

$$\omega_b(t) = \begin{cases} 2^b e^{-2} |\log t|^{-b}, & t \in \left(0, e^{-2} \right), \\ e^{-2}, & t \in \left[e^{-2}, +\infty \right) \end{cases}$$

and denote

$$\omega_b^{(1)} = \omega, \qquad \omega_b^{(j)} = \omega_b \circ \omega_b^{(j-1)}.$$

The following parameters shall also be introduced

$$\beta = \arctan \frac{1}{L}, \qquad \beta_1 = \arctan\left(\frac{\sin \beta}{4} \right), \qquad \lambda_1 = \frac{r_0}{1 + \sin \beta_1}$$

$$\rho_1 = \lambda_1 \sin \beta_1, \qquad a = \frac{1 - \sin \beta_1}{1 + \sin \beta_1}$$

$$\lambda_k = a\lambda_{k-1}, \qquad \rho_k = a\rho_{k-1}, \qquad \text{for every } k \geqslant 2,$$

$$d_k = \lambda_k - \rho_k, \qquad k \geqslant 1.$$

Let us denote here and in the sequel

$$E = \| \sigma_A^{(1)} - \sigma_A^{(2)} \|_{L^\infty(\Omega)}.$$

The following estimate for $\tilde{S}_{\mathcal{U}_K}(y, z)$ holds true.

Proposition 3.5. *(**Estimates of unique continuation**) If, for a positive number ε_0, we have*

$$\left| \tilde{S}_{\mathcal{U}_K}(y, z) \right| \le r_0^{2-n} \varepsilon_0, \quad \text{for every } (y, z) \in \left(D_0 \right)_{\frac{r_0}{3}} \times \left(D_0 \right)_{\frac{r_0}{3}}, \tag{3.12}$$

then the following inequality holds true for every $r \in (0, d_1]$

$$\left| \tilde{S}_{\mathcal{U}_K}\left(w_{\bar{h}}(P_{K+1}), w_{\bar{h}}(P_{K+1}) \right) \right| \le r_0^{2-n} C^{\bar{h}} (E + \varepsilon_0) \left(\omega_{1/C}^{(2K)} \left(\frac{\varepsilon_0}{E + \varepsilon_0} \right) \right)^{(1/C)^{\bar{h}}}, \tag{3.13}$$

where $P_{K+1} \in \Sigma_{K+1}$, $\bar{h} = \min \{ k \in \mathbb{N} \mid d_k \le r \}$, $w_{\bar{h}}(P_{K+1}) = P_{K+1} - \lambda_{\bar{h}} \nu(P_{K+1})$, ν is the exterior unit normal to ∂D_K and $C \ge 1$ depends on the a priori data only.

3.3. Proof of theorem 2.1

Proof. We denote by Λ_i the map $\Lambda_{\sigma_A^{(i)}}^{\Sigma}$, for $i = 1, 2$ and, for every $k \in \{0, ..., K\}$, the subscript j_k will be replaced by k. This should simplify the notation. Let us point out that

$$\left\| \left(\sigma_A^{(1)} - \sigma_A^{(2)} \right) \right\|_{L^\infty(\Omega)} \le \bar{A} \, \| \gamma^{(1)} - \gamma^{(2)} \|_{L^\infty(\Omega)},$$

where

$$\gamma^{(i)} = \sum_{j=1}^{N} \gamma_j^{(i)} \chi_{D_j}(x), \qquad i = 1, 2,$$

therefore (2.12) trivially follows from

$$\| \gamma^{(1)} - \gamma^{(2)} \|_{L^\infty(\Omega)} \le C \, \| \Lambda_1 - \Lambda_2 \|_*, \tag{3.14}$$

which we shall prove. Moreover we shall denote

$$\varepsilon = \| \Lambda_1 - \Lambda_2 \|_*, \qquad \delta_k = \| \tilde{\gamma}^{(1)} - \tilde{\gamma}^{(2)} \|_{L^\infty(\mathcal{W}_k)}.$$

We start by recalling that for every $y, z \in D_0$ we have

$$\left\langle \left(\Lambda_1 - \Lambda_2 \right) \tilde{G}_1(\cdot, y), \tilde{G}_2(\cdot, z) \right\rangle = \int_\Omega \left(\tilde{\gamma}^{(1)} - \tilde{\gamma}^{(2)} \right) A(\cdot) \nabla \tilde{G}_1(\cdot, y) \cdot \nabla \tilde{G}_2(\cdot, z)$$

and that, for every $k \in \{1, ..., K\}$

$$\tilde{S}_{\mathcal{U}_{k-1}}(y, z) = \int_{\mathcal{U}_{k-1}} \left(\tilde{\gamma}^{(1)} - \tilde{\gamma}^{(2)} \right) \tilde{A}(\cdot) \nabla \tilde{G}_1(\cdot, y) \cdot \nabla \tilde{G}_2(\cdot, z),$$

therefore

$$\begin{aligned}
\left| \tilde{S}_{\mathcal{U}_{k-1}}(y, z) \right| &\le \varepsilon \, \| \tilde{G}_1(\cdot, y) \|_{H_{co}^{1/2}(\Sigma)} \, \| \tilde{G}_2(\cdot, z) \|_{H_{co}^{1/2}(\Sigma)} \\
&\quad + \delta_{k-1} \bar{A} \, \| \nabla \tilde{G}_1(\cdot, y) \|_{L^2(\mathcal{W}_{k-1})} \, \| \nabla \tilde{G}_2(\cdot, z) \|_{L^2(\mathcal{W}_{k-1})} \\
&\le C (\varepsilon + \delta_{k-1}) r_0^{2-n}, \qquad \text{for every } y, z \in \left(D_0 \right)_{r_0/3}, \tag{3.15}
\end{aligned}$$

where C depends on A, L, λ, $\bar{A}$ and n. Let $\rho_0 = \frac{r_0}{\bar{C}}$, where $\bar{C}$ is the constant introduced in theorem 3.4, let $r \in (0, d_2)$ and denote

$$w = P_k + \sigma \nu(P_k), \qquad \text{where } \sigma = a^{\bar{h}-1} \lambda_1,$$

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where $\nu(P_k)$ denotes the exterior unit normal to ∂D_k in P_k, then

$$\tilde{S}_{\mathcal{U}_{k-1}}(w, w) = I_1(w) + I_2(w), \tag{3.16}$$

where

$$I_1(w) = \int_{B_{\rho_0}(P_k) \cap D_k} \left(\gamma^{(1)} - \gamma^{(2)}\right) A(\cdot) \nabla \tilde{G}_1(\cdot, w) \cdot \nabla \tilde{G}_1(\cdot, w),$$

$$I_2(w) = \int_{\mathcal{U}_{k-1} \setminus \left(B_{\rho_0}(P_k) \cap D_k\right)} \left(\gamma^{(1)} - \gamma^{(2)}\right) A(\cdot) \nabla \tilde{G}_1(\cdot, w) \cdot \nabla \tilde{G}_1(\cdot, w)$$

and (see [A-V])

$$\left|I_2(w)\right| \leqslant CE\rho_0^{2-n}, \tag{3.17}$$

where C depends on λ, $\bar{A}$ and n only. To estimate $I_1(w)$ we recall theorem 3.4 which leads to

$$\left|I_1(w)\right| \geqslant C \left\{ \left|\gamma_k^{(1)} - \gamma_k^{(2)}\right| \int_{B_{\rho_0}(P_k) \cap D_k} \left|\nabla_x \Gamma(Jx, Jw)\right|^2 \right.$$

$$- \int_{B_{\rho_0}(P_k) \cap D_k} \left|A(x) \| \nabla_x \Gamma(Jx, Jw)\right| \frac{|x-w|^{1-n+\beta}}{\rho_0^\beta}$$

