Periodic perturbations with rotational symmetry of planar systems driven by a central force

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Abstract

We consider periodic perturbations of a central force field having a rotational symmetry, and prove the existence of nearly circular periodic orbits. We thus generalize, in the planar case, some previous bifurcation results obtained by Ambrosetti and Coti Zelati in [1]. Our results apply, in particular, to the classical Kepler problem.

MSC: 34C25

Keywords: Kepler problem; Periodic solutions; Rotational symmetric systems

1. Introduction

We study the planar system

$$\ddot{x} + g(|x|)\,x = \varepsilon \nabla_x V(t,x)\,, \qquad (S_\varepsilon)$$

where $g :]0,+\infty[\to \mathbb{R}$ is a continuously differentiable function and $V : \mathbb{R} \times (\mathbb{R}^2 \setminus \{0\}) \to \mathbb{R}$ is continuous, T-periodic in its first variable, and twice continuously differentiable in its second variable. The real number ε will be considered as a small parameter.

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A particularly important special case of (S_ε) is illustrated by the perturbed Kepler problem

$$\ddot{x} + \frac{c}{|x|^3}\, x = \varepsilon \nabla_x V(t,x)\,, \tag{K_ε}$$

where c is a positive constant. This kind of systems has been studied by plenty of authors, mainly with the aim of finding periodic solutions: see, e.g., [1,2,4–6,8–14,17,19]. Let us describe two of these results in detail.

In 1989, Ambrosetti and Coti Zelati [1] considered system (K_ε) in any dimension, and proved that periodic solutions of period $2T$ exist assuming the perturbing potential to be *even* in x, i.e., that

$$V(t,x) = V(t,-x)\,, \qquad \text{for every } t, x\,.$$

Their method of proof is variational, and provides a bifurcation result from a circular solution $x_*(t)$ of (K_0) having minimal period $\tau_* = 2T/n$, with n odd. (By *circular solution* we mean a solution whose orbit is a circle centered at the origin.) Indeed, they proved that at least three such solutions $x(t)$ exist, satisfying the symmetry property

$$x(t+T) = -x(t)\,, \qquad \text{for every } t\,.$$

Recently, the authors of the present paper considered in [10] the case when the perturbing force is *radially symmetric*, i.e., when $\nabla_x V(t,x) = p(t,|x|)\,x$, for some scalar function $p : \mathbb{R}\times\,]0,+\infty[\,\to \mathbb{R}$. It was proved that, if $x_*(t)$ is a circular solution having minimal period τ_*, and T/τ_* is a rational number n/m which is not an integer, then near this circular solution there are mT-periodic solutions $x(t)$ of system (K_ε) making exactly n rotations around the origin in their period time, provided that $|\varepsilon|$ is small enough. Moreover, $t \mapsto |x(t)|$ is T-periodic, and $t \mapsto \mathrm{Rot}(x,[t,t+T])$ is constant.

Let us recall here that, when $\gamma : [\tau_1,\tau_2] \to \mathbb{R}^2$ is a curve such that $\gamma(t) \neq 0$ for every t, writing $\gamma(t) = \rho(t)(\cos\theta(t), \sin\theta(t))$, all functions being continuous, the *rotation number* around the origin is defined as

$$\mathrm{Rot}\,(\gamma;[\tau_1,\tau_2]) = \frac{\theta(\tau_2)-\theta(\tau_1)}{2\pi}\,.$$

In this paper we assume that the perturbing force has a *rotational symmetry*, i.e., there is a rotation around the origin $\mathcal{R}$, of angle

$$\vartheta = \frac{2\pi}{m}\,,$$

with m being a positive integer, such that

$$\nabla_x V(t,\mathcal{R}x) = \mathcal{R}\nabla_x V(t,x)\,, \qquad \text{for every } t, x\,. \tag{1}$$

Notice that, if $m=1$, the above assumption will be trivially satisfied.

Let $\rho_* > 0$ be such that $g(\rho_*) > 0$. Then, the unperturbed equation (S_0) has a circular solution $x_*(t)$, with $|x_*(t)| = \rho_*$, rotating counter-clockwise, having minimal period

$$\tau_* = \frac{2\pi}{\sqrt{g(\rho_*)}} \,. \tag{2}$$

We will ask the quotient T/τ_* to be a rational number; more precisely, we assume that there exist some positive integers ℓ, n for which

$$\frac{mT}{\tau_*} = \frac{n}{\ell} \,. \tag{3}$$

We then look for solutions $x : \mathbb{R} \to \mathbb{R}^2 \setminus \{0\}$ such that

$$x(t + \ell T) = \mathcal{R}^n x(t) \,, \qquad \text{for every } t \,; \tag{4}$$

these solutions are $m\ell T$-periodic, and rotate n times around the origin in their period time. Notice that, in this case, $t \mapsto |x(t)|$ is ℓT-periodic and $t \mapsto \mathrm{Rot}(x, [t, t + \ell T])$ is constant, precisely equal to n/m.

