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A note on the Bieberbach conjecture for some classes of slice regular functions

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Abstract. *In this note we prove the Bieberbach conjecture for some classes of quaternionic functions, including quaternionic slice regular functions with specific geometric properties such as starlike and convex functions. At the same time, we investigate some interesting properties related to the concepts of starlikeness for these quaternionic slice regular functions. Furthermore, we introduce and obtain generalizations and results for the class of quaternionic Carathéodory functions, such as a version of Rogosinski's Theorem over the quaternions.*

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1. Introduction

Let $\mathbb{H}$ denote the algebra of quaternions. In the quaternionic setting it is possible (see [6, 7])) to introduce a notion of regularity for functions defined in any open ball $B(0,r) = \{q \in \mathbb{H} \ : \ |q| < r\}$ (and, more in general, in some axially symmetric slice domain[1]) of $\mathbb{H}$ which mostly resemble the one of analiticity in the complex case. For this class of regular quaternionic functions, called *slice regular*, many of the results valid for the holomorphic maps can be extended, but many other new phenomena may occur. In particular slice regular functions are characterized to be quaternionic analytic with coefficients on (say) the right; namely f is slice regular in $B(0,r) \subset \mathbb{H}$ if and only if there exists a quaternionic power series $\sum_n q^n a_n$ with $a_n \in \mathbb{H}$ for any $n \in \mathbb{N}$

[1]If we denote by $\mathbb{S}_\mathbb{H}$ the sphere of imaginary units of $\mathbb{H}$, i.e. $\mathbb{S}_\mathbb{H} = \{q \in \mathbb{H} : q^2 = -1\}$, then every non real element q can be written in a unique way as $q = x + yI_q$, with $I_q \in \mathbb{S}_\mathbb{H}$ and $x, y \in \mathbb{R}, y > 0$. We will refer to $x = \Re e(q)$ as the real part of q and $y = Im(q)$ as the imaginary part of q.

Definition 1.1. *Let $\Omega \subseteq \mathbb{H}$ be a domain in $\mathbb{H}$; we say that Ω is an* axially symmetric domain *if, for all $x + Iy \in \Omega$, the whole sphere $x + \mathbb{S}_\mathbb{H} y = \{x + Jy \ : \ J \in \mathbb{S}_\mathbb{H}\}$ is contained in Ω.*

converging in $B(0,r)$ and such that

$$f(q) = \sum_n q^n a_n.$$

The notions of derivative (and primitive) can be naturally introduced for slice regular functions, therefore, this class perfectly fits for the purposes of this paper. Indeed, we recall the following

DEFINITION 1.2. *Let Ω be an axially symmetric slice domain in $\mathbb{H}$ and let $f : \Omega \to \mathbb{H}$ be a slice regular function. For any $I \in \mathbb{S}_{\mathbb{H}}$ and any point $q = x + yI$ in Ω (with $x = \Re eq$ and $y = \Im mq$) we define the* Cullen derivative *of f at q as*

$$\partial_C f(x+yI) = f'(x+yI) := \frac{1}{2}\left(\frac{\partial}{\partial x} - I\frac{\partial}{\partial y}\right) f_I(x+yI)$$

where f_I is the restriction of f on $\Omega \cap \mathbb{C}_I$, where $\mathbb{C}_I$ is the slice $\{x + Iy,\ x, y \in \mathbb{R}\}$.

Since in $\mathbb{H}$ one can choose different imaginary units, it is also worth considering the following

DEFINITION 1.3. *Let Ω be an axially symmetric slice domain in $\mathbb{H}$ and let $f : \Omega \to \mathbb{H}$ be a slice regular function. We define the* spherical derivative *of f at $q \notin \mathbb{R} \cap \Omega$ as*

$$\partial_S f(q) := (q - \bar{q})^{-1}[f(q) - f(\bar{q})].$$

If f is a slice regular function on an axially symmetric slice domain Ω, for each $q_0 \in \Omega$ one can define the function $R_{q_0} f : \Omega \to \mathbb{H}$ which turns out to be slice regular and such that

$$f(q) = f(q_0) + (q - q_0) * R_{q_0} f(q); \tag{1}$$

moreover $R_{q_0} f(q_0) = \partial_C f(q_0)$ and $R_{q_0} f(\bar{q}_0) = \partial_S f(q_0)$

One of the most famous theorems in Complex Analysis which provides (sharp) estimates of the moduli of the coefficients of an analytic function is given in the proof of the Bieberbach Conjecture due to De Branges (see [1]). In the long history of attempts to prove the conjecture in the holomorphic setting, some important improvements have been obtained when considering some special subclasses of the class of univalent holomorphic or schlicht functions. In this paper we refer to the usual definitions of the sets of (normalized) holomorphic schlicht functions $\mathcal{S}$, of starlike functions $\mathcal{S}^*$, of convex $\mathcal{C}$ and Carathéodory functions $\mathcal{P}$ as in [2, 10] for complex holomorphic maps.

In the quaternionic setting some results in the direction of solving this conjecture for the corresponding slice regular subclasses has been done in [5], for slice regular functions which map each slice into the same slice. We begin by extending the previous result and proving the following proposition in which it is required for f to preserve just one slice.

PROPOSITION 1.4. *For a slice regular, univalent function defined on the quaternionic unit ball $f : B(0,1) \to \mathbb{H}$ (such that $f(0) = 0$ and $f'(0) = 1$) that preserves one slice and whose power series expansion is*

$$f(q) = q + q^2 a_2 + \ldots + q^n a_n + \ldots,$$

the following inequalities hold

$$|a_n| \leq n \quad \text{for any } n.$$

These inequalities are sharp.

