On some multicomponent quasilinear elliptic systems

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A R T I C L E   I N F O

In memory of our dearest friend Ioan Vrabie

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A B S T R A C T

The paper contains some Liouville theorems for second order multicomponent quasilinear elliptic systems of equations or inequalities in $\mathbb{R}^N$ associated to general nonlinearities. No assumptions on the behaviour of the solutions at infinity are assumed.

1. Introduction

Liouville theorems of quasilinear elliptic systems has been an active research area since the publication of the seminal works [13], [7]. Since then, a huge number of papers have been published on this topic. See for instance [17], [14].

One of the main reasons for studying Liouville's theorems, apart from their intrinsic interest, lies in the fact that they are a powerful tool to demonstrate existence results for the Dirichlet problem of associated problems in bounded domains of $\mathbb{R}^N$. See [8], [1], [3], for some results in this direction.

In this paper we prove some Liouville theorems of general quasilinear second order elliptic systems and inequalities in $\mathbb{R}^N$ in divergence form. The interested reader may refer to [11], [10] for earlier results related to this work and to [5] and [6] for more recent outcomes.

Perhaps the main interest about our analysis is that we do not assume any kind of variational structure on our problems. In this way we can apply to prove existence theorems for general systems. The simplest problem that we have in mind in the semilinear case (see also [15]) is the following,

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$$\begin{cases} -\Delta u = u^{p_1} v^{q_1} w^{r_1}, & \text{in } \mathbb{R}^N, \\ -\Delta v = u^{p_2} u^{q_2} w^{r_2}, & \text{in } \mathbb{R}^N, \\ -\Delta w = u^{q_3} v^{p_3} w^{q_3}, & \text{in } \mathbb{R}^N, \\ u > 0,\ v > 0,\ w > 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

and its quasilinear counterpart,

$$\begin{cases} -\Delta_p u = u^{p_1} v^{q_1} w^{r_1}, & \text{in } \mathbb{R}^N, \\ -\Delta_q v = u^{p_2} u^{q_2} w^{r_2}, & \text{in } \mathbb{R}^N, \\ -\Delta_r w = u^{q_3} v^{p_3} w^{q_3}, & \text{in } \mathbb{R}^N, \\ u > 0,\ v > 0,\ w > 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{1.2}$$

where as usual, $\Delta_p = \operatorname{div}(|\nabla(.)|^{p-2}\nabla(.))$, $p > 1$, and similarly for the other two operators. We emphasise that here, the exponents of the power nonlinearities are real numbers (they can be negative).

In this paper we concentrate our attention on the following problem

$$\begin{cases} -\operatorname{div}(\mathscr{A}_p(x,u,\nabla u)) \geq f(u,v,w) & \text{in } \mathbb{R}^N, \\ -\operatorname{div}(\mathscr{A}_q(x,v,\nabla v)) \geq g(u,v,w) & \text{in } \mathbb{R}^n, \\ -\operatorname{div}(\mathscr{A}_r(x,w,\nabla w)) \geq h(u,v,w) & \text{in } \mathbb{R}^n, \\ u \geq 0,\ v \geq 0,\ w \geq 0 & \text{in } \mathbb{R}^n, \end{cases} \tag{1.3}$$

which is a strong generalization of (1.1) and (1.2). In (1.3), $\operatorname{div}(\mathscr{A}_i(\cdot,\cdot,\nabla\cdot)$ are general operators satisfying a classical structure condition (see Definition 1 below for the precise assumption), and f, g, and h are nonnegative continuous functions on $\mathbb{R}^3_+ =]0,+\infty[\times]0,+\infty[\times]0,+\infty[$ with an unique hypothesis on their behaviour near the origin $(0,0,0)$.

We emphasise that our assumptions on the nonlinearities allow us to study singular systems. See Remark 4.

The results proved in this paper are a substantial generalization of those proved in [4] for the scalar case, and [5] and [6] for systems of two equations. However, to keep our exposition simple and transparent we bound our interest to systems with three equations. Extensions of the results contained in this paper can be proved for systems containing more than three equations or inequalities. This will be a subject of our future work.

We point out that not all Liouville theorems proved in this paper are sharp. On one hand, this is due to the complexity of the problem and, on the other hand, this paper can be considered just a beginning of more systematic study for this kind of systems. However, we emphasise that we do not assume any kind of behaviour of the solutions at infinity.

This paper is organised as follows. In the next section we fix some notations, hypotheses and definitions that we shall make throughout the paper. Section 3 is devoted to the key a priori estimates of the solutions of the problems under consideration. See in particular Lemma 10 and its consequences. Finally the last section contains the main result of this paper (see Theorem 16) and other related theorems.

2. Multicomponent quasilinear systems

In this paper we shall study system quasilinear inequalities whose differential operators are in divergence form, namely

$$Lu(x) = \operatorname{div}(\mathscr{A}(x,u,\nabla u)).$$

Our model case is the p-Laplace operator.

Definition 1. Let $\Omega \subset \mathbb{R}^N$ be an open set. Let $p > 1$ and $\mathscr{A}_p : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function. The function $\mathscr{A}_p$ is called S-p-C, *strongly-p-coercive*, if there exist two constants $a, \tilde{a} > 0$ such that

$$\text{(S-}p\text{-C)} \qquad (\mathscr{A}_p(x,t,w) \cdot w) \geq \tilde{a}|w|^p \geq a|\mathscr{A}_p(x,t,w)|^{p'} \quad \text{for all } (x,t,w) \in \Omega \times \mathbb{R} \times \mathbb{R}^N,$$

see [11] and [12] for details.

As it is well known, a multicomponent system is a system that involves more than two equations. In this section we state some preliminary facts about the problem

$$\begin{cases} -\operatorname{div}(\mathscr{A}_p(x,u,\nabla u)) \geq f(u,v,w) & \text{in } \Omega, \\ -\operatorname{div}(\mathscr{A}_q(x,v,\nabla v)) \geq g(u,v,w) & \text{in } \Omega, \\ -\operatorname{div}(\mathscr{A}_r(x,w,\nabla w)) \geq h(u,v,w) & \text{in } \Omega, \\ u \geq 0, \ v \geq 0, \ w \geq 0 & \text{in } \Omega. \end{cases} \tag{2.1}$$

Along the whole paper we shall use the following assumption.

H1. The functions $\mathscr{A}_p, \mathscr{A}_q, \mathscr{A}_r : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ are S-p-C, S-q-C *and* S-r-C respectively, $N > \max\{p,q,r\} \geq \min\{p,q,r\} > 1$ and the functions $f, g, h :]0,\infty[\times]0,\infty[\times]0,\infty[\to [0,\infty[$ are continuous.