$$\left. - \int_{B_{\rho_0}(P_k) \cap D_k} |A(x)| \frac{|x-w|^{2-2n+\beta}}{\rho_0^{2\beta}} \right\},$$

where C is a constant that depends on M, λ, α, $\bar{A}$ and n only. Therefore, by combining (3.15) together with (3.16) and (3.17), we obtain

$$\left|I_1(w)\right| \geqslant C \left\{ \left|\gamma_k^{(1)} - \gamma_k^{(2)}\right| \int_{B_{\rho_0}(P_k) \cap D_k} \frac{\left|J^2(x-w)\right|^2}{|J(x-w)|^{2n}} \right.$$

$$- \frac{E}{\rho_0^\beta} \int_{B_{\rho_0}(P_k) \cap D_k} \frac{\left|J^2(x-w)\right|}{|J(x-w)|^n} |x-w|^{1-n+\beta}$$

$$\left. - \frac{E}{\rho_0^{2\beta}} \int_{B_{\rho_0}(P_k) \cap D_k} |x-w|^{2(1-n)+\beta} \right\}.$$

Therefore

$$\left|I_1(w)\right| \geqslant C \left\{ \left|\gamma_k^{(1)} - \gamma_k^{(2)}\right| \int_{B_{\rho_0}(P_k) \cap D_k} |x-w|^{2(1-n)} \right.$$

$$- \frac{E}{\rho_0^\beta} \int_{B_{\rho_0}(P_k) \cap D_k} |x-w|^{2(1-n)+\beta}$$

$$\left. - \frac{E}{\rho_0^{2\beta}} \int_{B_{\rho_0}(P_k) \cap D_k} |x-w|^{2(1-n+\beta)} \right\}, \tag{3.18}$$

which leads to

$$\left| I_1(w) \right| \geqslant C_1 \left| \gamma_k^{(1)} - \gamma_k^{(2)} \right| \sigma^{2-n} - C_2 E \frac{\sigma^{2-n+\beta}}{\rho_0^\beta}, \tag{3.19}$$

where β is the number introduced in theorem 3.4 and C_1, C_2 are constants depending on M, λ, α, $\bar{A}$ and n only. By combining (3.19) together with (3.16) and (3.17) we obtain

$$C_1 \left| \gamma_k^{(1)} - \gamma_k^{(2)} \right| \sigma^{2-n} \leqslant \left| \tilde{S}_{\mathcal{U}_{k-1}}(w, w) \right| + C_2 E \frac{\sigma^{2-n+\beta}}{\rho_0^\beta} \tag{3.20}$$

and by proposition 3.5 and (3.15) we obtain

$$\left| \tilde{S}_{\mathcal{U}_{k-1}}(w, w) \right| \leqslant \sigma^{2-n} C^{\bar{h}} \left(E + \varepsilon + \delta_{k-1} \right) \left(\omega_{\frac{1}{C}} \left(\frac{\varepsilon + \delta_{k-1}}{E + \varepsilon + \delta_{k-1}} \right) \right)^{\left(\frac{1}{C}\right)^{\bar{h}}},$$

where $C \geqslant 1$ is a constant depending on A, L, $\bar{A}$, M, N, α, λ and n only, therefore

$$\left| \gamma_k^{(1)} - \gamma_k^{(2)} \right| \leqslant C^{\bar{h}} \left(\varepsilon + \delta_{k-1} + E \right) \left(\omega_{1/C}^{(2(k-1))} \right)^{\left(\frac{1}{C}\right)^{\bar{h}}} + C_2 E \left(\frac{\sigma}{\rho_0} \right)^\beta. \tag{3.21}$$

We need to estimate $C^{\bar{h}}$ and $\left(\frac{1}{C} \right)^{\bar{h}}$, where $C > 1$. It turns out that

$$C^{\bar{h}} \leqslant C^2 \left(\frac{d_1}{r} \right)^{-\frac{1}{\log_c a}}$$

$$\left(\frac{1}{C} \right)^{\bar{h}} \leqslant \frac{1}{C} \left(\frac{r}{d_1} \right)^{-\frac{1}{\log_c a}}, \tag{3.22}$$

therefore

$$\left| \gamma_k^{(1)} - \gamma_k^{(2)} \right| \leqslant C \left(\varepsilon + \delta_{k-1} + E \right) \left(\left(\frac{d_1}{r} \right)^C \left(\omega_{1/C}^{(2(k-1))} \right)^{\left(\frac{r}{d_1}\right)^C} + \left(\frac{r}{d_1} \right)^\beta \right). \tag{3.23}$$

By (3.23) we obtain for every $k \in \{1,...,K\}$

$$\delta_k \leqslant \delta_{k-1} + C \left(\varepsilon + \delta_{k-1} + E \right) \left(\omega_{1/C}^{(2(k+1))} \left(\frac{\varepsilon + \delta_{k-1}}{\varepsilon + \delta_{k-1} + E} \right) \right)^{\frac{1}{C}},$$

which leads to

$$\left\| \gamma^{(1)} - \gamma^{(2)} \right\|_{L^\infty(\Omega)} \leqslant C(\varepsilon + E) \left(\omega_{\frac{1}{C}}^{(N^2)} \left(\frac{\varepsilon}{\varepsilon + E} \right) \right)^{\frac{1}{C}},$$

therefore

$$E \leqslant C(\varepsilon + E) \left(\omega_{\frac{1}{C}}^{(N^2)} \left(\frac{\varepsilon}{\varepsilon + E} \right) \right)^{\frac{1}{C}}. \tag{3.24}$$

Assuming that $E > \varepsilon e^2$ (if this is not the case then the theorem is proven) we obtain

$$E \leqslant C\left(\frac{E}{e^2} + E\right)\left(\omega_{\frac{1}{C}}^{(N^2)}\left(\frac{\varepsilon}{E}\right)\right)^{\frac{1}{C}},$$

which leads to

$$\frac{1}{C} \leqslant \omega_{\frac{1}{C}}^{(N^2)}\left(\frac{\varepsilon}{E}\right),$$

therefore

$$E \leqslant \frac{1}{\omega_{\frac{1}{C}}^{(-N^2)}\left(\frac{1}{C}\right)}\,\varepsilon,$$

which concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

4. Proof of technical propositions

4.1. Proof of the asymptotic estimates

Whenever φ is a Lipschitz continuous function on $\mathbb{R}^{n-1}$, we shall denote by $Q^+_{\varphi,\,r}$ and $Q^-_{\varphi,\,r}$ the following sets

$$Q^+_{\varphi,\,r} = \left\{(x', x_n) \in Q_r \,\big|\, x_n > \varphi(x')\right\}, \tag{4.1}$$

$$Q^-_{\varphi,\,r} = \left\{(x', x_n) \in Q_r \,\big|\, x_n < \varphi(x')\right\}. \tag{4.2}$$

Let $0 < \mu < 1$ and $B^+ \in C^\mu(\overline{Q^+_{\varphi,\,r}})$, $B^- \in C^\mu(\overline{Q^-_{\varphi,\,r}})$ be symmetric, positive definite matrix valued functions and define

$$B(x) = \begin{cases} B^+(x), & x \in Q^+_{\varphi,\,r}, \\ B^-(x), & x \in Q^-_{\varphi,\,r}, \end{cases}$$

such that B satisfies the uniform ellipticity condition

$$\lambda_0^{-1}|\xi|^2 \leqslant B(x)\xi \cdot \xi \leqslant \lambda_0|\xi|^2, \quad \text{for almost every } x \in Q_r,$$
$$\text{for every } \xi \in \mathbb{R}^n, \tag{4.3}$$

where $\lambda_0 > 0$ is a constant.