The proof will follow from the so called "One-Slice-Preserved-Principle" which is stated and proved in Section 2; notice that the assumptions of Proposition 1.4 can equivalently be formulated as $f : B(0,1) \to \mathbb{H}$ such that

$$f(q) = q + q^2 a_2 + \ldots + q^n a_n + \ldots,$$

with $a_j \in \mathbb{C}_I := \mathbb{R} + I\mathbb{R}$ for any j and for a given $I \in \mathbb{S}$.

One of the aims of the present note is to give estimates of the moduli of the coefficients in the power series expansion for slice regular functions which have suitable properties of starlikeness; this will be the content of the next section, which contains several result in this direction.

Before continuing, we recall that several attempts to generalize the notion of starlikeness from the holomorphic case to the class of quaternionic valued functions have appeared (see [3, 4, 5]).

Since the class of quaternionic maps we investigated can be regarded as (right) quaternionic analytic functions, the characterization of the geometric aspects of starlikeness can be provided in terms of inequality conditions (see [9]). In particular, after considering the regular product $*$ for slice regular functions (see for instance [6]), we gave two different definitions of starlikeness for slice regular functions, the *geometrical starlikeness* and the *algebraic starlikeness* [9], these two definitions agree for special subclasses of functions.

DEFINITION 1.5. *Assume f is an injective slice regular function in the unit ball of $\mathbb{H}$ such that $f(0) = 0$. Then we say that f is* geometrically starlike *with respect to* 0 *if and only if, for any real r such that $0 \leq r < 1$, and for any real t with $0 \leq t \leq 1$*

$$(1-t)f(B(0,r)) \subseteq f(B(0,r)).$$

The previous definition has a deep geometrical meaning, and it is proved to have a nice interpretation in terms of the positivity of the real part of a suitable (according to the Splitting) standard Hermitian product $\langle \cdot | \cdot \rangle$ in $\mathbb{C}^2$ of an expression involving the Cullen ($\partial_C f$) and the Spherical ($\partial_S f$) derivatives of f together with f as stated in [9].

In complete analogy with the complex holomorphic case we can introduce the subclass of slice regular functions

$$\begin{aligned}\mathcal{S}_{\mathbb{H}} := \big\{ f : B(0,1) \to \mathbb{H},\ f \text{ injective, slice regular} \\ \text{and such that } f(0)=0 \text{ and } f'(0)=1 \big\}.\end{aligned}$$

With this notation we have proved in [9]

THEOREM 1.6. *A function $f : B(0,1) \to \mathbb{H}$ in $\mathcal{S}_{\mathbb{H}}$ is geometrically starlike with respect to* 0 *if and only if*

$$\Re e \left\{ q^{-1} \frac{\langle f(q) | \partial_S f(q) \rangle}{\langle \partial_C f(q) | \partial_S f(q) \rangle} \right\} \geq 0 \tag{2}$$

for any $q \in B(0,1) \setminus \{0\}$.

DEFINITION 1.7. *Assume f is a slice regular function in the unit ball of $\mathbb{H}$ such that $f(0)=0$ and $\partial_C f(q) \neq 0$. Then we say that f is* algebraically starlike *if and only if*

$$\Re e \left(q^{-1} f(q) * [\partial_C f(q)]^{-*} \right) \geq 0.$$

DEFINITION 1.8. *A slice regular function f is* algebraically convex *in the unit ball $B(0,1)$ of $\mathbb{H}$, if and only if $f \in \mathcal{S}_{\mathbb{H}}$ and*

$$\Re e \left(\partial_C (q \partial_C f)(q) * [\partial_C f(q)]^{-*} \right) > 0.$$

We'll use the following notations

$$\mathcal{GS}^*_{\mathbb{H}} := \{ f \in \mathcal{S}_{\mathbb{H}}, \text{geometrically starlike with respect to } 0 \};$$

$$\mathcal{AS}^*_{\mathbb{H}} := \{ f \in \mathcal{S}_{\mathbb{H}}, \text{algebraically starlike} \};$$

$$\mathcal{C}^*_{\mathbb{H}} := \{ f \in \mathcal{S}_{\mathbb{H}}, \text{algebraically convex} \}.$$

REMARK 1.9. One can equivalently formulate the conditions of geometric and algebraic starlikeness/convexity in terms of real partial derivatives of f since, as stated in [6, Paragraph 8.4], the following relations hold

$$\frac{\partial f}{\partial x_0}(q_0) = \partial_C f(q_0) \quad \frac{\partial f}{\partial x_1}(q_0) = I \partial_C f(q_0)$$

$$\frac{\partial f}{\partial x_2}(q_0) = J \partial_S f(q_0) \quad \frac{\partial f}{\partial x_3}(q_0) = K \partial_S f(q_0)$$

where x_0, x_1, x_2, x_3 are the real coordinates corresponding to the frame

$$(1,\ I,\ J,\ IJ := K),$$

with $I, J \in \mathbb{S}_{\mathbb{H}}$ such that $I \perp J$ as unitary vectors in $\mathbb{R}^3$ and $q_0 \in \mathbb{C}_I$.

Notice that, according to these notation, one can actually show that for a slice regular function f it turns out that the real gradient of f at q_0 can be expressed in the following way

$$\mathrm{grad}_{\mathbb{R}} f(q_0) = (\partial_C f(q_0), I\partial_C f(q_0), J\partial_S f(q_0), K\partial_S f(q_0)).$$

Remark 1.10. It is not difficult to observe that for slice preserving functions the condition of geometrically starlikeness is equivalent to the one of algebraic starlikeness. Indeed, the Cullen and Spherical derivatives of a slice preserving f are also slice preserving functions so that the complex Jacobian of f at q_0 is diagonal. It then turns out that $(Df_{q_0})^{-1}[f(q_0)] = f(q_0) \cdot \partial_C f(q_0)^{-1}$, and therefore $0 < \Re e[qf(q) * \partial_C f(q)^{-*}] = \Re e[qf(q) \cdot \partial_C f(q)^{-1}]$, since for slice preserving functions the $*$–product coincides with the usual one (see [6, Lemma 1.30]).

