



Generic identifiability of pairs of ternary forms

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Abstract

We prove that two general ternary forms of degrees $c \leq d$ are simultaneously identifiable only in the classical cases $(c, d) = (2, 2)$ and $(c, d) = (2, 3)$. We translate the problem into the study of a certain linear system on a projective bundle on the plane, and we apply techniques from projective and birational geometry to prove that the associated map is not birational.

Keywords Ternary forms · Identifiability · Projective bundle · Secant map

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1 Introduction

Many questions in mathematics are called “Waring problem”, after the name of the eighteenth century English mathematician Edward Waring. He was interested in decomposing a natural number as a sum of powers. Since then, Waring problems have to do with additive decompositions of different mathematical objects. For instance, given a degree d form, or equivalently, a homogeneous polynomial f , we can decompose

$$f = \ell_1^d + \cdots + \ell_k^d$$

as a sum of powers of linear forms. In a similar way, one can decompose a tensor as a sum of rank one tensors. Waring decompositions raise a huge interest in many different areas, as well illustrated in the textbook [14, Section 1.3].

There are several different questions that we may ask. For instance, what is the smallest possible number of summands, called the *Waring rank* of f , or how to compute all the decompositions. When they are infinitely many, one can study the variety parametrizing

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them. When they are finitely many, one may ask to bound their number. All these questions are widely open in their generality. One interesting problem is to understand when the decomposition is unique. In this case, we say that f is k -*identifiable*. Identifiability is a desirable property with many applications, as it gives a canonical form for f . Examples range from Signal processing to Complexity theory, from Phylogenetics to Algebraic statistics. A complete list of applications would be far too long, so we refer the reader to [14, Section 1.3].

When we work over the complex field, there is a dense open subset of the space of polynomials where all elements have the same rank—called the *generic rank*—and the same number of decompositions. In this paper we will use the words “generic” or “general” for properties that hold almost everywhere—more precisely, on a dense open subset. Classically, the problem was to classify all pairs (n, d) such that the general $f \in \mathbb{C}[x_0, \dots, x_n]_d$ is identifiable. General identifiability is expected to be a rare phenomenon. A few classical cases were known to be generically identifiable since the work of Hilbert and Sylvester. It took more than a century to get new results in this direction [17, 18], and the full classification has been completed in [11]. It turns out that there are infinitely many generically identifiable cases for binary forms, while there are only two sporadic cases for polynomials in three or more variables.

In this paper we focus on the version of the Waring problem concerning pairs of polynomials. It is a classical result that two general quadratic forms $f, g \in \mathbb{C}[x_0, \dots, x_n]_2$ can be simultaneously diagonalized. More precisely, there exist linear forms $\ell_1, \dots, \ell_{n+1}$ and scalars $\lambda_1, \dots, \lambda_{n+1}$ such that

$$\begin{cases} f = \ell_1^2 + \dots + \ell_{n+1}^2 \\ g = \lambda_1 \ell_1^2 + \dots + \lambda_{n+1} \ell_{n+1}^2. \end{cases} \tag{1}$$

A canonical form (1) with $n + 1$ summands is unique if and only if the discriminant of the pencil $\langle f, g \rangle$ does not vanish, hence the general pair of quadratic forms has only one simultaneous diagonalization.

We generalize decomposition (1) to pairs of forms of any degrees. For symmetry reasons, it is convenient not to distinguish f from g , so we will allow coefficients in the decomposition of f as well.

Definition 1 Let $f \in \mathbb{C}[x_0, \dots, x_n]_c$ and $g \in \mathbb{C}[x_0, \dots, x_n]_d$ be two homogeneous polynomials. A *Waring decomposition* of (f, g) consists of linear forms $\ell_1, \dots, \ell_k \in \mathbb{C}[x_0, \dots, x_n]_1$ and scalars $\lambda_1, \dots, \lambda_k, \mu_1, \dots, \mu_k \in \mathbb{C}$ such that

$$\begin{cases} f = \lambda_1 \ell_1^c + \dots + \lambda_k \ell_k^c \\ g = \mu_1 \ell_1^d + \dots + \mu_k \ell_k^d. \end{cases} \tag{2}$$

This kind of decompositions are also called *simultaneous decompositions*. Due to the presence of the scalars $\lambda_1, \dots, \lambda_k$ and μ_1, \dots, μ_k , each linear form depends essentially only on n conditions, so the we can regard (2) as a polynomial system with $\binom{c+n}{n} + \binom{d+n}{n}$ equations—given by the data f and g —and $k(n+2)$ unknowns—namely, the scalars $\lambda_1, \dots, \lambda_k, \mu_1, \dots, \mu_k$ and the linear forms ℓ_1, \dots, ℓ_k . We consider two decompositions of (f, g) to be equal if they differ just by the order of the k summands. The *Waring rank*, or *simultaneous rank* of (f, g) is the minimum number k such that there exists a simultaneous

decomposition (2) with k summands. A pair is k -simultaneously identifiable, or simply k -identifiable if it admits a unique simultaneous decomposition with k summands.

This problem fascinates mathematicians since a long time ago. In [16], London proved that the rank of two general ternary cubics is 6, instead of the expected number 5. Later, Scorza described London's result from a different perspective in [21]—see [5, Theorem 4.1] for a modern reference. In [23], Terracini computed the simultaneous rank of two general ternary forms of the same degree. As described in [2, Section 2.2], we can rephrase these results in modern language: if we call $\text{SV}_{1 \times n}^{1,d}$ the Segre-Veronese embedding of $\mathbb{P}^1 \times \mathbb{P}^n$ via the complete linear system of divisors of bidegree $(1, d)$, then London proved that $\text{SV}_{1 \times 2}^{1,3}$ is 5-defective, while Terracini showed that $\text{SV}_{1 \times 2}^{1,d}$ is not defective for $d \neq 3$. We had to wait until the work [3] to have a classification of all defective $\text{SV}_{1 \times n}^{1,d}$, so now we know the rank of two general forms of the same degree in any number of variables. However, when the two degrees are different, the problem of computing the rank is not yet solved.

Papers like [9, 13, 15] use simultaneous diagonalization of symmetric matrices to bound the rank of a third-order symmetric tensor. We believe that simultaneous decompositions like (2) could be applied in a similar way to study the rank of higher-order symmetric tensors.

In this paper we focus on identifiability. The guiding problem is the classification of all triples (n, c, d) such that the general pair of forms of degrees c and d in $n + 1$ variables is identifiable. In [7, Section 5], Ciliberto and Russo solve the case $n = 1$ of binary forms. They work in a slightly different language and phrase their statement in terms of geometric properties of a rational normal scroll. Their result applies to tuples of binary forms, not just pairs—see also [2, Theorem 1.3]. Roughly speaking, the general pair of binary forms is identifiable, as long as the polynomial system (2) is square and d is not too large compared to c . This reminds us what happens for the Waring problem for one polynomial: while generic identifiability is rare, there are infinitely many cases for binary forms.