It will be useful to introduce the auxiliary function

$$h(\rho) = 4g(\rho) + g'(\rho)\rho \,.$$

Here is our main result.

Theorem 1. *In the above setting, assume that* $g'(\rho_*) \neq 0$ *and*

$$h(\rho_*) \notin \left\{ \left(\frac{2\pi k}{\ell T} \right)^2 : k \in \mathbb{N} \setminus \{0\} \right\}. \tag{5}$$

Then, for any $\sigma > 0$ *there exists* $\bar{\varepsilon} > 0$ *such that, if* $|\varepsilon| \le \bar{\varepsilon}$*, system* (S_ε) *has at least four geometrically distinct* $m\ell T$*-periodic solutions* $x(t)$ *satisfying* (4)*. Two of them rotate counter-clockwise, and are such that, for some* θ_1, θ_2 *in* $[0, \tau[$,

$$|x(t) - x_*(t + \theta_i)| \le \sigma \,, \qquad \textit{for every } t \in \mathbb{R} \,. \tag{6}$$

The other two solutions rotate clockwise, and are such that, for some θ_3, θ_4 *in* $[0, \tau[$,

$$|x(t) - x_*(-t + \theta_i)| \le \sigma \,, \qquad \textit{for every } t \in \mathbb{R} \,. \tag{7}$$

When assuming (1), once a solution $x(t)$ has been found, also $\mathcal{R}^j x(t)$ is clearly a solution, with $j = 1, \dots, m-1$. In the above statement, we say that the solutions are *geometrically distinct* if they are not related to each other in this way.

When dealing with the perturbed Kepler problem (K_ε), the functions $h(\rho)$ and $g(\rho) = c\rho^{-3}$ coincide, and we have following.

Corollary 2. *If* (3) *is verified, and*

$$\frac{\ell T}{\tau_*} = \frac{n}{m} \notin \mathbb{N}\,, \tag{8}$$

then the same conclusion of Theorem 1 holds for system (K_ε).

Indeed, in this case we have that $h(\rho_*) = (2\pi/\tau_*)^2$, so that (5) follows directly from (8). Notice that the case $\ell = 1$, $m = 2$, and n odd was treated in [1, Theorem 2.2], showing the existence of *at least three* $2T$-periodic solutions near $x_*(t)$, a circular solution of the unperturbed problem having minimal period $\tau_* = 2T/n$, in an arbitrary dimensional space. In the planar case, there are at least four $m\ell T$-periodic solutions because it is possible to distinguish between two of them rotating clockwise and two counter-clockwise. On the other hand, when considering the radially symmetric situation as in [10], the integer number $m \geq 2$ can be chosen arbitrarily. Theorem 1 thus extends [10, Corollary 4].

Now, since the minimal period of the circular orbits of the Kepler problem is strictly increasing with their radius, we may expect that, taking *any* circular orbit $x_*(t)$ of the unperturbed system (K_0), the assumptions of Corollary 2 will be satisfied by many circular orbits which are arbitrarily near to it. This is indeed the case, and will be established in Theorem 6 below.

The paper is organized as follows. In Section 2 we provide the proof of Theorem 1 by the use of a bifurcation result, in a variational setting. At the end of the section we will make some remarks on our choice of dealing only with the problem in dimension 2. In Section 3 we consider a generalized Kepler problem, taking $g(\rho) = c\rho^{-\gamma}$, for some $\gamma > 0$, and we provide the existence of a myriad of periodic solutions near *any* circular solution of the unperturbed system. As an example of application we propose a gravitational model of a pulsating star having some rotational symmetry.

Let us also mention that all our results still hold when $V(t, x)$ is replaced by some $V(t, x, \varepsilon)$, with a smooth dependence on the parameter ε.

2. Proof of Theorem 1

Searching for $m\ell T$-periodic solutions, we will deal with the Hilbert space

$$H^1_{m\ell T} = \{x \in H^1(0, m\ell T) : x(0) = x(m\ell T)\}\,,$$

equipped with the usual scalar product

$$\langle v,\, w\rangle = \int_0^{m\ell T} \big[\langle v(t),\, w(t)\rangle + \langle \dot v(t),\, \dot w(t)\rangle\ \ dt\,.$$

The elements of this space will be identified to the $m\ell T$-periodic continuous functions with locally square integrable weak derivative.