The previous fact is also valid for slice regular functions that preserve only one slice and it will follow from the One-Slice Preserved Principle 3.3.

Closely related to the class $\mathcal{S}_{\mathbb{H}}$ is the class of Carathéodory functions

$$\mathcal{P}_{\mathbb{H}} := \{f : B(0,1) \to \mathbb{H},\ f \text{ slice regular and such that } \Re e f > 0; f(0) = 1\}.$$

In this paper we prove several results for the classes of functions just introduced; the results concerning the estimates of the moduli of the coefficients of the power series expansions are obtained separately but summarized in the following

Theorem 1.11. *If $f \in \mathcal{P}_{\mathbb{H}}$ and $f(q) = 1 + \sum_{n \geq 1} q^n a_n$, then $|a_n| \leq 2$ for any $n \in \mathbb{N}$.*

If $f \in \mathcal{AS}^*_{\mathbb{H}}$ and $f(q) = q + \sum_{n \geq 2} q^n a_n$, then $|a_n| \leq n$ for any $n \in \mathbb{N}$.

If $f \in \mathcal{C}^*_{\mathbb{H}}$ and $f(z) = q + \sum_{n \geq 2} q^n a_n$, then $|a_n| \leq 1$ for any $n \in \mathbb{N}$.

All the above-given inequalities are sharp.

The first statement of Theorem 1.11 can be found also in [11, Theorem 3]. Some other possibly related results, such as generalizations of One-Quarter Theorem or Area Theorem from the complex holomorphic to the quaternionic setting will be investigated in another paper.

First we want to investigate some relations between the different notions of algebraic and geometric starlikeness.

2. Relation between algebraic starlikeness and geometric starlikeness

In the particular case in which f preserves one slice we state the following

Proposition 2.1. *Let* $f \in \mathcal{GS}_{\mathbb{H}}$ *be such that it preserves the slice* $\mathbb{C}_I$. *Then* f *is algebraically starlike on the preserved slice* $\mathbb{C}_I$.

This is trivial since the fact that the slice is preserved implies that the restriction of the function on the slice is holomorphic and geometrically starlike and thus algebraically starlike, since these two concepts agree for holomorphic functions.

We also give the following result again for functions which preserve a slice. In what follows Ω denotes an axially symmetric domain which contains 0.

Proposition 2.2. *Let* $f : \Omega \to \mathbb{H}$ *be slice regular and such that* $f(0) = 0$. *If*

$$\Re e(q^{-1} f(q) * \partial_C f(q)^{-*}) \geq 0$$

for any q *in a prescribed slice, say* $\mathbb{C}_I$, *then it turns out*

$$\Re e(q^{-1} f(q) * \partial_C f(q)^{-*}) \geq 0$$

for any q *in* Ω.

Proof. Since f preserves the slice $\mathbb{C}_I$ then both $\partial_C f$ and $q^{-1}\partial_C f(q) * f^{-*}(q)$ preserve the same slice $\mathbb{C}_I$. For $q = x + Iy$ we can write

$$q^{-1}\partial_C f(q) * f^{-*}(q) = \sum_n q^n b_n$$

where $b_n \in \mathbb{C}_I$. We can write b_n as $\gamma_n + I\delta_n$ for all n, with γ_n and $\delta_n \in \mathbb{R}$. Since $q \in \mathbb{C}_I$ we get that

$$q^n = \alpha_n + I\beta_n; \;\; \alpha_n, \beta_n \in \mathbb{R}.$$

Thus,

$$\sum_n q^n b_n = \sum_n (\alpha_n + I\beta_n)(\gamma_n + I\delta_n).$$

The real part of $q^{-1}\partial_C f(q) * f^{-*}(q)$ is then $\sum_n (\alpha_n\gamma_n - \beta_n\delta_n)$ and it is positive since f is algebraically starlike on $\mathbb{C}_I$.

We now take $\widetilde{q} = \widetilde{x} + J\widetilde{y}$, and consider the corresponding $\widehat{q} = \widetilde{x} + I\widetilde{y}$ in $\mathbb{S}_{\widetilde{q}} \cap \mathbb{C}_I$. Now $\widetilde{q}^n = (\widetilde{\alpha}_n + J\widetilde{\beta}_n)$ and $\sum \widetilde{q}^n b_n = \sum(\widetilde{\alpha}_n + J\widetilde{\beta}_n)(\gamma_n + I\delta_n)$. Its real part is given by

$$\sum (\widetilde{\alpha}_n\gamma_n - \langle J, I\rangle_{\mathbb{R}} \widetilde{\beta}_n\delta_n),$$

since, if I and J are considered as unitary vectors in $\mathbb{R}^3$, it follows that

$$JI = -\langle J, I\rangle_{\mathbb{R}} + J \wedge I.$$

From $|\langle J, I\rangle_{\mathbb{R}}| \leq 1$, one concludes

$$\sum(\widetilde{\alpha}_n\gamma_n - \langle J, I\rangle_{\mathbb{R}}\widetilde{\beta}_n\delta_n) \geq \sum(\widetilde{\alpha}_n\gamma_n - \widetilde{\beta}_n\delta_n)$$

which is positive since it is equal to the real part of $\sum \widehat{q}^n b_n$. □

Combining Propositions 2.1 and 2.2 we then conclude that a geometrically starlike function f on a preserved slice is algebraically starlike in the entire unit ball.