The situation changes when there are more than two variables. As widely expected, for $n \geq 2$ generic identifiability is very uncommon. In more than a century, mathematicians have found only two instances in which the general pair of forms is identifiable. The case (1) of two general quadrics goes back at least to Weierstrass and it is generically identifiable in any number of variables. Beside that, there is the case of a general plane conic and a cubic, studied by Roberts in [20] and revisited in [19, Theorem 10.2]. The challenge is to prove that those are the only cases, or to find new exceptional ones. In this paper we solve the problem for ternary forms.

Theorem 2 *Let c and d be positive integers such that $c \leq d$. The general pair of ternary forms of degrees c and d is identifiable if and only if $(c, d) \in \{(2, 2), (2, 3)\}$.*

In the special case $d = c + 1$, Theorem 2 has been proven in [2, Theorem 5.1]. Despite the algebraic statement, our approach relies on projective and birational geometry. In Sect. 3, we underline the tight connection between decompositions and secant varieties and we translate the problem into a question about the degree of a certain rational map. Namely, the set of decompositions of a pair (f, g) is the fiber of the secant map of the projective bundle $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(c) \oplus \mathcal{O}_{\mathbb{P}^2}(d))$. In order to disprove identifiability, we show that the map is not birational. As in [2, Section 5], the first step is to bound the degree of such a map with the degree of a certain linear projection of X . Then we degenerate the associated

linear system and we restrict it to a suitable subvariety to prove that such a map cannot be birational.

When performing this kind of degenerations, it is common to bump into some arithmetic obstructions. We overcome this obstacle by distinguishing two cases and give two different arguments, in Sects. 4 and 5 respectively. For this reason the proof of Theorem 2 is split in two parts, namely Propositions 20 and 25.

2 Geometric setup

Secant varieties are a classical construction that dates back to the Italian school of algebraic geometry at the end of nineteenth century. In this section we recall the definition of the secants of a variety V embedded in some projective space \mathbb{P}^N . We will use such a topic in the case in which V is the variety parametrizing pairs of polynomials of simultaneous rank 1.

We work over the complex field \mathbb{C} . Let $\mathbb{G}(k - 1, N)$ be the Grassmannian of $(k - 1)$ -linear spaces in \mathbb{P}^N . Let $V \subset \mathbb{P}^N$ be a nondegenerate irreducible variety of dimension n and let

$$\Gamma_k(V) = \overline{\{(x_1, \dots, x_k, L) \in V \times \dots \times V \times \mathbb{G}(k - 1, N) \mid L = \langle x_1, \dots, x_k \rangle\}}.$$

Observe that $\Gamma_k(V)$ is birational to $V \times \dots \times V$, therefore it is irreducible of dimension kn . Let $\pi_2 : \Gamma_k(V) \rightarrow \mathbb{G}(k - 1, N)$ be the projection onto the last factor and set

$$S_k(V) = \pi_2(\Gamma_k(V)) = \overline{\{L \in \mathbb{G}(k - 1, N) \mid L \text{ is spanned by } k \text{ points of } V\}}.$$

Thanks to the Trisecant lemma—see for instance [6, Proposition 2.6]—the general $L \in S_k(V)$ meets V in exactly k points, so the general fiber of π_2 has dimension zero. Hence

$$\dim S_k(V) = \dim(\Gamma_k(V)) = kn.$$

We are ready to define the secant varieties of V .

Definition 3 Let $V \subset \mathbb{P}^N$ be a nondegenerate irreducible variety. The *abstract k -secant variety* of V is

$$\text{Sec}_k(V) = \overline{\{(x, L) \in \mathbb{P}^N \times \mathbb{G}(k - 1, N) \mid x \in L \text{ and } L \in S_k(V)\}}.$$

Let $p_1 : \text{Sec}_k(V) \rightarrow \mathbb{P}^N$ and $p_2 : \text{Sec}_k(V) \rightarrow S_k(V)$ be the two projections. The general fiber of p_2 is a linear space of dimension $k - 1$, therefore

$$\dim \text{Sec}_k(V) = \dim S_k(V) + k - 1 = kn + k - 1.$$

The *k -secant variety* of V is

$$\text{Sec}_k(V) = p_1(\text{Sec}_k(V)) = \overline{\bigcup_{x_1, \dots, x_k \in V} \langle x_1, \dots, x_k \rangle} \subset \mathbb{P}^N.$$

By construction, $\dim \text{Sec}_k(V) \leq \min\{\dim \text{Sec}_k(V), N\} = \min\{kn + k - 1, N\}$. The variety V is called *k -defective* if

$$\dim \text{Sec}_k(V) < \min\{kn + k - 1, N\}.$$

A classical result about secant varieties is Terracini’s lemma, proven in [22]—see [8, Section 2] for a modern reference.

Lemma 4 (Terracini) *Let $V \subset \mathbb{P}^N$ be a nondegenerate irreducible variety. If $x_1, \dots, x_r \in V$ are in general position and $z \in \langle x_1, \dots, x_r \rangle$ is a general point, then the embedded tangent space to $\text{Sec}_r(V)$ at z is*

$$\mathbb{T}_z \text{Sec}_r(V) = \langle \mathbb{T}_{x_1} V, \dots, \mathbb{T}_{x_r} V \rangle.$$

Terracini’s lemma allows us to link the study of secant varieties of V to the study of linear systems of divisors of V with imposed singularities. Indeed, let \mathcal{L} be the linear system on V of hyperplane sections. Then

$$\begin{aligned} \text{codim Sec}_r(V) &= \text{codim } \mathbb{T}_z \text{Sec}_r(V) \\ &= \dim\{H \subset \mathbb{P}^N \mid H \text{ is a hyperplane and } H \supset \mathbb{T}_z \text{Sec}_r(V)\} \\ &= \dim\{H \cap V \mid H \text{ is a hyperplane and } H \supset \mathbb{T}_z \text{Sec}_r(V)\} \\ &= \dim\{H \cap V \mid H \text{ is a hyperplane and } H \supset \mathbb{T}_{x_i} V \text{ for every } i \in \{1, \dots, r\}\} \\ &= \dim\{D \in \mathcal{L} \mid D \text{ is singular at } x_i \text{ for every } i \in \{1, \dots, r\}\}. \end{aligned}$$

Let us recall some definitions we will use when dealing with linear systems.

