THE COLLOCATION METHOD IN THE NUMERICAL SOLUTION OF BOUNDARY VALUE PROBLEMS FOR NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS. PART I: CONVERGENCE RESULTS*

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Abstract. We consider the numerical solution of boundary value problems for general neutral functional differential equations by the collocation method. The collocation method can be applied in two versions: the finite element method and the spectral element method. We give convergence results for the collocation method deduced by the convergence theory developed in [S. Maset, *Numer. Math.*, (2015), pp. 1–31] for a general discretization of an abstract reformulation of the problems. Such convergence results are then applied in Part II [S. Maset, *SIAM J. Numer. Anal.*, 53 (2015), pp. 2794–2821] of this paper to boundary values problems for a particular type of neutral functional differential equations, namely, differential equations with deviating arguments.

Key words. boundary value problems, neutral functional differential equation, collocation method

AMS subject classifications. 65Q20, 65L10, 65L60

1. Introduction.

In this paper, we are interested in the numerical solution, by the *collocation method*, of the *neutral functional differential equation boundary value problem* (BVP)

$$\begin{cases} y'(t) = F(t, y, y', p), \ t \in [a, b], \\ B(y, y', p) = 0, \end{cases} \tag{1.1}$$

where $F : [a, b] \times V \times U \times \mathbb{R}^{d_0} \to \mathbb{R}^d$ and $B : V \times U \times \mathbb{R}^{d_0} \to \mathbb{R}^d \times \mathbb{R}^{d_0}$ are given functionals and the pair $(y, p) \in V \times \mathbb{R}^{d_0}$ is unknown. Note that p is an unknown parameter which has to be determined along with the solution y of (1.1). Here, V is the space of continuous functions $[a, b] \to \mathbb{R}^d$ equipped with the norm

$$\|v\|_V = \max_{t\in[a,b]} \|v(t)\|_\infty, \ v \in V,$$

where $\|\cdot\|_\infty$ denotes the ∞-norm on $\mathbb{R}^d$, and U is a Banach space of integrable functions $[a, b] \to \mathbb{R}^d$ which is now defined.

1.1. The space U.

Let $\xi_0, \xi_1, \ldots, \xi_{m-1}, \xi_m \in [a, b]$ such that

$$a = \xi_0 < \xi_1 < \ldots < \xi_{m-1} < \xi_m = b.$$

We set

$$I_j := [\xi_{j-1}, \xi_j], \ j = 1, \ldots, m.$$

The role of the points ξ_j, $j = 0, \ldots, m$, is that of *breaking points* for the solution y of (1.1): in the convergence theorems given below, y will be supposed sufficiently smooth on each interval I_j, $j = 1, \ldots, m$.

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The space U is the space of the functions $u:[a,b]\to\mathbb{R}^d$ such that the restrictions $u|_{I_j}$, $j=1,\ldots,m$, are bounded and measurable. We consider the restrictions $u|_{I_j}$ as functions, not as classes of functions that are equal almost everywhere in the L^∞ space sense. Note that a function $u\in U$ has two $\mathbb{R}^d$-values at any point ξ_j, $j=1,\ldots,m-1$, one value coming from the restriction $u|_{I_j}$ and the other from $u|_{I_{j+1}}$. The norm of U is given by

$$\|u\|_U=\max_{j=1,\ldots,m}\ \sup_{t\in I_j}\|u|_{I_j}(t)\|_\infty,\ u\in U.$$

Equipped with this norm, U is a Banach space. This follows by the fact that the space of the bounded functions (not classes of functions) defined on a closed bounded interval is a Banach space when it is equipped with the sup norm. The subspace of the bounded and measurable functions is closed (since the limit in the sup norm of a convergent sequence of bounded and measurable functions is a bounded and measurable function) and so it is also a Banach space.

1.2. The abstract form. We make the following assumption about the functional F.

AF (assumption on F). For any $(v,u,\beta)\in V\times U\times\mathbb{R}^{d_0}$, we have

$$F(\cdot,v,u,\beta)\in U.$$

Under AF, we can introduce the operator $\mathcal{F}:V\times U\times\mathbb{R}^{d_0}\to U$ given by

$$\mathcal{F}(v,u,\beta)=F(\cdot,v,u,\beta),\ (v,u,\beta)\in V\times U\times\mathbb{R}^{d_0}$$

and so write the functional differential equation

$$y'(t)=F(t,y,y',p),\ t\in[a,b],$$

as

$$y'=\mathcal{F}(y,y',p).$$

Then, as described in [14], the BVP (1.1) can be considered as a particular instance of the following problem in abstract form.

PAF (problem in abstract form). Given

- a normed space $\mathbb{V}$ and Banach spaces $\mathbb{U}$, $\mathbb{A}$, and $\mathbb{B}$;
- operators $\mathfrak{F}:\mathbb{V}\times\mathbb{U}\times\mathbb{B}\to\mathbb{U}$ and $\mathfrak{B}:\mathbb{V}\times\mathbb{U}\times\mathbb{B}\to\mathbb{A}\times\mathbb{B}$;
- a linear operator $\mathfrak{G}:\mathbb{U}\times\mathbb{A}\to\mathbb{V}$,

find a pair $(v,\beta)\in\mathbb{V}\times\mathbb{B}$ such that $v=\mathfrak{G}(u,\alpha)$ for some $u\in\mathbb{U}$ and $\alpha\in\mathbb{A}$ satisfying

$$\begin{cases} u=\mathfrak{F}(\mathfrak{G}(u,\alpha),u,\beta),\\ \mathfrak{B}(\mathfrak{G}(u,\alpha),u,\beta)=0.\end{cases}$$

In other words, we have to find $(v,\beta)\in\mathbb{V}\times\mathbb{B}$ such that $v=\mathfrak{G}(u,\alpha)$ for some fixed point (u,α,β) of the operator $\Phi:X\to X$, where $X:=\mathbb{U}\times\mathbb{A}\times\mathbb{B}$, given by

$$\Phi(x)=(\mathfrak{F}(\mathfrak{G}(u,\alpha),u,\beta),(\alpha,\beta)-\mathfrak{B}(\mathfrak{G}(u,\alpha),u,\beta)),\ x=(u,\alpha,\beta)\in X.$$

For the BVP (1.1), we have $\mathbb{V}=V$, $\mathbb{U}=U$, $\mathbb{A}=\mathbb{R}^d$ (equipped with the ∞-norm), $\mathbb{B}=\mathbb{R}^{d_0}$ (equipped with the ∞-norm), $\mathfrak{F}=\mathcal{F}$, $\mathfrak{B}=B$ and $\mathfrak{G}=\mathcal{G}$, where $\mathcal{G}:U\times\mathbb{R}^d\to V$ is a *Green operator* for the differential equation

$$v'(t)=u(t),\ t\in[a,b], \tag{1.2}$$

with $u \in U$ given and $v \in V$ unknown. In other words, $\mathcal{G}$ is such that, for any $u \in U$,

$$\{v \in V : v \text{ is a solution of (1.2)}\} = \{\mathcal{G}(u, \alpha) : \alpha \in \mathbb{R}^d\}.$$

In this paper, we use the Green operator for (1.2) given by

$$\mathcal{G}(u, \alpha)(t) = \int_a^t u(s)ds + \alpha, \ (u, \alpha) \in U \times \mathbb{R}^d \text{ and } t \in [a, b]. \tag{1.3}$$

The solutions (v, β) of the instance (1.1) of PAF are the solutions (y, p) of (1.1). In the following, in agreement with the theory of the numerical solution of PAF presented below, solutions of (1.1) are denoted by (v^*, β^*).

1.3. Numerical solution of PAF. The numerical solution of PAF, in case of spaces $\mathbb{A}$ and $\mathbb{B}$ of finite dimension, is studied in [14], where a quite general discretization of the space X and the operator Φ are introduced. The contents of this subsection and next subsection are taken from [14].

The discretization of PAF consists in

- a finite-dimensional space $\widehat{\mathbb{U}}_K$;
- a linear bounded operator $\pi_K : \widehat{\mathbb{U}}_K \to \mathbb{U}$ called *prolongation*;
- a linear bounded operator $\rho_K : \mathbb{U} \to \widehat{\mathbb{U}}_K$ called *restriction.*

Given $\widehat{\mathbb{U}}_K$, π_K, and ρ_K, the finite-dimensional product space

$$\widehat{X}_K := \widehat{\mathbb{U}}_K \times \mathbb{A} \times \mathbb{B}$$

is considered as a discretization of $X = \widehat{\mathbb{U}} \times \mathbb{A} \times \mathbb{B}$ and the finite-dimensional operator (here for "finite-dimensional operator" we mean an operator on a finite-dimensional space)

$$\widehat{\Phi}_K := R_K \Phi P_K : \widehat{X}_K \to \widehat{X}_K$$

is considered as a discretization of Φ, where $P_K : \widehat{X}_K \to X$ and $R_K : X \to \widehat{X}_K$ are given by

$$P_K \widehat{x} = (\pi_K \widehat{u}, \alpha, \beta), \ \widehat{x} = (\widehat{u}, \alpha, \beta) \in \widehat{X}_K$$

and

$$R_K x = (\rho_K u, \alpha, \beta), \ x = (u, \alpha, \beta) \in X.$$

Here, K is a positive integer which has the role of "level of discretization": the larger K, the better the discretization.