Starting from the definition of algebraically starlikeness one observes that, if f is slice regular, also the function $q \mapsto q^{-1}f(q) * \partial_C f(q)^{-*}$ is slice regular and then the maximum principle on the real part of it ([6, Theorem 7.13]) holds and implies that for $f \in \mathcal{AS}^*_{\mathbb{H}}$ necessarily $\Re e\ [q^{-1}f(q) * \partial_C f(q)^{-*}] > 0$.

We can also assert that a function f is in $\mathcal{AS}^*_{\mathbb{H}}$ if and only if

$$\Re e\ [(q\partial_C f(q)) * f(q)^{-*}] > 0. \tag{3}$$

Furthermore, if $q_0 = x_0 + I_0 y_0 \in \Omega$, $J \perp I_0$ and $q \in \mathbb{C}_{I_0}$, thanks to the Splitting Lemma (see [6, Lemma 1.3]), one can write $f_{I_0}(q) = F_1(q) + F_2(q)J$, $\partial_C f_{|I_0}(q) = F_1'(q) + F_2'(q)J$ and, similarly, $R_{q_0} f_{|I_0}(q) = R_{1,q_0}(q) + R_{2,q_0}(q)J$, $\partial_C R_{q_0} f_{|I_0}(q) = R_{1,q_0}'(q) + R_{2,q_0}'(q)J$. Since

$$(q - q_0) * R_{q_0} f(q) = f(q) - f(q_0), \tag{4}$$

if $q \in \mathbb{C}_{I_0}$, then

$$(q - q_0)R_{1,q_0}(q) = F_1(q) - F_1(q_0)\,, \tag{5}$$

$$(q - q_0)R_{2,q_0}(q) = F_2(q) - F_2(q_0)\,, \tag{6}$$

for the identity principle of holomorphic functions. Once the Leibnitz rule is applied to (4) (see [6]), it follows that

$$R_{q_0} f(q) + (q - q_0) * \partial_C R_{q_0} f(q) = \partial_C f(q);$$

on the other hand, from the above given considerations, it follows that

$$F_1'(q) = R_{1,q_0}(q) + (q - q_0) \cdot R_{1,q_0}'(q)\,, \tag{7}$$

$$F_2'(q) = R_{2,q_0}(q) + (q - q_0) \cdot R_{2,q_0}'(q)\,. \tag{8}$$

Furthermore, using (1.16) and the representation formulae (1.19) and (1.20) as in [6, Section 1], we get

$$[(\partial_C f(q))^{-*}]_{|I_0} = [F_1'(q)\overline{F_1'(\overline{q})} + F_2'(q)\overline{F_2'(\overline{q})}]^{-1} \cdot [\overline{F_1'(\overline{q})} - F_2'(q)J]$$

and so the real part of $\left\{[q^{-1}f(q) * \partial_C f(q)^{-*}]_{|I_0}\right\}$ is

$$\Re e\left\{q^{-1}[F_1'(q)\overline{F_1'(\overline{q})} + F_2'(q)\overline{F_2'(\overline{q})}]^{-1} \cdot [F_1(q)\overline{F_1'(\overline{q})} + F_2(q)\overline{F_2'(\overline{q})}]\right\}.$$

Now, from (5), (6), (7) and (8), one obtains

$$\begin{aligned}
&\Re e\left\{q^{-1}[F_1'(q)\overline{F_1'(\overline{q})}+F_2'(q)\overline{F_2'(\overline{q})}]^{-1}\cdot[F_1(q)\overline{F_1'(\overline{q})}+F_2(q)\overline{F_2'(\overline{q})}]\right\}=\\
&=\Re e\left\{q^{-1}\left\{[R_{1,q_0}(q)+(q-q_0)R_{1,q_0}'(q)]\cdot\overline{[R_{1,q_0}(\overline{q})+(\overline{q}-q_0)R_{1,q_0}'(\overline{q})]}+\right.\right.\\
&\quad\left.+[R_{2,q_0}(q)+(q-q_0)R_{2,q_0}'(q)]\cdot\overline{[R_{2,q_0}(\overline{q})+(\overline{q}-q_0)R_{2,q_0}'(\overline{q})]}\right\}^{-1}\\
&\quad\left\{[(q-q_0)R_{1,q_0}(q)+F_1(q_0)]\cdot\overline{[R_{1,q_0}(\overline{q})+(\overline{q}-q_0)R_{1,q_0}'(\overline{q})]}+\right.\\
&\quad\left.\left.+[(q-q_0)R_{2,q_0}(q)+F_2(q_0)]\cdot\overline{[R_{2,q_0}(\overline{q})+(\overline{q}-q_0)R_{2,q_0}'(\overline{q})]}\right\}\right\}.
\end{aligned}\tag{9}$$

When $q=q_0$, the previous computations yield to

$$\begin{aligned}
&\Re e\left\{q_0^{-1}\left\{R_{1,q_0}(q_0)\cdot\overline{[R_{1,q_0}(\overline{q_0})+(\overline{q_0}-q_0)R_{1,q_0}'(\overline{q_0})]}+\right.\right.\\
&\qquad\left.+R_{2,q_0}(q_0)\cdot\overline{[R_{2,q_0}(\overline{q_0})+(\overline{q_0}-q_0)R_{2,q_0}'(\overline{q_0})]}\right\}^{-1}\\
&\qquad\left\{F_1(q_0)\cdot\overline{[R_{1,q_0}(\overline{q_0})+(\overline{q_0}-q_0)R_{1,q_0}'(\overline{q_0})]}+\right.\\
&\qquad\left.\left.+F_2(q_0)\cdot\overline{[R_{2,q_0}(\overline{q_0})+(\overline{q_0}-q_0)R_{2,q_0}'(\overline{q_0})]}\right\}\right\}.
\end{aligned}\tag{10}$$