In order to take into account the symmetry requirement (4), we will work on the Hilbert subspace

$$\mathcal{E} = \left\{x \in H^1_{m\ell T} : x(t + \ell T) = \mathcal{R}^n x(t),\ \text{for every } t \in [0, m\ell T]\right\}.$$

Here, we recall, $\mathcal{R}$ is a rotation around the origin of angle $2\pi/m$. Without loss of generality, we will assume it to be counter-clockwise, i.e., in complex notation, if $z \in \mathbb{R}^2$,

$$\mathcal{R}z = e^{\frac{2\pi i}{m}} z\,.$$

Moreover, since we are interested in solutions which never cross the origin, it will be worth introducing the open subset

$$\mathcal{E}_* = \left\{ x \in \mathcal{E} : x(t) \neq 0, \text{ for every } t \in [0, m\ell T] \right\}.$$

We look for the critical points of the functional $\varphi_\varepsilon : \mathcal{E}_* \to \mathbb{R}$ defined as

$$\varphi_\varepsilon(x) = \int_0^{m\ell T} \left[\tfrac{1}{2}|\dot{x}(t)|^2 - F(|x(t)|) + \varepsilon V(t, x(t)) \right] dt\,,$$

where

$$F(\rho) = \int_0^{\rho} s g(s)\, ds\,.$$

Indeed, we have the following information.

Lemma 3. *Under the assumption* (1)*, the critical points of the functional* φ_ε *correspond to* $m\ell T$*-periodic solutions of* (S_ε) *satisfying* (4)*.*

Proof. Let $x \in \mathcal{E}_*$ be a critical point of φ_ε, namely

$$\varphi_\varepsilon'(x)v = \int_0^{m\ell T} \left[\langle \dot{x}(t), \dot{v}(t) \rangle - \langle g(|x(t)|)x(t) - \varepsilon \nabla_x V(t, x(t)), v(t) \rangle \right] dt = 0\,,$$

for every $v \in \mathcal{E}$. Notice that, since both x and v belong to $\mathcal{E}$, using (1) we have that

$$\int_0^{m\ell T} \left[\langle \dot{x}(t), \dot{v}(t) \rangle - \langle g(|x(t)|)x(t) - \varepsilon \nabla_x V(t, x(t)), v(t) \rangle \right] dt =$$

$$= \sum_{j=0}^{m-1} \int_{j\ell T}^{(j+1)\ell T} \left[\langle \dot{x}(t), \dot{v}(t) \rangle - \langle g(|x(t)|)x(t) - \varepsilon \nabla_x V(t, x(t)), v(t) \rangle \right] dt$$

$$= \sum_{j=0}^{m-1} \int_0^{\ell T} \Big[\langle \dot{x}(t + j\ell T), \dot{v}(t + j\ell T) \rangle - g(|x(t + j\ell T)|) \langle x(t + j\ell T), v(t + j\ell T) \rangle$$
$$- \varepsilon \langle \nabla_x V(t + j\ell T, x(t + j\ell T)), v(t + j\ell T) \rangle \Big] dt$$

$$= m \int_0^{\ell T} \big[\langle \dot{x}(t), \dot{v}(t) \rangle - \langle g(|x(t)|)x(t) - \varepsilon \nabla_x V(t, x(t)), v(t) \rangle \big] \, dt \, .$$

Hence,

$$\int_0^{\ell T} \big[\langle \dot{x}(t), \dot{v}(t) \rangle - \langle g(|x(t)|)x(t) - \varepsilon \nabla_x V(t, x(t)), v(t) \rangle \big] \, dt = 0 \, ,$$

for every $v \in H^1_{\ell T}$, since such a function $v(t)$ can be extended on $[0, m\ell T]$ to a function belonging to the space $\mathcal{E}$. We then deduce that $\dot{x}(t)$ has a weak derivative on $[0, \ell T]$, which almost everywhere coincides with $\psi(t) = -g(|x(t)|)x(t) + \varepsilon \nabla_x V(t, x(t))$, a continuous function. Therefore, $x(t)$ is twice continuously differentiable on $[0, \ell T]$, and it is a classical solution of the differential equation (S_ε) on $[0, \ell T]$. Since $x(t + \ell T) = \mathcal{R}^n x(t)$ and (1) holds, the same conclusion is reached on every interval $[j\ell T, (j+1)\ell T]$, with $j = 1, \dots, m-1$. Being $\psi(t)$ continuous on the whole real line, we see that $x(t)$ is indeed a classical $m\ell T$-periodic solution of (S_ε). $\square$

In order to state the abstract perturbation result, let us consider the symmetric bilinear form $\varphi_0''(x)$ which associates to every pair of vectors $v, w \in \mathcal{E}$ the real number

$$\begin{aligned} \varphi_0''(x)(v, w) &= \lim_{\lambda \to 0} \frac{\varphi_0'(x + \lambda w)v - \varphi_0'(x)v}{\lambda} \\ &= \int_0^{m\ell T} \Big[\langle \dot{v}, \dot{w} \rangle - \frac{g'(|x|)}{|x|} \langle x, v \rangle \langle x, w \rangle - g(|x|) \langle v, w \rangle \Big] \, dt \end{aligned}$$

By the Riesz representation theorem, we associate to this bilinear form a bounded selfadjoint operator in $\mathcal{E}$, still denoted by $\varphi_0''(x)$, such that, for every $v, w \in \mathcal{E}$,

$$\langle \varphi_0''(x)v, w \rangle = \varphi_0''(x)(v, w).$$

Let us now recall a well-established perturbation result which will be used in our proof (see, e.g., [3,7,15,16,18]).