Definition 5 Let p be a point on a variety V defined by the ideal I_p and let $m \in \mathbb{N}$. The *point of multiplicity m* supported at p is the 0-dimensional subscheme of V defined by I_p^m . A point of multiplicity 2 is also called a *double point*.

Since we are going to work with systems of plane curves, we introduce the notation we use. We denote by

$$L_2^d(2^r, 1^t)$$

the vector space of degree d ternary forms vanishing at r fixed general points with multiplicity at least 2 and vanishing at t further points in general position. Its projectivization

$$\mathcal{L}_2^d(2^r, 1^t) = \mathbb{P}(L_2^d(2^r, 1^t))$$

is the linear system of degree d plane curves singular at r fixed general points and containing t base points in general position. More generally, given a linear system $\mathcal{L} = \mathbb{P}(L)$ on a variety V , we denote by $\mathcal{L}(2^r, 1^t)$ the linear subsystem of \mathcal{L} consisting of divisors singular at r fixed general points and vanishing at t further points in general position. When \mathcal{L} is the linear system embedding $V \subset \mathbb{P}^N$, that is, the linear system of hyperplane sections of V , Lemma 4 tells us that

$$\text{codim Sec}_r(V) = \dim L(2^r). \tag{3}$$

It is natural to ask for the dimension of the vector space $L_2^d(2^r, 1^t)$. When the base points have arbitrary multiplicity, that can be very hard to compute. However, when the multiplicities are at most 2, the answer is provided by the celebrated Alexander–Hirschowitz’ theorem. Such result is a landmark in the fields and holds in a projective space of any dimension, but here we only recall the weaker version on the plane, that is known at least since [4].

Theorem 6 *The dimension of $L_2^d(2^r)$ is $\max\left\{0, \binom{d+2}{2} - 3r\right\}$, unless $(d, r) \in \{(2, 2), (4, 5)\}$. In these exceptional cases we have $\dim L_2^2(2^2) = \dim L_2^4(2^5) = 1$.*

Since simple points in general position always impose independent conditions, Theorem 6 gives us a formula to compute $\dim L_2^d(2^r, 1^t)$. When handling such systems, we will need some control on their singularities. The following is a slight generalization of a result proven in [1].

Lemma 7 *Let d, r and t be natural numbers such that $(d, r, t) \neq (6, 9, 0)$. If*

$$\binom{d-1}{2} \geq r \text{ and } \binom{d+2}{2} - 1 \geq 3r + t,$$

then the general element of $L_2^d(2^r, 1^t)$ is irreducible, has exactly r ordinary double points and it is smooth elsewhere.

Proof By Theorem 6, the hypothesis $\binom{d+2}{2} - 1 \geq 3r + t$ guarantees that $L_2^d(2^r, 1^t)$ is not empty. We argue by induction on t . For $t = 0$, the claim is proven in [1, Theorem 3.2]. Now assume that $t \geq 1$. The system $L_2^6(2^9, 1^1)$ is empty by Theorem 6, so we assume that $(d, r) \neq (6, 9)$. By induction hypothesis, $L_2^d(2^r, 1^{t-1})$ contains a nonempty open subset U consisting of irreducible curves with exactly r ordinary double points. Imposing a further simple point in the base locus corresponds to taking a hyperplane section of $L_2^d(2^r, 1^{t-1}) \subset \mathbb{P}(\mathbb{C}[x_0, x_1, x_2]_d)$; if the hyperplane is general, the intersection with U is non-empty. □

There is one last thing we need to recall before we move to the next section. Let V be a projective variety and let Z be a subvariety of V . Consider a linear system $\mathcal{L} = \mathbb{P}(L)$ on V and let $\mathcal{L}_Z = \mathbb{P}(L_Z)$ be the complete linear system on Z given by $\mathbb{P}(H^0 \mathcal{O}_Z(D|_Z))$ for a general element $D \in \mathcal{L}$. Then there is an exact sequence of vector spaces

$$0 \rightarrow L \cap I_Z \rightarrow L \rightarrow L_Z. \tag{4}$$

The image of the rightmost map is denoted by $L|_Z$ and its projectivization is $L|_Z = \{D|_Z \mid D \in \mathcal{L}\}$. Sequence (4) is called *Castelnuovo exact sequence*.

3 Problem reduction

In this section we formalize the problem and we present some simplifications. Let us start by considering the Waring problem for one polynomial. Degree d ternary forms of Waring rank 1 are parametrized by the d -Veronese surface

$$V_2^d = \{[\ell^d] \mid \ell \in \mathbb{C}[x_0, x_1, x_2]_1\} \subset \mathbb{P} \binom{n+2}{2}^{-1} = \mathbb{P}(\mathbb{C}[x_0, x_1, x_2]_d)$$

embedded in the space of all forms of degree d . Then the rank of the general ternary form is

$$\min \left\{ r \in \mathbb{N} \mid \text{Sec}_r(\mathbb{V}_2^d) = \mathbb{P}^{\binom{d+2}{2}-1} \right\}.$$

Thanks to Lemma 4 and Eq. (3), the latter equals

$$\min \{ r \in \mathbb{N} \mid \dim L_2^d(2^r) = 0 \}.$$

This allows us to use geometric techniques to study Waring decompositions. For instance, we can apply Theorem 6 to compute the rank of the general ternary form of degree d .

Moreover, the set of decompositions with k summands of a polynomial $f \in \mathbb{P}^{\binom{d+2}{2}-1}$ is the fiber of the secant map $p_1 : \text{Sec}_k(\mathbb{V}_2^d) \rightarrow \mathbb{P}^{\binom{d+2}{2}-1}$ over f —see Definition 3.

We consider a similar construction for simultaneous decompositions. Let c and d be positive integers such that $c \leq d$. The variety parametrizing pairs of polynomials of degrees c and d and simultaneous rank 1 is

$$X = \{ [a_1 \ell^c, a_2 \ell^d] \mid \ell \in \mathbb{C}[x_0, x_1, x_2]_1 \text{ and } a_1, a_2 \in \mathbb{C} \} \subset \mathbb{P}(\mathbb{C}[x_0, x_1, x_2]_c \oplus \mathbb{C}[x_0, x_1, x_2]_d).$$

The map

$$\pi : X \rightarrow \mathbb{P}^2 = \mathbb{P}(\mathbb{C}[x_0, x_1, x_2]_1)$$

sending $[a_1 \ell^c, a_2 \ell^d]$ to $[\ell]$ gives X the structure of a projective bundle over the plane, where each fiber is isomorphic to \mathbb{P}^1 . Therefore X is a threefold. For the basic definitions and notions concerning projective bundles, we refer the reader to [12, Chapter II.7] or [10, Chapter 9]. It turns out that X is the projectivization of the following rank two vector bundle on \mathbb{P}^2 :

$$X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(c) \oplus \mathcal{O}_{\mathbb{P}^2}(d)) \subset \mathbb{P}^N = \mathbb{P}(\mathbb{C}[x_0, x_1, x_2]_c \oplus \mathbb{C}[x_0, x_1, x_2]_d),$$

where $N = \binom{c+2}{2} + \binom{d+2}{2} - 1$. The immersion $X \subset \mathbb{P}^N$ is the tautological embedding. Namely, the very ample divisor T_X associated with the immersion—called the *tautological divisor*—is the only divisor class on X such that

$$\pi_* \mathcal{O}_X(T_X) = \mathcal{O}_{\mathbb{P}^2}(c) \oplus \mathcal{O}_{\mathbb{P}^2}(d).$$

We recall that T_X is *unisecant*, that is, each fiber of π intersects T_X in exactly one point.