Definition 2. The set of three functions $(u,v,w) \in W^{1,p}_{\mathrm{loc}}(\Omega) \times W^{1,q}_{\mathrm{loc}}(\Omega) \times W^{1,r}_{\mathrm{loc}}(\Omega)$ is called a *weak solution of* (3.2) if

$$f(u,v,w), g(u,v,w), h(u,v,w) \in L^1_{\mathrm{loc}}(\Omega),$$
$$|\mathscr{A}_p(\cdot,u,\nabla u)|^{p'}, |\mathscr{A}_q(\cdot,v,\nabla v)|^{q'}, |\mathscr{A}_r(\cdot,w,\nabla w)|^{r'} \in L^1_{\mathrm{loc}}(\Omega),$$

and the following inequalities hold

$$\int_\Omega (\mathscr{A}_p(x,u,\nabla u) \cdot \nabla \phi_1) \geq \int_\Omega f(u,v,w)\phi_1,$$
$$\int_\Omega (\mathscr{A}_q(x,v,\nabla v) \cdot \nabla \phi_2) \geq \int_\Omega g(u,v,w)\phi_2,$$
$$\int_\Omega (\mathscr{A}_r(x,w,\nabla w) \cdot \nabla \phi_3) \geq \int_\Omega h(u,v,w)\phi_3,$$

for all non-negative functions $\phi_1, \phi_2, \phi_3 \in C^1_0(\Omega)$.

We say that a weak solution (u,v,w) is *trivial* if either $u = 0$ or $v = 0$ or $w = 0$ a.e. in Ω.

Moreover in what follows we shall assume that f, g, h satisfy the following hypotheses.

(f_0) *there exist* $p_1 \geq 0$, $q_1 \geq 0$, $r_1 \geq 0$ *such that*

$$\liminf_{\tau+\xi+\zeta \to 0} \frac{f(\tau,\xi,\zeta)}{\tau^{p_1}\xi^{q_1}\zeta^{r_1}} > 0 \quad \text{(possibly infinity)},$$

(g_0) *there exist* $p_2 \geq 0$, $q_2 \geq 0$, $r_2 \geq 0$ *such that*

$$\liminf_{\tau+\xi+\zeta\to 0} \frac{g(\tau,\xi,\zeta)}{\tau^{p_2}\xi^{q_2}\zeta^{r_2}} > 0 \quad \text{(possibly infinity)},$$

(h_0) *there exist* $p_3 \geq 0$, $q_3 \geq 0$ *and* $r_3 \geq 0$ *such that*

$$\liminf_{\tau+\xi+\zeta\to 0} \frac{h(\tau,\xi,\zeta)}{\tau^{p_3}\xi^{q_3}\zeta^{r_3}} > 0 \quad \text{(possibly infinity)}.$$

Remark 3. The conditions (that we do not assume along the paper)

$$\max\{p_2,p_3\} > 0, \quad \max\{q_1,q_3\} > 0, \quad \max\{r_1,r_2\} > 0, \tag{2.2}$$

$$\max\{q_1,r_1\} > 0, \quad \max\{p_2,r_2\} > 0, \quad \max\{p_3,q_3\} > 0, \tag{2.3}$$

assure that the system is a genuinely strongly coupled. Indeed, for instance, in the case that f,g are pure powers, that is $f(\tau,\xi,\zeta) = \tau^{p_1}\xi^{q_1}\zeta^{r_1}$ and $g(\tau,\xi,\zeta) = \tau^{p_2}\xi^{q_2}\zeta^{r_2}$, if $r_1 = r_2 = 0$ (i.e. (2.2) does not hold), then the system can be decoupled, since the first two equations of (2.1) do not depend on w.

Remark 4. The assumptions (f_0), (g_0), (h_0) cover also the case of singular nonlinearities. Indeed, if for instance $f(\tau,\eta,\zeta) = \frac{\xi^\beta\zeta^\gamma}{\tau^\alpha}$, with $\alpha,\beta,\gamma > 0$, then (f_0) is fulfilled with $p_1 = 0$, $q_1 = \beta$ and $r_1 = \gamma$. Indeed in this case we have,

$$\lim_{\tau+\xi+\zeta\to 0} \frac{f(\tau,\xi,\zeta)}{\tau^{p_1}\xi^{q_1}\zeta^{r_1}} = +\infty.$$

For some results applicable to the singular case, see Lemma 11 and Proposition 14 below.

By B_R we denote the euclidean ball in $\mathbb{R}^N$ of radius R centred at the origin, while A_R stands for the ring $A_R := B_R \setminus \overline{B_{R/2}}$. In the rest of the paper, for sake of simplicity, by $\inf_A u$ we denote the essential infimum of u on the set A, that is $\operatorname{essinf}_A u$

We recall the following classical,

Proposition 5 *(Weak Harnack inequality). Suppose that* $\mathscr{A}_p$ *is S-p-C and* $\Omega \subseteq \mathbb{R}^N$ *is an open set. Then there exist* $\sigma > 0$ *and* $c_H > 0$ *such that for any weak solution* u *of*

$$\begin{cases} -\operatorname{div}(\mathscr{A}_p(x,u,\nabla u)) \geq 0 & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \end{cases}$$

for any $R > 0$ *such that* $B_{2R} \subset \Omega$, *we have*

(WH) $\quad \left(\frac{1}{|B_R|}\int_{B_R} u^\sigma\right)^{1/\sigma} \leq c_H \inf_{B_{R/2}} u.$

For a proof see for instance [12], [16].

Remark 6. The hypothesis S-p-C can be relaxed by requiring that the operators are of weak-p-C type and an Harnack type estimate holds. See [4] and [2]. However in this paper, for sake of brevity, we shall deal only with the S-p-C case.

The following is a direct consequence of (WH). See Proposition 1 and Remark 2 of [6].

Proposition 7. *If* (WH) *holds for the three nonnegative functions u, v and w, then* (WH) *also holds for $u+v+w$. Furthermore, there exists a positive constant C independent of u, v, w, such that,*

$$\inf_{B_R}(u+v+w) \leq C\left(\inf_{B_{R/2}} u + \inf_{B_{R/2}} v + \inf_{B_{R/2}} w\right),$$

for all $R>0$ such that $B_{2R} \subset \Omega$.

Lemma 8 *(Lemma 3.1 [4]). Let $u:\mathbb{R}^N \to [0,\infty)$ be a function such that $\inf_{\mathbb{R}^N} u = 0$. Assume that* (WH) *holds with exponent $\sigma>0$, then for all $\varepsilon>0$*

$$\lim_{R\to\infty} \frac{|A_R \cap T_\varepsilon^u|}{|A_R|} = 1, \qquad \lim_{R\to\infty} \frac{|B_R \cap T_\varepsilon^u|}{|B_R|} = 1,$$

where $T_\varepsilon^u = \{x \in \mathbb{R}^N \,:\, u(x) < \varepsilon\}$ and $A_R = B_R \setminus \overline{B_{R/2}}$.