Theorem 4.1. *Let $k > 0$, $r > 0$ and $0 < \alpha < 1$ be fixed numbers. Moreover, let B be a matrix as above. Assume that $\varphi \in C^{1,\alpha}(B'_r)$ and let $U \in H^1(Q_r)$ be a solution to*

$$\mathrm{div}\left(\left(1 + (k-1)\chi_{Q^+_{\varphi,\,r}}\right)B\,\nabla U\right) = 0. \tag{4.4}$$

Suppose α' satisfies at the same time $0 < \alpha' \leqslant \mu$ and $\alpha' < \frac{\alpha}{(\alpha+1)n}$. Then, there exists a positive constant C such that for any $\rho \leqslant \frac{r}{2}$ and for any $x \in Q_{r-2\rho}$, the following estimate holds true

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$$\| \nabla U \|_{L^{\infty}\left(Q_{\rho}(x)\right)} + \rho^{\alpha'} \, |\nabla U|_{\alpha', Q_{\rho}(x) \cap Q_{\varphi,r}^{+}} + \rho^{\alpha'} \, |\nabla U|_{\alpha', Q_{\rho}(x) \cap Q_{\varphi,r}^{-}}$$

$$\leqslant \frac{C}{\rho^{1+n/2}} \, \| U \|_{L^{2}\left(Q_{2\rho}(x)\right)}, \tag{4.5}$$

where C depends on $\| \varphi \|_{C^{1,\alpha}(B_{r}')}$, k, α, α', n, λ_0, $\| B^{+} \|_{C^{\alpha}(\overline{Q_{\varphi,r}^{+}})}$ and $\| B^{-} \|_{C^{\alpha}(\overline{Q_{\varphi,r}^{-}})}$ only.

Proof. For the proof we refer to [Li-Vo theorem 1.1], where the authors, among various results, obtain piecewise $C^{1,\alpha'}$ estimates for solutions to divergence form elliptic equations with piecewise Hölder continuous coefficients (see also [Li-Ni]).

We fix $l \in \{1,...,K-1\}$. There exists a rigid transformation of coordinates under which $P_{l+1} = 0$ and

$$\Sigma_l \cap Q_{\frac{r_0}{3}} = \left\{ x \in Q_{\frac{r_0}{3}} \middle| x_n = \varphi(x') \right\}, \tag{4.6}$$

$$D_{j_{l+1}} \cap Q_{\frac{r_0}{3}} = \left\{ x \in Q_{\frac{r_0}{3}} \middle| x_n > \varphi(x') \right\}, \tag{4.7}$$

$$D_{j_l} \cap Q_{\frac{r_0}{3}} = \left\{ x \in Q_{\frac{r_0}{3}} \middle| x_n < \varphi(x') \right\}, \tag{4.8}$$

where φ is a $C^{1,\alpha}$ function on $B_{\frac{r_0}{3}}'$ satisfying

$$\varphi(0) = |\nabla\varphi(0)| = 0, \qquad \| \varphi \|_{C^{1,\alpha}}\left(B_{r_0}' \right) \leqslant M r_0. \tag{4.9}$$

Moreover, up to a possible replacement of γ with $\frac{\gamma}{\gamma_{j_l}}$, we can assume that $\gamma|_{D_{j_l}} = 1$ and $\gamma|_{D_{j_{l+1}}} = k$, where k is a real number which satisfies

$$\bar{\gamma} \leqslant k \leqslant \bar{\gamma}^{-2}. \tag{4.10}$$

Let τ be a C^{∞} function on $\mathbb{R}$ such that $0 \leqslant \tau \leqslant 1$, $\tau(s) = 1$ for every $s \in (-1, 1)$, $\tau(s) = 0$ for every $s \in \mathbb{R} \backslash (-2, 2)$ and $|\tau'(s)| \leqslant 2$ for every $s \in \mathbb{R}$.

We introduce

$$r_1 = \frac{r_0}{3} \min \left\{ \frac{1}{2}(8\,M)^{-\frac{1}{\alpha}}, \frac{1}{4} \right\} \tag{4.11}$$

and we consider the following change of variable $\xi = \Phi(x)$ given by

$$\begin{cases} \xi' = x', \\ \xi_n = x_n - \varphi(x')\tau\left(\frac{|x'|}{r_1}\right)\tau\left(\frac{x_n}{r_1}\right). \end{cases} \tag{4.12}$$

It can be verified that the map Φ is a $C^{1,\alpha}(\mathbb{R}^n, \mathbb{R}^n)$ and it satisfies the following properties

$$\Phi\left(\Sigma_l \cap Q_{r_1} \right) = \left\{ x \in Q_{r_1} \middle| x_n = 0 \right\}, \tag{4.13}$$

$$\Phi(x) = x, \quad \text{for every } x \in \mathbb{R}^n \backslash Q_{2r_1}, \tag{4.14}$$

$$C^{-1} \, |x_1 - x_2| \leqslant \left| \Phi(x_1) - \Phi(x_2) \right|$$
$$\leqslant C \, |x_1 - x_2|, \quad \text{for every } x_1, x_2 \in \mathbb{R}^n, \tag{4.15}$$

$$|\Phi(x) - x| \leqslant \frac{C}{r_0^\alpha} |x|^{1+\alpha}, \quad \text{and} \tag{4.16}$$

$$|D\Phi(x) - I| \leqslant \frac{C}{r_0^\alpha} |x|^\alpha, \quad \text{for every} \quad x \in \mathbb{R}^n, \tag{4.17}$$

where $C > 1$, is a constant depending on M and α only and I denotes the identity matrix.

Let $y_n \in (-\frac{n}{2}, 0)$ and $y = y e_n$. We set

$$\eta = \Phi(y), \tag{4.18}$$

$$G_B(\xi, \eta) = G\Big(\Phi^{-1}(\xi), \Phi^{-1}(\eta)\Big), \tag{4.19}$$

$$J(\xi) = (D\Phi)\Big(\Phi^{-1}(\xi)\Big), \tag{4.20}$$

$$\sigma_B(\xi) = \frac{1}{\det J(\xi)} J(\xi) \gamma\Big(\Phi^{-1}(\xi)\Big) A\Big(\Phi^{-1}(\xi)\Big)(J(\xi))^T, \tag{4.21}$$

we have that $G_B(\cdot, \eta)$ is a solution to

$$\begin{cases} \operatorname{div}\Big(\tilde{\sigma}\, \nabla_\xi G_B(\cdot, \eta)\Big) = -\delta(\cdot - \eta), & \text{in} \quad \Omega, \\ G_B(\cdot, \eta) = 0, & \text{on} \quad \partial\Omega. \end{cases} \tag{4.22}$$

We have

$$\sigma_B(\xi) = \Big(1 + (k-1)\chi^+(\xi)\Big)B(\xi), \quad \text{for any} \ \xi \in Q_n, \tag{4.23}$$

where χ^+ is the characteristic function of $\mathbb{R}_+^n$ and

$$B(\xi) = \frac{1}{\det J(\xi)} J(\xi) A\Big(\Phi^{-1}(\xi)\Big)(J(\xi))^T. \tag{4.24}$$

Furthermore, we have that B is of class C^α and

$$\|B\|_{C^{0,\alpha}(\Omega)} \leqslant C, \tag{4.25}$$

where $C > 0$ is a constant depending on M, α, λ, $\bar{A}$ only. We also have that $B(0) = A(0)$.