The situation is similar to the case of the Veronese variety. According to [2, Section 2.2], the set of decompositions with k summands of $(f, g) \in \mathbb{P}^N$ is the fiber of the secant map $p_1 : \text{Sec}_k(X) \rightarrow \mathbb{P}^N$ over (f, g) . For the generic pair of polynomials, in order to have finitely many decompositions, we have to assume that $\dim \text{Sec}_k(X) = \dim \mathbb{P}^N$. This leads to the following definition.

Definition 8 Let c and d be positive integers such that $c \leq d$. We say that (c, d) is a *perfect case* if there exists $k \in \mathbb{N}$ such that $\dim \text{Sec}_k(X) = \dim \mathbb{P}^N$. This is equivalent to

$3k + k - 1 = \binom{c+2}{2} + \binom{d+2}{2} - 1$, so (c, d) is perfect if and only if $\binom{c+2}{2} + \binom{d+2}{2}$ is a multiple of 4. In this case

$$k = \frac{\binom{c+2}{2} + \binom{d+2}{2}}{4} \in \mathbb{N}. \tag{5}$$

In order to have an idea of how frequent they are, we list here all perfect cases for $1 \leq c \leq d \leq 10$.

- (1, 5) (1, 8) (2, 2) (2, 3) (2, 10)
- (3, 3) (3, 10) (4, 5) (4, 8) (5, 9)
- (6, 6) (6, 7) (7, 7) (8, 9) (10, 10).

If (c, d) is not a perfect case, then $\dim \text{Sec}_r(X) \neq \dim \mathbb{P}^N$ for every $r \in \mathbb{N}$. In this case the generic fiber of the map $p_1 : \text{Sec}_r(X) \rightarrow \mathbb{P}^N$ cannot be zero-dimensional and therefore the general pair of polynomials is not identifiable. Even if (c, d) is a perfect case, the general fiber of p_1 may have positive dimension. If this happens, then the general point of \mathbb{P}^N has no preimages under p_1 , therefore the general pair of ternary forms has no simultaneous decompositions with r summands.

There is another family for which we can easily disprove identifiability by looking at the Waring rank of the higher-degree polynomial of the pair.

Proposition 9 *Let (c, d) be a perfect case and let k be the number defined in (5). If $\left\lceil \frac{1}{3} \binom{d+2}{2} \right\rceil > k$, then X is k -defective. In particular, the general pair of ternary forms of degrees c and d is not identifiable.*

Proof Let $(f, g) \in \mathbb{P}^N$ be a general pair of ternary forms of degrees c and d . Without loss of generality we assume that $d \neq 2$ and $d \neq 4$. Indeed, there are no perfect cases satisfying our hypothesis for these values of d . Since the values $d = 2$ and $d = 4$ are the only exceptions of Theorem 6, the number $\left\lceil \frac{1}{3} \binom{d+2}{2} \right\rceil$ is the Waring rank of g . As every simultaneous decomposition of (f, g) gives a decomposition of g , the simultaneous rank of (f, g) is at least the rank of g . By hypothesis, the latter is strictly greater than k . This means that the general pair has no decomposition with k summands. The image of the secant map $p_1 : \text{Sec}_k(X) \rightarrow \mathbb{P}^N$ has dimension smaller than N , hence X is k -defective. \square

Remark 10 In light of our observations, from now on we always suppose that $\binom{c+2}{2} + \binom{d+2}{2}$ is a multiple of 4 and that k is the natural number defined in (5). Without loss of generality, we also assume that $p_1 : \text{Sec}_k(X) \rightarrow \mathbb{P}^N$ is *dominant*—namely, that its image has dimension N . Under these assumptions, the domain and the image of p_1 have the same dimension, hence the general fiber of p_1 has dimension 0—we say that p_1 is *generically finite*. In other words, X is not k -defective and $\text{Sec}_k(X) = \mathbb{P}^N$. Thanks to Proposition 9, this tells us we can work under the assumption that $\left\lceil \frac{1}{3} \binom{d+2}{2} \right\rceil \leq k$. This implies that $\binom{d+2}{2} \leq 3k$, that is

$$d^2 + 3d \leq 3c^2 + 9c + 4.$$

Under this hypothesis, the only perfect cases with $c \leq 2$ are (2, 2) and (2, 3), the special cases appearing in the statement of Theorem 2. As we have already observed, they are known to be identifiable since the late nineteenth century. For this reason, from now on we suppose that $c \geq 3$. Thanks to [2, Theorem 5.1], we further suppose that $d \neq c + 1$. Under these assumptions, the only cases left in the range $1 \leq c \leq d \leq 10$ are

$$(4, 8) \quad (5, 9) \quad (6, 6) \quad (7, 7) \quad (10, 10).$$

Notice that we excluded the perfect case (3, 3). The case of two plane cubics has been classically studied in [16, 21]. If $c = d = 3$, then $k = 5$, $N = 19$ and X is 5-defective—see also [5, Remark 4.2]. In other words, $\dim \text{Sec}_5(X) < 19$, thus the secant map $p_1 : \text{Sec}_5(X) \rightarrow \mathbb{P}^{19}$ cannot be birational.

We deal with the map $p_1 : \text{Sec}_k(X) \rightarrow \mathbb{P}^N$, dominant and generically finite. Our goal is to prove that p_1 is not birational, that is, $\deg(p_1) \geq 2$. The following result, proven in [2, Theorem 5.2], allows us to reduce the problem.