From now on, if not otherwise specified, for $\varepsilon>0$, we set

$$T_\varepsilon = \{x \in \mathbb{R}^N \,:\, u(x)+v(x)+w(x) < \varepsilon\}.$$

3. General a priori estimates

A slight variation of the proof of Theorem 2 of [6] gives the following.

Theorem 9 *(General a priori estimates). Assume that* **H1** *holds. There exist positive constants,*

$$c_1 = c_1(\mathscr{A}_p) > 0, c_2 = c_2(\mathscr{A}_q) > 0, c_3 = c_3(\mathscr{A}_r) > 0$$

such that if (u,v,w) is a weak solution of (2.1), then for all $R>0$ such that $B_{2R} \subset\subset \Omega$, we have

$$\begin{aligned}
\frac{1}{|B_R|}\int_{B_R} f(u,v,w) &\leq c_1 R^{-p}\left(\inf_{B_R} u\right)^{p-1}, \\
\frac{1}{|B_R|}\int_{B_R} g(u,v,w) &\leq c_2 R^{-q}\left(\inf_{B_R} v\right)^{q-1}, \\
\frac{1}{|B_R|}\int_{B_R} h(u,v,w) &\leq c_3 R^{-r}\left(\inf_{B_R} w\right)^{r-1}.
\end{aligned} \tag{3.1}$$

In what follows we shall concentrate our attention to the problem,

$$\begin{cases}
-\operatorname{div}(\mathscr{A}_p(x,u,\nabla u)) \geq f(u,v,w) & \text{in } \mathbb{R}^N, \\
-\operatorname{div}(\mathscr{A}_q(x,v,\nabla v)) \geq g(u,v,w) & \text{in } \mathbb{R}^N, \\
-\operatorname{div}(\mathscr{A}_r(x,w,\nabla w)) \geq h(u,v,w) & \text{in } \mathbb{R}^N, \\
\inf_{\mathbb{R}^N} u = \inf_{\mathbb{R}^N} v = \inf_{\mathbb{R}^N} w = 0.
\end{cases} \tag{3.2}$$

The following is the key of our results.

Lemma 10. *Assume that* **H1** *holds. Let* (u, v, w) *be a non trivial weak solution of* (3.2).

If (f_0) *holds, then for all* $\varepsilon > 0$ *sufficiently small and* $R > 0$ *sufficiently large the following estimate holds*

$$(\inf_{A_R \cap T_\varepsilon} u)^{p_1 - p + 1} (\inf_{A_R \cap T_\varepsilon} v)^{q_1} (\inf_{A_R \cap T_\varepsilon} w)^{r_1} \leq c R^{-p}, \tag{3.3}$$

where $T_\varepsilon = \{x \in \mathbb{R}^N \,:\, u(x) + v(x) + w(x) < \varepsilon\}$ *and* $A_R = B_R \setminus \overline{B_{R/2}}$.

Similarly, if (g_0) *and* (h_0) *hold we have, respectively*

$$(\inf_{A_R \cap T_\varepsilon} u)^{p_2} (\inf_{A_R \cap T_\varepsilon} v)^{q_2 - q + 1} (\inf_{A_R \cap T_\varepsilon} w)^{r_2} \leq c R^{-q}, \tag{3.4}$$

and

$$(\inf_{A_R \cap T_\varepsilon} u)^{p_3} (\inf_{A_R \cap T_\varepsilon} v)^{q_3} (\inf_{A_R \cap T_\varepsilon} w)^{r_3 - r + 1} \leq c R^{-r}. \tag{3.5}$$

Proof. Let us prove (3.3), the proof of (3.4) and (3.5) being similar. From (f_0) it follows that for a suitable $c > 0$ there exists $\epsilon > 0$ such that,

$$f(\tau, \xi, \zeta) \geq c \tau^{p_1} \xi^{q_1} \zeta^{r_1} \quad \text{for} \;\; \epsilon > \tau, \xi, \zeta > 0. \tag{3.6}$$

From the first inequality of (3.1) we obtain

$$\int_{B_R} f(u, v, w) \leq c R^{-p} |A_R| (\inf_{B_R} u)^{p-1} \leq c R^{-p} |A_R| (\inf_{A_R \cap T_\varepsilon} u)^{p-1}. \tag{3.7}$$

On the other hand using (3.6), we have

$$\int_{B_R} f(u, v, w) \geq \int_{A_R \cap T_\varepsilon} f(u, v, w) \geq c \int_{A_R \cap T_\varepsilon} u^{p_1} v^{q_1} w^{r_1},$$

hence,

$$\int_{A_R \cap T_\varepsilon} u^{p_1} v^{q_1} w^{r_1} \leq c R^{-p} |A_R| (\inf_{A_R \cap T_\varepsilon} u)^{p-1}. \tag{3.8}$$

Therefore

$$(\inf_{A_R \cap T_\varepsilon} u)^{p_1} (\inf_{A_R \cap T_\varepsilon} v)^{q_1} (\inf_{A_R \cap T_\varepsilon} w)^{r_1} \leq c R^{-p} \frac{|A_R|}{|A_R \cap T_\varepsilon|} (\inf_{A_R \cap T_\varepsilon} u)^{p-1} \tag{3.9}$$

and so, by Proposition 7 and by Lemma 8 applied to the function $u + v + w$,

$$(\inf_{A_R \cap T_\varepsilon} u)^{p_1 - p + 1} (\inf_{A_R \cap T_\varepsilon} v)^{q_1} (\inf_{A_R \cap T_\varepsilon} w)^{r_1} \leq c R^{-p},$$

for R sufficiently large. □

From the above lemma we immediately deduce the following Liouville theorem.

Lemma 11. *Assume that* **H1** *holds. If* (f_0) *holds with* $p_1 = q_1 = r_1 = 0$ *or* (g_0) *holds with* $p_2 = q_2 = r_2 = 0$ *or* (h_0) *holds with* $p_3 = q_3 = r_3 = 0$ *then* (3.2) *has no non trivial weak solution.*

Proof. Without loss of generality assume that (f_0) holds with $p_1 = q_1 = r_1 = 0$. By contradiction, let (u, v, w) be a non-trivial weak solution with $\inf_{\mathbb{R}^N} u = \inf_{\mathbb{R}^N} v = \inf_{\mathbb{R}^N} w = 0$. With the same notation of Lemma 10, from (3.3) we have

$$\varepsilon^{-(p-1)} \leq (\inf_{A_R \cap T_\varepsilon} u)^{-(p-1)} \leq cR^{-p},$$

and letting $R \to +\infty$ we get a contradiction. □

Since Lemma 11 holds, we realize that the following assumption is natural.

H2. (f_0) is fulfilled with $p_1, q_1, r_1 \geq 0$ and at least one of them positive,
(g_0) is fulfilled with $p_2, q_2, r_2 \geq 0$ and at least one of them positive,
(h_0) is fulfilled with $p_3, q_3, r_3 \geq 0$ and at least one of them positive.