Given a fixed point $\widehat{x}_K^* = (\widehat{u}_K^*, \alpha_K^*, \beta_K^*) \in \widehat{X}_K$ of $\widehat{\Phi}_K$, which can be computed by a standard nonlinear system solver,

$$P_K \widehat{x}_K^* = (\pi_K \widehat{u}_K^*, \alpha_K^*, \beta_K^*) \in X$$

is considered as an approximation of a fixed point of Φ and (v_K^*, β_K^*), where

$$v_K^* = \mathfrak{G}(\pi_K \widehat{u}_K^*, \alpha_K^*)$$

is considered as an approximation of a solution of PAF.

Remark 1.1. The discretization presented above is called a *primary discretization.* In [14], also a *secondary discretization* is considered. It consists in the substitution of the operators $\mathfrak{F}$ and $\mathfrak{B}$ with operators $\mathfrak{F}_K$ and $\mathfrak{B}_K$, respectively, whose values can be exactly computed. The motivation for this replacement is that, in the case of BVPs (1.1), for integro-differential equations

$$y'(t) = F(t,y,y',p) = f\left(t, y(t), \int_{\theta_1(t)}^{\theta_2(t)} k\left(t,s,y(s),y'(s)\right)ds, p\right) \tag{1.4}$$

or integral boundary conditions

$$B(y,y',p) = g\left(y(a), y(b), \int_a^b w(s,y(s))ds, p\right) = 0,$$

quadrature rules could be used for approximating the integrals. On the other hand, the replacement of $\mathfrak{F}$ with $\mathfrak{F}_K$ is not necessary for differential equations with deviating arguments

$$y'(t) = F(t,y,y',p) = f(t, y(t), y(\theta_1(t)), \ldots, y(\theta_k(t)), y'(\vartheta_1(t)), \ldots, y'(\vartheta_l(t)), p), \tag{1.5}$$

and the replacement of $\mathfrak{B}$ with $\mathfrak{B}_K$ is not necessary in case of multipoint boundary conditions

$$B(y,y',p) = g(y(a), y(b), y(t_1), \ldots, y(t_q), p) = 0.$$

In this paper we do not consider a secondary discretization in order to avoid being overwhelmed by too many details.

1.4. Convergence analysis. Throughout this work, we consider any product space as equipped with the norm given by the sum of the norm of the factor spaces.

Regarding the operators $\mathfrak{F}$, $\mathfrak{B}$ and the linear operator $\mathfrak{G}$ in PAF, we make the following assumptions:

A$\mathfrak{F}\mathfrak{B}$ (assumption on $\mathfrak{F}$ and $\mathfrak{B}$). $\mathfrak{F}$ and $\mathfrak{B}$ are Fréchet-differentiable at any point $(v_0, u_0, \beta_0) \in \mathbb{V} \times \mathbb{U} \times \mathbb{B}$.

A$\mathfrak{G}$ (assumption on $\mathfrak{G}$). $\mathfrak{G}$ is bounded.

Since A$\mathfrak{F}\mathfrak{B}$ and A$\mathfrak{G}$ hold, the operator Φ is Fréchet-differentiable at any point $x_0 = (u_0, \alpha_0, \beta_0) \in X$ and the Fréchet-derivative $D\Phi(x_0)$ is given by

$$D\Phi(x_0)\,x = \left(D\mathfrak{F}(v_0, u_0, \beta_0)(v,u,\beta), (\alpha, \beta) - D\mathfrak{B}(v_0, u_0, \beta_0)(v,u,\beta)\right),$$
$$x = (u, \alpha, \beta) \in X,$$

where $v_0 = \mathfrak{G}(u_0, \alpha_0)$ and $v = \mathfrak{G}(u, \alpha)$. Here and in the following the Fréchet-derivative of an operator A at a point y is denoted by $DA(y)$.

Let $x^* = (u^*, \alpha^*, \beta^*)$ be a fixed point of Φ and let (v^*, β^*), where $v^* = \mathfrak{G}(u^*, \alpha^*)$, be the relevant solution of PAF (recall that each solution of PAF is obtained by a fixed point of Φ; see the earlier page immediately after the definition of PAF).

We set $D^*\Phi := D\Phi(x^*)$ and make the following two assumptions:

Ax^*1 (assumption on x^* number 1). There exist $r_0 > 0$ and $L \geq 0$ such that

$$\|D\Phi(x) - D^*\Phi\| \leq L\|x - x^*\|_X, \;\; x \in \overline{B}(x^*, r_0).$$

Ax^*2. The linear bounded operator $I_X - D^*\Phi$ is invertible.

In Ax^*1 and in the following, we use the rule that the norm of an element y in the space Y is denoted by $\|y\|_Y$ and the norm of a linear bounded operator A is denoted by $\|A\|$. Moreover, we denote by $\overline{B}(y, r)$ the closed ball of center y and radius r.

We give two theorems concerning how x^* and (v^*, β^*) can be approximated by the approximations $P_K \widehat{x}^*_K$ and (v^*_K, β^*_K), respectively, obtained by some fixed point of $\widehat{\Phi}_K$. The theorems consider two particular situations known as the *simple case* and the *splitting case*, which are now introduced.

Let

$$Z := \mathbb{V} \times \mathbb{A} \times \mathbb{B}$$

and let $\Lambda : X \to Z$ be the linear bounded operator given by

$$\Lambda x = (\mathfrak{G}\,(u, 0)\,, \alpha, \beta)\,,\ \ x = (u, \alpha, \beta) \in X.$$

Definition 1.2. *The* simple case *holds whenever, for any $x \in X$, we can factorize $D\Phi(x)$ as*

$$D\Phi(x) = \Sigma(x)\Lambda,$$

where $\Sigma(x) : Z \to X$ is a linear bounded operator.

Definition 1.3. *The* splitting case *holds whenever there exists a splitting*

$$D^*\Phi = \Gamma^* + \Sigma^*\Lambda \tag{1.6}$$

of $D^*\Phi$, where $\Gamma^* : X \to X$ and $\Sigma^* : Z \to X$ are linear bounded operators, such that $I_X - \Gamma^*$ is invertible and, for any positive integer K, $I_X - P_K R_K \Gamma^*$ is invertible.

If the splitting case holds, we set, for any positive integer K,

$$\mu_K := \|\,(I_X - P_K R_K \Gamma^*_K)^{-1}\,\|. \tag{1.7}$$

The splitting case includes the nilpotency case now defined.

Definition 1.4. *The* nilpotency case *holds whenever there exist a splitting* (1.6) *of $D^*\Phi$ and a positive integer c such that $(\Gamma^*)^c = 0$ and, for any positive integer K, $(P_K R_K \Gamma^*)^c = 0$.*

The simple case holds for nonneutral BVPs (1.1), i.e., when $F(t, y, y', p) = F(t, y, p)$ and $B(y, y', p) = B(y, p)$. It also holds for BVPs for neutral integro-differential equations (1.4) with nonneutral boundary conditions $B(y, y', p) = B(y, p)$ (see [16]). The nilpotency case holds for BVPs for neutral differential equations with deviating arguments (1.5) and nonneutral boundary conditions, under a condition on the neutral deviating arguments $\vartheta_1, \ldots, \vartheta_l$ (see Part II [15]).

Now, we are ready to give the convergence theorems. We set

$$\lambda_K := \max\{\|\pi_K \rho_K\|, 1\}, \tag{1.8}$$

$$e^*_K := (\pi_K \rho_K - I_{\mathbb{U}})u^*, \tag{1.9}$$

where e^*_K is called the *consistency error of the (primary) discretization.* Moreover, for an operator $A : X \to X$, we denote by $A_{\mathbb{U}} : X \to \mathbb{U}$ the $\mathbb{U}$-component of A.

Finally, only for the simple case, we introduce the following condition CSC.

CSC (condition simple case). There exist $r_2 > 0$ and, for any positive integer K, $\sigma_K \geq 0$ such that

$$\|(\pi_K \rho_K - I_{\mathbb{U}})(D\Phi(x) - D^*\Phi)_{\mathbb{U}}\| \leq \sigma_K \|x - x^*\|_X, \; x \in \overline{B}(x^*, r_1).$$

and

$$\sigma_K = O(1), \; K \to \; \infty.$$

THEOREM 1.5 (the simple case). *Assume the simple case,*

$$\lim_{K\to\infty} \|(\pi_K \rho_K - I_{\mathbb{U}})(D^*\Phi)_{\mathbb{U}}\| = 0 \tag{1.10}$$

and

$$\lim_{K\to\infty} \begin{cases} \lambda_K \cdot \|e_K^*\|_{\mathbb{U}} & \\ \|e_K^*\|_{\mathbb{U}} & \text{if CSC holds} \end{cases} = 0. \tag{1.11}$$

(One has to read the lower row after {, instead of the upper one, if CSC holds.) Then, there exists a positive integer $\widehat{K}$ such that, for any positive integer $K \geq \widehat{K}$, $\widehat{\Phi}_K$ has a fixed point $\widehat{x}_K^*$ such that

$$\|P_K \widehat{x}_K^* - x^*\|_X = O(\|e_K^*\|_{\mathbb{U}}), \; K \to \infty. \tag{1.12}$$