Theorem 11 *Let $V \subset \mathbb{P}^N$ be a nondegenerate irreducible variety of dimension n and let $r \in \mathbb{N}$. Assume that the secant map $p_1 : \text{Sec}_r(V) \rightarrow \mathbb{P}^N$ is dominant and generically finite. Let $z \in \text{Sec}_{r-1}(V)$ be a general point. Consider the projection $\varphi : \mathbb{P}^N \dashrightarrow \mathbb{P}^n$ from the embedded tangent space $\mathbb{T}_z \text{Sec}_{r-1}(V)$. Then $\varphi|_V : V \dashrightarrow \mathbb{P}^n$ is dominant and generically finite, and $\deg(\varphi|_V) \leq \deg(p_1)$.*

We will apply Theorem 11 in the case when V is the projective bundle X and r is the number k defined in (5). In order to prove that $\deg(p_1) \geq 2$ it is enough to prove that $\deg(\varphi|_X) \geq 2$. We want to understand the linear system associated with $\varphi|_X$. The map φ described by Theorem 11 is the projection from the linear space $\mathbb{T}_z \text{Sec}_{k-1}(X)$. Once again we apply Lemma 4 to deduce that the linear system associated with $\varphi|_X$ is

$$\begin{aligned} & \{H \cap X \mid H \subset \mathbb{P}^N \text{ is a hyperplane and } H \supset \mathbb{T}_z \text{Sec}_{k-1}(X)\} \\ & = \{H \cap X \mid H \subset \mathbb{P}^N \text{ is a hyperplane and } H \supset \mathbb{T}_{x_i}(X) \text{ for every } i \in \{1, \dots, k-1\}\}. \end{aligned}$$

Keeping in mind that $H \supset \mathbb{T}_{x_i}(X)$ if and only if $H \cap X$ is singular at x_i , we are ready to define the linear system we are interested in.

Definition 12 Let $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(c) \oplus \mathcal{O}_{\mathbb{P}^2}(d)) \subset \mathbb{P}^N$. Let Σ be a 0-dimensional subscheme of X consisting of $k - 1$ points of multiplicity 2 in general position. We denote by

$$\mathcal{L} := \mathbb{P}(H^0 \mathcal{O}_X(T_X) \cap I_\Sigma)$$

the linear system of tautological divisors containing the subscheme Σ .

The linear system \mathcal{L} on X induces the rational map $\varphi|_X$. As we stressed in Remark 10, we work under the assumption that X is not k -defective, hence the map $p_1 : \text{Sec}_k(X) \rightarrow \mathbb{P}^N$ is dominant and generically finite. Thanks to Theorem 11, the map $\varphi|_X : X \dashrightarrow \mathbb{P}^3$ is also dominant and generically finite, so the dimension of \mathcal{L} is 3. Our task is to bound the degree of $\varphi|_X$.

A standard approach to work with linear systems is to degenerate them. In our case, we will pick some of the points of Σ in special position, rather than working with

general points. When we perform this kind of degenerations, we have to make sure that we can control the degree of the associated map.

Lemma 13 *Let $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(c) \oplus \mathcal{O}_{\mathbb{P}^2}(d)) \subset \mathbb{P}^N$ and fix $x_1, \dots, x_{k-1} \in X$. Let $\tilde{\Sigma}$ be the 0-dimensional subscheme of X consisting of $k - 1$ points of multiplicity 2 supported at x_1, \dots, x_{k-1} . Let $\tilde{\mathcal{L}} = \mathbb{P}(H^0 \mathcal{O}_X(T_X) \cap I_{\tilde{\Sigma}})$ and call $\tilde{\varphi}_{|X}$ the associated rational map. If $\dim \tilde{\mathcal{L}} = \dim \mathcal{L} = \dim X$, then $\deg(\tilde{\varphi}_{|X}) \leq \deg(\varphi_{|X})$.*

This follows from the more general [2, Lemma 5.4] and guarantees that we are allowed to degenerate some of the points of Σ in special position, as long as our degeneration does not change the dimension of the linear system. In our case, some of the points of Σ will belong to a given surface $Z \subset X$. In order to pick a suitable Z , consider the bundle morphism

$$\pi : X \rightarrow \mathbb{P}^2$$

and recall that the Picard group of X has rank 2. We choose as generators the tautological divisor T_X and the divisor $\pi^*(h)$, where $h \subset \mathbb{P}^2$ is a line. We set

$$Z = \{[0, \ell^d] \mid \ell \in \mathbb{C}[x_0, x_1, x_2]_1\} \subset X.$$

It is the section of π corresponding to the quotient

$$\mathcal{O}_{\mathbb{P}^2}(c) \oplus \mathcal{O}_{\mathbb{P}^2}(d) \rightarrow \mathcal{O}_{\mathbb{P}^2}(d) \rightarrow 0, \tag{6}$$

see [12, Exercise II.7.8]. In particular, Z is smooth and irreducible and the restriction $\pi|_Z : Z \rightarrow \mathbb{P}^2$ is an isomorphism. The tautological linear system on X embeds Z as a d -Veronese surface. The class of Z is

$$Z \sim T_X - c\pi^*(h), \tag{7}$$

see for instance [10, Proposition 9.13].¹

In order to bound $\deg(\varphi_{|X})$, we want to restrict the map to a suitable subvariety. We will show that this restriction does not increase the degree, provided that such a subvariety is not contained in the contracted locus of $\varphi_{|X}$.

Definition 14 The *contracted locus* of a rational map is the union of all positive-dimensional fibers. We denote by $\Delta \subset X$ the contracted locus of $\varphi_{|X}$.

Lemma 15 *Let $f : V \dashrightarrow W$ be a rational map between smooth irreducible varieties. Assume that f is dominant and generically finite. Let S be a subvariety of V . If S is not contained in the contracted nor in the indeterminacy locus of f , then $\deg(f|_S) \leq \deg(f)$.*

Proof Let $A = \{p \in W \mid f^{-1}(p) \text{ is finite}\}$. By hypothesis A is a nonempty open subset of W . Let $B \subset V$ be the indeterminacy locus of f and let $U = f^{-1}(A) \setminus B$. Then U is a nonempty open subset of V . By construction, the restriction $f_U : U \rightarrow f(U)$ is a finite surjective morphism and $\deg(f_U) = \deg(f)$. By [12, Exercise III.9.3(a)], f_U is flat. By

¹ The statement of [10, Proposition 9.13] is about a subbundle, instead of a quotient bundle. The reason is that [8, 12] use different conventions, as explained on [10, p. 324].

hypothesis $S \cap U \neq \emptyset$, so $S \cap U$ is a dense open subset of S . The flatness of f_U implies that $\deg(f_{U|S}) \leq \deg(f_U)$, so

$$\deg(f_{|S}) \leq \deg(f_{U|S}) \leq \deg(f_U) = \deg(f). \quad \square$$

Now we know that we can bound the degree by specializing some of the base points to Z and then restricting the map. Before we proceed, we have to understand a bit better the base points of our linear system \mathcal{L} . Although Σ is zero-dimensional, the base locus of \mathcal{L} contains many curves.