We denote

$$\sigma_{A(0)}(\xi) = \Big(1 + (k-1)\chi^+(\xi)\Big)A(0) \tag{4.26}$$

and we refer to G_0 as the Green's function solution to

$$\begin{cases} \operatorname{div}\Big(\sigma_{A(0)}(\cdot) \nabla G_0(\cdot, y)\Big) = -\delta(\cdot - y), & \text{in} \quad \Omega, \\ G_0(\cdot, y) = 0, & \text{on} \quad \partial\Omega. \end{cases} \tag{4.27}$$

We then define

$$R(\xi, \eta) = G_B(\xi, \eta) - G_0(\xi, \eta). \tag{4.28}$$

Lemma 4.2. *For every $\xi \in B_{\frac{r_1}{4}}$ and $\eta_n \in (-\frac{r_1}{4}, 0)$ we have that*

$$\left| R\left(\xi, e_n \eta_n \right) \right| + \left| \xi - e_n \eta_n \right| \left| \nabla_\xi R(\xi, \eta) \right| \leqslant \frac{c}{r_1^\beta} \left| \xi - e_n \eta_n \right|^{\beta+2-n}, \qquad (4.29)$$

where $\beta \in (0, \alpha^2]$ depends on α and n only and C depends on $M, \bar{\gamma}, \lambda, \bar{A}$ only.

Proof. It is easy to check that R in (4.28) satisfies

$$\begin{cases} \mathrm{div}_\xi\left(\sigma_B(\cdot) \nabla_\xi R(\,\cdot\,, \eta) \right) = -\mathrm{div}_\xi\left(\left(\sigma_B(\cdot) - \sigma_{A(0)}(\cdot) \right) \nabla_\xi G_0(\,\cdot\,, \eta) \right), & \text{in} \qquad \Omega, \\ R(\,\cdot\,, \eta) = 0, & \text{on} \qquad \partial\Omega. \end{cases} \qquad (4.30)$$

By the representation formula over Ω we have that R in (4.28) satisfies

$$R(\xi, \eta) = \int_\Omega \left(\sigma_B(\zeta) - \sigma_{A(0)}(\zeta) \right) \nabla_\zeta G_0(\zeta, \eta) \cdot \nabla \tilde{G}(\zeta, \xi) d\zeta. \qquad (4.31)$$

We consider $\xi \in Q_{\frac{r_1}{2}}$ and $\eta = e_n \eta_n$ and we split R as the sum of the following integrals

$$R_1(\xi, \eta) = \int_{\Omega \setminus Q_{r_1}} \left(\sigma_B(\zeta) - \sigma_{A(0)}(\zeta) \right) \nabla_\zeta G_0(\zeta, \eta) \cdot \nabla \tilde{G}(\zeta, \xi) d\zeta, \qquad (4.32)$$

$$R_2(\xi, \eta) = \int_{Q_{r_1}} \left(\sigma_B(\zeta) - \sigma_{A(0)}(\zeta) \right) \nabla_\zeta G_0(\zeta, \eta) \cdot \nabla \tilde{G}(\zeta, \xi) d\zeta. \qquad (4.33)$$

By the bounds (2.8)–(2.10) and by combining the Schwartz inequality with the Caccioppoli inequality we get

$$\left| R_1(\xi, \eta) \right| \leqslant \frac{C}{r_1^2} \left\| G_0(\,\cdot\,, \eta) \right\|_{L^2\left(\Omega \setminus Q_{3r_1/4} \right)} \left\| G_B(\,\cdot\,, \eta) \right\|_{L^2\left(\Omega \setminus Q_{3r_1/4} \right)}, \qquad (4.34)$$

where $C > 0$ depends on $M, \alpha, \bar{\gamma}, \lambda$ and $\bar{A}$ only. By the standard behaviour (3.3) of the Green's functions at hand, it follows that

$$\left| R_1(\xi, \eta) \right| \leqslant C r_1^{2-n}, \qquad (4.35)$$

where $C > 0$ depends on $M, \alpha, \bar{\gamma}, \lambda$ and $\bar{A}$ only. Moreover being $B(0) = A(0)$, it follows that (2.9) and (4.25) lead to

$$\left| \sigma_B(\xi) - \sigma_{A(0)}(\xi) \right| \leqslant \max\{1, k\}(|B(\xi) - A(0)|) \leqslant \frac{C}{r_1^\alpha} |\xi|^\alpha, \qquad (4.36)$$

for any $\xi \in Q_{r_1}$, where C depends on $M, \alpha, \bar{A}$ and $\bar{\gamma}$ only. By (3.3) and by theorem 4.1 we have that

$$\left| \nabla_\zeta G_0(\zeta, \xi) \right| \leqslant C|\zeta - \xi|^{1-n}, \quad \text{for every } \zeta, \xi \in Q_{r_1}, \qquad (4.37)$$

where C depends on $M, \alpha, \bar{A}$ and $\bar{\gamma}$ only. By (4.15) and the same arguments used above, we infer that

$$\left| \nabla_\zeta G_B(\zeta, \xi) \right| \leqslant C|\zeta - \xi|^{1-n}, \quad \text{for every } \zeta, \xi \in Q_{r_1}, \qquad (4.38)$$

where C depends on $M, \alpha, \bar{A}$ and $\bar{\gamma}$ only. We denote

$$I_1 = \int_{B_{4h}} |\zeta|^\alpha |\zeta - \xi|^{1-n} |\zeta - \eta|^{1-n} d\eta \qquad (4.39)$$

and

$$I_2 = \int_{\mathbb{R}^n \setminus B_{4h}} |\zeta|^\alpha |\zeta - \xi|^{1-n} |\zeta - \eta|^{1-n} d\eta. \tag{4.40}$$

By (4.36)–(4.38) we have that

$$\left| R_2(\xi, \eta) \right| \leqslant \frac{C}{r_1^\alpha} (I_1 + I_2). \tag{4.41}$$

Let us denote now $h = |\xi - \eta|$ and consider the following change of variables $\zeta = hw$; we set $t = \frac{\xi}{h}$ and $s = \frac{\eta}{h}$; it follows that for any t, $s \in \mathbb{R}^n$ we have that $|t - s| = 1$. We obtain

$$I_1 \leqslant 4^\alpha h^{\alpha + 2 - n} \int_{B_4} |t - w|^{1-n} |s - w|^{1-n} dw. \tag{4.42}$$

Let us now set

$$F(t, s) = \int_{B_4} |t - w|^{1-n} |s - w|^{1-n} dw. \tag{4.43}$$

From standard bounds (see for instance, [Mi chapter 2]) we have that

$$F(t, s) \leqslant C, \tag{4.44}$$

where C depends on n only. Hence

$$I_1 \leqslant C h^{\alpha + 2 - n}, \tag{4.45}$$

where C depends on n only. We consider now integral I_2 and we recall that $\eta = e_n \eta_n$, where $\eta_n \in (-\frac{n}{2}, 0)$ and $\xi \in Q_{\frac{r}{2}}^+$, hence we have

$$|\eta| = -\eta_n \leqslant -\eta_n + \xi_n \leqslant |\xi - \eta| = h, \tag{4.46}$$

which leads to

$$|\xi| \leqslant |\xi - \eta| + |\eta| \leqslant 2h. \tag{4.47}$$

On the other hand, we have that for any $\zeta \in \mathbb{R}^n \setminus B_{4h}$

$$|\zeta| \leqslant |\zeta - \eta| + |\eta| \leqslant |\zeta - \eta| + \frac{1}{4} |\zeta|, \tag{4.48}$$

hence we get

$$\frac{3}{4} |\zeta| \leqslant |\zeta - \eta|. \tag{4.49}$$

and

$$\frac{1}{2} |\zeta| \leqslant |\xi - \zeta|, \quad \text{for any } \zeta \in \mathbb{R}^n \setminus B_{4h}. \tag{4.50}$$