If, using the Splitting Lemma with respect to the choice of J orthogonal to $\mathbb{C}_{I_0}$, we call $\mathcal{R}=(\mathcal{R}_1,\mathcal{R}_2)$ where $\mathcal{R}_1(q_0)=(\overline{q_0}-q_0)R_{1,q_0}'(\overline{q_0})$ and $\mathcal{R}_2(q_0)=(\overline{q_0}-q_0)R_{2,q_0}'(\overline{q_0})$, then the previous expression becomes

$$\Re e(q^{-1}\partial_C f*f(q)^{-*}|_{q=q_0})=\Re e\left\{q_0^{-1}\frac{\langle f(q_0)|\partial_S f(q_0)\rangle+\langle f(q_0)|\mathcal{R}(q_0)\rangle}{\langle\partial_C f(q_0)|\partial_S f(q_0)\rangle+\langle\partial_C f(q_0)|\mathcal{R}(q_0)\rangle}\right\}.$$

Remark 2.3. Notice that if q_0 is real then $\mathcal{R}_1(q_0)=\mathcal{R}_2(q_0)=0$. Hence for a continuity argument applied to the real part of a slice regular function we can say that an algebraically starlike function is geometrically starlike in an axially symmetric and slice neighborhood of the real axis.

From (7) and (8) we have that

$$\mathcal{R}(q_0)=\partial_C f(\overline{q_0})-\partial_S f(q_0)\tag{11}$$

and

$$\mathcal{R}(\overline{q_0})=\partial_C f(q_0)-\partial_S f(\overline{q_0}).\tag{12}$$

We recall the following result proved in [8]

PROPOSITION 2.4. *Given an axially symmetric domain* $\Omega \subset \mathbb{H}$*, let* $f : \Omega \to \mathbb{H}$ *in* $\mathcal{S}_{\mathbb{H}}$*, then the spherical derivative is constant on any sphere* $\mathbb{S}_q = \{x + Jy\ ,\ J \in \mathbb{S}_{\mathbb{H}}\}$ *with* $q = x + I_q y$*.*

Notice that, in general, the Cullen derivative is not constant over a sphere $\mathbb{S}_q$. and thus so is $\mathcal{R}$.

Furthermore, thanks to Proposition 2.4 we have $\partial_S f(\overline{q_0}) = \partial_S f(q_0)$, therefore, whenever f is in $\mathcal{GS}^*_{\mathbb{H}}$ and its Spherical and Cullen derivatives at each point differ of a sufficiently small amount, then f is also algebraically starlike.

For instance, this is the case when $\max\{||\partial_C f(q)||, ||\partial_S f(q)||\}$ is controlled by a sufficiently small but positive ε.

We would like to find a manageable expression for the value of $\mathcal{R}$ at q_0 (or $\overline{q_0}$). From (4) we recall that

$$\begin{aligned} f(q) - f(q_0) &= (q - q_0) * [R_{q_0} f(q_0) + (q - q_0) * R_{q_0} R_{q_0} f(q)] = \\ &= (q - q_0) * R_{q_0} f(q_0) + (q - q_0)^{*2} * R_{q_0} R_{q_0} f(q). \end{aligned}$$

We compute the previous formula in $q = \overline{q_0}$.

$$\begin{aligned} f(\overline{q_0}) - f(q_0) &= (\overline{q_0} - q_0) * R_{q_0} f(q_0) + [(q - q_0)^{*2} * R_{q_0} R_{q_0} f(q)]_{|q=\overline{q_0}} = \\ &= -2I\Im m(q_0)\partial_C f(q_0)) + [(q - q_0)^{*2} * R_{q_0} R_{q_0} f(q)]_{|q=\overline{q_0}}. \end{aligned}$$

Therefore

$$\partial_S f(q_0) = \partial_C f(q_0) - [2I\Im m(q_0)]^{-1}[(q - q_0)^{*2} * R_{q_0} R_{q_0} f(q)]_{|q=\overline{q_0}}$$

and from (11) and (12) we get

$$-[2I\Im m(q_0)]^{-1}[(q - q_0)^{*2} * R_{q_0} R_{q_0} f(q)]_{|q=\overline{q_0}} = \mathcal{R}(\overline{q_0}).$$

Notice that

$$0 \le |(f * g)(x_0 + I_0 y_0)| \le |f(x_0 + I_0 y_0)| \cdot \max_{I \in \mathbb{S}_{\mathbb{H}}} |g(x_0 + I y_0)|$$

and hence

$$\max_{I \in \mathbb{S}_{\mathbb{H}}} |(f * g)(x_0 + I y_0)| \le \max_{I \in \mathbb{S}_{\mathbb{H}}} |f(x_0 + I y_0)| \cdot \max_{I \in \mathbb{S}_{\mathbb{H}}} |g(x_0 + I y_0)|.$$

We can find an estimate for $|\mathcal{R}|$ and $|R_{q_0} R_{q_0} f(q)|$, using the Cauchy estimates on the slice $\mathbb{C}_{I_0}$ containing q_0 (see [6, p. 131]). In the sequel we will denote any quaternion $q = x + I_0 y \in \mathbb{C}_{I_0}$ as the complex number $z = x + iy$, in particular q_0 will be replaced by z_0 and U_{I_0} will be the open ball in $\mathbb{C}_{I_0}$

centered in z_0 with a suitable radius r such that U_{I_0} is in $B(0,1)$. Thus $\partial U_{I_0} = \{z \in \mathbb{C}_{I_0} : |z - z_0| = r\}$.