Moreover, for the approximation (v_K^*, β_K^*) of (v^*, β^*), we have the two estimates

$$\|(v_K^*, \beta_K^*) - (v^*, \beta^*)\|_{\mathbb{V}\times\mathbb{B}} = O(\|e_K^*\|_{\mathbb{U}}), \; K \to \infty, \tag{1.13}$$

and

$$\begin{aligned} &\|(v_K^*, \beta_K^*) - (v^*, \beta^*)\|_{\mathbb{V}\times\mathbb{B}} \\ &= O\left(\lambda_K \cdot \|\mathfrak{G}(e_K^*, 0)\|_{\mathbb{V}}\right) + \begin{cases} O(\lambda_K \cdot \|e_K^*\|_{\mathbb{U}}^2) & \\ O(\|e_K^*\|_{\mathbb{U}}^2) & \text{if CSC holds,} \end{cases} \quad K \to \infty. \end{aligned} \tag{1.14}$$

Finally, if $\{\widehat{x}_K\}$ is a sequence of fixed points of $\widehat{\Phi}_K$ such that $\widehat{x}_K$ is eventually different from $\widehat{x}_K^*$, then

$$\frac{1}{\|P_K \widehat{x}_K - x^*\|_X} = \begin{cases} O(\lambda_K) & \\ O(1) & \text{if CSC holds,} \end{cases} \quad K \to \infty, \tag{1.15}$$

and

$$\frac{1}{\|\widehat{x}_K - \widehat{x}_K^*\|_{\widehat{X}_K}} = \begin{cases} O(\max\{\|\pi_K\|, 1\} \cdot \lambda_K) & \\ O(\max\{\|\pi_K\|, 1\}) & \text{if CSC holds,} \end{cases} \quad K \to \infty. \tag{1.16}$$

In the previous theorem and in the following ones, "$\widehat{x}_K$ is eventually different from $\widehat{x}_K^*$" means that there exists a positive integer K_0 such that, for $K \geq K_0$, $\widehat{x}_K \neq \widehat{x}_K^*$.

THEOREM 1.6 (the splitting case). *Assume the splitting case,*

$$\lim_{K\to\infty} \mu_K \cdot \|(\pi_K \rho_K - I_{\mathbb{U}})(\Sigma^*\Lambda)_{\mathbb{U}}\| = 0, \tag{1.17}$$

(1.18) $$\lim_{K\to\infty} \mu_K \cdot \|(\pi_K\rho_K - I_{\mathbb{U}})(\Gamma^*(I_X - \Gamma^*)^{-1}\Sigma^*\Lambda)_{\mathbb{U}}\| = 0$$

and

(1.19) $$\lim_{K\to\infty} \mu_K^2\lambda_K \cdot \|e_K^*\|_{\mathbb{U}} = 0,$$

Then, there exists a positive integer $\widehat{K}$ *such that, for any positive integer* $K \geq \widehat{K}$, $\widehat{\Phi}_K$ *has a fixed point* $\widehat{x}_K^*$ *such that*

(1.20) $$\|P_K\widehat{x}_K^* - x^*\|_X = O(\mu_K \cdot \|e_K^*\|_{\mathbb{U}}),\ K \to \infty.$$

Moreover, for the approximation (v_K^*, β_K^*) *of* (v^*, β^*)*, we have the estimate*

(1.21) $$\|(v_K^*, \beta_K^*) - (v^*, \beta^*)\|_{\mathbb{V}\times\mathbb{B}} = O(\mu_K \cdot \|e_K^*\|_{\mathbb{U}}),\ K \to \infty.$$

Finally, if $\{\widehat{x}_K\}$ *is a sequence of fixed points of* $\widehat{\Phi}_K$ *such that* $\widehat{x}_K$ *is eventually different from* $\widehat{x}_K^*$*, then*

(1.22) $$\frac{1}{\|P_K\widehat{x}_K - x^*\|_X} = O(\mu_K\lambda_K),\ K \to \infty,$$

and

(1.23) $$\frac{1}{\|\widehat{x}_K - \widehat{x}_K^*\|_{\widehat{X}_K}} = O\left(\max\{\|\pi_K\|, 1\} \cdot \mu_K\lambda_K\right),\ K \to \infty.$$

Observe that if the nilpotency case holds, then

(1.24) $$\mu_K = O(\lambda_K^{c-1}),\ K \to \infty,$$

in Theorem 1.6.

1.5. Aim of the paper. In this paper, we study the collocation method for the particular instance of problem PAF given by the BVP (1.1). The collocation method is a particular instance of the general discretization introduced in subsection 1.3 and, as a consequence, the two previous convergence theorems apply. We study both forms with which the collocation method can be used, namely, the finite element method (FEM) and the spectral element method (SEM).

The plan of the paper is the following. In section 2, we introduce the collocation method as a particular instance of the general discretization previously introduced. In sections 3 and 4, we give the convergence theorems for FEM and SEM, respectively. Finally, in section 5, we conclude describing the contents of Part II [15], where the convergence results for FEM and SEM will be applied to BVPs for neutral differential equations with deviating arguments and nonneutral boundary conditions.

1.6. State of the art. We conclude this introductory part by examining the current literature on the numerical solution of BVPs for functional differential equations and by stressing the advances provided with the present paper.

In the current numerical literature the form (1.1) has not ever been considered, since the literature concentrates on BVPs for integro-differential equations (of special types like Fredholm or Volterra) or differential equations with deviating arguments. Actually, there are also the papers [5] and [13], where BVPs for general second order nonneutral functional differential equations are considered, but the proposed methods have only order two. For a complete picture of the literature see [14].

There are several papers in the literature dealing with BVPs for neutral and nonneutral Fredholm integro-differential equations. As a selected example, we cite [11], where the BVP

$$\begin{cases} y'(t) = f(t, y(t)) + \int_a^b k(t, s, y(s), y'(s))ds, \ t \in [a, b], \\ B(y) = 0, \end{cases}$$

is solved by using a collocation method (in the FEM version) based on ν Legendre nodes. The uniform convergence order $\nu + 1$ is proved. In the present paper, we lay the foundations for showing that the uniform convergence order $\nu + 1$ can be obtained for general integro-differential equations (1.4) and for ν general collocation nodes, whenever the quadrature rule based on such nodes has order greater than ν. This result is new in the literature and it is given in [16].

We also remark here that in [11] (and also in [12], [9], and [10]), discretizations of the integral by quadrature rules are not considered. The same is done in the present paper (see the last sentence in Remark 1.1).

There are many papers in the literature dealing with BVPs for nonneutral differential equations with deviating arguments. As a selected example, we cite [3], where the BVP

$$\begin{cases} y'(t) = f(t, y(t), y(\theta_1(t)), \dots, y(\theta_k(t))), \ t \in [a, b], \\ B(y) = 0, \end{cases}$$

is solved by using a collocation method (in the FEM version) based on ν nodes. The uniform convergence order ν is proved. As in the case of integro-differential equations, in this paper we lay the foundations for showing in Part II [15] that the uniform convergence order $\nu + 1$ can be obtained, whenever the quadrature rule based on the collocation nodes has order greater than ν. Indeed, in the case of ν Gaussian collocation points, the order $\nu + 1$ is proved in [8] for the special BVP

$$\begin{cases} y'(t) = a(t)y(t) + b(t)y((t - \tau)\mathrm{mod}1) + g(t), \ t \in [0, 1], \\ y(0) = y(1), \end{cases}$$

which is related to the computation of the periodic solution of nonneutral linear delay differential equations (here the functions a, b, g are periodic of period 1). Our result of the uniform order $\nu + 1$ is much more general and it is valid for general nonneutral BVP (1.1) and general collocation nodes.

There are only four papers dealing with the numerical solution of BVPs for neutral differential equations with deviating arguments, namely, [7], [1], [4], and [6]. The paper [7] deals with the BVP

$$\begin{cases} y'(t) + dy'((t - \tau)\mathrm{mod}1) = a(t)y(s) + b(t)y((t - \tau)\mathrm{mod}\ 1) + g(t), \ t \in [0, 1], \\ y(0) = y(1), \end{cases}$$

which is related to the computation of a periodic solution of neutral linear delay differential equations, and solves it by using forward finite difference approximations for the derivatives. As a consequence, only order one is obtained. The paper [1] considers and describes methods for BVPs for neutral differential equations with deviating arguments, but the concrete examples of equations presented and the convergence analysis are confined to the nonneutral case. The paper [4] addresses the numerical solution by collocation (FEM version) of the BVP

$$\begin{cases} y'(t) = Tf(y(t), y\left(\left(t - \frac{\tau}{T}\right) \bmod 1\right), \frac{1}{T}y'\left(\left(t - \frac{\tau}{T}\right) \bmod 1\right), \ t \in [0,1], \\ y(0) = y(1), \\ b(y,T) = 0, \end{cases}$$

which is related to the computation of a periodic solution with unknown period T of neutral nonlinear delay differential equations (the boundary condition $b(y,T) = 0$ is the so-called phase condition and it is used to select a single solution). This paper contains many numerical experiments, but not a convergence analysis. Finally, the paper [6] considers difficult BVPs for the neutral differential equation with state-dependent deviating arguments arising in Wheeler–Feynman electrodynamics. Like the previous paper, it contains numerical experiments, but it does not give proofs of convergence. In the present paper, we lay the foundations for the proof of convergence given in Part II [15] of the collocation method, both in the FEM version and the SEM version, as applied to BVPs for neutral functional differential equations (1.5) and for general collocation nodes, under a condition on the neutral deviating arguments $\vartheta_1, \ldots, \vartheta_l$. Finally, we also remark that the SEM version of the collocation method has not yet been considered in literature, even confined to the nonneutral case.