Let us introduce the number,

$$\begin{aligned} D := & q_3 r_2 (p-1-p_1) + p_3 r_1 (q-1-q_2) + p_2 q_1 (r-1-r_3) \\ & - (p-1-p_1)(q-1-q_2)(r-1-r_3) + p_3 q_1 r_2 + p_2 q_3 r_1. \end{aligned}$$

Lemma 12. *Assume that* **H1**, (f_0), (g_0), (h_0) *hold. Let* (u, v, w) *be a non-trivial weak solution of* (3.2). *For* $\varepsilon > 0$, *we set* $T_\varepsilon = \{x \in \mathbb{R}^N : u(x) + v(x) + w(x) < \varepsilon\}$.

If $q_3 r_2 < (q-1-q_2)(r-1-r_3)$, $q_2 < q-1$, $r_3 < r-1$, *then for all* $\varepsilon > 0$ *sufficiently small and* $R > 0$ *sufficiently large, we have*

$$(\inf_{A_R \cap T_\varepsilon} u)^D \leq cR^{-\theta_u}, \tag{3.10}$$

where

$$\theta_u := r r_1 (q-1-q_2) + q q_1 (r-1-r_3) + p[(q-1-q_2)(r-1-r_3) - q_3 r_2] + q q_3 r_1 + r q_1 r_2. \tag{3.11}$$

Furthermore, if $D > 0$ *then for* R *large we obtain*

$$\int_{B_R} f(u, v, w) \leq cR^{N-p-\frac{p-1}{D}\theta_u}. \tag{3.12}$$

If $p_3 r_1 < (p-1-p_1)(r-1-r_3)$, $p_1 < p-1$, $r_3 < r-1$, *then for all* $\varepsilon > 0$ *sufficiently small and* $R > 0$ *sufficiently large, we have*

$$(\inf_{A_R \cap T_\varepsilon} v)^D \leq cR^{-\theta_v}, \tag{3.13}$$

where

$$\theta_v := r r_2 (p-1-p_1) + p p_2 (r-1-r_3) + q[(p-1-p_1)(r-1-r_3) - p_3 r_1] + p p_3 r_2 + r p_2 r_1. \tag{3.14}$$

Furthermore, if $D > 0$ *for* R *large we obtain*

$$\int_{B_R} g(u, v, w) \leq cR^{N-q-\frac{q-1}{D}\theta_v}. \tag{3.15}$$

If $p_2q_1 < (p-1-p_1)(q-1-q_2)$, $p_1 < p-1$, $q_2 < q-1$, then for all $\varepsilon > 0$ sufficiently small and $R > 0$ sufficiently large, we have

$$(\inf_{A_R\cap T_\varepsilon} w)^D \le cR^{-\theta_w}, \tag{3.16}$$

where

$$\theta_w := qq_3(p-1-p_1) + pp_3(q-1-q_2) + r[(p-1-p_1)(q-1-q_2) - p_2q_1] + pp_2q_3 + qp_3q_1. \tag{3.17}$$

Furthermore, if $D > 0$, for R large we

$$\int_{B_R} h(u,v,w) \le cR^{N-r-\frac{r-1}{D}\theta_w}. \tag{3.18}$$

Proof. In order to prove (3.10), we distinguish seven cases.

Case 1. $q_1 > 0$ *and* $q_3 > 0$. By (3.4) and (3.3), we have

$$(\inf_{A_R\cap T_\varepsilon} u)^{p_2} \le cR^{-q}\frac{(\inf_{A_R\cap T_\varepsilon} v)^{q-1-q_2}}{(\inf_{A_R\cap T_\varepsilon} w)^{r_2}} \le cR^{-q-p\frac{q-1-q_2}{q_1}}\frac{(\inf_{A_R\cap T_\varepsilon} u)^{(p-1-p_1)(q-1-q_2)/q_1}}{(\inf_{A_R\cap T_\varepsilon} w)^{r_2+r_1\frac{q-1-q_2}{q_1}}},$$

therefore

$$(\inf_{A_R\cap T_\varepsilon} u)^{p_2-\frac{(p-1-p_1)(q-1-q_2)}{q_1}} \le cR^{-q-p\frac{q-1-q_2}{q_1}}(\inf_{A_R\cap T_\varepsilon} w)^{-r_2-\frac{r_1(q-1-q_2)}{q_1}}. \tag{3.19}$$

Similarly, (3.4) and (3.5) imply

$$(\inf_{A_R\cap T_\varepsilon} w)^{r_2-\frac{(q-1-q_2)(r-1-r_3)}{q_3}} \le cR^{-q-r\frac{q-1-q_2}{q_3}}(\inf_{A_R\cap T_\varepsilon} u)^{-p_2-\frac{p_3(q-1-q_2)}{q_3}}, \tag{3.20}$$

for all $\varepsilon > 0$ and R sufficiently large. Combining together (3.19) and (3.20), we get

$$\begin{aligned}(\inf_{A_R\cap T_\varepsilon} u)^{p_2-\frac{(p-1-p_1)(q-1-q_2)}{q_1}+\frac{p_2q_3+p_3(q-1-q_2)}{(q-1-q_2)(r-1-r_3)-r_2q_3}\left(r_2+r_1\frac{q-1-q_2}{q_1}\right)} \le \\ \le R^{-q-p\frac{q-1-q_2}{q_1}-\frac{qq_3+(q-1-q_2)r}{(q-1-q_2)(r-1-r_3)-r_2q_3}\left(r_2+r_1\frac{q-1-q_2}{q_1}\right)},\end{aligned}$$

that is, by $q_3r_2 < (q-1-q_2)(r-1-r_3)$,

$$(\inf_{A_R\cap T_\varepsilon} u)^{(q-1-q_2)D} \le cR^{(q-1-q_2)[-p(q-1-q_2)(r-1-r_3)+pr_2q_3-qq_1(r-1-r_3)-qq_3r_1-rr_2q_1-rr_1(q-1-q_2)]}.$$

Hence (3.10) holds, being $q_2 < q-1$.