Theorem 4. *Let an open subset $\mathcal{E}_*$ of a Hilbert space $\mathcal{E}$ be the domain of a family of twice continuously differentiable functionals $\varphi_\varepsilon : \mathcal{E}_* \to \mathbb{R}$, depending smoothly on ε. Let $\mathcal{Z} \subseteq \mathcal{E}_*$ be a compact manifold without boundary such that*

(i) $\varphi_0'(x) = 0$, *for every $x \in \mathcal{Z}$;*
(ii) $\varphi_0''(x)$ *is a Fredholm operator of index* 0, *for every $x \in \mathcal{Z}$;*
(iii) $T_x \mathcal{Z} = \ker \varphi_0''(x)$, *for every $x \in \mathcal{Z}$.*

Then, there exists $\bar{\varepsilon} > 0$ such that, if $|\varepsilon| \le \bar{\varepsilon}$, the functional φ_ε has at least $\mathrm{cat}(\mathcal{Z})$ *critical points near $\mathcal{Z}$.*

In the above statement, we have denoted by $\mathrm{cat}(\mathcal{Z})$ the Lusternik–Schnirelmann category of the topological space $\mathcal{Z}$ with respect to itself, namely the least integer k such that $\mathcal{Z}$ can be covered by k closed subsets of $\mathcal{Z}$ which are contractible to a point in $\mathcal{Z}$.

We will apply Theorem 4 separately for the clockwise and for the counter-clockwise rotating solutions. Let us deal with those last ones, taking as $\mathcal{Z}$ the manifold of circular solutions with radius ρ_* of the unperturbed system (S_0) rotating counter-clockwise. (The other case is completely analogous.) Clearly enough, $\mathcal{Z} \subseteq \mathcal{E}_*$. Once we choose $x_*(t)$, one of such circular solutions, $\mathcal{Z}$ can be parametrized as a curve $\gamma : [0, 2\pi] \to \mathcal{E}_*$, e.g.,

$$\gamma(\theta)(t) = x_*(t + \theta) .$$

Hence, $\mathcal{Z}$ is homeomorphic to S^1, and $\mathrm{cat}(\mathcal{Z}) = 2$. Let us now verify the assumptions of Theorem 4.

Condition (i) holds since every $x \in \mathcal{Z}$ is a solution of (S_0), and hence $\varphi_0'(x) = 0$.

In order to prove condition (ii) notice that, taking $v, \xi \in \mathcal{E}$, one has $\varphi_0''(x)v = \xi$ if and only if

$$\int_0^{m\ell T} \left[\langle \dot v - \dot \xi, \dot w \rangle - \left\langle \frac{g'(|x|)}{|x|} \langle x,\ v \rangle x + g(|x|)v + \xi, w \right\rangle \right] dt = 0 ,$$

for every $w \in \mathcal{E}$. Setting $u = \xi - v$, with the same argument used to prove Lemma 3 we can then deduce that u is twice continuously differentiable, and satisfies the differential equation

$$\ddot u - u = \frac{g'(|x|)}{|x|} \langle x, v \rangle x + (g(|x|) + 1)\, v . \tag{9}$$

By the Ascoli–Arzelà Theorem it can be proved that the linear operator $P : \mathcal{E} \to \mathcal{E}$, taking every $v \in \mathcal{E}$ into the unique solution $u \in \mathcal{E}$ of (9), is a compact operator. Being $\varphi_0''(x) = \mathrm{Id} + P$, the Fredholm Alternative Theorem guarantees that $\varphi_0''(x)$ has a finite dimensional kernel and a closed image. Since it is selfadjoint, $\ker \varphi_0''(x) = [\mathrm{Im}\, \varphi_0''(x)]^\perp$, and we conclude that $\varphi_0''(x)$ is a Fredholm operator of index zero.

Finally, in order to verify (iii), we have to prove that $T_x\mathcal{Z} \supseteq \ker \varphi_0''(x)$ for every $x \in \mathcal{Z}$ (the other inclusion is always true because x is a critical point). So, we choose $x \in \mathcal{Z}$ and $v \in \ker \varphi_0''(x)$. Without loss of generality, we assume that $x(0) = (\rho_*, 0)$ so that, by (3), using the complex notation,

$$x(t) = \rho_* e^{\frac{2\pi i}{\tau_*} t} = \rho_* e^{\frac{2\pi n i}{m\ell T} t} .$$