Lemma 16 *Let $x \in X$ and let $D \subset X$ be a divisor. If $D \sim T_x$ and D is singular at x , then $D \supset \pi^*(\pi(x))$.*

Proof Since the class T_x is unisecant, $D \cdot \pi^*(\pi(x)) = 1$, so the only possibility for $\text{mult}_x D$ to be greater than 1 is that D contains $\pi^*(\pi(x))$. \square

Concerning our degeneration approach, in order to work, we need to choose carefully the number of points we are going to degenerate. For this reason, we need to check a simple arithmetic property .

Lemma 17 *Let $c, d \in \mathbb{N}$. If $\binom{c+2}{2} + \binom{d+2}{2}$ is a multiple of 4, then*

$$3d^2 + 9d - c^2 - 3c - 12 \equiv 0 \pmod{8}.$$

Proof By hypothesis there exists $t \in \mathbb{N}$ such that $\binom{c+2}{2} + \binom{d+2}{2} = 4t$. This means that

$$c^2 + 3c + d^2 + 3d + 4 = 8t \Rightarrow 3d^2 + 9d + 3c^2 + 9c + 12 = 24t,$$

hence $3d^2 + 9d - c^2 - 3c - 12 = 8(3t - 3) - 4c(c + 3) \equiv 0 \pmod{8}$. \square

We are actually interested in the class of $3d^2 + 9d - c^2 - 3c - 12$ modulo 16. By Lemma 17, there are two possibilities: either $3d^2 + 9d - c^2 - 3c - 12 \equiv 0 \pmod{16}$ or $3d^2 + 9d - c^2 - 3c - 12 \equiv 8 \pmod{16}$. We use two different strategies to deal with these two cases.

4 The first case

The goal of this section is to prove Theorem 2 when

$$3d^2 + 9d - c^2 - 3c - 12 \equiv 0 \pmod{16}. \quad (8)$$

This is accomplished in Proposition 20. We start by setting up the degeneration we need. Under hypothesis (8) we define the integer

$$s_1 = \frac{3d^2 + 9d - c^2 - 3c - 12}{16}.$$

We want to make sure that $s_1 \in \{0, \dots, k - 1\}$. Thanks to Remark 10 we work under the assumption $d \geq c \geq 3$, so

$$s_1 \geq \frac{2c^2 + 6c - 12}{16} \geq 0.$$

Remark 10 also allows us to assume that $d^2 + 3d \leq 3c^2 + 9c + 4$, hence

$$\begin{aligned} k - 1 - s_1 &= \frac{c^2 + 3c + d^2 + 3d - 4}{8} - \frac{3d^2 + 9d - c^2 - 3c - 12}{16} \\ &= \frac{3c^2 + 9c - d^2 - 3d + 4}{16} \geq 0. \end{aligned}$$

Consider the linear system \mathcal{L} on X introduced in Definition 12. Since $s_1 \in \{0, \dots, k - 1\}$, we can degenerate s_1 of the $k - 1$ base points of \mathcal{L} in special position.

Definition 18 Let $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(c) \oplus \mathcal{O}_{\mathbb{P}^2}(d)) \subset \mathbb{P}^N$ and let $Z \subset X$ be the section defined by (6). Let z_1, \dots, z_{s_1} be general points of Z and let $x_{s_1+1}, \dots, x_{k-1}$ be general points of X ; in particular, $x_{s_1+1}, \dots, x_{k-1} \notin Z$. Let Σ_1 be the 0-dimensional subscheme of X consisting of $k - 1$ points of multiplicity 2 supported at $z_1, \dots, z_{s_1}, x_{s_1+1}, \dots, x_{k-1}$. Similarly to Definition 12, we define $L_1 := H^0 \mathcal{O}_X(T_X) \cap I_{\Sigma_1}$ and $\mathcal{L}_1 := \mathbb{P}(L_1)$. We call φ_1 the associated rational map.

In order to be able to perform the computations on \mathcal{L}_1 , we need to prove that this specialization does not increase the dimension of the linear system. In our case, the Castelnuovo exact sequence (4) becomes

$$0 \rightarrow L_1 \cap I_Z \rightarrow L_1 \rightarrow (L_1)_Z. \tag{9}$$

Now we are in position to describe both the right and the left hand sides of sequence (9) and to find their dimensions, thereby computing $\dim(\mathcal{L}_1)$.

Lemma 19 *In the specialization of Definition 18, we have*

- (1) $L_1 \cap I_Z \cong L_2^c(2^{k-1-s_1}, 1^{s_1})$ and it has dimension 1,
- (2) $(L_1)_Z \cong L_2^d(2^{s_1}, 1^{k-1-s_1})$ and it has dimension 3,
- (3) $\dim(L_1) = 4$,
- (4) $(L_1)_Z = L_{1|Z}$.

Proof We will use the bundle morphism $\pi : X \rightarrow \mathbb{P}^2$ to translate the question on our linear systems in terms of linear systems on the plane. Recall that π restricts to an isomorphism between Z and \mathbb{P}^2 . Let us start by showing that $L_1 \cap I_Z$ is isomorphic to a vector subspace of $L_2^c(2^{k-1-s_1}, 1^{s_1})$ of dimension at most 1. Take $D \in \mathcal{L}_1$ containing Z . Then there exists a divisor D' such that $D = Z + D'$. From (7) we obtain $D' \sim c\pi^*(h)$, so it projects to a plane curve of degree c . Notice that the general D has multiplicity 2 at $z_1, \dots, z_{s_1}, x_{s_1+1}, \dots, x_{k-1}$, while Z has multiplicity 1 at z_1, \dots, z_{s_1} and does not contain $x_{s_1+1}, \dots, x_{k-1}$. It follows that

D' has multiplicity 1 at z_1, \dots, z_{s_1} and multiplicity 2 at $x_{s_1+1}, \dots, x_{k-1}$. The correspondence $D \mapsto D'$ is an isomorphism, therefore, after the projection on \mathbb{P}^2 , we can regard elements of $\mathbb{P}(L_1 \cap I_Z)$ as plane curves of degree c singular at $\pi(x_{s_1+1}), \dots, \pi(x_{k-1})$ and passing through $\pi(z_1), \dots, \pi(z_{s_1})$. Hence $L_1 \cap I_Z$ is a vector subspace of $L_2^c(2^{s_1}, 1^{k-1-s_1})$. The latter has dimension 1 by Theorem 6, hence $\dim(L_1 \cap I_Z) \leq 1$.