Case 2. $q_1 = 0$, $q_3 > 0$ *and* $r_1 > 0$. Therefore, as in the previous case, by (3.4) and (3.5), we obtain (3.20). Therefore by using (3.3), from (3.20) we have,

$$\begin{aligned}(\inf_{A_R\cap T_\varepsilon} u)^{p_1-p+1} &\le cR^{-p}(\inf_{A_R\cap T_\varepsilon} w)^{-r_1} \\ &\le cR^{-p+r_1\frac{qq_3+r(q-1-q_2)}{r_2q_3-(q-1-q_2)(r-1-r_3)}}(\inf_{A_R\cap T_\varepsilon} u)^{r_1\frac{p_2q_3+p_3(q-1-q_2)}{r_2q_3-(q-1-q_2)(r-1-r_3)}}.\end{aligned}$$

That is

$$(\inf_{A_R} u)^D \le cR^{-p[(q-1-q_2)(r-1-r_3)-r_2q_3]-r_1[qq_3+r(q-1-q_2)]}.$$

Case 3. $q_1 = 0$, $q_3 > 0$ and $r_1 = 0$. In this case (3.3) reads as

$$(\inf_{A_R\cap T_\varepsilon} u)^{p_1-p+1} \le cR^{-p},$$

which, taking into account that $(q-1-q_2)(r-1-r_3) - q_3r_2 > 0$, implies the claim.

Case 4. $q_1 > 0$, $q_3 = 0$, and $p_3 > 0$. Arguing as in the Case 1., by (3.4) and (3.3) we get (3.19) as before. On the other hand since $r_3 < r-1$, from (3.5) we have

$$\inf_{A_R\cap T_\varepsilon} w \ge cR^{\frac{r}{r-1-r_3}} (\inf_{A_R\cap T_\varepsilon} u)^{\frac{p_3}{r-1-r_3}}. \tag{3.21}$$

Hence, combining together (3.19) and (3.21), it follows that

$$(\inf_{A_R\cap T_\varepsilon} u)^{p_2-\frac{(p-1-p_1)(q-1-q_2)}{q_1}+\frac{p_3}{r-1-r_3}\left(r_2+r_1\frac{q-1-q_2}{q_1}\right)} \le R^{-q-p\frac{q-1-q_2}{q_1}-\frac{r}{r-1-r_3}\left(r_2+r_1\frac{q-1-q_2}{q_1}\right)},$$

which, since $q_1(r-1-r_3) > 0$, implies (3.10).

Case 5. $q_1 > 0$, $q_3 = 0$, and $p_3 = 0$. Arguing as in the Case 1., by (3.4) and (3.3) we get (3.19) as before. While, from (3.5) we have

$$\inf_{A_R\cap T_\varepsilon} w \ge cR^{\frac{r}{r-1-r_3}},$$

which plugged in (3.19) yields the claim.

Case 6. $q_1 = 0$, $q_3 = 0$ and $r_1 > 0$. From (3.5) and then (3.3) we have

$$(\inf_{A_R\cap T_\varepsilon} u)^{p_3} \le cR^{-r}(\inf_{A_R\cap T_\varepsilon} w)^{r-1-r_3} \le cR^{-r}\left(R^{-\frac{p}{r_1}}(\inf_{A_R\cap T_\varepsilon} u)^{\frac{p-1-p_1}{r_1}}\right)^{r-1-r_3}. \tag{3.22}$$

From (3.22) we deduce

$$(\inf_{A_R\cap T_\varepsilon} u)^{p_3-\frac{(p-1-p_1)(r-1-r_3)}{r_1}} \le cR^{-r-p\frac{p-1-p_1}{r_1}},$$

which, together the fact that $q-1-q_2 > 0$, implies the claim.

Case 7. $q_1 = 0$, $q_3 = 0$ and $r_1 = 0$. As in the case 3.

To complete the proof it remains to show (3.12). Inequality (3.12) is an immediate consequence of (3.1) and (3.10), namely

$$\int_{B_R} f(u,v,w) \le cR^{-p}|A_R|(\inf_{A_R\cap T_\varepsilon} u)^{p-1}$$
$$\le cR^{N-p-\frac{p-1}{D}\{rr_1(q-1-q_2)+qq_1(r-1-r_3)+p[(q-1-q_2)(r-1-r_3)-q_3r_2]+qq_3r_1+rq_1r_2\}}.$$

The proof of (3.13)–(3.18) follows the same pattern as above. So we omit the details. □

Lemma 13. *Assume that* **H1** *and* **H2** *hold. Let (u, v, w) be a non-trivial weak solution of (3.2). Then there exists a constant $c > 0$ such that for $R > 0$ sufficiently large the following inequalities hold,*

$$\left(\int_{B_R} f(u,v,w)\right)^{\frac{p_1}{p-1}} \left(\int_{B_R} g(u,v,w)\right)^{\frac{q_1}{q-1}} \left(\int_{B_R} h(u,v,w)\right)^{\frac{r_1}{r-1}} \le cR^{\alpha_1 - N} \int_{A_R} f(u,v,w), \tag{3.23}$$

$$\left(\int_{B_R} f(u,v,w)\right)^{\frac{p_2}{p-1}} \left(\int_{B_R} g(u,v,w)\right)^{\frac{q_2}{q-1}} \left(\int_{B_R} h(u,v,w)\right)^{\frac{r_2}{r-1}} \le cR^{\alpha_2 - N} \int_{A_R} g(u,v,w), \tag{3.24}$$

$$\left(\int_{B_R} f(u,v,w)\right)^{\frac{p_3}{p-1}} \left(\int_{B_R} g(u,v,w)\right)^{\frac{q_3}{q-1}} \left(\int_{B_R} h(u,v,w)\right)^{\frac{r_3}{r-1}} \le cR^{\alpha_3 - N} \int_{A_R} h(u,v,w), \tag{3.25}$$

where

$$\begin{aligned} \alpha_1 &:= \tfrac{N-p}{p-1}p_1 + \tfrac{N-q}{q-1}q_1 + \tfrac{N-r}{r-1}r_1, \\ \alpha_2 &:= \tfrac{N-p}{p-1}p_2 + \tfrac{N-q}{q-1}q_2 + \tfrac{N-r}{r-1}r_2, \\ \alpha_3 &:= \tfrac{N-p}{p-1}p_3 + \tfrac{N-q}{q-1}q_3 + \tfrac{N-r}{r-1}r_3. \end{aligned} \tag{3.26}$$

Proof. Let us prove (3.24). Assume that $p_2 > 0$ (the case $r_2 > 0$ is similar). From (f_0), (g_0), (h_0), it follows that for a suitable $c > 0$ there exists $\epsilon > 0$ such that,

$$f(\tau,\xi,\zeta) \ge c\tau^{p_1}\xi^{q_1}\zeta^{r_1}, \quad g(\tau,\xi,\zeta) \ge c\tau^{p_2}\xi^{q_2}\zeta^{r_2}, \quad h(\tau,\xi,\zeta) \ge c\tau^{p_3}\xi^{q_3}\zeta^{r_3}, \quad \text{for } \epsilon > \tau,\xi,\zeta > 0. \tag{3.27}$$