By combining (4.49) together with (4.50), we obtain

$$I_2 \leqslant \left(\frac{8}{3} \right)^{1-n} \int_{\mathbb{R}^n \setminus B_{4h}} |\zeta|^{\alpha + 2 - 2n} d\zeta \leqslant C h^{\alpha + 2 - n}, \tag{4.51}$$

where C depends on α and n only. By combining (4.35), (4.41), (4.45) and (4.51) we obtain

$$|R(\xi, \eta)| \leqslant \frac{C}{r_1^\alpha} h^{\alpha+2-n}, \tag{4.52}$$

where C depends on M, α, $\bar{A}$, n and $\bar{\gamma}$ only. Let us fix $\xi \in B_{\frac{r_1}{4}}^+$, $\eta_n \in (-r_1/4, 0)$ and consider the cylinder

$$Q = B'_{\frac{h}{8}}(\xi') \times \left(\xi_n, \xi_n + \frac{h}{8} \right). \tag{4.53}$$

Observing that $h = |\xi - (0, \eta_n e_n)| \leqslant \frac{r_1}{2}$ we deduce that $Q \subset Q_{\frac{r_1}{2}}^+$. Moreover $Q \subset Q_{\frac{h}{4}(\xi)}$ and $\xi \in \partial Q$, then by choosing for instance $\alpha' = \frac{1}{2} \min \left\{ \alpha, \frac{\alpha}{(\alpha+1)n} \right\}$ in the statement of theorem 4.1 and observing that $(0, \eta_n e_n) \notin Q_{\frac{h}{2}}(\xi)$, by (4.5) we obtain the following bound for the seminorm

$$\left| \nabla_\xi G_B(\cdot, e_n \eta_n) \right|_{\alpha', Q} \leqslant \left| \nabla_\xi G_B(\cdot, e_n \eta_n) \right|_{\alpha', Q_{\frac{h}{4}(\xi)} \cap Q_{\frac{r_1}{2}}}$$

$$\leqslant Ch^{-\alpha'-1-n/2} \| \nabla_\xi G_B(\cdot, e_n \eta_n) \|_{L^2\left(Q_{\frac{h}{2}(\xi)} \right)}, \tag{4.54}$$

where C depends on M, α, $\bar{A}$, n and $\bar{\gamma}$ only. Furthermore by observing that for any $\tilde{\xi} \in Q_{\frac{h}{2}(\xi)}$ we have that $|\tilde{\xi} - (0, e_n \eta_n)| \geqslant \frac{h}{2}$ and by (3.3) we have

$$\left| \nabla_\xi \tilde{G}(\cdot, e_n \eta_n) \right|_{\alpha', Q} \leqslant Ch^{\alpha'+1-n}, \tag{4.55}$$

where C depends on M, α, $\bar{A}$, n and $\bar{\gamma}$ only. By analogous argument we may also infer that

$$\left| \nabla_\xi G_0(\cdot, e_n \eta_n) \right|_{\alpha', Q} \leqslant Ch^{\alpha'+1-n}, \tag{4.56}$$

where C depends on M, α, $\bar{A}$, n and $\bar{\gamma}$ only. Hence by (4.28), (4.55) and (4.56) we obtain

$$\left| \nabla_\xi R(\cdot, e_n \eta_n) \right|_{\alpha', Q} \leqslant Ch^{\alpha'+1-n}, \tag{4.57}$$

where C depends on M, α, $\bar{A}$, n and $\bar{\gamma}$ only. We recall the following interpolation inequality (see for instance [A-S], proposition 8.3)

$$\| \nabla_\xi R(\cdot, e_n \eta_n) \|_{L^\infty(Q)} \leqslant \| R(\cdot, e_n \eta_n) \|_{L^\infty(Q)}^{\frac{\alpha'}{1+\alpha'}} \left| \nabla_\xi R(\cdot, e_n \eta_n) \right|_{\alpha', Q}^{\frac{1}{1+\alpha'}}, \tag{4.58}$$

where C depends on M, α, $\bar{A}$, n and $\bar{\gamma}$ only. By the above estimate and (4.52) we get

$$\left| \nabla_\xi R\left(\xi, e_n \eta_n \right) \right| \leqslant \frac{C}{r_1^\beta} h^{\beta+1-n}, \quad \text{for every } \xi \in B_{\frac{r_1}{4}}^+ \text{ and } \eta \in \left(-\frac{r_1}{4}, 0 \right), \tag{4.59}$$

where C depends on M, α, $\bar{A}$, n and $\bar{\gamma}$ only. The thesis follows with $\beta = \frac{\alpha'^2}{1+\alpha'}$. $\qquad \square$

Proof of theorem 3.4. We first assume that the auxiliary hypothesis that $A(0) = I$ is fulfilled and denote with $H(\xi, \eta)$ the half space fundamental solution of the operator $\mathrm{div}_\xi((1 + (k-1))\chi^+(\xi)I(\xi)\nabla_\xi)$ which has the following explicit form

$$H(\xi, \eta) = \begin{cases} \dfrac{1}{k}\Gamma(\xi, \eta) + \dfrac{k-1}{k(k+1)}\Gamma\left(\xi, \eta^*\right), & \text{if } \xi_n, \eta_n > 0, \\[2mm] \dfrac{2}{k+1}\Gamma(\xi, \eta), & \text{if } \xi_n\eta_n < 0, \\[2mm] \Gamma(\xi, \eta) + \dfrac{1-k}{k+1}\Gamma\left(\xi, \eta^*\right), & \text{if } \xi_n, \eta_n < 0, \end{cases} \qquad (4.60)$$

where Γ is the distribution introduced in (3.9) and for any $\xi = (\xi', \xi_n)$ we denote $\xi^* = (\xi', -\xi_n)$. Let $\eta_n \in (-\frac{n}{4}, 0)$, then we have

$$\begin{cases} \text{div}_\xi\Big((1 + (k-1))\chi^+(\xi)I(\xi)\nabla_\xi\big(G_0(\xi, e_n\eta_n) - H(\xi, e_n\eta_n)\big)\Big) = 0, & \text{in } Q_{\frac{r_1}{2}}, \\[2mm] \left|\big(G_0(\xi, e_n\eta_n) - H(\xi, e_n\eta_n)\big)\right| \leqslant Cr_1^{n-2}, & \text{on } \partial Q_{\frac{r_1}{2}}. \end{cases}$$

Hence by the maximum principle we can infer that

$$\| G_0(\,\cdot\,, e_n\eta_n) - H(\,\cdot\,, e_n\eta_n)\|_{L^\infty\left(Q_{\frac{r_1}{2}}\right)} \leqslant Cr_1^{n-2} \qquad (4.61)$$

and by theorem 4.1 we deduce that

$$\| \nabla_\xi G_0(\,\cdot\,, e_n\eta_n) - \nabla_\xi H(\,\cdot\,, e_n\eta_n)\|_{L^\infty\left(Q_{\frac{r_1}{4}}\right)} \leqslant Cr_1^{n-1}. \qquad (4.62)$$

We now consider $x \in \Phi^{-1}(B_{\frac{r_1}{4}}^+)$ and $y_n \in (-\frac{n}{2}, 0)$, then we observe that being $\Phi(y) = y$ we have that

$$|\Phi(y)| = |\Phi(y) - \Phi(0)| \leqslant |\Phi(y) - \Phi(x)|. \qquad (4.63)$$

Moreover, by (4.15) and the above estimate we have

$$C^{-1}|x| \leqslant |\Phi(x)| \leqslant |\Phi(x) - \Phi(y)| + |\Phi(y)| \leqslant C|x - y|. \qquad (4.64)$$