2. The collocation method.

Since AF holds, the BVP (1.1) becomes a particular instance of PAF. We assume A$\mathfrak{F}\mathfrak{B}$ for this particular instance, i.e., we assume the following:

A$\mathcal{F}B$ (assumption on $\mathcal{F}$ and B). $\mathcal{F}$ and B are Fréchet-differentiable at any point $(v_0, u_0, \beta_0) \in V \times U \times \mathbb{R}^{d_0}$.

Regarding A$\mathfrak{G}$, we observe that it holds for this particular instance of PAF, since the Green operator $\mathcal{G}$ given in (1.3) is bounded.

2.1. The space $\widehat{\mathbb{U}}_K$ and the operators π_K and ρ_K.

For the particular instance of PAF given by the BVP (1.1), the *collocation method* is a particular discretization included in the general discretization described in subsection 1.3. Hence, in order to introduce such a method, we have to specify, for any positive integer K (level of discretization), a finite-dimensional space $\widehat{\mathbb{U}}_K$, a prolongation π_K, and a restriction ρ_K.

Recall the breaking points ξ_j, $j = 0, 1, \ldots, m$, introduced in subsection 1.1. Over any interval $I_j = [\xi_{j-1}, \xi_j]$, $j = 1, \ldots, m$, we introduce a mesh

$$\Omega_{K,j} := \left\{ t_{K,j,0}, t_{K,j,1}, \ldots, t_{K,j,N_{K,j}-1}, t_{K,j,N_{K,j}} \right\},$$

where

$$\xi_{j-1} = t_{K,j,0} < t_{K,j,1} < \cdots < t_{K,j,N_{K,j}-1} < t_{K,j,N_{K,j}} = \xi_j.$$

We set

$$h_{K,j,n+1} := t_{K,j,n+1} - t_{K,j,n}, \ j = 1, \ldots, m \text{ and } n = 0, \ldots, N_{K,j} - 1,$$

$$h_K := \max_{\substack{j=1,\ldots,m \\ n=0,\ldots,N_{K,j}-1}} h_{K,j,n+1}, \tag{2.1}$$

$$N_K := \sum_{j=1}^{m} N_{K,j}.$$

We also introduce ν_K distinct nodes $c_{K,i} \in [0,1]$, $i = 1,\dots,\nu_K$, and denote by $\ell_{K,i}$, $i = 1,\dots,\nu_K$, the relevant Lagrange coefficients. For $j = 1,\dots,m$, $n = 0,\dots,N_{K,j}-1$, and $i = 1,\dots,\nu$, we set

$$t_{K,j,i}^{n+1} := t_{K,j,n} + c_{K,i} h_{K,j,n+1}.$$

We can now define the finite-dimensional space $\widehat{\mathbb{U}}_K$, the prolongation π_K, and the restriction ρ_K.

- The space $\widehat{\mathbb{U}}_K$ is given by

$$\widehat{\mathbb{U}}_K = \widehat{U}_K := \left(\mathbb{R}^d\right)^{\nu_K N_K}.$$

  An element $\widehat{u} \in U_K$ is denoted in the following manner:

$$\widehat{u} = \left(\widehat{u}_{j,i}^{n+1}\right)_{\substack{j=1,\dots,m \\ n=0,\dots,N_{K,j}-1 \\ i=1,\dots,\nu_K}},$$

  where $\widehat{u}_{j,i}^{n+1} \in \mathbb{R}^d$ is interpreted as a value at the point $t_{K,j,i}^{n+1}$. The norm on $\widehat{U}_K$ is given by

$$\|\widehat{u}\|_{\widehat{U}_K} = \max_{\substack{j=1,\dots,m \\ n=0,\dots,N_{K,j}-1 \\ i=1,\dots,\nu_K}} \|\widehat{u}_{j,i}^{n+1}\|_\infty,\ u \in \widehat{U}_K.$$

- The prolongation $\pi_K : \widehat{\mathbb{U}}_K = \widehat{U}_K \to \mathbb{U} = U$ is defined as follows: for $\widehat{u} \in \widehat{U}_K$ and $j = 1,\dots,m$, the function $\pi_K \widehat{u}\ |_{I_j}$ is such that

$$\begin{aligned}
&\pi_K \widehat{u}\ |_{I_j} \left(t_{K,j,n} + c h_{K,j,n+1}\right) = \sum_{i=1}^{\nu_K} \ell_{K,i}(c)\, \widehat{u}_{j,i}^{n+1}, \\
&n = 0,1,\dots,N_{K,j}-1 \text{ and } c \in [0,1), \\
&\pi_K \widehat{u}\ |_{I_j}(\xi_j) = \lim_{t \uparrow \xi_j} \pi_K \widehat{u}\ |_{I_j}(t) = \sum_{i=1}^{\nu_K} \ell_{K,i}(1)\, \widehat{u}_{j,i}^{N_{K,j}}.
\end{aligned}$$

  In other words, in any subinterval $[t_{K,j,n}, t_{K,j,n+1})$ of I_j, $n = 0,1,\dots,N_{K,j}-1$, the function $\pi_K \widehat{u}\ |_{I_j}$ coincides with the $\mathbb{R}^d$- valued interpolation polynomial relevant to the nodes $t_{K,j,i}^{n+1}$ and the values $\widehat{u}_{j,i}^{n+1}$, $i = 1,\dots,\nu_K$. In the last point ξ_j of I_j, the function $\pi_K \widehat{u}\ |_{I_j}$ is prolonged by continuity from $[t_{K,j,N_{K,j}-1}, \xi_j)$.

Remark 2.1. The reason for not considering U as the space W of the functions $u : [a,b] \to \mathbb{R}^d$ such that the restrictions $u|_{I_j}$, $j = 1,\dots,m$, are *continuous* is the fact that, in general, we have

$$\lim_{t \uparrow t_{K,j,n+1}} \pi_K \widehat{u}\ |_{I_j}(t) \neq \pi_K \widehat{u}\ |_{I_j}(t_{K,j,n+1}) \tag{2.2}$$

at any internal mesh point $t_{K,j,n+1}$, $n = 0,\dots,N_{K,j}-2$.

- The restriction $\rho_K : \mathbb{U} = U \to \widehat{\mathbb{U}}_K \to \widehat{U}_K$ is such that for $u \in U$, $j = 1,\dots,m$, $n = 0,\dots N_{K,j}-1$, and $i = 1,\dots,\nu_K$, we have

$$(\rho_K u)_{j,i}^{n+1} = u|_{I_j}(t_{K,j,i}^{n+1}).$$

2.2. The operator $\widehat{\Phi}_K$. Now, we pass to describe the finite-dimensional operator

$$\widehat{\Phi}_K = R_K \Phi P_K : \widehat{X}_K \to \widehat{X}_K,$$

where

$$\widehat{X}_K = \widehat{\mathbb{U}}_K \times \mathbb{A} \times \mathbb{B} = \widehat{U}_K \times \mathbb{R}^d \times \mathbb{R}^{d_0},$$

which is a discretization of the operator $\Phi : X = \mathbb{U} \times \mathbb{A} \times \mathbb{B} = U \times \mathbb{R}^d \times \mathbb{R}^{d_0} \to X$. We recall that, given a fixed point $\widehat{x}_K^* = (\widehat{u}_K^*, \alpha_K^*, \beta_K^*)$ of $\widehat{\Phi}_K$, $P_K x_K^* = (\pi_K \widehat{u}_K^*, \alpha_K^*, \beta_K^*)$ is an approximation of a fixed point of Φ and the pair (v_K^*, β_K^*), where $v_K^* = \mathcal{G}(\pi_K \widehat{u}_K^*, \alpha_K^*)$, is an approximation of a solution of (1.1).