In a similar way, now we prove that $(L_1)_Z$ is a vector subspace of $L_2^d(2^{s_1}, 1^{k-1-s_1})$ of dimension at most 3. Elements of $(\mathcal{L}_1)_Z$ have class $T_{X|Z}$, are singular at s_1 general points and pass through $k - 1 - s_1$ simple base points in general position. Indeed, by Lemma 16 the base locus $\text{Bs}(\mathcal{L}_1)$ contains not only Σ_1 , but also $k - 1$ general fibers. Since Z is a section of π , each of these fibers intersects Z in one point. Therefore curves in $(\mathcal{L}_1)_Z := \mathbb{P}((L_1)_Z)$ are not only singular at z_1, \dots, z_{s_1} , but they also contain $k - 1 - s_1$ simple base points in general position, namely the intersections of Z with the fibers $\pi^{-1}(\pi(x_{s_1+1})), \dots, \pi^{-1}(\pi(x_{k-1}))$.

Now we show that the linear system $|T_{X|Z}|$ corresponds isomorphically to $|dh|$ via the morphism π . Denote by $c_1(\mathcal{O}_{\mathbb{P}^2}(c) \oplus \mathcal{O}_{\mathbb{P}^2}(d))$ and $c_2(\mathcal{O}_{\mathbb{P}^2}(c) \oplus \mathcal{O}_{\mathbb{P}^2}(d))$ the first and the second Chern classes of the line bundle $\mathcal{O}_{\mathbb{P}^2}(c) \oplus \mathcal{O}_{\mathbb{P}^2}(d)$ —a good reference on Chern classes is [10, Chapter 5]. Recall that by the Whitney formula we have

$$c_1(\mathcal{O}_{\mathbb{P}^2}(c) \oplus \mathcal{O}_{\mathbb{P}^2}(d)) = (c + d)h \text{ and } c_2(\mathcal{O}_{\mathbb{P}^2}(c) \oplus \mathcal{O}_{\mathbb{P}^2}(d)) = cd(\text{pt}),$$

where pt indicates the class of a point of \mathbb{P}^2 . The fundamental relation

$$\begin{aligned} T_X^2 &\equiv T_X \cdot \pi^*(c_1(\mathcal{O}_{\mathbb{P}^2}(c) \oplus \mathcal{O}_{\mathbb{P}^2}(d))) - \pi^*(c_2(\mathcal{O}_{\mathbb{P}^2}(c) \oplus \mathcal{O}_{\mathbb{P}^2}(d))) \\ &= (c + d)T_X \cdot \pi^*(h) - cd\pi^*(\text{pt}), \end{aligned}$$

discussed for instance in [12, Section A.3], implies that

$$T_X \cdot Z = T_X^2 - c\pi^*(h) \cdot T_X = d\pi^*(h) \cdot T_X - cd\pi^*(\text{pt}).$$

It follows that $(\pi_{1Z})_* T_X \cdot Z \sim dh$, therefore $(L_1)_Z$ is isomorphic to a vector subspace of $L_2^d(2^{s_1}, 1^{k-1-s_1})$. Again by Theorem 6, the latter has dimension 3, so $\dim((L_1)_Z) \leq 3$. Since $\dim(L_1 \cap I_Z) \leq 1$ and $\dim((L_1)_Z) \leq 3$, exact sequence (9) gives

$$\dim L_1 \leq \dim(L_1 \cap I_Z) + \dim((L_1)_Z) \leq 1 + 3 = 4.$$

On the other hand, \mathcal{L}_1 is a degeneration of $\mathcal{L} = \mathbb{P}(L)$, so $\dim L_1 \geq \dim L \geq 4$ by semicontinuity. This proves the third claim.

As a consequence, $L_1 \cap I_Z$ and $(L_1)_Z$ have indeed dimension 1 and 3, respectively, otherwise $\dim(L_1)$ would be smaller than 4. This proves the first two claims.

We are left to prove the fourth part. Since $\dim(L_1) - \dim(L_1 \cap I_Z) = \dim(L_1)_Z$, the right-most arrow in the sequence (9) is surjective, hence $(L_1)_Z = L_{1|Z}$. \square

Now we know that the specialization we introduced in Definition 18 preserves the dimension of the linear system. In other words, $\dim \mathcal{L}_1 = \dim \mathcal{L}$, so the codomain of φ_1 is \mathbb{P}^3 and we can take advantage of Lemma 13. The following result proves Theorem 2 under hypothesis (8).

Proposition 20 *The map $\varphi_1 : X \dashrightarrow \mathbb{P}^3$ associated with \mathcal{L}_1 is not birational.*

Proof We want to apply Lemma 15 and show that $\varphi_{1|Z} : Z \dashrightarrow \mathbb{P}^2$ is not birational. For this purpose, the first thing we need is to show that $Z \not\subset \Delta$. Assume by contradiction that

$Z \subset \Delta$. By Lemma 19(2), the image of Z is a nondegenerate plane curve $Y \subset \mathbb{P}^2$, and images of divisors in $\mathcal{L}_{1|Z}$ are line sections of Y . This implies that the general element of $\mathcal{L}_{1|Z}$ is reducible. It follows by Lemma 19(4) that the general element of $\mathcal{L}_2^d(2^{s_1}, 1^{k-1-s_1})$ is reducible, and this contradicts Lemma 7. The case $(6, 9, 0)$ cannot happen because $(2, 6)$ is not perfect. Moreover, since $\dim L_1 = 4$ by Lemma 19(3), we see that Z is not in the base locus of \mathcal{L}_1 by Lemma 19(1), hence it is not contained in the indeterminacy locus of φ_1 .

Now we only have to prove that $\varphi_{1|Z} : Z \dashrightarrow \mathbb{P}^2$ is not birational. It suffices to show that the general element of $\mathcal{L}_2^d(2^{s_1}, 1^{k-1-s_1})$ is not a rational curve. Thanks to Lemma 7, the general element of $\mathcal{L}_2^d(2^{s_1}, 1^{k-1-s_1})$ is irreducible and is singular at exactly s_1 ordinary double points, so it has genus

$$\frac{(d-1)(d-2)}{2} - s_1 = \frac{5d^2 - 33d + c^2 + 3c + 28}{16}.$$

Under the assumptions we made in Remarks 10, together with hypothesis (8), it is not restrictive to suppose that $c \geq 4$ and $d \geq 6$. Then we can bound the genus as

$$\frac{5d^2 - 33d + c^2 + 3c + 28}{16} \geq \frac{5d^2 - 33d + 56}{16} > 0.$$

□

5 The second case

The goal of this section is to prove Theorem 2 when

$$3d^2 + 9d - c^2 - 3c - 12 \equiv 8 \pmod{16}. \tag{10}$$

This is accomplished in Proposition 25. Under the assumptions we made in Remark 10, the only case satisfying (10) with $c \leq 7$ is $(3, 3)$ which, as we have already observed, cannot be identifiable. Therefore in this section we assume that $c \geq 8$. As in Sect. 4, we start by setting up the degeneration we need. Under hypothesis (10) we can define the integer

$$s_2 = \frac{3d^2 + 9d - c^2 - 3c - 4}{16}.$$

Just as we did in Sect. 4 for s_1 , it is easy to check that $s_2 \in \{0, \dots, k-1\}$. Consider the linear system \mathcal{L} introduced in Definition 12. We degenerate s_2 of the $k-1$ base points of \mathcal{L} in special position.