By combining the above estimate with (4.16), we infer that

$$|\Phi(x) - x| \leqslant \frac{C}{r_0^\alpha}|x|^{1+\alpha} \leqslant \frac{C}{r_0^\alpha}|x - e_ny_n|^{1+\alpha}, \qquad (4.65)$$

where C depends on M and α only. Let $\{A_k\}_{k\geqslant 1}$ be a regularizing sequence for A obtained by convolution with a sequence of mollifiers, then we have that

$$\| A_k\|_{C^1(\Omega)} \leqslant 2\bar{A}, \quad \text{for any } k \in \mathbb{N} \qquad (4.66)$$

and A_k satisfies (2.10), with $A = A_k$, $k \in \mathbb{N}$. Let us introduce the following function

$$F_k: B_{r_0}\backslash\left\{e_ny_n\right\} \to \mathbb{R} \qquad (4.67)$$

$$z \mapsto\; < A_k(z)(z - e_ny_n),\, (z - e_ny_n) >^{\frac{2-n}{2}}, \qquad (4.68)$$

where $<\cdot\,,\,\cdot>$ denotes the Euclidean scalar product of vectors in $\mathbb{R}^n$. Given $z_1, z_2 \in B_{r_0}\backslash\{e_ny_n\}$ by the mean-value theorem, there exists t_k, $0 < t_k < 1$ such that

$$\left| F_k(z_1) - F_k(z_2) \right| \leqslant C \left| z_1 - z_2 \right| \left(\left| \left\langle A_k(z_{t_k})(z_{t_k} - e_n y_n), (z_{t_k} - e_n y_n) \right\rangle^{\frac{1-n}{2}} \right| \right.$$
$$+ \left| \left\langle A_k(z_{t_k})(z_{t_k} - e_n y_n), (z_{t_k} - e_n y_n) \right\rangle^{-\frac{n}{2}} \right|$$
$$\left. \times \left| \left\langle \sum_{i=1}^{n} \partial_{z_i} Ak(z_{t_k})(z_{t_k} - e_n y_n), (z_{t_k} - e_n y_n) \right\rangle \right| \right),$$

where $z_{t_k} = z_1 + t_k(z_2 - z_1)$ and where C depends on depends on M, α, $\bar{A}$ and n only. Let us denote with Γ_k the fundamental solution introduced in (3.9) associated to the matrix A_k. We choose $z_1 = \Phi(x)$ and $z_2 = x$, then we have

$$\left| \Gamma_k\left(\Phi(x), e_n y_n \right) - \Gamma_k(x, e_n y_n) \right| \leqslant C \left| \Phi(x) - x \right| \left| x - e_n y_n + t_k(\Phi(x) - x) \right|^{1-n},$$

where C depends on M, α, $\bar{A}$, λ and n only. By (4.65) and the triangle inequality we deduce that for any $x \in D_{j_{l+1}} \cap B_{\frac{r_0}{4C^{1/\alpha}}}$

$$\left| x - e_n y_n - t_k(\Phi(x) - x) \right| \geqslant \left| x - e_n y_n \right| - \left| t_k \right| \left\| \Phi(x) - x \right| \tag{4.69}$$

$$\geqslant \left| x - e_n y_n \right| - \left| x - e_n y_n \right|^{1+\alpha} \geqslant \frac{1}{2} \left| x - e_n - y_n \right|. \tag{4.70}$$

Finally combining the above estimates and (4.65) we obtain

$$\left| \Gamma_k\left(\Phi(x), e_n y_n \right) - \Gamma_k(x, e_n y_n) \right| \leqslant C \left| x - e_n y_n \right|^{2-n+\alpha}, \tag{4.71}$$

where C depends on M, α, λ, $\bar{A}$ and n only. Now since A_k converges uniformly to A in $\bar{\Omega}$ we can infer that

$$\left| \Gamma\left(\Phi(x), e_n y_n \right) - \Gamma(x, e_n y_n) \right| \leqslant C, \left| x - e_n y_n \right|^{2-n+\alpha}, \tag{4.72}$$

for $x \in \Phi^{-1}(B_{\frac{r_1}{4}}^+)$, where C depends on M, α, λ, $\bar{A}$ and n only. By (4.61), (4.62) and (4.72) we have

$$\left| G_0\left(\Phi(x), e_n y_n \right) - H(x, e_n y_n) \right| \leqslant \left| G_0\left(\Phi(x), e_n y_n \right) - H\left(\Phi(x), e_n y_n \right) \right|$$
$$+ \left| H\left(\Phi(x), e_n y_n \right) - H(x, e_n y_n) \right|$$
$$\leqslant \frac{C}{r_0^\alpha} \left| x - e_n y_n \right|^{\alpha+2-n} \tag{4.73}$$

and

$$\left| \nabla G_0\left(\Phi(x), e_n y_n \right) - \nabla H(x, e_n y_n) \right| \leqslant \frac{C}{r_0^\alpha} \left| x - e_n y_n \right|^{\alpha+1-n}, \tag{4.74}$$

for $x \in \Phi^{-1}(B_{\frac{r_1}{4}}^+)$, where C depends on M, λ, $\bar{\gamma}$, α and n only. Moreover, by lemma 4.2 and (4.15) and recalling that $\Phi(y) = y$, we get

$$\left| R\left(\Phi(x), e_n \eta_n \right) \right| + \left| x - e_n \eta_n \right| \left\| \nabla_\xi R(\Phi(x), \eta) \right| \leqslant \frac{c}{r_1^\beta} \left| \xi - e_n \eta_n \right|^{\beta+2-n}, \tag{4.75}$$

for $x \in \Phi^{-1}(B_{\frac{r_1}{4}}^+)$, C depends on M, λ, $\bar{\gamma}$, α and n only. Gathering (4.73)–(4.75) and recalling that

$$G\left(\bar{x}, e_n y_n\right) = G_0\left(\Phi(\bar{x}), e_n y_n\right) + R\left(\Phi(\bar{x}), e_n y_n\right) \tag{4.76}$$

we first find that

$$\left| G\left(\bar{x}, e_n y_n\right) - \frac{2}{1+k}\Gamma\left(\bar{x}, e_n y_n\right) \right| \leq \frac{C}{r_0^\beta}\left|\bar{x} - e_n y_n\right|^{\beta+2-n}, \tag{4.77}$$

$$\left| \nabla_x G\left(\bar{x}, e_n y_n\right) - \frac{2}{1+k}\nabla_x\Gamma\left(\bar{x}, e_n y_n\right) \right| \leq \frac{C}{r_0^\beta}\left|\bar{x} - e - n y_n\right|^{\beta+1-n}, \tag{4.78}$$

for a.e. $\bar{x} \in D_{j_{l+1}} \cap B_{\frac{r_0}{(4C)^{1/\alpha}}}$ and $y_n \in (-r_1/(4C)^{1/(\alpha)}, 0)$, where C depends on M, λ, $\bar{\gamma}$, $\bar{A}$, α and n only. The thesis then follows for the case $A(0) = I$.