For $\widehat{u} \in \widehat{U}_K$ and $\alpha \in \mathbb{R}^d$, we introduce the functions $\mu = \pi_K \widehat{u} \in U$ and $\eta = \mathcal{G}(\mu, \alpha) \in V$. We have, for $j = 1, \ldots, m$,

$$\begin{aligned}
&\mu|_{I_j}(t_{K,j,n} + c h_{K,j,n+1}) = \sum_{i=1}^{\nu_K} \ell_{K,i}(c)\, \widehat{u}_{j,i}^{n+1}, \\
&n = 0, \ldots N_{K,j} - 1 \text{ and } c \in [0,1), \\
&\mu(\xi_j) = \sum_{i=1}^{\nu_K} l_{K,i}(1)\, \widehat{u}_{j,i}^{N_{K,j}},
\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}
&\eta(a) = \eta(\xi_0) = \alpha, \\
&\eta(t_{K,j,n}) = \eta(\xi_{j-1}) + \sum_{q=0}^{n-1} h_{K,j,q+1} \sum_{i=1}^{\nu_K} b_{K,i}(1) \widehat{u}_{j,i}^{q+1}, \\
&n = 0, \ldots, N_{K,j}, \\
&\eta(t_{K,j,n} + c h_{K,j,n+1}) = \eta(t_{K,j,n}) + h_{K,j,n+1} \sum_{i=1}^{\nu_K} b_{K,i}(c)\, \widehat{u}_{j,i}^{n+1}, \\
&n = 0, \ldots, N_{K,j} - 1 \text{ and } c \in [0,1],
\end{aligned} \tag{2.4}$$

where, for $i = 1, \ldots, \nu_K$,

$$b_{K,i}(c) = \int_0^c \ell_{K,i}(\gamma)\, d\gamma, \ c \in [0,1].$$

Therefore, for $(\widehat{u}, \alpha, \beta) \in \widehat{X}_K$, we have

$$\widehat{\Phi}_K(\widehat{u}, \alpha, \beta) = (\widehat{u}^\circ, \alpha^\circ, \beta^\circ),$$

where $\widehat{u}^\circ \in \widehat{U}_K$ has components

$$\begin{aligned}
&(\widehat{u}^\circ)_{j,i}^{n+1} = \mathcal{F}(\eta, \mu, \beta)|_{I_j}(t_{K,j,i}^{n+1}), \\
&j = 1, \ldots, m, \ n = 0, \ldots, N_{K,j} - 1, \text{ and } i = 1, \ldots, \nu_K,
\end{aligned}$$

and $(\alpha^\circ, \beta^\circ) \in \mathbb{R}^d \times \mathbb{R}^{d_0}$ is given by

$$(\alpha^\circ, \beta^\circ) = (\alpha, \beta) - B(\eta, \mu, \beta)$$

with μ and η defined in (2.3) and (2.4), respectively.

The fixed point equation

$$\widehat{\Phi}_K(\widehat{u}, \alpha, \beta) = (\widehat{u}, \alpha, \beta)$$

corresponds to a nonlinear system of $d\nu_K N_K + d + d_0$ scalar equations into $d\nu_K N_K + d + d_0$ scalar unknowns. It can be solved by a standard nonlinear algebraic equation solver.

2.3. The linear operator $\pi_K \rho_K$. Finally, we pass to describe the linear operator $\pi_K \rho_K : \mathbb{U} = U \to \mathbb{U} = U$.

For $u \in U$ and $j = 1, \ldots, m$, we have

$$\begin{aligned}
&\pi_K \rho_K u \,|_{I_j} (t_{K,j,n} + c h_{K,j,n+1}) = \sum_{i=1}^{\nu_K} \ell_{K,i}(c) \cdot u|_{I_j}(t_{K,j,i}^{n+1}), \\
&n = 0, \ldots, N_{K,j} - 1 \text{ and } c \in [0,1), \\
&\pi_K \rho_K u \,|_{I_j} (\xi_j) = \sum_{i=1}^{\nu_K} \ell_{K,i}(1) \cdot u|_{I_j}(t_{K,j,i}^{N_{K,j}}).
\end{aligned}$$

In other words, $\pi_K \rho_K$ is the Lagrange piecewise interpolation operator over the meshes $\Omega_{K,j}$, $j = 1, \ldots, m$.

Observe that $\|\pi_K \rho_K\| = L_{\nu_K - 1}$, where

$$L_{\nu_K - 1} := \max_{c \in [0,1]} \sum_{i=1}^{\nu_K} |\ell_{K,i}(c)|$$

is the Lebesgue constant relevant to the nodes $c_{K,i}$, $i = 1, \ldots, \nu_K$. Hence, we have

$$\lambda_K = L_{\nu_K - 1}, \tag{2.5}$$

where λ_K is defined in (1.8). Moreover, we also observe that

$$\|\pi_K\| = L_{\nu_K - 1}. \tag{2.6}$$

2.4. FEM and SEM. There are two possibilities to obtain better and better approximations of fixed points of Φ, and then better and better approximations of solutions of (1.1), by the collocation method:

- the FEM, where ν_K is fixed and $h_K \to 0$, as $K \to \infty$, with h_K defined in (2.1);
- the SEM, where the meshes $\Omega_{K,j}$, $j = 1, \ldots, m$, are fixed and $\nu_K \to \infty$, as $K \to \infty$.

2.5. The bases for the convergence analysis. In the next two sections we give, for FEM and SEM, convergence theorems directly obtained from Theorems 1.5 and 1.6 in subsection 1.4. Here, in this subsection, we prepare the basis for FEM and SEM convergence analyses.

For a continuous function $f : I \to \mathbb{R}^d$, where I is an interval of $\mathbb{R}$, we define the *modulus of continuity* of f as

$$\omega(f, \delta) := \sup_{t,s\in I \text{ and } |t-s|\leq\delta} \|f(t) - f(s)\|_\infty, \ \delta \geq 0.$$

We recall the following facts concerning the error of the Lagrange interpolation of a continuous function $f : [t_0, t_0 + h] \to \mathbb{R}^d$. Let $p_{\nu-1}$ be the $\mathbb{R}^d$-valued Lagrange interpolation polynomial of f relevant to the nodes $t_i = t_0 + c_i h$, $i = 1, \dots, \nu$, where $c_i \in [0, 1]$. Moreover, let $L_{\nu-1}$ be the Lebesgue constant relevant to the nodes c_i, $i = 1, \dots, \nu$. Finally, let

$$E_{\nu-1}(f) := \max_{t\in[t_0,t_0+h]} \|p^*_{\nu-1}(t) - f(t)\|_\infty$$

be the error of the best uniform approximation $p^*_{\nu-1}$ of f with $\mathbb{R}^d$-valued polynomials of degree not larger than $\nu - 1$.

The following facts hold:

- $$\max_{t\in[t_0,t_0+h]} \|p_{\nu-1}(t) - f(t)\|_\infty \leq L_{\nu-1} \cdot \omega(f, h); \tag{2.7}$$
- if f is of class C^ν, then
$$\max_{t\in[t_0,t_0+h]} \|p_{\nu-1}(t) - f(t)\|_\infty \leq \frac{1}{\nu!} \cdot \max_{t\in[t_0,t_0+h]} \|f^{(\nu)}(t)\|_\infty \cdot h^\nu; \tag{2.8}$$
- $$\max_{t\in[t_0,t_0+h]} \|p_{\nu-1}(t) - f(t)\|_\infty \leq (1 + L_{\nu-1}) \cdot E_{\nu-1}(f); \tag{2.9}$$
- $$E_{\nu-1}(f) \leq 6\omega\left(f, \frac{h}{2(\nu-1)}\right); \tag{2.10}$$
- if f is of class C^k, where k is a positive integer, and $\nu \geq k$, then
$$E_{\nu-1}(f) \leq \left(\frac{h}{2}\right)^k \frac{c_k}{(\nu-1)^k} \max_{t\in[t_0,t_0+h]} \|f^{(k)}(t)\|_\infty, \tag{2.11}$$
where $c_k > 0$ depends only on k;
- if f is real analytic, then
$$E_{\nu-1}(f) \leq c e^{-\gamma(\nu-1)}, \tag{2.12}$$
where $c > 0$ and $\gamma > 0$ depend only on f.

The first is an easy exercise. The second follows by the well-known standard error formula for the Lagrange interpolation. The third is well known. The fourth, the fifth and the sixth can be obtained by [17, p. 22, Corollary 1.4.1], [2, p. 11, Theorem 1.10], and [2, p. 12, Theorem 1.12], respectively.

Let $x^* = (u^*, \alpha^*, \beta^*)$ be a fixed point of Φ and let (v^*, β^*), where $v^* = \mathcal{G}(u^*, \alpha^*)$, be the relevant solution of (1.1). Related to the fixed point x^*, we assume Ax^*1 and Ax^*2 and introduce some *regularity conditions*, which are used in the convergence theorems. In defining them, we use the A_U notation for the U-component of an operator $A : X \to X$, $X = U \times \mathbb{R}^d \times \mathbb{R}^{d_0}$.

Remark 2.2. The approximation $\pi_K \widehat{u}_K^*$ of the solution u^* is a discontinuous piecewise polynomial function (see Remark 2.1). However, we observe that if 0 and 1 are included in the nodes $c_{K,i}$, $i = 1, \dots, \nu$, and the operator $\mathcal{F}$ has values in W (see Remark 2.1), then the approximation $\pi_K \widehat{u}_K^*$ of u^* belongs to W. Therefore, our convergence analysis also includes the case of a continuous piecewise approximation of the solution.