Definition 21 Let $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}(c) \oplus \mathcal{O}_{\mathbb{P}^2}(d)) \subset \mathbb{P}^N$ and let $Z \subset X$ be the section given by (6). Let z_1, \dots, z_{s_2} be general points of Z and let $x_{s_2+1}, \dots, x_{k-1}$ be general points of X . Let Σ_2 be the 0-dimensional subscheme of X consisting of $k-1$ points of multiplicity 2 supported at $z_1, \dots, z_{s_2}, x_{s_2+1}, \dots, x_{k-1}$. We define $L_2 := H^0 \mathcal{O}_X(T_X) \cap I_{\Sigma_2}$ and $\mathcal{L}_2 = \mathbb{P}(L_2)$. We call φ_2 the associated rational map.

Again, our first concern is to check that the degeneration presented in Definition 21 does not increase the dimension of the linear system. In other words, we want to prove that $\dim(L_2) = 4$.

Lemma 22 *In the specialization of Definition 21, we have*

- (1) $(L_2)_Z \cong L_2^d(2^{s_2}, 1^{k-1-s_2})$ and it has dimension 2,
- (2) $L_2 \cap I_Z \cong L_2^c(2^{k-1-s_2}, 1^{s_2})$ and it has dimension 2,
- (3) $\dim(L_2) = 4$,
- (4) $(L_2)_Z = L_{2|Z}$.

Proof The proof goes exactly as in Lemma 19. We only have to check that both $L_2^d(2^{s_2}, 1^{k-1-s_2})$ and $L_2^c(2^{k-1-s_2}, 1^{s_2})$ have dimension 2. □

Remark 23 As a byproduct of Lemmas 19 and 22, we obtain that $\dim \mathcal{L} = 3$, even without the assumption that p_1 is dominant and generically finite. This means that, whenever (c, d) is a perfect case, $k - 1$ general double points impose independent conditions on $\mathcal{O}_X(T_X)$. By Lemma 4 and equation (3), X is not $(k - 1)$ -defective.

The main difference between situation (8) and situation (10) is that in this second case $(\mathcal{L}_2)_Z$ induces a map to \mathbb{P}^1 , instead of \mathbb{P}^2 . Therefore $Z \subset \Delta$, in this setting. This means that we cannot apply Lemma 15 to Z , but rather we will find another suitable subvariety. Observe that if $T \in \mathbb{P}(L_2 \cap I_Z)$, then $T \sim Z + V$ for some element $V \sim c\pi^*(h)$. By Lemma 22(2), there is a pencil of such V 's.

Lemma 24 *Let B be a general element of $\mathcal{L}_2^c(2^{k-1-s_2}, 1^{s_2})$ and $V = \pi^*B \subset X$. Let T be a general element of \mathcal{L}_2 . Then*

- (1) V is irreducible, it is not contained in the contracted locus Δ and $\overline{\varphi_2(V)} = \mathbb{P}^2$.
- (2) $T \not\supset V$ and $T \cap V \not\subset \Delta$. Moreover $\varphi_2(T \cap V) = \mathbb{P}^1$.

Proof Since V corresponds to a general element of $\mathcal{L}_2^c(2^{k-1-s_2}, 1^{s_2})$, it is irreducible by Lemma 7. By construction we have $Z \subset \Delta$. As $\text{codim } \Delta \geq 1$, we can choose V so that $V \not\subset \Delta$. Moreover, we can also assume that V is not contained in the indeterminacy locus of φ_2 . Hence $\dim(\varphi_2(V)) = 2$. Observe that $V + Z \in \mathcal{L}_2$, so $\overline{\varphi_2(V \cup Z)} = \mathbb{P}^2$. Since $\dim(\varphi_1(Z)) = 1$, the image of V is \mathbb{P}^2 . This completes the proof of the first claim.

Now assume by contradiction that the general element T of \mathcal{L}_2 contains V . Then $L_2 \cap I_Z$ would have only one element, up to scalar. Namely, $L_2 \cap I_Z = \langle V + Z \rangle$. This contradicts Lemma 22(2). Since T does not contain V , the intersection is a curve on V . If such a curve was contained in Δ for a general $T \in \mathcal{L}_2$, then $V \subset \Delta$, in contradiction to part (1). Finally, $T \cap V$ is an element of $\mathcal{L}_{2|V}$. AS $\overline{\varphi_2(V)} = \mathbb{P}^2$, we have $\overline{\varphi_2(T \cap V)} = \mathbb{P}^1$. □

We are interested in $T \cap V$. This is not an irreducible curve, because both T and V contain the $k - 1$ fibers of the base locus of $\mathcal{L}_2^c(2^{k-1-s_2}, 1^{s_2})$ via π . However, $T \sim T_X$ is a uniseccant divisor, that is T either contains a fiber of π or T intersects a fiber precisely in one point, hence there exists a unique horizontal component of $T \cap V$. We define C to be such an irreducible component. As T is general, $C \not\subset \Delta$ and C is not contained in the indeterminacy locus of φ_2 . Our strategy to bound $\deg(\varphi_2)$ is to restrict the map to C .

Proposition 25 *The map $\varphi_2 : X \dashrightarrow \mathbb{P}^3$ associated with \mathcal{L}_2 is not birational.*

Proof As in Lemma 24, we define B as a general element of $\mathcal{L}_2^c(2^{k-1-s_2}, 1^{s_2})$ and T a general element of \mathcal{L}_2 . We want to apply Lemma 15 and show that $\varphi_{2|C} : C \dashrightarrow \mathbb{P}^1$ is not birational. We only need to prove that C is not a rational curve. Since $\pi|_T$ is birational, it restricts to a birational map between C and B , hence they have the same genus. By Lemma 7, the curve B has only ordinary double points, so its genus is

$$\binom{c-1}{2} - (k-1-s_2) = \frac{5c^2 + d^2 - 33c + 3d + 20}{16} \geq \frac{6c^2 - 30c + 20}{16} \geq 1$$

for every $c \geq 8$. □

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