To treat the general case when $A(0) \neq I$, we introduce the fundamental solution $H_{A(0)}$ of the operator $\text{div}_\xi((1 + (k-1))\chi^+(\xi)A(0)\nabla_\xi)$. We set $\sigma_I(\xi) = (1 + (k-1))\chi^+(\xi)I$ and recall that $\sigma_{A(0)}(\xi) = (1 + (k-1))\chi^+(\xi)A(0)$. Let us introduce the linear change of variable

$$L: \mathbb{R}^n \to \mathbb{R}^n \tag{4.79}$$

$$\xi \mapsto L\xi := R\sqrt{A^{-1}(0)}\,\xi, \tag{4.80}$$

where R is the planar rotation in $\mathbb{R}^n$ that rotates the unit vector $\frac{v}{\|v\|}$, where $v = \sqrt{A(0)}\,e_n$, to the nth standard unit vector e_n and such that

$$R\Big|_{(\pi)^\perp} \equiv Id\Big|_{(\pi)^\perp},$$

where π is the plane in $\mathbb{R}^n$ generated by e_n, v and $(\pi)^\perp$ denotes the orthogonal complement of π in $\mathbb{R}^n$. For this choice of L we have

(i) $A(0) = L^{-1} \cdot (L^{-1})^T$,
(ii) $(L\xi) \cdot e_n = \frac{1}{\|v\|}\xi \cdot e_n$,

which leads to

$$\sigma_{A(0)}(\xi) = L^{-1}\sigma_I(L\xi)\left(L^{-1}\right)^T,$$

i.e. $L^{-1}: x \mapsto \xi$ is the linear change of variables that maps $\sigma_I(x)$ into $\sigma_{A(0)}(\xi)$. Therefore the fundamental solution for the operator $\text{div}_\xi((1 + (k-1))\chi^+(\xi)A(0)\nabla_\xi)$ turns out to be

$$H_{A(0)}(\xi, \eta) = \begin{cases} |J|\left(\dfrac{1}{k}\Gamma(L\xi, L\eta) + \dfrac{k-1}{k(k+1)}\Gamma\left(L\xi, L^*\eta\right)\right), & \text{if } \xi_n, \eta_n > 0, \\[3mm] |J|\left(\dfrac{2}{k+1}\Gamma(L\xi, L\eta)\right), & \text{if } \xi_n\eta_n < 0, \\[3mm] |J|\left(\Gamma(L\xi, L\eta) + \dfrac{1-k}{k+1}\Gamma\left(L\xi, L^*\eta\right)\right), & \text{if } \xi_n, \eta_n < 0, \end{cases} \tag{4.81}$$

where $|J|$ denotes the determinant of matrix $J = \sqrt{A^{-1}(0)}$, matrix $L^* = \{l_{i,j}^*\}_{i,j=1}^n$ is such that $l_{i,j}^* = l_{i,j}$ for $i = 1,...,n-1$, $j = 1,...,n$ and $l_{n,j}^* = -l_{n,j}$ for $j = 1...,n$. In particular we have that when $\xi_n\eta_n < 0$

$$H_{A(0)}(\xi, \eta) = \frac{2\,|J|}{k+1} < A^{-1}(0)(\xi - \eta, \xi - \eta) >^{\frac{2-n}{2}}.$$

Hence for the case $A(0) \neq I$ (4.77) and (4.78) shall be replaced by

$$\left| G(\bar{x}, e_n y_n) - \frac{2}{1+k} < A^{-1}(0)(\bar{x} - e_n y_n), (\bar{x} - e_n y_n) >^{\frac{2-n}{2}} \right| \leqslant \frac{C}{r_0^\beta} |\bar{x} - e_n y_n|^{2-n+\beta},$$

$$\left| \nabla_x G(\bar{x}, e_n y_n) - \frac{2}{1+k} \nabla_x < A^{-1}(0)(\bar{x} - e_n y_n), (\bar{x} - e_n y_n) >^{\frac{2-n}{2}} \right| \leqslant \frac{C}{r_0^\beta} |\bar{x} - e_n y_n|^{\beta+1-n},$$

for $\bar{x} \in D_{j_{l+1}} \cap B_{\frac{r_0}{(4C)^{1/\alpha}}}$ and $y_n \in (-r_1/(4C)^{1/(\alpha)}, 0)$, where C depends on M, λ, $\bar{\gamma}$, $\bar{A}$, α and n only. Hence the thesis follows also for the general case. $\qquad\square$

4.2. Proof of unique continuation estimates

Let P_1, D_0 Ω_0, $(D_0)_r$ and $\tilde{G}_i$, for $i = 1, 2$ be as in subsection 3.2.1. Let us fix $k \in \{2,...N\}$ and recall that there exist $j_1,...j_K \in \{2,...N\}$ such that

$$D_{j_1} = D_1,... D_{j_K} = D_k.$$

We recall that

$$\mathcal{W}_K = \bigcup_{i=0}^{K} D_{j_i}, \qquad \mathcal{U}_k = \Omega_0 \backslash \overline{\mathcal{W}_K}, \quad \text{when } k \geqslant 0$$

$(D_{j_0} = D_0)$ and for any $y, z \in \mathcal{W}_K$

$$\tilde{S}_{\mathcal{U}_K}(y, z) = \int_{\mathcal{U}_K} \left(\tilde{\sigma}_A^{(1)} - \tilde{\sigma}_A^{(2)} \right) \nabla \tilde{G}_1(\,\cdot\,, y) \cdot \nabla \tilde{G}_2(\,\cdot\,, z), \quad \text{when } k \geqslant 0.$$

The proof of proposition 3.5 is a straight forward consequence of the following result (see [A-V], proof of proposition 4.6).

Proposition 4.3. *Let v be a weak solution to*

$$\text{div}(\tilde{\sigma}\, \nabla v) = 0, \quad \text{in } \mathcal{W}_k,$$

where $\tilde{\sigma}$ is either equal to $\tilde{\sigma}_A^{(1)}$ or to $\tilde{\sigma}_A^{(2)}$. Assume that, for given positive numbers ε_0 and E_0, v satisfies

$$|v(x)| \leqslant \varepsilon_0 r_0^{2-n}, \quad \text{for every } x \in (D_0)_{\frac{r_0}{3}}, \tag{4.1}$$

and

$$|v(x)| \leqslant E_0 (r_0 d(x))^{1-n/2}, \quad \text{for every } x \in \mathcal{W}_k, \tag{4.2}$$

where $d(x) = \text{dist}(x, \Sigma_{k+1})$. Then the following inequality holds true for every $r \in (0, d_1]$

$$\left| v\left(w_{\bar{h}}(P_{k+1}) \right) \right| \leqslant r_0^{2-n} C^{\bar{h}} (E_0 + \varepsilon_0) \left(\omega_{1/C}^{(k)}\left(\frac{\varepsilon_0}{E_0 + \varepsilon_0} \right) \right)^{(1/C)^{\bar{h}}}. \tag{4.3}$$

Proof. We observe that the proof of this result follows the same line of the argument used in [A-V proof of proposition 4.4] which is independent from the presence of isotropy/anisotropy in $\tilde{\sigma}$. In fact their proof is based on an argument of unique continuation which requires $\tilde{\sigma}$ to be

Lipschitz continuous and the interfaces between each domain D_j to contain a $C^{1,\alpha}$ portion, therefore we simply recall [A-V], proof of proposition 4.4 for a complete proof of this proposition. Here we simply recall for sake of completeness the main fact proven in [A-V], proof of proposition 4.4. By defining the quantities

$$r_1 = \frac{r_0}{4}, \qquad \bar{\rho} = \frac{r_1}{128\sqrt{1 + L^2}},$$

let $y_m \in D_m$ be a point 'near the portion' Σ_{m+1} of the interface between D_m and D_{m+1} defined by

$$y_m = P_{m+1} - \frac{r_1}{32}\nu(P_{m+1}),$$

where $P_{m+1} \in \Sigma_{m+1}$. Their main point is the proof of the following fact

$$\|v\|_{L^\infty(B_{\bar{\rho}}(y_m))} \leqslant r_0^{2-n} C^{m+1}(E_0 + \varepsilon_0)\omega_{\frac{1}{C}}^{(m+1)}\left(\frac{\varepsilon_0}{E_0 + \varepsilon_0}\right), \tag{4.4}$$

where $\bar{\rho}$ has been chosen above so that $B_{\bar{\rho}}(y_m) \subset D_m$. The proof of the above inequality is done by induction. The so-called argument of *global propagation of smallness* is used there to prove (4.4) for $m = 0$. We refer to [A-R-R-V], theorem 5.3 for a complete treatment of this topic. The rest of the proof is based on the *three sphere inequality*, therefore we simply refer to [A-V], proof of proposition 4.4 for this.