By (3.1), and Lemma 8 applied to the function $u + v + w$, for R large we have

$$\begin{aligned} \int_{B_R} f(u,v,w) &\le cR^{-p}|A_R|(\inf_{B_R} u)^{p-1} \le cR^{-p}|A_R| \left(\frac{1}{|A_R \cap T_\varepsilon|} \int_{A_R \cap T_\varepsilon} u^{p_2}\right)^{\frac{p-1}{p_2}} \\ &\le cR^{-p}|A_R|^{1-\frac{p-1}{p_2}} \left[\frac{\int_{A_R\cap T_\varepsilon} u^{p_2}v^{q_2}w^{r_2}}{(\inf_{A_R\cap T_\varepsilon} v)^{q_2}(\inf_{A_R\cap T_\varepsilon} w)^{r_2}}\right]^{\frac{p-1}{p_2}} \\ &\le cR^{-p}|A_R|^{1-\frac{p-1}{p_2}} \left[\frac{\int_{A_R\cap T_\varepsilon} g(u,v,w)}{(\inf_{A_R\cap T_\varepsilon} v)^{q_2}(\inf_{A_R\cap T_\varepsilon} w)^{r_2}}\right]^{\frac{p-1}{p_2}} \\ &\le cR^{-p-\frac{p-1}{p_2}\left(\frac{qq_2}{q-1}+\frac{rr_2}{r-1}\right)}|A_R|^{1-\frac{p-1}{p_2}\left(1-\frac{q_2}{q-1}-\frac{r_2}{r-1}\right)} \cdot \\ &\qquad \cdot \left[\frac{\int_{A_R} g(u,v,w)}{\left(\int_{B_R} g(u,v,w)\right)^{\frac{q_2}{q-1}} \left(\int_{B_R} h(u,v,w)\right)^{\frac{r_2}{r-1}}}\right]^{\frac{p-1}{p_2}}. \end{aligned}$$

This completes the proof of (3.24).

If $p_2 = r_2 = 0$ then, by hypothesis **H2** we have that $q_2 > 0$ and the inequality follows the same pattern. Namely, by (3.1), (g_0) and Lemma 8, we have

$$\int_{B_R} g(u,v,w) \le cR^{-q}|A_R|(\inf_{B_R} v)^{q-1} \le cR^{-q}|A_R| \left(\frac{1}{|A_R \cap T_\varepsilon|} \int_{A_R\cap T_\varepsilon} v^{q_2} \right)^{\frac{q-1}{q_2}}$$
$$\le cR^{-q+N(1-\frac{q-1}{q_2})} \left(\int_{A_R\cap T_\varepsilon} g(u,v,w) \right)^{\frac{q-1}{q_2}}.$$

Since the proof of the remaining inequalities is similar to the one given above, we omit the details. □

4. Liouville theorems

In the previous results on a priori estimates as well as in the Liouville theorems we assume that $\inf_{\mathbb{R}^N} u = \inf_{\mathbb{R}^N} v = \inf_{\mathbb{R}^N} w = 0$. This hypothesis is quite natural. Indeed, when dealing with the prototype system

$$\begin{cases} -\Delta_p u \ge f(u,v,w) & \text{in } \mathbb{R}^N, \\ -\Delta_q v \ge g(u,v,w) & \text{in } \mathbb{R}^N, \\ -\Delta_r w \ge h(u,v,w) & \text{in } \mathbb{R}^N, \\ u \ge 0,\ v \ge 0,\ w \ge 0, \end{cases} \tag{4.1}$$

we see that, by changing the unknowns and the nonlinearities as follows,

$$\begin{aligned} &\tilde u := u - \inf u,\ \tilde v := v - \inf v,\ \tilde w := w - \inf w, \\ &\tilde f(\tau,\xi,\eta) := f(\tau + \inf u, \xi + \inf v, \eta + \inf w), \\ &\tilde g(\tau,\xi,\eta) := g(\tau + \inf u, \xi + \inf v, \eta + \inf w), \\ &\tilde h(\tau,\xi,\eta) := h(\tau + \inf u, \xi + \inf v, \eta + \inf w), \end{aligned} \tag{4.2}$$

we deduce that $(\tilde u, \tilde v, \tilde w)$ solves the problem,

$$\begin{cases} -\Delta_p \tilde u \ge \tilde f(\tilde u,\tilde v,\tilde w) & \text{in } \mathbb{R}^N, \\ -\Delta_q \tilde v \ge \tilde g(\tilde u,\tilde v,\tilde w) & \text{in } \mathbb{R}^N, \\ -\Delta_r \tilde w \ge \tilde h(\tilde u,\tilde v,\tilde w) & \text{in } \mathbb{R}^N, \\ \inf \tilde u = 0,\ \inf \tilde v = 0,\ \inf \tilde w = 0. \end{cases} \tag{4.3}$$

On the other hand if (u,v,w) is a weak solution of (4.1) then, roughly speaking, the infima of u, v and w must be a zero of f, g, and h:

$$f(\inf u, \inf v, \inf w) = 0,\ g(\inf u, \inf v, \inf w) = 0,\ h(\inf u, \inf v, \inf w) = 0.$$

Indeed we have,

Proposition 14. *Let (u,v,w) be a weak solution of (4.1), then as $(\tau,\xi,\eta) \to (\inf u, \inf v, \inf w)$ we have*

$$\liminf f(\tau,\xi,\eta) = 0,\ \liminf g(\tau,\xi,\eta) = 0,\ \liminf h(\tau,\xi,\eta) = 0.$$

Proof. Suppose that $\liminf f(\tau,\xi,\eta) = c > 0$ (possibly $+\infty$) as $(\tau,\xi,\eta) \to (\inf u, \inf v, \inf w)$. A simple translation argument as in (4.2) shows that we can consider (u,v,w) a solution of (4.1) such that $\inf_{\mathbb{R}^N} u =$

$\inf_{\mathbb{R}^N} v = \inf_{\mathbb{R}^N} w = 0$ and (f_0) is fulfilled with $p_1 = q_1 = r_1 = 0$. With the same notation of Lemma 10, from (3.3) we have

$$\varepsilon^{-(p-1)} \leq (\inf_{A_R \cap T_\varepsilon} u)^{-(p-1)} \leq cR^{-p},$$

and letting $R \to +\infty$ we get a contradiction. □

In the proof of the following results, very often we prove for instance that, $\int_{\mathbb{R}^N} f(u,v,w) = 0$. This information is enough to deduce that the solution is trivial. Indeed we the following lemma.

Lemma 15. *Assume that* **H1**, (f_0), (g_0) *and* (h_0) *hold. Let* (u,v,w) *be a weak solution of* (3.2). *If* $f(u(x),v(x),w(x)) = 0$ *or* $g(u(x),v(x),w(x)) = 0$ *or* $h(u(x),v(x),w(x)) = 0$ *for a.a.* $x \in \mathbb{R}^N$, *then either* $u = 0$ *or* $v = 0$ *or* $w = 0$ *a.e. in* $\mathbb{R}^N$.