We want to prove that $v \in T_x\mathcal{Z}$. Taking $\theta \in \mathbb{R}$ such that $x(t) = x_*(t + \theta)$, i.e., $x = \gamma(\theta)$, we notice that

$$T_x\mathcal{Z} = \left\{ \mu \gamma'(\theta) : \mu \in \mathbb{R} \right\} = \{ i \nu x : \nu \in \mathbb{R} \}.$$

Particularizing equation (9) with $\xi = 0$, we see that any $v \in \ker \varphi_0''(x)$ satisfies the differential equation

$$-\ddot{v} = g(|x|)\, v + \frac{g'(|x|)}{|x|} \langle v, x \rangle\, x\,. \tag{10}$$

Here, as well, using the complex notation we write $v : \mathbb{R} \to \mathbb{C}$, and since we look for $m\ell T$-periodic solutions, we have the corresponding Fourier series

$$v(t) \sim \sum_{k \in \mathbb{Z}} c_k e^{\frac{2\pi k i}{m\ell T} t},$$

with $c_k \in \mathbb{C}$. Moreover, since $v(t + \ell T) = \mathcal{R}^n v(t)$, it has to be

$$\sum_{k \in \mathbb{Z}} c_k e^{\frac{2\pi k i}{m\ell T}(t+\ell T)} = e^{\frac{2\pi n i}{m}} \sum_{k \in \mathbb{Z}} c_k e^{\frac{2\pi k i}{m\ell T} t},$$

so that

$$c_k e^{\frac{2\pi k i}{m}} = c_k e^{\frac{2\pi n i}{m}}, \qquad \text{for every } k \in \mathbb{Z},$$

hence

$$\text{either } c_k = 0\,, \text{ or } k \in n + m\mathbb{Z}\,. \tag{11}$$

Now, we write equation (10) as

$$\sum_{k \in \mathbb{Z}} \left(\frac{2\pi k}{m\ell T}\right)^2 c_k e^{\frac{2\pi k i}{m\ell T} t} = g(\rho_*) \sum_{k \in \mathbb{Z}} c_k e^{\frac{2\pi k i}{m\ell T} t} + g'(\rho_*) \left\langle \sum_{k \in \mathbb{Z}} c_k e^{\frac{2\pi k i}{m\ell T} t}, \rho_*\, e^{\frac{2\pi n i}{m\ell T} t} \right\rangle e^{\frac{2\pi n i}{m\ell T} t}.$$

Recalling that, by (3),

$$g(\rho_*) = \left(\frac{2\pi}{\tau_*}\right)^2 = \left(\frac{2\pi n}{m\ell T}\right)^2,$$

and setting

$$\alpha = \frac{g'(\rho_*)\rho_*}{2} \left(\frac{m\ell T}{2\pi}\right)^2,$$

we get

$$\sum_{k \in \mathbb{Z}} (k^2 - n^2) c_k e^{\frac{2\pi k i}{m\ell T} t} = 2\alpha \left\langle \sum_{k \in \mathbb{Z}} c_k e^{\frac{2\pi k i}{m\ell T} t}, e^{\frac{2\pi n i}{m\ell T} t} \right\rangle e^{\frac{2\pi n i}{m\ell T} t}.$$

Remark that, having assumed $g'(\rho_*) \neq 0$, it follows that $\alpha \neq 0$, as well. Since $\langle z_1, z_2 \rangle = \mathrm{Re}(z_1 z_2^*)$ and $\mathrm{Re}(z) = \frac{1}{2}(z + z^*)$, we have

$$\sum_{k\in\mathbb{Z}}(k^2-n^2)c_k e^{\frac{2\pi ki}{m\ell T}t} = 2\alpha\,\mathrm{Re}\left(\sum_{k\in\mathbb{Z}} c_k e^{\frac{2\pi(k-n)i}{m\ell T}t}\right) e^{\frac{2\pi ni}{m\ell T}t}$$
$$= \alpha\sum_{k\in\mathbb{Z}}\left[c_k e^{\frac{2\pi ki}{m\ell T}t} + c_k^* e^{\frac{2\pi(2n-k)i}{m\ell T}t}\right].$$

We then get

$$\sum_{k\in\mathbb{Z}}\left(k^2-n^2-\alpha\right)c_k e^{\frac{2\pi ki}{m\ell T}t} = \alpha\sum_{k\in\mathbb{Z}} c_k^* e^{\frac{2\pi(2n-k)i}{m\ell T}t},$$

which, after a shift in the indices, becomes

$$\sum_{k\in\mathbb{Z}}\left(k^2-n^2-\alpha\right)c_k e^{\frac{2\pi ki}{m\ell T}t} = \alpha\sum_{k\in\mathbb{Z}} c_{2n-k}^* e^{\frac{2\pi ki}{m\ell T}t}.$$

Hence, we must have that

$$\left(k^2-n^2-\alpha\right)c_k = \alpha c_{2n-k}^*, \qquad \text{for every } k\in\mathbb{Z}.$$