Acknowledgments

References

[A] Alessandrini G 1988 Stable determination of conductivity by boundary measurements *Appl. Anal.* **27** 153–72

[A1] Alessandrini G 1990 Singular solutions of elliptic equations and the determination of conductivity by boundary measurements *J. Differ. Equ.* **84** 252–72

[A2] Alessandrini G 1991 Determining conductivity by boundary measurements, the stability issue *Applied and Industrial Mathematics* ed R Spigler (Dordrecht: Kluwer) pp 317–24

[A3] Alessandrini G 2007 Open issues of stability for the inverse conductivity problem *J. Inv. Ill-Posed Problems* **15** 1–10

[A-R-R-V] Alessandrini G, Rondi L, Rosset E and Vessella S 2009 The stability for the Cauchy problem for elliptic equations (topical review) *Inverse Problems* **25** 123004

[A-G] Alessandrini G and Gaburro R 2001 Determining conductivity with special anisotropy by boundary measurements *SIAM J. Math. Anal.* **33** 153–71

[A-G1] Alessandrini G and Gaburro R 2009 The local Calderón problem and the determination at the boundary of the conductivity *Commun. Partial Differ. Equ.* **34** 918–36

[A-S] Alessandrini G and Sincich E 2013 Cracks with impedance, stable determination from boundary data *Indiana Univ. Math. J.* **62** 947–89

[A-V] Alessandrini G and Vessella S 2005 Lipschitz stability for the inverse conductivity problem *Adv. Appl. Math.* **35** 207–41

[A-La-P] Astala K, Lassas M and Päivärinta L 2005 Calderón inverse problem for anisotropic conductivity in the plane *Commun. Partial Differ. Equ.* **30** 207–24

[B-B-R] Barceló J A, Barceló T and Ruiz A 2001 Stability of the inverse conductivity problem in the plane for less regular conductivities *J. Differ. Equ.* **173** 231–70

[B-F-R] Barceló T, Faraco D and Ruiz A 2007 Stability of Calderón inverse conductivity problem in the plane *J. Math. Pures Appl.* **88** 522–56

[Be] Belishev M I 2003 The Calderón problem for two-dimensional manifolds by the BC-method *SIAM J. Math. Anal.* **35** 172–82

[B-dH-Q] Beretta E, De Hoop M V and Qiu L 2013 Lipschitz stability of an inverse boundary value problem for a Schrödinger type equation *SIAM J. Math. Anal.* **45** 679–99

[Be-Fr] Beretta E and Francini E 2011 Lipschitz stability for the electrical impedance tomography problem: the complex case *Commun. Partial Differ. Equ.* **36** 1723–49

[Be-Fr-V] Beretta E, Francini E and Vessella S 2014 Uniqueness and Lipschitz stability for the identification of Lamé parameters from boundary measurements *Inverse Problems and Imaging* **8** 611–44

[Bo] Borcea L 2002 Electrical impedance tomography *Inverse Problems* **18** R99–R136

[C] Calderón A P 1980 On an inverse boundary value problem *Seminar on Numerical Analysis and its Applications to Continuum Physics (Rio de Janeiro)* pp 65–73 (Rio de Janeiro: Brazilian Mathematical Society) (Reprinted in 2006) *Comput. Appl. Math.* **25** 133–138

[Ca-G-R] Caro P, García A and Reyes J M 2013 Stability for the Calderón problem for less regular conductivities *J. Differ. Equ.* **254** 469–92 Caro P, García A and Reyes J M 2013 *Inventiones Math.* **178** 119–71

[Ch-I-N] Cheney M, Isaacson D and Newell J C 1999 Electrical impedance tomography *SIAM Rev.* **41** 85–101

[D-Ke-S-U] Dos Santos Ferreira D, Kenig C, Salo M and Uhlmann G 2009 Limiting Carleman weights and anisotropic inverse problems *Inventiones Math.* **178** 119–71

[D-Ku-L-S] Dos Santos Ferreira D, Kurylev Y, Lassas M and Salo M 2014 The Calderón problem in transversally anisotropic geometries *J. Eur. Math. Soc.* at press (arXiv:1305.1273)

[F-K-R] Faraco D, Kurylev Y and Ruiz A 2013 G-convergence, Dirichlet to Neumann maps and invisibility *J. Funct. Anal.* **267** 24782576

[G-L] Gaburro R and Lionheart W R B 2009 Recovering Riemannian metrics in monotone families from boundary data *Inverse Problems* **25** 045004

[H-S] Henkin G and Santacesaria M 2010 On an inverse problem for anisotropic conductivity in the plane *Inverse Problems* **26** 095011

[K] Kwon K 2004 Identification of anisotropic anomalous region in inverse problems *Inverse Problems* **20** 1117–36

[K-Vo1] Kohn R and Vogelius M 1984 Identification of an unknown conductivity by means of measurements at the boundary *SIAM-AMS Proc.* **14** 113–23

[K-Vo2] Kohn R and Vogelius M 1985 Determining conductivity by boundary measurements II. interior results *Commun. Pure Appl. Math.* **38** 643–67

[L] Lionheart W R B 1997 Conformal uniqueness results in anisotropic electrical impedance imaging *Inverse Problems* **13** 125–34

[La-U] Lassas M and Uhlmann G 2001 On determining a Riemannian manifold from the Dirichlet-to-Neumann map *Ann. Sci. École Norm. Super.* **34** 771–87

[LU] Lee J M and Uhlmann G 1989 Determining anisotropic real-analytic conductivities by boundary measurements *Commun. Pure Appl. Math.* **42** 1097–112

[Liu] Liu L 1997 Stability estimates for the two-dimensional inverse conductivity problem *PhD Thesis* University of Rochester, New York

[Lit-St-W] Littman W, Stampacchia G and Weinberger H W 1963 Regular points for elliptic equations with discontinuous coefficients *Ann. Scuola Norm. Pisa C1 Sci.* **3** 43–77

[Li-Ni] Li Y Y and Niremberg L 2003 Estimates for elliptic systems from composite material *Commun. Pure Appl. Math.* **56** 892–925

[Li-Vo] Li Y Y and Vogelius M 2000 Gradient estimates for solutions to divergence form elliptic equations with discontinuous coefficients *Arch. Rational Mech. Anal.* **153** 91–151

[Ma] Mandache N 2001 Exponential instability in an inverse problem for the Schrödeinger equation *Inverse Problems* **17** 1435–44

[Mi] Miranda C 1970 *Partial Differential Equations of Elliptic Type* 2nd edn (Berlin: Springer)

[Na] Nachman A 1995 Global uniqueness for a two dimensional inverse boundary value problem *Ann. Math.* **142** 71–96

[N-S] Novikov R G and Santacesaria M 2010 A global stability estimate for the Gel'fand–Calderòn inverse problem in two dimensions *J. Inverse Ill-Posed Probl.* **18** 765–85

[S] Sylvester J 1990 An anisotropic inverse boundary value problem *Commun. Pure. Appl. Math.* **43** 201–32

[S-U] Sylvester J and Uhlmann G 1987 A global uniqueness theorem for an inverse boundary valued problem *Ann. Math.* **125** 153–69

[U] Uhlmann G 2009 Electrical impedance tomography and Calderón's problem (topical review) *Inverse Problems* **25** 123011