Proof. Suppose that $f(u(x),v(x),w(x)) = 0$ for a.a. $x \in \mathbb{R}^N$. Thanks to Proposition 7 we can apply Lemma 8 to the function $u+v+w$. Hence by (f_0) we get,

$$\begin{aligned}(\inf_{B_R} u)^{p_1}(\inf_{B_R} v)^{q_1}(\inf_{B_R} w)^{r_1} &\leq \\ &\leq \frac{1}{|A_R \cap T_\varepsilon|}\int_{A_R \cap T_\varepsilon} u^{p_1}v^{q_1}w^{r_1} \leq c\frac{1}{|A_R \cap T_\varepsilon|}\int_{A_R} f(u,v,w) = 0,\end{aligned}$$

for R sufficiently large and $\varepsilon > 0$, where $T_\varepsilon = \{x \in \mathbb{R}^N : u(x)+v(x)+w(x) < \varepsilon\}$. Using (WH) on u or v or w, we conclude that either $u = 0$ or $v = 0$ or $w = 0$ a.e. in $\mathbb{R}^N$. The proof of the remaining cases $g(u(x),v(x),w(x)) = 0$ or $h(u(x),v(x),w(x)) = 0$ for a.a. $x \in \mathbb{R}^N$ is similar. □

Theorem 16. *Assume that* **H1**, (f_0), (g_0) *and* (h_0) *hold. Assume that one of the following hypothesis holds*

a) $q_3r_2 < (q-1-q_2)(r-1-r_3)$, $q_2 < q-1$, $r_3 < r-1$ *and* $D\frac{N-p}{p-1} < \theta_u$ *(where* θ_u *is defined in* (3.11)*); or*
b) $p_3r_1 < (p-1-p_1)(r-1-r_3)$, $p_1 < p-1$, $r_3 < r-1$, *and* $D\frac{N-q}{q-1} < \theta_v$ *(where* θ_v *is defined in* (3.14)*); or*
c) $p_2q_1 < (p-1-p_1)(q-1-q_2)$, $p_1 < p-1$, $q_2 < q-1$, *and* $D\frac{N-r}{r-1} < \theta_w$ *(where* θ_w *is defined in* (3.17)*).*

Then problem (3.2) *has no non-trivial solution.*

Proof. Suppose that (u,v,w) is a non-trivial weak solution of (3.2). Assume that a) is fulfilled. The proof in the remaining case is similar.

<u>*Case* $D = 0$.</u> If $D = 0$, letting $R \to \infty$ in inequality (3.10) of Lemma 12, we reach the contradiction $1 \leq 0$.

<u>*Case* $D < 0$.</u> By (3.10) we get for all $\varepsilon >$ and R sufficiently large

$$\inf_{A_R \cap T_\varepsilon} u \geq R^{-\frac{1}{D}\{rr_1(q-1-q_2)+qq_1(r-1-r_3)+p[(q-1-q_2)(r-1-r_3)-q_3r_2]+qq_3r_1+rq_1r_2\}}.$$

By letting $R \to \infty$ we reach again a contradiction.

Case $D > 0$. As proved in Lemma 12, by (3.1) and (3.10), we have for all $\varepsilon > 0$ and $R > 0$ sufficiently large

$$\int_{B_R} f(u,v,w) \le cR^{-p}|A_R|(\inf_{A_R\cap T_\varepsilon} u)^{p-1}$$
$$\le cR^{N-p-\frac{p-1}{D}\{rr_1(q-1-q_2)+qq_1(r-1-r_3)+p[(q-1-q_2)(r-1-r_3)-q_3r_2]+qq_3r_1+rq_1r_2\}}.$$

Hence, letting $R \to \infty$ it results that $f(u,v,w) = 0$ a.e. in $\mathbb{R}^N$. An application of Lemma 15, yields a contradiction concluding the proof. □

The estimates in Lemma 13 can be used to recover a *critical case*[1] as the following results shows.

Theorem 17. *Assume that* **H1**, (f_0), (g_0) *and* (h_0) *hold with* $p_1 = r_1 = p_2 = q_2 = q_3 = r_3 = 0$ *and* $p_3q_1r_2 - (p-1)(q-1)(r-1) > 0$.

If

$$N\left[1-\frac{(p-1)(q-1)(r-1)}{p_3q_1r_2}\right] \le \max\left\{p + r\frac{p-1}{p_3} + q\frac{(p-1)(r-1)}{p_3r_2},\; q + p\frac{q-1}{q_1} + r\frac{(p-1)(q-1)}{p_3q_1},\; r + q\frac{r-1}{r_2} + p\frac{(q-1)(r-1)}{q_1r_2}\right\}, \tag{4.4}$$

then problem (3.2) *has no non-trivial solution.*

Proof. Let (u,v,w) be a non-trivial weak solution of (3.2). If

$$p + r\frac{p-1}{p_3} + q\frac{(p-1)(r-1)}{p_3r_2} = \max\left\{p + r\frac{p-1}{p_3} + q\frac{(p-1)(r-1)}{p_3r_2},\; q + p\frac{q-1}{q_1} + r\frac{(p-1)(q-1)}{p_3q_1},\; r + q\frac{r-1}{r_2} + p\frac{(q-1)(r-1)}{q_1r_2}\right\}$$

Without loss of generality we prove the theorem only when

$$N\left[1-\frac{(p-1)(q-1)(r-1)}{p_3q_1r_2}\right] = p + r\frac{p-1}{p_3} + q\frac{(p-1)(r-1)}{p_3r_2}.$$

From (3.25), since $p_3 > 0$, we have

$$\int_{B_R} f(u,v,w)\left(\int_{B_R} g(u,v,w)\right)^{\frac{q_3(p-1)}{p_3(q-1)}}\left(\int_{B_R} h(u,v,w)\right)^{\frac{r_3(p-1)}{p_3(r-1)}}$$
$$\le cR^{N\left[1-\frac{p-1}{p_3}\left(1-\frac{q_3}{q-1}-\frac{r_3}{r-1}\right)\right]-p-\frac{p-1}{p_3}\left(\frac{qq_3}{q-1}+\frac{rr_3}{r-1}\right)}\left(\int_{A_R} h(u,v,w)\right)^{\frac{p-1}{p_3}}. \tag{4.5}$$

[1] Here we refer to the *critical case* when in the nonexistence condition (4.4) the equality sign is allowed. See also Remark 18.