For the simple case (recall Definition 1.2), we introduce the following two regularity conditions:

- CRS1 (condition regularity simple case number 1). For any $x_0 \in X$, the function

$$w_{S1}(x_0) := (D^* \Phi)_U x_0 \in U$$

is such that the restrictions $w_{S1}(x_0)|_{I_j}$, $j = 1, \dots, m$, are continuous and there exists a function $\sigma_{S1} : [0, +\infty) \to [0, +\infty)$ independent of x_0 such that

$$\max_{j=1,\dots,m} \omega(w_{S1}(x_0)|_{I_j}, \delta) \leq \sigma_{S1}(\delta) \cdot \|x_0\|_X, \ \delta \geq 0.$$

- CRS2. There exists $r_{S2} > 0$ such that, for any $x \in \overline{B}(x^*, r_{S2})$ and $x_0 \in X$, the function

$$w_{S2}(x, x_0) = (D\Phi(x) - D^* \Phi)_U x_0 \in U$$

is such that the restrictions $w_{S2}(x, x_0)|_{I_j}$, $j = 1, \dots, m$, are continuous and there exists a function $\sigma_{S2} : [0, +\infty) \to [0, +\infty)$ independent of x and x_0 such that

$$\max_{j=1,\dots,m} \omega(w_{S2}(x, x_0)|_{I_j}, \delta) \leq \sigma_{S2}(\delta) \cdot \|x - x^*\|_X \cdot \|x_0\|_X, \ \delta \geq 0.$$

For the splitting case (recall Definition 1.3), we consider these regularity conditions:

- CRT1 (condition regularity splitting Case number 1). For any $x_0 \in X$, the function

$$w_{T1}(x_0) := (\Sigma^* \Lambda)_U x_0 \in U$$

is such that the restrictions $w_{T1}(x_0)|_{I_j}$, $j = 1, \dots, m$, are continuous and there exists a function $\sigma_{T1} : [0, +\infty) \to [0, +\infty)$ independent of x_0 such that

$$\max_{j=1,\dots,m} \omega(w_{T1}(x_0)|_{I_j}, \delta) \leq \sigma_{T1}(\delta) \cdot \|x_0\|_X, \ \delta \geq 0.$$

- CRT2. For any $x_0 \in X$, the function

$$w_{T2}(x_0) := (\Gamma^* (I_X - \Gamma^*)^{-1} \Sigma^* \Lambda)_U x_0 \in U$$

is such that the restrictions $w_{T2}(x_0)|_{I_j}$, $j = 1, \dots, m$, are continuous and there exists a function $\sigma_{T2} : [0, +\infty) \to [0, +\infty)$ independent of x_0 such that

$$\max_{j=1,\dots,m} \omega(w_{T2}(x_0)|_{I_j}, \delta) \leq \sigma_{T2}(\delta) \cdot \|x_0\|_X, \ \delta \geq 0.$$

3. Convergence analysis for FEM. In case of FEM, the positive integer ν_K and the nodes $c_{K,i}$, $i = 1, \dots, \nu_K$, are independent of K. Therefore, in this section, we write ν, c_i, ℓ_i, and b_i instead of ν_K, $c_{K,i}$, $\ell_{K,i}$, and $b_{K,i}$.

Observe that in case of FEM, we have

$$\lambda_K = \|\pi_K\| = L_{\nu-1} = O(1), \ K \to \infty \tag{3.1}$$

(recall (2.5) and (2.6)). Moreover, for the consistency error

$$e_K^* = (\pi_K \rho_K - I_U)u^*$$

given in (1.9), we have, with h_K the maximum stepsize defined in (2.1),

$$\|e_K^*\|_U = O(h_K^\nu), \ K \to \infty, \tag{3.2}$$

whenever the restrictions $u^*|_{I_j}$, $j = 1, \dots, m$, are of class C^ν. This follows by (2.8).

Here are the convergence theorems for FEM. The first considers the simple case and the second the splitting case.

THEOREM 3.1 (the simple case). *Assume the simple case,* CRS1 *and*

$$\lim_{\delta \to 0} \sigma_{S1}(\delta) = 0, \tag{3.3}$$

where σ_{S1} appears in CRS1*. If the restrictions $u^*|_{I_j}$, $j = 1, \dots, m$, are of class C^p, where p is the order of the composite interpolatory quadrature rule based on the nodes c_i, $i = 1, \dots, \nu$, then there exists a positive integer $\widehat{K}$ such that for any positive integer $K \geq \widehat{K}$, $\widehat{\Phi}_K$ has a fixed point $\widehat{x}_K^*$ such that*

$$\|P_K \widehat{x}_K^* - x^*\|_X = O(h_K^\nu), \ K \to \infty. \tag{3.4}$$

Moreover, for the approximation (v_K^*, β_K^*) of (v^*, β^*), we have the estimate

$$\|(v_K^*, \beta_K^*) - (v^*, \beta^*)\|_{V \times \mathbb{R}^{d_0}} = O\left(h_K^{\min\{\nu+1, p\}}\right), \ K \to \infty. \tag{3.5}$$

Finally, if $\{\widehat{x}_K\}$ is a sequence of fixed points of $\widehat{\Phi}_K$ such that $\widehat{x}_K$ is eventually different from $\widehat{x}_K^*$, then

$$\frac{1}{\|P_K \widehat{x}_K - x^*\|_X} = O(1), \ K \to \infty, \tag{3.6}$$

and

$$\frac{1}{\|\widehat{x}_K - \widehat{x}_K^*\|_{\widehat{X}_K}} = O(1), \ K \to \infty. \tag{3.7}$$

Proof. The proof is an application of Theorem 1.5. First, we have to check the conditions (1.10) and (1.11) in that theorem.

The condition (1.10) follows by (2.7) and (3.3). The condition (1.11) (upper line after {) follows by (3.1) and (3.2) (observe that $p \geq \nu$ and so the restrictions $u^*|_{I_j}$, $j = 1, \dots, m$, which are of class C^p, are also of class C^ν).

Now, we can apply Theorem 1.5.

The conclusion (3.4) follows by (1.12) and (3.2). The conclusions (3.6) and (3.7) follow by (1.15) and (1.16) (upper line after {), respectively, and (3.1).

It remains to prove that the conclusion (3.5) holds. Below, we show that

$$\|\mathcal{G}(e_K^*,0)\|_V = O\left(h_K^{\min\{\nu+1,p\}}\right),\ K\to\infty, \tag{3.8}$$

holds. By (1.14) (upper line after {), (3.8), (3.1), and (3.2), we obtain (3.5) (observe that $p \leq 2\nu$ and then $\min\{\nu+1,p\} \leq 2\nu$: the first term in the right-hand side of (1.14) is $O(h_K^{\min\{\nu+1,p\}})$, and the second term is $O(h_K^{2\nu})$).

Proof of (3.8). We write

$$\epsilon := \mathcal{G}(e_K^*,0) = \eta - v,$$

where $\eta := \mathcal{G}(\pi_K\rho_K u^*,0)$ and $v := \mathcal{G}(u^*,0)$.

By recalling (2.4) we have, for $j=1,\dots,m$,

$$\begin{aligned}
&\eta(a) = \eta(\xi_0) = 0,\\
&\eta\left(t_{K,j,n}\right) = \eta(\xi_{j-1}) + \sum_{q=0}^{n-1} h_{K,j,q+1}\sum_{i=1}^{\nu} b_i(1)\cdot u^*|_{I_j}(t_{K,j,i}^{q+1}),\\
&n=0,\dots,N_{K,j},\\
&\eta\left(t_{K,j,n}+ch_{K,j,n+1}\right) = \eta(t_{K,j,n}) + h_{K,j,n+1}\sum_{i=1}^{\nu} b_i\left(c\right)\cdot u^*|_{I_j}(t_{K,j,i}^{q+1}),\\
&n=0,\dots,N_{K,j}-1 \text{ and } c\in[0,1].
\end{aligned}$$

Moreover, we have, for $j=1,\dots,m$,

$$\begin{aligned}
&v(a) = v(\xi_0) = 0,\\
&v\left(t_{K,j,n}\right) = v(\xi_{j-1}) + \int_{\xi_{j-1}}^{t_{K,j,n}} u^*|_{I_j}\left(s\right)ds,\\
&n=0,\dots,N_{K,j},\\
&v\left(t_{K,j,n}+ch_{K,j,n+1}\right) = v(t_{K,n,j}) + h_{K,j,n+1}\int_0^c u^*|_{I_j}\left(t_{K,j,n}+\gamma h_{K,j,n+1}\right)d\gamma,\\
&n=0,\dots,N_{K,j}-1 \text{ and } c\in[0,1].
\end{aligned}$$

Hence, for $j=1,\dots,m$,

$$\begin{aligned}
&\epsilon(a) = \epsilon(\xi_0) = 0,\\
&\epsilon\left(t_{K,j,n}\right) = \epsilon(\xi_{j-1}) + \sum_{q=0}^{n-1} h_{K,j,q+1}\sum\nolimits_{i=1}^{\nu} b_i(1)u^*|_{I_j}(t_{K,j,i}^{q+1}) - \int_{\xi_{j-1}}^{t_{K,j,n}} u^*|_{I_j}\left(s\right)ds,\\
&n=0,\dots,N_{K,j},\\
&\epsilon\left(t_{K,j,n}+ch_{K,j,n+1}\right) = \epsilon(t_{K,j,n}) + h_{K,j,n+1}\int_0^c e_K^*|_{I_j}\left(t_{K,j,n}+\gamma h_{K,j,n+1}\right)d\gamma,\\
&n=0,\dots,N_{K,j}-1 \text{ and } c\in[0,1].
\end{aligned}$$

(3.9)

Since the composite quadrature rule based on the nodes c_i, $i = 1 \ldots \nu$, has order p, we have

$$\max_{\substack{j=1,\ldots,m \\ n=0,\ldots,N_{K,j}}} \|\epsilon\left(t_{K,j,n}\right)\|_\infty = O(h_K^p),\ K \to \infty.$$