$$\begin{aligned} R_{z_0}R_{z_0}f(z) &= \frac{1}{2\pi I_0}\int_{\partial U_{I_0}} \frac{R_{z_0}R_{z_0}f(s)ds}{s-z} = \\ &= \frac{1}{2\pi I_0}\int_{\partial U_{I_0}} \frac{1}{s-z}\left[\frac{f(s)-f(z_0)}{(s-z_0)^2} - \frac{R_{z_0}f(z_0)}{(s-z_0)}\right] ds = \\ &= \frac{1}{2\pi I_0}\int_{\partial U_{I_0}} \frac{1}{s-z}\left[\frac{R_{z_0}f(s)-R_{z_0}f(z_0)}{s-z_0}\right] ds = \\ &= \frac{1}{2\pi I_0}\int_{\partial U_{I_0}} \frac{1}{s-z}\left[\frac{R_{z_0}f(s)-f'(z_0)}{s-z_0}\right] ds = \\ &= \frac{1}{2\pi I_0}\int_{\partial U_{I_0}} \left[\frac{f(s)-f(z_0)-f'(z_0)(s-z_0)}{(s-z)(s-z_0)^2}\right] ds. \end{aligned}$$

So if we call $f(s) - f(z_0) - f'(z_0)(s - z_0) = R_2$ and compute the previous expression in $\overline{z_0}$ we get

$$|\mathcal{R}| \leq \frac{r \max_{\partial U_{I_0}} |R_2|}{2|2y_0 - r||y_0|},$$

where $y_0 = \Im m(z_0)$.

We can find another estimate of $|R_{q_0}R_{q_0}f(q)|$, using the Cauchy estimates on the slice $\mathbb{C}_{I_0}$ containing q_0. With the same notations as above thus we have that

$$\begin{aligned} R_{z_0}R_{z_0}f(z) &= \frac{1}{2\pi I_0}\int_{\partial U_{I_0}} \frac{1}{z-z_0}\cdot\left[\frac{1}{(s-z)} - \frac{1}{(s-z_0)}\right] R_{z_0}f(s)ds = \\ &= \frac{1}{2\pi I_0}\int_{\partial U_{I_0}} \frac{1}{z-z_0}\cdot\left[\frac{ds(f(s)-f(z_0))}{(s-z)(s-z_0)} - \frac{ds(f(s)-f(z_0))}{(s-z_0)^2}\right] = \\ &= \frac{1}{2\pi I_0}\int_{\partial U_{I_0}} \left[\frac{(f(s)-f(z_0))ds}{(s-z)(s-z_0)^2}\right]. \end{aligned}$$

So if we compute the previous expression in $\overline{z_0}$ we get

$$|R_{z_0}R_{z_0}f(\overline{z_0})| \leq \frac{\max_{\partial U_{I_0}} 2|f|}{r|2y_0 - r|},$$

where $y_0 = \Im m(z_0)$.

Both these inequalities lead to conclude that in an axially symmetric neighborhood Ω' of the real axis in Ω one can control the modulus of $\mathcal{R}$ and therefore assert that if f is geometrically starlike in Ω then f is also algebraically starlike in Ω'.

3. One-Slice Preserved Principle and related results

For *typically real* functions, i.e. (not necessarily) injective slice regular functions with real coefficients it is easy to prove that the Bieberbach Conjecture holds (we refer the interested reader to [2]). More precisely, a typically real quaternionic function is a slice regular function (in $B(0,1)$) such that $f(q) \in \mathbb{R}$ if and only if $q \in \mathbb{R}$ with $f(0) = 0$ and $f'(0) = 1$. We'll denote this class of functions with $\mathcal{T}_{\mathbb{H}}$. We also observe that for a typically real quaternionic function

$$f(q) = \sum_n q^n a_n$$

it turns out that $f(\overline{q}) = \overline{f(q)}$. We then consider the sets

$$\mathcal{S}_{\mathbb{H}}^{\mathbb{R}} := \left\{ f \in \mathcal{S}_{\mathbb{H}} \ : \ f(\overline{q}) = \overline{f(q)} \right\}$$

and observe that

$$\mathcal{S}_{\mathbb{H}}^{\mathbb{R}} = \mathcal{S}_{\mathbb{H}} \cap \mathcal{T}_{\mathbb{H}}.$$

Clearly, if $f \in \mathcal{T}_{\mathbb{H}}$, then $\partial_S f(q)$ is real. Viceversa we can prove

Proposition 3.1. *Given an axially symmetric domain $\Omega \subset \mathbb{H}$, let $f : \Omega \to \mathbb{H}$ be slice regular and have real spherical derivative. If f preserves one slice $\mathbb{C}_I$ and $f(0) = 0$, then f is typically real.*

Proof. From the hypothesis of preserving the slice $\mathbb{C}_I$, it follows that the coefficients of the power series expansion of f are all in $\mathbb{C}_I$; furthermore, if $q = x + Iy \in \mathbb{C}_I$, $f(q) - f(\overline{q}) \in \mathbb{C}_I$ and $\overline{f(\overline{q})} - \overline{f(q)} \in \mathbb{C}_I$, since $\partial_S f(q) \in \mathbb{R}$. In particular, it turns out that

$$f(q) - f(\overline{q}) = \sum_n (q^n - \overline{q}^n) a_n = 2Iy\partial_S f(q) = \overline{f(\overline{q})} - \overline{f(q)} = \sum_n \overline{a_n}(q^n - \overline{q}^n)$$

and then $a_n = \overline{a_n}$ for any $n \geq 1$. Finally $a_0 = f(0) = 0$, by assumption. □

We also have

Proposition 3.2. *Let $f(q) = q + \sum_{n\geq 2} q^n a_n$ belong to $\mathcal{T}_{\mathbb{H}}$, then $|a_n| \leq n$ for any n.*