From (3.24), since $r_2 > 0$, we have

$$\int_{B_R} h(u,v,w) \left(\int_{B_R} f(u,v,w) \right)^{\frac{p_2(r-1)}{r_2(p-1)}} \left(\int_{B_R} g(u,v,w) \right)^{\frac{q_2(r-1)}{r_2(q-1)}} \tag{4.6}$$
$$\leq cR^{N\left[1-\frac{r-1}{r_2}\left(1-\frac{p_2}{p-1}-\frac{q_2}{q-1}\right)\right]-r-\frac{r-1}{r_2}\left(\frac{pp_2}{p-1}+\frac{qq_2}{q-1}\right)} \left(\int_{A_R} g(u,v,w) \right)^{\frac{r-1}{r_2}}.$$

From (3.23), since $q_1 > 0$, we have

$$\int_{B_R} g(u,v,w) \left(\int_{B_R} f(u,v,w) \right)^{\frac{p_1(q-1)}{q_1(p-1)}} \left(\int_{B_R} h(u,v,w) \right)^{\frac{r_1(q-1)}{q_1(r-1)}} \tag{4.7}$$
$$\leq cR^{N\left[1-\frac{q-1}{q_1}\left(1-\frac{p_1}{p-1}-\frac{r_1}{r-1}\right)\right]-q-\frac{q-1}{q_1}\left(\frac{pp_1}{p-1}+\frac{rr_1}{r-1}\right)} \left(\int_{A_R} f(u,v,w) \right)^{\frac{q-1}{q_1}}.$$

By applying first (4.5), then (4.6) and (4.7), and taking into account that $p_1 = r_1 = p_2 = q_2 = q_3 = r_3 = 0$, we get

$$\begin{aligned}
\int_{B_R} f(u,v,w) &\leq cR^{N\left(1-\frac{p-1}{p_3}\right)-p} \left(\int_{A_R} h(u,v,w) \right)^{\frac{p-1}{p_3}} \\
&\leq cR^{N\left(1-\frac{p-1}{p_3}\right)-p+\left[N\left(1-\frac{r-1}{r_2}\right)-r\right]\frac{p-1}{p_3}} \left(\int_{A_R} g(u,v,w) \right)^{\frac{(p-1)(r-1)}{p_3 r_2}} \\
&\leq cR^{N\left[1-\frac{(p-1)(q-1)(r-1)}{p_3 q_1 r_2}\right]-p-r\frac{p-1}{p_3}-q\frac{(p-1)(r-1)}{p_3 r_2}} \left(\int_{A_R} f(u,v,w) \right)^{\frac{(p-1)(q-1)(r-1)}{p_3 q_1 r_2}} \\
&= c\left(\int_{A_R} f(u,v,w) \right)^{\frac{(p-1)(q-1)(r-1)}{p_3 q_1 r_2}} \leq c\left(\int_{B_R} f(u,v,w) \right)^{\frac{(p-1)(q-1)(r-1)}{p_3 q_1 r_2}}.
\end{aligned}$$

Hence, $f(u,v,w) \in L^1(\mathbb{R}^N)$ and letting $R \to \infty$ we get $f(u,v,w) = 0$ in $\mathbb{R}^N$. Therefore, we conclude that $u = 0$, or $v = 0$, or $w = 0$ a.e. in $\mathbb{R}^N$, which contradicts our assumption.

The remaining cases can be proved similarly. □

Remark 18. Under the hypothesis $p_1 = r_1 = p_2 = q_2 = q_3 = r_3 = 0$, $D = p_3 q_1 r_2 - (p-1)(q-1)(r-1) > 0$ the condition "a) or b) or c)" in Theorem 16 can be formulated by (4.4) with the strict inequality. In this sense Theorem 17 is an improvement of Theorem 16.

The following improves, from several points of view, an earlier result obtained in [9, see Remark 2.2].

Corollary 19. *Assume that* **H1**, (f_0), (g_0) *and* (h_0) *hold with* $p = q = r$, $p_1 = r_1 = p_2 = q_2 = q_3 = r_3 = 0$ *and* $p_3 q_1 r_2 - (p-1)^3 > 0$. *If*

$$\left(p_3q_1r_2-(p-1)^3\right)\frac{N-p}{p(p-1)}-(p-1)^2\le \max\left\{q_1(r_2+p-1),r_2(p_3+p-1),p_3(q_1+p-1)\right\}, \tag{4.8}$$

then problem (3.2) *has no non-trivial solution.*

Theorem 20. *Assume that* **H1**, (f_0), (g_0) *and* (h_0) *hold with* $p_1>p-1$, $q_2>q-1$ *and* $r_3>r-1$.

If

$$N\ge \min\left\{p_1\frac{N-p}{p-1}+q_1\frac{N-q}{q-1}+r_1\frac{N-r}{r-1},\quad p_2\frac{N-p}{p-1}+q_2\frac{N-q}{q-1}+r_2\frac{N-r}{r-1},\right.$$
$$\left. p_3\frac{N-p}{p-1}+q_3\frac{N-q}{q-1}+r_3\frac{N-r}{r-1}\right\}, \tag{4.9}$$

then problem (3.2) *has no non-trivial solution* (u,v,w).

Proof. Without loss of generality assume that

$$\alpha_1=p_1\frac{N-p}{p-1}+q_1\frac{N-q}{q-1}+r_1\frac{N-r}{r-1}=\min\left\{p_1\frac{N-p}{p-1}+q_1\frac{N-q}{q-1}+r_1\frac{N-r}{r-1},\right.$$
$$\left. p_2\frac{N-p}{p-1}+q_2\frac{N-q}{q-1}+r_2\frac{N-r}{r-1},\quad p_3\frac{N-p}{p-1}+q_3\frac{N-q}{q-1}+r_3\frac{N-r}{r-1}\right\}.$$

From (3.23) that for $R\ge R_0$, we have

$$c_1\left(\int\limits_{B_R} f(u,v,w)\right)^{\frac{p_1}{p-1}}\le cR^{\alpha_1-N}\int\limits_{A_R} f(u,v,w) \tag{4.10}$$

where $c_1:=\left(\int_{B_{R_0}} g(u,v,w)\right)^{\frac{q_1}{q-1}}\left(\int_{B_{R_0}} h(u,v,w)\right)^{\frac{r_1}{r-1}}>0$ for R_0 large enough. Inequality (4.10) implies

$$c_1\left(\int\limits_{B_R} f(u,v,w)\right)^{\frac{p_1-p+1}{p-1}}\le cR^{\alpha_1-N}. \tag{4.11}$$

Now if $N>\alpha_1$ the claim follows form (4.11) by letting $R\to+\infty$. While if $N=\alpha_1$, from (4.11) it follows that $f(u,v,w)\in L^1(\mathbb{R}^N)$, hence necessarily $\int_{A_R} f(u,v,w)\to 0$, as $R\to+\infty$, which plugged in (4.10) implies that $f(u,v,w)=0$. □

Remark 21. A simple analysis of the proof of the previous result shows that the claim still holds if one of the following conditions is satisfied,

1. $N\ge\alpha_1$, $p_1\ge p-1$ and at least one inequality is strict,
   or
2. $N\ge\alpha_2$, $q_2\ge q-1$ and at least one inequality is strict,
   or
3. $N\ge\alpha_3$, $r_3\ge r-1$ and at least one inequality is strict.

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