Taking $k=n$, we immediately observe that $c_n=-c_n^*$, i.e., $\mathrm{Re}(c_n)=0$. On the other hand, by a change of indices, we have that

$$\left((n+j)^2-n^2-\alpha\right)c_{n+j} = \alpha c_{n-j}^*, \qquad \text{for every } j\in\mathbb{Z}. \tag{12}$$

So, also

$$\left((n-j)^2-n^2-\alpha\right)c_{n-j} = \alpha c_{n+j}^*, \qquad \text{for every } j\in\mathbb{Z},$$

which is equivalent to

$$\alpha c_{n+j} = \left((n-j)^2-n^2-\alpha\right)c_{n-j}^*, \qquad \text{for every } j\in\mathbb{Z}. \tag{13}$$

We now solve the system (12)–(13). Since $\alpha\neq 0$, substitution yields

$$\left(j^2-\alpha+2nj\right)\left(j^2-\alpha-2nj\right)c_{n-j}^* = \alpha^2 c_{n-j}^*,$$

hence

$$\left[\left(j^2-\alpha\right)^2-4n^2j^2-\alpha^2\right]c_{n-j}^* = 0,$$

and finally

$$j^2[j^2-2\alpha-4n^2]c_{n-j}^* = 0, \qquad \text{for every } j\in\mathbb{Z}. \tag{14}$$

From (11) we know that $c_{n-j}=0$ when $j\notin m\mathbb{Z}$. Let now $j=mp$, for some $p\in(\mathbb{Z}\setminus\{0\})$; since

$$2\alpha + 4n^2 = \left(\frac{m\ell T}{2\pi}\right)^2 \left(4g(\rho_*) + g'(\rho_*)\rho_*\right) = \left(\frac{m\ell T}{2\pi}\right)^2 h(\rho_*)\,,$$

from (14) we deduce that

$$\left[p^2 - \left(\frac{\ell T}{2\pi}\right)^2 h(\rho_*)\right] c^*_{n-mp} = 0\,,$$

and assumption (5) leads to $c_{n-mp} = 0$. We then conclude that

$$c_k = 0, \quad \text{ for every } k \neq n.$$

Therefore we have proved that, if $v \in \ker \varphi_0''(x)$, it has the form

$$v(t) = c_n e^{\frac{2\pi i n}{m\ell T} t}\,,$$

where c_n is a pure imaginary complex number, hence it can be written as $c_n = i b_n$ with $b_n \in \mathbb{R}$. Therefore,

$$v(t) = i b_n e^{\frac{2\pi i n}{m\ell T} t} = i \frac{b_n}{\rho_*} x(t)\,.$$

Hence, $v \in T_x\mathcal{Z}$, and this concludes the proof that (iii) holds true.

Theorem 1 is thus completely proved. $\square$

To end this section, let us make some considerations on our choice of restricting the problem to the dimension 2. If one considers system (S_ε) in a higher dimension $d \geq 3$, a natural generalization of our symmetry condition (1) would be to assume the existence of a linear transformation $\mathcal{S} : \mathbb{R}^d \to \mathbb{R}^d$ for which

$$\nabla_x V(t, \mathcal{S}x) = \mathcal{S}\nabla_x V(t, x)\,, \qquad \text{for every } t, x\,. \tag{15}$$

When looking for bifurcations from the manifold of the circular solutions of (S_0) having a prescribed period τ_*, we may consider the functional $\varphi_\varepsilon : \mathcal{E}_* \to \mathbb{R}$, as in the proof above, adapting the situation to the new space

$$\mathcal{E} = \left\{ x \in H^1_{m\ell T} : x(t + \ell T) = \mathcal{S}x(t), \text{ for every } t \in [0, m\ell T] \right\}.$$

Clearly, we need the whole manifold of circular solutions with minimal period τ_* to be contained in $\mathcal{E}$. We claim that, in this case, $\mathcal{S}$ has to be either the identity, or minus the identity.

Indeed, take a circular solution in a plane generated by two orthonormal vectors $\vec{e}_1$ and $\vec{e}_2$, e.g.,

$$x(t) = \rho_* \left(\left(\cos \frac{2\pi}{\tau_*} t\right) \vec{e}_1 + \left(\sin \frac{2\pi}{\tau_*} t\right) \vec{e}_2 \right).$$

It has to belong to $\mathcal{E}$, hence

$$\left(\cos\frac{2\pi}{\tau_*}(t+\ell T)\right)\vec{e}_1+\left(\sin\frac{2\pi}{\tau_*}(t+\ell T)\right)\vec{e}_2=\left(\cos\frac{2\pi}{\tau_*}t\right)\mathcal{S}\vec{e}_1+\left(\sin\frac{2\pi}{\tau_*}t\right)\mathcal{S}\vec{e}_2\,,$$

for every t. We thus see that the vectors $\mathcal{S}\vec{e}_1$ and $\mathcal{S}\vec{e}_2$ must lie on the subspace generated by $\vec{e}_1$ and $\vec{e}_2$. Moreover, $|\mathcal{S}\vec{e}_1|=|\mathcal{S}\vec{e}_2|=1$.