Moreover, by (3.2), we have

$$\max_{\substack{j=1,\ldots,m \\ n=0,\ldots,N_{K,j}-1 \\ c\in[0,1]}} \left\| h_{K,j,n+1} \int_0^c e_K^*|_{I_j} \left(t_{K,j,n} + \gamma h_{K,j,n+1}\right) d\gamma \right\|_\infty = O(h_K^{\nu+1}),\ K \to \infty.$$

Thus, by (3.9) we obtain the estimate (3.8). □

THEOREM 3.2 (the splitting case). *Assume the splitting case,* CRT**1***,*

$$\mu_K \cdot \sigma_{T1}(h_K) \to 0,\ K \to \infty, \tag{3.10}$$

CRT**2***, and*

$$\mu_K \cdot \sigma_{T2}(h_k) \to 0,\ K \to \infty. \tag{3.11}$$

where σ_{T1} and σ_{T2} appear in CRT**1** *and* CRT**2***, respectively, and μ_K is given in* (1.7)*. If the restrictions $u^*|_{I_j}$, $j = 1, \ldots, m$, are of class C^ν, and*

$$\mu_K^2 h_K^\nu \to 0,\ K \to \infty, \tag{3.12}$$

then there exists a positive integer $\widehat{K}$ such that for any positive integer $K \geq \widehat{K}$, $\widehat{\Phi}_K$ has a fixed point $\widehat{x}_K^*$ such that

$$\|P_K \widehat{x}_K^* - x^*\|_X = O(\mu_K h_K^\nu),\ K \to \infty. \tag{3.13}$$

Moreover, for the approximation (v_K^*, β_K^*) of (v^*, β^*), we have the estimate

$$\|(v_K^*, \beta_K^*) - (v^*, \beta^*)\|_{V \times \mathbb{R}^{d_0}} = O(\mu_K h_K^\nu),\ K \to \infty. \tag{3.14}$$

Finally, if $\{\widehat{x}_K\}$ is a sequence of fixed points of $\widehat{\Phi}_K$ such that $\widehat{x}_K$ is eventually different from $\widehat{x}_K^*$, then

$$\frac{1}{\|P_K \widehat{x}_K - x^*\|_X} = O(\mu_K),\ K \to \infty, \tag{3.15}$$

and

$$\frac{1}{\|\widehat{x}_K - \widehat{x}_K^*\|_{\widehat{X}_K}} = O(\mu_K),\ K \to \infty. \tag{3.16}$$

Proof. The proof is an application of Theorem 1.6. First, we have to check the conditions (1.17), (1.18), and (1.19) in that theorem.

The condition (1.17) follows by (3.10) and (2.7). The condition (1.18) follows by (3.11) and (2.7). The condition (1.19) follows by (3.12), (3.1), and (3.2).

Now, we can apply Theorem 1.6.

The conclusions (3.13) and (3.14) follow by (1.20) and (1.21), respectively, and (3.2). The conclusions (3.15) and (3.16) follow by (1.22) and by (1.23), respectively, and (3.1). □

Note that if the nilpotency case holds, then, by (1.24), we have

$$\mu_K = O(1),\ K \to \infty,$$

in Theorem 3.2.

4. Convergence analysis for SEM. In the case of SEM, the meshes $\Omega_{K,j}$, $j = 1, \ldots, m$, are independent of K. Therefore, in this section, we write

$$\Omega_j = \left\{ t_{j,0}, t_{j,1}, \ldots, t_{j,N_j-1}, t_{j,N_j} \right\},$$

instead of $\Omega_{K,j}$, $h_{j,n+1}$ instead of $h_{K,j,n+1}$, $n = 0, 1, \ldots, N_j$, and h instead of h_K.

Remark 4.1. We do not confine SEM to the simplest case of meshes

$$\Omega_j = \{\xi_{j-1}, \xi_j\}, \ j = 1 \ldots, m.$$

The reason is that for BVPs of neutral differential equations with deviating arguments (1.5), under a condition on the neutral deviating arguments $\vartheta_1, \ldots, \vartheta_l$, the nilpotency case holds if the maximum stepsize h is less than a certain value (independent of K). This is illustrated in Part II [15].

In the case of SEM, we have

$$\lambda_K = \|\pi_K\| = L_{\nu_K - 1} \to +\infty, \ K \to \infty, \tag{4.1}$$

and the less rapid growth for L_{ν_K-1} is in case of Chebyshev nodes, where

$$L_{\nu_K-1} = O(\log \nu_K), \ K \to \infty. \tag{4.2}$$

Moreover, we have

$$\|e_K^*\|_U = O\left(\frac{L_{\nu_K-1}}{(\nu_K - 1)^q} \right), \ K \to \infty, \tag{4.3}$$

whenever the restrictions $u^*|_{I_j}$, $j = 1, \ldots, m$, are of class C^q, and

$$\|e_K^*\|_U = O\left(L_{\nu_K-1} \mathrm{e}^{-\gamma(\nu_K-1)} \right), \ K \to \infty, \tag{4.4}$$

for some $\gamma > 0$, whenever the restrictions $u^*|_{I_j}$, $j = 1 \ldots, m$, are real analytic. The estimates (4.3) and (4.4) follow by (2.9) with (2.11) and (2.9) with (2.12), respectively.

Here are the convergence theorems for SEM.

THEOREM 4.2 (the simple case). *Assume the simple case,* CRS1,

$$L_{\nu_K-1} \cdot \sigma_{S1} \left(\frac{h}{2(\nu_K - 1)} \right) \to 0, \ K \to \infty, \tag{4.5}$$

CRS2, *and*

$$L_{\nu_K-1} \cdot \sigma_{S2} \left(\frac{h}{2(\nu_K - 1)} \right) = O(1), \ K \to \infty, \tag{4.6}$$

where σ_{S1} and σ_{S2} appear in CRS1 *and* CRS2, *respectively. The following two results hold:*

- Ⓐ *If the restrictions $u^*|_{I_j}$,$j = 1, \ldots, m$, are of class C^q, where q is a positive integer such that*

$$\frac{L_{\nu_K-1}}{(\nu_K - 1)^q} \to 0, \ K \to \infty, \tag{4.7}$$

then there exists a positive integer $\widehat{K}$ *such that, for any positive integer* $K \geq \widehat{K}$, $\widehat{\Phi}_K$ *has a fixed point* $\widehat{x}_K^*$ *such that*

$$\|P_K\widehat{x}_K^* - x^*\|_X = O\left(\frac{L_{\nu_K-1}}{(\nu_K-1)^q}\right), \ K \to \infty. \tag{4.8}$$

Moreover, for the approximation (v_K^*, β_K^*) *of* (v^*, β^*)*, we have the estimate*

$$\|(v_K^*, \beta_K^*) - (v^*, \beta^*)\|_{V\times\mathbb{R}^{d_0}} = O\left(\frac{L_{\nu_K-1}}{(\nu_K-1)^q}\right), \ K \to \infty. \tag{4.9}$$

Finally, if $\{\widehat{x}_K\}$ *is a sequence of fixed points of* $\widehat{\Phi}_K$ *such that* $\widehat{x}_K$ *is eventually different from* $\widehat{x}_K^*$*, then*

$$\frac{1}{\|P_K\widehat{x}_K - x^*\|_X} = O(1), \ K \to \infty, \tag{4.10}$$

and

$$\frac{1}{\|\widehat{x}_K - \widehat{x}_K^*\|_{\widehat{X}_K}} = O(L_{\nu_K-1}), \ K \to \infty. \tag{4.11}$$

Ⓑ *If the restrictions* $u^*|_{I_j}$, $j = 1, \ldots, m$*, are real analytic and*

$$\text{for any } \gamma > 0: \quad L_{\nu_K-1}\mathrm{e}^{-\gamma(\nu_K-1)} \to 0, \ K \to \infty, \tag{4.12}$$

then we have the same conclusions as in Ⓐ *with the estimates* (4.8) *and* (4.9) *improved to*

$$\|P_K\widehat{x}_K^* - x^*\|_X = O\left(L_{\nu_K-1}\mathrm{e}^{-\gamma(\nu_K-1)}\right), \ K \to \infty, \tag{4.13}$$

and

$$\|(v_K^*, \beta_K^*) - (v^*, \beta^*)\|_{V\times\mathbb{R}^{d_0}} = O\left(L_{\nu_K-1}\mathrm{e}^{-\gamma(\nu_K-1)}\right), \ K \to \infty, \tag{4.14}$$

respectively, for some $\gamma > 0$.

Proof. The proof is an application of Theorem 1.5. We have to check the conditions (1.10) and (1.11) in that theorem.

The condition (1.10) follows by (4.5) and (2.9)–(2.10). Observe that (4.6) and (2.9)–(2.10) imply CSC. The condition (1.11) (case CSC holds) follows by (4.3) and (4.7) for situation Ⓐ and by (4.4) and (4.12) for situation Ⓑ.

Now, we can apply Theorem 1.5.