Indeed the proof of the inequalities $|a_n| \leq n$ in the case of a slice regular quaternionic function $f \in \mathcal{T}_{\mathbb{H}}$ can be repeated *verbatim* as in the holomorphic case (a first version of this fact for $\mathcal{S}_{\mathbb{H}}^{\mathbb{R}}$ appeared in [5]). However for this class of functions the Bieberbach conjecture can be proved true by a direct application of the following very general One-Slice-Preserved Principle

Proposition 3.3 (One-Slice-Preserved Principle). *Let Ω be a slice domain. Assume $f : \Omega \to \mathbb{H}$ is a slice regular function with the additional property of preserving one slice $\mathbb{C}_I$, i.e. such that $f(\mathbb{C}_I \cap \Omega) \subseteq \mathbb{C}_I$, with $I \in \mathbb{S}_{\mathbb{H}}$ an imaginary unit. Then all the properties and results concerning the coefficients of the power expansion of f are precisely the same as if f is considered a complex holomorphic function $f : \mathbb{C}_I \cap \Omega \to \mathbb{C}_I$.*

Proof. This is so since, for the Splitting Lemma (see e.g. [7]), the restriction of any slice regular function f to $\mathbb{C}_I \cap \Omega$ can be split as $F + GJ$ with $J \perp I$, $J^2 = -1$ and F and $F, G : \mathbb{C}_I \cap \Omega \to \mathbb{C}_I$ holomorphic in the variable $z = x + Iy$. The assumption $f(\mathbb{C}_I \cap \Omega) \subseteq \mathbb{C}_I$ then implies that $G \equiv 0$ and so the restriction of f along $\mathbb{C}_I \cap \Omega$ can be regarded as the holomorphic function F; furthermore, the identity principle and the uniqueness of power expansion of a holomorphic function, guarantees that the coefficients of f are exactly the same of F. $\square$

Remark 3.4. We want to observe here that if a slice regular function $f : \Omega \to \mathbb{H}$ takes a slice $\mathbb{C}_I \cap \Omega$ into another slice $\mathbb{C}_J$, then f is necessarily constant, and thus there exists a $K \in \mathbb{S}_{\mathbb{H}}$ such that $f_{|\mathbb{C}_K} \subset \mathbb{C}_K$. Moreover if f preserves two slices $\Omega \cap \mathbb{C}_I$ and $\Omega \cap \mathbb{C}_J$ with $I \neq \pm J$ then f preserves each slice (see e.g. [7]) and therefore it is slice-preserving and all the coefficients of its power expansion are real.

Similarly to the complex holomorphic case, we introduce the following class of quaternionic functions

$$\mathcal{P}_{\mathbb{H}}^{\mathbb{R}} := \left\{ f \in \mathcal{P}_{\mathbb{H}} \ : \ f(\overline{q}) = \overline{f(q)} \right\}$$

and prove the remarkable relation between typically real quaternionic functions and Carathéodory quaternionic functions, (in the complex holomorphic case this result is known as *Rogosinski's Theorem*)

Lemma 3.5 (Rogosinski). *If $f \in \mathcal{T}_{\mathbb{H}}$, then*

$$\varphi_f(q) = (1 - q^2) q^{-1} f(q)$$

belongs to $\mathcal{P}_{\mathbb{H}}^{\mathbb{R}}$; conversely if $\varphi \in \mathcal{P}_{\mathbb{H}}^{\mathbb{R}}$ then

$$f_\varphi(q) = (1 - q^2)^{-*} q \varphi(q)$$

belongs to $\mathcal{T}_{\mathbb{H}}$.

Proof. Both the functions φ_f and f_φ are slice-preserving thus, along each slice $B_I := B(0,1) \cap \mathbb{C}_I$, their restrictions are holomorphic and, for the analogous result in the complex holomorphic case, they are such that respectively $\varphi_f \in \mathcal{T}_{\mathbb{H}}$ and $f_\varphi \in \mathcal{P}_{\mathbb{H}}^{\mathbb{R}}$. $\square$

We begin with a uniform estimate for the coefficients of the expansions of any function in $\mathcal{P}_{\mathbb{H}}$. This result can be regarded as the natural generalization and extension of the analogous result in the complex holomorphic case and a different proof is also given in [12].

THEOREM 3.6 (Carathéodory). *If* $\varphi \in \mathcal{P}_{\mathbb{H}}$ *and*

$$\varphi(q) = 1 + \sum_{n=1}^{+\infty} q^n c_n,$$

then $|c_n| \leq 2$ *for* $n \geq 1$*. The inequality is sharp for each* n*.*

Proof. As in [10], given $0 < r < 1$, for $\varphi \in \mathcal{P}_{\mathbb{H}}$, if I is in $\mathbb{S}_{\mathbb{H}}$ consider the integral

$$\mathcal{I}_n(r) := \frac{1}{2\pi I} \int_{q=re^{It}} [2 - q^n - q^{-n}]\ q^{-1}\varphi(q)\mathrm{d}q\,.$$

By the Residue Theorem (see [13] and [6, Proposition 6.10]), we have

$$\mathcal{I}_n(r) = 2 - c_n\,.$$

On the other hand

$$\lim_{r\to 1^-} \mathcal{I}_n(r) = \lim_{r\to 1^-} \frac{2}{\pi}\int_0^{2\pi} (2 - r^n e^{nIt} - r^{-n}e^{-nIt})\varphi(re^{It})\mathrm{d}t\,;$$

now, combining the fact that

$$\lim_{r\to 1^-}(2 - r^n e^{nIt} - r^{-n}e^{-nIt}) = \sin^2\frac{t}{2} \quad \text{and} \quad \Re e\varphi(re^{It}) > 0$$

we conclude that $\Re e(2 - c_n) \geq 0$ or $\Re e(c_n) \leq 2$.