As a consequence of the above argument, the linear function $\mathcal{S}$ is an isometry, and it transforms every 2-dimensional subspace into itself. It then follows by linearity that also every 1-dimensional subspace, being the intersection of two 2-dimensional subspaces, is transformed into itself. (We have used here the fact that $d\geq 3$.) Then, for every $x\in\mathbb{R}^d$, it has to be $\mathcal{S}x=\pm x$.

Let now $\{\vec{e}_1,\dots,\vec{e}_d\}$ be the canonical basis of $\mathbb{R}^d$. By the above, the matrix associated to $\mathcal{S}$ is diagonal, with entries ±1. Assume by contradiction, e.g., that $\mathcal{S}\vec{e}_1=\vec{e}_1$ and $\mathcal{S}\vec{e}_2=-\vec{e}_2$. Then $\mathcal{S}(\vec{e}_1+\vec{e}_2)=\vec{e}_1-\vec{e}_2$, while it should be $\mathcal{S}(\vec{e}_1+\vec{e}_2)=\pm(\vec{e}_1+\vec{e}_2)$. We thus conclude that either all diagonal entries are equal to 1, or they are all equal to -1. The claim is thus proved.

We remark once more that Ambrosetti and Coti Zelati in [1] have indeed considered perturbations of Keplerian type problems in the d-dimensional space assuming $\mathcal{S}=-\mathrm{Id}$, hence requiring the potential $V(t,x)$ to be an even function of x.

3. A myriad of periodic solutions

As a variant of system (K_ε), we may consider the generalized Kepler problem

$$\ddot{x}+\frac{c}{|x|^\gamma}x=\varepsilon\nabla_x V(t,x)\,, \qquad (\widetilde{K}_\varepsilon)$$

where c is a positive constant, and $\gamma\neq 0$. We assume again that (1) holds, where $\mathcal{R}$ is a rotation around the origin of angle $2\pi/m$. We fix a circular orbit $x_*(t)$ of the unperturbed system $(\widetilde{K}_0)$, with radius ρ_* and minimal period τ_*, and we assume (3), for some positive integers ℓ,n. In this case, the function $h(\rho)$ coincides with $(4-\gamma)g(\rho)=(4-\gamma)c\rho^{-\gamma}$, and we have the following immediate consequence of Theorem 1.

Corollary 5. *If* (3) *is verified, and*

$$(4-\gamma)\left(\frac{\ell T}{\tau_*}\right)^2\notin\left\{k^2:k\in\mathbb{N}\setminus\{0\}\right\}, \qquad (16)$$

then the same conclusion of Theorem 1 holds for system $(\widetilde{K}_\varepsilon)$.

Notice that Corollary 2 is a direct consequence of Corollary 5. However, condition (16) could be verified even if $m=1$, in which case no symmetry is required on the perturbing term. This surely happens, e.g., if $\gamma\geq 4$ or, assuming $\gamma<4$ and recalling (3), when $\sqrt{4-\gamma}$ is an irrational number.

One could ask whether a condition like (16) is really needed. Indeed, such a condition is not found in the paper by Ambrosetti and Coti Zelati [1]. However, there is an unclear point in their proof, when they deduce $(1.14)_k$ from (1.13), claiming that $n\pm n(4-\gamma)^{1/2}$ cannot be an integer when $\gamma\neq 3$. It could be possible that a weaker nondegenerate condition would guarantee the existence of periodic solutions, but we will not investigate further this point.

Nevertheless, we will now prove the following rather surprising result.

Theorem 6. *Assume $\gamma > 0$ and, whenever $\gamma = 3$, then $m \geq 2$. For any circular solution $x_*(t)$ of the unperturbed system $(\widetilde{K}_0)$, for any positive integer N and any $\sigma > 0$, there exists $\bar{\varepsilon} > 0$ such that, if $|\varepsilon| \leq \bar{\varepsilon}$, equation $(\widetilde{K}_\varepsilon)$ has at least $4N$ periodic solutions $x(t)$, whose orbits are near the one of $x_*(t)$, in the sense of* (6) *or* (7).

Proof. Making ρ vary near ρ_*, we find infinitely many of its values for which $mT/\tau(\rho)$ is a rational number, i.e.,

$$\frac{mT}{\tau(\rho)} = \frac{n(\rho)}{\ell(\rho)}\,, \tag{17}$$

for some positive integers $\ell(\rho)$, $n(\rho)$. We would like to apply Corollary 5 when ρ_* and τ_* are replaced by ρ, $\tau(\rho)$, respectively. To this aim, we need to prove that condition (16) is satisfied, i.e., that

$$(4-\gamma)\left(\frac{\ell(\rho)T}{\tau(\rho)}\right)^2 \notin \left\{k^2 : k \in \mathbb{N} \setminus \{0\}\right\}\,, \tag{18}$$

for several values of ρ near ρ_*.