For situation Ⓐ, the conclusions (4.8) and (4.9) follow by (1.12) and (1.13), respectively, and (4.3). For situation Ⓑ, the conclusions (4.13) and (4.14) follow by (1.12) and (1.13), respectively, and (4.4). The conclusions (4.10) and (4.11) follow by (1.15) and (1.16) (case CSC holds), respectively, and (4.1). ☐

Theorem 4.3 (the splitting case). *Assume the splitting case,* CRT**1***,*

$$\mu_K L_{\nu_K-1} \cdot \sigma_{T1}\left(\frac{h}{2(\nu_K-1)}\right) \to 0, \ K \to \infty, \tag{4.15}$$

CRT**2***, and*

$$\mu_K L_{\nu_K-1} \cdot \sigma_{T2}\left(\frac{h}{2(\nu_K-1)}\right) \to 0, \ K \to \infty, \tag{4.16}$$

where σ_{T1} *and* σ_{T2} *appear in* CRT**1** *and* CRT**2***, respectively, and* μ_K *is given in* (1.7). *The following two results hold:*

Ⓐ *If the restrictions* $u^*|_{I_j}$, $j = 1, \ldots, m$, *are of class* C^q, *where* q *is a positive integer such that*

$$\frac{\mu_K^2 L_{\nu_K-1}^2}{(\nu_K - 1)^q} \to 0, \ K \to \infty, \tag{4.17}$$

then there exists a positive integer $\widehat{K}$ *such that, for any positive integer* $K \geq \widehat{K}$, $\widehat{\Phi}_K$ *has a fixed point* $\widehat{x}_K^*$ *such that*

$$\|P_K \widehat{x}_K^* - x^*\|_X = O\left(\frac{\mu_K L_{\nu_K-1}}{(\nu_K - 1)^q}\right), \ K \to \infty. \tag{4.18}$$

Moreover, for the approximation (v_K^*, β_K^*) *of* (v^*, β^*), *we have the estimate*

$$\|(v_K^*, \beta_K^*) - (v^*, \beta^*)\|_{V \times \mathbb{R}^{d_0}} = O\left(\frac{\mu_K L_{\nu_K-1}}{(\nu_K - 1)^q}\right), \ K \to \infty. \tag{4.19}$$

Finally, if $\{\widehat{x}_K\}$ *is a sequence of fixed points of* $\widehat{\Phi}_K$ *such that* $\widehat{x}_K$ *is eventually different from* $\widehat{x}_K^*$, *then*

$$\frac{1}{\|P_K \widehat{x}_K - x^*\|_X} = O(\mu_K L_{\nu_K-1}), \ K \to \infty, \tag{4.20}$$

and

$$\frac{1}{\|\widehat{x}_K - \widehat{x}_K^*\|_{\widehat{X}_K}} = O(\mu_K L_{\nu_K-1}^2), \ K \to \infty. \tag{4.21}$$

Ⓑ *If the restrictions* $u^*|_{I_j}$, $j = 1, \ldots, m$, *are real analytic and*

$$\text{for any } \gamma > 0: \quad \mu_K^2 L_{\nu_K-1}^2 \mathrm{e}^{-\gamma(\nu_K-1)} \to 0, \ K \to \infty, \tag{4.22}$$

then we have the same conclusions as in Ⓐ *with the estimates* (4.18) *and* (4.19) *improved to*

$$\|P_K \widehat{x}_K^* - x^*\|_X = O\left(\mu_K L_{\nu_K-1} \mathrm{e}^{-\gamma(\nu_K-1)}\right), \ K \to \infty, \tag{4.23}$$

and

$$\|(v_K^*, \beta_K^*) - (v^*, \beta^*)\|_{V \times \mathbb{R}^{d_0}} = O\left(\mu_K L_{\nu_K-1} \mathrm{e}^{-\gamma(\nu_K-1)}\right), \ K \to \infty, \tag{4.24}$$

respectively, for some $\gamma > 0$.

Proof. The proof is an application of Theorem 1.6. We have to check the conditions (1.17), (1.18), and (1.19) in that theorem.

The condition (1.17) follows by (4.15) and (2.9)–(2.10). The condition (1.18) follows by (4.16) and (2.9)–(2.10). For situation Ⓐ, the condition (1.19) follows by (4.1), (4.3), and (4.17). For situation Ⓑ, it follows by (4.1), (4.4), and (4.22).

Now, we can apply Theorem 1.6.

For situation Ⓐ, the conclusions (4.18) and (4.19) follow by (1.20) and (1.21), respectively, and (4.3). For situation Ⓑ, the conclusions (4.23) and (4.24) follow by (1.20) and (1.21), respectively, and (4.4). The conclusions (4.20) and (4.21) follow by (1.22) and (1.23), respectively, and (4.1). □

Note that if the nilpotency case holds, then, by (1.24), we have

$$\mu_K = O(L_{\nu_K-1}^{c-1}),\ K \to \infty, \tag{4.25}$$

in Theorem 4.3.

Remark 4.4. Theorem 4.2 for the simple case does not use the estimate (1.14) since, for SEM, this estimate is not better than the estimate (1.13). Moreover, Theorem 4.2 for the simple case is not a corollary of Theorem 4.3 for the splitting case, obtained when $\Gamma^* = 0$ and so when the nilpotency case holds with $c = 1$: for

$$\mu_K = O(L_{\nu_K-1}^{c-1}) = O(1),\ K \to \infty,$$

compare (4.17) and (4.22) with (4.7) and (4.12), respectively, and compare (4.20) and (4.21) with (4.10) and (4.11), respectively. The better results obtained by Theorem 4.2 are due to the introduction of the regularity condition CRS2.

Remark 4.5. If the solution u^* is such that for any $j = 1, \ldots, m$, the restriction $u^*|_{I_j}$ is C^∞ or real analytic, the previous convergence theorems say that the SEM version of collocation exhibits an infinite order of convergence whenever

$$L_{\nu_K-1} = O((\nu_K - 1)^p) \text{ and } \mu_K L_{\nu_K-1} = O((\nu_K - 1)^p),\ K \to \infty,$$

for some positive integer p. This happens, for example, when the nilpotency case holds and Chebyshev nodes $c_{K,i}$, $i = 1, \ldots, \nu$, are used (see (4.2) and (4.25)).

Here "infinite order of convergence"means faster than any convergence

$$(\nu_K - 1)^{-c}, K \to \infty,$$

where $c > 0$. If the restrictions $u^*|_{I_j}$, $j = 1 \ldots, m$, are real analytic, and then the exponential estimates (4.13)–(4.14)–(4.23)–(4.24) hold, we have the so-called spectral accuracy (see [18]).

We remark also that the FEM version of collocation has only a finite order of convergence $O(h_K^\nu)$ or $O(h_K^{\min\{\nu+1,p\}})$, $K \to \infty$, in the same situation of restrictions $u^*|_{I_j}$, $j = 1, \ldots, m$, of class C^∞ or real analytic.

5. Conclusions. In Part II [15] of this paper, we apply the convergence results presented here to BVPs for neutral differential equations with deviating arguments (1.5)

$$y'(t) = f(t, y(t), y(\theta_1(t)), \ldots, y(\theta_k(t)), y'(\vartheta_1(t)), \ldots, y'(\vartheta_l(t)), p)$$

and nonneutral boundary conditions. For these problems, the simple case holds if there are no neutral deviating arguments and the nilpotency case holds under a condition on the neutral deviating arguments.

For such BVPs, we have to verify the assumptions AF, A$\mathcal{F}B$, Ax^*1, Ax^*2, and, for the simple case, the regularity conditions CRS1 and CRS2, and, for the nilpotency case, the regularity conditions CRT1 and CRT2. Finally, in order to apply the convergence theorems for FEM and SEM, we also find an estimate of the convergence to zero of the functions σ_{S1}, σ_{S2}, σ_{T1}, and σ_{T2} appearing in the regularity conditions.

The aim of Part II is to replace the convergence results of Part I, which are a little bit abstract and based on assumptions involving the Fréchet-derivative $D\Phi$ of the operator Φ, with practical convergence results based on simple assumptions involving the function f and the deviating arguments $\theta_1, \ldots, \theta_k, \vartheta_1, \ldots, \vartheta_l$, assumptions that are easy to check. This task is far from trivial.

Moreover, after this nontrivial work, we obtain really the following new, important and concrete results for BVPs of neutral differential equations with deviating arguments:

(1) In the case of nonneutral equations, we show that for ν collocation nodes whose relevant quadrature rule has order $\nu + 1$, the order of convergence for FEM is $O\left(h^{\nu+1}\right)$, h being the maximum stepsize. This improves the $O\left(h^{\nu}\right)$ known in the literature (see [1] and [3]).

(2) In the case of neutral equations, we give, for the first time in the literature, a proof of convergence for methods of arbitrarily high order of convergence (both for FEM and SEM). The existing literature, reports only the proof of convergence of a method of order one (see [7]).

Part II applies the results of this paper to BVPs for neutral differential equations with deviating arguments. On the other hand, the application of such results to BVPs for neutral integro-differential equations is considered in [16].

Finally, we remark that the FEM convergence theorems, namely, Theorems 3.1 and 3.2, present *uniform error estimates* that are valid for all the points of the interval $[a, b]$. On the other hand, also *nodal error estimates*, i.e., error estimates that are valid at only the mesh points $t_{K,j,n+1}$, $j = 1, \ldots, m$, and $n = 0, 1, \ldots, N_{K,j} - 1$, are interesting because at these points we can have a higher order of convergence with respect to the uniform estimates. This phenomenon is called *nodal superconvergence*. By using the results in the paper [14], one can indeed study the nodal superconvergence and this will be done in a forthcoming paper.

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