If $\Re e\varphi(q) > 0$ we also have that $\Re e\varphi(qe^{It}) > 0$ with $0 \leq t \leq 2\pi$ and any $I \in \mathbb{S}_{\mathbb{H}}$.

Assume that $c_n = \rho_n e^{I_{c_n}\vartheta_n}$ and take $q_o \in B(0,1)$ in the same slice of c_n, hence $\Re e\varphi(q_o e^{I_{c_n}t}) > 0$ and $\varphi(q_o e^{I_{c_n}t}) = 1 + \sum_k {q_o}^k e^{I_{c_n}kt}c_k$ so that $\Re e(e^{I_{c_n}kt}c_k) \leq 2$; in particular for $k = n$ if $t = -\dfrac{\vartheta_n}{n}$ we have that $|c_n| = \rho_n \leq 2$ for any $n \geq 1$.

The Cayley transformation

$$\varphi(q) = (1+q)(1-q)^{-1} = 1 + 2q + 2q^2 + \ldots 2q^n + \ldots$$

shows that the estimate is sharp. □

4. Some results for the Bieberbach conjecture in $\mathbb{H}$

The Bieberbach conjecture for complex holomorphic schlicht functions has been investigated in several ways by many authors until, after almost seventy years, it has been finally proved by De Branges in 1985. In fact, in 1916 Bieberbach proved that if $f \in \mathcal{S}$ with $f(z) = 1 + \sum_{n\geq 2} z^n a_n$ then $|a_2| \leq 2$, and stated the conjecture that $|a_n| \leq n$ (known as the *Bieberbach conjecture*) with equality if and only if f is a rotation of the function

$$\kappa(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \ldots + nz^n + \ldots$$

called *Koebe function*.

In order to prove the conjecture, some important steps have been done when considering some special subclasses of the class of schlicht (holomorphic) functions, such as typically real functions or starlike functions.

We'll follow similar steps and prove general results in the case of some classes of quaternionic slice regular functions.

After Bieberbach conjecture was proved for the class of typically real functions, Loewner (1917) and Nevanlinna (1921) independently proved the Bieberbach conjecture for starlike functions and our aim in the present section is to extend the analogous result for starlike slice regular functions as introduced and studied in [9].

We are now in the position to prove the Bieberbach conjecture for this class of slice regular functions $\mathcal{AS}^*_{\mathbb{H}}$. A similar result has been obtained in [12, Theorem 4.3], using different techniques.

PROPOSITION 4.1. *Given $f \in \mathcal{AS}^*_{\mathbb{H}}$, whose power series expansion is*

$$f(q) = q + q^2 a_2 + \ldots + q^n a_n + \ldots$$

the following inequalities hold

$$|a_n| \leq n \quad \text{for any } n.$$

These inequalities are sharp.

Proof. Assume $f \in \mathcal{AS}^*_{\mathbb{H}}$, so we can write $f(q) = q + q^2 a_2 + \ldots + q^n a_n + \ldots$. Then

$$q\partial_C f(q) * f(q)^{-*} = 1 + qc_1 + q^2 c_2 + \ldots + q^n c_n + \ldots$$

and $|c_n| \leq 2$ for $n \geq 1$. Hence

$$q\partial_C f(q) = \sum_{n=1}^{+\infty} nq^n a_n = q\partial_C f(q) * f(q)^{-*} * f(q) =$$

$$= \left(1 + \sum_{n=1}^{+\infty} q^n c_n\right) * \left(\sum_{n=1}^{+\infty} q^n a_n\right) = \sum_{n=1}^{+\infty} q^n s_n$$

with $s_n = a_n + c_1 a_{n-1} + \ldots + c_{n-1}$. Therefore

$$(n-1)|a_n| = |c_1 a_{n-1} + \ldots + c_{n-1}| \leq 2(|a_{n-1}| + \ldots + |a_2| + 1).$$

For $n = 2$, this yealds $|a_2| \leq 2$; so $2|a_3| \leq 2(|a_2| + 1) \leq 6$ or $|a_3| \leq 3$. Assume that $|a_k| \leq k$ for $1 \leq k \leq m$. From

$$(n-1)|a_n| = |c_1 a_{n-1} + \ldots + c_{n-1}| \leq 2(|a_{n-1}| + \ldots + |a_2| + 1)$$

we have

$$m|a_{m+1}| \leq 2(m + (m-1) + \ldots + 2 + 1) = 2m(m+1)/2$$

or $|a_{m+1}| \leq m + 1$.

The Koebe function $\kappa(q) = q + 2q^2 + 3q^3 + \ldots + nq^n + \ldots$ and all its distorted versions, namely $q \mapsto \sum_n nq^n e^{I_n \theta_n}$ with $I_n \in \mathbb{S}_{\mathbb{H}}$, $\theta_n \in \mathbb{R}$ for all n, guarantee the sharpness of the inequalities. □

Finally, for algebraically convex functions we have the following

Proposition 4.2. *Let g be an algebraically convex function whose power series expansion at 0 is $\sum_n q^n a_n$ then*

$$|a_n| \leq 1 \; \forall n \in \mathbb{N}.$$

These inequalities are sharp.

Proof. Since $q\partial_C g(q) = \sum_n nq^n a_n$ is algebraically starlike it follows that $|na_n| \leq n$ and so $|a_n| \leq 1$ for any n in $\mathbb{N}$. The function $g(q) = q + q^2 + \cdots + q^n + \cdots$ shows that these estimates are sharp. □

For geometrically starlike slice regular functions, that preserve one slice $\mathbb{C}_I$, the Bieberbach conjecture is a particular case of the One-Slice-Preserved Principle.

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