If $\gamma \geq 4$, then condition (18) is surely satisfied. Assume then $\gamma < 4$, and consider the number $\kappa = \sqrt{4-\gamma}\,/m$. Then, by (17), condition (18) is equivalent to

$$\kappa\,\frac{\ell(\rho)mT}{\tau(\rho)} = \kappa\, n(\rho) \notin \mathbb{N}\,. \tag{19}$$

If κ is not rational, condition (19) is surely satisfied. Otherwise, let $\kappa = p/q \in \mathbb{Q}$, for some relatively prime positive integers p, q. We claim that q cannot be equal to 1. Indeed, if it were $q = 1$, i.e., $\gamma = 4 - (mp)^2$, since m, p are positive integers, we would have two possibilities: if $m = p = 1$, then $\gamma = 3$, in which case we contradict the assumption that $m \geq 2$; on the other hand, if $m \neq 1$ or $p \neq 1$, then $\gamma \leq 0$, contradicting the assumption that $\gamma > 0$.

Now, among all the fractions in (17) we can select only those for which $n(\rho)$ is relatively prime with q. There are infinitely many of them, and

$$\kappa\, n(\rho) = p\,\frac{n(\rho)}{q} \notin \mathbb{N}\,.$$

We have then proved that condition (19) is verified by infinitely many values of ρ near ρ_*, so that the assumptions of Corollary 5 are satisfied.

Take N of these values of ρ satisfying the conditions (17) and (18) above, denote them by $\rho_1, \ldots, \rho_N$, and let $x_*^1(t), \ldots, x_*^N(t)$ be some circular solutions of $(\widetilde{K}_0)$ having those radii. By Corollary 5, there are N positive numbers $\bar{\varepsilon}_1, \ldots, \bar{\varepsilon}_N$ such that, if $|\varepsilon| \leq \bar{\varepsilon}_j$, equation $(\widetilde{K}_\varepsilon)$ has at least four periodic solutions $x_j(t)$ whose orbit is near the one of $x_*^j(t)$. If $|\varepsilon|$ is small enough, all these solutions must be different from one another, thus proving the result. □

As an illustrative example of application of Corollary 5 and Theorem 6, we consider the motion of a body having zero mass under the gravitational attraction of $m+1$ bodies, in the following configuration. The first of them is fixed at the origin, and has a mass equal to M_0. The

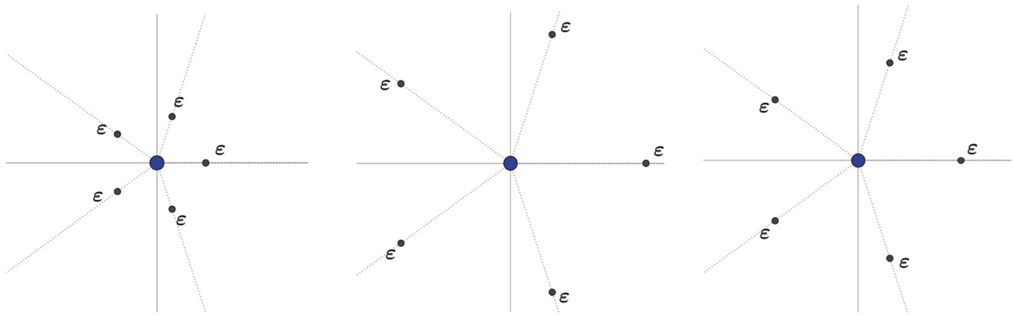

Fig. 1. The case $m = 5$, at three different times.

other m bodies move along the lines passing through the origin with angular directions $2\pi j/m$, with $j = 1, \dots, m$, and have small masses equal to the same value ε. We assume that these m small masses move symmetrically and periodically in time (see Fig. 1).

Precisely, let $p : \mathbb{R} \to [0, +\infty[$ be a T-periodic function, and denote by $P : \mathbb{R} \to \mathbb{R}^2$ the vector-valued function $P(t) = (p(t), 0)$. The equation of motion for our zero-mass body will then be of the type (K_ε), with $c = GM_0$ (here, G is the universal gravitational constant), and

$$V(t,x) = \sum_{j=1}^{m} \frac{G}{|x - \mathcal{R}^j P(t)|}\,.$$

As usual, $\mathcal{R}$ denotes the rotation of angle $2\pi/m$ about the origin. It is easily seen that (1) is satisfied, so that Corollary 5 (or Corollary 2) and Theorem 6 can be applied, providing the existence of plenty of periodic orbits.

Acknowledgment

This work has been partially supported by the Istituto Nazionale di Alta Matematica (INdAM).

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